Christian Nunez

Stanley Huddy

Differential Equations

12 December 2022

Imaginary Coefficients

In the peer reviewed article *The linear differential equations with complex constant coefficients and Schrödinger equations* by Soon-Mo Jung and Jaiok Roh they demonstrate a possible solution to a second-order inhomogeneous linear differential equations

$$y''(x) + \alpha y'(x) + \beta y(x) = r(x)$$

with complex constant coefficients, by looking at some characteristics of approximate solutions to these kind of differential equations. As a practical example to this method that was applied to the time-independent Schrödinger's equations that is commonly used in quantum mechanics to describe the behavior of electrons when there is no observer.

In order to prove the existence of solution to a second-order inhomogeneous linear differential equation with complex constant coefficients, $y''(x) + \alpha y'(x) + \beta y(x) = r(x)$, where α and β are complex-valued constants and r is a continuous function with a complex-valued output. Denoting λ and μ are the roots to the characteristic equation, $x^2 + \alpha x + \beta = 0$, where p is equal to the real part of λ and q is equal to the real part of μ. What was done was that a method to show the existence of a solution to these types of differential equations was made, then in order to prove that this method works it was proven for all possible cases of p and q that could arise in a problem.

In Theorem 2.1, they show, assuming that both p and q are positive numbers, if one lets I be any open interval between $-\infty$ and ∞. For a second-order continuous differential equation with complex constants y and a continuous function with complex values r, we define the following functions with x being a value inside I:

$$g(x) = y'(x) - \mu y(x), \quad z(x) = \lim_{s\to b}\left(g(s)e^{-\lambda(s-x)} - e^{\lambda x}\int_x^s r(t)e^{-\lambda t}dt\right)$$

With this the following exists for any value of x inside I:

$$\int_x^b r(t)e^{-\lambda t}dt, \quad \int_x^b z(t)e^{-\mu t}dt, \quad \lim_{s\to b} g(s)e^{-\lambda s}, \quad \lim_{s\to b} y(s)e^{-\mu s}$$

They explain that, given that $\varepsilon \geq 0$, assuming that y satisfies the following inequality for all x inside I:

$$|y''(x) + \alpha y'(x) + \beta y(x) - r(x)| \leq \varepsilon$$

Then, there should exist a complex-valued solution u to the differential equation for all values of x inside I. This is shown in the following equation.

$$|y(x) - u(x)| \leq \frac{\varepsilon}{pq}$$

To demonstrate this method he first differentiates z with respect to x, in order to find that z is a first-order differential equation when in terms of g, but a second-order differential equation when in terms of y for any x inside I. This is shown below.

$$z'(x) = \lambda z(x) + r(x)$$

Then, in order to show that g is a solution to the general form of the second-order inhomogeneous differential equation y and in turn showing that it is a solution to z. Turned g into the form of a first-order differential equation and plugged in g that was previous defined in terms of y and doing this found that it could be turned into the the form of the second-order differential equation defined in terms of y.

$$g'(x) = \lambda g(x) + r(x) \ = \ y''(x) + \alpha y'(x) + \beta y(x) - r(x)$$

Since these are equal that means that g the solution to the first-order differential equation g is also the solution to the second-order differential equation y. Now, since previously z is shown to be a second-order differential equation when in terms of y, this means that g is also a solution to z. Then, assuming that the differential equation $y \leq \varepsilon$ we get

$$|g'(x) = \lambda g(x) + r(x)| \;=\; |y''(x) + \alpha y'(x) + \beta y(x) - r(x)| \leq \varepsilon$$

for when x is inside I.

Since, it has already been shown that z is a twice differential linear inhomogeneous function equation with complex constant coefficients and g is a solution to this differential equation we may proceed to see if $|z(x) \;-\; g(x)| \;\leq\; \frac{\varepsilon}{pq}$. With this we show that λ exists as a solution for the differential equation. The following is the work shown in the article.

$$\begin{aligned}
|z(x) - g(x)| &= \left| e^{\lambda x} \lim_{s \to b} \left(g(s)e^{-\lambda s} - g(x)e^{-\lambda x} - \int_x^s r(t)e^{-\lambda t} dt \right) \right| \\
&= e^{px} \lim_{s \to b} \left| \int_x^s \left(g(t)e^{-\lambda l} \right)' dt - \int_x^s r(t)e^{-\lambda t} dt \right| \\
&\leq e^{px} \lim_{s \to b} \int_x^s \left| e^{-\lambda t} \right| \left| g'(t) - \lambda g(t) - r(t) \right| dt \\
&\leq \frac{\varepsilon}{p} \lim_{s \to b} \left(1 - e^{-p(s-x)} \right) \leq \frac{\varepsilon}{p}.
\end{aligned}$$

In order to show that μ exists as a solution to the differential equation as well. A similar process was used however instead of using z and g they defined for all x inside I solution to the differential equation y:

$$u(x) = \lim_{s \to b} \left(y(s)e^{-\mu(s-x)} - e^{\mu x} \int_x^s z(t)e^{-\mu t} dt \right)$$

Similarly, he differentiates this variable to get

$$u'(x) = \mu u(x) + z(x), \text{ for all x inside } I$$

Then, using the definition of z and the definition of u' just found you can show that the function u' is a second-order differential equation for all x inside I

$$u''(x) + \alpha u'(x) + \beta u(x) = r(x)$$

with this implying that u is a solution to a differential equation.

With this we can prove the existence of the solution μ with a similar method as in proving λ, however what was proven in this case was $|y(x) - u(x)| \leq \frac{\varepsilon}{pq}$, because u is a solution to y just like g is a solution to z. It seems it was done this way so that the math would end up less messy to prove that the inequality is true. The following is the work shown for this proof.

$$\begin{aligned}
|y(x) - u(x)| &= \left| y(x) - \lim_{s\to b} \left(y(s)e^{-\mu(s-x)} - e^{\mu x} \int_x^s z(t)e^{-\mu l} dt \right) \right| \\
&= |e^{\mu x}| \lim_{s\to b} \left| y(x)e^{-\mu x} - y(s)e^{-\mu s} + \int_x^s z(t)e^{-\mu l} dt \right| \\
&= e^{qx} \lim_{s\to b} \left| \int_x^s z(t)e^{-\mu l} dt - \int_x^s \left(y(t)e^{-\mu l}\right)' dt \right| \\
&= e^{qx} \lim_{s\to b} \left| \int_x^s e^{-\mu l} \left(z(t) - y'(t) + \mu y(t)\right) dt \right| \\
&= e^{qx} \lim_{s\to b} \left| \int_x^s e^{-\mu l} (z(t) - g(t)) dt \right| \leq e^{qx} \lim_{s\to b} \int_x^s e^{-qt} \frac{\varepsilon}{p} dt \\
&\leq \frac{\varepsilon}{pq} \lim_{s\to b} \left(1 - e^{-q(\sigma - x)}\right) \leq \frac{\varepsilon}{pq}, \qquad \text{for all } x \text{ inside } I.
\end{aligned}$$

The article then processes in Theorem 2.2, to prove the rest of the possible cases of p and q that could occur. They start off by using same definitions and assumptions of I, $y(x)$, $r(x)$, $g(x)$ and $z(x)$ and used in Theorem 2.1. However, they assume that p and q are nonnegative numbers now, so with this they show that a solution u, with an input of a value in I and an output of a number in the complex plane, to the differential equation exists for all x inside I when the inequality for its respective case of p and q is true.

$$|y(x) - u(x)| \leq \begin{cases} \frac{\varepsilon}{pq} & (\text{ for } p, q > 0), \\ \frac{\varepsilon}{q}(b - a) & (\text{ for } p = 0, q > 0), \\ \frac{\varepsilon}{2}(b - a)^2 & (\text{ for } p = q = 0) \end{cases}$$

For when $p = 0$ and $q > 0$, when doing the same procedure to prove λ as in Theorem 2.1, the following was found for all x in I

$$|z(x) - g(x)| \leq (b - x)\varepsilon$$

The was also done to prove the existence of μ to get

$$\begin{aligned} |y(x) - u(x)| &= e^{qx} \lim_{s\to b} \left| \int_x^s e^{-\mu l}(z(t) - g(t))dt \right| \\ &\leq e^{qx} \lim_{s\to b} \int_x^s e^{-qt}\varepsilon(b-t)dt \leq \frac{(b-a)\varepsilon}{q}, \quad \text{for all } x \text{ inside } I. \end{aligned}$$

Works Cited

Abell, Martha L., and James P. Braselton. "Chapter 4 - Higher Order Equations." *Introductory Differential Equations (Fourth Edition)*, edited by AbellBraselton, Fourth Edition, Academic P, 2014, pp. 131–225, https://doi.org/https://doi.org/10.1016/B978-0-12-417219-7.00004-1.

Jung, Soon-Mo, and Jaiok Roh. "The linear differential equations with complex constant coefficients and Schrödinger equations." *Applied Mathematics Letters*, vol. 66, 2017, pp. 23–29. https://doi.org/https://doi.org/10.1016/j.aml.2016.11.003.