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Differential Equations

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Imaginary Coefficients

In the peer reviewed article *The linear differential equations with complex constant coefficients and Schrödinger equations* by Soon-Mo Jung and Jaiok Roh they demonstrate a possible solution to a second-order inhomogeneous linear differential equations, $y''(x) + \alpha y'(x) + \beta y(x) = r(x)$, with complex constant coefficients, by looking at some characteristics of approximate solutions to these kind of differential equations. As a practical example to this method that was applied to the time-independent Schrödinger's equations that is commonly used in quantum mechanics to describe the behavior of electrons when there is no observer.

In order to prove the existence of solution to a second-order inhomogeneous linear differential equation with complex constant coefficients, $y''(x) + \alpha y'(x) + \beta y(x) = r(x)$, where α and β are complex-valued constants and r is a continuous function with a complex-valued output. Denoting λ and μ are the roots to the characteristic equation, $x^2 + \alpha x + \beta = 0$, where p = $\Re(\lambda)$ and q = $\Re(\mu)$. What was done was that they gave a proof for a method to show the existence of a solution to these types of differential equations for all possible values of p and q.

In Theorem 2.1, they show, assuming that both p and q are positive numbers, if one lets I be any open interval between $-\infty$ and ∞. For a second-order continuous differential equation with complex constants y and a continuous function with complex values r, we define the following functions with x being a value inside I:

$$g(x) = y'(x) - \mu y(x), \quad z(x) = \lim_{s \to b} \left(g(s)e^{-\lambda(s-x)} - e^{\lambda x} \int_x^s r(t)e^{-\lambda t} dt \right)$$

With this the following exists for any value of x inside I:

$$\int_x^b r(t)e^{-\lambda t}dt, \quad \int_x^b z(t)e^{-\mu t}dt, \quad \lim_{s\to b} g(s)e^{-\lambda s}, \quad \lim_{s\to b} y(s)e^{-\mu s}$$

They explain that, given that $\varepsilon \geq 0$, assuming that y satisfies the following inequality for all x inside I:

$$|y''(x) + \alpha y'(x) + \beta y(x) - r(x)| \leq \varepsilon$$

Then, there should exist a complex-valued solution u to the differential equation for all values of x inside I. This is shown in the following equation.

$$|y(x) - u(x)| \leq \frac{\varepsilon}{pq}$$

To demonstrate this method he first differentiates z with respect to x, in order to find that z is a first-order differential equation when in terms of g, but a second-order differential equation when in terms of y for any x inside I. This is shown below.

$$z'(x) = \lambda z(x) + r(x)$$

Then, in order to show that g is a solution to the general form of the second-order inhomogeneous differential equation y and in turn showing that it is a solution to z. Turned g into the form of a first-order differential equation and plugged in g that was previous defined in terms of y and doing this found that it could be turned into the the form of the second-order differential equation defined in terms of y.

$$g'(x) = \lambda g(x) + r(x) \ = \ y''(x) + \alpha y'(x) + \beta y(x) - r(x)$$

Since these are equal that means that g the solution to the first-order differential equation g is also the solution to the second-order differential equation y. Now, since previously z is shown to be a second-order differential equation when in terms of y, this means that g is also a solution to z.

Then, assuming that the differential equation $y \leq \varepsilon$ we get

$$|g'(x) = \lambda g(x) + r(x)| \ = \ |y''(x) + \alpha y'(x) + \beta y(x) - r(x)| \leq \varepsilon$$

for when x is inside I.

Since, it has already been shown that z is a second-order inhomogeneous linear differential equation with complex constant coefficients and g is a solution to this differential equation we may proceed to see if $|z(x) - g(x)| \leq \varepsilon$. With this we show that λ exists as a solution for the differential equation. The following is the work show in the article.

$$
\begin{aligned}
|z(x) - g(x)| &= \left| e^{\lambda x} \lim_{s \to b} \left(g(s)e^{-\lambda s} - g(x)e^{-\lambda x} - \int_x^s r(t)e^{-\lambda t} dt \right) \right| \\
&= e^{px} \lim_{s \to b} \left| \int_x^s \left(g(t)e^{-\lambda l} \right)' dt - \int_x^s r(t)e^{-\lambda t} dt \right| \\
&\leq e^{px} \lim_{s \to b} \int_x^s \left| e^{-\lambda t} \right| \left| g'(t) - \lambda g(t) - r(t) \right| dt \\
&\leq \frac{\varepsilon}{p} \lim_{s \to b} \left(1 - e^{-p(s-x)} \right) \leq \frac{\varepsilon}{p}.
\end{aligned}
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