Mathematics for the Modern World:

Standards for a Mathematically Literate Society

April 2021

CENTER FOR
CURRICULUM
REDESIGN

Prepared by the Center for Curriculum Redesign's Maths standards team:

Maya Bialik, Emma Smith Zbarsky, Tina Cardone, Charles Fadel
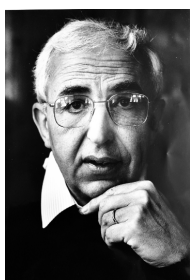
With generous and critical help from:

Rachael Whitney-Smith (ACARA), Veronica Mariño Salazar (U. of Los Andes), Janet Slesinski (CCR), Carrie Ott (CCR), Will Morony (ACARA), Neil Marshall (NZQA), Andrew Lorimer-Derham (ACARA), David Leigh-Lancaster (State of Victoria), Sharyn Sweeney (CCR), Zbigniew Marciniak (Warsaw U.), Sanjoy Mahajan (MIT), Julian Patricio Mariño Von Hildebrand (U. of Los Andes), and the team at Cambridge Mathematics (UK).

CCR is *deeply grateful*, for their trust and support, to:

Table of Contents

Section 1: Math for All

The Modern World

The modern world for which schools must prepare students bears little resemblance to the world for which our education systems were designed. The first challenge is the lack of a cohesive vision for systemic improvement[1]; without a cohesive plan, each effort, even if successful, gets lost in the noise of all of the other efforts. The Center for Curriculum Redesign endeavors to answer the question, "What should students learn for the 21st Century?" by defining the Knowledge, Skills, Character, and Meta-Learning abilities that will prepare students for their personal, civic, and professional lives.[2] And yet, even with a set of goals in mind, it is extremely challenging to overcome the inertia in the field of education due to the confluence of many forces perpetuating the present system.[3]

Society has undergone massive changes in recent times, which have transformed the way math is used and thus must be taught. Much of the procedural aspects of mathematics are no longer performed by humans. The phrase "you won't have a calculator with you every day" has become patently false, as we now all carry with us not only calculators but supercomputers and libraries and teachers in our pockets *and in the cloud*. Yet curriculum remains firmly rooted in its tradition and history, extending all the way back to ancient times.

Further, math is a discipline that builds on itself in a fundamental way; it is not possible to simply skip learning multiplication, for example, and continue to Algebra. If students do not learn what they are meant to at a given time, they fall behind, which causes them to continue to fall behind.

On the other hand, many occupations that did not used to require math are beginning to incorporate quantitative methods into their expectations. Figure 1[4] below shows the relative importance of math based on O*NET and Bureau of Labor Statistics data about importance and number of people employed in jobs that require different traditional knowledge areas. Most important are speaking and reading comprehension, understandably. But following not far behind are Math and writing, which are almost equal. After that, there is a sharp drop off to the rest of the subjects.

---

[1] See "*Four-Dimensional Education*" (2015)

[2] See "*Knowledge for the Age of Artificial Intelligence*" (2018) and "*Four-Dimensional Education*" (2015)

[3] See "Overcoming System Inertia in Education Reform" (2017)

[4] https://curriculumredesign.org/onetexplorer_raw/ Purple represents data from 2016; Orange represents the projection for 2026

However, being "well-versed" in math is very poorly defined. Does it mean being able to do mental math or calculus? In Figure 2, each dot represents an occupation, and is plotted on the graph to indicate how important math is, and the level of math required. A 6 in importance corresponds to deriving a complex equation, and a 1 corresponds to adding two numbers.
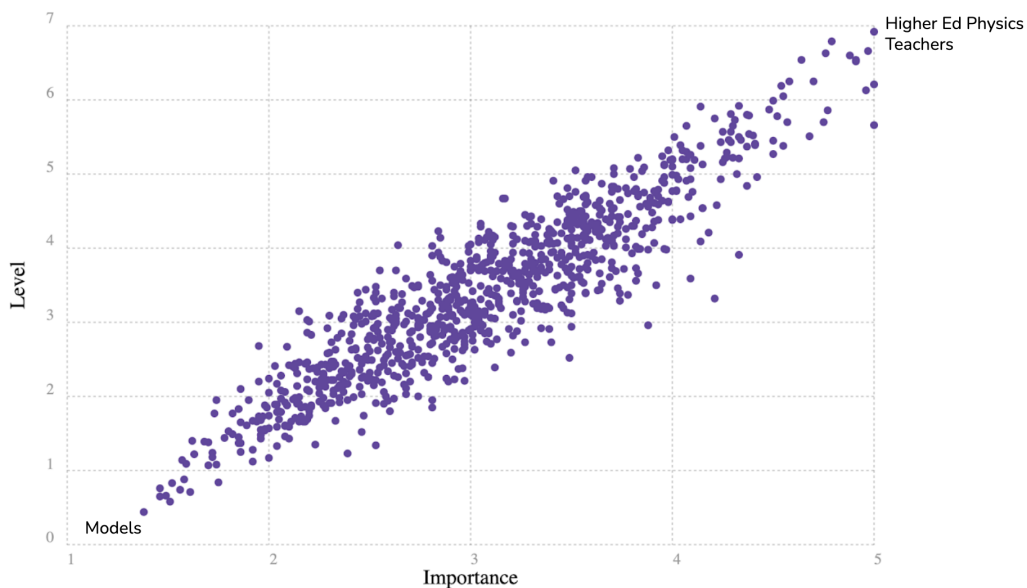
The first thing we can see is that there is a clear positive correlation, that is, the more important math is to a profession, the more likely it will be a higher level of math. But the other important thing to

notice is that **the majority of occupations requiring math, require a medium level of math, roughly corresponding to "Analyze data to determine areas with the highest output".** Mathematics is a wide and varied field, and in preparing students for the workforce, it is important to not get swept up in traditional notions of mathematics education, but rather, design for the world of the future.

The current moment adds one more layer to the challenge at hand; Covid-19 has forced schools to figure out how to continue education with many students not physically present at school. In many cases, this necessarily means less instructional time. So, how can we decide what to prioritize given that we simply cannot cover everything?

Even in normal times, there has been a vast chasm between the intended curriculum and the implemented curriculum, due to the insufficient time available for all of the assigned (and ever-expanding!) content. The situation has only been exacerbated by COVID-19. Some jurisdictions have taken steps to offer guidance on which topics to trim and which to focus on. In this paper, CCR has taken this one step further and fundamentally re-imagined math standards as a whole.

Although this document focuses exclusively on standards, CCR recognizes that a change in standards alone is not enough to result in meaningful change. While we offer some starting points, it will be crucial to build upon this work with aligned curriculum, courseware, and assessment.

Goals for All

For whom is the mathematics curriculum designed? If one were to look solely at how mathematics is treated in schools, they might conclude that math is really only needed for those who continue on to a STEM (Science, Technology, Engineering, Mathematics) career, and the only reason it is taught to *everyone* is in order to identify those that may be able to utilize it professionally. The curriculum, in terms of its structure, prepares students for Calculus, which students are pressured to take. Why?

> Calculus is the quintessence of high school success; it represents prestige for parents. Guidance counselors often suggest that Calculus is the key to university admission. The College Board, which administers the AP exams, suggests it's a way to purchase inexpensive university credits. Politicians think that Calculus enrollment is a simple way to measure school quality, which causes schools to push more students into Calculus. Reform organizations like the National Math and Science Initiative make increasing Calculus enrollment a goal ... *without asking whether it's a sensible goal*.[5]

---

[5] https://www.forbes.com/sites/johnewing/2020/02/15/should-i-take-calculus-in-high-school/?sh=c77252776253

Worries about inequities are focused on the students who "fail to" make it in a STEM track, and those students, rather than receiving a math education that would prepare them for how *they* might *actually* use math, receive remedial math instruction, still attempting to push them as far as possible along the Calculus track (and further solidifying their distaste for anything described as "math").

Besides the inherent judgment of non-STEM careers to be inferior to STEM careers, there are a number of other problems with this conceptualization. CCR believes that there *is* value to mathematics education for all, and in fact, ***it is crucial that *all* students internalize the important takeaways of math*, without necessarily mastering all the procedures.** By identifying the ultimate goals of each item in a set of math standards, we can begin to trace which content is only necessary for those going into STEM and begin to make room for the mathematics required for *all citizens of the 21st century.*

So what *should* be the takeaways of math education by those who do *not* go on to a STEM career (the vast majority of students)? The word "literacy" has been adopted from discussions about the reading and writing skills that all citizens need to participate in society, to refer to a set of minimal skills within any given particular modality[6] that are absolutely necessary for all citizens to possess (and thus the responsibility of one's school to instill). So what are the minimal necessary skills for all citizens when it comes to math?

> Quantitatively literate citizens need to know more than formulas and equations. They need a predisposition to look at the world through mathematical eyes, to see the benefits (and risks) of thinking quantitatively about commonplace issues, and to approach complex problems with confidence in the value of careful reasoning. Quantitative literacy empowers people by giving them tools to think for themselves, to ask intelligent questions of experts, and to confront authority confidently.[7]

The OECD defined[8] Mathematical Literacy as:

> "…an individual's capacity to [bullet point structure added]
> - identify and understand the role that mathematics plays in the world,
> - to make well-founded judgements and to use and
> - engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen."

---

[6] Including Information Literacy, Environmental Literacy, etc.
[7] Lynn Steen, Mathematics and Democracy
[8] Organisation for Economic Co-operation and Development. (2013).Pisa 2012 assessment and analytical framework: Mathematics, reading, science, problem solving and financial literacy. OECD Publishing. doi:10.1787/9789264190511-en

These are both great descriptions of what we hope students will take away from Mathematics education, yet *math curricula continue to get bogged down in technical/procedural details of Mathematics.*

This is not unique to mathematics — all disciplines fall in this trap; for instance, in the traditional sense of the word literacy, some language curricula have fallen into a similar trap of focusing on drilling grammar rules. We believe that rather than drilling grammar rules, a more useful approach would be to instill in students an understanding of linguistics so they can recognize and understand grammars in many different languages and settings if they wish. Similarly, we believe the goal of mathematics education is to teach students the skills they need to navigate the world through a mathematical lens.

Many mathematicians and some math educators will stress the importance of learning the technical details in order to gain a deeper understanding with the same certainty that parents stress eating vegetables before having dessert. But *is* procedural learning truly a necessary prerequisite to attain deeper mathematical learning?

A consensus is growing that the answer is no. Programs and curricula that aim to teach advanced topics of math, such as calculus and topology, to small children are cropping up, showing that the long-held wisdom about the inaccessibility of higher-level math may be based on flawed reasoning.[9] Namely, it is true that for those going into a STEM track, it is best to learn the prerequisites to their full depth before moving forward. But for the majority of students, who aren't on the STEM track, it is absolutely possible and necessary that they are exposed to a broader swath of the field of math, so that they gain an understanding of its many uses and value, without necessarily being able to reproduce it all themselves with no support and under time pressure. In life, after all, one can always use a calculator or software, look things up online, or ask an expert; *the trick is knowing what to type, what to search, and whom to ask.*

The traditional, pragmatic approach to mathematics education is not only counterproductive to students' conception of the field of mathematics, but it also does not align with the true use of mathematics in industry. In professions that use mathematics as part of their work but are not

---

[9] e.g.

Luba Vangelova (2014) "5-Year-Olds Can Learn Calculus: Why playing with algebraic and calculus concepts—rather than doing arithmetic drills—may be a better way to introduce children to math". *The Atlantic.*

Carraher, D. W., Schliemann, A. D., & Schwartz, J. L. (2008). Early algebra is not the same as algebra early. *Algebra in the early grades*, *235*, 272.

Fluck, A., Ranmuthugala, D., Chin, C., & Penesis, I. (2012). Calculus in elementary school: an example of ICT-based curriculum transformation. *Journal of Computers in Mathematics and Science Teaching*, *31*(2), 159-174.

Patricia Clark Kenschaft (2005). Racial equality requires teaching elementary school teachers more mathematics. Notices of the AMS, 52, #2, p 208-212.

centered around it, such as nursing, studies have shown that professionals use "more efficient, informal strategies that are typical of the situation" and many times those who do not score well on traditional tests of school mathematics are extremely accurate in their specific implementations.[10]

Even for jobs truly focused around mathematics, the on-the-job math is *not* a natural extension of school math; "In school, the professor formulates the problem and you solve it—you hope. In industry, you formulate the problem and the software solves it—you hope".[11] Conrad Wolfram describes four categories in Math education: Recognizing where mathematics is applicable, Translating practical problems into mathematical problems, Solving mathematical problems, and Interpreting and evaluating the outcomes. Of these, "Solving" is the only one that gets "systematically addressed in mathematics education, and that this is exactly the step that is increasingly carried out by computers!"[12]

Such a heavy focus on math in industry also neglects the importance of mathematical literacy for everyday life and for a democratic society. In 2020, the world witnessed firsthand how the understanding or lack thereof by policy makers of exponentials ("deceiving then explosive") and complex systems ("small changes, big impact") led to very different approaches to handling the Covid outbreak and the vaccine distribution. Because of how different "school math" can be from real world math, there is a long-standing challenge to teach math such that it **transfers** *into the real world*.

We believe a large part of the problem is that "school math" does not pay enough attention to the translation steps between the real world and math, and back. Part of the issue is that this gets into the very perception of what *is* a math problem, which teachers see effortlessly due to their embedded expertise (aka "curse of knowledge"[13]), and thus rarely give students the chance to practice this crucial step.

By focusing on the formal difficulty of the mathematical content, experts miss the important goal of preparing adults who can apply basic math concepts to everyday situations. While the math itself might be basic, the applications can be quite varied and far from straightforward! "Opportunity to Learn" (OTL) refers to the coverage of topics that a given student is exposed to; according to this construct, the greater the number of topics, the greater is their "opportunity". But if in their rush to

---

[10] Gravemeijer, K., Stephan, M., Julie, C., Lin, F. L., & Ohtani, M. (2017). What mathematics education may prepare students for the society of the future?. International Journal of Science and Mathematics Education, 15(1), 105-123.
[11] Damlamian, A., Rodrigues, J. F., & Strässer, R. (Eds.). (2013). *Educational interfaces between mathematics and industry: Report on an ICMI-ICIAM-study* (Vol. 16). Springer Science & Business Media. p.41
[12] Gravemeijer, K., Stephan, M., Julie, C., Lin, F. L., & Ohtani, M. (2017). What mathematics education may prepare students for the society of the future?. *International Journal of Science and Mathematics Education*, *15*(1), 105-123.

[13] e.g. Birch, S. A. (2005). When knowledge is a curse: Children's and adults' reasoning about mental states. *Current directions in psychological science*, *14*(1), 25-29.

"cover" more content, students do not practice applying their knowledge to real-world situations, they have not gotten the opportunities they need to use math in their lives.

As discussed above, the current system seems to be designed to optimize for students who are going on to STEM careers. The majority, of course, "fail to do so", but at least they "had the opportunity" - so the reasoning goes. There is a movement to cover Algebra as early as possible, as well as the widespread belief that tracking students by ability is equivalent to denying them the opportunity to have a STEM career.

There are several things wrong with this picture:

1. **An assumption that non-STEM math curriculum is inherently *worse* than STEM track mathematics.** One article from the National Council of Teachers of Mathematics (NCTM) states this belief clearly: "*tracking prevents students access to a high-quality mathematics curriculum, to effective teaching and learning, to high expectations, and to the necessary supports needed to maximize their learning potential.*"[14] Yet, none of the qualities listed are inherent to a non-STEM track. In fact, if non-STEM tracks truly *are* inferior in their curriculum, pedagogy, expectations, and supports, then *that* is the problem that needs to be addressed, not the tracking itself.

2. **Coverage of topics is not a causal factor in better mathematical learning.** It appears that the initial studies, which found a benefit to learning algebra earlier, suffered from selection bias: "stronger math students take algebra in eighth grade, and although they indeed may benefit academically from the course, that does not mean that weaker students will also benefit from taking algebra earlier."[15] In fact, studies that have controlled for selection bias have found that pushing students who are not ready to cover algebra had a "decidedly harmful" effect.[16]

3. **The notion of "coverage" is poorly defined.** When curriculum is aligned to standards, there is a lot of ambiguity in the exact nature of the alignment. Standards are written to be broad, and teachers are often the ones breaking them down into learning objectives. Does the standard get tagged as soon as a piece of it gets mentioned, or only once the whole thing has been practiced and tested? Perhaps a standard can be touched on at first and then explored in more depth later, or explored in depth at first but reviewed in more challenging contexts later. Perhaps a standard is worked on in depth,

---

[14] [Initiating Critical Conversations on the Discontinuation of Tracking](#)
[15] https://www.brookings.edu/wp-content/uploads/2016/06/2013-brown-center-report-web-3.pdf
[16] Charles T. Clotfelter, Helen F. Ladd, and Jacob L. Vigdor, The Aftermath of Accelerating Algebra Evidence from a District Policy Initiative (Washington, DC: National Center for Analysis of Longitudinal Data in Education Research, American Institutes for Research, 2012)

but then never addressed again, and most students completely forget it by the end of the year. None of this richness is captured in the word "coverage" and yet it is the most common way to describe the relationship between curriculum and the standards it aligns to.

4. **The current system isn't the best way to maximize the number of students that go into STEM.** According to a report by the Mathematical Association of America, the rush to Calculus leads to a focus on procedure at the cost of conceptual understanding. "In some sense, the worst preparation a student heading toward a career in science or engineering could receive is one that rushes toward accumulation of problem-solving abilities in calculus while short-changing the broader preparation needed for success beyond calculus."[17]

This entire line of thinking ultimately stems from a very achievement-focused view of education, with K-12 merely being a time to create the best possible university application. And unfortunately, in the current system, Calculus is the gatekeeper and barometer for all students graduating high school and hoping to make it into a selective (or even non-selective) university. That means students who may be very talented in their chosen field but never got to Calculus are often kept back from achieving their full potential, even though Calculus will not figure into their career at all. Instead of a goal of Quantitative Literacy, the goal becomes, in the US, the SATs[18] and Calculus, and working backwards from that goal leads to the current state of most curricula around the world.

Conclusion

What would mathematical literacy for everyone look like? It would mean that every adult, no matter their occupation, is able to meaningfully use mathematics knowledge to understand and make decisions in their lives, relying on experts when needed (e.g. financial advisors, lawyers, etc.). Officials would be able to interpret and ask probing questions about the impact of healthcare policy, college students would be able to reason about the salary implications of choosing a particular major, patients and their doctors would understand the risk involved in a procedure, and *everyone* would wear a mask during a pandemic.

---

[17] Insights and Recommendations from the MAA National Studyof College Calculus
https://www.maa.org/sites/default/files/pdf/cspcc/InsightsandRecommendations.pdf
[18] Or Baccalaureate, or Maturite, or Abitur, etc. etc.

Section 2: CCR's Approach to Math Standards Design

Challenges to Standards Design

Timing and Bloat

Research has shown that humans have a strong tendency to find solutions that involve adding something, and very reluctant to finding solutions that involve removing something[19]. The politicization of education, and "groupthink" on a global scale, only reinforce this tendency. Based on our work with standards, there appears to be an underlying form of risk aversion that follows this reasoning: what if we remove something and don't realize all the interconnections it had, and our removal results in the loss of learning integrity?

The CCR team made very careful and deliberate choices about how to simplify and tweak the standards, using among other tools, graphs of interconnections (as described in section "Threads - Direct Connections" further down in this document). The goal was to be precise, cutting as if with a scalpel, rather than removing entire sections as if with a chainsaw. The idea is that removal is necessary, and it is also necessary to make sure what is being removed is not "load-bearing" to the overall structure.

Another aspect of choice, besides careful and precise curation and removal, is to *de-emphasize* some topics for the benefits of others. This was done by rebalancing the ultimate learning objectives of each standard[20].

The process of Standards design "typically involves consultation, sometimes wide-ranging consultation with subject-matter experts, curriculum specialists, child developmentalists, teachers, employers, and parents. The result is likely to be a list that is either too long or too short."[21] This seemingly paradoxical statement is actually quite profound; it implies that the main issue in standards design has to do with the level of granularity and their structure, as well as the politics of acceptance by the public and teachers.

---

[19] Adams, G. S., Converse, B. A., Hales, A. H., & Klotz, L. E. (2021). People systematically overlook subtractive changes. *Nature*, *592*(7853), 258-261. https://www.nature.com/articles/s41586-021-03380-y#article-info
[20] explained further down in the Given | Asked | Can section under "Decision-Making"
[21] Bereiter, C. (2005). *Education and mind in the knowledge age*. Routledge.

When many different people come together speaking slightly different languages with differing assumptions, it is challenging to come up with one list that covers everything they all value. As a result, the items on the list are either very large and abstract, attempting to encompass everything by meaning many different things to many people, or very small, attempting to describe and enumerate exactly what each expectation is, so that even people with fundamentally different starting positions all reach the same understanding.

In order to deal with this challenge, CCR's system does not rely on the *number* of standards as an indicator. Rather, we have divided our intended timing appropriately, across all of the standards in each grade. In this way, each grade adds up to 100%, and each standard is clearly addressed to the appropriate degree in comparison to the other standards that year. This let us choose where we were placing emphasis without relying on a proxy like the number of standards.

The graph below summarizes the way each grade's math content is distributed across Branches. 100% represents the total time devoted to math during a single grade.[22] We used this visualization to make sure we were living up to our goals, namely adding Statistics and Probability as well as Discrete / Computational throughout the curriculum, slowly replacing Arithmetic with Algebra through eighth grade, and minimizing it in ninth grade, due to the fact that *Algebra is naturally used extensively across the other four branches by this point.*[23]

---
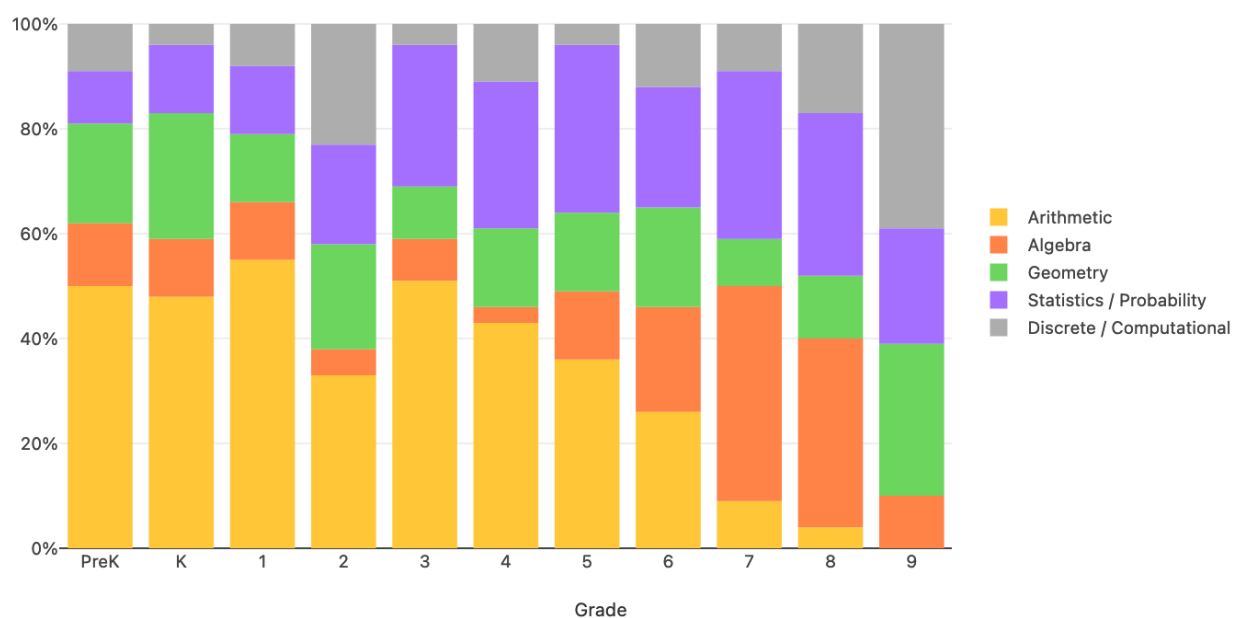
[22] This will remain the case for the rest of the paper
[23] Grade 9 may still change depending on the CCR math recommendations for grades 10-12. In particular, *higher level maths in grades 10-12 are likely to necessitate a greater focus on Algebra.*

This demonstrates an overall pattern of moving from foundational arithmetic to more and more meaningful application: first to algebra, and then to the other branches.

Standards vs. Curriculum

The space between setting goals for learning and designing learning experiences can be an ambiguous one. In fact, different countries often make their delineations differently. At CCR, we have drawn the lines according to what best reflects where we believe the design decisions need to be made, and we have separated the space into three layers: Standards, Curriculum, and Courseware.

To make decisions about Standards, we consider the historical context of the field and difficulties encountered, recent paradigm shifts and insights from the Learning Sciences, updates based on modern needs (e.g. the need for information literacy in math), and the general goals and functions of the subject in the broader educational fabric (See Appendix 1). We begin by comparing leading standards frameworks around the world and choosing the one that would be our starting point (for Math, that was ACARA). We also consider the big picture goals of which essential content and core concepts should be covered.

To make decisions about curriculum, we begin with content and concepts, linking them to the standards and breaking standards down into learning progressions. We then consider the potential for interdisciplinarity, for embedding competencies (See Appendix 2), and for assessments, including projects. Finally, to make decisions about Courseware, we consider the student perspective, formative assessments, gradual release of responsibility, spacing & interleaving, and other pedagogical decisions.

Different students have different needs

As discussed above, the math standards must be refocused to be aimed at the majority of students, who will not go on to study STEM, but *will* need to understand how math applies to their lives. However, this creates a tension because those students who will go into a STEM career need to go into greater depth, which of course should be encouraged.

One way we have captured this in our structure is by creating what we've called "Extension" standards (those that begin with [EXT] in their Name). This is to help support schools who would like to create more differentiation in their classrooms by providing students who have finished their work with more challenging material which is still on the same topic as the rest of the class. On the other end, we also provide certain Learning Objectives (those that begin with [LO] in the Name) when we believe

there is a particularly important way to break down a dense standard into smaller standards or even activities.

Neither the list of EXTs nor LOs is exhaustive in the current version of the standards. Teachers and curriculum designers working with these standards are encouraged to make their own additions to this list, in order to further scaffold and extend the standards.

Furthermore, our standards reach a point after ninth grade where they reach a "trifurcation", i.e. get divided into three tracks. There is no value judgment placed on any of the tracks; rather the purpose is to give each student the math that would be most helpful to them. For that reason, we recommend classes aimed at three groups:

- Those going to study STEM in university
- Those going into a vocational career
- Those going to university for Humanities or Arts

Depending on their goals, students would be best served by classes that treat math in very different ways. For the students going on to STEM, classes would typically resemble what we have traditionally seen in math classrooms: going into great technical depth and preparing students for exams. For those going into a vocational career, there are certain other maths that are very important, and would need to be studied carefully, in the natural, applied, contexts of the vocations. Finally, for those going into Humanities and Arts, it is still important to cover math, but the goal is to prepare students to be critical *consumers* of math and math claims, rather than its *producers*.

Consuming vs. Producing Mathematics

This distinction, between consuming and producing mathematics, is key to understanding how students will be expected to learn what appear to be more advanced topics. It is true that if one learns math to such a degree that they can produce it, they will have learned enough to be able to consume (i.e. interpret) math when they come across it. This is rooted in a long tradition of constructivist pedagogy.

However, what we are seeing is that the large portion of students who n*ever do* master math, are also unable to interpret it; in other words, they are not math literate. It seems that when we "shoot for the moon" (all students producing math), we do not "land among the stars" (at least most students properly interpreting math). Instead, a small portion of students land on the moon, and the majority are alienated from mathematics altogether.

What if, instead of trying to make all students produce math to whatever degree they are able, we refocused instead on teaching all students to *interpret* math (i.e. making everyone mathematically

literate). Like increasing the use of technology, this shift has the potential to extend how advanced a mathematical topic can be covered, and puts math literacy in its rightful place as the goal for all students.

There is also reason to believe that exposing students to more advanced content by asking them to interpret something they could not themselves produce could lead to better results than keeping a focus only on content students are able to produce. For example, in traditional literacy (reading), it has been shown that when students listen to text at a reading level higher than that which they could read on their own, their reading capabilities increase *more* than when they practice reading at the level they are able.[24]

In Math, there is the case where a superintendent asked several schools to delay math instruction until sixth grade. They still practiced measuring, counting, and estimating, but the rest of the time was spent instead on practicing speaking fluently about various familiar topics. Their math ability was then tested in sixth grade, and the results were striking[25]:

> ...kids who received just one year of arithmetic, in sixth grade, performed at least as well on standard calculations and much better on story problems than kids who had received several years of arithmetic training. This was all the more remarkable because of the fact that those who received just one year of training were from the poorest neighborhoods — the neighborhoods that had previously produced the poorest test results.

It is therefore quite plausible that if the goal is for students to become mathematically literate, they must practice what is involved in interpreting math, and that doing so will also open up possibilities for meaningfully exposing students to higher level mathematics topics.

Appreciation

When discussing the purpose of math education with mathematics experts, CCR has noted that experts cannot help but see the beauty in what they imagine is being taught. Some students do see

---

[24] Hogan, T., Bridges, M. S., Justice, L. M., & Cain, K. (2011). Increasing higher level language skills to improve reading comprehension. Special Education and Communication Disorders Faculty Publications.
[25] L. P. Benezet (1935/1936). The teaching of Arithmetic: The Story of an Experiment. Originally published in Journal of the National Education Association in three parts. Vol. 24, #8, pp 241-244; Vol. 24, #9, p 301-303; & Vol. 25, #1, pp 7-8. as cited in Gray, Peter (2010) When Less Is More: The Case for Teaching Less Math in School. *Psychology Today*. https://www.psychologytoday.com/us/blog/freedom-learn/201003/when-less-is-more-the-case-teaching-less-math-in-school

this beauty and go on to a STEM profession, but the majority would be surprised and confused to learn that anyone considers math beautiful.

One of the purposes of K-12 schooling is to expose students to the range of types of ideas that are out there. This means that there are some topics that CCR recommends including at an "appreciate" level. This is even more passive than the "interpret" level, because it requires students to simply understand what's explained to them, and later in life, recognize the topic whenever it is relevant.

That makes the "appreciate" level particularly difficult to assess and thus difficult to describe via standards. For this reason, CCR has included many such topics as Sample Topics that go with existing standards. It is then up to each teacher to decide what they feel comfortable with and passionate about, in order to properly transfer to their students their enthusiasm for math, rather than their fear. For example, a formal education in Discrete Mathematics might include a thorough dissection of the Bridges of Konigsberg Problem, whereas CCR's standards contain a standard called "Does a solution exist?", which features with it the Sample Topics of the Bridges of Konigsberg and the Traveling Salesman problems.

Nickname	Given	Asked	Students can
Does a solution exist?	a network problem	does a solution exist	reason about the existence of a solution, a proof that there isn't one (e.g. including the pigeonhole principle), or that they cannot determine whether or not a solution exists

Goals of this Standards Design Process

To make improvements to a set of standards, it is important to first establish what constitutes an improvement. In other words, what are the goals we are trying to move toward? CCR embarked in 2012 on a decades-long journey, starting with a Mathematics conference[26], which covered the history of Mathematics[27]. From that conference and two subsequent colloquia, CCR authored "Recommendations for PISA 2021"[28], which highlighted several emphases needed in this revision of standards. They are (slightly reformulated):

1. Rigorous foundation in Arithmetic
2. Mathematical ways of thinking

---

[26]https://curriculumredesign.org/global-maths-conference-affirms-the-need-to-deeply-redesign-mathematics-standards/
[27] The history of Mathematics is important to highlight difficult transitions for humankind, which might be reflected in younger students as well: base 10 numeration vs base 60; zero as a quantity and non-quantity; imprecision of probabilities explained by Maths (the language of precision!); imaginary numbers; etc.
[28]https://curriculumredesign.org/wp-content/uploads/Recommendations-for-PISA-Maths-2021-FINAL-EXTENDED-VERSION-WITH-EXAMPLES-CCR.pdf

3. Decision Making: Real Scenarios and Recognition
4. More, and different, Statistics and Probability
5. Modern Math: Discrete and Computational Branch
6. Using Technology as a tool
7. Careful treatment of points of cognitive density

Below, we will go into detail on each one.

Rigorous foundation in Arithmetic

When one thinks about the kinds of math used in everyday life, the most practical applications of an understanding of numbers, one is really describing a firm foundation in arithmetic (up to fractions, percentages, and ratios, i.e. proportional reasoning). The importance of this is clear; people must be able to understand what the number in their bank account means, how many cars they need for a trip for 20 people, and so on. With some exceptions, it appears that the current math standards systems successfully accomplish this goal.

However, looking just a little deeper, it becomes apparent that for many, arithmetic added up to learning procedural algorithms, not gaining an embodied understanding of Number (nor of algorithms, for that matter). Research talks a lot about "Number Sense" and yet different scholars define it in different ways. A definition that seems to have gathered a consensus is "the ability to perceive, manipulate and understand numerosities".[29] But what does this mean in practice?

We believe that a source of confusion is that Number Sense means different things at different levels. In the very early stages, it means subitizing, learning the names of the numbers and attributing them to the sizes of groups of objects. However, as students progress, Number Sense begins to mean an *embodied* understanding of Number that allows them to work with numbers intuitively. This includes being able to properly use approximation (dropping irrelevant levels of detail while preserving the relevant ones) as well as estimation (predicting a numerical quantity without calculating all of the necessary information). In everyday applications, approximation and estimation are crucial to efficiently apply mathematical knowledge, and yet they often get neglected in mathematics standards to make room for more formal algorithms.

This number-based intuition is also what allows students to eventually develop a critical orientation toward their more formal mathematical calculations as well; in other words, to have a general understanding of the kinds of answers they are expecting, and to instinctively compare the answer they arrive at with the type of answer they were expecting.

---

[29] https://www.cambridgemaths.org/Images/espresso_4_early_number_sense.pdf

As students move on from counting to addition and subtraction, and on to multiplication and division, they must develop an embodied understanding of Number in order to properly hone this intuition. That involves experimenting with numbers and units in many different representations, so as to triangulate the concepts in an embodied way. This culminates in an embodied understanding of Proportional Reasoning, which allows students to fluently manipulate not just numbers but relationships between numbers.

This is a point of great cognitive density and importance. Research has described it thus:[30]

> Proportional reasoning has been described as the cornerstone of many higher-level mathematics topics, including algebra, and the capstone of elementary school topics, such as number and measurement (Lesh et al.,1988). It is essential in many subjects beyond mathematics, including science, economics, and geography (Akatugba & Wallace, 2009; Boyer, Levine, & Huttenlocher, 2008; Howe, Nunes, & Bryant, 2011); is one of the most commonly used applications of mathematics in everyday life; and is essential in many professions, for example, architecture, nursing, and pharmacy. Ahl, Moore, and Dixon (1992) described proportional reasoning as a pervasive activity that transcends topical barriers in adult life (p. 81), and yet Lamon (2007) estimated that close to 90% of the adult population are not proportional thinkers. The importance of developing proportional reasoning is widely recognised (Howe et al.,2011 ; Jones, Taylor, & Broadwell, 2009 ; Staples & Truxaw, 2012), however, its development requires a relatively long period of time (Siemon, Izard, Breed, & Virgona, 2006). The National Council of Teachers of Mathematics (NCTM) (1989) argued that its development is so important that it merits whatever time and effort that must be expended to assure its careful development (p. 82), and yet it is rarely explicit in curriculum documents and many teachers lack sufficient pedagogical content knowledge to teach it (Kastberg, D'Ambrosio, & Lynch-Davis, 2012 ; Lobato, Orrill, Druken, & Jacobson, 2011; Sowder et al., 1998.

Further, Proportional reasoning is also increasingly important in a modern world where citizens must understand differences in scale, for example, the difference between a millionaire and billionaire, or between linear and exponential models for global warming and population growth. Yet students struggle greatly with mastering Proportional Reasoning[31]:

> Research literature abounds with examples of students' proportional reasoning difficulties. These include the inability to distinguish between proportional and non-proportional situations or to identify relationships that are multiplicative; the tendency to use additive or absolute thinking (i.e.,considering a quantity by itself rather than its value in relation to other quantities) in situations of proportion that require multiplicative or relative thinking; the use of multiplicative thinking in situations requiring additive thinking; ignoring some data, such as

---

[30] Hilton, A., Hilton, G., Dole, S., & Goos, M. (2016). Promoting middle school students' proportional reasoning skills through an ongoing professional development programme for teachers. *Educational Studies in Mathematics*, *92*(2), 193-219.
[31] *ibid.*

denominators in equations; inappropriate application of algorithms, such as cross multiplication; and incorrect patterning strategies (Cramer & Post,1993; Lamon, 1993; Leshetal., 1988; Misailidou & Williams, 2003; Nabors, 2003; Tourniaire & Pulos, 1985; Van de Walle, Karp, & Bay-Williams, 2010; Van Dooren, De Bock, & Verschaffel, 2010).)

For these reasons, Proportional Reasoning is a key landmark to work toward and to work on within Arithmetic.

The foundation built in the early grades must be strong enough to support further learning on top of it. Whether it's more transferring the same arithmetic to different contexts or transferring it to higher level math, students' understanding must be firm enough to support it.

However, there is an important caveat here. By allowing ourselves to take "contributing to a necessary foundation for future math" as a valid reason for including certain standards, we are introducing a loophole to our otherwise tightly controlled conditions for inclusion. After all, any mathematician can look at *nearly any* standard and correctly claim that it contributes to a necessary foundation for higher level math. It is therefore crucial to avoid a slippery slope, keeping the majority of standards included on other merits and only a small minority that must rely on this reason.

Mathematical ways of thinking

In our paper synthesizing the research on Knowledge[32], we describe the way that a certain type of learning leads to concepts being crystallized that can then be transferred to and used in many different contexts. These concepts are particular to each field or discipline, and even to each branch.

Part of the big picture goals of teaching a given discipline is to teach students who are not going to become experts in that discipline to still benefit from the type of thinking that discipline instills. In other words, while each student will only develop expertise in one, maybe two, disciplines, we want each student to be able to "think like" a mathematician, notice when mathematics can be used, and interact intelligently with mathematical objects they come into contact with.

A large part of "thinking like a mathematician" is seeing the opportunity for and the influence of math in all facets of life. All experts can be said to have a lens (or a hammer) that they apply to everything they encounter. For mathematics, this means the ability to "mathematize" the world, and the ability to interpret mathematics back into the context of the world. For this reason, many of CCR's standards begin with a scenario and ask students to do math, or begin with math and ask students to interpret the implications, rather than beginning and ending within the world of mathematics.

---

[32] https://curriculumredesign.org/wp-content/uploads/CCR_Knowledge_FINAL_January_2018.pdf

There is also a common belief that learning mathematics teaches one to think logically, critically, or rigorously. In our paper examining this claim[33], we were unable to find evidence that mathematics, as it is currently taught, leads to an increase in higher order thinking skills. The most likely scenario is that this is true for a minority of students (those who fall in love with math and go on to develop expertise in it), which would explain why mathematicians believe math education to have this effect.

There is reason to believe that mathematics *is* particularly well suited to logical reasoning, since it deals with situations where the context is abstracted away and all that's left are the logical relationships. However, cognitive science has shown that this actually makes logical reasoning much harder, since we use contextual clues to help ground our logical reasoning.[34] That said, as long as these ways of thinking are built up in other classes, mathematics can be a crucial place for these to be practiced in their extreme form, which can help crystallize the work that was started in more contextualized disciplines.

In our Knowledge paper we created a structure of knowledge that identified Essential Content and then identified Core Concept at the different tiers of knowledge. So, in this case, Mathematics the discipline would have a set of Core Concepts which are the cognitive power tools that mathematicians use when they are "thinking like a mathematician". Because Math is the Discipline level, these Core Concepts are referred to as DCCs. In addition, each of the branches (Arithmetic, Geometry, Algebra, Statistics, Probability, and Discrete/Computational) would have its own Core Concepts which have far reaching applications, yet stem from a specific branch.
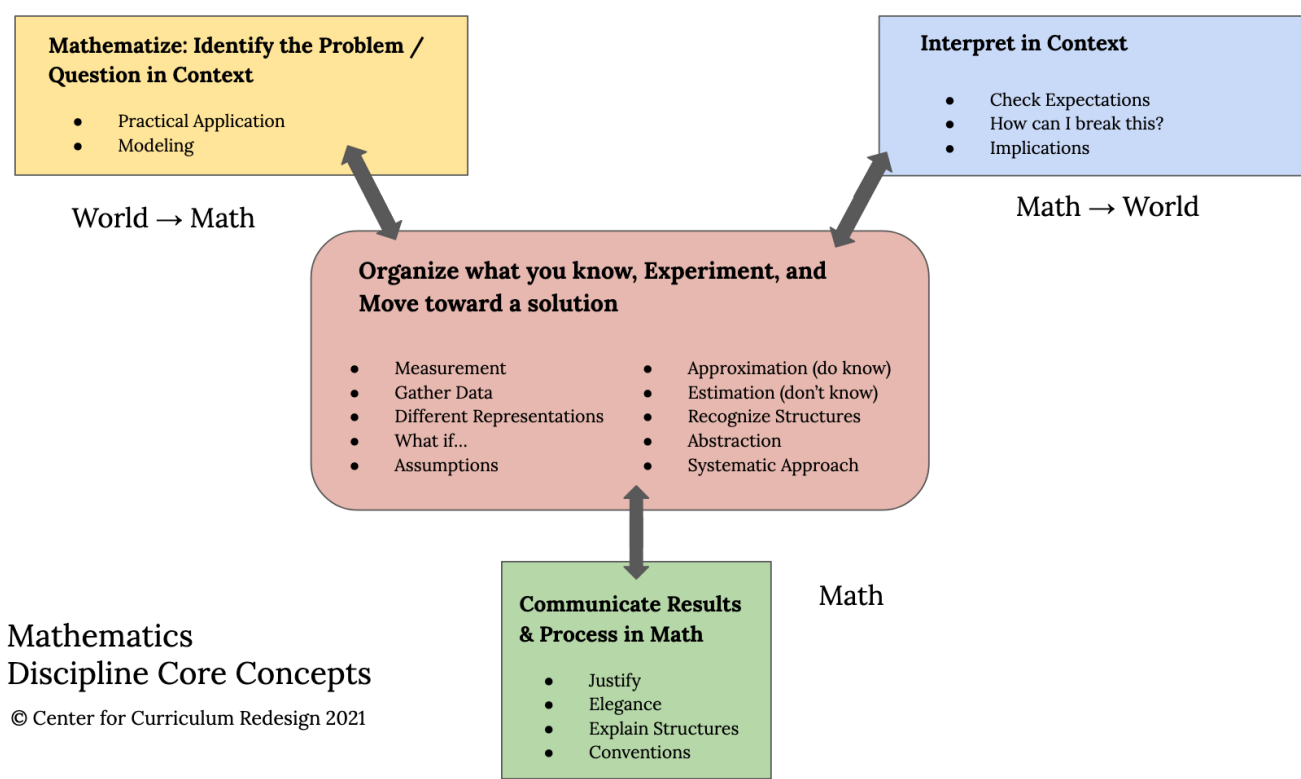
It is crucial that the DCCs be intentionally interwoven throughout the mathematics curriculum in order to make sure *all* students absorb these ways of thinking, not just a small minority who are predisposed to it. Often, these kinds of concepts are presented as a problem solving "process." Although we agree that these concepts are closely related to what happens in the mind of someone who is problem solving, we believe it is crucial not to imply any sort of linearity or mindlessness (i.e. "just follow the process") when it comes to describing DCCs. Refer to Appendix 2 for a description of each DCC.

Our framework places at its center 10 of the 19 DCCs into one central fluid step which can be described as "Organize what you know, experiment, and move toward a solution". The other nine are split between "Identifying the problem or question in context" (i.e. turning the world into math), "Interpreting in context" (i.e. turning math into the world), and "Communicating results and process

---

[33] https://curriculumredesign.org/wp-content/uploads/CCR-Maths-Thinking-Skills.pdf
[34] For example, the Wason selection task fools most people when it is done with abstract letters and numbers, but is perfectly solvable when placed into a context of human laws (e.g. drinking age).

in math" (i.e. formalizing the fluid experimentation in terms of math structures and conventions).

Mathematize: Identify the Problem / Question in Context
- Practical Application
- Modeling

World → Math

Interpret in Context
- Check Expectations
- How can I break this?
- Implications

Math → World

Organize what you know, Experiment, and Move toward a solution
- Measurement
- Gather Data
- Different Representations
- What if...
- Assumptions

- Approximation (do know)
- Estimation (don't know)
- Recognize Structures
- Abstraction
- Systematic Approach

Math

Mathematics
Discipline Core Concepts

© Center for Curriculum Redesign 2021

Communicate Results & Process in Math
- Justify
- Elegance
- Explain Structures
- Conventions

Each of these DCCs[35] makes an appearance across branches and across grades in our standards set, as can be seen below. The vertical axis here represents the percent of the school year devoted to each standard tagged with each DCC; 100% refers to one full school year.[36]

---

[35] An asterisk simply denotes that a given Thread came from our original Core Concepts documents.
[36] According to the OECD's Education at a Glance report, the average amount of time spent on math instruction is about 45 minutes per day. As a rough approximation then, 1% in our framework corresponds to roughly two class periods.
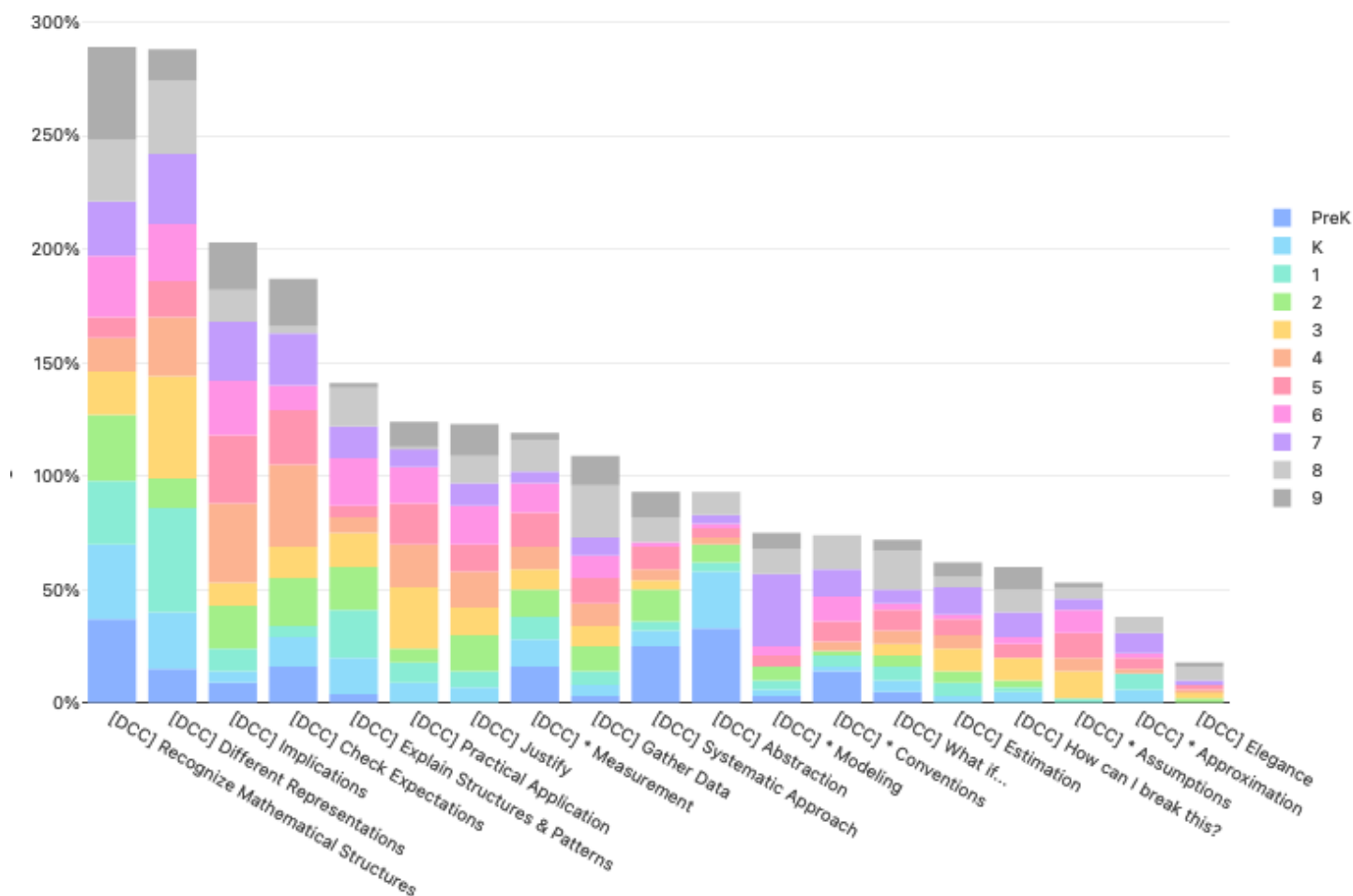
This visualization also lets us see when certain DCCs are weighted more heavily toward the earlier or later grades. For example, Abstraction is weighted more heavily toward the earlier grades when students are learning the basics of numbers, whereas Implications are weighted more heavily toward the later grades, when students have more tools at their disposal. Later in this paper, we will go over the ways DCCs show up differently across each Branch.

Decision making: real scenarios and recognition

An emphasis CCR wanted to make sure to explore in its standards was on using math for decision making. For this reason, one big change we made was, whenever it was reasonable, to begin the standard by giving students a scenario in the real world, realistic data, or familiar events (rather than "raw" math[37]) and then ask students to interpret their mathematical conclusions in the original

---

[37] Sometimes called "naked math"

context. Far from being mere "word problems", these scenarios are meant to reflect as much as possible the way that this math may manifest in students' lives.

The graph below shows the distribution of the types of Applications. The most common type is a Scenario[38], followed by "Raw" Math -- in other words, standards which are not grounded in any application outside of Math. Finally, certain Applications were primarily well-suited for one Branch above all others, i.e. Manipulatives for Arithmetic and Given Data for Statistics / Probability.
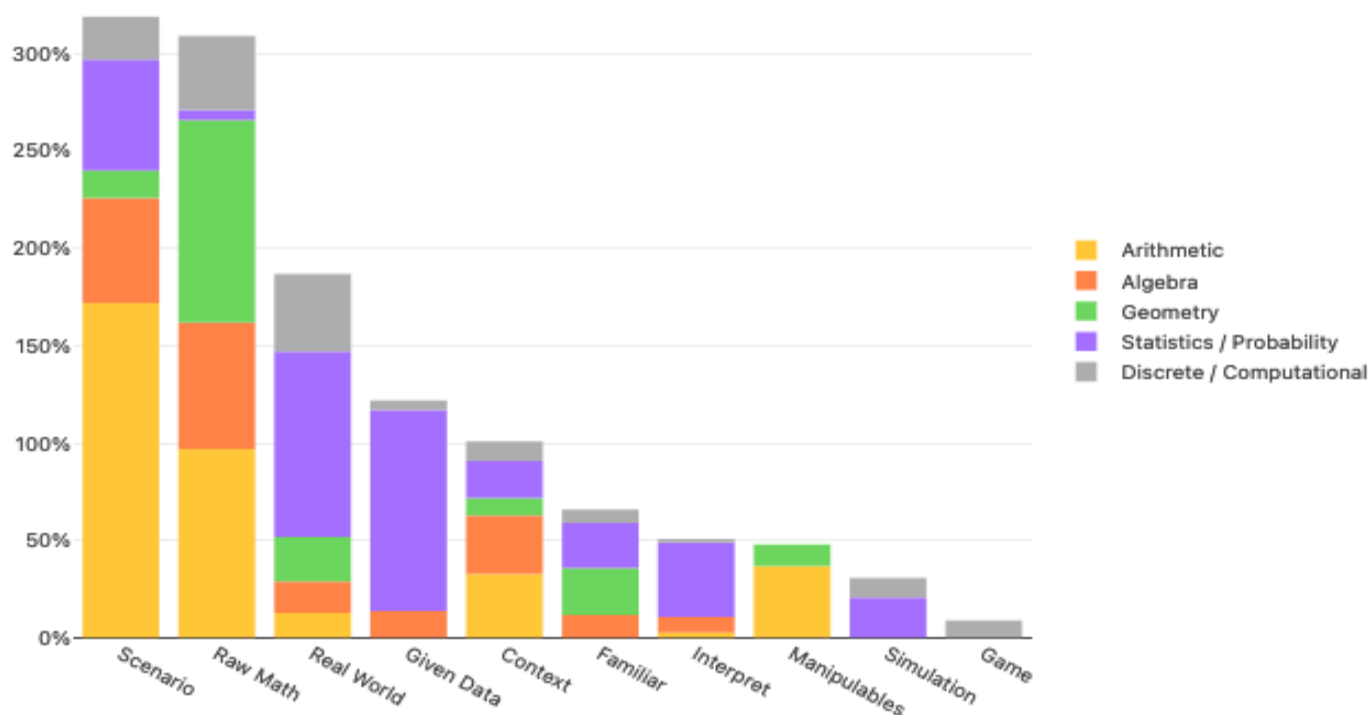
The other big change was to be methodical about the leaps we are asking students to make. While traditional standards do not specify the leap a student must make, only what they should be able to do, the CCR standards specifically describe what students are given and what they are asked, *separately from what they should be able to do*. This allows us to be precise with the moments where it's important that teachers do not "give away" the translation steps, which come so naturally to them.

In terms of the DCCs, these translation steps take place in the top two boxes: Identifying a problem or question in context and Interpreting the result of the math in context. The habits of mind listed there serve as a useful framework for tracking what kinds of leaps students are being asked to make in the general sense, in addition to the specific leaps captured in the space between the Given, the Asked, and the Can of each standard.

---

[38] more applied than Context, but not quite as applied as Real-World

In a traditional version of the following standard, students may be asked to identify the dependent and independent variables in a scenario. Common Core, for instance, has articulated the corresponding standard as "Represent and analyze quantitative relationships between dependent and independent variables."[39]

However, by taking one step back and asking students to express the relationship mathematically, we make space for the students to *realize the need for* conceptualizing the relationship as between an independent and a dependent variable, so that when they encounter relationships in the wild, they are ready to apply this thinking proactively, without prompting from a teacher who will not be there.

Nickname	Given	Asked	Students can
Variables	a scenario involving a mathematical relationship	to express the relationship mathematically	identify and name an independent and a dependent variable in the relationship.

Or for example, instead of directing students to make a two way table, we present them with a realistic data set or scenario and ask them to find a particular conditional probability; it is then up to the students to realize that this situation calls for a two way table.

Nickname	Given	Asked	Students can
Conditional Probability by Grouping	a realistic data set broken down along two category types or a description of a scenario with odds	to find some conditional probability	build and/or interpret a two way table with appropriate groups to determine the probability

This allows the standards to directly target the behaviors that are intended to transfer, such as in this one:

Nickname	Given	Asked	Students can
Pierce Obfuscation: Graphing	a misleading graphical presentation of data from a scenario (e.g. not labeling axes, different scales, one is log, starting not at zero...)	to interpret	point out misleading factors and explain how those factors affected their interpretation and arrival at appropriate conclusions

In summary, the Given, Asked, Can structure results in the following benefits:

---

[39] http://www.corestandards.org/Math/Content/6/EE/C/9/

GIVEN	ASKED	CAN
Scaffolding: We are able to open a whole range of possible goals by using tools that are not paper and pencil. This makes previously inaccessible complex math ideas become accessible.	**Recognizing:** Separating the Asked from the Can allows us to specify a condition in which the question does not give away which math is needed to solve it, so students can actually practice recognizing situations and deciding which math to use and how. This is a crucial often overlooked step for Transfer.	**Transfer:** By isolating the student actions, we are able to reflect on exactly what they are being asked to practice. What they practice will be transferred, and what they don't will not, so this helps us double check the emphasis of each standard.
Interpret: Most traditional standards are written with the goal of students taking on the role of producer (as a future expert). By specifying what is given, we are creating the opportunity for Interpretation-heavy standards, rather than just Production-heavy ones. In other words, we can give students an outcome of math, and let them practice interpreting it, as this is going to be the main application of what they are learning in their lives.	**Mental Lift:** By precisely describing what is asked of the students, we are able to enforce an amount of mental lift that the student must be able to do. Very often, teachers feel pressure to break things down into very small pieces so that students can do each piece, but do not recognize that breaking down work they did was crucial for students to learn how to do themselves.	**Approach:** Without describing it in the Asked, we are able to specify *how* we intend for students to apply the math they are learning by describing it in the Can.
Authentic: Many of our standards start by giving a real world authentic scenario, including their "noisy" presentations. If teachers feel the need to break these down, that's perfectly fine, but the goal of putting it all back together is properly highlighted.		**Overshooting:** By specifying the Can we can make sure it describes *only* the parts we specifically want to enforce. The Asked can be more broad to allow students to practice a range of things, but the Can specifies which of these things is actually necessary to complete the standard.
		Action: The Can formulation forces a description of actions, which helps us avoid passive actions such as "understanding".

Adding more, and different, Statistics and Probability

Statistics and Probability have been historically underrepresented in mathematics curricula due to their more recent advances and more applied approaches. Yet in a very real sense, Probability and Statistics are just as natural as quantity and shape, in that humans must naturally reason with uncertainty in their day to day lives. The figure below[40] shows how math education begins with these three conceptions -- quantity, shape, and uncertainty -- and then slowly weave them together, creating the foundation for higher level mathematics.

---

[40] Inspired by Dehaene, S. (2011). The number sense: How the mind creates mathematics. OUP USA.
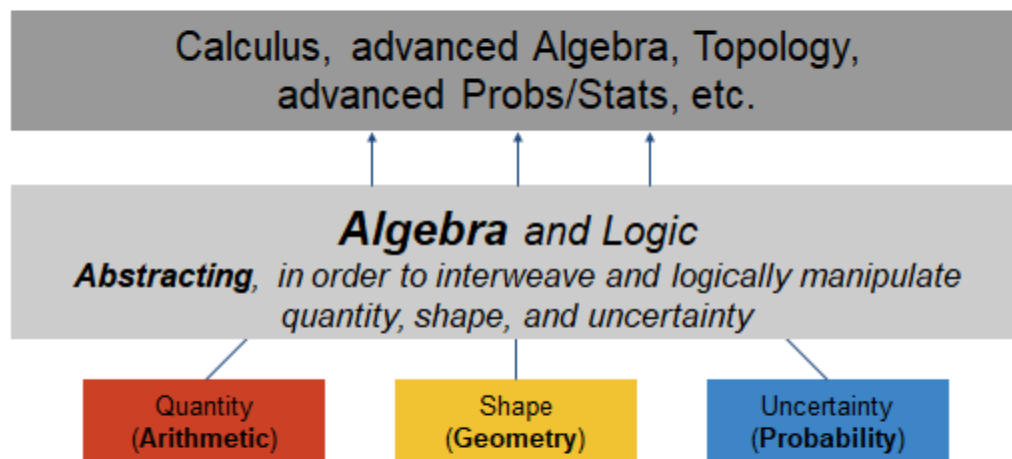
Innately to Mammals: manipulating quantity, shape, uncertainty

Adapted from Stanislas Dehaene, CNRS

On our own first pass, the same thing happened even to the CCR team; we began with Arithmetic, Geometry, and Algebra, and found ourselves trying to make room for anything else we wanted to add. So the second time through, we took the reverse approach. We allotted times to Probability and Statistics standards according to what we believed they necessitated, and then found ways to reduce time spent on the other branches.

Three important themes emerged from the Statistics and Probability standards we were adding. First, we added a Thread for a Bayesian approach to probability, to complement the existing Frequentist approach. Second, we made sure to spend a lot of time on interpreting graphs and the claims they supposedly support. Much of Statistics in particular has been off limits due to the intensity of the rigor it takes to *produce* a statistical output, however, in the PreK-9 curriculum we are not training students to be statisticians; we are training them to be statistically *literate*. Therefore, they will get exposure to concepts and practice interpreting statistical claims, instead of focusing on reproducing calculations. Finally, third, we are adding in some content about Combinatorics; the underlying calculations of Combinatorics are quite simple (basic arithmetic operations), but the process of thinking through mathematizing a combinatorial situation in order to do the calculations is far from easy.

Modern Math: Discrete/Computational Branch

Another large change we have made is adding modern branches and subjects of mathematics, which have not been traditionally included in curriculum because they are too new and typically seen as too difficult to grasp. However, exposure to the high level ideas of these branches that have relevance for the modern world is crucial for students who are going to be informed citizens in the 21st century.[41]

The new branch is focused on Discrete/Computational Math, and covers subjects such as algorithms, network graphs[42], game theory, and complex systems. Again, it is not crucial that students master the advanced computations necessary for these fields, rather that they gain exposure to the ideas that shape the world they live in.

Using technology as a tool

There is a long-standing debate in Mathematics education about the use of technology. On the one hand, if students are not given the opportunity to understand for themselves the foundational aspects of Number, their manipulation of number using tools is going to be tenuous. However, research has shown that students who are given calculators from an early age do *not* appear to develop a reliance on them.[43] Further, calculators can be used for many purposes, including creative exploration, manipulating concepts without getting stuck in procedures, and allowing teachers to detect misconceptions in student understanding.

In fact, it can be a useful thought experiment to consider providing students with calculators, graphing software, and other tools and see what mathematics is left for them to do. This environment would most closely mimic the environment they will find themselves in outside of a school context, in the real world, where we want their learning to transfer to.

There is also evidence that providing students with technology has a motivating effect.[44]

---

[41] https://curriculumredesign.org/wp-content/uploads/Recommendations-for-PISA-Maths-2021-FINAL-EXTENDED-VERSION-WITH-EXAMPLES-CCR.pdf

[42] Aka Graph Theory, but we avoid this term because it is confusing within the preK-12 curriculum with "graphs" and "graphing"

[43] Ruthven, K. (2009). Towards a calculator-aware number curriculum. *Mediterranean Journal of Mathematics Education*, *8*(1), 111-124.

[44] https://www.slideshare.net/OECDEDU/equations-and-inequalities-making-mathematics-accessible-to-all

Using a computer during mathematics lessons is associated with higher motivation for learning mathematics

*Change in intrinsic motivation for mathematics associated with using a computer in mathematics class*
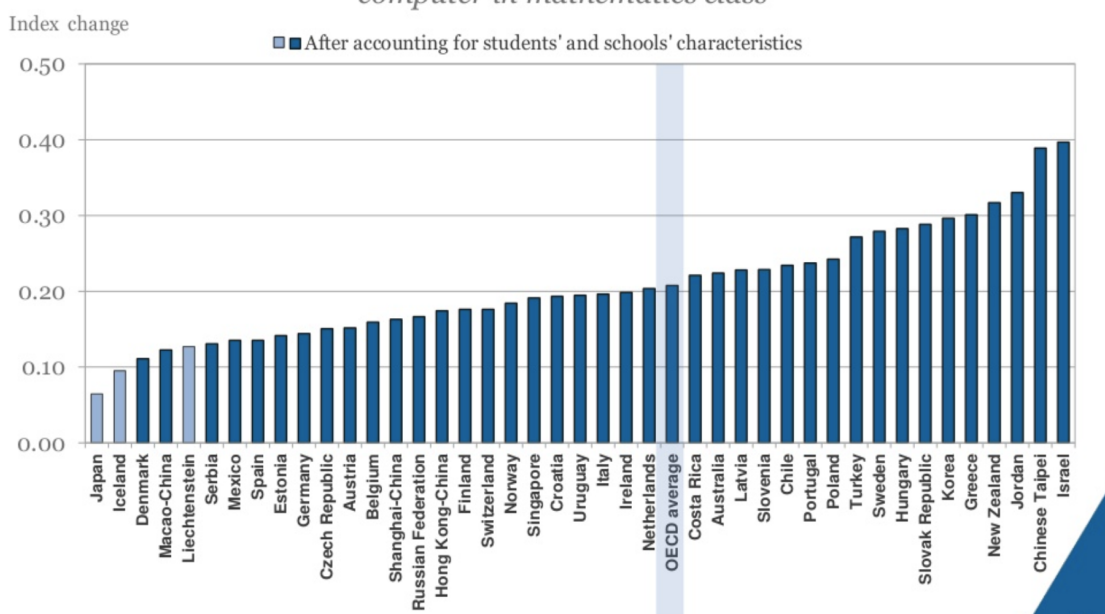
Along the same lines, there are some tools[45] which we want students to comfortably use in their lives to do mathematics, and yet we do not teach students to use them properly. A great example of this is spreadsheets. Spreadsheets let students manipulate numbers and patterns and see the direct result of their manipulation in real time.[46] Therefore, we were very cognizant of supporting the moments in the curriculum where technology could allow students to reach a greater depth or application of a math concept, while still building up the foundational maths that underlie an understanding of Number.

In the graph below, the vertical axis represents the estimated time spent on standards, and the horizontal axis represents grade level. The graph shows how much time is spent on standards that involve the use of technology, and which Branches these standards are in. Statistics and Probability seems to be evenly spread from third to seventh grade, and Algebra exhibits a steady increase from grade 6 to grade 8.

---

[45] In High School, more advanced tools such as MATLAB can be considered
[46] There is even one curriculum which reproduces all of K-12 mathematics with spreadsheets: whatifmath.org
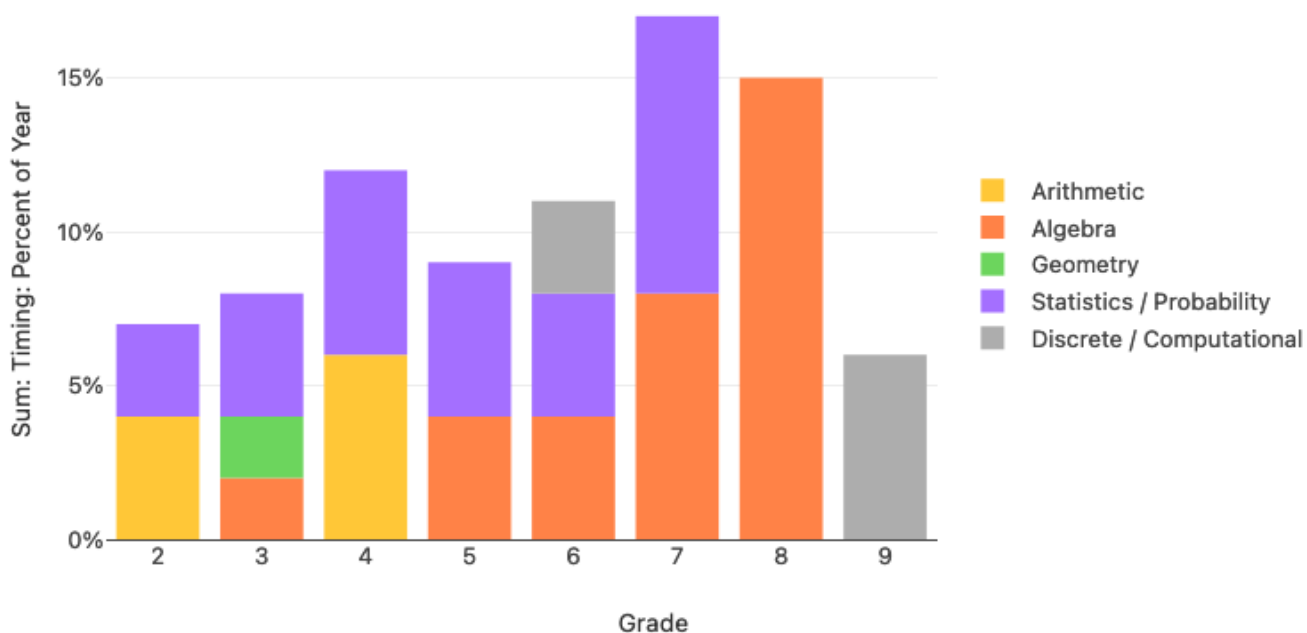
Careful treatment of points of cognitive density

Traditional mathematics curricula often spend too much time on teaching students to produce mathematical procedures, and do not leave room to dig sufficiently into moments of high cognitive density, such as:

- **Leaps in abstraction**, requiring the student to have already built up multiple concepts in their minds from previous years to be able to apply and combine them in new ways. For example, CCR requires students to consider the concept of a "solution space" with inequalities on the number line, far before they apply that idea to algebraic relations; by the time they get to that point, they will not need to *overcome the misconception* of a variable as a single unknown in order to understand the variable as representing the many possibilities that would make the expression true.

- **Topics where** the procedure is relatively straightforward, but *the concept is complex*. In this set of standards, CCR paid close attention to the order and emphasis of such concepts and topics. For example, the mean is relatively simple to calculate and thus is often taught relatively early. But research suggests that many students struggle to grasp the implications of the mean as a concept (as opposed to median, for example) and thus end up struggling to apply the concept accurately outside of a classroom setting (for instance, by assuming that all distributions are symmetrical, and thus underappreciating the importance of the median). To

implement suggestions from research, the CCR standards place the idea and calculation of a statistical mean in its context among other measures like shape and spread such that it can be discussed within the bigger discussion of a "statistical toolkit" instead of it being taught formally, too early, and overemphasized.[47]

Standing on the Shoulders of Giants

A fundamental principle underlying all of the work at CCR is taking the best of what exists in each category and use it as a starting point for further improvement. This ensures that we can remain a small and nimble team that is nevertheless able to accomplish large projects, without reinventing the wheel.

We began our process of math standards design by comparing standards sets from across the world to find the ones that were most comprehensive and succinct, as well as well-aligned to our goals of concept connections, interdisciplinary connections, and competency connections. From this analysis, we decided to start with the standards of the Australian Curriculum Authority (ACARA) as our base.[48]
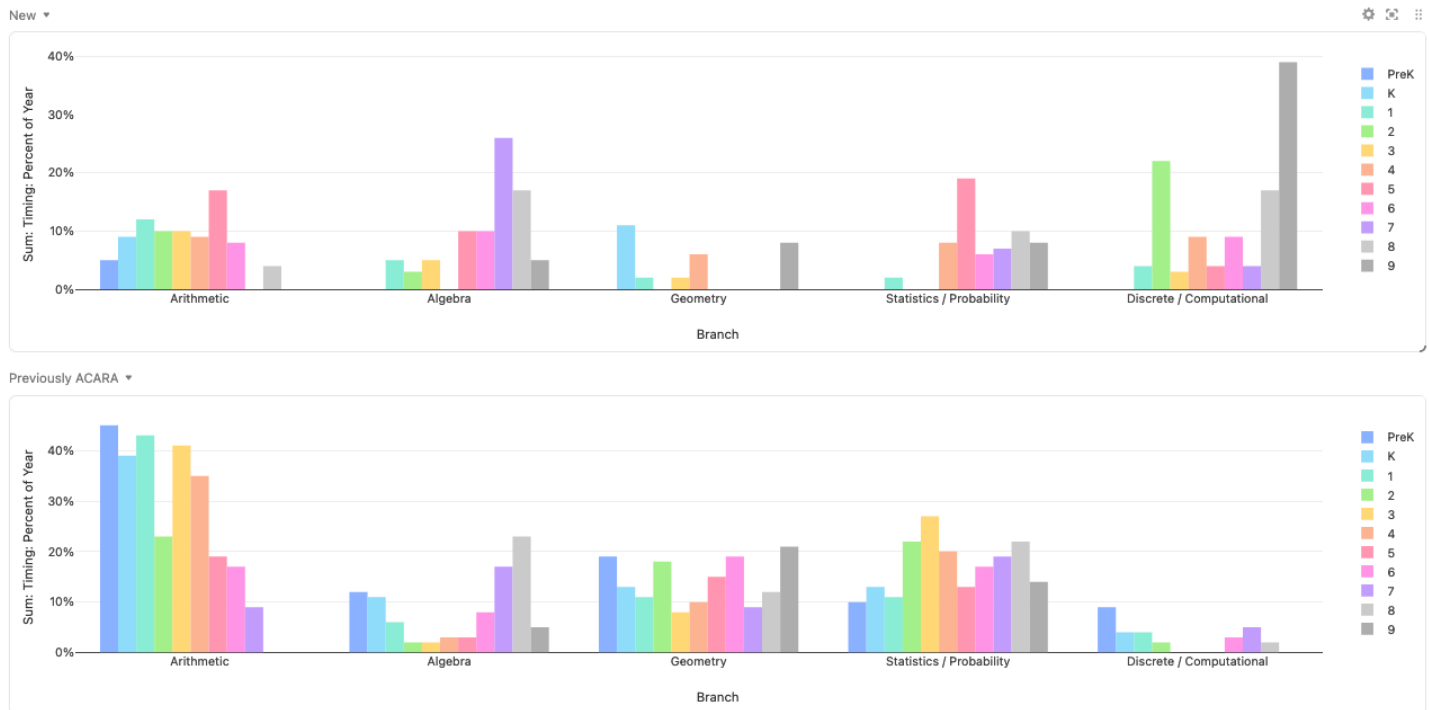
In addition to conducting our own literature reviews, we worked closely with Cambridge Mathematics [49], who have been conducting in depth mapping of mathematics curriculum for years. This helped us determine the areas that most needed to be improved upon based on a synthesis of findings about the best ways to teach mathematics. We were then able to focus our efforts on the areas that needed the most change.

The graph below shows the standards that A) originated with CCR, and B) originated with ACARA.

---

[47] https://www.cambridgemaths.org/espresso/view/concepts-of-the-mean/
[48] https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/
[49] https://www.cambridgemaths.org/

Perhaps unsurprisingly (though still importantly), the standards we kept the most were ones that were more foundational: lower grades Arithmetic and Geometry. Branches where we made the most changes were Algebra (adding modeling, specifically with exponentials), Statistics/Probability, and the new branch, Discrete/Computational .

Threads Across the Standards

The primary organizational feature of CCR's standards is that every standard is in careful connection with every other standard, across a multitude of dimensions. To do this, we reviewed the relevant research[50] to connect the standards to each other directly as well as indirectly, and mapped the standards appropriately.

Direct Connections: Coming From and Leading Up To

By tagging standards to each other, we were able to go one step further than any standards we had seen, by making concrete claims about how earlier standards are meant to prepare students for later

---

[50] e.g. https://www.cambridgemaths.org/ and
https://www.education.vic.gov.au/Documents/school/teachers/teachingresources/discipline/maths/mynumfreport.pdf

ones. This allows us to create visualizations to represent these interconnections. For example, the graph below represents all the standards in the Geometry branch, and color represents grade.
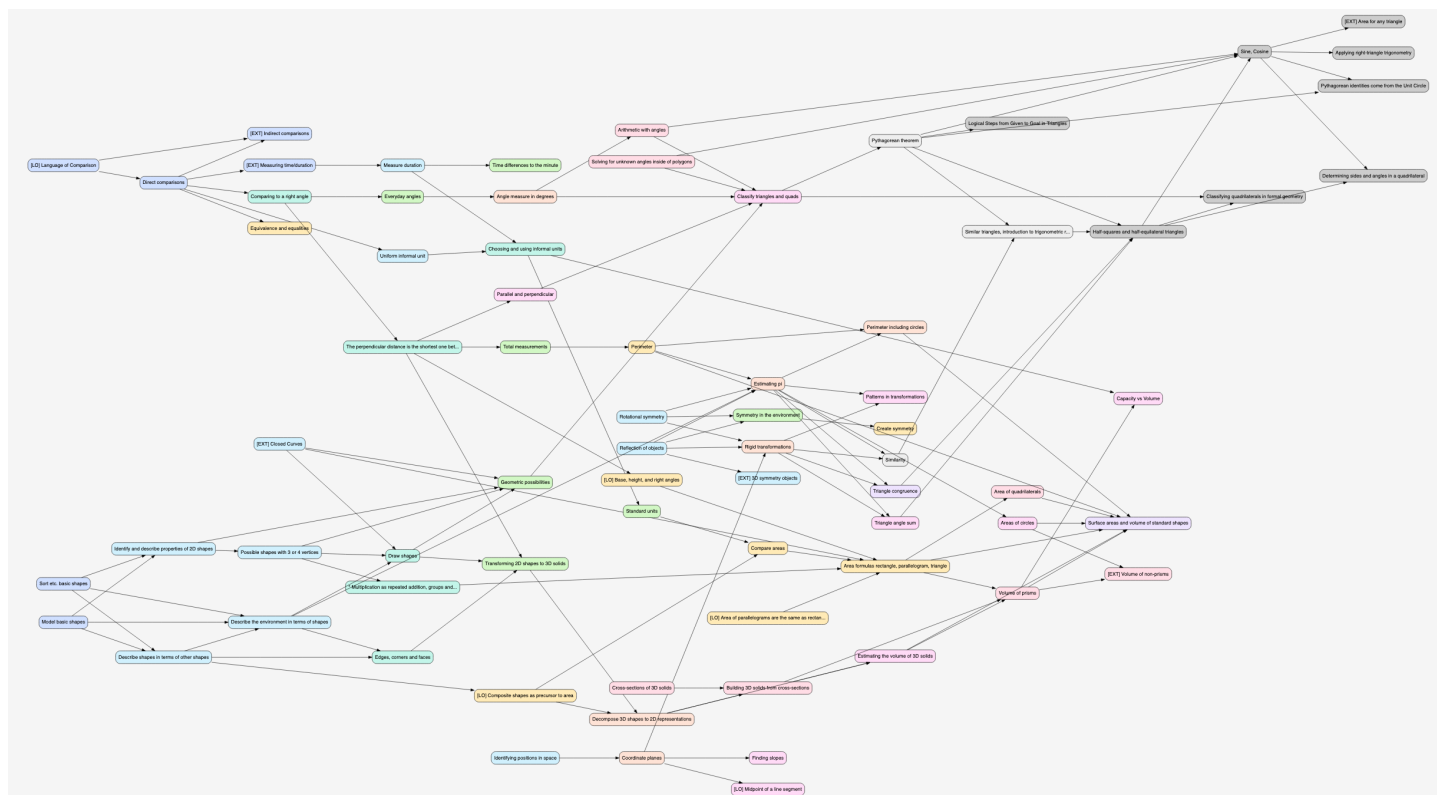
This lets us make sure that students are properly prepared for standards, and that all our progressions are logical. For example in this case, we can see that there are two disconnected entrances into Geometry. Upon inspection it is clear that one comprises all the "Measurement" standards, and the other comprises the "Shape" standards. Some of the standards at the intersection of the two are "Discovering Pi" and "Measuring Perimeter."

These networks of standards can highlight not only entire Branches but also *specific moments we want to make sure to focus on*. The BCC Thread "Growth", for example, was one that was reasoned deserved extra consideration, so we created the following graph to represent the progression:
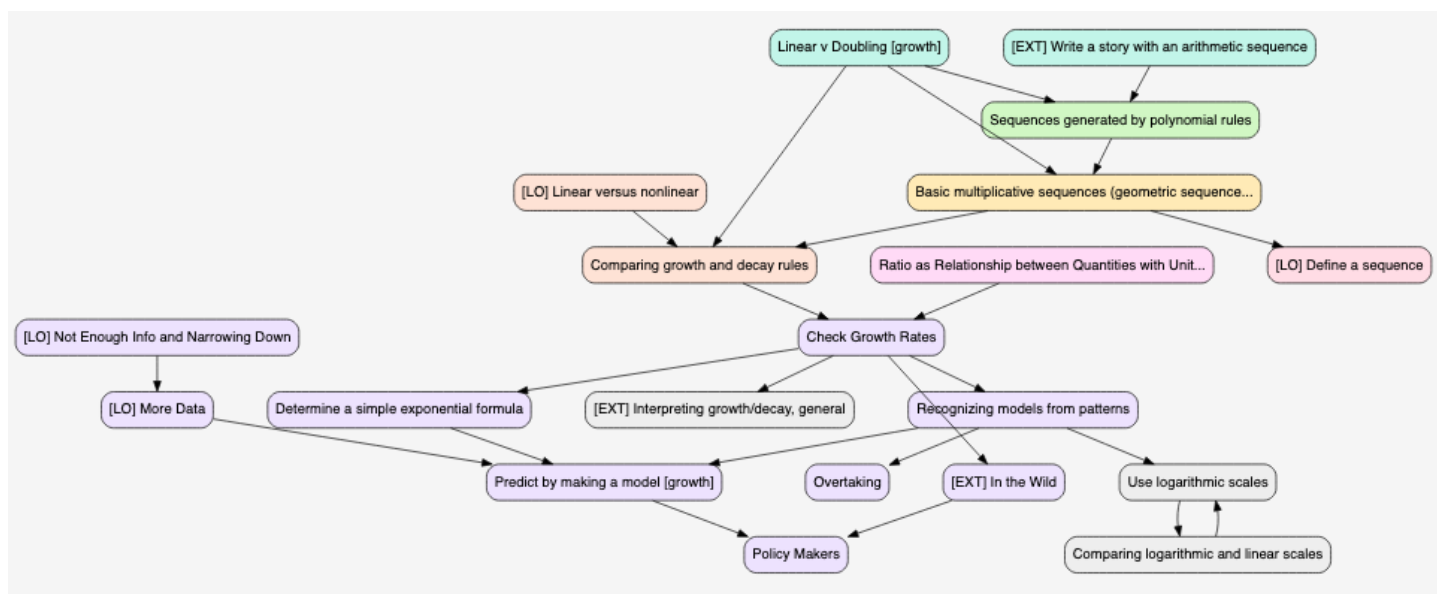
Tracing standards directly lets curriculum designers and teachers know where the students are coming from (in other words, what knowledge they should build off of), and where they are going (what knowledge they are preparing their students to learn). This should fundamentally shape how one sees each standard -- not as a singular entity but as a stop in many overlapping routes through Mathematics.

Indirect Connections: Threads

The other way that CCR standards are connected to each other is via Threads, which we define as any important idea that we track across standards. That wide of a definition means we've already mentioned a few threads, most notably Branches and Discipline-level Core Concepts. But anything at all can be a Thread; for example, we also discussed standards that are based in the Real World - which were tracked using Threads as well.[51] In this way, standards are connected with many overlapping Threads that are woven through them, to create an emergent understanding that is carefully constructed to be coherent from many different perspectives.

The different branches receive different treatment across the grades. Below, we can see the recommended emphases on each of the branches from the National Council on Teaching Mathematics juxtaposed with the data from our standards, showing the relative emphases. Although NCTM includes Measurement as its own branch and we include Computational/Discrete, the general patterns hold up: there is less and less Number/Arithmetic as there is more and more Algebra, and Geometry remains relatively stable throughout. However, in our standards we put far less emphasis

---

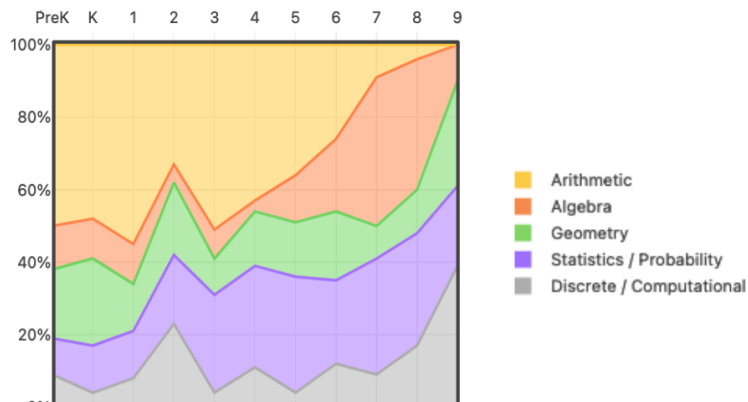[51] These threads were Scenario, Real World, and Interpretation

on Geometry overall, and greater emphasis on Statistics and Probability as well as adding Discrete/Computational as a whole new branch.

NCTM: CCR:
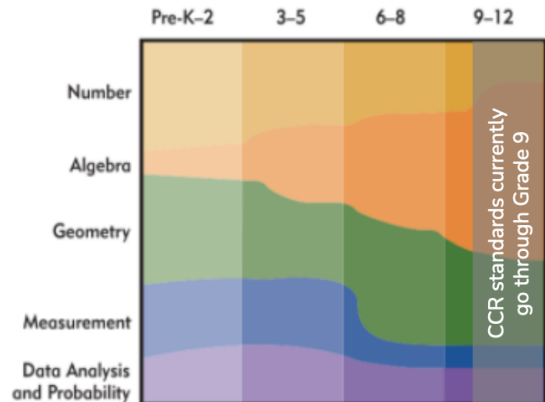
The Content Standards should receive different emphases across the grade bands.

The CCR Content Standards' distribution by branch across grades

Further, a significant portion of CCR standards (47/348) are categorized into multiple branches. This encourages connections to be made between pieces of Mathematics that have been traditionally kept apart for logistical reasons.

We will now discuss each CCR Branch as it appears in the CCR standards[52], some of the most important Threads that run through each, followed by a discussion about the connections between the branches.
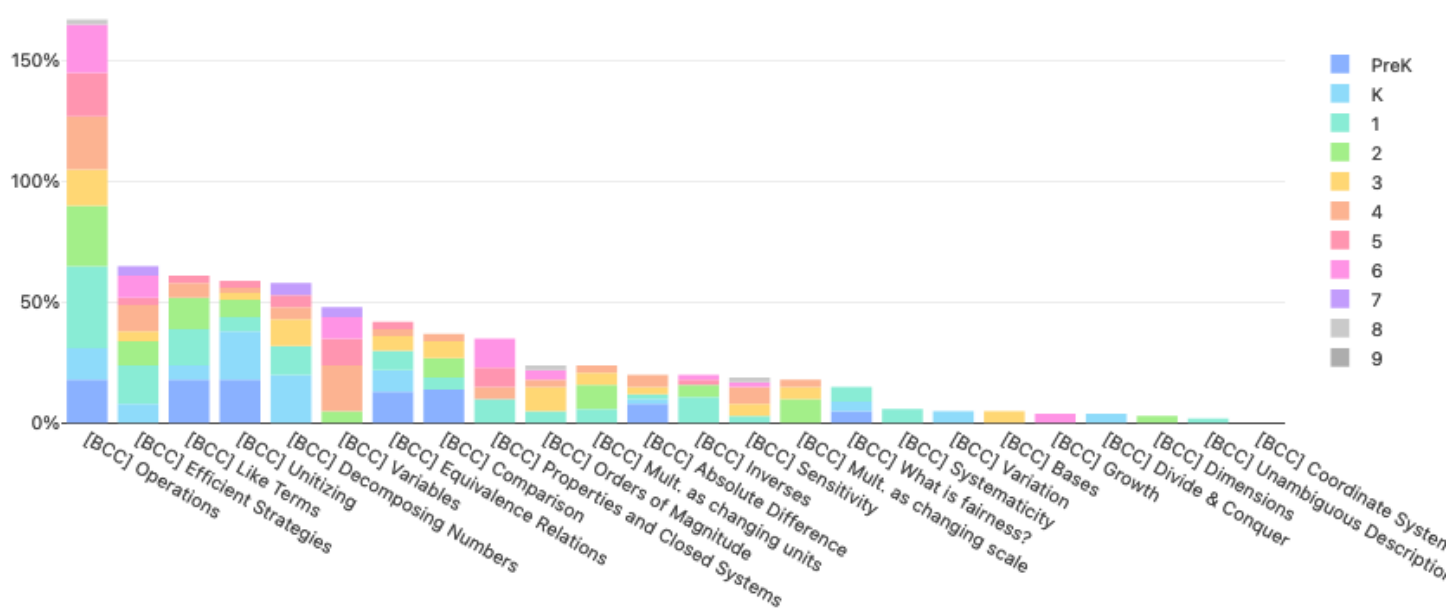
Arithmetic

Core Concepts and Content Threads

The most emphasized DCCs in Arithmetic are Check Expectations, Recognize Mathematical Structures, and Different Representations. This reflects CCR's commitment to highlighting the thinking skills math can instill and to letting students practice seeing math in the world around them.


[52] The order the branches are presented in is not meaningful.
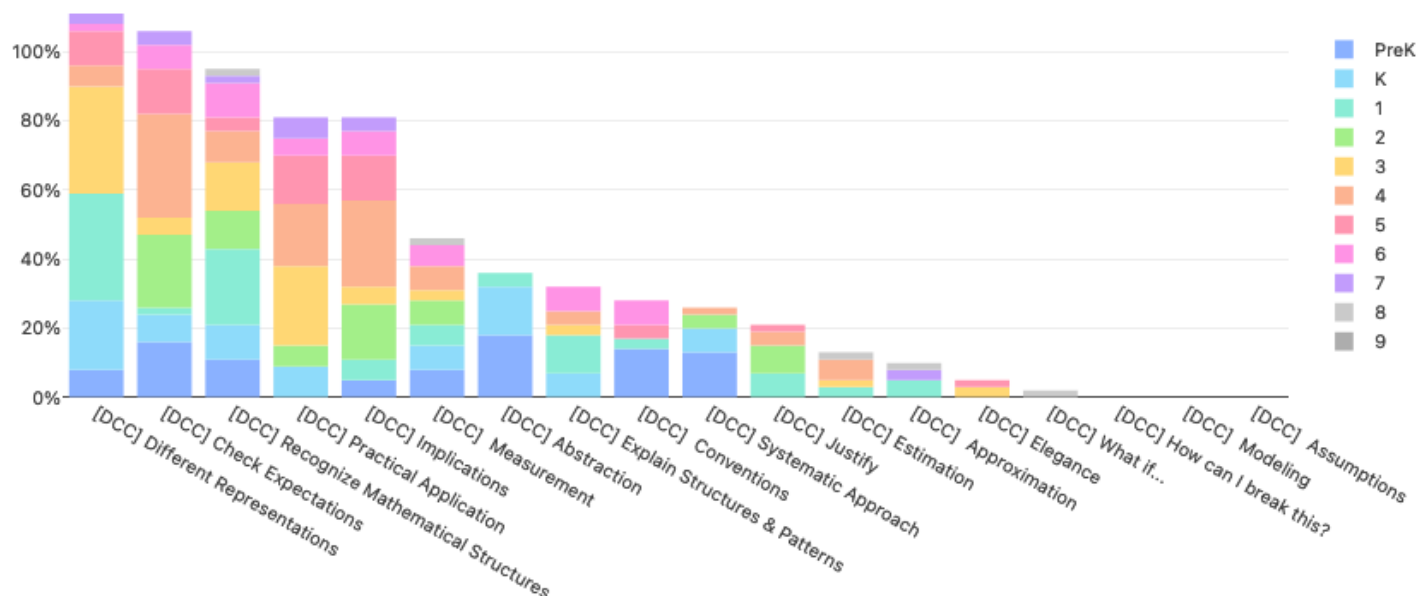
The most emphasized BCC in Arithmetic is Operations (by a very large margin). There are 44 standards tagged Operations and together they account for over a full year and a half of math curriculum. However, it is important to note that the Core Concept of Operations is not simply the act of computation, but rather it is about the exploration of the way that operations work, i.e. the ways mathematical objects can be combined to make new mathematical objects.

The most common LPs in Arithmetic are Proportional Reasoning and Multiplicative Thinking, since these are the places where Arithmetic content tends to come together. Next, Additive Thinking and Spatial Reasoning (followed by Unitizing and Counting) are also significant LPs, clearly emphasized more in the lower grades. There are several LPs that feature more prominently in other Branches which make an appearance in the standards tagged with Arithmetic.
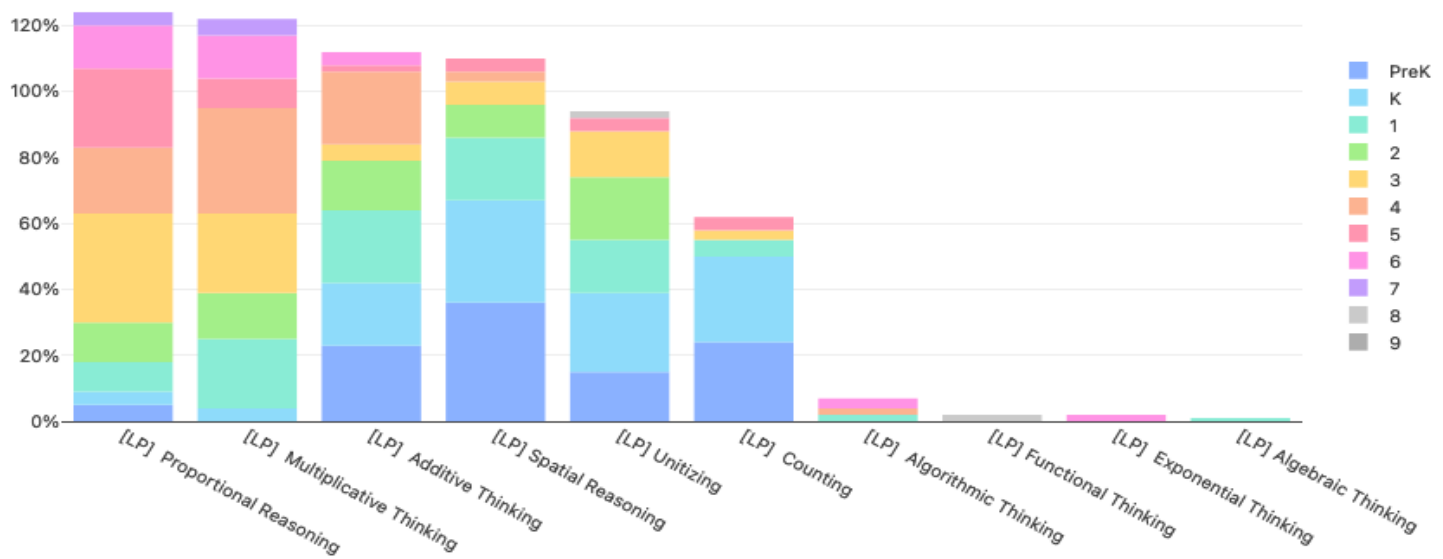
The most emphasized Content Threads in Arithmetic are Fractions, Decimals, and the Number line, showing the more Content-oriented description of Proportional Reasoning and Multiplicative Thinking.
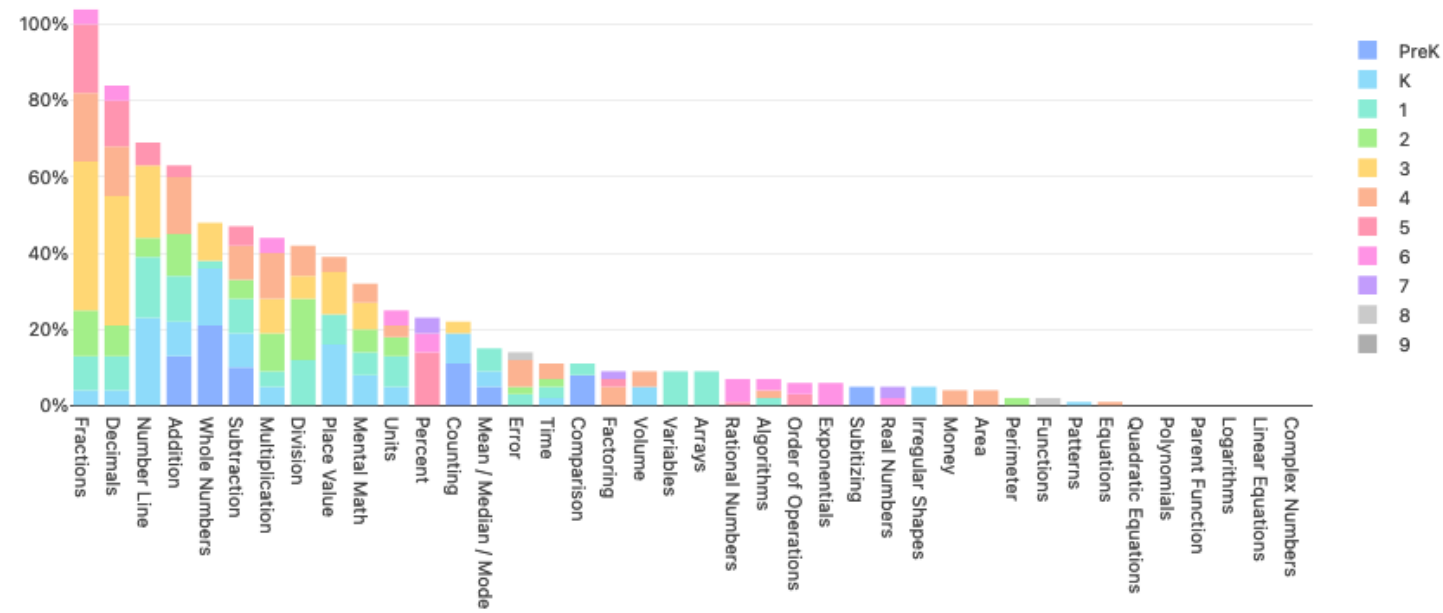
Overall Narrative

The purpose of the Arithmetic branch is to build up students' conception and intuition around Number based on their natural understanding of quantity. That means we begin with concrete counting, grounded in manipulatives and spatial representation, and slowly move our way from counting to addition (and its connection to subtraction), then to multiplication (and its connection to division). Throughout all this, the connection to different kinds of representations of Number are emphasized.[53]

Division is introduced both in terms of dividing one unit *and* a way to divide a collection of units. This sets students up for more fluent understanding of proportionality,[54] probability, as well as higher level Algebraic reasoning with Unitizing. At this point, Arithmetic begins to be used as a tool to be applied: both to realistic scenarios and to other branches of math.

Algebra

Core Concepts and Content Threads

The most emphasized DCCs in Algebra are Recognizing Mathematical Structures and Explaining Structures and Patterns. This reflects CCR's conceptualization of Algebra as related to the structure of mathematics (rather than, say, the procedures associated with manipulating expressions).
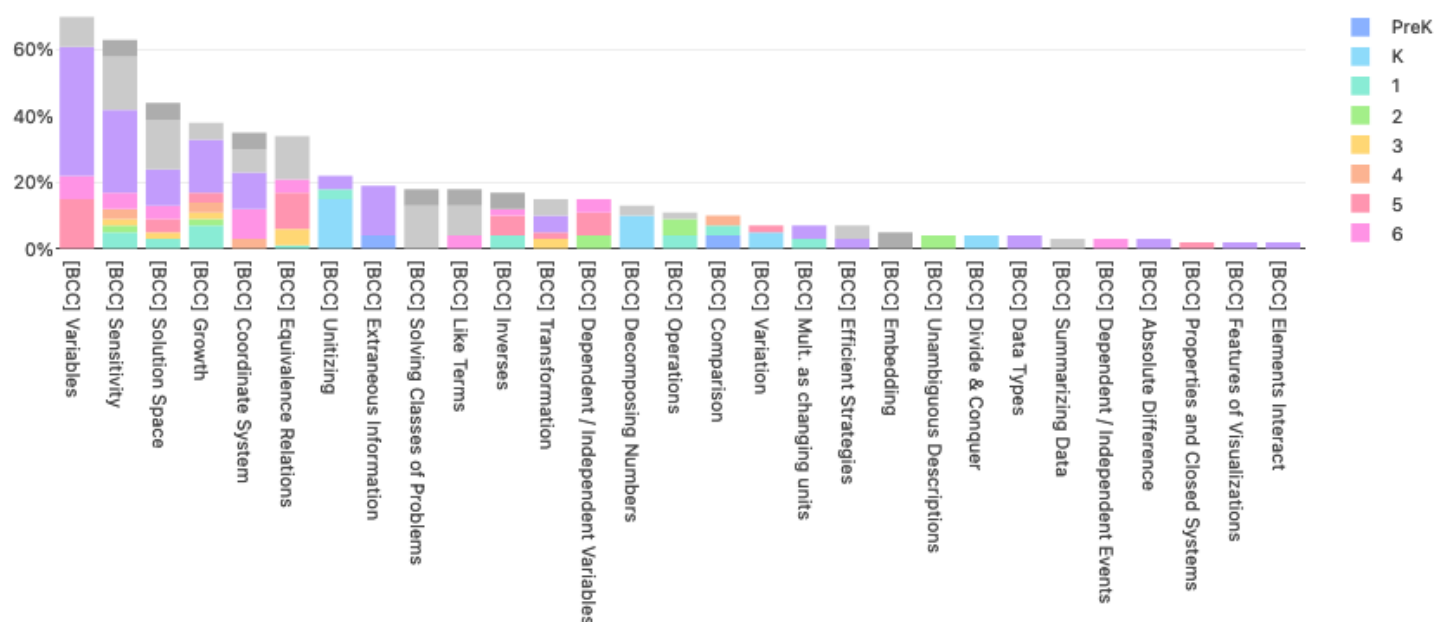

The most emphasized BCCs are Variables, followed by Sensitivity and Solution Space. Variables, of

---

[53] e.g. manipulatives, number lines, arrays, etc.

[54] which can be represented with fractions, decimals, percents, rates, etc.

course, represent the fundamental new idea introduced in Algebra. Sensitivity is a Core Concept we describe as: "when we change parameters (digits, coefficients, input values), we want to determine what effect that has on the overall value or solution space. If there are several parameters, we want to determine which changes have the largest or smallest impact." This encompasses standards that ask students to consider an algebraic expression as a statement of relationships between variables, and how manipulating those variables affects the relationships.

The Solution Space BCC refers to CCR's focus on the conceptual jump from using variables to represent an unknown number that we can solve for (such as 3+x=7) to using variables to represent unknowns in the broader sense of modeling relationships. Rather than introduce this idea suddenly and subtly, the CCR standards underscore its existence in earlier grades to prepare students for this broader way of thinking.
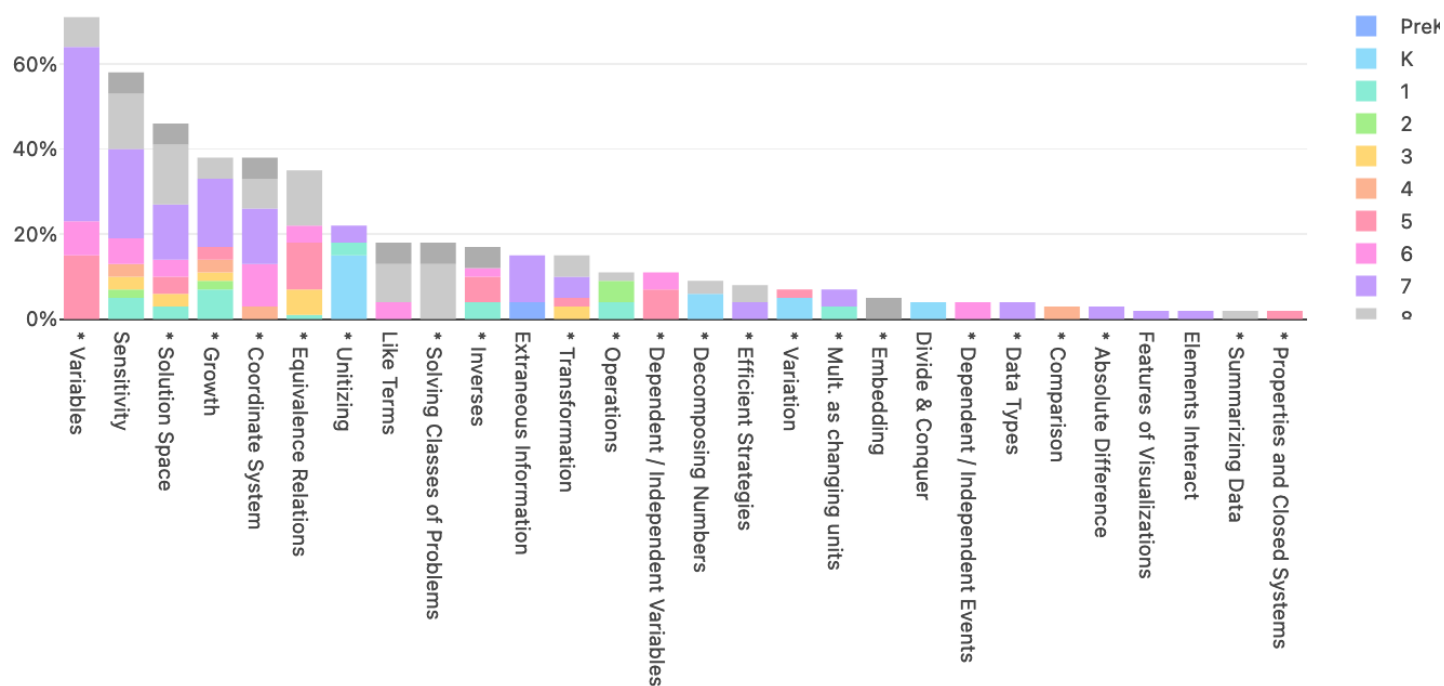
The most common LP in Algebra is Functional Thinking (though heavily weighted toward later grades), followed by Spatial Reasoning and Algebraic Thinking.
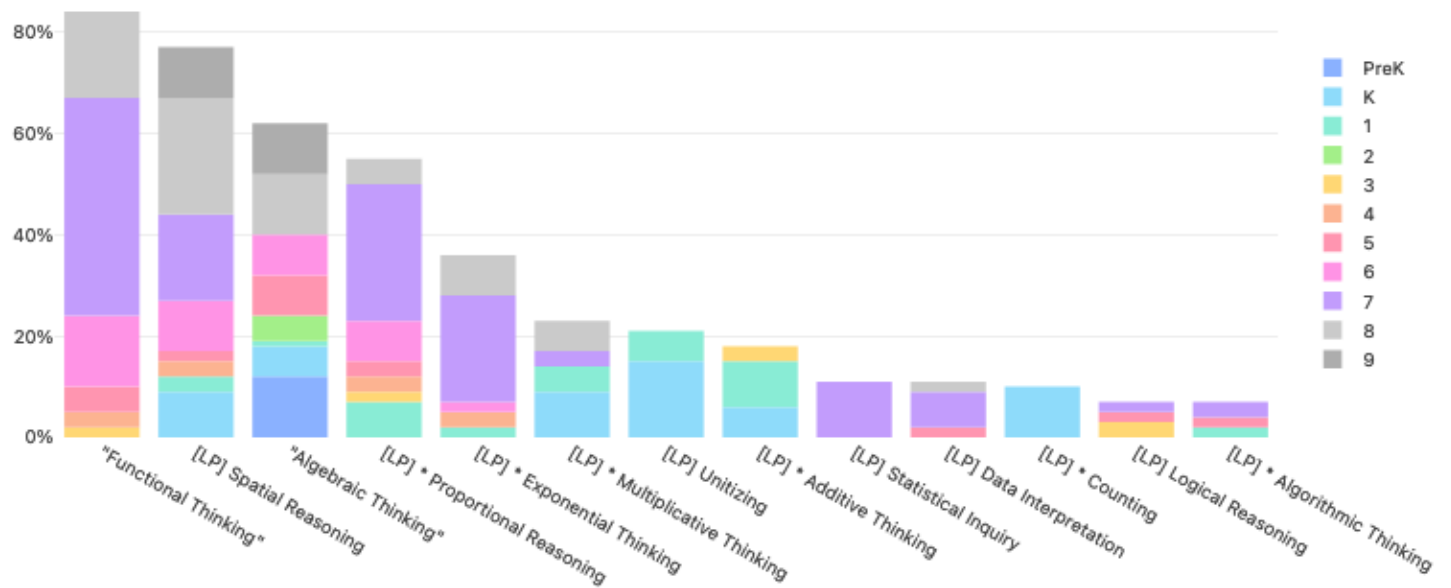
The most emphasized Content Thread in Algebra is Graphs, followed by Exponentials and Functions. All three of these topics are heavily weighted toward the later years, and notably the next most common Content Threads -- Patterns and Number Sequences -- exhibit the reverse pattern.
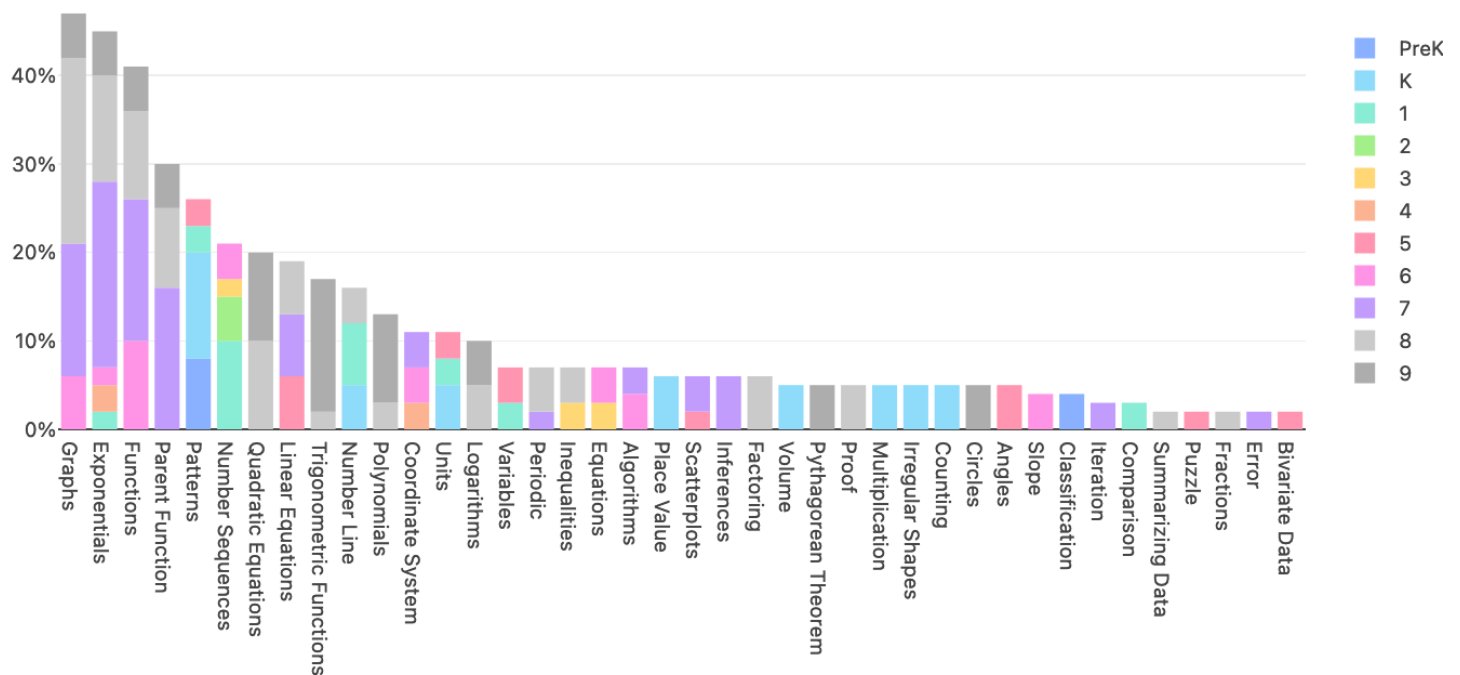
Overall Narrative

Algebra as a branch can be an elusive one to pin down, because it means different things to different people and at different ages.[55] Recent research on Early Algebra education urges curriculum developers to start Algebraic thinking as young as Kindergarten.[56] We begin introducing Algebraic thinking at the beginning of our standards, in PreK with identifying features of objects for the purposes of sorting, classifying, and creating patterns and continue to hone this study of pattern with number.

This leads to a focus on unknown quantities and thus on equivalence by examining how relationships can be transformed while maintaining either equality or inequality. This is designed to take the focus away from symbolic manipulation and onto the structural elements of the mathematical objects they are manipulating.

Once the basics are covered, the focus shifts to modeling relationships: describing covariance as growth and decay, and paying close attention to making choices about the type of model used (linear, polynomial, exponential, etc.). Rather than algebraic manipulation, the focus is on recognizing the underlying parent function along with its parameters and transformations using different representations.
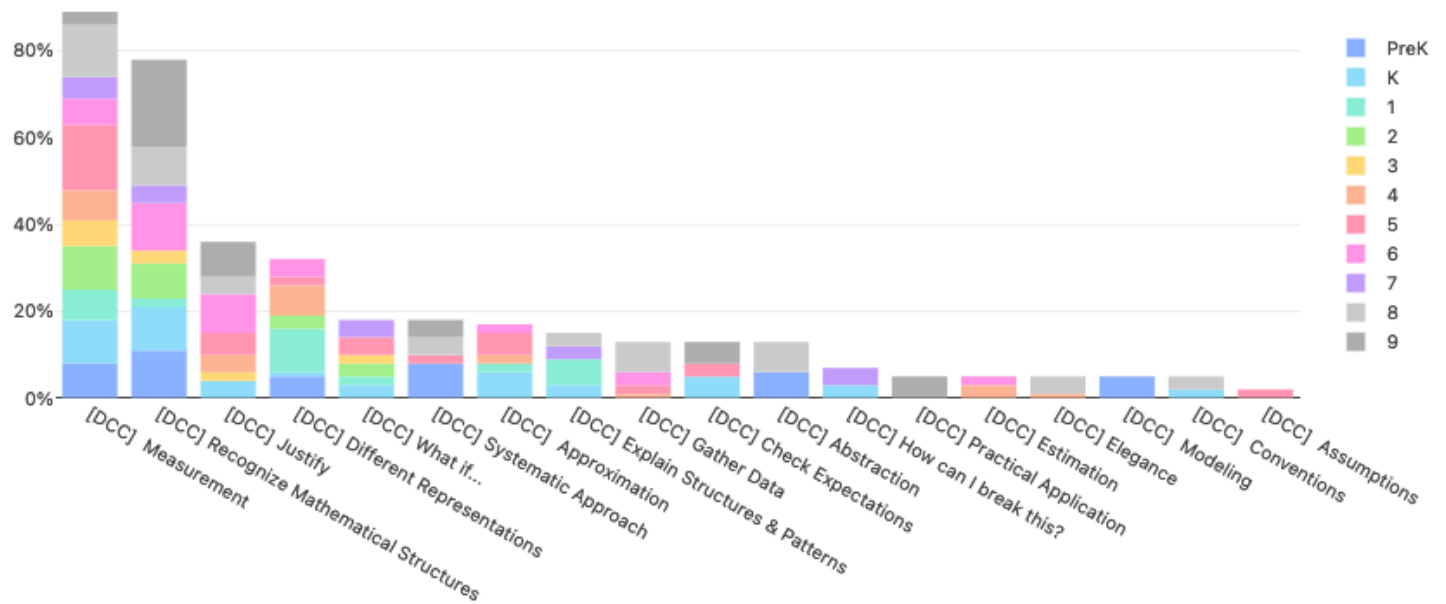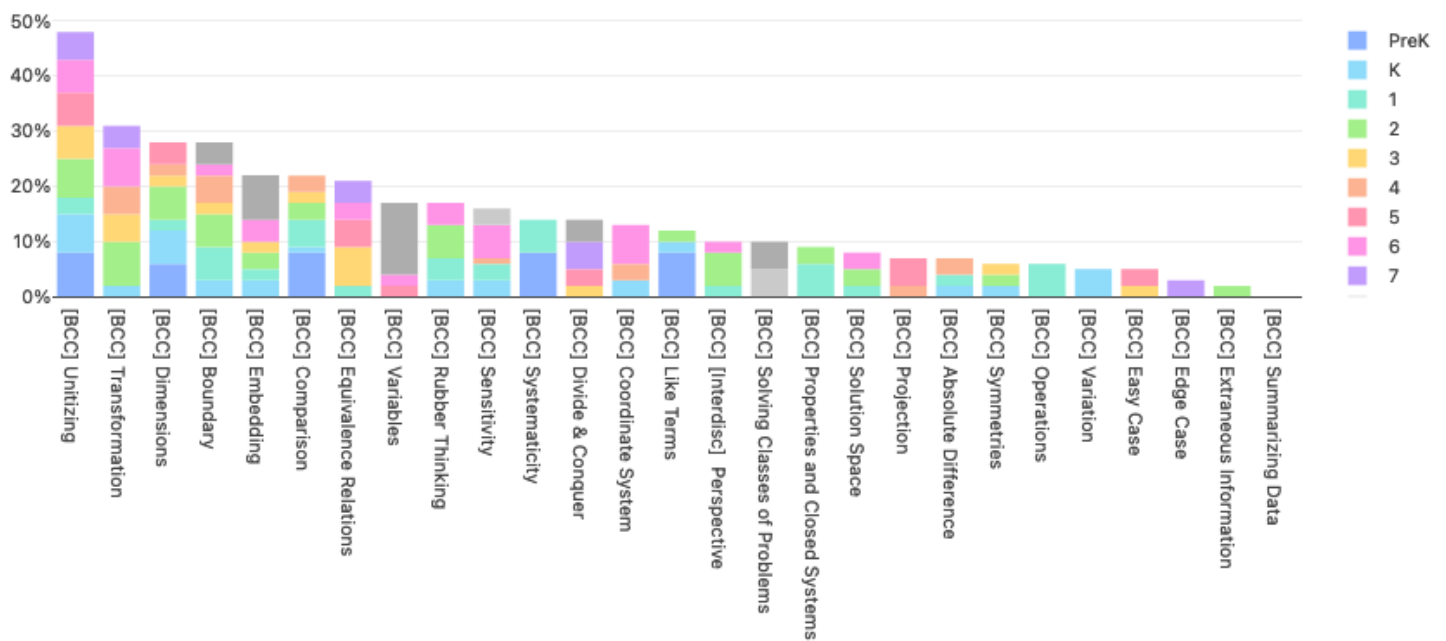
Geometry

Core Concepts and Content Threads

The most emphasized DCC in Geometry was Measurement, followed by Recognize Mathematical Structures. Those were followed by Justify and Different Representations.

---

[55] Different definitions we examined included "Doing and Undoing", "Functional Thinking", "Seeing mathematical Structure", "Generalizing Arithmetic" and more.
[56] e.g. NGA & CCSSO, 2010 as cited in Blanton, M., Stephens, A., Knuth, E., Gardiner, A. M., Isler, I., & Kim, J. S. (2015). The development of children's algebraic thinking: The impact of a comprehensive early algebra intervention in third grade. *Journal for research in Mathematics Education*, *46*(1), 39-87.

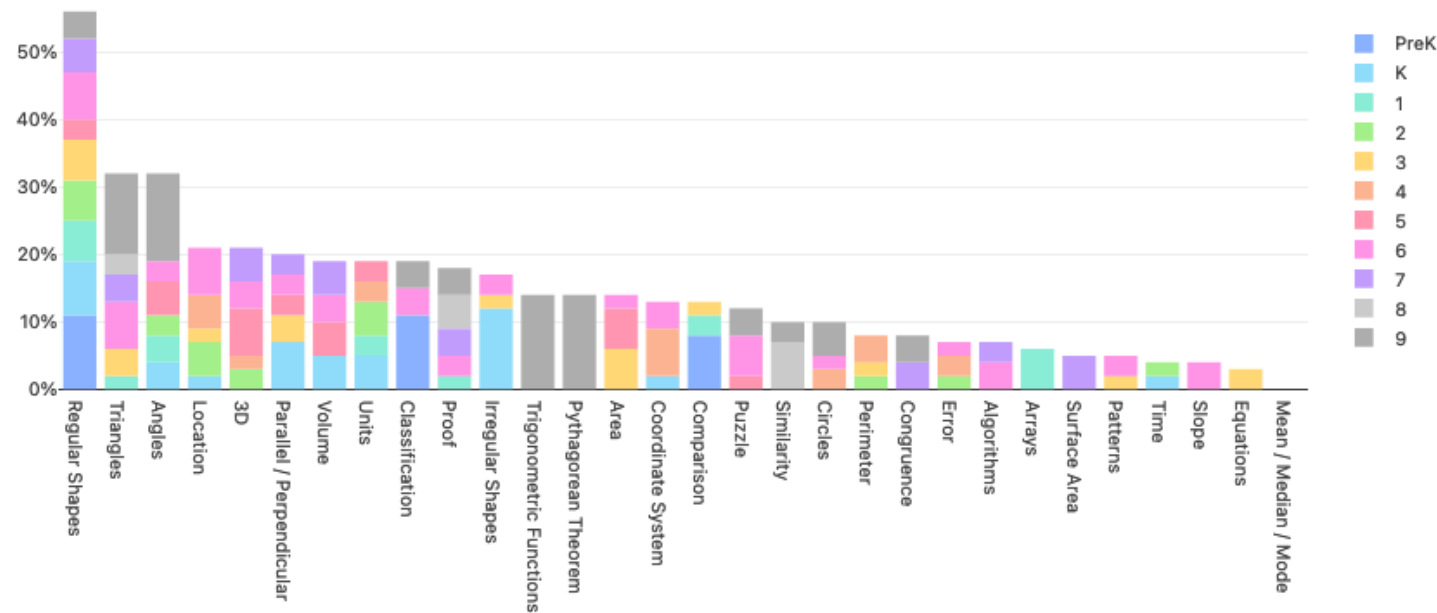The most emphasized BCC in Geometry was Unitizing, followed by Transformation.

The most emphasized LPs in Geometry was Spatial Reasoning.
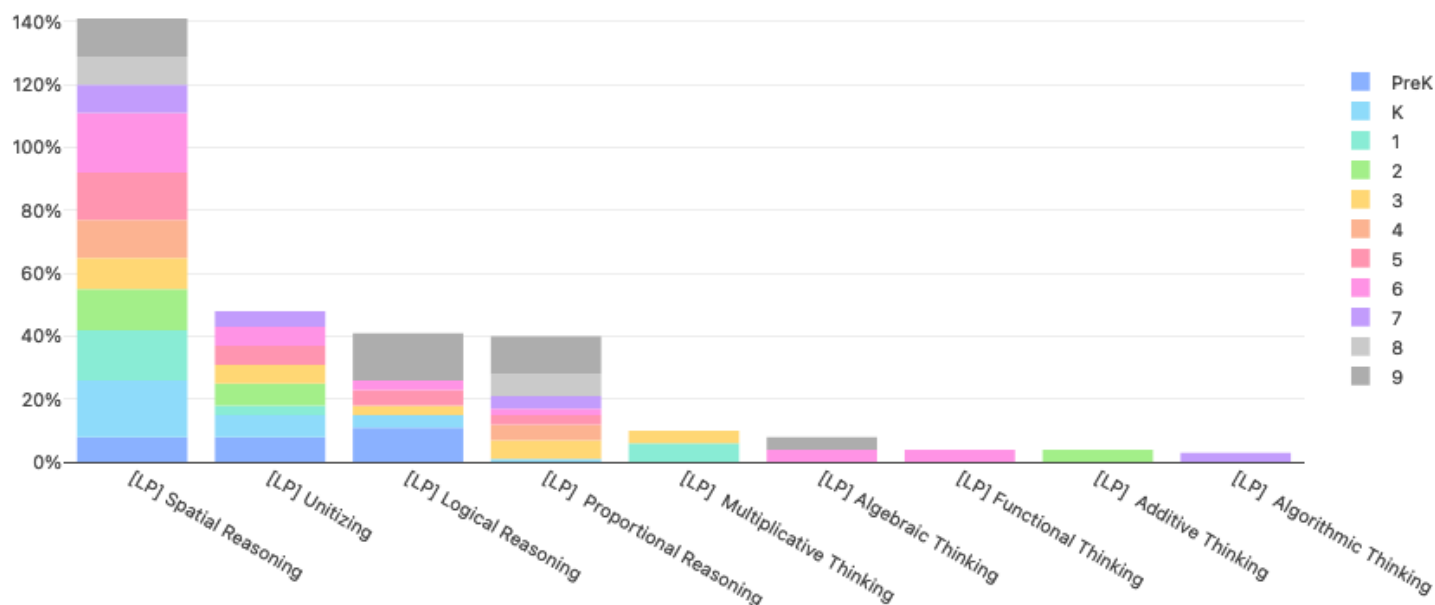
The most emphasized Content Thread in Geometry was Regular Shapes.

Overall Narrative

The power of Geometry shines most when it is being used as measurement — translating the world into mathematical units — or when it is illustrating a mathematical object in a spatial representation.

In particular, the importance of the Coordinate System connecting Algebra and Geometry is key for students to develop a deep understanding of either. Geometry is also often used as a vehicle for teaching proof; in the CCR standards, there are some instances of this, however, *formal* proof is avoided, while the idea of logical reasoning is interwoven throughout the branches and disciplines, and geometry is stressed as a tool to understand the world.

The CCR Geometry standards begin with familiarizing students with the idea of shape, and how to see it in the world around them. They continue the idea of decomposing objects and shapes into simpler shapes, while also building up basic Geometry ideas such as area, perimeter, angles, transformations etc., as much as possible connecting it with Algebra and the other branches.

Probability / Statistics

Core Concepts and Content Threads
The most emphasized DCCs in Probability / Statistics are Different Representations and Implications.
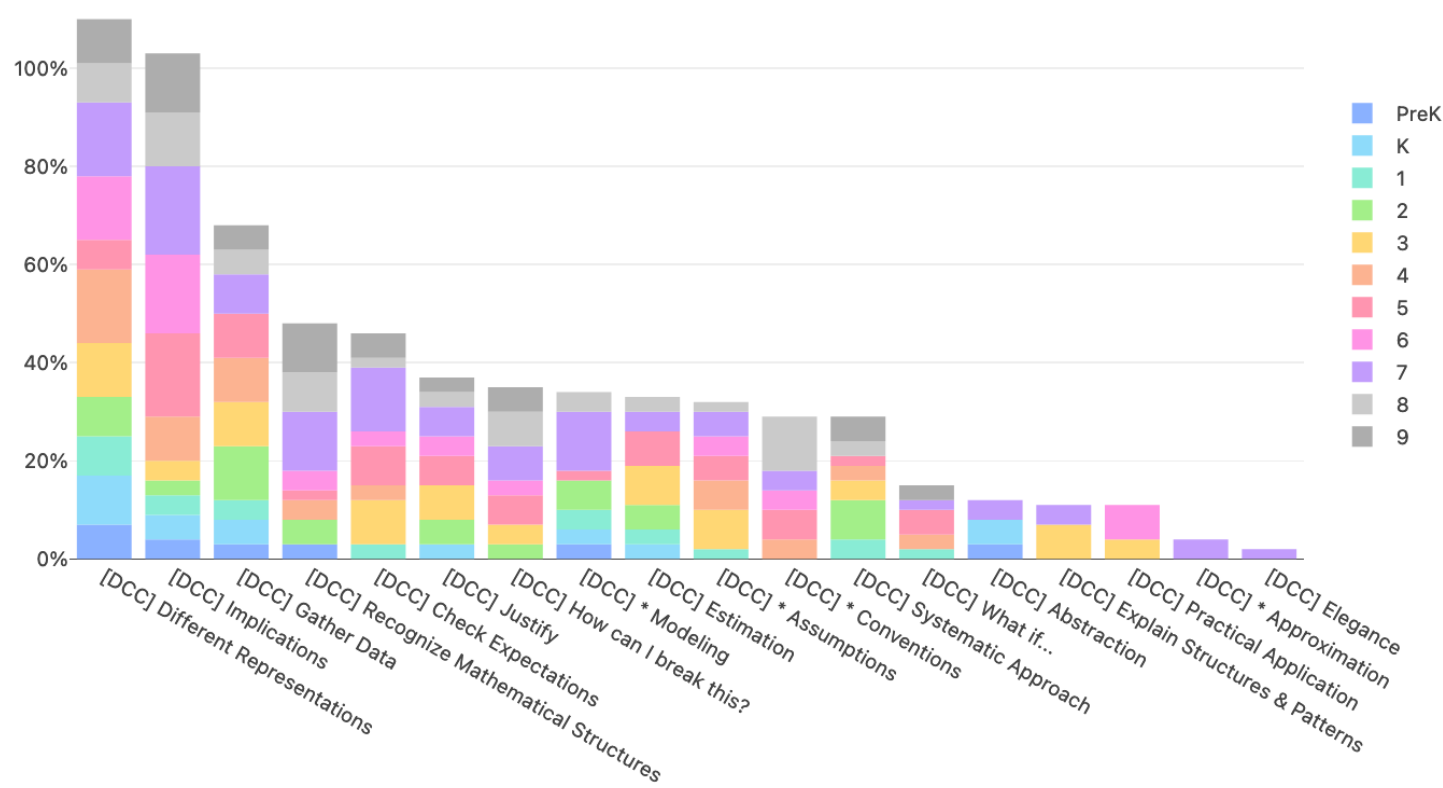
The most common BCC in Probability / Statistics is Summarizing Data.
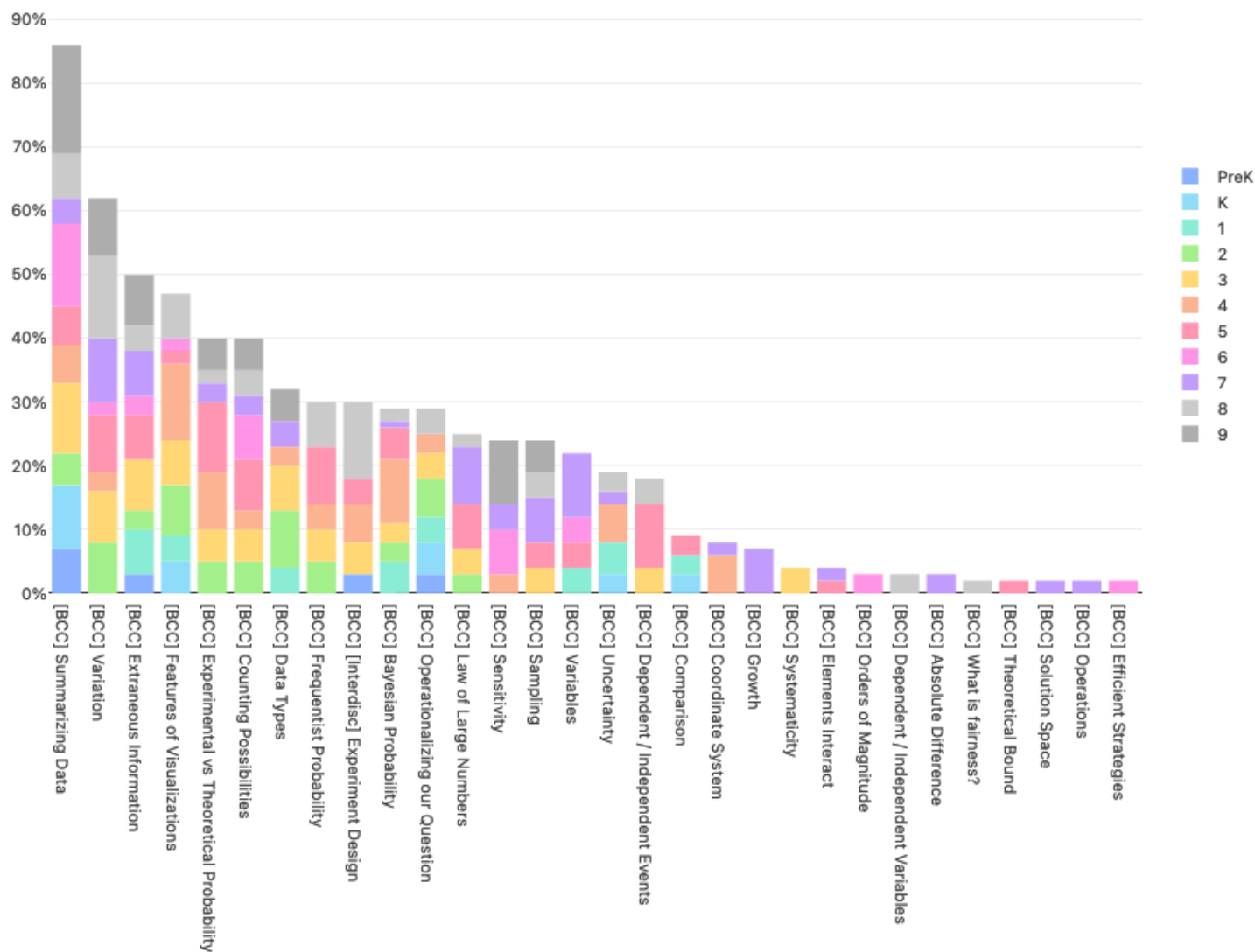
The most common Learning Progressions in Probability / Statistics are Data Construction & Interpretation, followed by Statistical Inquiry, Probabilistic Reasoning, Bayesian Thinking, and Functional Thinking. While one may think of Functional Thinking as specific to Algebra, our close focus on understanding Data through Algebra boosted the role of Functional Thinking in Statistics.
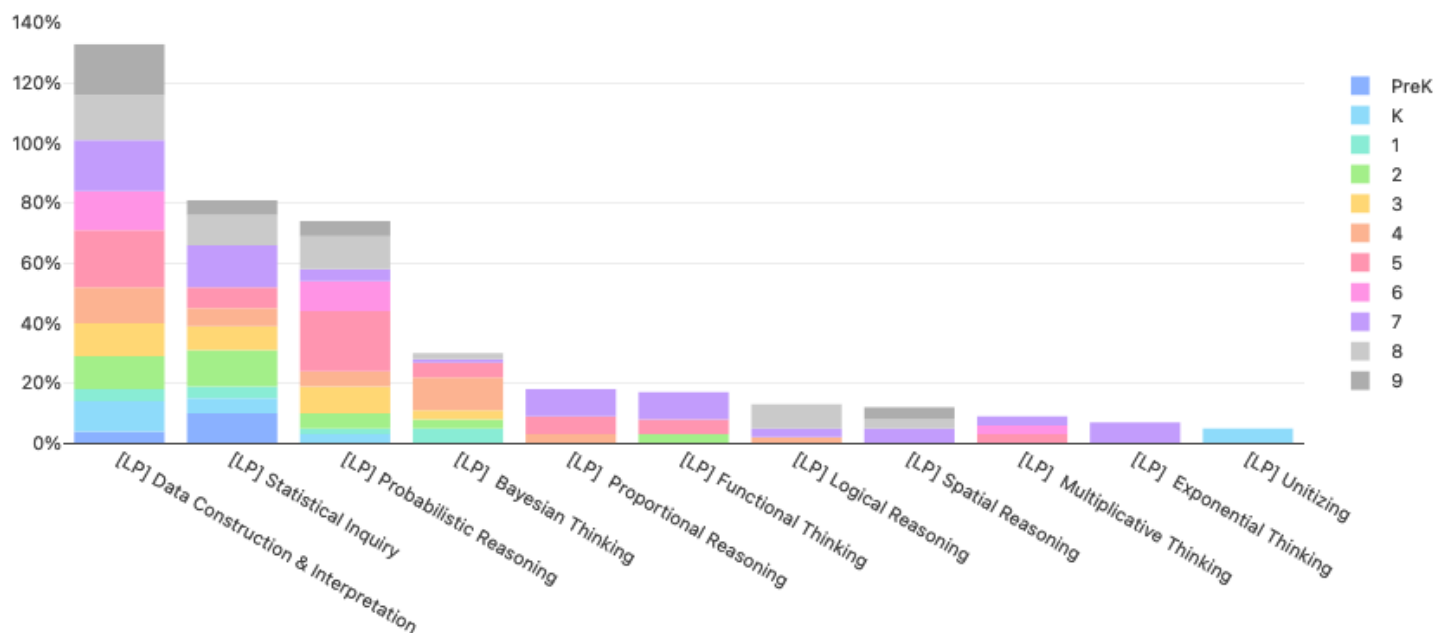
The most common Content Threads in Probability / Statistics are Graphs, followed by Chance and Inferences.
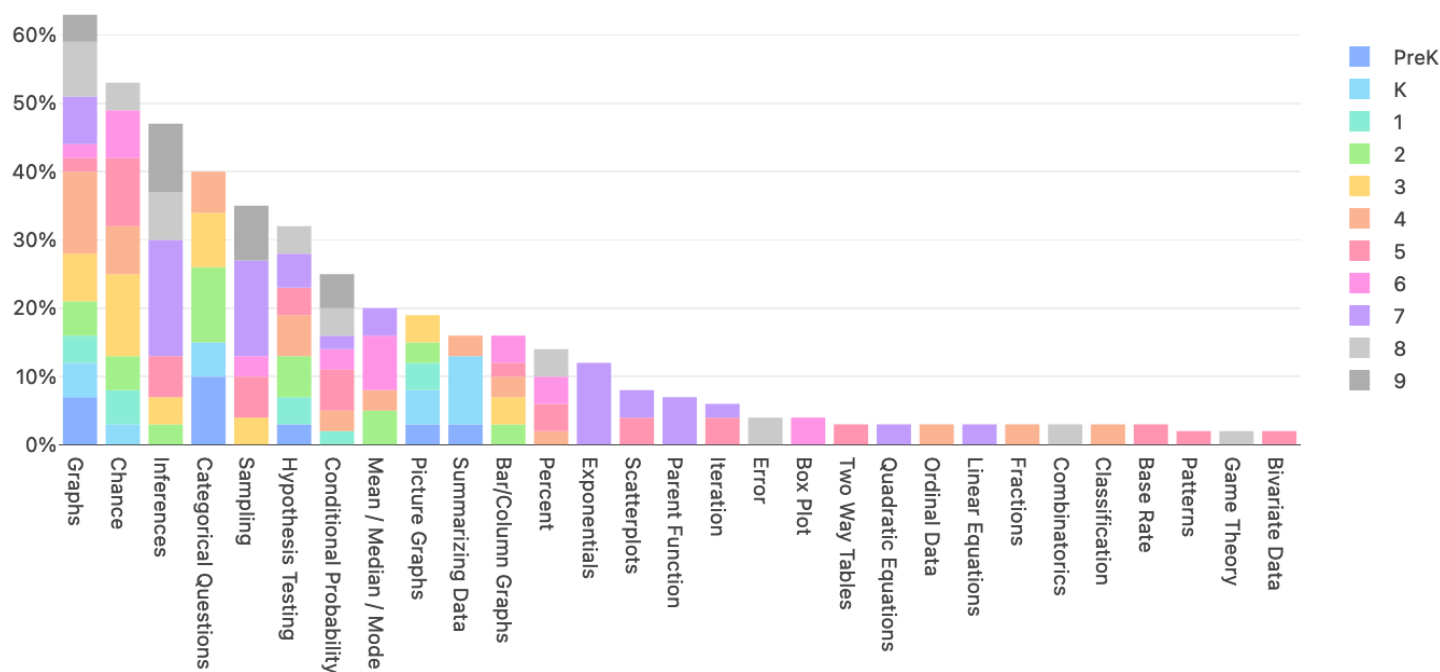
Overall Narrative

Probability and Statistics exist in the traditional math curriculum, but often as somewhat of an afterthought, often as an option only and in high school, and with the focus usually on students

producing the math rather than interpreting it. Since this is the Branch most closely tied to the way students will encounter math in their everyday lives as citizens, and the ways in which they will need to be "mathematically literate" beyond everyday tasks, CCR has, to the extent possible, removed the emphasis on production, and *placed an emphasis on interpretation instead.* Of particular importance is practicing thinking critically about the inquiry process that led to a given statistical claim, as well as interpreting the implications of such claims in the context of what they represent in the world.

When it comes to production, research suggests that the idea of the "mean" is very often misunderstood. For that reason, CCR recommends introducing the mode first, followed by the median, and only then the mean, making sure to highlight its importance as it relates to various possible shapes of a graph[57], as well as comparing groups of different sizes, and the different ways of approaching division.

The new thread added to the traditional Probability standards is the Bayesian Reasoning thread. We introduce the idea of updating one's opinion in the early grades, and then build on this idea once students begin exploring the quantification of uncertainty with probability.

Discrete / Computational

Core Concepts and Content Threads
The most emphasized DCC in Discrete / Computational is Explain Structures, followed by Justify and Systematic Approach.

---
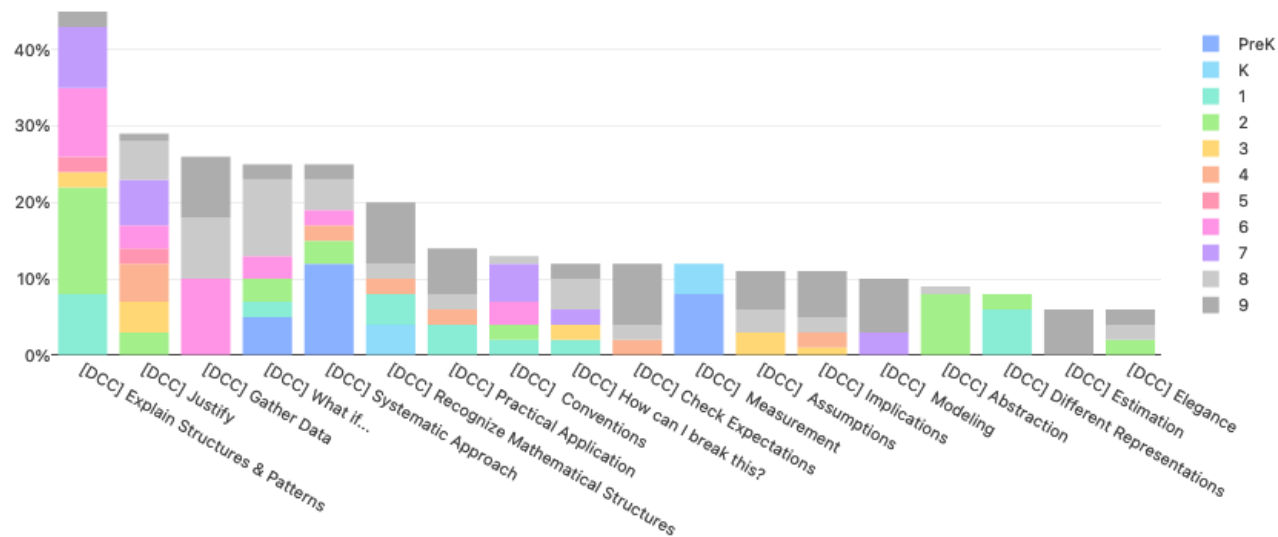
[57] Considering it as a "balance point"

The most emphasized BCCs in Discrete / Computational are Systematicity, Unambiguous Descriptions, and Divide and Conquer.
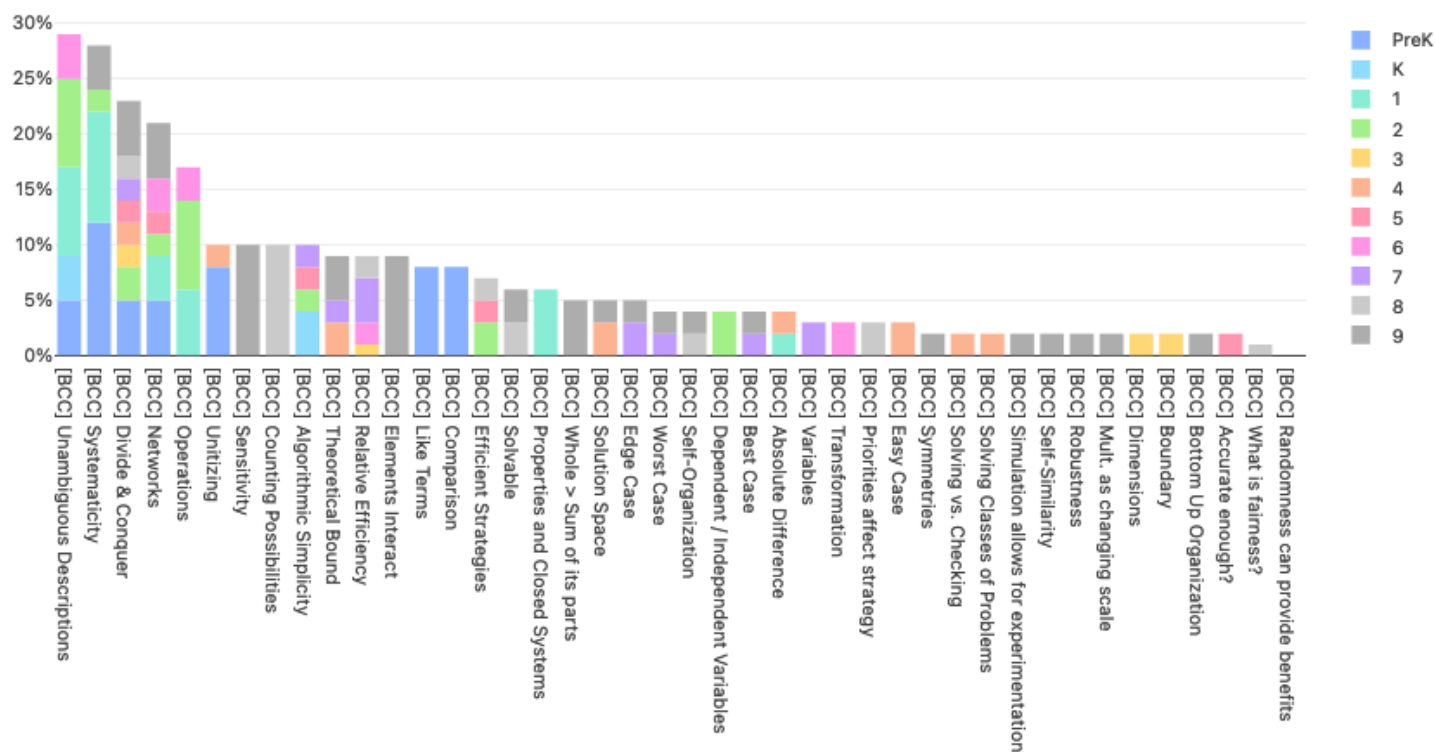
Unsurprisingly, the most emphasized Content thread in Discrete / Computational is Algorithms, and the most emphasized LP is Algorithmic Thinking.

The chart shows a stacked bar graph with y-axis from 0% to 90% and grade levels PreK, K, 1, 2, 3, 4, 5, 6, 7, 8, 9 in the legend. The x-axis categories are: [LP] * Algorithmic Thinking, [LP] Spatial Reasoning, [LP] * Multiplicative Thinking, [LP] Logical Reasoning, [LP] * Additive Thinking, [LP] Unitizing, "Functional Thinking", [LP] * Proportional Reasoning, [LP] Probabilistic Reasoning, Systems Thinking, [LP] * Counting.

Overall Narrative

This branch of mathematics was added to the traditional branches in order to make sure that modern subjects and topics got covered, with a particular emphasis on the broader learning of Computational Thinking[58].

This is notably different from teaching students algorithms to follow. In fact, it is instead teaching students to see algorithms all around them, including the ones they follow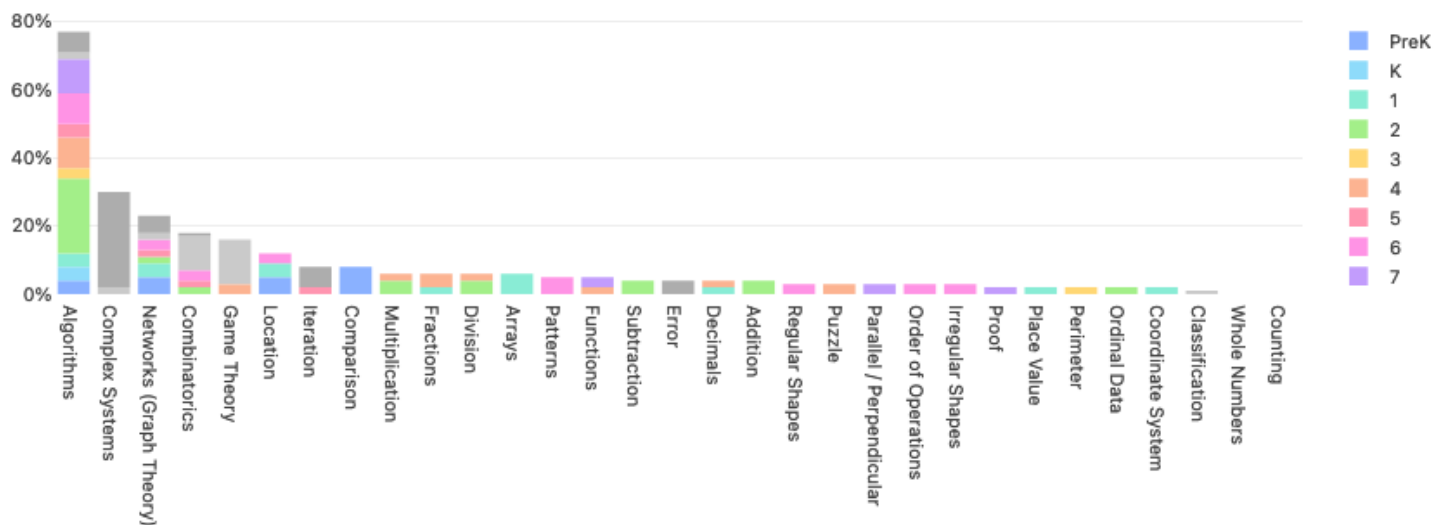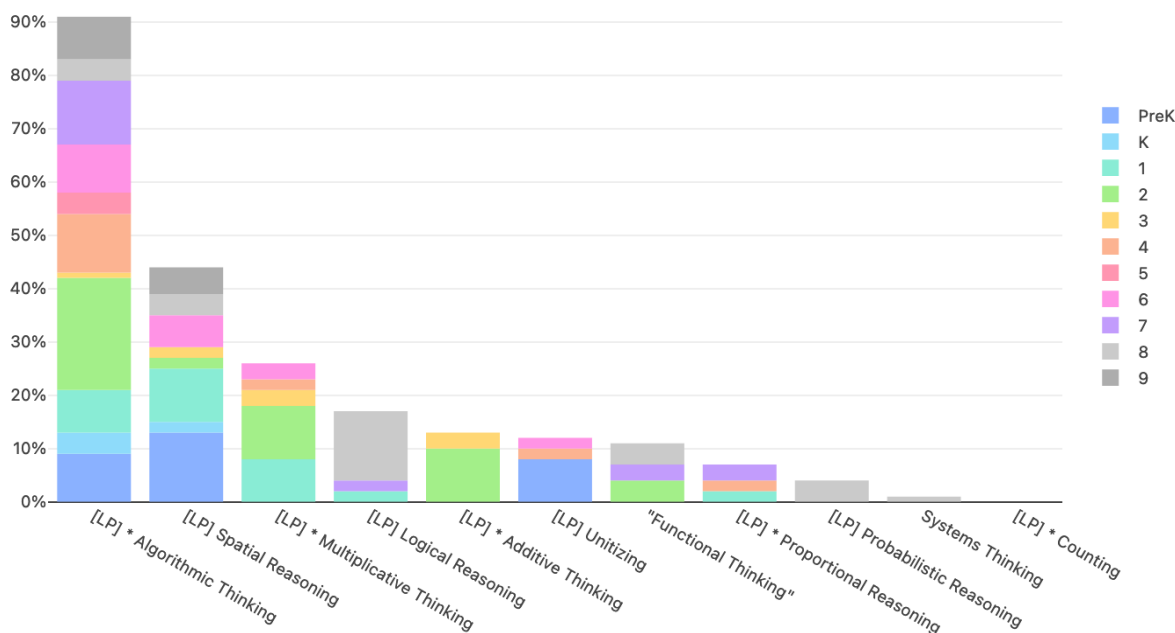 for other math problems. For that reason, we have several standards[59] that take an Arithmetic or Geometric topic that has been covered and hopefully mastered, and review it from the standpoint of dissecting the algorithm students are choosing. This both helps students think in terms of algorithms and helps solidify other learning.

The other goal of this branch was to introduce students to some new mathematical ideas that the curriculum has not traditionally covered up until this point. In addition to Algorithms, these include Complex Systems, Game Theory, and Networks. Some of this content would be cumbersome to teach with traditional paper and pencil, and thus many of them suggest the use of technologies such as simulators; however the concepts underneath the procedures are both accessible and important.

One of the common pushbacks is that these subjects are "university-level", which is patently false: they are only taught in universities because traditional mathematics has taken up all the time

---

[58] Computational Thinking is also one of CCR's Themes, which means it is covered from many different angles across disciplines.
[59] e.g. Counting, Odd/Even, Parallel/Perpendicular, Comparing decimals, etc.

available. Algebraically, they are often simpler than grades 11-12 algebra, and rely more on logical reasoning. Finally, as discussed above, research suggests that exposing students to more advanced content by asking them to interpret something they could not themselves produce produces better results than keeping a focus only on content students are able to produce. These content areas often involve fascinating mathematical objects and models to interpret, without getting bogged down in the specifics necessary to produce them.

Inter-Branch Connections

One of the strengths of the CCR standards is the ability to make connections across branches. The graph below illustrates only the standards that belong in more than one branch, and how much time they are designed to take up.
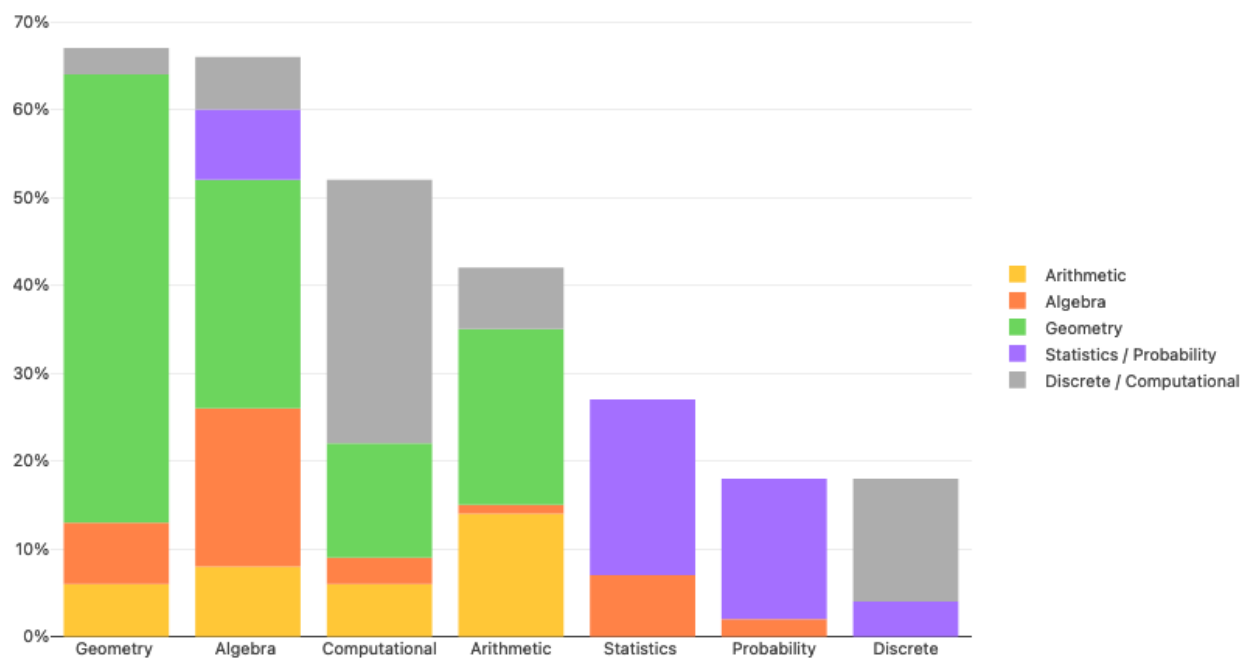
Salient curricular points

- Algebra can be said to be a generalized form of Arithmetic, so there are many overlaps between the two. In a sense, as students progress through PreK-9, more and more of the time they would have spent on Arithmetic is instead spent on Algebra.

- From a Geometry perspective, the Measurement thread closely ties the idea of number with the quantities they represent. More broadly, Geometry offers many of the differing representations used to establish a strong foundation in Arithmetic.

- Algebra and Geometry come together using the essential tool of the Coordinate Plane. Remembering to connect these different representations as often as possible is important to create in students a rich, flexible, understanding.

- The Thread of "Unitizing" neatly connects Arithmetic, Algebra, and Geometry, as it deals with the abstract idea of deliberately choosing what one's unit will be.

- The Discrete / Computational branch allows us to approach the algorithms students learn to combine numbers in a more mindful way, namely, by articulating the algorithms themselves, vs. learning to follow a single "standard" algorithm blindly.

- Combinatorics is a probability topic only because of what is involved in mathematizing combinatorics problems. The actual procedures performed are all Arithmetic, so there is a natural relationship there where Combinatorics can serve as a useful context for practicing some Arithmetic.

- Combinatorics scenarios can also provide a useful grounding context for Algebra, since the number of possibilities grows in different relationships to the number of options. In Discrete Mathematics, Graphs obey various Combinatorics patterns.[60]

Conclusion

There will always be gaps between the intended curriculum, the implemented curriculum, and the achieved curriculum. Standards are the starting point for this inevitable game of telephone; although they cannot perfectly determine how they will be interpreted and deployed, standards are still crucial in setting a clear vision for what is important for students to be able to do.

By formulating them crisply with the Given, Asked, Can structure, connecting them to each other and to Threads, CCR has created a modern, comprehensive, and cohesive set of mathematical goals that are worth working toward.

As the world collectively learned during the pandemic, there are always people who will resist top-down enforcement of laws, even when those laws are designed to save lives.[61] Mathematics standards, though less directly tied to life and death, are still crucial for individual and collective development. The more meaningful these standards can be to students, the less resistance there will be from students, and the more mathematically literate we will be as a society. The challenge is for leaders to take the initial step away from the known, and into the innovative.

---

[60] https://en.wikipedia.org/wiki/Metcalfe's_law
[61] https://datastudio.google.com/reporting/1RX1twzFniNjjOqd_HzuSqS_05rYD0zYH/page/i58TB

Appendix 1: Goals and Functions of Learning Maths

K-12 Math functions:		Thesis	Antithesis	Synthesis
Everybody needs these skills	**Everyday Math and Number Sense**	Financial literacy, etc.	Very very limited actually needed.[62]	Teach Arithmetic (through proportional reasoning) more thoroughly
	Teach Critical Thinking	Citizens have to make decisions based on math. We are even bombarded with stats and figures in our entertainment and from our peers!	Doesn't seem to really be happening with math as it is taught now. The people who learned problem solving through math are those that went on to become mathematicians and push for everyone to learn as they did. There is a selective bias in the conversation, as we are not hearing from the people who sharpened their thinking in other subjects and not math.	Refocus on interpreting Math (especially Stats), not producing it. Refocus on the more or less basic kinds that actually appear in the news, not technical procedures (noisy data).
	Teach Problem Solving	We all need to be able to formulate questions and think systematically and abstractly, because it's more efficient and will help in all aspects of life. There are also more jobs that will demand this, as automatable things get automated.		Consider what we are actually asking students to do that we think will result in them learning problem solving for decision making. This involves a lot of translating world/language into math and math into the world/language, and teaching via visualization. (CC: Mathematizing)
Training	**Prepare Experts for STEM jobs**	There is a public conversation about large skills gaps.	There might not be large skills gaps (stats on this upon request), and even if there are, we shouldn't treat everyone like they're going to become mathematicians. Also it seems like many STEM jobs also use pretty basic math. Academic Math ("Pure Math", proofs etc.) is essentially a different discipline from Math (Applied, fundamental/basic).	Math that goes beyond what is needed for non-STEM jobs and being an informed citizen should be an elective like Art and Music. The fundamentals should be taught much more rigorously and the rest should be taught to give a basic understanding of its existence and usefulness. Also math can be intertwined where relevant in other subjects

---

[62] See Patton, J. R., Cronin, M. E., Bassett, D. S., & Koppel, A. E. (1997). A life skills approach to mathematics instruction: Preparing students with learning disabilities for the real-life math demands of adulthood. Journal of Learning Disabilities, 30, 178-187

				(interdisciplinarity).
Employers to select employees	**Proxy for Grit**	(After IQ) grit/ conscientiousness is the most important determiner of success, and according to O*NET the most important quality in employees.	After adjusting for IQ and socioeconomic status, GPA measures grit perfectly fine. And we aren't even considering grit across different subjects. Maybe a brilliant artist just doesn't care about math and can't get himself to care when he sees no point, but would be super gritty with meaningful art assignments.	Not a good reason for math content, but should be integrated in competencies as resilience and meta-learning.
	Proxy for IQ/ Cognitive Aptitude	For employers to select employees that will be successful at their jobs. Even in elementary school, can't be classified as Gifted/Talented unless your math scores are high.	IQ tests are taboo but we still use various methods to sort people by cognitive ability. Math is one of these ways. But there are much more direct and efficient ways (that also aren't IQ tests) that can do this job and not spend 1700 hours of childhood on math.	Not a good reason to keep math as a barrier. (High level math is required for everyone, including those (the majority) who only use low level math).
	Weeding out process		Weeding out and then lamenting that too many people are weeded out is not a coherent strategy and wasteful (to have our feet on the gas and the brake). We will need to figure out technological or other ways to sort through applicants.	
Political	**Show off as a country (Optics)**	The main impetus for math change seems to be comparisons with other countries.	Isn't it more embarrassing to fail? Also showing off is not a great reason for forcing kids to do something.	This is kind of performative and can lead to unproductive curriculum decisions that sacrifice quality for optics.
	Tradition	Learned automatically so why focus	Inertia, political courage	It is becoming more dangerous not to change, but that is hard to demonstrate until it is too late.

Appendix 2: Competencies

Although Competencies pertain to curriculum and courseware development more directly than to content Standards, they often accompany Standards as a "mathematical practices", "elaborations", etc. CCR has researched, both top-down (academic papers) and bottom-up (expert teachers' opinions), the Competencies that would be most conducively taught via mathematics. The consensus recommendation is:

Discipline	Skills				Character						Meta-Learning	
	Creativity	Critical thinking	Communication	Collaboration	Mindfulness	Curiosity	Courage	Resilience	Ethics	Leadership	Metacognition	Growth Mindset
Mathematics		Core									Core	

The color legend means:
- **Dark Green:** Top 4: Math is relied on to address explicitly and demonstrably (and Core implies just that).
- **Light green:** Middle 4: Math should incorporate with a certain frequency (for instance, the appreciation of various types of curves (cardiods, cycloids, etc.) relates to Curiosity)
- **Grey:** Math should incorporate ad hoc when appropriate (for instance, Ethics when teaching Algorithms)

Appendix 3: Sample Core Concept Descriptions

Core Concepts were created in collaboration with ACARA between the years of 2018 and 2020. There is a list of Core Concepts for each Branch, as well as Core Concepts at the Math (i.e. Discipline) level.

The focus of these descriptions is to make sure we have been comprehensive and coherent, and to help us be consistent in tagging standards. The language in these definitions is **<u>NOT</u>** meant to be presented directly to students; in the balance of accessibility and precision, these descriptions have been optimized for precision. A sample set is presented below. For a complete list of Core Concepts, please contact info@curriculumredesign.org

- **[DCC] Different Representations:** Different representations of the same set of mathematical objects can facilitate the recognition of desired information about a mathematical structure. This includes both symbolic and visual/illustrative representations.

- **[DCC] Estimation:** Often it's not as important to get one specific answer as it is to properly estimate the answer. This can make calculation much simpler and can help us set and check our expectations.

- **[BCC] Bayesian Probability:** The probability of an event can be thought of as quantifying the degree to which we should reasonably believe that the event will come to pass. In this interpretation, 0 indicates an event is impossible, and 1 indicates an event is guaranteed. A probability cannot be less than 0 because we cannot believe anything to be *less* likely than impossible. A probability cannot be more than 1 because we can never believe something to be *more* than absolutely certain.

- **[BCC] Sensitivity:** If we change parameters (digits, coefficients, input values), we want to determine what effect that has on the overall value or solution space. If there are several parameters, we want to determine which changes have the largest or smallest impact.

- **[BCC] Operations:** Mathematical objects can be combined to make a new mathematical object. (e.g. addition, subtraction, multiplication, division, cross product, dot product, set union, set intersection, logarithm).