A Polyhedral Study on Unit Commitment with a Single Type of Binary Variables

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Efficient power production scheduling is a crucial concern for power system operators aiming to minimize operational costs. Previous mixed-integer linear programming formulations for unit commitment (UC) problems have primarily used two or three types of binary variables. The investigation of strong formulations with a single type of binary variables has been limited, as it is believed to be challenging to derive strong valid inequalities using fewer binary variables, and the reduction of number of binary variables is often accompanied by a compromise in tightness. To address these issues, this paper considers a formulation for unit commitment using a single type of binary variables and develops strong valid inequality families to enhance the tightness of the formulation. Conditions under which these strong valid inequalities serve as facet-defining inequalities for the single-generator UC polytope are provided. For those large-size valid inequality families, the existence of efficient separation algorithms for determining the most violated inequalities is also discussed. The effectiveness of the proposed single-binary formulation and strong valid inequalities is demonstrated through computational experiments on network-constrained UC problems. The results indicate that the strong valid inequalities presented in this paper are effective in solving UC problems and can also be applied to UC formulations that contain more than one type of binary variables.

Key words: Unit commitment, polyhedral study, strong valid inequalities, convex hull

1. Introduction

With the increased prevalence of extreme weather events such as droughts, wildfires, and flooding around the world in recent years, power consumption in many areas hit an all-time high in the summer of 2022. In addition, the continued retirements of coal-fired generating plants, relatively high coal prices, and lower-than-average coal stocks at power plants have limited coal consumption (US EIA 2022). As a result, the efficient production and distribution of electricity have been identified as the biggest concern for power producers around the globe.

The unit commitment (UC) problem, which involves the scheduling of power generators, has been a challenging optimization problem in the power industry for many years. It often needs to be solved multiple times per day by system operators (Xavier et al. 2021). Because of such practical needs, it has received considerable attention over the past decades. The UC problem involves scheduling a group of generators at a possibly minimal operational cost, subject to their physical and system constraints over a finite time horizon. Physical constraints specify the technical properties of generators, and they may vary depending on the type of generation unit, such as hydro, thermal, and wind units (Van Ackooij et al. 2018). Unless explicitly stated otherwise, all subsequent discussions are about thermal units.

The most common physical constraints for thermal units are the ramp-up/-down rate, generation lower/upper bound, and minimum up/down time constraints. System constraints typically include the load (demand) requirement, system reserve constraint, and transmission flow limit. All generation units are coupled by the system constraints to ensure the reliability of the entire system. The operational cost comprises various components, including the generation (fuel) cost and the startup and shutdown costs of generators. The generation cost is a significant component (Padhy 2004), and it is generally assumed to be an increasing quadratic convex function of the generation amount (Takriti and Birge 2000). Start-up and shut-down costs are incurred each time the status of a generator changes (Sen and Kothari 1998).

Many different UC problems can arise depending on the structure of the electrical power system. The ability to solve UC problems efficiently can have a great impact on both society and individual consumers. Even small improvements in the quality of solutions for UC problems can affect the electricity price over large regions and lead to millions of U.S. dollars of savings per day (Damcı-Kurt et al. 2016). Although the single-generator UC (single-UC) problem with only physical constraints and a quadratic generation cost function is proven to be polynomial-time solvable (Frangioni and Gentile 2006b), general UC problems with system constraints have not yet been solved satisfactorily. They are often formulated as large-scale mixed-integer programming problems. Such problems are typically NP-hard and difficult to solve when the problem sizes are large (Zheng et al. 2015, Knueven et al. 2020b, Tejada-Arango et al. 2020). For example, a UC problem with an arbitrary number of generators that contain only minimum up/down, generation lower/upper bound, and demand constraints is classified as NP-hard even considering a single operational period (Bendotti et al. 2019). In a day-ahead deregulated electricity market, an independent system operator (ISO) is expected to determine the generation schedule for a power system within a very short time. Such a generation schedule involves hundreds of thermal units, thousands of transmission lines, and 24–48 operating hours. Therefore, over the past decades, numerous approaches have been devised from both formulation and algorithm perspectives in order to solve UC problems efficiently.

Priority list and heuristic algorithms are among the earliest solution approaches used to solve UC problems (Kazarlis et al. 1996). The former lists all units by their operational costs and then economically dispatches the system load to generators by a pre-determined order. Heuristic algorithms, such as genetic algorithms and simulated annealing, are also frequently adopted as they can be converted to work on parallel computers. However, these approaches usually lead to suboptimal solutions. Dynamic programming (DP)-based approaches, in contrast, are exact solution techniques for UC problems and are widely applied in the early period (Wang and Shahidehpour 1993, Baldick 1995). Using these approaches, UC problems are decomposed by time, and the state for each time period is represented by the combinations of units. However, the number of states increases dramatically as the number of units and time periods grows. Therefore, these approaches are also integrated with heuristic methods to reduce the search space (Ouyang and Shahidehpour 1991). Recently, Frangioni and Gentile (2006b) and Guan et al. (2018) propose DP algorithms to solve single-UC problems and provide theoretical complexity results. They show that single-UC problems with only physical constraints and suitable generation cost functions, such as quadratic cost functions, can be solved in polynomial time. In addition, given the large size of UC problems, Lagrangian relaxation (LR)-based approaches are proposed (Muckstadt and Koenig 1977, Abdul-Rahman et al. 1996, Takriti and Birge 2000, Lu and Shahidehpour 2005). These approaches relax system constraints, such as load requirements, and integrate them into the objective function through Lagrangian multipliers. The resulting problem is then decomposed into subproblems either by units or by time periods, and these subproblems are solved iteratively until the primal-dual gap is difficult to shrink. However, LR-based approaches suffer from slow and unsteady convergence, and the feasibility of the final solution cannot be guaranteed (Ma and Shahidehpour 1999). To overcome the convergence problem, quadratic terms are incorporated into the objective function to penalize the violation of demand constraints and improve its convexity, resulting in augmented Lagrangian relaxation based approaches. A comprehensive examination of solution approaches for UC problems can be found in Van Ackooij et al. (2018).

The extensive advancement of mixed-integer linear programming (MILP) solvers has led to the widespread application of MILP-based approaches in formulating and solving UC problems. MILP-based approaches can guarantee convergence to the optimal solution while providing a flexible and accurate modeling framework. Moreover, the optimality gap is easy to obtain. ISOs are therefore increasingly adopting MILP-based approaches over LR-based approaches to solve large-scale UC problems (Hedman et al. 2009, Wu 2011, Ostrowski et al. 2012, Li et al. 2021).

Two primary factors are generally considered in evaluations of MILP formulations for UC problems: *compactness* and *tightness*. Compactness refers to the size of the problem, which can be quantified by the number of constraints, decision variables, or nonzero coefficients; tightness refers to the proximity of the linear programming (LP) relaxation of the problem to the convex hull of its feasible region (Morales-España et al. 2013, Knueven et al. 2020b). For UC problems, compactness can be achieved by reducing the number of integer variables in an MILP formulation, as fewer integer variables may lead to a reduction in the number of nodes of the search tree for the branch-and-cut method. Note that for UC formulations, all integer variables are binary. In terms of the number of binary variables used, UC formulations can be categorized into two categories. A single-binary formulation uses a single set of binary variables to denote the on/off status of all units. A three-binary formulation uses two additional sets of binary variables to represent the start-up and shut-down decisions. As an important variant of the three-binary formulation, a two-binary formulation can be obtained by expressing the shut-down variables in terms of the on/off and start-up variables. For a UC problem, a three-binary formulation can generally be tighter than a single-binary formulation because the addition of the start-up and shut-down binary variables in the three-binary formulation may facilitate the improvement of the physical constraints, potentially leading to a better LP bound. However, the large size of its search tree and the difficulty of solving the subproblem at each branching node may increase the solution time. On the other hand, a single-binary formulation uses fewer binary variables, which reduce the size of the search tree and may decrease the solution time. Nevertheless, it usually has a weaker LP bound than the three-binary formulation. Tightness can be achieved by deriving strong valid inequalities for the MILP formulation to tighten its LP relaxation. Most strong valid inequalities are obtained by studying the physical constraints of a single generator (see, e.g., Lee et al. 2004, Rajan and Takriti 2005, Morales-España et al. 2013, Damcı-Kurt et al. 2016, Pan and Guan 2016, and Bendotti et al. 2018).

Three-binary formulations for UC problems are the most widely studied. Garver (1962) is the first to propose an MILP formulation for a UC problem. In this three-binary formulation, the generation cost function is assumed to be linear with respect to the generation amount. Arroyo and Conejo (2000) introduce a three-binary formulation for a single-UC problem. They approximate the exponential start-up cost function and the nonconvex generation cost function using stairwise and piecewise functions, respectively. Chang et al. (2001) present a three-binary formulation for a short-term hydro scheduling UC problem. Chang et al. (2004) put forward a new three-binary formulation for UC problems, and they approximate the cubic generation cost function using a piecewise linear one with three breakpoints. Li and Shahidehpour (2005) compare the LR-based approach with the MILP-based approach in solving a price-based UC problem with various types of generators based on the formulation of Chang et al. (2004). Their numerical results indicate that the MILP-based approach exhibits superior performance on small scale problems compared with the LR-based approach, and the MILP formulation must be tightened to improve its performance on large-scale problems. Ostrowski et al. (2012) consider the formulation of Arroyo and Conejo (2000) and replace their minimum up/down constraints with those of Rajan and Takriti (2005) because the latter can reduce the computational time significantly. By studying the physical constraints of a single generator, they derive a class of strong valid inequalities to tighten the MILP formulation. Their computational results indicate that their tightened formulation is more effective than Carrión and Arroyo's (2006) single-binary formulation. Morales-España et al. (2013) propose an alternative three-binary formulation based on the formulation of Ostrowski et al. (2012). The generation cost function is represented as a linear function with respect to the generation amount. They introduce generation limit constraints to substitute for those in Ostrowski et al. (2012). They show that the resulting formulation has smaller size and better LP bound than that of Ostrowski et al. (2012). Morales-España et al. (2015) establish a three-binary formulation based on that of Morales-España et al. (2013) by considering different start-up/shut-down trajectories, which are ignored in conventional research. Damcı-Kurt et al. (2016) conduct a polyhedral study of the physical constraints based on the work of Ostrowski et al. (2012). They derive a convex hull for the two-period case and strong valid inequalities for the multi-period case to tighten the MILP formulation. Because the number of these strong valid inequalities can be exponential, polynomial separation algorithms are provided to apply them in the solution process. Computational results demonstrate that this formulation outperforms the strong formulation of Ostrowski et al. (2012). Atakan et al. (2018) develop a state-transition formulation for UC problems based on the formulations of Ostrowski et al. (2012) and Morales-España et al. (2013). Transmission constraints are not considered in their formulation. Their test results demonstrate that the proposed formulation has a shorter computational time for long-horizon problems than the formulations of Ostrowski et al. (2012) and Morales-España et al. (2013). Gentile et al. (2017) provide the convex hull description for the single-UC polytope considering the minimum up/down time constraints, start-up/shut-down capabilities constraints, and generation limit constraints.

Other variants of three-binary formulations have also received considerable attention. Rajan and Takriti (2005) study a single-UC problem with minimum up/down times constraints using three types of binary variables. They provide a complete description of the convex hull of the polytope. Based on Rajan and Takriti's three-binary formulation, Pan et al. (2016) derive several families of strong valid inequalities for UC problems with gas turbine generators. Their strong valid inequalities are facet-defining for the polytope of physical constraints under specific conditions. Pan and Guan (2016) conduct a polyhedral study of physical constraints based on the formulation of Pan et al. (2016). They derive the complete convex hull descriptions for the two- and three-period polytopes under different parameter settings. They also develop strong valid inequalities for the multi-period case and provide polynomial-time separation algorithms for exponentially large valid inequality families. Bendotti et al. (2018) analyze the minimum up/down polytope of multiple generators based on the formulation of Pan and Guan (2016); their generation cost function is linear in the generation amount. They obtain *up-set* and *interval up-set* valid inequalities to accelerate the branch-and-cut algorithm. However, given a fractional solution, the problems of separating these two types of inequalities are NP-complete and NP-hard, respectively. Pan et al. (2022) perform a polyhedral study of a single generator by incorporating fuel constraints. They prove that the single-UC problem with a fuel constraint is NP-hard, and they derive strong valid inequalities to improve the computational performance. Dupin (2017) present two main formulations for a UC problem with min-stop ramping constraints based on the definitions of the so-called *state* and *level* variables. The two formulations are compared and exhibit an isomorphism.

Studies on single-binary formulations are limited. Lee et al. (2004) investigate the minimum up/down polytope using a single type of binary variables. They give a complete convex hull description of the polytope, obtain valid inequalities, and design an efficient separation procedure for using these valid inequalities. Carrión and Arroyo (2006) propose a single-binary MILP formulation for UC problems. They approximate the generation cost function and the exponential start-up cost function using linear functions as in Arroyo and Conejo (2000). They also establish new minimum up/down constraints. They then compare the proposed formulation with the three-binary formulation of Arroyo and Conejo (2000), as well as with its variant in which one type of binary variables in Arroyo and Conejo (2000) is relaxed. They computationally demonstrate that their single-binary formulation outperforms the other two formulations significantly. Frangioni and Gentile (2006a) derive perspective cuts for the mixed-integer quadratic programming problem with semi-continuous variables. They test the effectiveness of these cuts by solving a single-binary UC formulation with a quadratic generation cost function. These cuts can substantially improve the performance of the branch-and-cut method. However, their formulation does not consider the ramp-up/-down, system reserve, and transmission constraints. Frangioni et al. (2009) apply the perspective cuts of Frangioni and Gentile (2006a) to provide a new piecewise linear approximation of the generation cost function for a short-term UC problem with hydro and thermal generators. Brandenberg et al. (2017) consider the summed start-up cost across all time periods in a single-binary UC formulation by introducing a single continuous variable for each unit. They derive the H-representation of its epigraph and provide an exact linear separation algorithm.

Generally, enhancing the tightness of a formulation for a UC problem necessitates the inclusion of additional variables or constraints, which may lead to an increase in the solution time since the solver needs to solve larger LP subproblems repeatedly; conversely, improving the compactness of a formulation often comes at the expense of weakening tightness, resulting in a weak lower bound (Morales-España et al. 2013, Knueven et al. 2020b). Therefore, in practice, tightness and compactness must be balanced. In most cases, improvement in tightness for UC formulations is preferred over that in compactness because of the potential reduction in solution time, despite the increased complexity resulting from additional binary variables (Hedman et al. 2009, Ostrowski et al. 2012). Moreover, a lack of binary variables for start-up/shut-down decisions makes it difficult to generate strong valid inequalities (Ostrowski et al. 2012). Thus, few studies examine formulations with a single type of binary variables and derive strong valid inequalities to improve their tightness. To bridge this gap, this paper studies a single-binary formulation for a UC problem with thermal units and derives strong valid inequalities to speed up the solution process. The studied single-binary formulation also offers an alternative solution approach for UC problems, which can be leveraged in both theoretical research and practical applications. The main contributions of this study are summarized as follows:

- Through an investigation of the physical constraints, we provide a complete convex hull description of the two-period single-UC polytope of the single-binary formulation.
- We develop strong valid inequality families for the multi-period single-UC polytope, and we derive the conditions under which the strong valid inequalities are facet-defining. We also develop efficient separation algorithms for determining a most violated inequality in each valid inequality family.
- We demonstrate the effectiveness of our strong valid inequalities in tightening our single-binary formulation through computational experiments. The results indicate that our strong valid inequalities are effective in solving UC problems and can also be applied to UC formulations that contain more than one type of binary variables.

The rest of the paper is organized as follows. Section 2 describes the UC problem under study, presents a single-binary MILP formulation for it, and introduces the single-UC polytope. Section 3 provides the complete description of the convex hull for the two-period case and discusses its importance for solving our UC problem. Section 4 presents various strong valid inequalities to tighten the single-UC polytope and discusses the existence of efficient separation algorithms. Section 5 reports the results of a computational study conducted to assess the effectiveness of our strong valid inequalities in tightening the single-binary MILP formulation and speeding up the solution process. Section 6 concludes the paper and offers suggestions for future research. All mathematical proofs are provided in Online Appendix A. Some additional computational results are provided in Online Appendix B.

2. MILP Model for Unit Commitment

In this section, we first present an MILP model for the UC problem with thermal units, and then present some important properties of the single-generator case. In the UC problem being studied, a system operator plans the generation schedule of a set of generators $\mathcal{G}$ for a number of time periods at minimal operating costs while satisfying physical and system constraints. The system includes a set of buses $\mathcal{B}$ and a set of transmission lines $\mathcal{E}$ that link the buses, allowing surplus power to be distributed. Each bus can be equipped with multiple generators and is responsible for the load requirement of a geographical region. Surplus power at one bus can be transferred to neighboring buses through transmission lines to satisfy the load requirements of other regions. The power flow on each transmission line should not exceed the line's capacity. To ensure the reliability of the power supply, some generation capacity should be reserved for outages. All of the generators should operate without violating their physical configurations. Every time a generator starts up or shuts down, a fixed cost is incurred. For each time period, an operational cost is incurred depending on the generation amount and the online/offline status of the generators.

We let $\mathbb{R}^n$ denote the n-dimensional real vector space, $\mathbb{R}^n_+$ denote the n-dimensional nonnegative real vector space, and $\mathbb{B}^n$ denote the n-dimensional binary vector space. Given any nonnegative integers a and b, we let $[a,b]_{\mathbb{Z}}$ denote the set of all integers between a and b; that is, $[a,b]_{\mathbb{Z}} = \{a, a+1, \ldots, b\}$ if $a \leq b$, and $[a,b]_{\mathbb{Z}} = \emptyset$ if $a > b$.

Let T be the number of time periods in the operation horizon. For each generator $g \in \mathcal{G}$, let $L^g > 0$ and $\ell^g > 0$ be the minimum up and minimum down time requirements, respectively. That is, once the generator starts up, it must stay online for at least L^g time periods, and once it shuts down, it must stay offline for at least ℓ^g time periods. For each $g \in \mathcal{G}$, let $\overline{C}^g$ and $\underline{C}^g$ be the generation upper and lower bounds, where $\overline{C}^g > \underline{C}^g > 0$. For each $g \in \mathcal{G}$, let $V^g > 0$ be the maximum change in the generation amount between two consecutive online time periods, i.e., the ramp-up rate is assumed to be equal to the ramp-down rate. For each $g \in \mathcal{G}$, let $\overline{V}^g > 0$ be the start-up/shut-down ramp limit, i.e., the start-up ramp limit is assumed to be equal to the shut-down ramp limit. Thus, when a generator g is online, its generation amount should be within the range $[\underline{C}^g, \overline{C}^g]$. When the generator starts up, its generation amount in the start-up period should be within the range $[\underline{C}^g, \overline{V}^g]$. When the generator shuts down, its generation amount in the previous time period should also be within the range $[\underline{C}^g, \overline{V}^g]$. We assume that $\overline{V}^g + V^g \leq \overline{C}^g$ for all $g \in \mathcal{G}$. This condition guarantees that a generator can ramp up at its full rate V^g for at least one period after it starts up. We also assume that $\underline{C}^g < \overline{V}^g < \underline{C}^g + V^g$ for all $g \in \mathcal{G}$, which holds in most industrial settings as indicated by Morales-España et al. (2015), Damcı-Kurt et al. (2016), Pan et al. (2016), and Gentile et al. (2017). For each bus $b \in \mathcal{B}$, let $\mathcal{G}_b$ be the set of generators at bus b (note: $\bigcup_{b \in \mathcal{B}} \mathcal{G}_b = \mathcal{G}$ and

$\mathcal{G}_b \cap \mathcal{G}_{b'} = \emptyset$ for all $b, b' \in \mathcal{B}$ such that $b \neq b'$). The other parameters of our model are defined as follows:

- $f^g(\cdot)$: The generation cost function for generator g (for each $g \in \mathcal{G}$, $f^g(\cdot)$ is a non-decreasing convex piecewise linear function with a fixed number of linear segments).
- c^g: The fixed cost incurred if generator g is online ($c^g \geq 0$ for all $g \in \mathcal{G}$).
- ϕ^g: The fixed start-up cost of generator g ($\phi^g \geq 0$ for all $g \in \mathcal{G}$).
- ψ^g: The fixed shut-down cost of generator g ($\psi^g \geq 0$ for all $g \in \mathcal{G}$).
- d_t^b: The load (demand) at bus b in time period t ($d_t^b \geq 0$ for all $t \in [1, T]_{\mathbb{Z}}$ and $b \in \mathcal{B}$).
- C_e: The capacity limit of transmission line e ($C_e \geq 0$ for all $e \in \mathcal{E}$).
- K_e^b: Line flow distribution factor for the flow on transmission line e contributed by the net injection at bus b ($K_e^b \geq 0$ for all $e \in \mathcal{E}$ and $b \in \mathcal{B}$).
- r_t: The system reserve factor of time period t ($r_t \geq 0$ for all $t \in [1, T]_{\mathbb{Z}}$).

Here, the non-decreasing convex piecewise linear generation cost function $f^g(x)$ is used to approximate the convex quadratic cost function $a^g x^2 + b^g x$; see Carrión and Arroyo (2006) and Pan et al. (2022) for similar approximations. Our model has the following decision variables:

- x_t^g: The generation amount of generator $g \in \mathcal{G}$ in period $t \in [1, T]_{\mathbb{Z}}$.
- y_t^g: The online/offline status of generator $g \in \mathcal{G}$ in period $t \in [1, T]_{\mathbb{Z}}$, where $y_t^g = 1$ if g is online in period t, and $y_t^g = 0$ otherwise.
- u_t^g: The start-up cost of generator $g \in \mathcal{G}$ in period $t \in [1, T]_{\mathbb{Z}}$.
- v_t^g: The shut-down cost of generator $g \in \mathcal{G}$ in period $t \in [1, T]_{\mathbb{Z}}$.

Variables x_t^g, u_t^g, and v_t^g are continuous, whereas variable y_t^g is binary. We assume that the values of $y^g_{-\max\{L^g,\ell^g\}+1}, y^g_{-\max\{L^g,\ell^g\}+2}, \ldots, y^g_{-1}, y^g_0$, and x_0^g (for all $g \in \mathcal{G}$) are given as initial conditions. The UC problem is formulated as follows:

$$
\begin{aligned}
\text{Problem 1:}\ \min\ & \textstyle\sum_{g\in\mathcal{G}}\sum_{t=1}^{T}(u_t^g + v_t^g + c^g y_t^g + f^g(x_t^g)) && \text{(1a)}\\
\text{s.t.}\ & -y_{t-1}^g + y_t^g - y_k^g \leq 0,\ \forall t \in [-L^g+2, T]_{\mathbb{Z}},\ \forall k \in [t, \min\{T, t+L^g-1\}]_{\mathbb{Z}},\ \forall g \in \mathcal{G}, && \text{(1b)}\\
& y_{t-1}^g - y_t^g + y_k^g \leq 1,\ \forall t \in [-\ell^g+2, T]_{\mathbb{Z}},\ \forall k \in [t, \min\{T, t+\ell^g-1\}]_{\mathbb{Z}},\ \forall g \in \mathcal{G}, && \text{(1c)}\\
& -x_t^g + \underline{C}^g y_t^g \leq 0,\ \forall t \in [1, T]_{\mathbb{Z}},\ \forall g \in \mathcal{G}, && \text{(1d)}\\
& x_t^g - \overline{C}^g y_t^g \leq 0,\ \forall t \in [1, T]_{\mathbb{Z}},\ \forall g \in \mathcal{G}, && \text{(1e)}\\
& x_t^g - x_{t-1}^g \leq V^g y_{t-1}^g + \overline{V}^g(1 - y_{t-1}^g),\ \forall t \in [1, T]_{\mathbb{Z}},\ \forall g \in \mathcal{G}, && \text{(1f)}\\
& x_{t-1}^g - x_t^g \leq V^g y_t^g + \overline{V}^g(1 - y_t^g),\ \forall t \in [1, T]_{\mathbb{Z}},\ \forall g \in \mathcal{G}, && \text{(1g)}\\
& u_t^g \geq \phi^g(y_t^g - y_{t-1}^g),\ \forall t \in [1, T]_{\mathbb{Z}},\ \forall g \in \mathcal{G}, && \text{(1h)}\\
& v_t^g \geq \psi^g(y_{t-1}^g - y_t^g),\ \forall t \in [1, T]_{\mathbb{Z}},\ \forall g \in \mathcal{G}, && \text{(1i)}\\
& \textstyle\sum_{g\in\mathcal{G}} x_t^g = \sum_{b\in\mathcal{B}} d_t^b,\ \forall t \in [1, T]_{\mathbb{Z}}, && \text{(1j)}
\end{aligned}
$$

$$\textstyle\sum_{g\in\mathcal{G}} \overline{C}^g y_t^g \geq (1+r_t) \sum_{b\in\mathcal{B}} d_t^b,\ \forall t \in [1,T]_{\mathbb{Z}}, \tag{1k}$$

$$-C_e \leq \textstyle\sum_{b\in\mathcal{B}} K_e^b \big(\sum_{g\in\mathcal{G}_b} x_t^g - d_t^b\big) \leq C_e,\ \forall t \in [1,T]_{\mathbb{Z}}, \forall e \in \mathcal{E}, \tag{1l}$$

$$y_t^g \in \{0,1\}, x_t^g \geq 0, u_t^g \geq 0, v_t^g \geq 0,\ \forall t \in [1,T]_{\mathbb{Z}}, \forall g \in \mathcal{G}. \tag{1m}$$

Objective function (1a) minimizes the total cost, which includes the start-up costs, shut-down costs, and fixed and variable generation costs. Constraint (1b) states the minimum up requirement for generator g. It requires generator g to stay online in periods $[t, \min\{T, t+L^g-1\}]_{\mathbb{Z}}$ if it starts up in period t. Constraint (1c) states the minimum down requirement for generator g. It requires generator g to stay offline in periods $[t, \min\{T, t+\ell^g-1\}]_{\mathbb{Z}}$ if it shuts down in period t. Constraints (1d) and (1e) ensure that the generation amount of generator g in period t is 0 if the generator is offline and is within the range $[\underline{C}^g, \overline{C}^g]$ if the generator is online. Constraints (1f) and (1g) guarantee that generator g ramps up/down within its limit V^g between two consecutive online time periods. They also guarantee that generator g ramps up by no more than $\overline{V}^g$ units when it starts up and ramps down by no more than $\overline{V}^g$ units when it shuts down. Constraint (1h) and objective function (1a), together with the nonnegativity constraint of u_t^g, imply that the start-up cost for generator g in period t is ϕ^g if the generator starts up in period t, and 0 otherwise. Constraint (1i) and the objective function (1a), together with the nonnegativity constraint of v_t^g, imply that the shut-down cost for generator g in period t is ψ^g if the generator shuts down in period t, and 0 otherwise. Constraint (1j) is the load balance constraint in period t, which requires the total generation amount to satisfy the total demand in the period. Constraint (1k) is the system reserve requirement, which requires the total generation capacity of all online generators to exceed the load requirement by a system reserve factor to deal with demand variations. Constraint (1l) states the transmission flow limit. In the distribution process, a bus b contributes a factor K_e^b of its net injection $\sum_{g\in\mathcal{G}_b} x_t^g - d_t^b$ to each transmission line e, and constraint (1l) requires the absolute value of the total net injection contributed by all buses to each transmission line to stay below its capacity limit to prevent it from being overloaded; see Ma and Shahidehpour (1999), Shahidehpour et al. (2002, p. 290), and Xavier et al. (2021) for similar settings. Constraint (1m) states the nonnegativity and binary requirements of the decision variables. Note that Problem 1 uses only a single type of binary variables, y_t^g, and thus is a single-binary formulation. However, in the optimal solution, the continuous variables u_t^g and v_t^g have only two possible values. Hence, Problem 1 can also be formulated using two additional vectors of discrete variables. Because the polytope $\mathrm{conv}(\mathcal{P})$ that we are analyzing in this paper is independent of u_t^g and v_t^g (see the definition of set $\mathcal{P}$ below), the polytope is a single-binary polytope regardless of whether u_t^g and v_t^g are declared as continuous or discrete. Note also that the objective function of Problem 1 is piecewise linear. Following the literature (see, e.g., Arroyo and Conejo 2000), Problem 1 can be converted into an MILP.

Bendotti et al. (2019) consider a UC problem in which there is a linear generation cost, a minimum demand requirement, no ramp-up, ramp-down, start-up, and shut-down limits, no system reservation requirement, no transmission flow limit, and some initial conditions. They prove that the problem is strongly NP-hard. It is easy to verify that their NP-hardness proof remains valid when applied to our UC problem. Thus, Problem 1 is also NP-hard in the strong sense.

In Problem 1, constraints (1b)–(1g) specify the physical properties of the generators. Constraints (1h) and (1i) determine the start-up and shut-down costs, respectively. Once the y_t^g values are determined for all $g \in \mathcal{G}$ and $t \in [1,T]_{\mathbb{Z}}$, the u_t^g and v_t^g values can be easily obtained by these constraints. Constraints (1j)–(1l) are the coupling constraints, or system constraints, that link all of the generators. Because of the scale and complexity of the UC problem, one way of reducing the solving time is to decompose the problem into smaller subproblems with one subproblem corresponding to each generator (see Knueven et al. 2020a). For example, in the Lagrangian relaxation method, the coupling constraints can be integrated into the objective function through Lagrangian multipliers, and the resulting problem is decomposed into subproblems that contain only the physical constraints (Baldick 1995, Takriti and Birge 2000). Thus, most improvements in UC models result from studying the properties of an individual generator's feasible region (Knueven et al. 2018). Moreover, strong valid inequalities for the physical constraints are valid for Problem 1 and can be used to tighten its linear relaxation. A tighter linear relaxation can often improve computational efficiency by reducing the amount of enumeration required to find and prove an optimal solution (Knueven et al. 2020b). Hence, in the mathematical analysis presented in Sections 3 and 4, we focus on deriving strong valid inequalities for the physical constraints for the generators in Problem 1. Because all of the generators have the same set of physical constraints, it suffices to concentrate on the physical constraints of a single generator, and the results obtained can be applied to all of other generators. Therefore, in the following analysis, the superscript g in the parameters and decision variables is dropped.

Denote $\mathbf{x} = (x_1, \ldots, x_T)$ and $\mathbf{y} = (y_1, \ldots, y_T)$. Thus, the vector $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^T \times \mathbb{B}^T$ contains the generation amount and on/off status of the generator in the T time periods. The set of $(\mathbf{x}, \mathbf{y})$ values that satisfy the physical constraints of Problem 1 is given as

$$
\begin{aligned}
\mathcal{P} = \big\{ (\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^T \times \mathbb{B}^T :& \\
& -y_{t-1} + y_t - y_k \le 0,\ \forall t \in [2,T]_{\mathbb{Z}},\ \forall k \in [t, \min\{T, t+L-1\}]_{\mathbb{Z}}, && \text{(2a)}\\
& y_{t-1} - y_t + y_k \le 1,\ \forall t \in [2,T]_{\mathbb{Z}},\ \forall k \in [t, \min\{T, t+\ell-1\}]_{\mathbb{Z}}, && \text{(2b)}\\
& -x_t + \underline{C} y_t \le 0,\ \forall t \in [1,T]_{\mathbb{Z}}, && \text{(2c)}\\
& x_t - \overline{C} y_t \le 0,\ \forall t \in [1,T]_{\mathbb{Z}}, && \text{(2d)}\\
& x_t - x_{t-1} \le V y_{t-1} + \overline{V}(1 - y_{t-1}),\ \forall t \in [2,T]_{\mathbb{Z}}, && \text{(2e)}\\
& x_{t-1} - x_t \le V y_t + \overline{V}(1 - y_t),\ \forall t \in [2,T]_{\mathbb{Z}} \big\}. && \text{(2f)}
\end{aligned}
$$

Here, the assumptions $\overline{C} > \underline{C} > 0$, $V > 0$, $\overline{V} + V \leq \overline{C}$, and $\underline{C} < \overline{V} < \underline{C} + V$ remain valid. Note that inequalities (2a)–(2f) in $\mathcal{P}$ are the same as inequalities (1b)–(1g) in Problem 1 for a specific generator g, except that t is restricted to the range $[2, T]_{\mathbb{Z}}$ in (2a), (2b), (2e), and (2f) (i.e., constraints dependent on the initial conditions are not included in $\mathcal{P}$).

Let $\text{conv}(\mathcal{P})$ denote the convex hull of $\mathcal{P}$, and we refer to $\text{conv}(\mathcal{P})$ as the single-UC polytope. Note that $\text{conv}(\mathcal{P})$ is full dimensional, which is shown in Appendix A.3. Obviously, a valid inequality for $\mathcal{P}$ is also valid for Problem 1 for any generator g. Hence, the strong valid inequalities developed for $\text{conv}(\mathcal{P})$ can be used to tighten the formulation of Problem 1. The following two lemmas provide some important properties of $\mathcal{P}$.

LEMMA 1. *Consider any point $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}$ and $t \in [2, T]_{\mathbb{Z}}$. (i) If $y_t = 0$, then $y_{t-j} - y_{t-j-1} \leq 0$ for all $j \in [0, \min\{t-2, L-1\}]_{\mathbb{Z}}$. (ii) If $y_t = 1$, then there exists at most one $j \in [0, \min\{t-2, L\}]_{\mathbb{Z}}$ such that $y_{t-j} - y_{t-j-1} = 1$.*

LEMMA 2. *Denote $\mathbf{x}' = (x'_1, \ldots, x'_T)$ and $\mathbf{y}' = (y'_1, \ldots, y'_T)$. Let*

$$\begin{aligned}
\mathcal{P}' = \big\{(\mathbf{x}', \mathbf{y}') \in \mathbb{R}^T_+ \times \mathbb{B}^T :\ & \\
& -y'_{T-t+2} + y'_{T-t+1} - y'_{T-k+1} \leq 0,\ \forall t \in [2, T]_{\mathbb{Z}},\ \forall k \in [t, \min\{T, t+L-1\}]_{\mathbb{Z}}, \\
& y'_{T-t+2} - y'_{T-t+1} + y'_{T-k+1} \leq 1,\ \forall t \in [2, T]_{\mathbb{Z}},\ \forall k \in [t, \min\{T, t+\ell-1\}]_{\mathbb{Z}}, \\
& -x'_{T-t+1} + \underline{C} y'_{T-t+1} \leq 0,\ \forall t \in [1, T]_{\mathbb{Z}}, \\
& x'_{T-t+1} - \overline{C} y'_{T-t+1} \leq 0,\ \forall t \in [1, T]_{\mathbb{Z}}, \\
& x'_{T-t+1} - x'_{T-t+2} \leq V y'_{T-t+2} + \overline{V}(1 - y'_{T-t+2}),\ \forall t \in [2, T]_{\mathbb{Z}}, \\
& x'_{T-t+2} - x'_{T-t+1} \leq V y'_{T-t+1} + \overline{V}(1 - y'_{T-t+1}),\ \forall t \in [2, T]_{\mathbb{Z}}\big\}.
\end{aligned}$$

Then, $\mathcal{P} = \mathcal{P}'$.

Lemma 1 states a relationship among the y_t variables. This relationship, derived from the minimum up time requirement, allows us to simplify the validity proofs of the inequality families presented in Section 4. Lemma 2 states that if variables x_t and y_t are replaced by x_{T-t+1} and y_{T-t+1}, respectively, then the set $\mathcal{P}$ remains unchanged. This property enables us to show that if a given inequality family is known to be valid and facet-defining, then the corresponding "mirror image" of that inequality family is also valid and facet-defining.

3. The Two-period Convex Hull

In this section, we investigate the properties of the set $\mathcal{P}$ when there are only two periods. Then, we demonstrate that the strong valid inequalities resulting from our investigation can be used not only to tighten the single-UC polytope $\mathrm{conv}(\mathcal{P})$ but also to derive other forms of strong valid inequalities for $\mathrm{conv}(\mathcal{P})$.

Consider any two consecutive periods $t-1$ and t, where $t \in [2, T]_{\mathbb{Z}}$. Denote

$$
\begin{aligned}
\mathcal{P}_2 = \big\{ (x_{t-1}, x_t, y_{t-1}, y_t) \in \mathbb{R}^2_+ \times \mathbb{B}^2 :& \\
& -x_i + \underline{C} y_i \leq 0,\ \forall i \in \{t-1, t\}, && \text{(3a)}\\
& x_i - \overline{C} y_i \leq 0,\ \forall i \in \{t-1, t\}, && \text{(3b)}\\
& x_t - x_{t-1} \leq V y_{t-1} + \overline{V}(1 - y_{t-1}), && \text{(3c)}\\
& x_{t-1} - x_t \leq V y_t + \overline{V}(1 - y_t) \big\}. && \text{(3d)}
\end{aligned}
$$

Note that when $t = 2$, the set $\mathcal{P}_2$ is the same as the set $\mathcal{P}$ with $T = 2$. Note also that when $T = 2$, inequalities (2a) and (2b) become redundant, which significantly simplifies the set $\mathcal{P}_2$. Let $\mathrm{conv}(\mathcal{P}_2)$ denote the convex hull of $\mathcal{P}_2$. Pan and Guan (2016) have considered the two-period case of their two-binary formulation of the single-UC problem and have provided a convex hull description of its polytope. The following description of our convex hull can be obtained by projecting out the start-up variable from Pan and Guan's two-period convex hull via Fourier-Motzkin elimination:

$$
\begin{aligned}
\mathrm{conv}(\mathcal{P}_2) = \big\{ (x_{t-1}, x_t, y_{t-1}, y_t) \in \mathbb{R}^4 :& \\
& y_i \leq 1,\ \forall i \in \{t-1, t\}, && \text{(4a)}\\
& \underline{C} y_i \leq x_i \leq \overline{C} y_i,\ \forall i \in \{t-1, t\}, && \text{(4b)}\\
& x_{t-1} \leq \overline{V} y_{t-1} + (\overline{C} - \overline{V}) y_t, && \text{(4c)}\\
& x_t \leq (\overline{C} - \overline{V}) y_{t-1} + \overline{V} y_t, && \text{(4d)}\\
& x_t - x_{t-1} \leq (\underline{C} + V) y_t - \underline{C} y_{t-1}, && \text{(4e)}\\
& x_t - x_{t-1} \leq \overline{V} y_t - (\overline{V} - V) y_{t-1}, && \text{(4f)}\\
& x_{t-1} - x_t \leq (\underline{C} + V) y_{t-1} - \underline{C} y_t, && \text{(4g)}\\
& x_{t-1} - x_t \leq \overline{V} y_{t-1} - (\overline{V} - V) y_t \big\}. && \text{(4h)}
\end{aligned}
$$

Damcı-Kurt et al. (2016, Sec. 2.2) have also considered a formulation of the two-period ramp-up single-UC problem without the use of start-up variables and have derived the convex hull of the feasible set by projecting out the start variables via Fourier-Motzkin elimination. Our inequalities (4a), (4b), (4d), (4e), and (4f) are identical to the constraints in Damcı-Kurt et al. (2016)'s convex hull.

Note that for every $t \in [2, T]_{\mathbb{Z}}$, any inequality in (4a)–(4h) is valid for $\mathrm{conv}(\mathcal{P})$. Note also that inequalities (4c)–(4h) do not exist in the description of $\mathcal{P}$. Hence, they can be added to the constraint set of $\mathcal{P}$ to tighten the linear relaxation of $\mathcal{P}$. In particular, because $\overline{V} y_t - (\overline{V} - V) y_{t-1} \leq$

$\underline{V}y_{t-1} + \overline{V}(1 - y_{t-1})$ for any $y_t \leq 1$, the right-hand side of (4f) is no greater than the right-hand side of (2e), and thus inequality (4f) dominates inequality (2e) and can effectively tighten the linear relaxation of $\mathcal{P}$. Similarly, inequality (4h) dominates inequality (2f) and can effectively tighten the linear relaxation of $\mathcal{P}$. Therefore, the following inequality families can be used as valid inequalities for conv($\mathcal{P}$):

$$x_t \leq \overline{V}y_t + (\overline{C} - \overline{V})y_{t+1},\ \forall t \in [1, T-1]_{\mathbb{Z}}; \tag{5}$$

$$x_t \leq (\overline{C} - \overline{V})y_{t-1} + \overline{V}y_t,\ \forall t \in [2, T]_{\mathbb{Z}}; \tag{6}$$

$$x_t - x_{t-1} \leq (\underline{C} + V)y_t - \underline{C}y_{t-1},\ \forall t \in [2, T]_{\mathbb{Z}}; \tag{7}$$

$$x_t - x_{t-1} \leq \overline{V}y_t - (\overline{V} - V)y_{t-1},\ \forall t \in [2, T]_{\mathbb{Z}}; \tag{8}$$

$$x_t - x_{t+1} \leq (\underline{C} + V)y_t - \underline{C}y_{t+1},\ \forall t \in [1, T-1]_{\mathbb{Z}}; \tag{9}$$

$$x_t - x_{t+1} \leq \overline{V}y_t - (\overline{V} - V)y_{t+1},\ \forall t \in [1, T-1]_{\mathbb{Z}}. \tag{10}$$

These valid inequalities provide upper bounds on the generation amount x_t for each time period t, upper bounds on $x_t - x_{t-1}$ for each pair of consecutive time periods t and $t-1$, and upper bounds on $x_t - x_{t+1}$ for each pair of consecutive time periods t and $t+1$.

Inequalities (5)–(10) also enable us to develop other strong valid inequalities for conv($\mathcal{P}$). We demonstrate this by presenting a strong valid inequality derived from (7). Consider any point $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}$. For any $k \in [1, T-1]_{\mathbb{Z}}$ and any $t \in [k+1, T]_{\mathbb{Z}}$, because inequality (7) is valid for conv($\mathcal{P}$), we have

$$\sum_{\tau=t-k+1}^{t} (x_\tau - x_{\tau-1}) \leq \sum_{\tau=t-k+1}^{t} \left[(\underline{C} + V)y_\tau - \underline{C}y_{\tau-1}\right] = V \sum_{\tau=t-k+1}^{t} y_\tau + \underline{C} \sum_{\tau=t-k+1}^{t} (y_\tau - y_{\tau-1}),$$

which implies that

$$x_t - x_{t-k} \leq V \sum_{\tau=t-k+1}^{t} y_\tau + \underline{C}y_t - \underline{C}y_{t-k}. \tag{11}$$

If $y_t = 1$, then $\sum_{\tau=t-k+1}^{t} y_\tau \leq k y_t$, and by (11), $x_t - x_{t-k} \leq (\underline{C} + kV)y_t - \underline{C}y_{t-k}$. If $y_t = 0$, then by (2c) and (2d), $-x_{t-k} \leq -\underline{C}y_{t-k}$ and $x_t = 0$, which also imply that $x_t - x_{t-k} \leq (\underline{C} + kV)y_t - \underline{C}y_{t-k}$. Thus, in both cases,

$$x_t - x_{t-k} \leq (\underline{C} + kV)y_t - \underline{C}y_{t-k}. \tag{12}$$

Hence, (12) is a valid inequality for conv($\mathcal{P}$) for any $k \in [1, T-1]_{\mathbb{Z}}$ and $t \in [k+1, T]_{\mathbb{Z}}$. It is worth noting that inequality (19) presented in Proposition 9 in Section 4.2 is reduced to inequality (12) when $m = 0$ and $\mathcal{S} = \emptyset$. Therefore, inequality (12) is a special case of the facet-defining valid inequality (19).

4. Multi-period Strong Valid Inequalities

When there are more than two periods, the set $\mathcal{P}$ is significantly more complex than the set $\mathcal{P}_2$ because it involves not only more time periods but also minimum up/down constraints. In this section, we present a collection of strong valid inequalities that can effectively enhance the tightness of Problem 1. We provide the validity proofs for these inequalities, and we identify the conditions under which these inequalities are facet-defining for $\mathrm{conv}(\mathcal{P})$. For each family of valid inequalities, we also show that for any given point $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{2T}$, an efficient separation algorithm exists for determining a most violated inequality.

4.1. Valid Inequalities with a Single Continuous Variable

In this subsection, we present strong valid inequalities that provide upper bounds on the generation amount x_t for each time period t. Families of such inequalities appear in pairs. The first family consists of inequalities for which the upper bound on x_t depends mainly on the values of $y_{t-s} - y_{t-s-1}$ for some $s \geq 0$, and the second family consists of inequalities for which the upper bound on x_t depends mainly on the values of $y_{t+s} - y_{t+s+1}$ for some $s \geq 0$. The following proposition presents a pair of such inequality families.

PROPOSITION 1. *Consider any $\mathcal{S} \subseteq [0, \min\{L-1, T-2, \lfloor (\overline{C} - \overline{V})/V \rfloor\}]_{\mathbb{Z}}$. For any $t \in [1, T]_{\mathbb{Z}}$ such that $t \geq s+2$ for all $s \in \mathcal{S}$, the inequality*

$$x_t \leq \overline{C} y_t - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \tag{13}$$

is valid and facet-defining for $\mathrm{conv}(\mathcal{P})$*. For any $t \in [1, T]_{\mathbb{Z}}$ such that $t \leq T - s - 1$ for all $s \in \mathcal{S}$, the inequality*

$$x_t \leq \overline{C} y_t - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \tag{14}$$

is valid and facet-defining for $\mathrm{conv}(\mathcal{P})$*.*

In Proposition 1, inequalities (13) and (14) provide upper bounds on the generation amount x_t. These upper bounds can be explained as follows. Let $s_{\max}$ denote the largest element of $\mathcal{S}$. The condition "$\mathcal{S} \subseteq [0, \min\{L-1, T-2, \lfloor (\overline{C} - \overline{V})/V \rfloor\}]_{\mathbb{Z}}$" implies that $s_{\max} \leq L-1$, which in turn implies that there is at most one startup and at most one shutdown during the time interval $[t - s_{\max}, t]$, and that there is at most one shutdown and at most one startup during the time interval $[t+1, t+s_{\max}+1]$. Consider the situation in which a generator starts up in period $t - s_1$, stays online until period $t + s_2$, and shuts down in period $t + s_2 + 1$, where $s_1, s_2 \in [0, s_{\max}]_{\mathbb{Z}}$, $t - s_1 \geq 2$, and $t + s_2 + 1 \leq T$. Then, $y_{t-s_1-1} = 0$, $y_{t-s_1} = y_{t-s_1+1} = \cdots = y_{t+s_2} = 1$, and $y_{t+s_2+1} = 0$. If $s_1 \in \mathcal{S}$ and none of the time periods in $\{t - s \geq 2 : s \in \mathcal{S}\}$ is a shut-down period, then the right-hand side of inequality (13) becomes $\overline{C} - (\overline{C} - \overline{V} - s_1 V)$. This upper bound limits the value of x_t to be no more than $\overline{V} + s_1 V$, which is smaller than the generation upper bound $\overline{C}$. Similarly, if $s_2 \in \mathcal{S}$ and none of the periods in $\{t + s + 1 \leq T : s \in \mathcal{S}\}$ is a start-up period, then the right-hand side of inequality (14) becomes $\overline{C} - (\overline{C} - \overline{V} - s_2 V)$. This upper bound limits the value of x_t to be no more than $\overline{V} + s_2 V$, which is smaller than the generation upper bound $\overline{C}$.

In Proposition 1, the set $\mathcal{S}$ only contains elements that are less than L. The following proposition states that under certain conditions, inequalities (13) and (14) remain valid and facet-defining when $\mathcal{S}$ contains some elements that are greater than or equal to L.

PROPOSITION 2. *Consider any integers α, β, and $s_{\max}$ such that (a) $L \leq s_{\max} \leq \min\{T-2, \lfloor(\overline{C} - \overline{V})/V\rfloor\}$, (b) $0 \leq \alpha < \beta \leq s_{\max}$, and (c) $\beta = \alpha + 1$ or $s_{\max} \leq L + \alpha$. Let $\mathcal{S} = [0, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. For any $t \in [s_{\max} + 2, T]_{\mathbb{Z}}$, inequality (13) is valid and facet-defining for $\operatorname{conv}(\mathcal{P})$. For any $t \in [1, T - s_{\max} - 1]_{\mathbb{Z}}$, inequality (14) is valid and facet-defining for $\operatorname{conv}(\mathcal{P})$.*

EXAMPLE 1. Let $T = 16$, $\overline{C} = 80$, $\underline{C} = 8$, $L = \ell = 5$, $\overline{V} = 15$, and $V = 10$. Then, $\lfloor(\overline{C} - \overline{V})/V\rfloor = 6$. By Proposition 1, we obtain the following pair of valid inequalities if we set $\mathcal{S} = \{0, 2, 4\}$ and $t = 8$:

$$\begin{cases} x_8 \leq 25y_3 - 25y_4 + 45y_5 - 45y_6 + 65y_7 + 15y_8; \\ x_8 \leq 15y_8 + 65y_9 - 45y_{10} + 45y_{11} - 25y_{12} + 25y_{13}. \end{cases}$$

By Proposition 2, we obtain the following pair of valid inequalities if we set $\mathcal{S} = \{0, 1, 2, 5, 6\}$ (i.e., $\alpha = 2$, $\beta = 5$, and $s_{\max} = 6$) and $t = 8$:

$$\begin{cases} x_8 \leq 5y_1 + 10y_2 - 15y_3 + 45y_5 + 10y_6 + 10y_7 + 15y_8; \\ x_8 \leq 15y_8 + 10y_9 + 10y_{10} + 45y_{11} - 15y_{13} + 10y_{14} + 5y_{15}. \end{cases}$$

The next proposition extends Proposition 1 and presents another two families of strong valid inequalities.

PROPOSITION 3. *Consider any set $\mathcal{S} \subseteq [0, \min\{L-1, T-3, \lfloor(\overline{C} - \overline{V})/V\rfloor\}]_{\mathbb{Z}}$ and any real number η such that $0 \leq \eta \leq \min\{L-1, (\overline{C} - \overline{V})/V\}$. For any $t \in [1, T-1]_{\mathbb{Z}}$ such that $t \geq s + 2$ for all $s \in \mathcal{S}$, the inequality*

$$x_t \leq (\overline{C} - \eta V)y_t + \eta V y_{t+1} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \tag{15}$$

is valid for $\operatorname{conv}(\mathcal{P})$. For any $t \in [2, T]_{\mathbb{Z}}$ such that $t \leq T - s - 1$ for all $s \in \mathcal{S}$, the inequality

$$x_t \leq (\overline{C} - \eta V)y_t + \eta V y_{t-1} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \tag{16}$$

is valid for $\operatorname{conv}(\mathcal{P})$. Furthermore, inequalities (15) and (16) are facet-defining for $\operatorname{conv}(\mathcal{P})$ when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L - 1 \in \mathcal{S}$.

When $t \neq T$, inequality (15) is a generalization of inequality (13). Specifically, the right-hand side of (15) differs from the right-hand side of (13) by $\eta V y_{t+1} - \eta V y_t$, and this difference is zero if $\eta = 0$. Similarly, when $t \neq 1$, inequality (16) is a generalization of inequality (14), and the right-hand side of (16) differs from the right-hand side of (14) by $\eta V y_{t-1} - \eta V y_t$. In Proposition 3, the set $\mathcal{S}$ only contains elements that are less than L. The following proposition, which extends Proposition 2, states that under certain conditions, inequalities (15) and (16) remain valid and facet-defining when $\mathcal{S}$ contains some elements that are greater than or equal to L.

PROPOSITION 4. *Consider any real number η such that $0 \leq \eta \leq \min\{L-1, (\overline{C}-\overline{V})/V\}$ and any integers α, β, and $s_{\max}$ such that (a) $L \leq s_{\max} \leq \min\{T-3, \lfloor(\overline{C}-\overline{V})/V\rfloor\}$, (b) $0 \leq \alpha < \beta \leq s_{\max}$, and (c) $\beta = \alpha+1$ or $s_{\max} \leq L+\alpha$. Let $\mathcal{S} = [0,\alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. For any $t \in [s_{\max}+2, T-1]_{\mathbb{Z}}$, inequality (15) is valid for $\operatorname{conv}(\mathcal{P})$. For any $t \in [2, T-s_{\max}-1]_{\mathbb{Z}}$, inequality (16) is valid for $\operatorname{conv}(\mathcal{P})$. Furthermore, (15) and (16) are facet-defining for $\operatorname{conv}(\mathcal{P})$ when $\eta \in \{0, (\overline{C}-\overline{V})/V\}$ or $\eta = L-1 \in \mathcal{S}$.*

EXAMPLE 2 (CONTINUATION OF EXAMPLE 1). In Example 1, if we set $\eta = 2.5$, $\mathcal{S} = \{0,2,4\}$, and $t = 8$, then by Proposition 3, we obtain the following pair of valid inequalities:

$$\begin{cases} x_8 \leq 25y_3 - 25y_4 + 45y_5 - 45y_6 + 65y_7 - 10y_8 + 25y_9; \\ x_8 \leq 25y_7 - 10y_8 + 65y_9 - 45y_{10} + 45y_{11} - 25y_{12} + 25y_{13}. \end{cases}$$

Note that the right-hand sides of the first and second inequalities differ from those in the first pair of inequalities in Example 1 by $\eta V y_{t+1} - \eta V y_t$ (i.e., $25y_9 - 25y_8$) and $\eta V y_{t-1} - \eta V y_t$ (i.e., $25y_7 - 25y_8$), respectively. If we set $\eta = 2.5$, $\mathcal{S} = \{0,1,2,5,6\}$ (i.e., $\alpha = 2$, $\beta = 5$, and $s_{\max} = 6$), and $t = 8$, then by Proposition 4, we obtain the following pair of valid inequalities:

$$\begin{cases} x_8 \leq 5y_1 + 10y_2 - 15y_3 + 45y_5 + 10y_6 + 10y_7 - 10y_8 + 25y_9; \\ x_8 \leq 25y_7 - 10y_8 + 10y_9 + 10y_{10} + 45y_{11} - 15y_{13} + 10y_{14} + 5y_{15}. \end{cases}$$

Similarly, the right-hand sides of the first and second inequalities differ from those in the second pair of inequalities in Example 1 by $\eta V y_{t+1} - \eta V y_t$ (i.e., $25y_9 - 25y_8$) and $\eta V y_{t-1} - \eta V y_t$ (i.e., $25y_7 - 25y_8$), respectively.

The next proposition also extends Proposition 1 and presents another two families of strong valid inequalities.

PROPOSITION 5. *Consider any $\mathcal{S} \subseteq [1, \min\{L, T-2, \lfloor(\overline{C}-\overline{V})/V\rfloor\}]_{\mathbb{Z}}$ and any real number η such that $0 \leq \eta \leq \min\{L, (\overline{C}-\overline{V})/V\}$. For any $t \in [2,T]_{\mathbb{Z}}$ such that $t \geq s+2$ for all $s \in \mathcal{S}$, the inequality*

$$x_t \leq (\overline{V}+\eta V)y_t + (\overline{C}-\overline{V}-\eta V)y_{t-1} - \sum_{s\in\mathcal{S}}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1}) \tag{17}$$

is valid for $\operatorname{conv}(\mathcal{P})$. For any $t \in [1, T-1]_{\mathbb{Z}}$ such that $t \leq T-s-1$ for all $s \in \mathcal{S}$, the inequality

$$x_t \leq (\overline{V}+\eta V)y_t + (\overline{C}-\overline{V}-\eta V)y_{t+1} - \sum_{s\in\mathcal{S}}(\overline{C}-\overline{V}-sV)(y_{t+s}-y_{t+s+1}) \tag{18}$$

is valid for $\mathrm{conv}(\mathcal{P})$. *Furthermore, inequalities* (17) *and* (18) *are facet-defining for* $\mathrm{conv}(\mathcal{P})$ *when* $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ *or* $\eta = L \in \mathcal{S}$.

In Proposition 5, the set $\mathcal{S}$ only contains elements that are less than or equal to L. The following proposition states that under certain conditions, inequalities (17) and (18) remain valid and facet-defining when $\mathcal{S}$ contains some elements that are greater than L.

PROPOSITION 6. *Consider any integers* α, β, *and* $s_{\max}$ *such that (a)* $L + 1 \leq s_{\max} \leq \min\{T - 2, \lfloor(\overline{C} - \overline{V})/V\rfloor\}$, *(b)* $1 \leq \alpha < \beta \leq s_{\max}$, *and (c)* $\beta = \alpha + 1$ *or* $s_{\max} \leq L + \alpha$. *Let* $\mathcal{S} = [1, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. *For any* $t \in [s_{\max} + 2, T]_{\mathbb{Z}}$, *inequality* (17) *is valid for* $\mathrm{conv}(\mathcal{P})$. *For any* $t \in [1, T - s_{\max} - 1]_{\mathbb{Z}}$, *inequality* (18) *is valid for* $\mathrm{conv}(\mathcal{P})$. *Furthermore,* (17) *and* (18) *are facet-defining for* $\mathrm{conv}(\mathcal{P})$ *when* $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ *or* $\eta = L \in \mathcal{S}$.

EXAMPLE 3 (CONTINUATION OF EXAMPLE 1). In Example 1, if we set $\eta = 2.5$, $\mathcal{S} = \{1, 3, 5\}$, and $t = 8$, then by Proposition 5, we obtain the following pair of valid inequalities:

$$\begin{cases} x_8 \leq 15y_2 - 15y_3 + 35y_4 - 35y_5 + 55y_6 - 15y_7 + 40y_8; \\ x_8 \leq 40y_8 - 15y_9 + 55y_{10} - 35y_{11} + 35y_{12} - 15y_{13} + 15y_{14}. \end{cases}$$

If we set $\eta = 2.5$, $\mathcal{S} = \{1, 2, 5, 6\}$ (i.e., $\alpha = 2$, $\beta = 5$, and $s_{\max} = 6$), and $t = 8$, then by Proposition 6, we obtain the following pair of valid inequalities:

$$\begin{cases} x_8 \leq 5y_1 + 10y_2 - 15y_3 + 45y_5 + 10y_6 - 15y_7 + 40y_8; \\ x_8 \leq 40y_8 - 15y_9 + 10y_{10} + 45y_{11} - 15y_{13} + 10y_{14} + 5y_{15}. \end{cases}$$

Propositions 1–6 present different families of valid inequalities. For each family of valid inequalities and any given point $(\mathbf{x}, \mathbf{y})$ with non-binary y values, it is important to have an efficient separation algorithm that can identify a most violated inequality in the family, if such a violated inequality exists. In Propositions 1, 3, and 5, the number of combinations of $\mathcal{S}$ is exponential in T. Furthermore, in Propositions 3 and 5, η is a real value. However, the next proposition states that given any point $(\mathbf{x}, \mathbf{y})$ with non-binary y values, a most violated inequality in each of the inequality families stated in Propositions 1, 3, and 5 can be determined in linear time.

PROPOSITION 7. *For any given point* $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2T}_{+}$, *the most violated inequalities* (13)–(14), (15)–(16), *and* (17)–(18) *in Propositions 1, 3, and 5, respectively, can be determined in* $O(T)$ *time if such violated inequalities exist.*

In Propositions 2, 4, and 6, the number of combinations of α, β, $s_{\max}$, and t is $O(T^4)$. Furthermore, in Propositions 4 and 6, η is a real value. However, the next proposition states that given any point $(\mathbf{x}, \mathbf{y})$ with non-binary y values, a most violated inequality in each of the inequality families stated in Propositions 2, 4, and 6 can be determined in $O(T^3)$ time.

PROPOSITION 8. *For any given point* $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2T}_{+}$, *the most violated inequalities* (13)–(14), (15)–(16), *and* (17)–(18) *in Propositions 2, 4, and 6, respectively, can be determined in* $O(T^3)$ *time if such violated inequalities exist.*

4.2. Valid Inequalities with Two Continuous Variables

In this subsection, we present strong valid inequalities that provide upper bounds on $x_t - x_{t-k}$ (respectively $x_t - x_{t+k}$) for each pair of time periods t and $t-k$ (respectively t and $t+k$). The following proposition presents a pair of such inequality families.

PROPOSITION 9. *Consider any $k \in [1, T-1]_{\mathbb{Z}}$ such that $\overline{C} - \underline{C} - kV > 0$, any $m \in [0, k-1]_{\mathbb{Z}}$, and any $\mathcal{S} \subseteq [0, \min\{k-1, L-m-1\}]_{\mathbb{Z}}$. For any $t \in [k+1, T-m]_{\mathbb{Z}}$, the inequality*

$$x_t - x_{t-k} \leq (\underline{C} + (k-m)V)y_t + V\sum_{i=1}^{m} y_{t+i} - \underline{C}y_{t-k} - \sum_{s\in\mathcal{S}}(\underline{C} + (k-s)V - \overline{V})(y_{t-s} - y_{t-s-1}) \tag{19}$$

is valid for $\mathrm{conv}(\mathcal{P})$. *For any $t \in [m+1, T-k]_{\mathbb{Z}}$, the inequality*

$$x_t - x_{t+k} \leq (\underline{C} + (k-m)V)y_t + V\sum_{i=1}^{m} y_{t-i} - \underline{C}y_{t+k} - \sum_{s\in\mathcal{S}}(\underline{C} + (k-s)V - \overline{V})(y_{t+s} - y_{t+s+1}) \tag{20}$$

is valid for $\mathrm{conv}(\mathcal{P})$. *Furthermore, (19) and (20) are facet-defining for* $\mathrm{conv}(\mathcal{P})$ *when $m = 0$ and $s \geq$* $\min\{k-1, 1\}$ *for all $s \in \mathcal{S}$.*

In Proposition 9, the number of combinations of $\mathcal{S}$, t, k, and m is exponential in T. Thus, the sizes of the inequality families (19) and (20) are exponential in T. However, the next proposition states that given any point $(\mathbf{x}, \mathbf{y})$ with non-binary y values, the most violated inequalities (19) and (20) can be determined in polynomial time.

PROPOSITION 10. *For any given point $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2T}_{+}$, the most violated inequalities (19) and (20) can be determined in $O(T^3)$ time if such violated inequalities exist.*

PROPOSITION 11. *Consider any $k \in [1, T-1]_{\mathbb{Z}}$ such that $\overline{C} - \underline{C} - kV > 0$, any $m \in [0, k-1]_{\mathbb{Z}}$, and any $\mathcal{S} \subseteq [0, \min\{k-1, L-m-2\}]_{\mathbb{Z}}$. For any $t \in [k+1, T-m-1]_{\mathbb{Z}}$, the inequality*

$$\begin{aligned} x_t - x_{t-k} \leq\ & (\underline{C} + (k-m)V - \overline{V})y_{t+m+1} + V\sum_{i=1}^{m} y_{t+i} + \overline{V}y_t - \underline{C}y_{t-k} \\ & - \sum_{s\in\mathcal{S}}(\underline{C} + (k-s)V - \overline{V})(y_{t-s} - y_{t-s-1}) \end{aligned} \tag{21}$$

is valid and facet-defining for $\mathrm{conv}(\mathcal{P})$. *For any $t \in [m+2, T-k]_{\mathbb{Z}}$, the inequality*

$$\begin{aligned} x_t - x_{t+k} \leq\ & (\underline{C} + (k-m)V - \overline{V})y_{t-m-1} + V\sum_{i=1}^{m} y_{t-i} + \overline{V}y_t - \underline{C}y_{t+k} \\ & - \sum_{s\in\mathcal{S}}(\underline{C} + (k-s)V - \overline{V})(y_{t+s} - y_{t+s+1}) \end{aligned} \tag{22}$$

is valid and facet-defining for $\mathrm{conv}(\mathcal{P})$.

In Proposition 11, the number of combinations of $\mathcal{S}$, t, k, and m is exponential in T. Thus, the sizes of the inequality families (21) and (22) are exponential in T. However, the next proposition states that given any point $(\mathbf{x}, \mathbf{y})$ with non-binary y values, the most violated inequalities (21) and (22) can be determined in polynomial time.

PROPOSITION 12. *For any given point* $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_{+}^{2T}$, *the most violated inequalities* (21) *and* (22) *can be determined in* $O(T^3)$ *time if such violated inequalities exist.*

5. Computational Experiments

We conduct a computational study to evaluate the effectiveness of our strong valid inequalities in tightening the proposed single-binary MILP formulation for the UC problem. In Section 5.1, we describe the problem instances that we use in this computational study. In Section 5.2, we present the computational results.

All of the computational experiments are performed on a computer node with Intel(R) Xeon(R) CPU E5-2699 v3 at 2.30GHz and 16 cores. The addressable memory is 32GB. IBM ILOG CPLEX 22.1 is used as the MILP solver to run all of the experiments. The MILP solver is called through its Python application programming interface under the default settings. Note that the performance of a mathematical programming formulation is affected by the inherent random component of the heuristic process used in solvers (Tejada-Arango et al. 2020). Thus, to accurately evaluate the effectiveness of our strong valid inequalities, "traditional branch-and-cut" is chosen to be the search strategy.

5.1. Test Instances

We conduct three computational experiments. These experiments are based on a network-constrained UC problem. Recall that in Sections 3 and 4, the superscript g was omitted when we focused on deriving strong valid inequalities for the polytope $\text{conv}(\mathcal{P})$ that consists of a single generator. In the test instances of these three experiments, we reinstate the superscript g in the strong valid inequalities that are used to tighten the UC formulations. Thus, when a family of strong valid inequality is added to a UC formulation in these three experiments, it will be added to each generator with its corresponding parameters at the same time. In all three experiments, the non-decreasing convex piecewise cost function is obtained by approximating the given quadratic cost function $a^g x^2 + b^g x$. We apply the method developed by Frangioni et al. (2009) to perform this piecewise linear approximation, using nine line segments with the x-coordinates of the break-points spread evenly between the lower bound $\underline{C}$ and the upper bound $\overline{C}$.

In the first experiment, we use the data obtained from Ostrowski et al. (2012) and Pan and Guan (2016). Because of the absence of transmission flow data in this data set, the transmission constraint (1l) is not considered in this experiment. The removal of the transmission constraint does not have a major impact on our computational study because we focus primarily on evaluating the effectiveness of the strong valid inequalities in tightening the single-binary formulation.

Table 1 Generator Data (Ostrowski et al. 2012, Pan and Guan 2016)

Generator Type	$\underline{C}^g$ (MW)	$\overline{C}^g$ (MW)	L^g (h)	ℓ^g (h)	V^g (MW/h)	$\overline{V}^g$ (MW/h)	ϕ^g (\$/h)	ψ^g (\$/h)	a^g (\$/MW^{2}h)	b^g (\$/MWh)	c^g (\$/h)
1	150	455	8	8	91	180	2000	2000	0.00048	16.19	1000
2	150	455	8	8	91	180	2000	2000	0.00031	17.26	970
3	20	130	5	5	26	35	500	500	0.00200	16.6	700
4	20	130	5	5	26	35	500	500	0.00211	16.5	680
5	25	162	6	6	32.4	40	700	700	0.00398	19.7	450
6	20	80	3	3	16	28	150	150	0.00712	22.26	370
7	25	85	3	3	17	33	200	200	0.00079	27.74	480
8	10	55	1	1	11	15	60	60	0.00413	25.92	660

Table 2 Problem Instances (Ostrowski et al. 2012, Pan and Guan 2016)

Instance	Number of generators								Total no. of generators
	Type 1	Type 2	Type 3	Type 4	Type 5	Type 6	Type 7	Type 8	
1	12	11	0	0	1	4	0	0	28
2	13	15	2	0	4	0	0	1	35
3	15	13	2	6	3	1	1	3	44
4	15	11	0	1	4	5	6	3	45
5	15	13	3	7	5	3	2	1	49
6	10	10	2	5	7	5	6	5	50
7	17	16	1	3	1	7	2	4	51
8	17	10	6	5	2	1	3	7	51
9	12	17	4	7	5	2	0	5	52
10	13	12	5	7	2	5	4	6	54
11	46	45	8	0	5	0	12	16	132
12	40	54	14	8	3	15	9	13	156
13	50	41	19	11	4	4	12	15	156
14	51	58	17	19	16	1	2	1	165
15	43	46	17	15	13	15	6	12	167
16	50	59	8	15	1	18	4	17	172
17	53	50	17	15	16	5	14	12	182
18	45	57	19	7	19	19	5	11	182
19	58	50	15	7	16	18	7	12	183
20	55	48	18	5	18	17	15	11	187

Table 3 System Load—Percentage of Total Generation Capacity (Ostrowski et al. 2012, Pan and Guan 2016)

Period	1	2	3	4	5	6	7	8	9	10	11	12
System Load	71%	65%	62%	60%	58%	58%	60%	64%	73%	80%	82%	83%

Period	13	14	15	16	17	18	19	20	21	22	23	24
System Load	82%	80%	79%	79%	83%	91%	90%	88%	85%	84%	79%	74%

The system contains eight types of generators. Table 1 contains the data of these eight generator types. The generation cost function for generator g is $a^g x^2 + b^g x$, where the values of a^g and b^g are provided in the 10th and 11th columns, respectively, of the table. The data set comprises 20 test instances, as shown in Table 2. For each instance, the operation horizon is set equal to 24 hours, i.e., $T = 24$, and the system reserve factor is set equal to 3% for all periods, i.e., $r_t = 0.03$ for all $t \in [1, T]_{\mathbb{Z}}$. The system load $\sum_{b \in \mathcal{B}} d_t^b$ in each period t is shown in Table 3, and it is expressed as a percentage of the total generation capacity $\sum_{g \in \mathcal{G}} \overline{C}^g$.

In this experiment, we compare the following two formulations:

$$\begin{aligned} \text{F1:} \quad & \text{minimize} && \text{objective function (1a)} \\ & \text{subject to} && \text{constraints (1b)–(1k), (1m).} \end{aligned}$$

$$\begin{aligned} \text{F1-X:} \quad & \text{minimize} && \text{objective function (1a)} \\ & \text{subject to} && \text{constraints (1b)–(1k), (1m);} \\ & && \text{constraints (5)–(10);} \\ & && \text{cutting planes (13)–(22).} \end{aligned}$$

Formulation F1 is the original formulation of Problem 1 with the transmission constraint (1l) removed. In F1-X, the strong valid inequality families (5)–(10) obtained from the two-period single-UC polytope are added to the formulation as constraints, and the multi-period strong valid inequality families (13)–(22) derived in Section 4 are added to the user cut pool of the CPLEX optimizer and are applied at any stage of the optimization. Note that each of the inequality families (13)–(22) contains a large number of inequalities. Thus, for each of these inequality families, only some of the inequalities are added to F1-X as user cuts , because the use of separation algorithms through the callbacks of CPLEX will slow down the solution process. This is attributable to the frequent invocation of the separation algorithms during the solution process. Specifically, for each of the inequality families (13)–(22), $\mathcal{S}$ is restricted to the empty set and the set that contains all of the elements in its range, and the other parameters such as t, k, and m are allowed to take any values in their respective ranges such that the inequality obtained is facet-defining for $\text{conv}(\mathcal{P})$. For example, for inequality family (19), we consider each $k \in [1, T-1]_{\mathbb{Z}}$, $m = 0$, $\mathcal{S} = \{\emptyset, [0, \min\{k-1, L-1\}]_{\mathbb{Z}}\}$, and $t \in [k+1, T]_{\mathbb{Z}}$ such that $s \geq \min\{k-1, 1\}$ for all $s \in \mathcal{S}$.

In the second experiment, we use the same data as in the first experiment. We compare the effectiveness of our strong valid inequalities with that of the valid inequalities in Pan and Guan (2016) in tightening Pan and Guan's formulation. We also test the effectiveness of our strong valid inequalities when they are combined with the valid inequalities in Pan and Guan (2016). To do so, we solve the following four formulations:

$$\begin{aligned} \text{F2:} \quad & \text{minimize} && \text{objective function (38a) in Pan and Guan (2016)} \\ & \text{subject to} && \text{constraints (38b)–(38i), (38k) in Pan and Guan (2016).} \end{aligned}$$

F2-X: minimize objective function (38a) in Pan and Guan (2016)
subject to constraints (38b)–(38i), (38k) in Pan and Guan (2016);
constraints (5)–(10) in this paper;
cutting planes (13)–(22) in this paper.

F2-Y: minimize objective function (38a) in Pan and Guan (2016)
subject to constraints (38b)–(38i), (38k) in Pan and Guan (2016);
constraints (2d)–(2g) in Pan and Guan (2016);
cutting planes (4)–(7), (10)–(13), (24d), (24h)–(24i),
(24o)–(24r), (28)–(36) in Pan and Guan (2016).

F2-Z: minimize objective function (38a) in Pan and Guan (2016)
subject to constraints (38b)–(38i), (38k) in Pan and Guan (2016);
constraints (2d)–(2g) in Pan and Guan (2016);
constraints (5)–(10) in this paper;
cutting planes (4)–(7), (10)–(13), (24d), (24h)–(24i),
(24o)–(24r), (28)–(36) in Pan and Guan (2016).
cutting planes (13)–(22) in this paper.

Formulation F2 is the two-binary UC formulation in Pan and Guan (2016), except that the transmission constraint (38j) has been excluded. In formulation F2-X, the valid inequalities (5)–(10) are added as constraints, and the valid inequalities (13)–(22) are added as user cuts in the same way as in the first experiment. In formulation F2-Y, the strong valid inequalities in Pan and Guan (2016) are used the same way as in Pan and Guan's computational study to tighten formulation F2. Specifically, valid inequalities in the two-period convex hull, (2d)–(2g), are added as constraints, and other valid inequalities are added as user cuts. For inequality families that have an exponential size, the $\mathcal{S}$ set is restricted to the empty set and the set that contains all of the elements in its range. The other parameters, such as t, m, and n, are allowed to take any values in their respective ranges such that the inequality obtained is facet-defining for $\text{conv}(\mathcal{P})$. Formulation F2-Z contains all the valid inequalities in formulations F2-X and F2-Y.

The third experiment examines a network-constrained UC problem based on the modified IEEE 118-bus system. The system comprises 54 thermal generators, 118 buses, and 186 transmission lines. System data such as $\overline{C}^g$, $\underline{C}^g$, L^g, ℓ^g, a^g, b^g, c^g, etc., as well as the load amount of each load bus, are obtained from http://motor.ece.iit.edu/data/SCUC_118/. Each instance has a 24-hour operation horizon, i.e., $T = 24$. The system reserve factor of each time period is set equal to 3%, as in the first experiment. The maximum hourly load of the system is randomly generated from a uniform distribution on $[0.5 \sum_{g \in \mathcal{G}} \overline{C}^g, \sum_{g \in \mathcal{G}} \overline{C}^g]$. The maximum hourly load of each load bus is then obtained by allocating the maximum hourly load of the system to each load bus in proportion to their load amounts. For each load bus, the loads in different time periods are then obtained by following a daily load profile such that the maximum load of the day is equal to the maximum hourly load. This daily load profile is obtained from `https://www.pjm.com/markets-and-operations/data-dictionary`, which was generated based on the average values of the actual hourly electricity demand over 30 days in the western market. Twenty instances with randomly generated loads are created using this method. Each instance is solved using the following formulations:

F1$^+$:	minimize	objective function (1a)
	subject to	constraints (1b)–(1m).

F1$^+$-X:	minimize	objective function (1a)
	subject to	constraints (1b)–(1m);
		constraints (5)–(10);
		cutting planes (13)–(22).

Formulations F1$^+$ and F1$^+$-X resemble formulations F1 and F1-X, respectively, in the first experiment, with the transmission constraint (1l) reinstated. In F1$^+$-X, the strong valid inequalities (5)–(10) and (13)–(22) are added as constraints and user cuts, respectively, in the same way as in the first experiment.

5.2. Computational Results

In this subsection, we report the computational results of the three experiments described in Section 5.1. In these experiments, each test instance is executed once using each of the formulations in the experiment, and the time limit for each execution is set to one hour. Tables 4–8 summarize the computational results. In these tables, the "# var," "# bin var," and "# cstr" columns report the total number of decision variables, the number of binary decision variables, and the number of constraints (excluding nonnegativity and binary constraints), respectively, in the formulation. The "IGap" columns report the root node integrality gaps of the different formulations, where IGap is given as $|Z^* - Z_{\text{LP}}|/Z^* \times 100\%$, where Z^* is the best objective function value obtained by solving the formulations in the experiment and Z_{LP} is the optimal objective function value of the LP relaxation of the formulation concerned. Note that for each strong formulation, the objective function value of its LP relaxation (i.e., Z_{LP}) is obtained such that there is no violation of all valid inequality families. For example, to obtain Z_{LP} for F1-X, we solve its LP relaxation with a subset of valid inequalities (13)–(22) added as constraints first. When an optimal solution is obtained, the separation algorithms are invoked to check for any violation of valid inequalities (13)–(22). If there exists a violation, the most violated inequality will be added to the LP problem as constraints and the new problem is solved again to obtain an optimal solution. This process is repeated until there is no violation of valid inequalities (13)–(22). Thus, the obtained solution satisfies valid inequalities (13)–(22), and its objective function value Z_{LP} is used to calculate the integrality gap. This integrality gap measures the tightness of the formulation. To evaluate the effectiveness of the strong valid inequalities in tightening the formulation, we report the percentage reduction in integrality gap in the "Pct. reduction" columns, where

$$\text{Pct. reduction} = \frac{\text{IGap}_{\text{without cuts}} - \text{IGap}_{\text{with cuts}}}{\text{IGap}_{\text{without cuts}}} \times 100\%,$$

$\text{IGap}_{\text{without cuts}}$ is the integrality gap of the formulation without our derived strong valid inequalities, and $\text{IGap}_{\text{with cuts}}$ is the integrality gap of the current formulation. The "CPU time [TGap]" columns report the computational time (in seconds) required to solve the instance to optimality (with a default optimality gap of 0.01%). Instances that could not be solved to optimality within one hour are marked with "**," and the terminating gaps of those instances are reported (enclosed in square brackets). The "# nodes" columns report the number of branch-and-cut nodes explored. The "# user cuts" columns report the number of user cuts added to each formulation.

The computational results of the first experiment are presented in Tables 4 and 5. The integrality gaps generated by formulation F1-X are considerably smaller than those generated by formulation F1, particularly for large instances (i.e., instances 11–20, where the total number of generators exceeds 100 and the number of binary decision variables exceeds 2000). This suggests that formulation F1-X is tighter than formulation F1. Using formulation F1, CPLEX is able to solve only one of the 20 test instances to optimality within one hour. In contrast, using formulation F1-X, CPLEX is able to solve 15 instances to optimality within the same time limit. For instances that cannot be solved to optimality using formulation F1-X, the terminating gaps are all within 0.05% and are much smaller than those using formulation F1. Formulation F1-X tends to explore fewer nodes than formulation F1, and the number of user cuts added by F1-X in the solution process is small compared with the total number of constraints in formulations F1 and F1-X. These results demonstrate that our proposed strong valid inequalities can significantly tighten the single-binary formulation of the network-constrained UC problem and thus speed up the solution process.

Tables 6 and 7 present the computational results of the second experiment, in which four formulations, namely F2, F2-X, F2-Y, and F2-Z, are used to solve the network-constrained UC problem. Using formulation F2, CPLEX solves only two of the 20 instances within the one-hour time limit.

Table 4 Performance of MIP Formulations in the First Experiment

Instance	# var	# bin var	# cstr		CPU time [TGap]		# nodes		# user cuts
			F1	F1-X	F1	F1-X	F1	F1-X	F1-X
1	3360	672	19422	23286	666.8	38.6	289339	11563	211
2	4200	840	24514	29344	** [0.06%]	301.8	1009424	137851	367
3	5280	1056	29556	35628	** [0.03%]	231.2	596629	53267	554
4	5400	1080	29304	35514	** [0.01%]	1335.1	1584979	494770	299
5	5880	1176	32812	39574	** [0.06%]	703.7	481348	149614	525
6	6000	1200	31562	38462	** [0.07%]	190.5	604379	94790	520
7	6120	1224	33610	40648	** [0.05%]	3194.2	463539	903454	445
8	6120	1224	32932	39970	** [0.09%]	662.4	504831	171291	518
9	6240	1248	34346	41522	** [0.09%]	1630.6	456636	425148	556
10	6480	1296	34356	41808	** [0.09%]	2119.2	410349	433936	697
11	15840	3168	87526	105742	** [0.12%]	** [0.04%]	274195	255185	1006
12	18720	3744	102148	123676	** [0.09%]	** [0.02%]	212690	279909	1923
13	18720	3744	102174	123702	** [0.10%]	927.5	172698	59414	1457
14	19800	3960	113314	136084	** [0.12%]	** [0.01%]	88150	123673	2899
15	20040	4008	109194	132240	** [0.17%]	** [0.01%]	114417	150664	1566
16	20640	4128	112994	136730	** [0.11%]	** [0.01%]	240391	234442	2214
17	21840	4368	120156	145272	** [0.13%]	660.0	177862	15758	1208
18	21840	4368	119936	145052	** [0.13%]	2880.9	207181	130559	1533
19	21960	4392	120816	146070	** [0.09%]	1958.5	208972	54786	2312
20	22440	4488	122468	148274	** [0.12%]	920.3	257848	19072	1198

Table 5 The Strength of LP Relaxations of MIP Formulations in the First Experiment

IGap	Instance	1	2	3	4	5	6	7	8	9	10
	F1	0.45%	0.39%	0.41%	0.36%	0.51%	0.61%	0.34%	0.55%	0.51%	0.64%
	F1-X	0.20%	0.14%	0.08%	0.06%	0.05%	0.04%	0.07%	0.06%	0.06%	0.04%
	Instance	11	12	13	14	15	16	17	18	19	20
	F1	0.32%	0.32%	0.40%	0.38%	0.50%	0.29%	0.45%	0.42%	0.37%	0.43%
	F1-X	0.06%	0.02%	0.02%	0.03%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%
Pct. reduction	Instance	1	2	3	4	5	6	7	8	9	10
	F1-X	55.7%	63.8%	80.4%	82.3%	90.0%	93.3%	78.7%	89.4%	89.1%	93.1%
	Instance	11	12	13	14	15	16	17	18	19	20
	F1-X	82.6%	94.0%	95.2%	90.9%	96.4%	94.1%	96.1%	95.9%	94.1%	96.1%

Table 6 Performance of MIP Formulations in the Second Experiment

Instance	# var	# bin var	# cstr				CPU time [TGap]				# nodes				# user cuts		
			F2	F2-X	F2-Y	F2-Z	F2	F2-X	F2-Y	F2-Z	F2	F2-X	F2-Y	F2-Z	F2-X	F2-Y	F2-Z
1	3304	1960	12300	16164	14876	18096	710.8	26.0	33.8	17.3	365652	12926	25641	11508	168	98	203
2	4130	2450	15350	20180	18570	22595	** [0.07%]	819.0	316.2	292.0	900272	577704	321040	213721	294	126	307
3	5192	3080	19354	25426	13402	28462	** [0.02%]	134.2	104.5	234.3	666918	57024	44507	86655	494	252	439
4	5310	3150	19842	26052	23982	29157	** [0.01%]	222.7	1675.7	497.4	760760	260255	1732340	505512	203	130	234
5	5782	3430	21556	28318	26064	31699	** [0.05%]	127.1	458.9	406.6	516797	65305	309318	244711	448	243	463
6	5900	3500	22098	28998	26698	32448	1908.6	219.9	147.1	151.2	375616	268065	213464	151584	515	202	475
7	6018	3570	22458	29496	27150	33015	** [0.04%]	** [0.01%]	417.0	564.4	618447	3089892	357940	456912	291	132	317
8	6018	3570	22496	29534	27188	33053	** [0.05%]	419.0	238.9	72.5	526007	240221	147181	27169	407	208	344
9	6136	3640	22896	30072	27680	33660	** [0.05%]	930.8	1034.6	985.1	434524	440662	495795	397759	535	261	517
10	6372	3780	23846	31298	28814	35024	** [0.08%]	1252.6	** [0.02%]	2206.9	310975	455369	1538372	787578	449	233	507
11	15576	9240	58012	76228	70156	85336	** [0.12%]	** [0.03%]	** [0.03%]	1036.6	180807	256996	570824	184432	806	353	836
12	18408	10920	68630	90158	82982	100922	** [0.08%]	** [0.01%]	1739.3	**[0.01%]	108631	285196	276194	382696	1484	815	1763
13	18408	10920	68630	90158	82982	100922	** [0.09%]	733.4	887.2	1205.6	84351	44990	92311	125922	1420	810	1603
14	19470	11550	72312	95082	87492	106467	** [0.11%]	** [0.01%]	** [0.01%]	1187.2	62170	179443	412241	69854	2285	1570	2619
15	19706	11690	73482	96528	88846	108051	** [0.13%]	539.7	180.6	438.4	50839	15365	6643	26615	1410	919	1717
16	20296	12040	75640	99376	91464	111244	** [0.10%]	2790.1	** [0.01%]	1498.7	142997	191482	249999	157638	1690	762	1630
17	21476	12740	80014	105130	96758	117688	** [0.10%]	444.3	180.4	244.3	70436	14938	7444	7710	1294	757	1753
18	21476	12740	80026	105142	96770	117700	** [0.10%]	1686.3	239.2	225.3	41514	148584	24456	7093	1515	1067	1923
19	21594	12810	80450	105704	97286	118331	** [0.09%]	1685.8	242.5	632.2	93893	136577	19084	51614	1703	1050	1977
20	22066	13090	82264	108070	99468	120973	** [0.12%]	293.2	245.7	187.7	62970	19072	10870	4069	1299	970	1746

Using formulation F2-X, which includes our proposed valid inequalities, CPLEX is able to solve 16 instances to optimality. Using formulation F2-Y, which includes the valid inequalities developed by Pan and Guan (2016), CPLEX is also able to solve 16 instances to optimality. The integrality gaps and CPU time of formulations F2-X and F2-Y are significantly smaller than those of formulation F2, whereas the integrality gaps and CPU time of formulation F2-X are comparable to those of formulation F2-Y.

Table 7 The Strength of LP Relaxations of MIP Formulations in the Second Experiment

	Instance	1	2	3	4	5	6	7	8	9	10
IGap	F2	0.45%	0.39%	0.38%	0.35%	0.47%	0.60%	0.34%	0.52%	0.48%	0.63%
	F2-X	0.20%	0.14%	0.07%	0.06%	0.05%	0.04%	0.07%	0.06%	0.06%	0.04%
	F2-Y	0.20%	0.14%	0.08%	0.05%	0.05%	0.05%	0.08%	0.06%	0.05%	0.05%
	F2-Z	0.20%	0.14%	0.07%	0.05%	0.05%	0.04%	0.07%	0.06%	0.05%	0.04%
	Instance	11	12	13	14	15	16	17	18	19	20
	F2	0.32%	0.31%	0.36%	0.34%	0.47%	0.29%	0.41%	0.40%	0.34%	0.41%
	F2-X	0.05%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%
	F2-Y	0.06%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.01%	0.02%	0.01%
	F2-Z	0.05%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.01%	0.02%	0.01%
Pct. reduction	Instance	1	2	3	4	5	6	7	8	9	10
	F2-X	55.7%	64.0%	81.2%	82.2%	89.3%	93.2%	79.4%	88.8%	88.3%	93.0%
	F2-Y	55.7%	63.9%	80.3%	84.3%	90.3%	92.0%	77.5%	88.1%	89.0%	92.4%
	F2-Z	55.7%	64.0%	81.2%	84.3%	90.3%	93.5%	79.4%	89.0%	89.0%	93.0%
	Instance	11	12	13	14	15	16	17	18	19	20
	F2-X	84.0%	93.7%	94.7%	94.6%	96.1%	94.5%	95.6%	95.7%	93.2%	95.7%
	F2-Y	82.5%	92.1%	94.5%	94.1%	96.4%	92.0%	96.2%	96.7%	94.4%	96.3%
	F2-Z	84.0%	93.9%	94.8%	94.7%	96.5%	94.5%	96.2%	96.7%	94.4%	96.4%

This demonstrates that strong valid inequalities developed for a single-binary formulation can be used for a formulation with more than a single type of binary variables and can achieve comparable effectiveness. Comparing the results presented in Tables 6 and 7 with the results presented in Tables 4 and 5, we observe that formulations F1-X, F2-X, and F2-Y have similar performance. This shows that a single-binary formulation has a similar performance as a formulation that uses more than a single type of binary variables when strong valid inequalities are added to these formulations. It also shows that the strong valid inequalities obtained from our single-binary formulation have an effectiveness similar to those strong valid inequalities obtained from a formulation that uses more than a single type of binary variables. However, formulation F2-Z outperforms formulations F1-X, F2-X, and F2-Y, as CPLEX can solve 19 instances to optimality with F2-Z, and the instance that is not solved optimally has a TGap of only 0.01%. This indicates that our strong valid inequalities derived from a single-binary formulation can be used in conjunction with those valid inequalities obtained from a formulation with more than a single type of binary variables to achieve better performance.

Tables 8 and 9 present the computational results of the third experiment, in which formulations $F1^+$ and $F1^+$-X are used to solve the network-constrained UC problem based on the modified IEEE 118-bus system. The integrality gaps generated by formulation $F1^+$-X are 44.1% to 66.3% smaller than those generated by formulation $F1^+$. Using formulation $F1^+$, CPLEX is able to solve only one of the 20 instances to optimality within one hour. In contrast, using formulation $F1^+$-X, CPLEX is able to solve 17 instances to optimality within the same time limit. Formulation $F1^+$-X explores fewer nodes than formulation $F1^+$, and the number of user cuts added by $F1^+$-X in the solution process is quite small. These results demonstrate the effectiveness of the strong formulation $F1^+$-X. Some additional computational results on formulations $F1^+$ and $F1^+$-X using a more congested demand setting and a less congested demand setting are presented in Online Appendix B, which show that formulation $F1^+$-X is more effective when the demand is more congested.

Table 8 Performance of MIP Formulations in the Third Experiment

Instance	# var	# bin var	# cstr		CPU time [TGap]		# nodes		# user cuts
			$F1^+$	$F1^+$-X	$F1^+$	$F1^+$-X	$F1^+$	$F1^+$-X	$F1^+$-X
1	6372	1296	36124	43576	** [0.15%]	571.7	109080	14069	316
2					** [0.06%]	2324.7	97424	84034	474
3					** [0.02%]	2251.6	158482	31631	445
4					** [0.11%]	2803.9	131942	43108	544
5					** [0.01%]	288.5	175810	7220	394
6					** [0.17%]	1653.5	116598	24813	441
7					** [0.27%]	1800.0	122958	43624	545
8					** [0.11%]	1799.3	180499	22310	591
9					** [0.13%]	851.4	250053	19961	464
10					2938.4	850.7	181549	18889	449
11					** [0.08%]	2425.3	196002	59158	463
12					** [0.15%]	** [0.05%]	287079	44675	619
13					** [0.14%]	3351.3	164549	82738	563
14					** [0.17%]	1969.8	235344	37421	486
15					** [0.30%]	** [0.03%]	266093	111169	696
16					** [0.10%]	1666.2	112045	42146	438
17					** [0.13%]	461.9	139242	13578	448
18					** [0.14%]	2433.3	139375	61670	531
19					** [0.15%]	** [0.02%]	169572	133348	742
20					** [0.21%]	3298.6	158536	50471	615

6. Conclusions

This paper considers a UC formulation with a single type of binary variables. By analyzing the physical constraints of a single generator, we obtain the convex hull description of the two-period

Table 9 The Strength of LP Relaxations of MIP Formulations in the Third Experiment

IGap	Instance	1	2	3	4	5	6	7	8	9	10
	$F1^+$	0.85%	1.03%	0.76%	0.93%	0.86%	0.88%	1.04%	0.92%	0.91%	0.80%
	$F1^+$-X	0.36%	0.39%	0.39%	0.52%	0.33%	0.36%	0.35%	0.42%	0.38%	0.40%
	Instance	11	12	13	14	15	16	17	18	19	20
	$F1^+$	0.90%	1.17%	0.87%	0.91%	1.02%	0.89%	0.84%	0.85%	0.95%	0.89%
	$F1^+$-X	0.40%	0.47%	0.37%	0.43%	0.44%	0.40%	0.40%	0.44%	0.39%	0.40%
Pct. reduction	Instance	1	2	3	4	5	6	7	8	9	10
	$F1^+$-X	57.4%	62.2%	49.0%	44.1%	61.9%	59.7%	66.3%	54.8%	58.8%	52.2%
	Instance	11	12	13	14	15	16	17	18	19	20
	$F1^+$-X	55.9%	60.0%	57.5%	52.2%	57.4%	55.6%	53.0%	48.4%	59.0%	55.6%

single-UC polytope, which can be used to tighten the original MILP formulation and derive other strong valid inequalities. For the multi-period single-UC polytope, we derive strong valid inequalities with one and two continuous variables. Conditions under which these valid inequalities are facet-defining for the multi-period single-UC polytope are provided. Because the number of inequalities in each valid inequality family is very large, efficient separation algorithms are provided to identify the most violated inequalities. The effectiveness of the proposed strong valid inequalities is demonstrated in solving network-constrained UC problems. Computational results show that our valid inequalities can speed up the solution process significantly. Moreover, these strong valid inequalities exhibit effectiveness comparable to two-binary valid inequalities and thus can be used to tighten two/three-binary formulations.

Various intriguing research directions can be pursued following this line of work. First, it would be interesting to investigate the complete convex hull descriptions of the multi-period single-UC polytopes, such as the three-period polytope, and derive strong valid inequalities with more than two continuous variables to further tighten Problem 1. In addition, the discussion of the single-UC polytope can be extended to different parameter settings, such as the case where $\overline{V} \geq \underline{C} + V$ or the case where $\overline{V} + V > \overline{C}$. Second, it would be appealing to incorporate different start-up/shut-down trajectories of generators into the physical constraints to accurately represent the operation of units and to conduct a polyhedral study on the obtained single-UC polytope to derive strong valid inequalities. Third, considering the demand and electricity price fluctuations that often occur in practice when dealing with UC problems, it would be interesting to formulate the corresponding stochastic UC problems to better reflect real-world scenarios. Fourth, given that different types of electrical generators (e.g., pumped storage hydro units) may have different physical constraints in addition to those considered in this paper, it would be interesting to derive strong valid inequalities for the UC problems with these specific generators. For example, consider the hydro UC problem described in Cheng et al. (2016) in which there are constraints that impose restrictions on the power output in the safe operating zones. By employing a similar approach to the derivation of inequality (12), we can obtain a valid inequality by deriving an upper bound for the quantity "$p_{i,t} - p_{i,t-k}$," where $p_{i,t}$ and $p_{i,t-k}$ are the power output of unit i in periods t and $t-k$, respectively. Fifth, as mentioned in Section 1, some studies have utilized LR-based methods to tackle complex UC problems by decomposing the multiple-generator problem into independent single-generator subproblems. The strong valid inequalities derived in this paper can be applied to these subproblems, thus enhancing their effectiveness. It would be useful to develop mathematical techniques that can integrate these valid inequalities into the decomposition procedure to improve its efficiency. Sixth, it would be intriguing to make a theoretical comparison between our strong valid inequalities and those derived in prior studies, such as Pan and Guan (2016) and Damcı-Kurt et al. (2016), to reveal the relationships and distinctions between them. We leave these issues for future research.

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Online Appendix

Appendix A: Mathematical Proofs

A.1. Proof of Lemma 1

Lemma 1. *Consider any point* $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}$ *and* $t \in [2, T]_{\mathbb{Z}}$. *(i) If* $y_t = 0$, *then* $y_{t-j} - y_{t-j-1} \leq 0$ *for all* $j \in [0, \min\{t-2, L-1\}]_{\mathbb{Z}}$. *(ii) If* $y_t = 1$, *then there exists at most one* $j \in [0, \min\{t-2, L\}]_{\mathbb{Z}}$ *such that* $y_{t-j} - y_{t-j-1} = 1$.

Proof. (i) Consider any $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}$ and any $t \in [2, T]_{\mathbb{Z}}$ such that $y_t = 0$. Suppose, to the contrary, that there exists $j \in [0, \min\{t-2, L-1\}]_{\mathbb{Z}}$ such that $y_{t-j} - y_{t-j-1} = 1$. Then, $t - j \in [2, T]_{\mathbb{Z}}$. Thus, by (2a), $y_k = 1$ for all $k \in [t-j, \min\{T, t-j+L-1\}]_{\mathbb{Z}}$. It is easy to check that $t \in [t-j, \min\{T, t-j+L-1\}]_{\mathbb{Z}}$. Hence, $y_t = 1$, which is a contradiction. Therefore, $y_{t-j} - y_{t-j-1} \leq 0$ for all $j \in [0, \min\{t-2, L-1\}]_{\mathbb{Z}}$.

(ii) Consider any $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}$ and any $t \in [2, T]_{\mathbb{Z}}$ such that $y_t = 1$. Suppose, to the contrary, that there exist $j_1, j_2 \in [0, \min\{t-2, L\}]_{\mathbb{Z}}$ such that $j_1 < j_2$ and $y_{t-j_1} - y_{t-j_1-1} = y_{t-j_2} - y_{t-j_2-1} = 1$. Because $t - j_2 \in [2, T]_{\mathbb{Z}}$, and $y_{t-j_2} - y_{t-j_2-1} = 1$, by (2a), $y_k = 1$ for all $k \in [t-j_2, \min\{T, t-j_2+L-1\}]_{\mathbb{Z}}$. Note that $t - j_1 - 1 \geq t - j_2$, $t - j_1 - 1 \leq T$, and $t - j_1 - 1 \leq t - 1 \leq t - j_2 + L - 1$. Thus, $t - j_1 - 1 \in [t-j_2, \min\{T, t-j_2+L-1\}]_{\mathbb{Z}}$. Hence, $y_{t-j_1-1} = 1$, which contradicts that $y_{t-j_1} - y_{t-j_1-1} = 1$. Therefore, there exists at most one $j \in [0, \min\{t-2, L\}]_{\mathbb{Z}}$ such that $y_{t-j} - y_{t-j-1} = 1$. $\square$

A.2. Proof of Lemma 2

Lemma 2. *Denote* $\mathbf{x}' = (x'_1, \ldots, x'_T)$ *and* $\mathbf{y}' = (y'_1, \ldots, y'_T)$*. Let*

$$\begin{aligned}
\mathcal{P}' = \big\{ &(\mathbf{x}', \mathbf{y}') \in \mathbb{R}^T_+ \times \mathbb{B}^T : \\
& -y'_{T-t+2} + y'_{T-t+1} - y'_{T-k+1} \leq 0, \forall t \in [2,T]_{\mathbb{Z}}, \forall k \in [t, \min\{T, t+L-1\}]_{\mathbb{Z}}, && \text{(EC.1a)} \\
& y'_{T-t+2} - y'_{T-t+1} + y'_{T-k+1} \leq 1, \forall t \in [2,T]_{\mathbb{Z}}, \forall k \in [t, \min\{T, t+\ell-1\}]_{\mathbb{Z}}, && \text{(EC.1b)} \\
& -x'_{T-t+1} + \underline{C} y'_{T-t+1} \leq 0, \forall t \in [1,T]_{\mathbb{Z}}, && \text{(EC.1c)} \\
& x'_{T-t+1} - \overline{C} y'_{T-t+1} \leq 0, \forall t \in [1,T]_{\mathbb{Z}}, && \text{(EC.1d)} \\
& x'_{T-t+1} - x'_{T-t+2} \leq V y'_{T-t+2} + \overline{V}(1 - y'_{T-t+2}), \forall t \in [2,T]_{\mathbb{Z}}, && \text{(EC.1e)} \\
& x'_{T-t+2} - x'_{T-t+1} \leq V y'_{T-t+1} + \overline{V}(1 - y'_{T-t+1}), \forall t \in [2,T]_{\mathbb{Z}} \big\}. && \text{(EC.1f)}
\end{aligned}$$

Then, $\mathcal{P} = \mathcal{P}'$*.*

Proof. First, consider any element $(\mathbf{x}, \mathbf{y})$ of $\mathcal{P}$, and we show that $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}'$.

Consider inequality (EC.1a) and any $t \in [2,T]_{\mathbb{Z}}$. Obviously, $(\mathbf{x}, \mathbf{y})$ satisfies (EC.1a) if $-y_{T-t+2} + y_{T-t+1} \leq 0$. Consider the case where $-y_{T-t+2} + y_{T-t+1} > 0$ (i.e., $y_{T-t+2} = 0$ and $y_{T-t+1} = 1$). Suppose, to the contrary, that $y_{T-k+1} = 0$ for some $k \in [t, \min\{T, t+L-1\}]_{\mathbb{Z}}$. Then, because $y_{T-k+1} = 0$ and $y_{T-t+1} = 1$, there exists $p \in [t, k-1]_{\mathbb{Z}}$ such that $y_{T-p} = 0$ and $y_{T-p+1} = 1$. This implies that $-y_{T-p} + y_{T-p+1} - y_{T-t+2} = 1$. Note that $T-p+1 \in [2,T]_{\mathbb{Z}}$ and $T-t+2 \in [T-p+1, \min\{T, (T-p+1)+L-1\}]_{\mathbb{Z}}$. Thus, $(\mathbf{x}, \mathbf{y})$ violates inequality (2a), which contradicts that $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}$. Hence, $y_{T-k+1} = 1$ for all $k \in [t, \min\{T, t+L-1\}]_{\mathbb{Z}}$. Thus, $(\mathbf{x}, \mathbf{y})$ satisfies inequality (EC.1a).

Consider inequality (EC.1b) and any $t \in [2,T]_{\mathbb{Z}}$. Obviously, $(\mathbf{x}, \mathbf{y})$ satisfies (EC.1b) if $y_{T-t+2} - y_{T-t+1} \leq 0$. Consider the case where $y_{T-t+2} - y_{T-t+1} > 0$ (i.e., $y_{T-t+2} = 1$ and $y_{T-t+1} = 0$). Suppose, to the contrary, that $y_{T-k+1} = 1$ for some $k \in [t, \min\{T, t+\ell-1\}]_{\mathbb{Z}}$. Then, because $y_{T-k+1} = 1$ and $y_{T-t+1} = 0$, there exists $p \in [t, k-1]_{\mathbb{Z}}$ such that $y_{T-p} = 1$ and $y_{T-p+1} = 0$. This implies that $y_{T-p} - y_{T-p+1} + y_{T-t+2} = 2$. Note that $T-p+1 \in [2,T]_{\mathbb{Z}}$ and $T-t+2 \in [T-p+1, \min\{T, (T-p+1)+\ell-1\}]_{\mathbb{Z}}$. Thus, $(\mathbf{x}, \mathbf{y})$ violates inequality (2b), which contradicts that $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}$. Hence, $y_{T-k+1} = 0$ for all $k \in [t, \min\{T, t+\ell-1\}]_{\mathbb{Z}}$. Thus, $(\mathbf{x}, \mathbf{y})$ satisfies inequality (EC.1b).

Consider inequalities (EC.1c) and (EC.1d). For any $t \in [1,T]_{\mathbb{Z}}$, because $(\mathbf{x}, \mathbf{y})$ satisfies inequalities (2c) and (2d), $(\mathbf{x}, \mathbf{y})$ also satisfies inequalities (EC.1c) and (EC.1d).

Consider inequality (EC.1e) and any $t \in [2,T]_{\mathbb{Z}}$. Because $T-t+2 \in [2,T]_{\mathbb{Z}}$, by (2f), $x_{T-t+1} - x_{T-t+2} \leq V y_{T-t+2} + \overline{V}(1 - y_{T-t+2})$. Thus, $(\mathbf{x}, \mathbf{y})$ satisfies inequality (EC.1e).

Consider inequality (EC.1f) and any $t \in [2,T]_{\mathbb{Z}}$. Because $T-t+2 \in [2,T]_{\mathbb{Z}}$, by (2e), $x_{T-t+2} - x_{T-t+1} \leq V y_{T-t+1} + \overline{V}(1 - y_{T-t+1})$. Thus, $(\mathbf{x}, \mathbf{y})$ satisfies inequality (EC.1f).

Summarizing the above analysis, we conclude that $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}'$.

Next, consider any element $(\mathbf{x}',\mathbf{y}')$ of $\mathcal{P}'$, and we show that $(\mathbf{x}',\mathbf{y}') \in \mathcal{P}$.

Consider inequality (2a) and any $t \in [2,T]_{\mathbb{Z}}$. Obviously, $(\mathbf{x}',\mathbf{y}')$ satisfies (2a) if $-y'_{t-1} + y'_t \leq 0$. Consider the case where $-y'_{t-1} + y'_t > 0$ (i.e., $y'_{t-1} = 0$ and $y'_t = 1$). Suppose, to the contrary, that $y'_k = 0$ for some $k \in [t, \min\{T, t+L-1\}]_{\mathbb{Z}}$. Then, because $y'_t = 1$ and $y'_k = 0$, there exists $p \in [t, k-1]_{\mathbb{Z}}$ such that $y'_p = 1$ and $y'_{p+1} = 0$. This implies that $-y'_{T-(T-p+1)+2} + y'_{T-(T-p+1)+1} - y'_{T-(T-t+2)+1} = 1$. Note that $T-p+1 \in [2,T]_{\mathbb{Z}}$ and $T-t+2 \in [T-p+1, \min\{T, (T-p+1)+L-1\}]_{\mathbb{Z}}$. Thus, $(\mathbf{x}',\mathbf{y}')$ violates inequality (EC.1a), which contradicts that $(\mathbf{x}',\mathbf{y}') \in \mathcal{P}'$. Hence, $y'_k = 1$ for all $k \in [t, \min\{T, t+L-1\}]_{\mathbb{Z}}$. Thus, $(\mathbf{x}',\mathbf{y}')$ satisfies inequality (2a).

Consider inequality (2b) and any $t \in [2,T]_{\mathbb{Z}}$. Obviously, $(\mathbf{x}',\mathbf{y}')$ satisfies (2b) if $y'_{t-1} - y'_t \leq 0$. Consider the case where $y'_{t-1} - y'_t > 0$ (i.e., $y'_{t-1} = 1$ and $y'_t = 0$). Suppose, to the contrary, that $y'_k = 1$ for some $k \in [t, \min\{T, t+\ell-1\}]_{\mathbb{Z}}$. Then, because $y'_t = 0$ and $y'_k = 1$, there exists $p \in [t, k-1]_{\mathbb{Z}}$ such that $y'_p = 0$ and $y'_{p+1} = 1$. This implies that $y'_{T-(T-p+1)+2} - y'_{T-(T-p+1)+1} + y'_{T-(T-t+2)+1} = 2$. Note that $T-p+1 \in [2,T]_{\mathbb{Z}}$ and $T-t+2 \in [T-p+1, \min\{T, (T-p+1)+\ell-1\}]_{\mathbb{Z}}$. Thus, $(\mathbf{x}',\mathbf{y}')$ violates inequality (EC.1b), which contradicts that $(\mathbf{x}',\mathbf{y}') \in \mathcal{P}'$. Hence, $y'_k = 0$ for all $k \in [t, \min\{T, t+\ell-1\}]_{\mathbb{Z}}$. Thus, $(\mathbf{x}',\mathbf{y}')$ satisfies inequality (2b).

Consider inequalities (2c) and (2d). For any $t \in [1,T]_{\mathbb{Z}}$, because $(\mathbf{x}',\mathbf{y}')$ satisfies inequalities (EC.1c) and (EC.1d), $(\mathbf{x}',\mathbf{y}')$ also satisfies inequalities (2c) and (2d).

Consider inequality (2e) and any $t \in [2,T]_{\mathbb{Z}}$. Because $T-t+2 \in [2,T]_{\mathbb{Z}}$, by (EC.1f), $x'_{T-(T-t+2)+2} - x'_{T-(T-t+2)+1} \leq V y'_{T-(T-t+2)+1} + \overline{V}(1 - y'_{T-(T-t+2)+1})$. Hence, $x'_t - x'_{t-1} \leq V y'_{t-1} + \overline{V}(1 - y'_{t-1})$. Thus, $(\mathbf{x}',\mathbf{y}')$ satisfies inequality (2e).

Consider inequality (2f) and any $t \in [2,T]_{\mathbb{Z}}$. Because $T-t+2 \in [2,T]_{\mathbb{Z}}$, by (EC.1e), $x'_{T-(T-t+2)+1} - x'_{T-(T-t+2)+2} \leq V y'_{T-(T-t+2)+2} + \overline{V}(1 - y'_{T-(T-t+2)+2})$. Hence, $x'_{t-1} - x'_t \leq V y'_t + \overline{V}(1 - y'_t)$. Thus, $(\mathbf{x}',\mathbf{y}')$ satisfies inequality (2f).

Summarizing the above analysis, we conclude that $(\mathbf{x}',\mathbf{y}') \in \mathcal{P}$. Therefore, $\mathcal{P} = \mathcal{P}'$. $\square$

A.3. Proof of Proposition 1

Proposition 1. *Consider any $\mathcal{S} \subseteq [0, \min\{L-1, T-2, \lfloor(\overline{C}-\overline{V})/V\rfloor\}]_{\mathbb{Z}}$. For any $t \in [1,T]_{\mathbb{Z}}$ such that $t \geq s+2$ for all $s \in \mathcal{S}$, the inequality*

$$x_t \leq \overline{C}y_t - \sum_{s\in\mathcal{S}}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1}) \tag{13}$$

is valid and facet-defining for $\mathrm{conv}(\mathcal{P})$. *For any $t \in [1,T]_{\mathbb{Z}}$ such that $t \leq T-s-1$ for all $s \in \mathcal{S}$, the inequality*

$$x_t \leq \overline{C}y_t - \sum_{s\in\mathcal{S}}(\overline{C}-\overline{V}-sV)(y_{t+s}-y_{t+s+1}) \tag{14}$$

is valid and facet-defining for $\mathrm{conv}(\mathcal{P})$.

Proof. We first prove that inequality (13) is valid and facet-defining for $\mathrm{conv}(\mathcal{P})$. Note that the proof of facet-defining of (13) here can also be used to prove the facet-defining of (13) in Proposition 2. For notational convenience, we define $s_{\max} = \max\{s : s \in \mathcal{S}\}$ if $\mathcal{S} \neq \emptyset$, and $s_{\max} = -1$ if $\mathcal{S} = \emptyset$.

Consider any $t \in [s_{\max}+2, T]_{\mathbb{Z}}$ (i.e., $t \in [1,T]_{\mathbb{Z}}$ such that $t \geq s+2$ for all $s \in \mathcal{S}$). To prove that linear inequalities (13) is valid for $\mathrm{conv}(\mathcal{P})$, it suffices to show that they are valid for $\mathcal{P}$. Consider any element $(\mathbf{x}, \mathbf{y})$ of $\mathcal{P}$. We show that $(\mathbf{x}, \mathbf{y})$ satisfies (13).

Case 1: $y_t = 0$. By Lemma 1(i), $y_{t-j} - y_{t-j-1} \leq 0$ for all $j \in [0, \min\{t-2, L-1\}]_{\mathbb{Z}}$. Because $\mathcal{S} \subseteq [0, L-1]_{\mathbb{Z}}$ and $s_{\max} \leq t-2$, we have $\mathcal{S} \subseteq [0, \min\{t-2, L-1\}]_{\mathbb{Z}}$. Thus, $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, \lfloor(\overline{C}-\overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C}-\overline{V}-sV \geq 0$ for all $s \in \mathcal{S}$. Thus, the right-hand side of inequality (13) is nonnegative. Because $y_t = 0$, by (2d), $x_t = 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (13).

Case 2: $y_t = 1$ and $y_{t-s'} - y_{t-s'-1} = 1$ for some $s' \in \mathcal{S}$. By Lemma 1(ii), there exists at most one $j \in [0, \min\{t-2, L\}]_{\mathbb{Z}}$ such that $y_{t-j} - y_{t-j-1} = 1$. Because $\mathcal{S} \subseteq [0, L]_{\mathbb{Z}}$ and $s_{\max} \leq t-2$, we have $\mathcal{S} \subseteq [0, \min\{t-2, L\}]_{\mathbb{Z}}$. This implies that $y_{t-s} - y_{t-s-1} \leq 0$, for all for all $s \in \mathcal{S} \setminus \{s'\}$. For any $s \in \mathcal{S}$, because $s \leq \lfloor(\overline{C}-\overline{V})/V\rfloor$, we have $\overline{C}-\overline{V}-sV \geq 0$. Thus, for any $s \in \mathcal{S} \setminus \{s'\}$, $(\overline{C}-\overline{V}-sV)(y_{t-s} - y_{t-s-1})$ is either zero or negative. Hence, $\sum_{s\in\mathcal{S}}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1}) \leq \overline{C}-\overline{V}-s'V$. Thus, the right-hand side of inequality (13) is at least $s'V + \overline{V}$. By (2e), $\sum_{\tau=t-s'}^{t}(x_\tau - x_{\tau-1}) \leq \sum_{\tau=t-s'}^{t} Vy_{\tau-1} + \sum_{\tau=t-s_j}^{t} \overline{V}(1-y_{\tau-1})$, which implies that $x_t - x_{t-s'-1} \leq s'V + \overline{V}$. Because $y_{t-s'} - y_{t-s'-1} = 1$, we have $y_{t-s'-1} = 0$. By (2d), $x_{t-s'-1} = 0$. Hence, $x_t \leq s'V + \overline{V}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (13).

Case 3: $y_t = 1$ and $y_{t-s} - y_{t-s-1} \neq 1$ for all $s \in \mathcal{S}$. In this case, $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S}$. For any $s \in \mathcal{S}$, because $s \leq \lfloor(\overline{C}-\overline{V})/V\rfloor$, we have $\overline{C}-\overline{V}-sV \geq 0$. Thus, the right-hand side of inequality (13) is at least $\overline{C}$. By (2d), $x_t \leq \overline{C}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (13).

Summarizing Cases 1–3, we conclude that $(\mathbf{x}, \mathbf{y})$ satisfies (13). Hence, (13) is valid for $\mathrm{conv}(\mathcal{P})$.

Consider any $t \in [s_{\max}+2, T]_{\mathbb{Z}}$. To prove that inequality (13) is facet-defining for $\text{conv}(\mathcal{P})$, it suffices to show that there exist $2T$ affinely independent points in $\text{conv}(\mathcal{P})$ that satisfy (13) at equality. Because $\mathbf{0} \in \text{conv}(\mathcal{P})$ and $\mathbf{0}$ satisfies (13) at equality, it suffices to create the remaining $2T-1$ nonzero linearly independent points. We denote these $2T-1$ points as $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ for $r \in [1,T]_{\mathbb{Z}} \setminus \{t\}$ and $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ for $r \in [1,T]_{\mathbb{Z}}$, and denote the qth component of $\bar{\mathbf{x}}^r$, $\bar{\mathbf{y}}^r$, $\hat{\mathbf{x}}^r$, and $\hat{\mathbf{y}}^r$ as $\bar{x}_q^r$, $\bar{y}_q^r$, $\hat{x}_q^r$, and $\hat{y}_q^r$, respectively. Let $\epsilon = \overline{V} - \underline{C} > 0$. We divide these $2T-1$ points into the following four groups:

(A1) For each $r \in [1,T]_{\mathbb{Z}} \setminus \{t\}$, we create a point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as follows:

$$\bar{x}_q^r = \begin{cases} \overline{C}, & \text{for } q \in [1,T]_{\mathbb{Z}} \setminus \{r\}; \\ \overline{C} - \epsilon, & \text{for } q = r; \end{cases}$$

and $\bar{y}_q^r = 1$ for all $q \in [1,T]_{\mathbb{Z}}$. It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (2a)–(2f). Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. It is also easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (13) at equality.

(A2) For each $r \in [1,t-1]_{\mathbb{Z}}$, we create a point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as follows: If $t-r-1 \notin \mathcal{S}$, then

$$\hat{x}_q^r = \begin{cases} \underline{C}, & \text{for } q \in [1,r]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [r+1,T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^r = \begin{cases} 1, & \text{for } q \in [1,r]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [r+1,T]_{\mathbb{Z}}. \end{cases}$$

If $t-r-1 \in \mathcal{S}$, then

$$\hat{x}_q^r = \begin{cases} 0, & \text{for } q \in [1,r]_{\mathbb{Z}}; \\ \overline{V} + (q-r-1)V, & \text{for } q \in [r+1,t]_{\mathbb{Z}}; \\ \overline{V} + (t-r-1)V, & \text{for } q \in [t+1,T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^r = \begin{cases} 0, & \text{for } q \in [1,r]_{\mathbb{Z}}; \\ 1, & \text{for } q \in [r+1,T]_{\mathbb{Z}}. \end{cases}$$

We first consider the case where $t-r-1 \notin \mathcal{S}$. In this case, it is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2a)–(2f) and is therefore in $\text{conv}(\mathcal{P})$. Note that in this case $\hat{x}_t^r = \hat{y}_t^r = 0$, and $t-s-1 \neq r$ for all $s \in \mathcal{S}$, which implies that $\hat{y}_{t-s}^r - \hat{y}_{t-s-1}^r = 0$ for all $s \in \mathcal{S}$. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (13) at equality. Next, we consider the case where $t-r-1 \in \mathcal{S}$. In this case, it is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2a) and (2b). For each $q \in [1,r]_{\mathbb{Z}}$, we have $\hat{x}_q^r = \hat{y}_q^r = 0$. For each $q \in [r+1,T]_{\mathbb{Z}}$, because $t-r-1 \in \mathcal{S}$, we have $t-r-1 \leq \lfloor (\overline{C} - \overline{V})/V \rfloor$, which implies that $\overline{V} + (t-r-1)V \leq \overline{C}$, which in turn implies that $\underline{C} \leq \hat{x}_q^r \leq \overline{C}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2c) and (2d). Note that $\hat{x}_q^r - \hat{x}_{q-1}^r = 0$ when $q \in [2,r]_{\mathbb{Z}}$, $\hat{x}_q^r - \hat{x}_{q-1}^r = \overline{V}$ when $q = r+1$, and $0 \leq \hat{x}_q^r - \hat{x}_{q-1}^r \leq V$ when $q \in [r+2,T]_{\mathbb{Z}}$. Thus, $-V\hat{y}_q^r - \overline{V}(1-\hat{y}_q^r) \leq \hat{x}_q^r - \hat{x}_{q-1}^r \leq V\hat{y}_{q-1}^r + \overline{V}(1-\hat{y}_{q-1}^r)$ for all $q \in [2,T]_{\mathbb{Z}}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2e) and (2f). Therefore, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Note that in this case $\hat{x}_t^r = \overline{V} + (t-r-1)V$, $\hat{y}_t^r = 1$, $\hat{y}_{t-s}^r - \hat{y}_{t-s-1}^r = 1$ when $s = t-r-1$, and $\hat{y}_{t-s}^r - \hat{y}_{t-s-1}^r = 0$ when $s \neq t-r-1$. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (13) at equality.

(A3) We create a point $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ by setting $\hat{x}_q^t = \overline{C}$ and $\hat{y}_q^t = 1$ for $q \in [1,T]_{\mathbb{Z}}$. It is easy to verify that $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (2a)–(2f). Thus, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) \in \operatorname{conv}(\mathcal{P})$. It is also easy to verify that $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (13) at equality.

(A4) For each $r \in [t+1,T]_{\mathbb{Z}}$, we create a point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as follows:

$$\hat{x}_q^r = \begin{cases} 0, & \text{for } q \in [1, r-1]_{\mathbb{Z}}; \\ \underline{C}, & \text{for } q \in [r, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^r = \begin{cases} 0, & \text{for } q \in [1, r-1]_{\mathbb{Z}}; \\ 1, & \text{for } q \in [r, T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2a)–(2f). Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \operatorname{conv}(\mathcal{P})$. It is also easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (13) at equality.

Table EC.1 shows a matrix with $2T-1$ rows, where each row represents a point created by this process. This matrix can be transformed into the matrix in Table EC.2 via the following Gaussian elimination process:

(i) For each $r \in [1,T]_{\mathbb{Z}} \setminus \{t\}$, the point with index r in group (B1), denoted $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point with index r in group (A1), and $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point in group (A3).

(ii) For each $r \in [1,t-1]_{\mathbb{Z}}$, the point with index r in group (B2), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $t-r-1 \notin \mathcal{S}$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) - (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $t-r-1 \in \mathcal{S}$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (A2), and $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point in group (A3).

(iii) The point in group (B3), denoted $(\underline{\hat{\mathbf{x}}}^t, \underline{\hat{\mathbf{y}}}^t)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^t, \underline{\hat{\mathbf{y}}}^t) = (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) - (\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$. Here, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point in group (A3), and $(\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$ is the point with index $t+1$ in group (A4).

(iv) For each $r \in [t+1,T]_{\mathbb{Z}}$, the point with index r in group (B4), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ if $r \neq T$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $r = T$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ and $(\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ are the points with indices r and $r+1$, respectively, in group (A4).

The matrix shown in Table EC.2 is lower triangular; that is, the position of the last nonzero component of a row of the matrix is greater than the position of the last nonzero component of the previous row. This implies that the $2T-1$ points in groups (A1)–(A4) are linearly independent. Therefore, inequality (13) is facet-defining for $\operatorname{conv}(\mathcal{P})$.

Next, we show that inequality (14) is valid and facet-defining for $\operatorname{conv}(\mathcal{P})$. Note that this proof can also be used to prove the validity and facet-defining of inequality (14) in Proposition 2. Denote $x'_t = x_{T-t+1}$ and $y'_t = y_{T-t+1}$ for $t \in [1,T]_{\mathbb{Z}}$. Because inequality (13) is valid and facet-defining for $\operatorname{conv}(\mathcal{P})$ for any $t \in [s_{\max}+2, T]_{\mathbb{Z}}$, the inequality

$$x'_{T-t+1} \leq \overline{C} y'_{T-t+1} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y'_{T-t+s+1} - y'_{T-t+s+2})$$

Table EC.1 A matrix with the rows representing $2T-1$ points in $\mathrm{conv}(\mathcal{P})$ that satisfy inequality (13) at equality

Group	Point	Index r	$\mathbf{x}$							$\mathbf{y}$						
			1	$\cdots$	$t-1$	t	$t+1$	$\cdots$	T	1	$\cdots$	$t-1$	t	$t+1$	$\cdots$	T
(A1)	$(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$	1	$\overline{C}-\epsilon$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	$\cdots$	1	1	1	$\cdots$	1
		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-1$	$\overline{C}$	$\cdots$	$\overline{C}-\epsilon$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	$\cdots$	1	1	1	$\cdots$	1
		$t+1$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}-\epsilon$	$\cdots$	$\overline{C}$	1	$\cdots$	1	1	1	$\cdots$	1
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		T	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}-\epsilon$	1	$\cdots$	1	1	1	$\cdots$	1
(A2)	$(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$	1 $\vdots$ $t-1$	(See Note EC.1-1)							(See Note EC.1-1)						
(A3)		t	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	$\cdots$	1	1	1	$\cdots$	1
(A4)		$t+1$	0	$\cdots$	0	0	$\underline{C}$	$\cdots$	$\underline{C}$	0	$\cdots$	0	0	1	$\cdots$	1
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T	0	$\cdots$	0	0	0	$\cdots$	$\underline{C}$	0	$\cdots$	0	0	0	$\cdots$	1

Note EC.1-1: For $r \in [1, t-1]_{\mathbb{Z}}$, the $\mathbf{x}$ and $\mathbf{y}$ vectors in group (A2) are given as follows:

$\hat{\mathbf{x}}^r = (\underbrace{\underline{C}, \ldots, \underline{C}}_{r \text{ terms}}, \underbrace{0, \ldots, 0}_{T-r \text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{1, \ldots, 1}_{r \text{ terms}}, \underbrace{0, \ldots, 0}_{T-r \text{ terms}})$ if $t-r-1 \notin \mathcal{S}$;

$\hat{\mathbf{x}}^r = (\underbrace{0, \ldots, 0}_{r \text{ terms}}, \underbrace{\overline{V}, \overline{V}+V, \overline{V}+2V, \ldots, \overline{V}+(t-r-1)V}_{t-r \text{ terms}}, \underbrace{\overline{V}+(t-r-1)V, \ldots, \overline{V}+(t-r-1)V}_{T-t \text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{0, \ldots, 0}_{r \text{ terms}}, \underbrace{1, \ldots, 1}_{T-r \text{ terms}})$ if $t-r-1 \in \mathcal{S}$.

Table EC.2 Lower triangular matrix obtained from Table EC.1 via Gaussian elimination

Group	Point	Index r	$\mathbf{x}$							$\mathbf{y}$						
			1	$\cdots$	$t-1$	t	$t+1$	$\cdots$	T	1	$\cdots$	$t-1$	t	$t+1$	$\cdots$	T
(B1)	$(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$	1	$-\epsilon$	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-1$	0	$\cdots$	$-\epsilon$	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$t+1$	0	$\cdots$	0	0	$-\epsilon$	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		T	0	$\cdots$	0	0	0	$\cdots$	$-\epsilon$	0	$\cdots$	0	0	0	$\cdots$	0
(B2)	$(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$	1	(Omitted)							1	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$								$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-1$								1	$\cdots$	1	0	0	$\cdots$	0
(B3)		t	(Omitted)							1	$\cdots$	1	1	0	$\cdots$	0
(B4)		$t+1$	(Omitted)							0	$\cdots$	0	0	1	$\cdots$	0
		$\vdots$								$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T								0	$\cdots$	0	0	0	$\cdots$	1

is valid and facet-defining for $\text{conv}(\mathcal{P}')$ for any $t \in [s_{\max} + 2, T]_{\mathbb{Z}}$. Let $t' = T - t + 1$. Then, the inequality

$$x'_{t'} \leq \overline{C} y'_{t'} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y'_{t'+s} - y'_{t'+s+1})$$

is valid and facet-defining for $\text{conv}(\mathcal{P}')$ for any $t' \in [1, T - s_{\max} - 1]_{\mathbb{Z}}$. Hence, by Lemma 2, inequality (14) is valid and facet-defining for $\text{conv}(\mathcal{P})$ for any $t \in [1, T - s_{\max} - 1]_{\mathbb{Z}}$. $\square$

A.4. Proof of Proposition 2

Proposition 2. *Consider any integers α, β, and $s_{\max}$ such that (a) $L \leq s_{\max} \leq \min\{T-2, \lfloor(\overline{C}-\overline{V})/V\rfloor\}$, (b) $0 \leq \alpha < \beta \leq s_{\max}$, and (c) $\beta = \alpha+1$ or $s_{\max} \leq L+\alpha$. Let $\mathcal{S} = [0,\alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. For any $t \in [s_{\max}+2, T]_{\mathbb{Z}}$, inequality (13) is valid and facet-defining for* $\operatorname{conv}(\mathcal{P})$*. For any $t \in [1, T-s_{\max}-1]_{\mathbb{Z}}$, inequality (14) is valid and facet-defining for* $\operatorname{conv}(\mathcal{P})$*.*

Proof. Consider any $t \in [s_{\max}+2, T]_{\mathbb{Z}}$. To prove that the linear inequality (13) is valid for $\operatorname{conv}(\mathcal{P})$ when $\mathcal{S} = [0,\alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$, it suffices to show that (13) is valid for $\mathcal{P}$ when $\mathcal{S} = [0,\alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. Consider any element $(\mathbf{x},\mathbf{y})$ of $\mathcal{P}$. We show that $(\mathbf{x},\mathbf{y})$ satisfies (13) when $\mathcal{S} = [0,\alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. We divide the analysis into four cases.

Case 1: $y_t = 0$ and $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, \lfloor(\overline{C}-\overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Thus, in this case, the right-hand side of inequality (13) is nonnegative. Because $y_t = 0$, by (2d), $x_t = 0$. Therefore, in this case, $(\mathbf{x},\mathbf{y})$ satisfies (13).

Case 2: $y_t = 0$ and $y_{t-s} - y_{t-s-1} > 0$ for some $s \in \mathcal{S}$. Let $\tilde{\mathcal{S}} = \{\sigma \in \mathcal{S} : y_{t-\sigma} - y_{t-\sigma-1} > 0\}$ and $v = |\tilde{\mathcal{S}}|$. Then, $v \geq 1$. Denote $\tilde{\mathcal{S}} = \{\sigma_1, \sigma_2, \ldots, \sigma_v\}$, where $\sigma_1 < \sigma_2 < \cdots < \sigma_v$. Note that $y_{t-\sigma_j-1} = 0$ and $y_{t-\sigma_j} = 1$ for $j = 1, \ldots, v$. Denote $\sigma_0 = -1$. Then for each $j = 1, \ldots, v$, there exists $\sigma'_j \in [\sigma_{j-1}+1, \sigma_j - 1]_{\mathbb{Z}}$ such that $y_{t-\sigma'_j-1} = 1$ and $y_{t-\sigma'_j} = 0$. Thus,

$$0 \leq \sigma'_1 < \sigma_1 < \sigma'_2 < \sigma_2 \cdots < \sigma'_v < \sigma_v \leq s_{\max}.$$

Because $y_{t-\sigma_v} - y_{t-\sigma_v-1} = 1$ and $t - \sigma_v \in [2,T]_{\mathbb{Z}}$, by (2a), $y_k = 1$ for all $k \in [t-\sigma_v, \min\{T, t-\sigma_v+L-1\}]_{\mathbb{Z}}$, which implies that $t - \sigma'_j \geq t - \sigma_v + L$ for $j = 1, \ldots, v$. Hence, for $j = 1, \ldots, v$, we have $\sigma'_j \leq \sigma_v - L$, which implies that

$$\sigma'_j \leq s_{\max} - L. \tag{EC.2}$$

If $\beta = \alpha+1$, then $\mathcal{S} = [0, s_{\max}]_{\mathbb{Z}}$, which implies that $\sigma'_j \in \mathcal{S}$ for $j = 1, \ldots, v$. If $\beta \neq \alpha+1$, then condition (c) of Proposition 2 implies that $s_{\max} \leq L+\alpha$, which, by (EC.2), implies that $\sigma'_j \leq \alpha$ for $j = 1, \ldots, v$. Thus, in both cases, $\sigma'_j \in \mathcal{S}$ for $j = 1, \ldots, v$. Because $y_t = 0$, by (2d), $x_t = 0$. Hence, the left-hand side of inequality (13) is 0. Because $\mathcal{S} \subseteq [0, \lfloor(\overline{C}-\overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Note that $\{\sigma'_1, \ldots, \sigma'_v\} \subseteq \mathcal{S} \setminus \tilde{\mathcal{S}}$ and $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S} \setminus \tilde{\mathcal{S}}$. Thus, $\sum_{s \in \mathcal{S} \setminus \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \leq \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma'_j V)(y_{t-\sigma'_j} - y_{t-\sigma'_j-1})$. Hence, the right-hand side of inequality (13) is

$$\begin{aligned}
&\overline{C} y_t - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \\
&= -\sum_{s \in \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) - \sum_{s \in \mathcal{S} \setminus \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \\
&\geq -\sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma_j V)(y_{t-\sigma_j} - y_{t-\sigma_j-1}) - \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma'_j V)(y_{t-\sigma'_j} - y_{t-\sigma'_j-1})
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{j=1}^{v}(\overline{C} - \overline{V} - \sigma_j V) + \sum_{j=1}^{v}(\overline{C} - \overline{V} - \sigma_j' V) \\
&= \sum_{j=1}^{v}(\sigma_j - \sigma_j')V \\
&> 0.
\end{aligned}
$$

Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (13).

Case 3: $y_t = 1$ and $y_{t-s} - y_{t-s-1} \le 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, \lfloor(\overline{C} - \overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \ge 0$ for all $s \in \mathcal{S}$. Thus, in this case, the right-hand side of inequality (13) is at least $\overline{C}$. By (2d), $x_t \le \overline{C}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (13).

Case 4: $y_t = 1$ and $y_{t-s} - y_{t-s-1} > 0$ for some $s \in \mathcal{S}$. Let $\tilde{\mathcal{S}} = \{\sigma \in \mathcal{S} : y_{t-\sigma} - y_{t-\sigma-1} > 0\}$ and $v = |\tilde{\mathcal{S}}|$. Then, $v \ge 1$. Denote $\tilde{\mathcal{S}} = \{\sigma_1, \sigma_2, \ldots, \sigma_v\}$, where $\sigma_1 < \sigma_2 < \cdots < \sigma_v$. Note that $y_{t-\sigma_j-1} = 0$ and $y_{t-\sigma_j} = 1$ for $j = 1, \ldots, v$. Then, for each $j = 2, \ldots, v$, there exists $\sigma_j' \in [\sigma_{j-1} + 1, \sigma_j - 1]_{\mathbb{Z}}$ such that $y_{t-\sigma_j'-1} = 1$ and $y_{t-\sigma_j'} = 0$. Thus,

$$0 \le \sigma_1 < \sigma_2' < \sigma_2 < \cdots < \sigma_v' < \sigma_v \le s_{\max}.$$

In addition, $y_k = 1$ for all $k \in [t - \sigma_1, t]_{\mathbb{Z}}$. Because $y_{t-\sigma_v} - y_{t-\sigma_v-1} = 1$ and $t - \sigma_v \in [2, T]_{\mathbb{Z}}$, by (2a), $y_k = 1$ for all $k \in [t - \sigma_v, \min\{T, t - \sigma_v + L - 1\}]_{\mathbb{Z}}$, which implies that $t - \sigma_j' \ge t - \sigma_v + L$ for $j = 2, \ldots, v$. Hence, for $j = 2, \ldots, v$, we have $\sigma_j' \le \sigma_v - L$, which implies that

$$\sigma_j' \le s_{\max} - L. \tag{EC.3}$$

If $\beta = \alpha + 1$, then $\mathcal{S} = [0, s_{\max}]_{\mathbb{Z}}$, which implies that $\sigma_j' \in \mathcal{S}$ for $j = 2, \ldots, v$. If $\beta \ne \alpha + 1$, then condition (c) of Proposition 2 implies that $s_{\max} \le L + \alpha$, which, by (EC.3) implies that $\sigma_j' \le \alpha$ for all $j = 2, \ldots, v$. Thus, in both cases, $\sigma_j' \in \mathcal{S}$ for $j = 2, \ldots, v$. By (2e),

$$\sum_{\tau=t-\sigma_1}^{t}(x_\tau - x_{\tau-1}) \le \sum_{\tau=t-\sigma_1}^{t} V y_{\tau-1} + \sum_{\tau=t-\sigma_1}^{t} \overline{V}(1 - y_{\tau-1}),$$

which implies that

$$x_t - x_{t-\sigma_1-1} \le \sum_{\tau=t-\sigma_1}^{t} V y_{\tau-1} + \sum_{\tau=t-\sigma_1}^{t} \overline{V}(1 - y_{\tau-1}) = \sigma_1 V + \overline{V}.$$

Because $y_{t-\sigma_1-1} = 0$, by (2d), $x_{t-\sigma_1-1} = 0$. Hence, $x_t \le \sigma_1 V + \overline{V}$; that is, the left-hand side of inequality (13) is at most $\sigma_1 V + \overline{V}$. Because $\mathcal{S} \subseteq [0, \lfloor(\overline{C} - \overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \ge 0$ for all $s \in \mathcal{S}$. Note that $\{\sigma_2', \ldots, \sigma_v'\} \subseteq \mathcal{S} \setminus \tilde{\mathcal{S}}$ and $y_{t-s} - y_{t-s-1} \le 0$ for all $s \in \mathcal{S} \setminus \tilde{\mathcal{S}}$. Thus, $\sum_{s \in \mathcal{S}\setminus\tilde{\mathcal{S}}}(\overline{C} - \overline{V} -$

$sV)(y_{t-s} - y_{t-s-1}) \leq \sum_{j=2}^{v}(\overline{C} - \overline{V} - \sigma'_j V)(y_{t-\sigma'_j} - y_{t-\sigma'_j-1})$. Hence, the right-hand side of inequality (13) is

$$\begin{aligned}
&\overline{C}y_t - \sum_{s\in\mathcal{S}}(\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \\
&= \overline{C} - \sum_{s\in\tilde{\mathcal{S}}}(\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) - \sum_{s\in\mathcal{S}\setminus\tilde{\mathcal{S}}}(\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \\
&\geq \overline{C} - \sum_{j=1}^{v}(\overline{C} - \overline{V} - \sigma_j V)(y_{t-\sigma_j} - y_{t-\sigma_j-1}) - \sum_{j=2}^{v}(\overline{C} - \overline{V} - \sigma'_j V)(y_{t-\sigma'_j} - y_{t-\sigma'_j-1}) \\
&= \overline{C} - \sum_{j=1}^{v}(\overline{C} - \overline{V} - \sigma_j V) + \sum_{j=2}^{v}(\overline{C} - \overline{V} - \sigma'_j V) \\
&= \sigma_1 V + \overline{V} + \sum_{j=2}^{v}(\sigma_j - \sigma'_j)V \\
&\geq \sigma_1 V + \overline{V}.
\end{aligned}$$

Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (13).

Summarizing Cases 1–4, we conclude that $(\mathbf{x}, \mathbf{y})$ satisfies (13). Hence, (13) is valid for $\text{conv}(\mathcal{P})$ when $\mathcal{S} = [0, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$.

It is easy to verify that the proof of facet-defining of inequality (13) in the proof of Proposition 1 remains valid when $\mathcal{S} = [0, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. Therefore, inequality (13) is facet-defining for $\text{conv}(\mathcal{P})$ under the conditions stated in Proposition 2.

It is also easy to verify that the proof of validity and facet-defining of inequality (14) in the proof of Proposition 1 remains valid when $\mathcal{S} = [0, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. Therefore, inequality (14) is valid and facet-defining for $\text{conv}(\mathcal{P})$ under the conditions stated in Proposition 2. $\square$

A.5. Proof of Proposition 3

Proposition 3. *Consider any set* $\mathcal{S} \subseteq [0, \min\{L-1, T-3, \lfloor(\overline{C}-\overline{V})/V\rfloor\}]_{\mathbb{Z}}$ *and any real number* η *such that* $0 \le \eta \le \min\{L-1, (\overline{C}-\overline{V})/V\}$. *For any* $t \in [1, T-1]_{\mathbb{Z}}$ *such that* $t \ge s+2$ *for all* $s \in \mathcal{S}$, *the inequality*

$$x_t \le (\overline{C} - \eta V) y_t + \eta V y_{t+1} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \tag{15}$$

is valid for $\mathrm{conv}(\mathcal{P})$. *For any* $t \in [2, T]_{\mathbb{Z}}$ *such that* $t \le T - s - 1$ *for all* $s \in \mathcal{S}$, *the inequality*

$$x_t \le (\overline{C} - \eta V) y_t + \eta V y_{t-1} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \tag{16}$$

is valid for $\mathrm{conv}(\mathcal{P})$. *Furthermore, inequalities* (15) *and* (16) *are facet-defining for* $\mathrm{conv}(\mathcal{P})$ *when* $\eta \in \{0, (\overline{C}-\overline{V})/V\}$ *or* $\eta = L-1 \in \mathcal{S}$.

Proof. We first prove that inequality (15) is valid and facet-defining for $\mathrm{conv}(\mathcal{P})$. Note that the proof of facet-defining of (15) here can also be used to prove the facet-defining of (15) in Proposition 4. For notational convenience, we define $s_{\max} = \max\{s : s \in \mathcal{S}\}$ if $\mathcal{S} \ne \emptyset$, and $s_{\max} = -1$ if $\mathcal{S} = \emptyset$.

Consider any $t \in [s_{\max}+2, T-1]_{\mathbb{Z}}$ (i.e., $t \in [1, T-1]_{\mathbb{Z}}$ such that $t \ge s+2$ for all $s \in \mathcal{S}$). To prove that the linear inequality (15) is valid for $\mathrm{conv}(\mathcal{P})$, it suffices to show that it is valid for $\mathcal{P}$. Consider any element $(\mathbf{x}, \mathbf{y})$ of $\mathcal{P}$. We show that $(\mathbf{x}, \mathbf{y})$ satisfies (15). We divide the analysis into three cases.

Case 1: $y_t = 0$. By Lemma 1(i), $y_{t-j} - y_{t-j-1} \le 0$ for all $j \in [0, \min\{t-2, L-1\}]_{\mathbb{Z}}$. Because $\mathcal{S} \subseteq [0, L-1]_{\mathbb{Z}}$ and $s_{\max} \le t-2$, we have $\mathcal{S} \subseteq [0, \min\{t-2, L-1\}]_{\mathbb{Z}}$. Thus, $y_{t-s} - y_{t-s-1} \le 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, \lfloor(\overline{C}-\overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \ge 0$ for all $s \in \mathcal{S}$. Thus, the right-hand side of (15) is at least $\eta V y_{t+1} \ge 0$. Because $y_t = 0$, by (2d), $x_t = 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (15).

Case 2: $y_t = 1$ and $y_{t-s'} - y_{t-s'-1} = 1$ for some $s' \in \mathcal{S}$. In this case, $y_{t-s'} = 1$ and $y_{t-s'-1} = 0$. Because $s' \le s_{\max} \le t-2$, we have $t - s' \in [2, T]_{\mathbb{Z}}$. By (2a), $y_k = 1$ for all $k \in [t-s', \min\{T, t-s'+L-1\}]_{\mathbb{Z}}$. Because $\mathcal{S} \subseteq [0, L-1]_{\mathbb{Z}}$, we have $s' \le L-1$, or equivalently $t - s' + L - 1 \ge t$, and thus $y_{t-s} = 1$ for all $s \in [0, s']_{\mathbb{Z}}$, which implies that $y_{t-s} - y_{t-s-1} = 0$ for all $s \in [0, s'-1]_{\mathbb{Z}}$. Because $s' \le t-2$, either $s' = t-2$ or $s' \le t-3$. If $s' = t-2$, then it does not exist any $s \in \mathcal{S}$ such that $s > s'$. If $s' \le t-3$, then $t - s' - 1 \in [2, T]_{\mathbb{Z}}$, and by Lemma 1(i), $y_{t-s'-j-1} - y_{t-s'-j-2} \le 0$ for all $j \in [0, \min\{t-s'-3, L-1\}]_{\mathbb{Z}}$, which implies that $y_{t-s} - y_{t-s-1} \le 0$ for all $s \in [s'+1, \min\{t-2, L+s'\}]_{\mathbb{Z}}$, which in turn implies that $y_{t-s} - y_{t-s-1} \le 0$ for all $s \in \mathcal{S}$ such that $s > s'$. Hence, $y_{t-s} - y_{t-s-1} \le 0$ for all $s \in \mathcal{S} \setminus \{s'\}$. Because $\mathcal{S} \subseteq [0, \lfloor(\overline{C}-\overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \ge 0$ for all $s \in \mathcal{S}$. Thus,

$$-\sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \ge -(\overline{C} - \overline{V} - s'V). \tag{EC.4}$$

Because $y_t = 1$, by (EC.4), the right-hand side of (15) is at least $s'V + \overline{V}$ when $y_{t+1} = 1$ and is at least $\overline{V} + (s' - \eta)V$ when $y_{t+1} = 0$. By (2d), $x_{t-s'-1} = 0$. By (2e), $\sum_{\tau=t-s'}^{t}(x_\tau - x_{\tau-1}) \leq \sum_{\tau=t-s'}^{t} Vy_{\tau-1} + \sum_{\tau=t-s'}^{t} \overline{V}(1 - y_{\tau-1})$, which implies that $x_t \leq s'V + \overline{V}$. If $y_{t+1} = 0$, then by (2d) and (2f), $x_{t+1} = 0$ and $x_t - x_{t+1} \leq Vy_{t+1} + \overline{V}(1 - y_{t+1})$, implying that $x_t \leq \overline{V}$. In addition, if $y_{t+1} = 0$, then because $y_k = 1$ for all $k \in [t - s', \min\{T, t - s' + L - 1\}]_{\mathbb{Z}}$, we have $t + 1 \geq t - s' + L$, which implies that $s' \geq L - 1 \geq \eta$. Thus, if $y_{t+1} = 0$, then $x_t \leq \overline{V} + (s' - \eta)V$. Hence, x_t is at most $s'V + \overline{V}$, and it is at most $\overline{V} + (s' - \eta)V$ when $y_{t+1} = 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (15).

Case 3: $y_t = 1$ and $y_{t-s} - y_{t-s-1} \neq 1$ for all $s \in \mathcal{S}$. In this case, $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, \lfloor(\overline{C} - \overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Thus, $\sum_{s\in\mathcal{S}}(\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \leq 0$. Because $y_t = 1$, the right-hand side of (15) is at least $\overline{C}$ when $y_{t+1} = 1$ and is at least $\overline{C} - \eta V \geq \overline{V}$ when $y_{t+1} = 0$ (as $\eta \leq (\overline{C} - \overline{V})/V$). By (2d), $x_t \leq \overline{C}$. If $y_{t+1} = 0$, then by (2d) and (2f), $x_{t+1} = 0$ and $x_t - x_{t+1} \leq Vy_{t+1} + \overline{V}(1 - y_{t+1})$, which imply that $x_t \leq \overline{V}$. Hence, x_t is at most $\overline{C}$, and it is at most $\overline{V}$ when $y_{t+1} = 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (15).

Summarizing Cases 1–3, we conclude that $(\mathbf{x}, \mathbf{y})$ satisfies (15). Hence, (15) is valid for $\mathrm{conv}(\mathcal{P})$.

We first show that $\mathrm{conv}(\mathcal{P})$ is full dimensional. As the $\mathrm{conv}(\mathcal{P})$ contains $2T$ decision variables, to show $\dim(\mathrm{conv}(\mathcal{P})) = 2T$, we need to find $2T + 1$ affinely independent points in $\mathrm{conv}(\mathcal{P})$. Because $\mathbf{0} \in \mathrm{conv}(\mathcal{P})$, it suffices to create the remaining $2T$ nonzero linearly independent points. We denote these $2T$ points as $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ and $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$, and denote the qth component of $\bar{\mathbf{x}}^r$, $\bar{\mathbf{y}}^r$, $\hat{\mathbf{x}}^r$, and $\hat{\mathbf{y}}^r$ as $\bar{x}_q^r, \bar{y}_q^r, \hat{x}_q^r$ and $\hat{y}_q^r$, respectively. Let $\epsilon = \overline{V} - \underline{C} > 0$.

(A1) For each $r \in [1, T]_{\mathbb{Z}}$, we create a point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as follows:

$$\bar{x}_q^r = \begin{cases} \underline{C} + \epsilon, & q \in [1, r]_{\mathbb{Z}}; \\ 0, & q \in [r+1, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\bar{y}_q^r = \begin{cases} 1, & q \in [1, r]_{\mathbb{Z}}; \\ 0, & q \in [r+1, T]_{\mathbb{Z}}. \end{cases}$$

It is easy to observe that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (2a)–(2f) and thus $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \mathrm{conv}(\mathcal{P})$ for $r \in [1, T]_{\mathbb{Z}}$.

(A2) For each $r \in [1, T]_{\mathbb{Z}}$, we create a point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as follows:

$$\hat{x}_q^r = \begin{cases} \underline{C}, & q \in [1, r]_{\mathbb{Z}}; \\ 0, & q \in [r+1, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^r = \begin{cases} 1, & q \in [1, r]_{\mathbb{Z}}; \\ 0, & q \in [r+1, T]_{\mathbb{Z}}. \end{cases}$$

It is easy to observe that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2a)–(2f) and thus $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \mathrm{conv}(\mathcal{P})$ for $r \in [1, T]_{\mathbb{Z}}$.

It is also easy to observe that the above $2T$ points are linearly independent. Therefore, the dimension of $\mathrm{conv}(\mathcal{P})$ is $2T$, i.e., $\mathrm{conv}(\mathcal{P})$ is full dimensional. Consider any $t \in [s_{\max} + 2, T - 1]_{\mathbb{Z}}$. To prove that inequality (15) is facet-defining for $\mathrm{conv}(\mathcal{P})$ when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L - 1 \in \mathcal{S}$, it suffices to show that there exist $2T$ affinely independent points in $\text{conv}(\mathcal{P})$ that satisfy (15) at equality when $\eta \in \{0, (\overline{C}-\overline{V})/V\}$ or $\eta = L-1 \in \mathcal{S}$. When $\eta = 0$, inequalities (15) become inequality (13), and by Proposition 1, it is facet-defining for $\text{conv}(\mathcal{P})$. Hence, in the following, we only consider the case where $\eta = (\overline{C}-\overline{V})/V$ or $\eta = L-1 \in \mathcal{S}$. Because $\mathbf{0} \in \text{conv}(\mathcal{P})$ and $\mathbf{0}$ satisfies (15) at equality, it suffices to create the remaining $2T-1$ nonzero linearly independent points. We denote these $2T-1$ points as $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ for $r \in [1,T]_{\mathbb{Z}} \setminus \{t\}$ and $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ for $r \in [1,T]_{\mathbb{Z}}$.

We divide these $2T-1$ points into the following five groups:

(A1) For each $r \in [1,T]_{\mathbb{Z}} \setminus \{t\}$, we create the same point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as in group (A1) in the proof of Proposition 1. Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (15) at equality.

(A2) For each $r \in [1,t-1]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (A2) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Consider the case where $t-r-1 \notin \mathcal{S}$. In this case, $\hat{x}_t^r = \hat{y}_t^r = \hat{y}_{t+1}^r = 0$. In addition, $t-s-1 \neq r$ for all $s \in \mathcal{S}$, which implies that $\hat{y}_{t-s}^r - \hat{y}_{t-s-1}^r = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (15) at equality. Next, consider the case where $t-r-1 \in \mathcal{S}$. In this case, $\hat{x}_t^r = \overline{V} + (t-r-1)V$ and $\hat{y}_t^r = \hat{y}_{t+1}^r = 1$. In addition, $\hat{y}_{t-s}^r - \hat{y}_{t-s-1}^r = 1$ when $s = t-r-1$, and $\hat{y}_{t-s}^r - \hat{y}_{t-s-1}^r = 0$ when $s \neq t-r-1$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (15) at equality.

(A3) We create a point $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ as follows: If $\eta = (\overline{C}-\overline{V})/V$, then

$$\hat{x}_q^t = \begin{cases} \overline{V}, \text{ for } q \in [1,t]_{\mathbb{Z}}; \\ 0, \ \text{ for } q \in [t+1,T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^t = \begin{cases} 1, \text{ for } q \in [1,t]_{\mathbb{Z}}; \\ 0, \text{ for } q \in [t+1,T]_{\mathbb{Z}}. \end{cases}$$

If $\eta = L-1 \in \mathcal{S}$, then

$$\hat{x}_q^t = \begin{cases} \overline{V}, \text{ for } q \in [t-L+1,t]_{\mathbb{Z}}; \\ 0, \ \text{ for } q \in [1,t-L]_{\mathbb{Z}} \cup [t+1,T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^t = \begin{cases} 1, \ q \in [t-L+1,t]_{\mathbb{Z}}; \\ 0, \ q \in [1,t-L]_{\mathbb{Z}} \cup [t+1,T]_{\mathbb{Z}}. \end{cases}$$

We first consider the case where $\eta = (\overline{C}-\overline{V})/V$. It is easy to verify that $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (2a)–(2f). Thus, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) \in \text{conv}(\mathcal{P})$. In this case, $\hat{x}_t^t = \overline{V}$, $\hat{y}_t^t = 1$, and $\hat{y}_{t+1}^t = 0$, and $\hat{y}_{t-s}^t - \hat{y}_{t-s-1}^t = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (15) at equality. Next, we consider the case where $\eta = L-1 \in \mathcal{S}$. In this case, for any $q \in [2,T]_{\mathbb{Z}}$, $\hat{y}_q^t - \hat{y}_{q-1}^t \leq 0$ if $q \neq t-L+1$, while $\hat{y}_q^t - \hat{y}_{q-1}^t = 1$ and $\hat{y}_k^t = 1$ for all $k \in [q, \min\{T, q+L-1\}]_{\mathbb{Z}}$ if $q = t-L+1$. Thus, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (2a). For any $q \in [2,T]_{\mathbb{Z}}$, $\hat{y}_{q-1}^t - \hat{y}_q^t \leq 0$ if $q \neq t+1$, while $\hat{y}_{q-1}^t - \hat{y}_q^t = 1$ and $\hat{y}_k^t = 0$ for all $k \in [q, \min\{T, q+\ell-1\}]_{\mathbb{Z}}$ if $q = t+1$. Thus, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (2b). It is easy to verify that $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (2c)–(2f). Thus, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) \in \text{conv}(\mathcal{P})$. Note that $\hat{x}_t^t = \overline{V}$, $\hat{y}_t^t = 1$, $\hat{y}_{t+1}^t = 0$, $\hat{y}_{t-s}^t - \hat{y}_{t-s-1}^t = 0$ for all $s \in \mathcal{S} \setminus \{L-1\}$, $\hat{y}_{t-L+1}^t - \hat{y}_{t-L}^t = 1$, and $(\overline{C} - \eta V) - (\overline{C} - \overline{V} - (L-1)V) = \overline{V}$. Thus, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (15) at equality.

(A4) We create a point $(\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$ by setting $\hat{x}_q^{t+1} = \overline{C}$ and $\hat{y}_q^{t+1} = 1$ for $q \in [1, T]_{\mathbb{Z}}$. It is easy to verify that $(\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$ satisfies (2a)–(2f). Thus, $(\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1}) \in \mathrm{conv}(\mathcal{P})$. It is also easy to verify that $(\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$ satisfies (15) at equality.

(A5) For each $r \in [t+2, T]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (A4) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \mathrm{conv}(\mathcal{P})$. It is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (15) at equality.

Table EC.3 shows a matrix with $2T-1$ rows, where each row represents a point created by this process. This matrix can be transformed into the matrix in Table EC.4 via the following Gaussian elimination process:

(i) For each $r \in [1, T]_{\mathbb{Z}} \setminus \{t\}$, the point with index r in group (B1), denoted $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point with index r in group (A1), and $(\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$ is the point in group (A5).

(ii) For each $r \in [1, t-1]_{\mathbb{Z}}$, the point with index r in group (B2), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $t - r - 1 \notin \mathcal{S}$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1}) - (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $t - r - 1 \in \mathcal{S}$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (A2), and $(\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$ is the point in group (A4).

(iii) The point in group (B3), denoted $(\underline{\hat{\mathbf{x}}}^t, \underline{\hat{\mathbf{y}}}^t)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^t, \underline{\hat{\mathbf{y}}}^t) = (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$. Here, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point in group (A3).

Table EC.3 A matrix with the rows representing $2T-1$ points in $\mathrm{conv}(\mathcal{P})$ that satisfy inequality (15) at equality

Group	Point	Index r	$\mathbf{x}$								$\mathbf{y}$							
			1	$\cdots$	$t-1$	t	$t+1$	$t+2$	$\cdots$	T	1	$\cdots$	$t-1$	t	$t+1$	$t+2$	$\cdots$	T
(A1)	$(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$	1	$\overline{C}-\epsilon$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	$\cdots$	1	1	1	1	$\cdots$	1
		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-1$	$\overline{C}$	$\cdots$	$\overline{C}-\epsilon$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	$\cdots$	1	1	1	1	$\cdots$	1
		$t+1$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}-\epsilon$	$\overline{C}$	$\cdots$	$\overline{C}$	1	$\cdots$	1	1	1	1	$\cdots$	1
		$t+2$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}-\epsilon$	$\cdots$	$\overline{C}$	1	$\cdots$	1	1	1	1	$\cdots$	1
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		T	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}-\epsilon$	1	$\cdots$	1	1	1	1	$\cdots$	1
(A2)	$(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$	1, $\vdots$, $t-1$	(See Note EC.3-1)								(See Note EC.3-1)							
(A3)		t	(See Note EC.3-2)								(See Note EC.3-2)							
(A4)		$t+1$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	$\cdots$	1	1	1	1	$\cdots$	1
(A5)		$t+2$	0	$\cdots$	0	0	0	$\underline{C}$	$\cdots$	$\underline{C}$	0	$\cdots$	0	0	0	1	$\cdots$	1
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T	0	$\cdots$	0	0	0	0	$\cdots$	$\underline{C}$	0	$\cdots$	0	0	0	0	$\cdots$	1

Note EC.3-1: For $r \in [1, t-1]_{\mathbb{Z}}$, the $\mathbf{x}$ and $\mathbf{y}$ vectors in group (A2) are given as follows:

$\hat{\mathbf{x}}^r = (\underbrace{\underline{C}, \ldots, \underline{C}}_{r \text{ terms}}, \underbrace{0, \ldots, 0}_{T-r \text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{1, \ldots, 1}_{r \text{ terms}}, \underbrace{0, \ldots, 0}_{T-r \text{ terms}})$ if $t-r-1 \notin \mathcal{S}$;

$\hat{\mathbf{x}}^r = (\underbrace{0, \ldots, 0}_{r \text{ terms}}, \underbrace{\overline{V}, \overline{V}+V, \overline{V}+2V, \ldots, \overline{V}+(t-r-1)V}_{t-r \text{ terms}}, \underbrace{\overline{V}+(t-r-1)V, \ldots, \overline{V}+(t-r-1)V}_{T-t \text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{0, \ldots, 0}_{r \text{ terms}}, \underbrace{1, \ldots, 1}_{T-r \text{ terms}})$ if $t-r-1 \in \mathcal{S}$.

Note EC.3-2: The $\mathbf{x}$ and $\mathbf{y}$ vectors in group (A3) are given as follows:

$\hat{\mathbf{x}}^t = (\underbrace{\overline{V}, \ldots, \overline{V}}_{t \text{ terms}}, \underbrace{0, \ldots, 0}_{T-t \text{ terms}})$ and $\hat{\mathbf{y}}^t = (\underbrace{1, \ldots, 1}_{t \text{ terms}}, \underbrace{0, \ldots, 0}_{T-t \text{ terms}})$ if $\eta = (\overline{C} - \overline{V})/V$; $\hat{\mathbf{x}}^t = (\underbrace{0, \ldots, 0}_{t-L \text{ terms}}, \underbrace{\overline{V}, \ldots, \overline{V}}_{L \text{ terms}}, \underbrace{0, \ldots, 0}_{T-t \text{ terms}})$ and $\hat{\mathbf{y}}^t = (\underbrace{0, \ldots, 0}_{t-L \text{ terms}}, \underbrace{1, \ldots, 1}_{L \text{ terms}}, \underbrace{0, \ldots, 0}_{T-t \text{ terms}})$ if $\eta = L-1 \in \mathcal{S}$.

Table EC.4 Lower triangular matrix obtained from Table EC.3 via Gaussian elimination

Group	Point	Index r	$\mathbf{x}$								$\mathbf{y}$							
			1	$\cdots$	$t-1$	t	$t+1$	$t+2$	$\cdots$	T	1	$\cdots$	$t-1$	t	$t+1$	$t+2$	$\cdots$	T
(B1)	$(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$	1	$-\epsilon$	$\cdots$	0	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-1$	0	$\cdots$	$-\epsilon$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	0	$\cdots$	0
		$t+1$	0	$\cdots$	0	0	$-\epsilon$	0	$\cdots$	0	0	$\cdots$	0	0	0	0	$\cdots$	0
		$t+2$	0	$\cdots$	0	0	0	$-\epsilon$	$\cdots$	0	0	$\cdots$	0	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		T	0	$\cdots$	0	0	0	0	$\cdots$	$-\epsilon$	0	$\cdots$	0	0	0	0	$\cdots$	0
(B2)	$(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$	1	(Omitted)								1	$\cdots$	0	0	0	0	$\cdots$	0
		$\vdots$									$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-1$									1	$\cdots$	1	0	0	0	$\cdots$	0
(B3)		t	(Omitted)								(See Note EC.4-1)							
(B4)		$t+1$	(Omitted)								1	$\cdots$	1	1	1	0	$\cdots$	0
(B5)		$t+2$	(Omitted)								0	$\cdots$	0	0	0	1	$\cdots$	0
		$\vdots$									$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T									0	$\cdots$	0	0	0	0	$\cdots$	1

Note EC.4-1: The $\mathbf{y}$ vector in group (B3) is given as follows:

$\underline{\hat{\mathbf{y}}}^t = (\underbrace{1,\ldots,1}_{t \text{ terms}}, \underbrace{0,\ldots,0}_{T-t \text{ terms}})$ if $\eta = (\overline{C} - \overline{V})/V$; $\underline{\hat{\mathbf{y}}}^t = (\underbrace{0,\ldots,0}_{t-L \text{ terms}}, \underbrace{1,\ldots,1}_{L \text{ terms}}, \underbrace{0,\ldots,0}_{T-t \text{ terms}})$ if $\eta = L - 1 \in \mathcal{S}$.

(iv) The point in group (B4), denoted $(\underline{\hat{\mathbf{x}}}^{t+1}, \underline{\hat{\mathbf{y}}}^{t+1})$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^{t+1}, \underline{\hat{\mathbf{y}}}^{t+1}) = (\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1}) - (\hat{\mathbf{x}}^{t+2}, \hat{\mathbf{y}}^{t+2})$. Here, $(\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$ is the point in group (A4), and $(\hat{\mathbf{x}}^{t+2}, \hat{\mathbf{y}}^{t+2})$ is the point with index $t+2$ in group (A5).

(v) For each $r \in [t+2, T]_{\mathbb{Z}}$, the point with index r in group (B5), denoted $(\underline{\hat{\mathbf{x}}}^{r}, \underline{\hat{\mathbf{y}}}^{r})$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^{r}, \underline{\hat{\mathbf{y}}}^{r}) = (\hat{\mathbf{x}}^{r}, \hat{\mathbf{y}}^{r}) - (\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ if $r \neq T$, and setting $(\underline{\hat{\mathbf{x}}}^{r}, \underline{\hat{\mathbf{y}}}^{r}) = (\hat{\mathbf{x}}^{r}, \hat{\mathbf{y}}^{r})$ if $r = T$. Here, $(\hat{\mathbf{x}}^{r}, \hat{\mathbf{y}}^{r})$ and $(\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ are the points with indices r and $r+1$, respectively, in group (A5).

The matrix shown in Table EC.4 is lower triangular; that is, the position of the last nonzero component of a row of the matrix is greater than the position of the last nonzero component of the previous row. This implies that the $2T-1$ points in groups (A1)–(A5) are linearly independent. Therefore, inequality (15) is facet-defining for $\mathrm{conv}(\mathcal{P})$.

Next, we show that inequality (16) is valid for $\mathrm{conv}(\mathcal{P})$ and is facet-defining for $\mathrm{conv}(\mathcal{P})$ when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L - 1 \in \mathcal{S}$. Note that this proof can also be used to prove the validity and facet-defining of inequality (16) in Proposition 4. Denote $x'_t = x_{T-t+1}$ and $y'_t = y_{T-t+1}$ for $t \in [1, T]_{\mathbb{Z}}$. Because inequality (15) is valid for $\mathrm{conv}(\mathcal{P})$ and is facet-defining for $\mathrm{conv}(\mathcal{P})$ when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L - 1 \in \mathcal{S}$ for any $t \in [s_{\max} + 2, T - 1]_{\mathbb{Z}}$, the inequality

$$x'_{T-t+1} \leq (\overline{C} - \eta V) y'_{T-t+1} + \eta V y'_{T-t} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y'_{T-t+s+1} - y'_{T-t+s+2})$$

is valid for $\mathrm{conv}(\mathcal{P}')$ and is facet-defining for $\mathrm{conv}(\mathcal{P}')$ when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L - 1 \in \mathcal{S}$ for any $t \in [s_{\max} + 2, T - 1]_{\mathbb{Z}}$. Let $t' = T - t + 1$. Then, the inequality

$$x'_{t'} \leq (\overline{C} - \eta V) y'_{t'} + \eta V y'_{t'-1} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y'_{t'+s} - y'_{t'+s+1})$$

is valid for $\mathrm{conv}(\mathcal{P}')$ and is facet-defining for $\mathrm{conv}(\mathcal{P}')$ when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L - 1 \in \mathcal{S}$ for any $t' \in [2, T - s_{\max} - 1]_{\mathbb{Z}}$. Hence, by Lemma 2, inequality (16) is valid for $\mathrm{conv}(\mathcal{P})$ and is facet-defining for $\mathrm{conv}(\mathcal{P})$ when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L - 1 \in \mathcal{S}$ for any $t \in [2, T - s_{\max} - 1]_{\mathbb{Z}}$. $\square$

A.6. Proof of Proposition 4

Proposition 4. *Consider any real number η such that $0 \le \eta \le \min\{L-1, (\overline{C}-\overline{V})/V\}$ and any integers α, β, and $s_{\max}$ such that (a) $L \le s_{\max} \le \min\{T-3, \lfloor(\overline{C}-\overline{V})/V\rfloor\}$, (b) $0 \le \alpha < \beta \le s_{\max}$, and (c) $\beta = \alpha+1$ or $s_{\max} \le L+\alpha$. Let $\mathcal{S} = [0,\alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. For any $t \in [s_{\max}+2, T-1]_{\mathbb{Z}}$, inequality (15) is valid for $\mathrm{conv}(\mathcal{P})$. For any $t \in [2, T-s_{\max}-1]_{\mathbb{Z}}$, inequality (16) is valid for $\mathrm{conv}(\mathcal{P})$. Furthermore, (15) and (16) are facet-defining for $\mathrm{conv}(\mathcal{P})$ when $\eta \in \{0, (\overline{C}-\overline{V})/V\}$ or $\eta = L-1 \in \mathcal{S}$.*

Proof. Consider any $t \in [s_{\max}+2, T-1]_{\mathbb{Z}}$. To prove that the linear inequality (15) is valid for $\mathrm{conv}(\mathcal{P})$ when $\mathcal{S} = [0,\alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$, it suffices to show that (15) is valid for $\mathcal{P}$ when $\mathcal{S} = [0,\alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. Consider any element $(\mathbf{x}, \mathbf{y})$ of $\mathcal{P}$. We show that $(\mathbf{x}, \mathbf{y})$ satisfies (15) when $\mathcal{S} = [0,\alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. We divide the analysis into four cases.

Case 1: $y_t = 0$ and $y_{t-s} - y_{t-s-1} \le 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, \lfloor(\overline{C}-\overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \ge 0$ for all $s \in \mathcal{S}$. Thus, in this case, the right-hand side of inequality (15) is nonnegative. Because $y_t = 0$, by (2d), $x_t = 0$. Therefore, $(\mathbf{x}, \mathbf{y})$ satisfies (15).

Case 2: $y_t = 0$ and $y_{t-s} - y_{t-s-1} > 0$ for some $s \in \mathcal{S}$. Let $\tilde{\mathcal{S}} = \{\sigma \in \mathcal{S} : y_{t-\sigma} - y_{t-\sigma-1} > 0\}$ and $v = |\tilde{\mathcal{S}}|$. Then, $v \ge 1$. Denote $\tilde{\mathcal{S}} = \{\sigma_1, \sigma_2, \ldots, \sigma_v\}$, where $\sigma_1 < \sigma_2 < \cdots < \sigma_v$. Note that $y_{t-\sigma_j-1} = 0$ and $y_{t-\sigma_j} = 1$ for $j = 1, \ldots, v$. Denote $\sigma_0 = -1$. Then, for each $j = 1, \ldots, v$, there exists $\sigma'_j \in [\sigma_{j-1}+1, \sigma_j - 1]_{\mathbb{Z}}$ such that $y_{t-\sigma'_j-1} = 1$ and $y_{t-\sigma'_j} = 0$. Thus,

$$0 \le \sigma'_1 < \sigma_1 < \sigma'_2 < \sigma_2 < \cdots < \sigma'_v < \sigma_v \le s_{\max}.$$

Because $y_{t-\sigma_v} - y_{t-\sigma_v-1} = 1$ and $t - \sigma_v \in [2, T]_{\mathbb{Z}}$, by (2a), $y_k = 1$ for all $k \in [t-\sigma_v, \min\{T, t-\sigma_v+L-1\}]_{\mathbb{Z}}$, which implies that $t - \sigma'_j \ge t - \sigma_v + L$ for $j = 1, \ldots, v$. Hence, for $j = 1, \ldots, v$, we have $\sigma'_j \le \sigma_v - L$, which implies that

$$\sigma'_j \le s_{\max} - L. \tag{EC.5}$$

If $\beta = \alpha+1$, then $\mathcal{S} = [0, s_{\max}]_{\mathbb{Z}}$, which implies that $\sigma'_j \in \mathcal{S}$ for $j = 1, \ldots, v$. If $\beta \ne \alpha+1$, then condition (c) of Proposition 4 implies that $s_{\max} \le L+\alpha$, which, by (EC.5), implies that $\sigma'_j \le \alpha$ for $j = 1, \ldots, v$. Thus, in both cases, $\sigma'_j \in \mathcal{S}$ for $j = 1, \ldots, v$. Because $y_t = 0$, by (2d), $x_t = 0$. Hence, the left-hand side of inequality (15) is 0. Because $\mathcal{S} \subseteq [0, \lfloor(\overline{C}-\overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \ge 0$ for all $s \in \mathcal{S}$. Note that $\{\sigma'_1, \ldots, \sigma'_v\} \subseteq \mathcal{S} \setminus \tilde{\mathcal{S}}$ and $y_{t-s} - y_{t-s-1} \le 0$ for all $s \in \mathcal{S} \setminus \tilde{\mathcal{S}}$. Thus, $\sum_{s \in \mathcal{S}\setminus\tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \le \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma'_j V)(y_{t-\sigma'_j} - y_{t-\sigma'_j-1})$. Hence, the right-hand side of inequality (15) is

$$\begin{aligned}
&(\overline{C} - \eta V) y_t + \eta V y_{t+1} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \\
&= \eta V y_{t+1} - \sum_{s \in \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) - \sum_{s \in \mathcal{S}\setminus\tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1})
\end{aligned}$$

$$
\begin{aligned}
&\geq \eta V y_{t+1} - \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma_j V)(y_{t-\sigma_j} - y_{t-\sigma_j-1}) - \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma'_j V)(y_{t-\sigma'_j} - y_{t-\sigma'_j-1}) \\
&= \eta V y_{t+1} - \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma_j V) + \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma'_j V) \\
&= \eta V y_{t+1} + \sum_{j=1}^{v} (\sigma_j - \sigma'_j) V \\
&> 0.
\end{aligned}
$$

Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (15).

Case 3: $y_t = 1$ and $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Thus, $\sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \leq 0$. Because $y_t = 1$, the right-hand side of (15) is at least $\overline{C}$ when $y_{t+1} = 1$ and is at least $\overline{C} - \eta V \geq \overline{V}$ when $y_{t+1} = 0$ (as $\eta \leq (\overline{C} - \overline{V})/V$). By (2d), $x_t \leq \overline{C}$. If $y_{t+1} = 0$, then by (2d) and (2f), $x_{t+1} = 0$ and $x_t - x_{t+1} \leq V y_{t+1} + \overline{V}(1 - y_{t+1})$, which imply that $x_t \leq \overline{V}$. Hence, x_t is at most $\overline{C}$, and it is at most $\overline{V}$ when $y_{t+1} = 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (15).

Case 4: $y_t = 1$ and $y_{t-s} - y_{t-s-1} > 0$ for some $s \in \mathcal{S}$. If $y_{t+1} = 1$, then inequality (15) becomes inequality (13), and by Proposition 1, $(\mathbf{x}, \mathbf{y})$ satisfies the inequality. In the following, we consider the case where $y_{t+1} = 0$. Let $\tilde{\mathcal{S}} = \{\sigma \in \mathcal{S} : y_{t-\sigma} - y_{t-\sigma-1} > 0\}$ and $v = |\tilde{\mathcal{S}}|$. Then, $v \geq 1$. Denote $\tilde{\mathcal{S}} = \{\sigma_1, \sigma_2, \ldots, \sigma_v\}$, where $\sigma_1 < \sigma_2 < \cdots < \sigma_v$. Note that $y_{t-\sigma_j-1} = 0$ and $y_{t-\sigma_j} = 1$ for $j = 1, \ldots, v$. Then, for each $j = 2, \ldots, v$, there exists $\sigma'_j \in [\sigma_{j-1} + 1, \sigma_j - 1]_{\mathbb{Z}}$ such that $y_{t-\sigma'_j-1} = 1$ and $y_{t-\sigma'_j} = 0$. Thus,

$$0 \leq \sigma_1 < \sigma'_2 < \sigma_2 < \cdots < \sigma'_v < \sigma_v \leq s_{\max}.$$

In addition, $y_k = 1$ for all $k \in [t - \sigma_1, t]_{\mathbb{Z}}$. Because $y_{t-\sigma_v} - y_{t-\sigma_v-1} = 1$ and $t - \sigma_v \in [2, T]_{\mathbb{Z}}$, by (2a), $y_k = 1$ for all $k \in [t - \sigma_v, \min\{T, t - \sigma_v + L - 1\}]_{\mathbb{Z}}$, which implies that $t - \sigma'_j \geq t - \sigma_v + L$ for $j = 2, \ldots, v$. Hence, for $j = 2, \ldots, v$, we have $\sigma'_j \leq \sigma_v - L$, which implies that

$$\sigma'_j \leq s_{\max} - L. \tag{EC.6}$$

If $\beta = \alpha + 1$, then $\mathcal{S} = [0, s_{\max}]_{\mathbb{Z}}$, which implies that $\sigma'_j \in \mathcal{S}$ for $j = 2, \ldots, v$. If $\beta \neq \alpha + 1$, then condition (c) of Proposition 4 implies that $s_{\max} \leq L + \alpha$, which, by (EC.6), implies that $\sigma'_j \leq \alpha$ for $j = 2, \ldots, v$. Thus, in both cases, $\sigma'_j \in \mathcal{S}$ for $j = 2, \ldots, v$. Because $y_{t+1} = 0$, by (2d) and (2f), $x_{t+1} = 0$ and $x_t - x_{t+1} \leq V y_{t+1} + \overline{V}(1 - y_{t+1})$, which imply that $x_t \leq \overline{V}$; that is, the left-hand side of inequality (15) is at most $\overline{V}$. Because $y_{t-\sigma_1} - y_{t-\sigma_1-1} = 1$ and $t - \sigma_1 \in [2, T]_{\mathbb{Z}}$, by (2a), $y_k = 1$ for all $k \in [t - \sigma_1, \min\{T, t - \sigma_1 + L - 1\}]_{\mathbb{Z}}$. Because $y_{t+1} = 0$, this implies that $t + 1 \geq t - \sigma_1 + L$, or equivalently, $L - 1 \leq \sigma_1$. Because $\eta \leq L - 1$, we have

$$\eta \leq \sigma_1. \tag{EC.7}$$

Because $\mathcal{S} \subseteq [0, \lfloor(\overline{C}-\overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C}-\overline{V}-sV \geq 0$ for all $s \in \mathcal{S}$. Note that $\{\sigma'_2, \ldots, \sigma'_v\} \subseteq \mathcal{S} \setminus \tilde{\mathcal{S}}$ and $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S} \setminus \tilde{\mathcal{S}}$. Thus, $\sum_{s\in\mathcal{S}\setminus\tilde{\mathcal{S}}}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1}) \leq \sum_{j=2}^{v}(\overline{C}-\overline{V}-\sigma'_j V)(y_{t-\sigma'_j}-y_{t-\sigma'_j-1})$. Hence, the right-hand side of inequality (15) is

$$
\begin{aligned}
&(\overline{C}-\eta V)y_t + \eta V y_{t+1} - \sum_{s\in\mathcal{S}}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1}) \\
&= (\overline{C}-\eta V) - \sum_{s\in\tilde{\mathcal{S}}}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1}) - \sum_{s\in\mathcal{S}\setminus\tilde{\mathcal{S}}}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1}) \\
&\geq (\overline{C}-\eta V) - \sum_{j=1}^{v}(\overline{C}-\overline{V}-\sigma_j V)(y_{t-\sigma_j}-y_{t-\sigma_j-1}) - \sum_{j=2}^{v}(\overline{C}-\overline{V}-\sigma'_j V)(y_{t-\sigma'_j}-y_{t-\sigma'_j-1}) \\
&= (\overline{C}-\eta V) - \sum_{j=1}^{v}(\overline{C}-\overline{V}-\sigma_j V) + \sum_{j=2}^{v}(\overline{C}-\overline{V}-\sigma'_j V) \\
&= \overline{V} + (\sigma_1-\eta)V + \sum_{j=2}^{v}(\sigma_j-\sigma'_j)V \\
&\geq \overline{V} + (\sigma_1-\eta)V \\
&\geq \overline{V},
\end{aligned}
$$

where the last inequality follows from (EC.7). Therefore, in this case, $(\mathbf{x},\mathbf{y})$ satisfies (15).

Summarizing Cases 1–4, we conclude that $(\mathbf{x},\mathbf{y})$ satisfies (15). Hence, (15) is valid for $\text{conv}(\mathcal{P})$ when $\mathcal{S} = [0,\alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$.

It is easy to verify that the proof of facet-defining of inequality (15) in the proof of Proposition 3 remains valid when $\mathcal{S} = [0,\alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. Therefore, inequality (15) is facet-defining for $\text{conv}(\mathcal{P})$ under the conditions stated in Proposition 4.

It is also easy to verify that the proof of validity and facet-defining of inequality (16) in the proof of Proposition 3 remains valid when $\mathcal{S} = [0,\alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. Therefore, inequality (16) is valid and facet-defining for $\text{conv}(\mathcal{P})$ under the conditions stated in Proposition 4. $\square$

A.7. Proof of Proposition 5

Proposition 5. *Consider any $\mathcal{S} \subseteq [1, \min\{L, T-2, \lfloor(\overline{C}-\overline{V})/V\rfloor\}]_{\mathbb{Z}}$ and any real number η such that $0 \le \eta \le \min\{L, (\overline{C}-\overline{V})/V\}$. For any $t \in [2,T]_{\mathbb{Z}}$ such that $t \ge s+2$ for all $s \in \mathcal{S}$, the inequality*

$$x_t \le (\overline{V}+\eta V)y_t + (\overline{C}-\overline{V}-\eta V)y_{t-1} - \sum_{s\in\mathcal{S}}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1}) \tag{17}$$

is valid for conv$(\mathcal{P})$. *For any $t \in [1, T-1]_{\mathbb{Z}}$ such that $t \le T-s-1$ for all $s \in \mathcal{S}$, the inequality*

$$x_t \le (\overline{V}+\eta V)y_t + (\overline{C}-\overline{V}-\eta V)y_{t+1} - \sum_{s\in\mathcal{S}}(\overline{C}-\overline{V}-sV)(y_{t+s}-y_{t+s+1}) \tag{18}$$

is valid for conv$(\mathcal{P})$. *Furthermore, inequalities (17) and (18) are facet-defining for* conv$(\mathcal{P})$ *when $\eta \in \{0, (\overline{C}-\overline{V})/V\}$ or $\eta = L \in \mathcal{S}$.*

Proof. We first prove that inequality (17) is valid and facet-defining for conv$(\mathcal{P})$. Note that the proof of facet-defining of (17) here can also be used to prove the facet-defining of (17) in Proposition 6. For notational convenience, we define $s_{\max} = \max\{s : s \in \mathcal{S}\}$ if $\mathcal{S} \neq \emptyset$, and $s_{\max} = 0$ if $\mathcal{S} = \emptyset$.

Consider any $t \in [s_{\max}+2, T]_{\mathbb{Z}}$ (i.e., $t \in [2,T]_{\mathbb{Z}}$ such that $t \ge s+2$ for all $s \in \mathcal{S}$). To prove that the linear inequality (17) is valid for conv$(\mathcal{P})$, it suffices to show that it is valid for $\mathcal{P}$. Consider any element $(\mathbf{x}, \mathbf{y})$ of $\mathcal{P}$. We show that $(\mathbf{x}, \mathbf{y})$ satisfies (17). We divide the analysis into three cases.

Case 1: $y_t = 0$. In this case, by (2d), $x_t = 0$. Thus, the left-hand side of (17) and the first term on the right-hand side of (17) are 0. Because $y_t = 0$, by Lemma 1(i), $y_{t-j} - y_{t-j-1} \le 0$ for all $j \in [0, \min\{t-2, L-1\}]_{\mathbb{Z}}$. Because $s_{\max} \le t-2$, we have $\mathcal{S} \subseteq [0, \min\{t-2, L\}]_{\mathbb{Z}}$. Thus, $y_{t-s} - y_{t-s-1} \le 0$ for all $s \in \mathcal{S} \setminus \{L\}$. Because $\eta \le (\overline{C}-\overline{V})/V$, the coefficient "$\overline{C}-\overline{V}-\eta V$" on the right-hand side of (17) is nonnegative. Because $\mathcal{S} \subseteq [1, \lfloor(\overline{C}-\overline{V})/V\rfloor]_{\mathbb{Z}}$, for any $s \in \mathcal{S}$, the coefficient "$\overline{C}-\overline{V}-sV$" on the right-hand side of (17) is also nonnegative. Hence, if $s_{\max} \le L-1$ or $y_{t-L} - y_{t-L-1} \le 0$, then the right-hand side of (17) is nonnegative. Now, consider the situation where $s_{\max} = L$ and $y_{t-L} - y_{t-L-1} > 0$. Then, $y_{t-L} = 1$ and $y_{t-L-1} = 0$. By (2a), $y_{t-1} = 1$. Thus, the right-hand side of (17) is at least $(\overline{C}-\overline{V}-\eta V)y_{t-1} - (\overline{C}-\overline{V}-LV)(y_{t-L} - y_{t-L-1}) = (\overline{C}-\overline{V}-\eta V) - (\overline{C}-\overline{V}-LV) = (L-\eta)V \ge 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (17).

Case 2: $y_t = 1$ and $y_{t-s'} - y_{t-s'-1} = 1$ for some $s' \in \mathcal{S}$. In this case, $y_{t-s'} = 1$ and $y_{t-s'-1} = 0$. Because $s_{\max} \le t-2$, we have $s' \le t-2$. If $s' = t-2$, then it does not exist any $s \in \mathcal{S}$ such that $s > s'$. If $s' \le t-3$, then $t-s'-1 \in [2,T]_{\mathbb{Z}}$, and by Lemma 1(i), $y_{t-s'-j-1} - y_{t-s'-j-2} \le 0$ for all $j \in [0, \min\{t-s'-3, L-1\}]_{\mathbb{Z}}$, which implies that $y_{t-s} - y_{t-s-1} \le 0$ for all $s \in [s'+1, \min\{t-2, L+s'\}]_{\mathbb{Z}}$. Thus, $y_{t-s} - y_{t-s-1} \le 0$ for all $s \in \mathcal{S}$ such that $s > s'$. Because $y_{t-s'} - y_{t-s'-1} = 1$ and $t-s' \in [2,T]_{\mathbb{Z}}$, by (2a), $y_k = 1$ for all $k \in [t-s', \min\{T, t-s'+L-1\}]_{\mathbb{Z}}$. This implies that $y_{t-s} - y_{t-s-1} = 0$ for all $s \in [1, s'-1]_{\mathbb{Z}}$ (as $s' \le L$). This in turn implies that $y_{t-s} - y_{t-s-1} = 0$ for all $s \in \mathcal{S}$ such that $s < s'$. Hence,

$y_{t-s} - y_{t-s-1} \le 0$ for all $s \in \mathcal{S} \setminus \{s'\}$. Because $\mathcal{S} \subseteq [1, \lfloor(\overline{C} - \overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \ge 0$ for all $s \in \mathcal{S}$. Thus,

$$-\sum_{s\in\mathcal{S}}(\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \ge -(\overline{C} - \overline{V} - s'V). \tag{EC.8}$$

Note that $t-1 \in [t-s', \min\{T, t-s'+L-1\}]_{\mathbb{Z}}$. Hence, $y_{t-1} = 1$. Because $y_t = 1$ and $y_{t-1} = 1$, by (EC.8), the right-hand side of inequality (17) is at least $s'V + \overline{V}$. By (2e), $\sum_{\tau=t-s'}^{t}(x_\tau - x_{\tau-1}) \le \sum_{\tau=t-s'}^{t} Vy_{\tau-1} + \sum_{\tau=t-s'}^{t} \overline{V}(1-y_{\tau-1})$, which implies that $x_t - x_{t-s'-1} \le s'V + \overline{V}$. Because $y_{t-s'-1} = 0$, we have $x_{t-s'-1} = 0$. Thus, $x_t \le s'V + \overline{V}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (17).

Case 3: $y_t = 1$ and $y_{t-s} - y_{t-s-1} \neq 1$ for all $s \in \mathcal{S}$. In this case, $y_{t-s} - y_{t-s-1} \le 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [1, \lfloor(\overline{C} - \overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \ge 0$ for all $s \in \mathcal{S}$. Thus, $\sum_{s\in\mathcal{S}}(\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \le 0$. The right-hand side of (17) is at least $\overline{V} + \eta V$ when $y_{t-1} = 0$, and is at least $\overline{C}$ when $y_{t-1} = 1$. If $y_{t-1} = 0$, then by (2d) and (2e), $x_{t-1} = 0$ and $x_t - x_{t-1} \le \overline{V}$, which imply that $x_t \le \overline{V}$, and hence, x_t is less than or equal to the right-hand side of (17). If $y_{t-1} = 1$, then by (2d), $x_t \le \overline{C}$, and hence, x_t is less than or equal to the right-hand side of (17). Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (17).

Summarizing Cases 1–3, we conclude that $(\mathbf{x}, \mathbf{y})$ satisfies (17). Hence, (17) is valid for $\mathrm{conv}(\mathcal{P})$.

Consider any $t \in [s_{\max}+2, T]_{\mathbb{Z}}$. To prove that inequality (17) is facet-defining for $\mathrm{conv}(\mathcal{P})$ when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L \in \mathcal{S}$, it suffices to show that there exist $2T$ affinely independent points in $\mathrm{conv}(\mathcal{P})$ that satisfy (17) at equality when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L \in \mathcal{S}$. Because $\mathbf{0} \in \mathrm{conv}(\mathcal{P})$ and $\mathbf{0}$ satisfies (17) at equality, it suffices to create the remaining $2T-1$ nonzero linearly independent points. We denote these $2T-1$ points as $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ for $r \in [1, T]_{\mathbb{Z}} \setminus \{t\}$ and $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ for $r \in [1, T]_{\mathbb{Z}}$, and denote the qth component of $\bar{\mathbf{x}}^r$, $\bar{\mathbf{y}}^r$, $\hat{\mathbf{x}}^r$, and $\hat{\mathbf{y}}^r$ as $\bar{x}_q^r$, $\bar{y}_q^r$, $\hat{x}_q^r$, and $\hat{y}_q^r$, respectively. Let $\epsilon = \overline{V} - \underline{C} > 0$. We divide these $2T-1$ points into the following five groups:

(A1) For each $r \in [1, T]_{\mathbb{Z}} \setminus \{t\}$, we create the same point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as in group (A1) in the proof of Proposition 1. Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \mathrm{conv}(\mathcal{P})$. It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (17) at equality.

(A2) For each $r \in [1, t-2]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (A2) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \mathrm{conv}(\mathcal{P})$. Consider the case where $t-r-1 \notin \mathcal{S}$. In this case, $\hat{x}_t^r = \hat{y}_t^r = \hat{y}_{t-1}^r = 0$. In addition, $t-s-1 \neq r$ for all $s \in \mathcal{S}$, which implies that $\hat{y}_{t-s}^r - \hat{y}_{t-s-1}^r = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (17) at equality. Next, consider the case where $t-r-1 \in \mathcal{S}$. In this case, $\hat{x}_t^r = \overline{V} + (t-r-1)V$ and $\hat{y}_t^r = \hat{y}_{t-1}^r = 1$. In addition, $\hat{y}_{t-s}^r - \hat{y}_{t-s-1}^r = 1$ when $s = t-r-1$, and $\hat{y}_{t-s}^r - \hat{y}_{t-s-1}^r = 0$ when $s \neq t-r-1$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (17) at equality.

(A3) We create a point $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ as follows: If $\eta = 0$, then

$$\hat{x}_q^{t-1} = \begin{cases} 0, & \text{for } q \in [0, t-1]_{\mathbb{Z}}; \\ \overline{V}, & \text{for } q \in [t, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^{t-1} = \begin{cases} 0, \text{ for } q \in [0, t-1]_{\mathbb{Z}}; \\ 1, \text{ for } q \in [t, T]_{\mathbb{Z}}. \end{cases}$$

If $\eta = (\overline{C} - \overline{V})/V$, then

$$\hat{x}_q^{t-1} = \begin{cases} \underline{C}, \text{ for } q \in [1, t-1]_{\mathbb{Z}}; \\ 0, \text{ for } q \in [t, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^{t-1} = \begin{cases} 1, \text{ for } q \in [1, t-1]_{\mathbb{Z}}; \\ 0, \text{ for } q \in [t, T]_{\mathbb{Z}}. \end{cases}$$

If $\eta = L \in \mathcal{S}$, then

$$\hat{x}_q^{t-1} = \begin{cases} 0, \text{ for } q \in [1, t-L-1]_{\mathbb{Z}} \cup [t, T]_{\mathbb{Z}}; \\ \overline{V}, \text{ for } q \in [t-L, t-1]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^{t-1} = \begin{cases} 0, \text{ for } q \in [1, t-L-1]_{\mathbb{Z}} \cup [t, T]_{\mathbb{Z}}; \\ 1, \text{ for } q \in [t-L, t-1]_{\mathbb{Z}}. \end{cases}$$

We first consider the case where $\eta = 0$. It is easy to verify that $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ satisfies (2a)–(2f). Thus, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1}) \in \operatorname{conv}(\mathcal{P})$. In this case, $\hat{x}_t^{t-1} = \overline{V}$, $\hat{y}_t^{t-1} = 1$, $\hat{y}_{t-1}^{t-1} = 0$, and $\hat{y}_{t-s}^{t-1} - \hat{y}_{t-s-1}^{t-1} = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ satisfies (17) at equality. Next, we consider the case where $\eta = (\overline{C} - \overline{V})/V$. It is easy to verify that $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ satisfies (2a)–(2f). Thus, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1}) \in \operatorname{conv}(\mathcal{P})$. In this case, $\hat{x}_t^{t-1} = \hat{y}_t^{t-1} = 0$, $\overline{C} - \overline{V} - \eta V = 0$, and $\hat{y}_{t-s}^{t-1} = \hat{y}_{t-s-1}^{t-1} = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ satisfies (17) at equality. Next, we consider the case where $\eta = L \in \mathcal{S}$. In this case, for any $q \in [2, T]_{\mathbb{Z}}$, $\hat{y}_q^{t-1} - \hat{y}_{q-1}^{t-1} \leq 0$ if $q \neq t - L$, while $\hat{y}_q^{t-1} - \hat{y}_{q-1}^{t-1} = 1$ and $\hat{y}_k^{t-1} = 1$ for all $k \in [q, \min\{T, q + L - 1\}]_{\mathbb{Z}}$ if $q = t - L$. Thus, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ satisfies (2a). For any $q \in [2, T]_{\mathbb{Z}}$, $\hat{y}_{q-1}^{t-1} - \hat{y}_q^{t-1} \leq 0$ if $q \neq t$, while $\hat{y}_{q-1}^{t-1} - \hat{y}_q^{t-1} = 1$ and $\hat{y}_q^{t-1} = 0$ for all $k \in [q, \min\{T, q + \ell - 1\}]_{\mathbb{Z}}$ if $q = t$. Thus, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ satisfies (2b). It is easy to verify that $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ satisfies (2c)–(2f). Hence, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1}) \in \operatorname{conv}(\mathcal{P})$. Note that $\hat{x}_t^{t-1} = \hat{y}_t^{t-1} = 0$, $\hat{y}_{t-1}^{t-1} = 1$, $\hat{y}_{t-s}^{t-1} - \hat{y}_{t-s-1}^{t-1} = 0$ for all $s \in \mathcal{S} \setminus \{L\}$, $\hat{y}_{t-L}^{t-1} - \hat{y}_{t-L-1}^{t-1} = 1$, and $\overline{C} - \overline{V} - \eta V = \overline{C} - \overline{V} - LV$. Thus, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ satisfies (17) at equality.

(A4) We create the same point $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ as in group (A3) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) \in \operatorname{conv}(\mathcal{P})$. It is easy to verify that $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (17) at equality.

(A5) For each $r \in [t+1, T]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (A4) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \operatorname{conv}(\mathcal{P})$. It is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (17) at equality.

Table EC.5 shows a matrix with $2T - 1$ rows, where each row represents a point created by this process. This matrix can be transformed into the matrix in Table EC.6 via the following Gaussian elimination process:

(i) For each $r \in [1, T]_{\mathbb{Z}} \setminus \{t\}$, the point with index r in group (B1), denoted $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point with index r in group (A1), and $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point in group (A4).

Table EC.5 A matrix with the rows representing $2T-1$ points in $\text{conv}(\mathcal{P})$ that satisfy inequality (17) at equality

Group	Point	Index r	$\mathbf{x}$								$\mathbf{y}$							
			1	$\cdots$	$t-2$	$t-1$	t	$t+1$	$\cdots$	T	1	$\cdots$	$t-2$	$t-1$	t	$t+1$	$\cdots$	T
(A1)	$(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$	1	$\overline{C}-\epsilon$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	$\cdots$	1	1	1	1	$\cdots$	1
		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-2$	$\overline{C}$	$\cdots$	$\overline{C}-\epsilon$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	$\cdots$	1	1	1	1	$\cdots$	1
		$t-1$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}-\epsilon$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	$\cdots$	1	1	1	1	$\cdots$	1
		$t+1$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}-\epsilon$	$\cdots$	$\overline{C}$	1	$\cdots$	1	1	1	1	$\cdots$	1
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		T	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}-\epsilon$	1	$\cdots$	1	1	1	1	$\cdots$	1
(A2)	$(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$	1, $\vdots$, $t-2$	(See Note EC.5-1)								(See Note EC.5-1)							
(A3)		$t-1$	(See Note EC.5-2)								(See Note EC.5-2)							
(A4)		t	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	$\cdots$	1	1	1	1	$\cdots$	1
(A5)		$t+1$	0	$\cdots$	0	0	0	$\underline{C}$	$\cdots$	$\underline{C}$	0	$\cdots$	0	0	0	1	$\cdots$	1
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T	0	$\cdots$	0	0	0	0	$\cdots$	$\underline{C}$	0	$\cdots$	0	0	0	0	$\cdots$	1

Note EC.5-1: For $r \in [1, t-2]_{\mathbb{Z}}$, the $\mathbf{x}$ and $\mathbf{y}$ vectors in group (A2) are given as follows: $\hat{\mathbf{x}}^r = (\underbrace{\underline{C}, \ldots, \underline{C}}_{r \text{ terms}}, \underbrace{0, \ldots, 0}_{T-r \text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{1, \ldots, 1}_{r \text{ terms}}, \underbrace{0, \ldots, 0}_{T-r \text{ terms}})$ if $t-r-1 \notin \mathcal{S}$;

$\hat{\mathbf{x}}^r = (\underbrace{0, \ldots, 0}_{r \text{ terms}}, \underbrace{\overline{V}, \overline{V}+V, \overline{V}+2V, \ldots, \overline{V}+(t-r-1)V}_{t-r \text{ terms}}, \underbrace{\overline{V}+(t-r-1)V, \ldots, \overline{V}+(t-r-1)V}_{T-t \text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{0, \ldots, 0}_{r \text{ terms}}, \underbrace{1, \ldots, 1}_{T-r \text{ terms}})$ if $t-r-1 \in \mathcal{S}$.

Note EC.5-2: The $\mathbf{x}$ and $\mathbf{y}$ vectors in group (A3) are given as follows: $\hat{\mathbf{x}}^{t-1} = (\underbrace{0, \ldots, 0}_{t-1 \text{ terms}}, \underbrace{\overline{V}, \ldots, \overline{V}}_{T-t+1 \text{ terms}})$ and $\hat{\mathbf{y}}^{t-1} = (\underbrace{0, \ldots, 0}_{t-1 \text{ terms}}, \underbrace{1, \ldots, 1}_{T-t+1 \text{ terms}})$ if $\eta = 0$;

$\hat{\mathbf{x}}^{t-1} = (\underbrace{\underline{C}, \ldots, \underline{C}}_{t-1 \text{ terms}}, \underbrace{0, \ldots, 0}_{T-t+1 \text{ terms}})$ and $\hat{\mathbf{y}}^{t-1} = (\underbrace{1, \ldots, 1}_{t-1 \text{ terms}}, \underbrace{0, \ldots, 0}_{T-t+1 \text{ terms}})$ if $\eta = (\overline{C} - \overline{V})/V$; $\hat{\mathbf{x}}^{t-1} = (\underbrace{0, \ldots, 0}_{t-L-1 \text{ terms}}, \underbrace{\overline{V}, \ldots, \overline{V}}_{L \text{ terms}}, \underbrace{0, \ldots, 0}_{T-t+1 \text{ terms}})$ and $\hat{\mathbf{y}}^{t-1} = (\underbrace{0, \ldots, 0}_{t-L-1 \text{ terms}}, \underbrace{1, \ldots, 1}_{L \text{ terms}}, \underbrace{0, \ldots, 0}_{T-t+1 \text{ terms}})$ if $\eta = L \in \mathcal{S}$.

Table EC.6 Lower triangular matrix obtained from Table EC.5 via Gaussian elimination

Group	Point	Index r	$\mathbf{x}$								$\mathbf{y}$							
			1	$\cdots$	$t-2$	$t-1$	t	$t+1$	$\cdots$	T	1	$\cdots$	$t-2$	$t-1$	t	$t+1$	$\cdots$	T
(B1)	$(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$	1	$-\epsilon$	$\cdots$	0	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-2$	0	$\cdots$	$-\epsilon$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	0	$\cdots$	0
		$t-1$	0	$\cdots$	0	$-\epsilon$	0	0	$\cdots$	0	0	$\cdots$	0	0	0	0	$\cdots$	0
		$t+1$	0	$\cdots$	0	0	0	$-\epsilon$	$\cdots$	0	0	$\cdots$	0	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		T	0	$\cdots$	0	0	0	0	$\cdots$	$-\epsilon$	0	$\cdots$	0	0	0	0	$\cdots$	0
(B2)	$(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$	1	(Omitted)								1	$\cdots$	0	0	0	0	$\cdots$	0
		$\vdots$									$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-2$									1	$\cdots$	1	0	0	0	$\cdots$	0
(B3)		$t-1$	(Omitted)								(See Note EC.6-1)							
(B4)		t	(Omitted)								1	$\cdots$	1	1	1	0	$\cdots$	0
(B5)		$t+1$	(Omitted)								0	$\cdots$	0	0	0	1	$\cdots$	0
		$\vdots$									$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T									0	$\cdots$	0	0	0	0	$\cdots$	1

Note EC.6-1: The $\mathbf{y}$ vector in group (B3) is given as follows:

$\underline{\hat{\mathbf{y}}}^{t-1} = (\underbrace{-1,\ldots,-1}_{t-1 \text{ terms}}, \underbrace{0,\ldots,0}_{T-t+1 \text{ terms}})$ if $\eta = 0$; $\underline{\hat{\mathbf{y}}}^{t-1} = (\underbrace{1,\ldots,1}_{t-1 \text{ terms}}, \underbrace{0,\ldots,0}_{T-t+1 \text{ terms}})$ if $\eta = (\overline{C} - \overline{V})/V$; $\underline{\hat{\mathbf{y}}}^{t-1} = (\underbrace{0,\ldots,0}_{t-L-1 \text{ terms}}, \underbrace{1,\ldots,1}_{L \text{ terms}}, \underbrace{0,\ldots,0}_{T-t+1 \text{ terms}})$ if $\eta = L \in \mathcal{S}$.

(ii) For each $r \in [1, t-2]_{\mathbb{Z}}$, the point with index r in group (B2), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $t-r-1 \notin \mathcal{S}$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) - (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $t-r-1 \in \mathcal{S}$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (A2), and $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point in group (A4).

(iii) The point in group (B3), denoted $(\underline{\hat{\mathbf{x}}}^{t-1}, \underline{\hat{\mathbf{y}}}^{t-1})$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^{t-1}, \underline{\hat{\mathbf{y}}}^{t-1}) = (\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1}) - (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ if $\eta = 0$, and setting $(\underline{\hat{\mathbf{x}}}^{t-1}, \underline{\hat{\mathbf{y}}}^{t-1}) = (\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ if $\eta = (\overline{C} - \overline{V})/V$ or $\eta = L \in \mathcal{S}$. Here, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ is the point in group (A3), and $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point in group (A4).

(iv) The point in group (B4), denoted $(\underline{\hat{\mathbf{x}}}^t, \underline{\hat{\mathbf{y}}}^t)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^t, \underline{\hat{\mathbf{y}}}^t) = (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) - (\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$. Here, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point in group (A4), and $(\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$ is the point with index $t+1$ in group (A5).

(v) For each $r \in [t+1, T]_{\mathbb{Z}}$, the point with index r in group (B5), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ if $r \neq T$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $r = T$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ and $(\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ are the points with indices r and $r+1$, respectively, in group (A5).

The matrix shown in Table EC.6 is lower triangular; that is, the position of the last nonzero component of a row of the matrix is greater than the position of the last nonzero component of the previous row. This implies that the $2T-1$ points in groups (A1)–(A5) are linearly independent. Therefore, inequality (17) is facet-defining for $\text{conv}(\mathcal{P})$.

Next, we show that inequality (18) is valid for $\text{conv}(\mathcal{P})$ and is facet-defining for $\text{conv}(\mathcal{P})$ when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L \in \mathcal{S}$. Note that this proof can also be used to prove the validity and facet-defining of inequality (18) in Proposition 6. Denote $x'_t = x_{T-t+1}$ and $y'_t = y_{T-t+1}$ for $t \in [1, T]_{\mathbb{Z}}$. Because inequality (17) is valid for $\text{conv}(\mathcal{P})$ and is facet-defining for $\text{conv}(\mathcal{P})$ when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L \in \mathcal{S}$ for any $t \in [s_{\max}+2, T]_{\mathbb{Z}}$, the inequality

$$x'_{T-t+1} \leq (\overline{V} + \eta V) y'_{T-t+1} + (\overline{C} - \overline{V} - \eta V) y'_{T-t+2} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y'_{T-t+s+1} - y'_{T-t+s+2})$$

is valid for $\text{conv}(\mathcal{P}')$ and is facet-defining for $\text{conv}(\mathcal{P}')$ when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L \in \mathcal{S}$ for any $t \in [s_{\max}+2, T]_{\mathbb{Z}}$. Let $t' = T - t + 1$. Then, the inequality

$$x'_{t'} \leq (\overline{V} + \eta V) y'_{t'} + (\overline{C} - \overline{V} - \eta V) y'_{t'+1} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y'_{t'+s} - y'_{t'+s+1})$$

is valid for $\text{conv}(\mathcal{P}')$ and is facet-defining for $\text{conv}(\mathcal{P}')$ when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L \in \mathcal{S}$ for any $t' \in [1, T - s_{\max} - 1]_{\mathbb{Z}}$. Hence, by Lemma 2, inequality (18) is valid for $\text{conv}(\mathcal{P})$ and is facet-defining for $\text{conv}(\mathcal{P})$ when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L \in \mathcal{S}$ for any $t \in [1, T - s_{\max} - 1]_{\mathbb{Z}}$. $\square$

A.8. Proof of Proposition 6

Proposition 6. *Consider any integers α, β, and $s_{\max}$ such that (a) $L+1 \le s_{\max} \le \min\{T-2, \lfloor(\overline{C}-\overline{V})/V\rfloor\}$, (b) $1 \le \alpha < \beta \le s_{\max}$, and (c) $\beta = \alpha+1$ or $s_{\max} \le L+\alpha$. Let $\mathcal{S} = [1,\alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. For any $t \in [s_{\max}+2, T]_{\mathbb{Z}}$, inequality (17) is valid for* $\text{conv}(\mathcal{P})$. *For any $t \in [1, T-s_{\max}-1]_{\mathbb{Z}}$, inequality (18) is valid for* $\text{conv}(\mathcal{P})$. *Furthermore, (17) and (18) are facet-defining for* $\text{conv}(\mathcal{P})$ *when $\eta \in \{0, (\overline{C}-\overline{V})/V\}$ or $\eta = L \in \mathcal{S}$.*

Proof. Consider any $t \in [s_{\max}+2, T]_{\mathbb{Z}}$. To prove that the linear inequality (17) is valid for $\text{conv}(\mathcal{P})$ when $\mathcal{S} = [1,\alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$, it suffices to show that (17) is valid for $\mathcal{P}$ when $\mathcal{S} = [1,\alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. Consider any element $(\mathbf{x}, \mathbf{y})$ of $\mathcal{P}$. We show that $(\mathbf{x}, \mathbf{y})$ satisfies (17) when $\mathcal{S} = [1,\alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. We divide the analysis into four cases.

Case 1: $y_t = 0$ and $y_{t-s} - y_{t-s-1} \le 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [1, \lfloor(\overline{C}-\overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \ge 0$ for all $s \in \mathcal{S}$. Because $\eta \le (\overline{C}-\overline{V})/V$, we have $\overline{C} - \overline{V} - \eta V \ge 0$. Thus, in this case, the right-hand side of inequality (17) is nonnegative. Because $y_t = 0$, by (2d), $x_t = 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (17).

Case 2: $y_t = 0$ and $y_{t-s} - y_{t-s-1} > 0$ for some $s \in \mathcal{S}$. Let $\tilde{\mathcal{S}} = \{\sigma \in \mathcal{S} : y_{t-\sigma} - y_{t-\sigma-1} > 0\}$ and $v = |\tilde{\mathcal{S}}|$. Then, $v \ge 1$. Denote $\tilde{\mathcal{S}} = \{\sigma_1, \sigma_2, \ldots, \sigma_v\}$, where $\sigma_1 < \sigma_2 < \cdots < \sigma_v$. Note that $y_{t-\sigma_j-1} = 0$ and $y_{t-\sigma_j} = 1$ for $j = 1, \ldots, v$. Denote $\sigma_0 = -1$. Then, for each $j = 1, \ldots, v$, there exists $\sigma'_j \in [\sigma_{j-1}+1, \sigma_j - 1]_{\mathbb{Z}}$ such that $y_{t-\sigma'_j-1} = 1$ and $y_{t-\sigma'_j} = 0$. Thus,

$$0 \le \sigma'_1 < \sigma_1 < \sigma'_2 < \sigma_2 < \cdots < \sigma'_v < \sigma_v \le s_{\max}.$$

Because $y_{t-\sigma_v} - y_{t-\sigma_v-1} = 1$ and $t - \sigma_v \in [2, T]_{\mathbb{Z}}$, by (2a), $y_k = 1$ for all $k \in [t-\sigma_v, \min\{T, t-\sigma_v+L-1\}]_{\mathbb{Z}}$, which implies that $t - \sigma'_j \ge t - \sigma_v + L$ for $j = 1, \ldots, v$. Hence, for $j = 1, \ldots, v$, we have $\sigma'_j \le \sigma_v - L$, which implies that

$$\sigma'_j \le s_{\max} - L. \tag{EC.9}$$

If $\beta = \alpha + 1$, then $\mathcal{S} = [1, s_{\max}]_{\mathbb{Z}}$, which implies that $\sigma'_j \in \mathcal{S}$ for $j = 2, \ldots, v$. If $\beta \ne \alpha + 1$, then condition (c) of Proposition 6 implies that $s_{\max} \le L + \alpha$, which, by (EC.9), implies that $\sigma'_j \le \alpha$ for $j = 1, \ldots, v$. Because $\sigma'_2 > \sigma_1 \ge 1$, we have $\sigma'_j \in \mathcal{S}$ for $j = 2, \ldots, v$. Thus, in both cases, $\sigma'_j \in \mathcal{S}$ for $j = 2, \ldots, v$. Because $y_{t-\sigma_1} - y_{t-\sigma_1-1} = 1$ and $t - \sigma_1 \in [2, T]_{\mathbb{Z}}$, by (2a), $y_k = 1$ for all $k \in [t-\sigma_1, \min\{T, t-\sigma_1+L-1\}]_{\mathbb{Z}}$. Because $y_t = 0$, , this implies that $t \ge t - \sigma_1 + L$, or equivalent, $\sigma_1 \ge L$. Because $\eta \le L$, we have

$$\eta \le \sigma_1. \tag{EC.10}$$

Because $y_t = 0$, by (2d), $x_t = 0$. Hence, the left-hand side of inequality (17) is 0. Because $s_{\max} \le \lfloor(\overline{C}-\overline{V})/V\rfloor$, we have $\overline{C} - \overline{V} - sV \ge 0$ for all $s \in \mathcal{S}$. Note that if $y_{t-1} = 0$, then $\sigma'_1 \ge 1$ and $\sigma'_1 \in \mathcal{S}$;

if $y_{t-1}=1$, then $\sigma'_1=0$ and $\sigma'_1 \notin \mathcal{S}$. Note that $\{\sigma'_2,\ldots,\sigma'_v\} \subseteq \mathcal{S}\setminus\tilde{\mathcal{S}}$ and $y_{t-s}-y_{t-s-1}\le 0$ for all $s\in\mathcal{S}\setminus\tilde{\mathcal{S}}$. Thus, $\sum_{s\in\mathcal{S}\setminus\tilde{\mathcal{S}}}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1}) \le \sum_{j=2}^{v}(\overline{C}-\overline{V}-\sigma'_jV)(y_{t-\sigma'_j}-y_{t-\sigma'_j-1})+(\overline{C}-\overline{V}-\sigma'_1V)(y_{t-\sigma'_1}-y_{t-\sigma'_1-1})(1-y_{t-1})$. Hence, the right-hand side of inequality (17) is

$$
\begin{aligned}
&(\overline{V}+\eta V)y_t+(\overline{C}-\overline{V}-\eta V)y_{t-1}-\sum_{s\in\mathcal{S}}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1})\\
&=(\overline{C}-\overline{V}-\eta V)y_{t-1}-\sum_{s\in\tilde{\mathcal{S}}}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1})-\sum_{s\in\mathcal{S}\setminus\tilde{\mathcal{S}}}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1})\\
&\ge(\overline{C}-\overline{V}-\eta V)y_{t-1}-\sum_{j=1}^{v}(\overline{C}-\overline{V}-\sigma_jV)(y_{t-\sigma_j}-y_{t-\sigma_j-1})-\sum_{j=2}^{v}(\overline{C}-\overline{V}-\sigma'_jV)(y_{t-\sigma'_j}-y_{t-\sigma'_j-1})\\
&\quad-(\overline{C}-\overline{V}-\sigma'_1V)(y_{t-\sigma'_1}-y_{t-\sigma'_1-1})(1-y_{t-1})\\
&=(\overline{C}-\overline{V}-\eta V)y_{t-1}-(\overline{C}-\overline{V}-\sigma'_1V)y_{t-1}-(\overline{C}-\overline{V}-\sigma_1V)+(\overline{C}-\overline{V}-\sigma'_1V)\\
&\quad-\sum_{j=2}^{v}(\overline{C}-\overline{V}-\sigma_jV)+\sum_{j=2}^{v}(\overline{C}-\overline{V}-\sigma'_jV)\\
&=(\sigma'_1-\eta)Vy_{t-1}+(\sigma_1-\sigma'_1)V+\sum_{j=2}^{v}(\sigma_j-\sigma'_j)V\\
&\ge(\sigma'_1-\eta)Vy_{t-1}+(\sigma_1-\sigma'_1)V\\
&\ge 0.
\end{aligned}
$$

where the last inequality follows from $y_{t-1}\in\{0,1\}$, $\sigma_1>\sigma'_1$, and (EC.10). Therefore, in this case, $(\mathbf{x},\mathbf{y})$ satisfies (17).

Case 3: $y_t=1$ and $y_{t-s}-y_{t-s-1}\le 0$ for all $s\in\mathcal{S}$. Because $\mathcal{S}\subseteq[0,\lfloor(\overline{C}-\overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C}-\overline{V}-sV\ge 0$ for all $s\in\mathcal{S}$. Thus, $\sum_{s\in\mathcal{S}}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1})\le 0$. Because $y_t=1$, the right-hand side of inequality (17) is at least $\overline{C}$ when $y_{t-1}=1$ and is at least $\overline{V}+\eta V\ge\overline{V}$ when $y_{t-1}=0$ (as $\eta\ge 0$). By (2d), $x_t\le\overline{C}$. If $y_{t-1}=0$, then by (2d) and (2e), $x_{t-1}=0$ and $x_t-x_{t-1}\le Vy_{t-1}+\overline{V}(1-y_{t-1})$, which imply that $x_t\le\overline{V}$. Hence, x_t is at most $\overline{C}$, and is at most $\overline{V}$ when $y_{t-1}=0$. Therefore, in this case, $(\mathbf{x},\mathbf{y})$ satisfies (17).

Case 4: $y_t=1$ and $y_{t-s}-y_{t-s-1}>0$ for some $s\in\mathcal{S}$. Let $\tilde{\mathcal{S}}=\{\sigma\in\mathcal{S}:y_{t-\sigma}-y_{t-\sigma-1}>0\}$ and $v=|\tilde{\mathcal{S}}|$. Then, $v\ge 1$. Denote $\tilde{\mathcal{S}}=\{\sigma_1,\sigma_2,\ldots,\sigma_v\}$, where $\sigma_1<\sigma_2<\cdots<\sigma_v$. Note that $y_{t-\sigma_j-1}=0$ and $y_{t-\sigma_j}=1$ for $j=1,\ldots,v$. Then, for each $j=2,\ldots,v$, there exists $\sigma'_j\in[\sigma_{j-1}+1,\sigma_j-1]_{\mathbb{Z}}$ such that $y_{t-\sigma'_j-1}=1$ and $y_{t-\sigma'_j}=0$. Thus,

$$1\le\sigma_1<\sigma'_2<\sigma_2<\cdots<\sigma'_v<\sigma_v\le s_{\max}.$$

Because $y_{t-\sigma_v}-y_{t-\sigma_v-1}=1$ and $t-\sigma_v\in[2,T]_{\mathbb{Z}}$, by (2a), $y_k=1$ for all $k\in[t-\sigma_v,\min\{T,t-\sigma_v+L-1\}]_{\mathbb{Z}}$, which implies that $t-\sigma'_j\ge t-\sigma_v+L$ for $j=2,\ldots,v$. Hence, for $j=2,\ldots,v$, we have $\sigma'_j\le\sigma_v-L$, which implies that

$$\sigma'_j\le s_{\max}-L. \tag{EC.11}$$

If $\beta = \alpha + 1$, then $\mathcal{S} = [1, s_{\max}]_{\mathbb{Z}}$, which implies that $\sigma'_j \in \mathcal{S}$ for $j = 2, \ldots, v$. If $\beta \neq \alpha + 1$, then condition (c) of Proposition 6 implies that $s_{\max} \leq L + \alpha$, which, by (EC.11), implies that $1 < \sigma'_j \leq \alpha$ for $j = 2, \ldots, v$. Thus, in both cases, $\sigma'_j \in \mathcal{S}$ for $j = 2, \ldots, v$. If $y_{t-1} = 0$, by (2d) and (2e), then $x_{t-1} = 0$ and $x_t - x_{t-1} \leq V y_{t-1} + \overline{V}(1 - y_{t-1}) = \overline{V}$, which implies that $x_t \leq \overline{V}$. In addition, there exists $\sigma'_1 \in [1, \sigma_1 - 1]_{\mathbb{Z}}$ such that $y_{t-\sigma'_1} = 0$, $y_{t-\sigma'_1-1} = 1$, and $\sigma'_1 \in \mathcal{S}$. If $y_{t-1} = 1$, then we have $y_k = 1$ for all $k \in [t - \sigma_1, t]_{\mathbb{Z}}$. Because $y_{t-\sigma_1-1} = 0$, by (2d) and (2e), we have $x_{t-\sigma_1-1} = 0$ and

$$\sum_{\tau=t-\sigma_1}^{t} (x_\tau - x_{\tau-1}) \leq \sum_{\tau=t-\sigma_1}^{t} V y_{\tau-1} + \sum_{\tau=t-\sigma_1}^{t} \overline{V}(1 - y_{\tau-1}),$$

which implies that

$$x_t - x_{t-\sigma_1-1} \leq \sum_{\tau=t-\sigma_1}^{t} V y_{\tau-1} + \sum_{\tau=t-\sigma_1}^{t} \overline{V}(1 - y_{\tau-1}) = \sigma_1 V + \overline{V}.$$

Thus, the left-hand side of inequality (17) is at most $\sigma_1 V + \overline{V}$. Because $\mathcal{S} \subseteq [0, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Note that $\{\sigma'_2, \ldots, \sigma'_v\} \subseteq \mathcal{S} \setminus \tilde{\mathcal{S}}$. Thus, $\sum_{s \in \mathcal{S} \setminus \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \leq \sum_{j=2}^{v} (\overline{C} - \overline{V} - \sigma'_j V)(y_{t-\sigma'_j} - y_{t-\sigma'_j-1})$. Hence, when $y_{t-1} = 0$, the right-hand side of inequality (17) is

$$\begin{aligned}
&(\overline{V} + \eta V) y_t + (\overline{C} - \overline{V} - \eta V) y_{t-1} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \\
&= \overline{V} + \eta V - \sum_{s \in \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) - \sum_{s \in \mathcal{S} \setminus \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \\
&\geq \overline{V} + \eta V - \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma_j V)(y_{t-\sigma_j} - y_{t-\sigma_j-1}) - \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma'_j V)(y_{t-\sigma'_j} - y_{t-\sigma'_j-1}) \\
&= \overline{V} + \eta V + \sum_{j=1}^{v} (\sigma_j - \sigma'_j) V \\
&\geq \overline{V} + \eta V \\
&\geq \overline{V}.
\end{aligned}$$

When $y_{t-1} = 1$, the right-hand side of inequality (17) is

$$\begin{aligned}
&(\overline{V} + \eta V) y_t + (\overline{C} - \overline{V} - \eta V) y_{t-1} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \\
&= \overline{C} - \sum_{s \in \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) - \sum_{s \in \mathcal{S} \setminus \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \\
&\geq \overline{C} - \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma_j V)(y_{t-\sigma_j} - y_{t-\sigma_j-1}) - \sum_{j=2}^{v} (\overline{C} - \overline{V} - \sigma'_j V)(y_{t-\sigma'_j} - y_{t-\sigma'_j-1}) \\
&= \overline{C} - (\overline{C} - \overline{V} - \sigma_1 V) - \sum_{j=2}^{v} (\overline{C} - \overline{V} - \sigma_j V) + \sum_{j=2}^{v} (\overline{C} - \overline{V} - \sigma'_j V) \\
&= \sigma_1 V + \overline{V} + \sum_{j=2}^{v} (\sigma_j - \sigma'_j) V \\
&\geq \sigma_1 V + \overline{V}.
\end{aligned}$$

Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (17).

Summarizing Cases 1–4, we conclude that $(\mathbf{x}, \mathbf{y})$ satisfies (17). Hence, (17) is valid for $\text{conv}(\mathcal{P})$.

It is easy to verify that the proof of facet-defining of inequality (17) in the proof of Proposition 5 remains valid when $\mathcal{S} = [1, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. Therefore, inequality (17) is facet-defining under the conditions stated in Proposition 6.

It is also easy to verify that the proof of validity and facet-defining of inequality (18) in the proof of Proposition 5 remains valid when $\mathcal{S} = [1, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. Therefore, inequality (18) is valid and facet-defining for $\text{conv}(\mathcal{P})$ under the conditions stated in Proposition 6. $\square$

A.9. Proof of Proposition 7

Proposition 7. *For any given point* $(\mathbf{x},\mathbf{y}) \in \mathbb{R}_+^{2T}$*, the most violated inequalities* (13)–(14)*,* (15)–(16)*, and* (17)–(18) *in Propositions 1, 3, and 5, respectively, can be determined in* $O(T)$ *time if such violated inequalities exist.*

Proof. Let $\hat{\eta}$ and $a_1, \ldots, a_6$ be any real numbers such that $\hat{\eta} \geq 0$. Let $\check{s}$ and $\hat{s}$ be any integers such that $0 \leq \check{s} \leq \hat{s} \leq \min\{T-2, \lfloor(\overline{C}-\overline{V})/V\rfloor\}$. Let $\check{t} = 1$ if $a_1 = a_2 = 0$, and let $\check{t} = 2$ otherwise. Let $\hat{t} = T$ if $a_5 = a_6 = 0$, and let $\hat{t} = T-1$ otherwise.

(i) Consider the following family of inequalities:

$$x_t \leq (a_1 + a_2\eta)y_{t-1} + (a_3 + a_4\eta)y_t + (a_5 + a_6\eta)y_{t+1} - \sum_{s \in \mathcal{S}}(\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}), \tag{EC.12}$$

where $\eta \in [0, \hat{\eta}]$, $\mathcal{S} \subseteq [\check{s}, \hat{s}]_{\mathbb{Z}}$, $t \in [\check{t}, \hat{t}]_{\mathbb{Z}}$, and $t \geq s+2$ for all $s \in \mathcal{S}$. Note that inequality family (13) in Proposition 1, inequality family (15) in Proposition 3, and inequality family (17) in Proposition 5 are special cases of this inequality family. Consider any given point $(\mathbf{x},\mathbf{y}) \in \mathbb{R}_+^{2T}$. We show that the set $\mathcal{S}$, the real number η, and the integer t corresponding to a most violated inequality (EC.12) can be determined in $O(T)$ time.

For any integer $t \leq T$, let

$$\theta(t) = \sum_{\tau=2}^{t} \max\{y_\tau - y_{\tau-1}, 0\}.$$

Then, for any $t \in [2, T]_{\mathbb{Z}}$,

$$\sum_{\substack{s \in [\check{s}, \hat{s}]_{\mathbb{Z}} \\ t-s \geq 2}} \max\{y_{t-s} - y_{t-s-1}, 0\} = \theta(t - \check{s}) - \theta(t - \hat{s} - 1) \tag{EC.13}$$

and

$$\sum_{\substack{s \in [\check{s}+1, \hat{s}+1]_{\mathbb{Z}} \\ t-s \geq 2}} \max\{y_{t-s} - y_{t-s-1}, 0\} = \theta(t - \check{s} - 1) - \theta(t - \hat{s} - 2). \tag{EC.14}$$

Furthermore, for any $t \in [2, T]_{\mathbb{Z}}$,

$$\begin{aligned}
&\sum_{\substack{s \in [\check{s}, \hat{s}]_{\mathbb{Z}} \\ t-s \geq 2}} s \max\{y_{t-s} - y_{t-s-1}, 0\} - \sum_{\substack{s \in [\check{s}+1, \hat{s}+1]_{\mathbb{Z}} \\ t-s \geq 2}} s \max\{y_{t-s} - y_{t-s-1}, 0\} \\
&\quad = \begin{cases} \check{s} \max\{y_{t-\check{s}} - y_{t-\check{s}-1}, 0\} - (\hat{s}+1) \max\{y_{t-\hat{s}-1} - y_{t-\hat{s}-2}, 0\}, & \text{if } 2 \leq t - \hat{s} - 1; \\ \check{s} \max\{y_{t-\check{s}} - y_{t-\check{s}-1}, 0\}, & \text{if } t - \hat{s} - 1 < 2 \leq t - \check{s}; \\ 0, & \text{if } t - \check{s} < 2. \end{cases}
\end{aligned}$$

Note that

$$\theta(t - \check{s}) - \theta(t - \check{s} - 1) = \begin{cases} \max\{y_{t-\check{s}} - y_{t-\check{s}-1}, 0\}, & \text{if } t - \check{s} \geq 2; \\ 0, & \text{if } t - \check{s} < 2. \end{cases}$$

and

$$\theta(t-\hat{s}-1)-\theta(t-\hat{s}-2)=\begin{cases}\max\{y_{t-\hat{s}-1}-y_{t-\hat{s}-2},0\}, & \text{if } t-\hat{s}-1\geq 2;\\ 0, & \text{if } t-\hat{s}-1<2.\end{cases}$$

Hence, for any $t\in[2,T]_{\mathbb{Z}}$,

$$\begin{aligned}&\sum_{\substack{s\in[\check{s},\hat{s}]_{\mathbb{Z}}\\ t-s\geq 2}} s\max\{y_{t-s}-y_{t-s-1},0\}-\sum_{\substack{s\in[\check{s}+1,\hat{s}+1]_{\mathbb{Z}}\\ t-s\geq 2}} s\max\{y_{t-s}-y_{t-s-1},0\}\\ &\qquad=\check{s}[\theta(t-\check{s})-\theta(t-\check{s}-1)]-(\hat{s}+1)[\theta(t-\hat{s}-1)-\theta(t-\hat{s}-2)].\end{aligned}\tag{EC.15}$$

For any $\eta\in[0,\hat{\eta}]$, $\mathcal{S}\subseteq[\check{s},\hat{s}]_{\mathbb{Z}}$, and $t\in[\check{t},\hat{t}]_{\mathbb{Z}}$ such that $t\geq s+2\ \forall s\in\mathcal{S}$, let

$$\tilde{v}(\eta,\mathcal{S},t)=x_t-(a_1+a_2\eta)y_{t-1}-(a_3+a_4\eta)y_t-(a_5+a_6\eta)y_{t+1}+\sum_{s\in\mathcal{S}}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1}).$$

If $\tilde{v}(\eta,\mathcal{S},t)>0$, then $\tilde{v}(\eta,\mathcal{S},t)$ is the amount of violation of inequality (EC.12). If $\tilde{v}(\eta,\mathcal{S},t)\leq 0$, then there is no violation of inequality (EC.12). For any $\eta\in[0,\hat{\eta}]$ and $t\in[\check{t},\hat{t}]_{\mathbb{Z}}$, let

$$v(\eta,t)=\max_{\mathcal{S}\subseteq[\check{s},\min\{\hat{s},t-2\}]_{\mathbb{Z}}}\{\tilde{v}(\eta,\mathcal{S},t)\}.$$

If $v(\eta,t)>0$, then $v(\eta,t)$ is the largest possible violation of inequality (EC.12) for this combination of η and t. If $v(\eta,t)\leq 0$, then the largest possible violation of inequality (EC.12) is zero for this combination of η and t.

Note that $\overline{C}-\overline{V}-sV\geq 0$ for any $s\in[\check{s},\hat{s}]_{\mathbb{Z}}$. Thus, for any given $\eta\in[0,\hat{\eta}]$ and $t\in[\check{t},\hat{t}]_{\mathbb{Z}}$, $\tilde{v}(\eta,\mathcal{S},t)$ is maximized when $\mathcal{S}$ contains all $s\in[\check{s},\min\{\hat{s},t-2\}]_{\mathbb{Z}}$ such that $y_{t-s}-y_{t-s-1}>0$ (if any). If it does not exist any $s\in[\check{s},\min\{\hat{s},t-2\}]_{\mathbb{Z}}$ such that $y_{t-s}-y_{t-s-1}>0$, then $\tilde{v}(\eta,\mathcal{S},t)$ is maximized when $\mathcal{S}=\emptyset$, and $v(\eta,t)=x_t-(a_1+a_2\eta)y_{t-1}-(a_3+a_4\eta)y_t-(a_5+a_6\eta)y_{t+1}$. Hence, for any $\eta\in[0,\hat{\eta}]$ and $t\in[\check{t},\hat{t}]_{\mathbb{Z}}$,

$$\begin{aligned}v(\eta,t)=\;&x_t-(a_1+a_2\eta)y_{t-1}-(a_3+a_4\eta)y_t-(a_5+a_6\eta)y_{t+1}\\ &+(\overline{C}-\overline{V})\sum_{\substack{s\in[\check{s},\hat{s}]_{\mathbb{Z}}\\ t-s\geq 2}}\max\{y_{t-s}-y_{t-s-1},0\}-V\sum_{\substack{s\in[\check{s},\hat{s}]_{\mathbb{Z}}\\ t-s\geq 2}} s\max\{y_{t-s}-y_{t-s-1},0\}.\end{aligned}$$

When $t=\check{t}$, we have

$$v(\eta,\check{t})=\begin{cases}x_1-(a_3+a_4\eta)y_1-(a_5+a_6\eta)y_2, & \text{if } \check{t}=1;\\ x_2-(a_1+a_2\eta)y_1-(a_3+a_4\eta)y_2-(a_5+a_6\eta)y_3 & \\ \qquad\qquad+(\overline{C}-\overline{V})\max\{y_2-y_1,0\}, & \text{if } \check{t}=2 \text{ and } \check{s}=0;\\ x_2-(a_1+a_2\eta)y_1-(a_3+a_4\eta)y_2-(a_5+a_6\eta)y_3, & \text{otherwise.}\end{cases}$$

For any $\eta \in [0,\hat{\eta}]$ and $t \in [\check{t}+1,\hat{t}]_{\mathbb{Z}}$,

$$\begin{aligned} v(\eta,t) - v(\eta,t-1) = (x_t - x_{t-1}) & \\ - (a_1 + a_2\eta)(y_{t-1} - y_{t-2}) - (a_3 + a_4\eta)(y_t - y_{t-1}) - (a_5 + a_6\eta)(y_{t+1} - y_t) & \\ + (\overline{C} - \overline{V}) \left[\sum_{\substack{s\in[\check{s},\hat{s}]_{\mathbb{Z}} \\ t-s\geq 2}} \max\{y_{t-s} - y_{t-s-1}, 0\} - \sum_{\substack{s\in[\check{s}+1,\hat{s}+1]_{\mathbb{Z}} \\ t-s\geq 2}} \max\{y_{t-s} - y_{t-s-1}, 0\} \right] & \\ - V \left[\sum_{\substack{s\in[\check{s},\hat{s}]_{\mathbb{Z}} \\ t-s\geq 2}} s \max\{y_{t-s} - y_{t-s-1}, 0\} - \sum_{\substack{s\in[\check{s}+1,\hat{s}+1]_{\mathbb{Z}} \\ t-s\geq 2}} (s-1) \max\{y_{t-s} - y_{t-s-1}, 0\} \right] &, \end{aligned}$$

which implies that

$$\begin{aligned} v(\eta,t) = v(\eta,t-1) + (x_t - x_{t-1}) & \\ - (a_1 + a_2\eta)(y_{t-1} - y_{t-2}) - (a_3 + a_4\eta)(y_t - y_{t-1}) - (a_5 + a_6\eta)(y_{t+1} - y_t) & \\ + (\overline{C} - \overline{V}) \left[\sum_{\substack{s\in[\check{s},\hat{s}]_{\mathbb{Z}} \\ t-s\geq 2}} \max\{y_{t-s} - y_{t-s-1}, 0\} - \sum_{\substack{s\in[\check{s}+1,\hat{s}+1]_{\mathbb{Z}} \\ t-s\geq 2}} \max\{y_{t-s} - y_{t-s-1}, 0\} \right] & \\ - V \sum_{\substack{s\in[\check{s}+1,\hat{s}+1]_{\mathbb{Z}} \\ t-s\geq 2}} \max\{y_{t-s} - y_{t-s-1}, 0\} & \\ - V \left[\sum_{\substack{s\in[\check{s},\hat{s}]_{\mathbb{Z}} \\ t-s\geq 2}} s \max\{y_{t-s} - y_{t-s-1}, 0\} - \sum_{\substack{s\in[\check{s}+1,\hat{s}+1]_{\mathbb{Z}} \\ t-s\geq 2}} s \max\{y_{t-s} - y_{t-s-1}, 0\} \right] &. \end{aligned}$$

Thus, by (EC.13), (EC.14), and (EC.15),

$$\begin{aligned} v(\eta,t) = v(\eta,t-1) + (x_t - x_{t-1}) & \\ - (a_1 + a_2\eta)(y_{t-1} - y_{t-2}) - (a_3 + a_4\eta)(y_t - y_{t-1}) - (a_5 + a_6\eta)(y_{t+1} - y_t) & \\ + (\overline{C} - \overline{V})[\theta(t-\check{s}) - \theta(t-\hat{s}-1) - \theta(t-\check{s}-1) + \theta(t-\hat{s}-2)] & \\ - V[\check{s}\theta(t-\check{s}) - (\check{s}-1)\theta(t-\check{s}-1) - (\hat{s}+1)\theta(t-\hat{s}-1) + \hat{s}\theta(t-\hat{s}-2)] & \end{aligned} \tag{EC.16}$$

for any $\eta \in [0,\hat{\eta}]$ and $t \in [\check{t}+1,\hat{t}]_{\mathbb{Z}}$. Note that $\tilde{v}(\eta,\mathcal{S},t)$ is linear in η. Thus, for any given t, $v(\eta,t)$ is maximized when $\eta = 0$ or $\eta = \hat{\eta}$. That is, the largest possible value of $v(\eta,t)$ is equal to $v(0,t)$ if $a_2y_{t-1} + a_4y_t + a_6y_{t+1} \geq 0$, and the largest possible value of $v(\eta,t)$ is equal to $v(\hat{\eta},t)$ if $a_2y_{t-1} + a_4y_t + a_6y_{t+1} < 0$. Hence, to determine the η and t values corresponding to the largest violation of inequality (EC.12), it suffices to determine $v(0,\check{t}), v(0,\check{t}+1),\ldots,v(0,\hat{t})$ and $v(\hat{\eta},\check{t}), v(\hat{\eta},\check{t}+1),\ldots,v(\hat{\eta},\hat{t})$. Algorithm 1 performs this computation.

In Algorithm 1, step 1 sets $\theta(t)$ to zero when $t \leq 1$. Steps 2–4 determine the $\theta(t)$ values recursively for $t = 2,3,\ldots,\hat{t}$. These steps require $O(T)$ time. Steps 5–16 consider the case $\eta = 0$ and

Algorithm 1 Determination of a most violated inequality (EC.12) for any given $(\mathbf{x},\mathbf{y}) \in \mathbb{R}_+^{2T}$

1: $\theta(t) \leftarrow 0 \;\forall\, t \in [-\hat{s},1]_{\mathbb{Z}}$
2: **for** $t = 2,\ldots,\hat{t}$ **do**
3: $\quad \theta(t) \leftarrow \theta(t-1) + \max\{y_t - y_{t-1}, 0\}$
4: **end for**
5: **for** $\eta = 0, \hat{\eta}$ **do**
6: $\quad$ **if** $\check{t} = 1$ **then**
7: $\quad\quad v(\eta,\check{t}) \leftarrow x_1 - (a_3 + a_4\eta)y_1 - (a_5 + a_6\eta)y_2$
8: $\quad$ **else if** $\check{s} = 0$ **then**
9: $\quad\quad v(\eta,\check{t}) \leftarrow x_2 - (a_1 + a_2\eta)y_1 - (a_3 + a_4\eta)y_2 - (a_5 + a_6\eta)y_3 + (\overline{C} - \overline{V})\max\{y_2 - y_1, 0\}$
10: $\quad$ **else**
11: $\quad\quad v(\eta,\check{t}) \leftarrow x_2 - (a_1 + a_2\eta)y_1 - (a_3 + a_4\eta)y_2 - (a_5 + a_6\eta)y_3$
12: $\quad$ **end if**
13: $\quad$ **for** $t = \check{t}+1,\ldots,\hat{t}$ **do**
14: $\quad\quad$
$$\begin{aligned} v(\eta,t) \leftarrow\; & v(\eta,t-1) + (x_t - x_{t-1}) \\ & - (a_1 + a_2\eta)(y_{t-1} - y_{t-2}) - (a_3 + a_4\eta)(y_t - y_{t-1}) - (a_5 + a_6\eta)(y_{t+1} - y_t) \\ & + (\overline{C} - \overline{V})[\theta(t-\check{s}) - \theta(t-\hat{s}-1) - \theta(t-\check{s}-1) + \theta(t-\hat{s}-2)] \\ & - V[\check{s}\theta(t-\check{s}) - (\check{s}-1)\theta(t-\check{s}-1) - (\hat{s}+1)\theta(t-\hat{s}-1) + \hat{s}\theta(t-\hat{s}-2)] \end{aligned}$$
15: $\quad$ **end for**
16: **end for**
17: $(\eta^*, t^*) \leftarrow \operatorname{argmax}_{(\eta,t)\in\{0,\hat{\eta}\}\times[\check{t},\hat{t}]_{\mathbb{Z}}}\{v(\eta,t)\}$
18: $\mathcal{S}^* \leftarrow \emptyset$
19: **for** $s = \check{s},\ldots,\min\{\hat{s}, t^*-2\}$ **do**
20: $\quad$ **if** $y_{t^*-s} - y_{t^*-s-1} > 0$ **then** $\mathcal{S}^* \leftarrow \mathcal{S}^* \cup \{s\}$
21: **end for**

the case $\eta = \hat{\eta}$. For each of these two η values, these steps first determine $v(\eta,\check{t})$, and then determine $v(\eta,\check{t}+1), v(\eta,\check{t}+2),\ldots,v(\eta,\hat{t})$ recursively using equation (EC.16). These steps require $O(T)$ time. Steps 17–21 identify a most violated inequality (EC.12) by setting the η and t values to $(\eta^*,t^*) = \operatorname{argmax}_{(\eta,t)\in\{0,\hat{\eta}\}\times[\check{t},\hat{t}]_{\mathbb{Z}}}\{v(\eta,t)\}$ and setting $\mathcal{S}$ equal to the set of s values such that $s \in [\check{s}, \min\{\hat{s}, t^*-2\}]_{\mathbb{Z}}$ and $y_{t^*-s} - y_{t^*-s-1} > 0$. These steps also require $O(T)$ time. Therefore, the total computational time of Algorithm 1 is $O(T)$.

(ii) Consider the following family of inequalities:

$$x_t \le (a_1 + a_2\eta)y_{t-1} + (a_3 + a_4\eta)y_t + (a_5 + a_6\eta)y_{t+1} - \sum_{s\in\mathcal{S}}(\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}), \qquad \text{(EC.17)}$$

where $\eta \in [0,\hat{\eta}]$, $\mathcal{S} \subseteq [\check{s},\hat{s}]_{\mathbb{Z}}$, $t \in [\check{t},\hat{t}]_{\mathbb{Z}}$, and $t \ge s+2$ for all $s \in \mathcal{S}$. Note that inequality family (14) in Proposition 1, inequality family (16) in Proposition 3, and inequality family (18) in Proposition 5 are special cases of this inequality family. Consider any given point $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{2T}$. Let $x'_t = x_{T-t+1}$ and $y'_t = y_{T-t+1}$ for $t \in [1, T]_{\mathbb{Z}}$. Inequality (EC.17) becomes

$$x'_{T-t+1} \leq (a_1 + a_2\eta)y'_{T-t+2} + (a_3 + a_4\eta)y'_{T-t+1} + (a_5 + a_6\eta)y'_{T-t} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y'_{T-t-s+1} - y'_{T-t-s}).$$

Letting $t' = T - t + 1$, this inequality becomes

$$x'_{t'} \leq (a_5 + a_6\eta)y'_{t'-1} + (a_3 + a_4\eta)y'_{t'} + (a_1 + a_2\eta)y'_{t'+1} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y'_{t'-s} - y'_{t'-s-1}). \quad \text{(EC.18)}$$

Let $\check{t}' = 1$ if $a_5 = a_6 = 0$, and let $\check{t}' = 2$ otherwise. Let $\hat{t}' = T$ if $a_1 = a_2 = 0$, and let $\hat{t}' = T - 1$ otherwise. From the analysis in part (i), the set $\mathcal{S} \subseteq [\check{s}, \hat{s}]$, the real number $\eta \in [0, \hat{\eta}]_{\mathbb{Z}}$, and the integer $t' \in [\check{t}', \hat{t}']_{\mathbb{Z}}$ with $t' \geq s + 2\ \forall s \in \mathcal{S}$ corresponding to a most violated inequality (EC.18) can be obtained in $O(T)$ time using Algorithm 1. Hence, the set $\mathcal{S} \subseteq [\check{s}, \hat{s}]$, the real number $\eta \in [0, \hat{\eta}]_{\mathbb{Z}}$, and the integer $t \in [\check{t}, \hat{t}]_{\mathbb{Z}}$ with $t \leq T - s - 1\ \forall s \in \mathcal{S}$ corresponding to a most violated inequality (EC.17) can be obtained in $O(T)$ time.

Summarizing (i) and (ii), we conclude that for any given point $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{2T}$, the most violated inequalities (13)–(14), (15)–(16), and (17)–(18) in Propositions 1, 3, and 5, respectively, can be determined in $O(T)$ time if such violated inequalities exist. $\square$

A.10. Proof of Proposition 8

Proposition 8. *For any given point* $(\mathbf{x},\mathbf{y}) \in \mathbb{R}_+^{2T}$, *the most violated inequalities* (13)–(14), (15)–(16), *and* (17)–(18) *in Propositions* 2, 4, *and* 6, *respectively, can be determined in* $O(T^3)$ *time if such violated inequalities exist.*

Proof. Let $\hat{\eta}$ and $a_1,\ldots,a_6$ be any real numbers such that $\hat{\eta} \geq 0$. Let $\check{s}$, $\check{s}_{\max}$, and $\hat{s}_{\max}$ be any integers such that $0 \leq \check{s} \leq \check{s}_{\max} \leq \hat{s}_{\max} \leq \min\{T-2, \lfloor(\overline{C}-\overline{V})/V\rfloor\}$. Let $\check{t}=1$ if $a_1=a_2=0$, and let $\check{t}=2$ otherwise. Let $\hat{t}=T$ if $a_5=a_6=0$, and let $\hat{t}=T-1$ otherwise.

(i) Consider the following family of inequalities:

$$x_t \leq (a_1+a_2\eta)y_{t-1} + (a_3+a_4\eta)y_t + (a_5+a_6\eta)y_{t+1} - \sum_{s\in\mathcal{S}}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1}), \qquad \text{(EC.19)}$$

where $\eta \in [0,\hat{\eta}]$, $\mathcal{S} = [\check{s},\alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$, $t \in [s_{\max}+2,\hat{t}]_{\mathbb{Z}}$, and α, β, and $s_{\max}$ are integers such that (a) $\check{s}_{\max} \leq s_{\max} \leq \hat{s}_{\max}$, (b) $\check{s} \leq \alpha < \beta \leq s_{\max}$, and (c) $\beta = \alpha+1$ or $s_{\max} \leq L+\alpha$. Note that inequality family (13) in Proposition 2, inequality family (15) in Proposition 4, and inequality family (17) in Proposition 6 are special cases of this inequality family. Consider any given point $(\mathbf{x},\mathbf{y}) \in \mathbb{R}_+^{2T}$. We show that the integers α, β, $s_{\max}$, t and the real number η corresponding to a most violated inequality can be determined in $O(T^3)$ time.

For any $\eta \in [0,\hat{\eta}]_{\mathbb{Z}}$, $s_{\max} \in [\check{s}_{\max},\hat{s}_{\max}]_{\mathbb{Z}}$, $\mathcal{S} \subseteq [\check{s}, s_{\max}]_{\mathbb{Z}}$, $t \in [s_{\max}+2,\hat{t}]_{\mathbb{Z}}$, let

$$\tilde{v}(\eta,\mathcal{S},t) = x_t - (a_1+a_2\eta)y_{t-1} - (a_3+a_4\eta)y_t - (a_5+a_6\eta)y_{t+1} + \sum_{s\in\mathcal{S}}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1}).$$

If $\tilde{v}(\eta,\mathcal{S},t) > 0$, then $\tilde{v}(\eta,\mathcal{S},t)$ is the amount of violation of inequality (EC.19). If $\tilde{v}(\eta,\mathcal{S},t) \leq 0$, then there is no violation of inequality (EC.19). Note that $\tilde{v}(\eta,\mathcal{S},t)$ is linear in η. Thus, for any given $\mathcal{S}$ and t, the function $\tilde{v}(\eta,\mathcal{S},t)$ is maximized at $\eta=0$ if $a_2y_{t-1}+a_4y_t+a_6y_{t+1} \geq 0$, and is maximized at $\eta=\hat{\eta}$ if $a_2y_{t-1}+a_4y_t+a_6y_{t+1} < 0$. For any $s_{\max} \in [\check{s}_{\max},\hat{s}_{\max}]_{\mathbb{Z}}$, $t \in [s_{\max}+2,\hat{t}]_{\mathbb{Z}}$, and $i \in [\check{s}, s_{\max}]_{\mathbb{Z}}$, let

$$v_1(s_{\max},t,i) = \begin{cases} \tilde{v}(0,[\check{s},i]_{\mathbb{Z}},t), \text{ if } a_2y_{t-1}+a_4y_t+a_6y_{t+1} \geq 0; \\ \tilde{v}(\hat{\eta},[\check{s},i]_{\mathbb{Z}},t), \text{ if } a_2y_{t-1}+a_4y_t+a_6y_{t+1} < 0; \end{cases}$$

that is,

$$v_1(s_{\max},t,i) = \begin{cases} x_t - a_1y_{t-1} - a_3y_t - a_5y_{t+1} \\ \qquad + \sum_{s=\check{s}}^{i}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1}), \quad \text{if } a_2y_{t-1}+a_4y_t+a_6y_{t+1} \geq 0; \\ x_t - (a_1+a_2\hat{\eta})y_{t-1} - (a_3+a_4\hat{\eta})y_t - (a_5+a_6\hat{\eta})y_{t+1} \\ \qquad + \sum_{s=\check{s}}^{i}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1}), \quad \text{if } a_2y_{t-1}+a_4y_t+a_6y_{t+1} < 0. \end{cases}$$

For any $s_{\max} \in [\check{s}_{\max},\hat{s}_{\max}]_{\mathbb{Z}}$, $t \in [s_{\max}+2,\hat{t}]_{\mathbb{Z}}$, and $j \in [\check{s}+1, s_{\max}]_{\mathbb{Z}}$, let

$$v_2(s_{\max},t,j) = \sum_{s=j}^{s_{\max}}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1}).$$

Note that

$$v_1(s_{\max},t,i)=\begin{cases} v_1(s_{\max},t,i-1)+(\overline{C}-\overline{V}-iV)(y_{t-i}-y_{t-i-1}), & \text{if } i\ge \check{s}+1;\\ x_t-a_1y_{t-1}-a_3y_t-a_5y_{t+1} & \\ \qquad\qquad +(\overline{C}-\overline{V}-\check{s}V)(y_{t-\check{s}}-y_{t-\check{s}-1}), & \text{if } i=\check{s} \text{ and } a_2y_{t-1}+a_4y_t+a_6y_{t+1}\ge 0;\\ x_t-(a_1+a_2\hat{\eta})y_{t-1}-(a_3+a_4\hat{\eta})y_t & \\ \quad -(a_5+a_6\hat{\eta})y_{t+1}+(\overline{C}-\overline{V}-\check{s}V)(y_{t-\check{s}}-y_{t-\check{s}-1}), & \text{if } i=\check{s} \text{ and } a_2y_{t-1}+a_4y_t+a_6y_{t+1}<0. \end{cases}$$

Thus, for each $s_{\max}$ and t, the values of $v_1(s_{\max},t,\check{s}),v_1(s_{\max},t,\check{s}+1),\ldots,v_1(s_{\max},t,s_{\max})$ can be determined recursively in $O(T)$ time. This implies that the $v_1(s_{\max},t,i)$ values (for all $s_{\max}$, t, and i) can be determined in $O(T^3)$ time. Similarly, the $v_2(s_{\max},t,j)$ values (for all $s_{\max}$, t, and j) can be determined in $O(T^3)$ time. For each $s_{\max}$, t, and j, let

$$\hat{v}_2(s_{\max},t,j)=\max_{\beta\in[j,s_{\max}]_{\mathbb{Z}}}\{v_2(s_{\max},t,\beta)\}$$

and

$$\hat{\beta}(s_{\max},t,j)=\operatorname*{argmax}_{\beta\in[j,s_{\max}]_{\mathbb{Z}}}\{v_2(s_{\max},t,\beta)\}.$$

Note that for each $s_{\max}$ and t, the values of $\hat{v}_2(s_{\max},t,\check{s}+1),\hat{v}_2(s_{\max},t,\check{s}+2),\ldots,\hat{v}_2(s_{\max},t,s_{\max})$ and $\hat{\beta}_2(s_{\max},t,\check{s}+1),\hat{\beta}_2(s_{\max},t,\check{s}+2),\ldots,\hat{\beta}_2(s_{\max},t,s_{\max})$ can be determined in $O(T)$ time by setting

$$\hat{v}_2(s_{\max},t,j)=\begin{cases}\max\{\hat{v}_2(s_{\max},t,j+1),v_2(s_{\max},t,j)\}, & \text{if } j\le s_{\max}-1;\\ v_2(s_{\max},t,s_{\max}), & \text{if } j=s_{\max};\end{cases}$$

and

$$\hat{\beta}(s_{\max},t,j)=\begin{cases}\hat{\beta}(s_{\max},t,j+1), & \text{if } \hat{v}_2(s_{\max},t,j+1)\ge v_2(s_{\max},t,j);\\ j, & \text{if } \hat{v}_2(s_{\max},t,j+1)< v_2(s_{\max},t,j).\end{cases}$$

This implies that the $\hat{v}_2(s_{\max},t,j)$ and $\hat{\beta}(s_{\max},t,j)$ values (for all $s_{\max}$, t, and j) can be determined in $O(T^3)$ time. Note that the condition "$\beta=\alpha+1$ or $s_{\max}\le L+\alpha$" in Proposition 8 implies that "$\mathcal{S}=[\check{s},s_{\max}]_{\mathbb{Z}}$" or "$s_{\max}\le L+\alpha$." For any $s_{\max}\in[\check{s}_{\max},\hat{s}_{\max}]_{\mathbb{Z}}$ and $t\in[s_{\max}+2,\hat{t}]_{\mathbb{Z}}$, if $\mathcal{S}=[\check{s},s_{\max}]_{\mathbb{Z}}$, then the largest possible amount of violation of inequality (EC.19) is equal to $v_1(s_{\max},t,s_{\max})$. For any $s_{\max}\in[\check{s}_{\max},\hat{s}_{\max}]_{\mathbb{Z}}$, $t\in[s_{\max}+2,\hat{t}]_{\mathbb{Z}}$, and $\alpha\in[\check{s},s_{\max}-1]_{\mathbb{Z}}$, if $s_{\max}\le L+\alpha$, then the largest possible amount of violation of inequality (EC.19) is equal to $v_1(s_{\max},t,\alpha)+v_2(s_{\max},t,\hat{\beta}(s_{\max},t,\alpha+1))=v_1(s_{\max},t,\alpha)+\hat{v}_2(s_{\max},t,\alpha+1)$.

To determine the most violated inequality (EC.19) that satisfies the conditions in Proposition 8, we first determine all $v_1(s_{\max},t,i)$, $v_2(s_{\max},t,j)$, $\hat{v}_2(s_{\max},t,j)$, and $\hat{\beta}(s_{\max},t,j)$ values, which requires $O(T^3)$ time. Next, we search for the $s_{\max}$ and t values such that $v_1(s_{\max},t,s_{\max})$ is the largest possible. This requires $O(T^2)$ time. Let $s^*_{\max}$ and t^* be the $s_{\max}$ and t values obtained, and let $\mathcal{S}^*=[\check{s},s^*_{\max}]_{\mathbb{Z}}$. Let $\eta^*=0$ if $a_2y_{t^*-1}+a_4y_{t^*}+a_6y_{t^*+1}\ge 0$, and $\eta^*=\hat{\eta}$ otherwise. Next, we search for the $s_{\max}$, t, and α values, where $\alpha\in[s_{\max}-L,s_{\max}-1]_{\mathbb{Z}}$, such that $v_1(s_{\max},t,\alpha)+\hat{v}_2(s_{\max},t,\alpha+1)$ is the largest possible. This requires $O(T^3)$ time. Let $s_{\max}^{**}$, t^{**}, and α^{**} be the $s_{\max}$, t, and α values obtained, and let $\mathcal{S}^{**} = [\check{s}, \alpha^{**}]_{\mathbb{Z}} \cup [\beta^{**}, s_{\max}^{**}]_{\mathbb{Z}}$, where $\beta^{**} = \hat{\beta}(s_{\max}^{**}, t^{**}, \alpha^{**}+1)$. Let $\eta^{**} = 0$ if $a_2 y_{t^{**}-1} + a_4 y_{t^{**}} + a_6 y_{t^{**}+1} \geq 0$, and $\eta^{**} = \hat{\eta}$ otherwise. If $v_1(s_{\max}^{*}, t^{*}, s_{\max}^{*}) > v_1(s_{\max}^{**}, t^{**}, \alpha^{**}) + \hat{v}_2(s_{\max}^{**}, t^{**}, \alpha^{**}+1)$, then a most violated inequality (EC.19) is obtained by setting $\mathcal{S} = \mathcal{S}^*$, $\eta = \eta^*$, and $t = t^*$. Otherwise, it is obtained by setting $\mathcal{S} = \mathcal{S}^{**}$, $\eta = \eta^{**}$, and $t = t^{**}$. The overall computational time of this process is $O(T^3)$.

(ii) Consider the following family of inequalities:

$$x_t \leq (a_1 + a_2\eta) y_{t-1} + (a_3 + a_4\eta) y_t + (a_5 + a_6\eta) y_{t+1} - \sum_{s\in\mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}), \tag{EC.20}$$

where $\eta \in [0, \hat{\eta}]$, $\mathcal{S} = [\check{s}, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$, $t \in [\check{t}, T - s_{\max} - 1]_{\mathbb{Z}}$, and α, β, and $s_{\max}$ are integers such that (a) $\check{s}_{\max} \leq s_{\max} \leq \hat{s}_{\max}$, (b) $\check{s} \leq \alpha < \beta \leq s_{\max}$, and (c) $\beta = \alpha + 1$ or $s_{\max} \leq L + \alpha$. Note that inequality family (14) in Proposition 2, inequality family (16) in Proposition 4, and inequality family (18) in Proposition 6 are special cases of this inequality family. Consider any given point $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{2T}$. Let $x'_t = x_{T-t+1}$ and $y'_t = y_{T-t+1}$ for $t \in [1, T]_{\mathbb{Z}}$. Inequality (EC.20) becomes

$$x'_{T-t+1} \leq (a_1 + a_2\eta) y'_{T-t+2} + (a_3 + a_4\eta) y'_{T-t+1} + (a_5 + a_6\eta) y'_{T-t} - \sum_{s\in\mathcal{S}} (\overline{C} - \overline{V} - sV)(y'_{T-t-s+1} - y'_{T-t-s}).$$

Letting $t' = T - t + 1$, this inequality becomes

$$x'_{t'} \leq (a_5 + a_6\eta) y'_{t'-1} + (a_3 + a_4\eta) y'_{t'} + (a_1 + a_2\eta) y'_{t'+1} - \sum_{s\in\mathcal{S}} (\overline{C} - \overline{V} - sV)(y'_{t'-s} - y'_{t'-s-1}). \tag{EC.21}$$

Let $\hat{t}' = T$ if $a_1 = a_2 = 0$, and let $\hat{t}' = T - 1$ otherwise. From the analysis in part (i), the integers α, β, $s_{\max}$, and $t' \in [s_{\max} + 2, \hat{t}']_{\mathbb{Z}}$ corresponding to a most violated inequality (EC.21) can be obtained in $O(T^3)$ time. Hence, the integers α, β, $s_{\max}$, and $t \in [\check{t}, T - s_{\max} - 1]_{\mathbb{Z}}$ corresponding to a most violated inequality (EC.20) can be obtained in $O(T^3)$ time.

Summarizing (i) and (ii), we conclude that for any given point $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{2T}$, the most violated inequalities (13)–(14), (15)–(16), and (17)–(18) in Propositions 2, 4, and 6, respectively, can be determined in $O(T^3)$ time if such violated inequalities exist. $\square$

A.11. Proof of Proposition 9

Proposition 9. *Consider any $k \in [1, T-1]_{\mathbb{Z}}$ such that $\overline{C} - \underline{C} - kV > 0$, any $m \in [0, k-1]_{\mathbb{Z}}$, and any $\mathcal{S} \subseteq [0, \min\{k-1, L-m-1\}]_{\mathbb{Z}}$. For any $t \in [k+1, T-m]_{\mathbb{Z}}$, the inequality*

$$x_t - x_{t-k} \leq (\underline{C} + (k-m)V)y_t + V\sum_{i=1}^{m} y_{t+i} - \underline{C}y_{t-k} - \sum_{s \in \mathcal{S}} (\underline{C} + (k-s)V - \overline{V})(y_{t-s} - y_{t-s-1}) \tag{19}$$

is valid for $\mathrm{conv}(\mathcal{P})$. *For any $t \in [m+1, T-k]_{\mathbb{Z}}$, the inequality*

$$x_t - x_{t+k} \leq (\underline{C} + (k-m)V)y_t + V\sum_{i=1}^{m} y_{t-i} - \underline{C}y_{t+k} - \sum_{s \in \mathcal{S}} (\underline{C} + (k-s)V - \overline{V})(y_{t+s} - y_{t+s+1}) \tag{20}$$

is valid for $\mathrm{conv}(\mathcal{P})$. *Furthermore,* (19) *and* (20) *are facet-defining for* $\mathrm{conv}(\mathcal{P})$ *when $m = 0$ and $s \geq \min\{k-1, 1\}$ for all $s \in \mathcal{S}$.*

Proof. We first prove that inequality (19) is valid and facet-defining for $\mathrm{conv}(\mathcal{P})$. For notational convenience, we define $s_{\max} = \max\{s : s \in \mathcal{S}\}$ if $\mathcal{S} \neq \emptyset$, and $s_{\max} = -1$ if $\mathcal{S} = \emptyset$. Consider any $t \in [k+1, T-m]_{\mathbb{Z}}$. To prove that the linear inequality (19) is valid for $\mathrm{conv}(\mathcal{P})$, it suffices to show that it is valid for $\mathcal{P}$. Consider any element $(\mathbf{x}, \mathbf{y})$ of $\mathcal{P}$. We show that $(\mathbf{x}, \mathbf{y})$ satisfies (19). We divide the analysis into three cases:

Case 1: $y_t = 0$. In this case, by (2c) and (2d), $-x_{t-k} \leq -\underline{C}y_{t-k}$ and $x_t = 0$. Thus, the left-hand side of (19) is at most $-\underline{C}y_{t-k}$ and the first term on the right-hand side of (19) is 0. Because $y_t = 0$ and $t \in [2, T]_{\mathbb{Z}}$, by Lemma 1(i), we have $y_{t-j} - y_{t-j-1} \leq 0$ for all $j \in [0, \min\{t-2, L-1\}]_{\mathbb{Z}}$. Because $\mathcal{S} \subseteq [0, \min\{k-1, L-m-1\}]_{\mathbb{Z}}$, $m \geq 0$ and, $t \geq k+1$, we have $\mathcal{S} \subseteq [0, \min\{t-2, L-1\}]_{\mathbb{Z}}$. Thus, $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, k-1]_{\mathbb{Z}}$ and $\underline{C} + V > \overline{V}$, for any $s \in \mathcal{S}$, the coefficient "$\underline{C} + (k-s)V - \overline{V}$" on the right-hand side of (19) is positive. Hence, the right-hand side of (19) is at least $-\underline{C}y_{t-k}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (19).

Case 2: $y_t = 1$ and $y_{t-s'} - y_{t-s'-1} = 1$ for some $s' \in \mathcal{S}$. In this case, $y_{t-s'} = 1$ and $y_{t-s'-1} = 0$. Because $y_t = 1$ and $t \in [2, T]_{\mathbb{Z}}$, by Lemma 1(ii), there exists at most one $j \in [0, \min\{t-2, L\}]_{\mathbb{Z}}$ such that $y_{t-j} - y_{t-j-1} = 1$. Because $\mathcal{S} \subseteq [0, \min\{k-1, L-m-1\}]_{\mathbb{Z}}$, $m \geq 0$ and, $t \geq k+1$, we have $\mathcal{S} \subseteq [0, \min\{t-2, L\}]_{\mathbb{Z}}$. Thus, $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S} \setminus \{s'\}$. Because $y_{t-s'} - y_{t-s'-1} = 1$ and $t - s' \in [2, T]_{\mathbb{Z}}$, by (2a), we have $y_\tau = 1$ for all $\tau \in [t-s', \min\{T, t-s'+L-1\}]_{\mathbb{Z}}$. Because $\mathcal{S} \subseteq [0, L-m-1]_{\mathbb{Z}}$, we have $t - s' + L - 1 \geq t + m$. Thus, $y_\tau = 1$ for all $\tau \in [t-s', t+m]_{\mathbb{Z}}$, which implies that $(\underline{C} + (k-m)V)y_t + V\sum_{i=1}^{m} y_{t+i} = \underline{C} + kV$. Because $\mathcal{S} \subseteq [0, k-1]_{\mathbb{Z}}$ and $\underline{C} + V > \overline{V}$, the coefficient "$\underline{C} + (k-s)V - \overline{V}$" on the right-hand side of (19) is positive for all $s \in \mathcal{S}$. Hence, the right-hand side of (19) is at least $\underline{C} + kV - \underline{C}y_{t-k} - (\underline{C} + (k-s')V - \overline{V}) = s'V + \overline{V} - \underline{C}y_{t-k}$. By (2e), $\sum_{\tau=t-s'}^{t} (x_\tau - x_{\tau-1}) \leq \sum_{\tau=t-s'}^{t} Vy_{\tau-1} + \sum_{\tau=t-s'}^{t} \overline{V}(1 - y_{\tau-1})$, which implies that $x_t - x_{t-s'-1} \leq s'V + \overline{V}$. Because $y_{t-s'-1} = 0$, by (2d), $x_{t-s'-1} = 0$. By (2c), $-x_{t-k} \geq -\underline{C}y_{t-k}$. Thus, $x_t - x_{t-k} \leq s'V + \overline{V} - \underline{C}y_{t-k}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (19).

Case 3: $y_t = 1$ and $y_{t-s} - y_{t-s-1} \le 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, k-1]_{\mathbb{Z}}$ and $\underline{C} + V > \overline{V}$, we have $\underline{C} + (k-s)V - \overline{V} > 0$ for each $s \in \mathcal{S}$. Hence, the term "$-\sum_{s \in \mathcal{S}}(\underline{C} + (k-s)V - \overline{V})(y_{t-s} - y_{t-s-1})$" on the right-hand side of inequality (19) is nonnegative. We divide our analysis into three subcases.

Case 3.1: $y_\tau = 0$ for some $\tau \in [t, t+m]_{\mathbb{Z}}$. Let $t' = \min\{\tau \in [t, t+m]_{\mathbb{Z}} : y_\tau = 0\}$. Then, $y_\tau = 1$ for all $\tau \in [t, t'-1]_{\mathbb{Z}}$. Thus, the right-hand side of (19) is at least $\underline{C} + (k-m)V + (t'-t-1)V - \underline{C}y_{t-k}$. Because $k - m \ge 1$ and $\underline{C} + V > \overline{V}$, the right-hand side of (19) is at least $\overline{V} + (t'-t-1)V - \underline{C}y_{t-k}$. By (2f), $\sum_{\tau=t+1}^{t'}(x_{\tau-1} - x_\tau) \le \sum_{\tau=t+1}^{t'} Vy_\tau + \sum_{\tau=t+1}^{t'} \overline{V}(1 - y_\tau)$, which implies that $x_t - x_{t'} \le \overline{V} + (t' - t - 1)V$. Because $y_{t'} = 0$, by (2d), $x_{t'} = 0$. By (2c), $-x_{t-k} \le -\underline{C}y_{t-k}$. Hence, $x_t - x_{t-k} \le \overline{V} + (t' - t - 1)V - \underline{C}y_{t-k}$. Thus, the left-hand side of (19) is less than or equal to the right-hand side.

Case 3.2: $y_\tau = 1$ for all $\tau \in [t, t+m]_{\mathbb{Z}}$ and $y_\tau = 0$ for some $\tau \in [t-k, t-1]_{\mathbb{Z}}$. In this case, the right-hand side of (19) is at least $\underline{C} + kV - \underline{C}y_{t-k}$. Let $t' = \max\{\tau \in [t-k, t-1]_{\mathbb{Z}} : y_\tau = 0\}$. Because $t' \ge t-k$ and $\underline{C} + V > \overline{V}$, the right-hand side of (19) is greater than $\overline{V} + (t - t' - 1)V - \underline{C}y_{t-k}$. By (2e), $\sum_{\tau=t'+1}^{t}(x_\tau - x_{\tau-1}) \le \sum_{\tau=t'+1}^{t} Vy_{\tau-1} + \sum_{\tau=t'+1}^{t} \overline{V}(1 - y_{\tau-1})$, which implies that $x_t - x_{t'} \le \overline{V} + (t - t' - 1)V$. Because $y_{t'} = 0$, by (2d), $x_{t'} = 0$. By (2c), $-x_{t-k} \le -\underline{C}y_{t-k}$. Hence, $x_t - x_{t-k} \le \overline{V} + (t - t' - 1)V - \underline{C}y_{t-k}$. Thus, the left-hand side of (19) is less than the right-hand side.

Case 3.3: $y_\tau = 1$ for all $\tau \in [t-k, t+m]_{\mathbb{Z}}$. In this case, the right-hand side of (19) is at least kV. By (2e), $\sum_{\tau=t-k+1}^{t}(x_\tau - x_{\tau-1}) \le \sum_{\tau=t-k+1}^{t} Vy_{\tau-1} + \sum_{\tau=t-k+1}^{t} \overline{V}(1 - y_{\tau-1})$, which implies that $x_t - x_{t-k} \le kV$. Thus, the left-hand side of (19) is less than or equal to the right-hand side.

In Cases 3.1–3.3, $(\mathbf{x}, \mathbf{y})$ satisfies (19). Summarizing Cases 1–3, we conclude that (19) is valid for $\text{conv}(\mathcal{P})$.

Consider any $t \in [k+1, T-m]_{\mathbb{Z}}$. To prove that inequality (19) is facet-defining for $\text{conv}(\mathcal{P})$ when $m = 0$ and $s \ge \min\{k-1, 1\}$ for all $s \in \mathcal{S}$, it suffices to show that there exist $2T$ affinely independent points in $\text{conv}(\mathcal{P})$ that satisfy (19) at equality when $m = 0$ and $s \ge \min\{k-1, 1\}$ for all $s \in \mathcal{S}$. Because $\mathbf{0} \in \text{conv}(\mathcal{P})$ and $\mathbf{0}$ satisfies (19) at equality, it suffices to create the remaining $2T - 1$ nonzero linearly independent points. We denote these $2T - 1$ points as $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ for $r \in [1, T]_{\mathbb{Z}} \setminus \{t-k\}$ and $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ for $r \in [1, T]_{\mathbb{Z}}$, and denote the qth component of $\bar{\mathbf{x}}^r$, $\bar{\mathbf{y}}^r$, $\hat{\mathbf{x}}^r$, and $\hat{\mathbf{y}}^r$ as $\bar{x}_q^r$, $\bar{y}_q^r$, $\hat{x}_q^r$, and $\hat{y}_q^r$, respectively. Let $\epsilon = \min\{\overline{V} - \underline{C}, \overline{C} - \underline{C} - kV\} > 0$. We divide these $2T - 1$ points into the following eight groups:

(A1) For each $r \in [1, t-k-1]_{\mathbb{Z}}$, we create a point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as follows:

$$\bar{x}_q^r = \begin{cases} \underline{C}, & \text{for } q \in [1, r-1]_{\mathbb{Z}}; \\ \underline{C} + \epsilon, & \text{for } q = r; \\ 0, & \text{for } q \in [r+1, T]; \end{cases}$$

and

$$\bar{y}_q^r = \begin{cases} 1, & \text{for } q \in [1, r]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [r+1, T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (2a)–(2f). Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \operatorname{conv}(\mathcal{P})$. Note that $\bar{x}_t^r = \bar{x}_{t-k}^r = \bar{y}_t^r = \bar{y}_{t-k}^r = 0$ and $m = 0$. Because $t - s - 1 \neq r$ for all $s \in \mathcal{S}$, we have $\bar{y}_{t-s}^r - \bar{y}_{t-s-1}^r = 0$ for all $s \in \mathcal{S}$. Hence, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (20) at equality.

(A2) For each $r \in [t-k+1, t-1]_{\mathbb{Z}}$, we create a point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as follows:

$$\bar{x}_q^r = \begin{cases} \underline{C}, & \text{for } q \in [1, t-1]_{\mathbb{Z}} \setminus \{r\}; \\ \underline{C} + \epsilon, & \text{for } q = r; \\ 0, & \text{for } q \in [t, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\bar{y}_q^r = \begin{cases} 1, & \text{for } q \in [1, t-1]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [t, T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (2a)–(2f). Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \operatorname{conv}(\mathcal{P})$. Note that $\bar{x}_t^r = \bar{y}_t^r = 0$, $\bar{x}_{t-k}^r = \underline{C}$, $\bar{y}_{t-k}^r = 1$, and $m = 0$. The existence of $r \in [t-k+1, t-1]_{\mathbb{Z}}$ implies that $k \geq 2$, which in turn implies that $s \geq 1$ for all $s \in \mathcal{S}$. Hence, $\bar{y}_{t-s}^r - \bar{y}_{t-s-1}^r = 0$ for all $s \in \mathcal{S}$. Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (20) at equality.

(A3) We create a point $(\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t)$ as follows:

$$\bar{x}_q^t = \begin{cases} \underline{C}, & \text{for } q \in [1, t-k-1]_{\mathbb{Z}}; \\ \underline{C} + (q - t + k)V + \epsilon, & \text{for } q \in [t-k, t]_{\mathbb{Z}}; \\ \underline{C} + kV, & \text{for } q \in [t+1, T]_{\mathbb{Z}}; \end{cases}$$

and $\bar{y}_q^t = 1$ for all $q \in [1, T]_{\mathbb{Z}}$. It is easy to verify that $(\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t)$ satisfies (2a)–(2d). Note that $\bar{x}_q^t - \bar{x}_{q-1}^t = 0$ when $q \in [2, t-k-1]_{\mathbb{Z}}$, $0 < \bar{x}_q^t - \bar{x}_{q-1}^t \leq V$ when $q \in [t-k, t]_{\mathbb{Z}}$, and $-\epsilon \leq \bar{x}_q^t - \bar{x}_{q-1}^t \leq 0$ when $q \in [t+1, T]_{\mathbb{Z}}$. Thus, $-V\bar{y}_q^t - \overline{V}(1 - \bar{y}_q^t) \leq \bar{x}_q^t - \bar{x}_{q-1}^t \leq V\bar{y}_{q-1}^t + \overline{V}(1 - \bar{y}_{q-1}^t)$ for all $q \in [2, T]_{\mathbb{Z}}$. Hence, $(\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t)$ satisfies (2e) and (2f). Therefore, $(\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t) \in \operatorname{conv}(\mathcal{P})$. Note that $\bar{x}_t^t = \underline{C} + kV + \epsilon$, $\bar{x}_{t-k}^t = \underline{C} + \epsilon$, $\bar{y}_t^t = \bar{y}_{t-k}^t = 1$, $m = 0$, and $\bar{y}_{t-s}^t - \bar{y}_{t-s-1}^t = 0$ for all $s \in \mathcal{S}$. Thus, $(\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t)$ satisfies (19) at equality.

(A4) For each $r \in [t+1, T]_{\mathbb{Z}}$, we create a point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as follows:

$$\bar{x}_q^r = \begin{cases} 0, & \text{for } q \in [1, r-1]_{\mathbb{Z}}; \\ \underline{C} + \epsilon, & \text{for } q = r; \\ \underline{C}, & \text{for } q \in [r+1, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\bar{y}_q^r = \begin{cases} 0, & \text{for } q \in [1, r-1]_{\mathbb{Z}}; \\ 1, & \text{for } q \in [r, T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (2a)–(2f). Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \operatorname{conv}(\mathcal{P})$. Note that $\bar{x}_t^r = \bar{x}_{t-k}^r = \bar{y}_t^r = \bar{y}_{t-k}^r = 0$, $m = 0$, and $\bar{y}_{t-s}^r - \bar{y}_{t-s-1}^r = 0$ for all $s \in \mathcal{S}$. Hence, $(\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t)$ satisfies (19) at equality.

(A5) For each $r \in [1, t-1]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (A2) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \operatorname{conv}(\mathcal{P})$. To show that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (19) at equality, we first consider the case where $t-r-1 \notin \mathcal{S}$. In this case, $\hat{x}^r_t = \hat{y}^r_t = 0$ and $m=0$. Because $t-k \leq t-s_{\max}-1 \leq r$, we have $\hat{x}^r_{t-k} = \underline{C}$ and $\hat{y}^r_{t-k} = 1$. Because $t-s-1 \neq r$ for all $s \in \mathcal{S}$, we have $\hat{y}^r_{t-s} - \hat{y}^r_{t-s-1} = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (19) at equality. Next, we consider the case where $t-r-1 \in \mathcal{S}$. In this case, $\hat{x}^r_t = \overline{V} + (t-r-1)V$, $\hat{y}^r_t = 1$, $\hat{x}^r_{t-k} = \hat{y}^r_{t-k} = 0$, and $m=0$. In addition, $\hat{y}^r_{t-s} - \hat{y}^r_{t-s-1} = 1$ when $s = t-r-1$, and $\hat{y}^r_{t-s} - \hat{y}^r_{t-s-1} = 0$ when $s \neq t-r-1$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (19) at equality.

(A6) We create a point $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ as follows:

$$\hat{\mathbf{x}}^t_q = \begin{cases} \underline{C}, & \text{for } q \in [1, t-k-1]_{\mathbb{Z}}; \\ \underline{C} + (q-t+k)V, & \text{for } q \in [t-k, t]_{\mathbb{Z}}; \\ \underline{C} + kV, & \text{for } q \in [t+1, T]_{\mathbb{Z}}; \end{cases}$$

and $\hat{y}^t_q = 1$ for all $q \in [1,T]_{\mathbb{Z}}$. It is easy to verify that $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (2a)–(2d). Note that $\hat{x}^t_q - \hat{x}^t_{q-1} = 0$ when $q \in [2, t-k]_{\mathbb{Z}}$, $\hat{x}^t_q - \hat{x}^t_{q-1} = V$ when $q \in [t-k+1, t]_{\mathbb{Z}}$, and $\hat{x}^t_q - \hat{x}^t_{q-1} = 0$ when $q \in [t+1, T]_{\mathbb{Z}}$. Thus, $-V\hat{y}^t_q - \overline{V}(1-\hat{y}^t_q) \leq \hat{x}^t_q - \hat{x}^t_{q-1} \leq V\hat{y}^t_{q-1} + \overline{V}(1-\hat{y}^t_{q-1})$ for all $q \in [2,T]_{\mathbb{Z}}$. Hence, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (2e) and (2f). Therefore, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) \in \mathrm{conv}(\mathcal{P})$. Note that $\hat{x}^t_t = \underline{C} + kV$, $\hat{x}^t_{t-k} = \underline{C}$, $\hat{y}^t_t = \hat{y}^t_{t-k} = 1$, $m=0$, and $\hat{y}^t_{t-s} - \hat{y}^t_{t-s-1} = 0$ for all $s \in \mathcal{S}$, Thus, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (19) at equality.

(A7) For each $r \in [t+1, T]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (A4) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \mathrm{conv}(\mathcal{P})$. Note that $\hat{x}^r_t = \hat{x}^r_{t-k} = \hat{y}^r_t = \hat{y}^r_{t-k} = 0$, $m=0$, and $\hat{y}^r_{t-s} - \hat{y}^r_{t-s-1} = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (19) at equality.

Table EC.7 shows a matrix with $2T-1$ rows, where each row represents a point created by this process. This matrix can be transformed into the matrix in Table EC.8 via the following Gaussian elimination process:

(i) For each $r \in [1, t-k-1]_{\mathbb{Z}}$, the point with index r in group (B1), denoted $(\underline{\mathbf{x}}^r, \underline{\mathbf{y}}^r)$, is obtained by setting $(\underline{\mathbf{x}}^r, \underline{\mathbf{y}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point with index r in group (A1), and $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (A5). Note that $t-r-1 \notin \mathcal{S}$ for all $r \in [1, t-k-1]_{\mathbb{Z}}$. Thus, when $r \leq t-k-1$, the point with index r in group (A5) is given by $\hat{x}^r_q = \underline{C}$ and $\hat{y}^r_q = 1$ for $q \in [1,r]_{\mathbb{Z}}$, and $\hat{x}^r_q = \hat{y}^r_q = 0$ for $q \in [r+1, T]_{\mathbb{Z}}$.

(ii) For each $r \in [t-k+1, t-1]_{\mathbb{Z}}$, the point with index r in group (B2), denoted $(\underline{\mathbf{x}}^r, \underline{\mathbf{y}}^r)$, is obtained by setting $(\underline{\mathbf{x}}^r, \underline{\mathbf{y}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point in group (A2), and $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ is the point with index $t-1$ in group (A5). Note that because $s \geq 1$ for all $s \in \mathcal{S}$, the point with index $t-1$ in group (A5) is given by $\hat{x}^{t-1}_q = \underline{C}$ and $\hat{y}^{t-1}_q = 1$ for $q \in [1, t-1]_{\mathbb{Z}}$, and $\hat{x}^{t-1}_q = \hat{y}^{t-1}_q = 0$ for $q \in [t, T]_{\mathbb{Z}}$.

(iii) The point in group (B3), denoted $(\underline{\mathbf{x}}^t, \underline{\mathbf{y}}^t)$, is obtained by setting $(\underline{\mathbf{x}}^t, \underline{\mathbf{y}}^t) = (\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t) - (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$. Here, $(\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t)$ is the point in group (A3), and $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point in group (A6).

Table EC.7 A matrix with the rows representing $2T-1$ linearly independent points in $\mathrm{conv}(\mathcal{P})$ satisfying inequality (19) at equality

Group	Point	Index r	$\mathbf{x}$													$\mathbf{y}$												
			1	$\cdots$	$t-k-1$	$t-k$	$t-k+1$	$\cdots$	$t-1$	t	$t+1$	$\cdots$	T	1	$\cdots$	$t-k-1$	$t-k$	$t-k+1$	$\cdots$	$t-1$	t	$t+1$	$\cdots$	T
(A1)		1	$\underline{C}+\epsilon$	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	1	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-k-1$	$\underline{C}$	$\cdots$	$\underline{C}+\epsilon$	0	0	$\cdots$	0	0	0	$\cdots$	0	1	$\cdots$	1	0	0	$\cdots$	0	0	0	$\cdots$	0
(A2)		$t-k+1$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}+\epsilon$	$\cdots$	$\underline{C}$	0	0	$\cdots$	0	1	$\cdots$	1	1	1	$\cdots$	1	0	0	$\cdots$	0
	$(\bar{\mathbf{x}}^r,\bar{\mathbf{y}}^r)$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-1$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}+\epsilon$	0	0	$\cdots$	0	1	$\cdots$	1	1	1	$\cdots$	1	0	0	$\cdots$	0
(A3)		t	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}+\epsilon$	$\underline{C}+V+\epsilon$	$\cdots$	$\underline{C}+(k-1)V+\epsilon$	$\underline{C}+kV+\epsilon$	$\underline{C}+kV$	$\cdots$	$\underline{C}+kV$	1	$\cdots$	1	1	1	$\cdots$	1	1	1	$\cdots$	1
(A4)		$t+1$	0	$\cdots$	0	0	0	$\cdots$	0	0	$\underline{C}+\epsilon$	$\cdots$	$\underline{C}$	0	$\cdots$	0	0	0	$\cdots$	0	0	1	$\cdots$	1
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	$\underline{C}+\epsilon$	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	1
(A5)		1																						
		$\vdots$	(See Note EC.7-1)											(See Note EC.7-1)										
		$t-1$																						
(A6)	$(\hat{\mathbf{x}}^r,\hat{\mathbf{y}}^r)$	t	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}+V$	$\cdots$	$\underline{C}+(k-1)V$	$\underline{C}+kV$	$\underline{C}+kV$	$\cdots$	$\underline{C}+kV$	1	$\cdots$	1	1	1	$\cdots$	1	1	1	$\cdots$	1
(A7)		$t+1$	0	$\cdots$	0	0	0	$\cdots$	0	0	$\underline{C}$	$\cdots$	$\underline{C}$	0	$\cdots$	0	0	0	$\cdots$	0	0	1	$\cdots$	1
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	$\underline{C}$	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	1

Note EC.7-1: For $r \in [1, t-1]_{\mathbb{Z}}$, the $\mathbf{x}$ and $\mathbf{y}$ vectors in group (A5) are given as follows:

$\hat{\mathbf{x}}^r = (\underbrace{\underline{C},\ldots,\underline{C}}_{r\text{ terms}},\underbrace{0,\ldots,0}_{T-r\text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{1,\ldots,1}_{r\text{ terms}},\underbrace{0,\ldots,0}_{T-r\text{ terms}})$ if $t-r-1 \notin \mathcal{S}$;

$\hat{\mathbf{x}}^r = (\underbrace{0,\ldots,0}_{r\text{ terms}},\underbrace{\overline{V},\overline{V}+V,\overline{V}+2V,\ldots,\overline{V}+(t-r-1)V}_{t-r\text{ terms}},\underbrace{\overline{V}+(t-r-1)V,\ldots,\overline{V}+(t-r-1)V}_{T-t\text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{0,\ldots,0}_{r\text{ terms}},\underbrace{1,\ldots,1}_{T-r\text{ terms}})$ if $t-r-1 \in \mathcal{S}$.

Table EC.8 Lower triangular matrix obtained from Table EC.7 via Gaussian elimination

Group	Point	Index r	$\mathbf{x}$											$\mathbf{y}$						
			1	$\cdots$	$t-k-1$	$t-k$	$t-k+1$	$\cdots$	$t-1$	t	$t+1$	$\cdots$	T	1	$\cdots$	$t-1$	t	$t+1$	$\cdots$	T
(B1)		1	ϵ	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-k-1$	0	$\cdots$	ϵ	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
(B2)		$t-k+1$	0	$\cdots$	0	0	ϵ	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
	$(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-2$	0	$\cdots$	0	0	0	$\cdots$	ϵ	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
(B3)		t	0	$\cdots$	0	ϵ	ϵ	$\cdots$	ϵ	ϵ	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
(B4)		$t+1$	0	$\cdots$	0	0	0	$\cdots$	0	0	ϵ	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		T	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	ϵ	0	$\cdots$	0	0	0	$\cdots$	0
(B5)		1	(Omitted)											1	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$												$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-1$												1	$\cdots$	1	0	0	$\cdots$	0
(B6)	$(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$	t	(Omitted)											1	$\cdots$	1	1	0	$\cdots$	0
(B7)		$t+1$	(Omitted)											0	$\cdots$	0	0	1	$\cdots$	0
		$\vdots$												$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T												0	$\cdots$	0	0	0	$\cdots$	1

(iv) For each $r \in [t+1, T]_{\mathbb{Z}}$, the point with index r in group (B4), denoted $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point with index r in group (A4), and $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point in group (A7).

(v) For each $r \in [1, t-1]_{\mathbb{Z}}$, the point with index r in group (B5), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $t-r-1 \notin \mathcal{S}$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) - (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $t-r-1 \in \mathcal{S}$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (A5), and $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point in group (A6).

(vi) The point in group (B6), denoted $(\underline{\hat{\mathbf{x}}}^t, \underline{\hat{\mathbf{y}}}^t)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^t, \underline{\hat{\mathbf{y}}}^t) = (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) - (\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$. Here, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point in group (A6), and $(\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$ is the point with index $t+1$ in group (A7).

(vii) For each $r \in [t+1, T]_{\mathbb{Z}}$, the point with index r in group (B7), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ if $r \neq T$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $r = T$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ and $(\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ are the points with indices r and $r+1$, respectively, in group (A7).

The matrix shown in Table EC.8 is lower triangular; that is, the position of the last nonzero component of a row of the matrix is greater than the position of the last nonzero component of the previous row. This implies that these $2T-1$ points in groups (A1)–(A7) are linearly independent. Therefore, inequality (19) is facet-defining for $\mathrm{conv}(\mathcal{P})$.

Next, we show that inequality (20) is valid for $\mathrm{conv}(\mathcal{P})$ and is facet-defining for $\mathrm{conv}(\mathcal{P})$ when $m = 0$ and $s \geq \min\{k-1, 1\}$ $\forall s \in \mathcal{S}$. Denote $x'_t = x_{T-t+1}$ and $y'_t = y_{T-t+1}$ for $t \in [1, T]_{\mathbb{Z}}$. Because inequality (19) is valid for $\mathrm{conv}(\mathcal{P})$ and is facet-defining for $\mathrm{conv}(\mathcal{P})$ when $m = 0$ and $s \geq \min\{k-1, 1\}$ $\forall s \in \mathcal{S}$ for any $t \in [k+1, T-m]_{\mathbb{Z}}$, the inequality

$$
\begin{aligned}
x'_{T-t+1} - x'_{T-t+k+1} \leq (\underline{C} + (k-m)V)y'_{T-t+1} + V\sum_{i=1}^{m} y'_{T-t-i+1} - \underline{C}y'_{T-t+k+1} \\
- \sum_{s \in \mathcal{S}} (\underline{C} + (k-s)V - \overline{V})(y'_{T-t+s+1} - y'_{T-t+s+2})
\end{aligned}
$$

is valid for $\mathrm{conv}(\mathcal{P}')$ and is facet-defining for $\mathrm{conv}(\mathcal{P}')$ when $m = 0$ and $s \geq \min\{k-1, 1\}$ $\forall s \in \mathcal{S}$ for any $t \in [k+1, T-m]_{\mathbb{Z}}$. Let $t' = T - t + 1$. Then, the inequality

$$
x'_{t'} - x'_{t'+k} \leq (\underline{C} + (k-m)V)y'_{t'} + V\sum_{i=1}^{m} y'_{t'-i} - \underline{C}y'_{t'+k} - \sum_{s \in \mathcal{S}} (\underline{C} + (k-s)V - \overline{V})(y'_{t'+s} - y'_{t'+s+1})
$$

is valid for $\mathrm{conv}(\mathcal{P}')$ and is facet-defining for $\mathrm{conv}(\mathcal{P}')$ when $m = 0$ and $s \geq \min\{k-1, 1\}$ $\forall s \in \mathcal{S}$ for any $t' \in [m+1, T-k]_{\mathbb{Z}}$. Hence, by Lemma 2, inequality (20) is valid for $\mathrm{conv}(\mathcal{P})$ and is facet-defining for $\mathrm{conv}(\mathcal{P})$ when $m = 0$ and $s \geq \min\{k-1, 1\}$ $\forall s \in \mathcal{S}$ for any $t \in [m+1, T-k]_{\mathbb{Z}}$. $\square$

A.12. Proof of Proposition 10

Proposition 10. *For any given point $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2T}_{+}$, the most violated inequalities (19) and (20) can be determined in $O(T^3)$ time if such violated inequalities exist.*

Proof. We first consider inequality (19). Consider any given $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2T}_{+}$. For notational convenience, denote $\hat{k} = \max\{k \in [1, T-1]_{\mathbb{Z}} : \overline{C} - \underline{C} - kV > 0\}$, and denote $\hat{s}_{km} = \min\{k-1, L-m-1\}$ for any $k \in [1,\hat{k}]_{\mathbb{Z}}$ and $m \in [0, k-1]_{\mathbb{Z}}$. For any $t \in [1, T]_{\mathbb{Z}}$, let

$$\theta(t) = \sum_{\tau=2}^{t} \max\{y_\tau - y_{\tau-1}, 0\}.$$

Then, for any $k \in [1,\hat{k}]_{\mathbb{Z}}$, $m \in [0,k-1]_{\mathbb{Z}}$, and $t \in [k+1, T-m]_{\mathbb{Z}}$,

$$\sum_{s=1}^{\hat{s}_{km}} \max\{y_{t-s} - y_{t-s-1}, 0\} = \sum_{\tau=t-\hat{s}_{km}}^{t-1} \max\{y_\tau - y_{\tau-1}, 0\} = \theta(t-1) - \theta(t - \hat{s}_{km} - 1). \tag{EC.22}$$

For any $k \in [1,\hat{k}]_{\mathbb{Z}}$, $m \in [0,k-1]_{\mathbb{Z}}$, $t \in [k+1, T-m]_{\mathbb{Z}}$, and $\mathcal{S} \subseteq [0, \hat{s}_{km}]_{\mathbb{Z}}$, let

$$\tilde{v}_{km}(\mathcal{S}, t) = x_t - x_{t-k} - (\underline{C} + (k-m)V)y_t - V\sum_{i=1}^{m} y_{t+i} + \underline{C}y_{t-k} + \sum_{s \in \mathcal{S}} (\underline{C} + (k-s)V - \overline{V})(y_{t-s} - y_{t-s-1}).$$

If $\tilde{v}_{km}(\mathcal{S}, t) > 0$, then $\tilde{v}_{km}(\mathcal{S}, t)$ is the amount of violation of inequality (19). If $\tilde{v}_{km}(\mathcal{S}, t) \le 0$, there is no violation of inequality (19). For any $k \in [1,\hat{k}]_{\mathbb{Z}}$, $m \in [0,k-1]_{\mathbb{Z}}$, and $t \in [k+1, T-m]_{\mathbb{Z}}$, let

$$v_{km}(t) = \max_{\mathcal{S} \subseteq [0, \hat{s}_{km}]_{\mathbb{Z}}} \{\tilde{v}_{km}(\mathcal{S}, t)\}.$$

If $v_{km}(t) > 0$, then $v_{km}(t)$ is the largest possible violation of inequality (19) for this combination of k, m, and t. If $v_{km}(t) \le 0$, the largest possible violation of inequality (19) is zero for this combination of k, m, and t. Because $\underline{C} + V > \overline{V}$, we have $\underline{C} + (k-s)V - \overline{V} > 0$ for all $k \in [1,\hat{k}]_{\mathbb{Z}}$, $s \in [0, \hat{s}_{km}]_{\mathbb{Z}}$, and $m \in [0,k-1]_{\mathbb{Z}}$. Thus, for any $k \in [1,\hat{k}]_{\mathbb{Z}}$, $m \in [0,k-1]_{\mathbb{Z}}$, and $t \in [k+1, T-m]_{\mathbb{Z}}$, $\tilde{v}_{km}(\mathcal{S}, t)$ is maximized when $\mathcal{S}$ contains all $s \in [0, \hat{s}_{km}]_{\mathbb{Z}}$ such that $y_{t-s} - y_{t-s-1} > 0$ (if any). If it does not exist any $s \in [0, \hat{s}]_{\mathbb{Z}}$ such that $y_{t-s} - y_{t-s-1} > 0$, then $\tilde{v}_{km}(\mathcal{S}, t)$ is maximized when $\mathcal{S} = \emptyset$, and $v_{km}(t) = x_t - x_{t-k} - (\underline{C} + (k-m)V)y_t - V\sum_{i=1}^{m} y_{t+i} + \underline{C}y_{t-k}$. Hence, for any $k \in [1,\hat{k}]_{\mathbb{Z}}$, $m \in [0,k-1]_{\mathbb{Z}}$, and $t \in [k+1, T-m]_{\mathbb{Z}}$,

$$v_{km}(t) = x_t - x_{t-k} - (\underline{C} + (k-m)V)y_t - V\sum_{i=1}^{m} y_{t+i} + \underline{C}y_{t-k} + \sum_{s=0}^{\hat{s}_{km}} (\underline{C} + (k-s)V - \overline{V}) \max\{y_{t-s} - y_{t-s-1}, 0\}.$$

Determining $\theta(t)$ for all $t \in [1, T]_{\mathbb{Z}}$ can be done recursively in $O(T)$ time by setting $\theta(1) = 0$ and setting $\theta(t) = \theta(t-1) + \max\{y_t - y_{t-1}, 0\}$ for $t = 2, \ldots, T$. Clearly, for each $k \in [1,\hat{k}]_{\mathbb{Z}}$ and each $m \in [0, k-1]_{\mathbb{Z}}$, the value of $v_{km}(k+1)$ can be determined in $O(T)$ time. For any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [k+2, T-m]_{\mathbb{Z}}$,

$$\begin{aligned} v_{km}(t) - v_{km}(t-1) = {} & (x_t - x_{t-1}) - (x_{t-k} - x_{t-k-1}) - (\underline{C} + (k-m)V)(y_t - y_{t-1}) \\ & - V\left[\sum_{i=1}^{m} y_{t+i} - \sum_{i=1}^{m} y_{t+i-1}\right] + \underline{C}(y_{t-k} - y_{t-k-1}) \\ & + (\underline{C} + kV - \overline{V})\left[\sum_{s=0}^{\hat{s}_{km}} \max\{y_{t-s} - y_{t-s-1}, 0\} - \sum_{s=0}^{\hat{s}_{km}} \max\{y_{t-s-1} - y_{t-s-2}, 0\}\right] \\ & - V\left[\sum_{s=0}^{\hat{s}_{km}} s \max\{y_{t-s} - y_{t-s-1}, 0\} - \sum_{s=0}^{\hat{s}_{km}} s \max\{y_{t-s-1} - y_{t-s-2}, 0\}\right] \\ = {} & (x_t - x_{t-1}) - (x_{t-k} - x_{t-k-1}) - (\underline{C} + (k-m)V)(y_t - y_{t-1}) \\ & - V(y_{t+m} - y_t) + \underline{C}(y_{t-k} - y_{t-k-1}) \\ & + (\underline{C} + kV - \overline{V})\left[\max\{y_t - y_{t-1}, 0\} - \max\{y_{t-\hat{s}_{km}-1} - y_{t-\hat{s}_{km}-2}, 0\}\right] \\ & - V\left[\sum_{s=1}^{\hat{s}_{km}} \max\{y_{t-s} - y_{t-s-1}, 0\} - \hat{s}_{km} \max\{y_{t-\hat{s}_{km}-1} - y_{t-\hat{s}_{km}-2}, 0\}\right]. \end{aligned}$$

This, together with (EC.22), implies that

$$\begin{aligned} v_{km}(t) = {} & v_{km}(t-1) + (x_t - x_{t-1}) - (x_{t-k} - x_{t-k-1}) - (\underline{C} + (k-m)V)(y_t - y_{t-1}) \\ & - V(y_{t+m} - y_t) + \underline{C}(y_{t-k} - y_{t-k-1}) \\ & + (\underline{C} + kV - \overline{V})\left[\max\{y_t - y_{t-1}, 0\} - \max\{y_{t-\hat{s}_{km}-1} - y_{t-\hat{s}_{km}-2}, 0\}\right] \\ & - V\left[\theta(t-1) - \theta(t - \hat{s}_{km} - 1) - \hat{s}_{km} \max\{y_{t-\hat{s}_{km}-1} - y_{t-\hat{s}_{km}-2}, 0\}\right]. \end{aligned}$$

Thus, for each $k \in [1, \hat{k}]_{\mathbb{Z}}$ and $m \in [0, k-1]_{\mathbb{Z}}$, the values of $v_{km}(k+1), v_{km}(k+2), \ldots, v_{km}(T-m)$ can be determined recursively in $O(T)$ time. Hence, the values of k, m, t and the set $\mathcal{S}$ corresponding to the largest possible violation of inequality (19) can be obtained in $O(T^3)$ time.

Next, we consider inequality (20). Consider any given $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{2T}$. Let $x'_t = x_{T-t+1}$ and $y'_t = y_{T-t+1}$ for $t \in [1, T]_{\mathbb{Z}}$. Inequality (20) becomes

$$\begin{aligned} x'_{T-t+1} - x'_{T-t-k+1} \le {} & (\underline{C} + (k-m)V)y'_{T-t+1} + V\sum_{i=1}^{m} y'_{T-t+i+1} - \underline{C}y'_{T-t-k+1} \\ & - \sum_{s \in \mathcal{S}} (\underline{C} + (k-s)V - \overline{V})(y'_{T-t-s+1} - y'_{T-t-s}). \end{aligned}$$

Letting $t' = T - t + 1$, this inequality becomes

$$\begin{aligned} x'_{t'} - x'_{t'-k} \le {} & (\underline{C} + (k-m)V)y'_{t'} + V\sum_{i=1}^{m} y'_{t'+i} - \underline{C}y'_{t'-k} \\ & - \sum_{s \in \mathcal{S}} (\underline{C} + (k-s)V - \overline{V})(y'_{t'-s} - y'_{t'-s-1}). \end{aligned} \tag{EC.23}$$

Because the values of k, m, t and the set $\mathcal{S}$ corresponding to the largest possible violation of inequality (19) can be obtained in $O(T^3)$ time, the values of k, m, t' and the set $\mathcal{S}$ corresponding to the largest possible violation of inequality (EC.23) can be obtained in $O(T^3)$ time. Hence, the values of k, m, t and the set $\mathcal{S}$ corresponding to the largest possible violation of inequality (20) can be obtained in $O(T^3)$ time. $\square$

A.13. Proof of Proposition 11

Proposition 11. *Consider any* $k \in [1, T-1]_{\mathbb{Z}}$ *such that* $\overline{C} - \underline{C} - kV > 0$, *any* $m \in [0, k-1]_{\mathbb{Z}}$, *and any* $\mathcal{S} \subseteq [0, \min\{k-1, L-m-2\}]_{\mathbb{Z}}$. *For any* $t \in [k+1, T-m-1]_{\mathbb{Z}}$, *the inequality*

$$\begin{aligned} x_t - x_{t-k} \le (\underline{C} + (k-m)V - \overline{V})y_{t+m+1} + V\sum_{i=1}^{m} y_{t+i} + \overline{V}y_t - \underline{C}y_{t-k} \\ - \sum_{s\in\mathcal{S}}(\underline{C} + (k-s)V - \overline{V})(y_{t-s} - y_{t-s-1}) \end{aligned} \tag{21}$$

is valid and facet-defining for $\operatorname{conv}(\mathcal{P})$. *For any* $t \in [m+2, T-k]_{\mathbb{Z}}$, *the inequality*

$$\begin{aligned} x_t - x_{t+k} \le (\underline{C} + (k-m)V - \overline{V})y_{t-m-1} + V\sum_{i=1}^{m} y_{t-i} + \overline{V}y_t - \underline{C}y_{t+k} \\ - \sum_{s\in\mathcal{S}}(\underline{C} + (k-s)V - \overline{V})(y_{t+s} - y_{t+s+1}) \end{aligned} \tag{22}$$

is valid and facet-defining for $\operatorname{conv}(\mathcal{P})$.

Proof. We first prove that inequality (21) is valid and facet-defining for $\operatorname{conv}(\mathcal{P})$. For notational convenience, we define $s_{\max} = \max\{s : s \in \mathcal{S}\}$ if $\mathcal{S} \neq \emptyset$, and $s_{\max} = -1$ if $\mathcal{S} = \emptyset$. Consider any $t \in [k+1, T-m-1]_{\mathbb{Z}}$. To prove that the linear inequality (21) is valid for $\operatorname{conv}(\mathcal{P})$, it suffices to show that it is valid for $\mathcal{P}$. Consider any element $(\mathbf{x}, \mathbf{y})$ of $\mathcal{P}$. We show that $(\mathbf{x}, \mathbf{y})$ satisfies (21). We divide the analysis into three cases:

Case 1: $y_t = 0$. In this case, by (2c) and (2d), $-x_{t-k} \le -\underline{C}y_{t-k}$ and $x_t = 0$. Thus, the left-hand side of (21) is at most $-\underline{C}y_{t-k}$. Because $y_t = 0$ and $t \in [2, T]_{\mathbb{Z}}$, by Lemma 1(i), $y_{t-j} - y_{t-j-1} \le 0$ for all $j \in [0, \min\{t-2, L-1\}]_{\mathbb{Z}}$. Because $\mathcal{S} \subseteq [0, \min\{k-1, L-m-2\}]_{\mathbb{Z}}$, $m \ge 0$, and $t \ge k+1$, we have $\mathcal{S} \subseteq [0, \min\{t-2, L-1\}]_{\mathbb{Z}}$. Thus, $y_{t-s} - y_{t-s-1} \le 0$ for all $s \in \mathcal{S}$. Because $m \le k-1$, $\mathcal{S} \subseteq [0, k-1]_{\mathbb{Z}}$, and $\underline{C} + V > \overline{V}$, the coefficients "$\underline{C} + (k-m)V - \overline{V}$" and "$\underline{C} + (k-s)V - \overline{V}$" on the right-hand side of (21) are positive for any $s \in \mathcal{S}$. Thus, the right-hand side of (21) is at least $-\underline{C}y_{t-k}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (21).

Case 2: $y_t = 1$ and $y_{t-s'} - y_{t-s'-1} = 1$ for some $s' \in \mathcal{S}$. In this case, $y_{t-s'} = 1$ and $y_{t-s'-1} = 0$. Because $y_t = 1$ and $t \in [2, T]_{\mathbb{Z}}$, by Lemma 1(ii), there exists at most one $j \in [0, \min\{t-2, L\}]_{\mathbb{Z}}$ such that $y_{t-j} - y_{t-j-1} = 1$. Because $\mathcal{S} \subseteq [0, \min\{k-1, L-m-2\}]_{\mathbb{Z}}$, $m \ge 0$, and $t \ge k+1$, we have $\mathcal{S} \subseteq [0, \min\{t-2, L\}]_{\mathbb{Z}}$. Thus, $y_{t-s} - y_{t-s-1} \le 0$ for all $s \in \mathcal{S} \setminus \{s'\}$. Because $y_{t-s'} - y_{t-s'-1} = 1$ and $t - s' \in [2, T]_{\mathbb{Z}}$, by (2a), we have $y_\tau = 1$ for all $\tau \in [t-s', \min\{T, t-s'+L-1\}]_{\mathbb{Z}}$. Because $\mathcal{S} \subseteq [0, L-m-2]_{\mathbb{Z}}$, we have $t - s' + L - 1 \ge t + m + 1$. Thus, $y_\tau = 1$ for all $\tau \in [t-s', t+m+1]_{\mathbb{Z}}$, which implies that $(\underline{C} + (k-m)V - \overline{V})y_{t+m+1} + V\sum_{i=1}^{m} y_{t+i} + \overline{V}y_t = \underline{C} + kV$. Because $\mathcal{S} \subseteq [0, k-1]_{\mathbb{Z}}$ and $\underline{C} + V > \overline{V}$, the coefficient "$\underline{C} + (k-s)V - \overline{V}$" on the right-hand side of inequality (21) is positive for all $s \in \mathcal{S}$. Hence, the right-hand side of (21) is at least $\underline{C} + kV - \underline{C}y_{t-k} - (\underline{C} + (k-s')V - \overline{V}) = s'V + \overline{V} - \underline{C}y_{t-k}$. By (2e), $\sum_{\tau=t-s'}^{t}(x_\tau - x_{\tau-1}) \le \sum_{\tau=t-s'}^{t} Vy_{\tau-1} + \sum_{\tau=t-s'}^{t} \overline{V}(1 - y_{\tau-1})$, which implies that $x_t - x_{t-s'-1} \leq s'V + \overline{V}$. Because $y_{t-s'-1} = 0$, by (2d), $x_{t-s'-1} = 0$. By (2c), $-x_{t-k} \leq -\underline{C}y_{t-k}$. Thus, $x_t - x_{t-k} \leq s'V + \overline{V} - \underline{C}y_{t-k}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (21).

Case 3: $y_t = 1$ and $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, k-1]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $\underline{C} + V > \overline{V}$, we have $\underline{C} + (k-m)V - \overline{V} > 0$ and $\underline{C} + (k-s)V - \overline{V} > 0$ for each $s \in \mathcal{S}$. Hence, the terms "$(\underline{C} + (k-m)V - \overline{V})y_{t+m+1}$" and "$-\sum_{s \in \mathcal{S}}(\underline{C} + (k-s)V - \overline{V})(y_{t-s} - y_{t-s-1})$" on the right-hand side of inequality (21) are nonnegative. We divide our analysis into three subcases.

Case 3.1: $y_\tau = 0$ for some $\tau \in [t, t+m+1]_{\mathbb{Z}}$. Let $t' = \min\{\tau \in [t, t+m+1]_{\mathbb{Z}} : y_\tau = 0\}$. Then, $y_\tau = 1$ for all $\tau \in [t, t'-1]_{\mathbb{Z}}$. Thus, the right-hand side of (21) is at least $(t'-t-1)V + \overline{V} - \underline{C}y_{t-k}$. By (2f), $\sum_{\tau=t+1}^{t'}(x_{\tau-1} - x_\tau) \leq \sum_{\tau=t+1}^{t'} Vy_\tau + \sum_{\tau=t+1}^{t'} \overline{V}(1-y_\tau)$, which implies that $x_t - x_{t'} \leq (t'-t-1)V + \overline{V}$. Because $y_{t'} = 0$, by (2d), $x_{t'} = 0$. By (2c), $-x_{t-k} \leq -\underline{C}y_{t-k}$. Hence, $x_t - x_{t-k} \leq (t'-t-1)V + \overline{V} - \underline{C}y_{t-k}$. Thus, the left-hand side of (21) is less than or equal to the right-hand side.

Case 3.2: $y_\tau = 1$ for all $\tau \in [t, t+m+1]_{\mathbb{Z}}$ and $y_\tau = 0$ for some $\tau \in [t-k, t-1]_{\mathbb{Z}}$. In this case, the right-hand side of (21) is at least $(\underline{C} + (k-m)V - \overline{V}) + mV + \overline{V} - \underline{C}y_{t-k} = \underline{C} + kV - \underline{C}y_{t-k}$. Let $t' = \max\{\tau \in [t-k, t-1]_{\mathbb{Z}} : y_\tau = 0\}$. Because $t' \geq t-k$ and $\underline{C} + V > \overline{V}$, the right-hand side of (21) is greater than $\overline{V} + (t-t'-1)V - \underline{C}y_{t-k}$. By (2e), $\sum_{\tau=t'+1}^{t}(x_\tau - x_{\tau-1}) \leq \sum_{\tau=t'+1}^{t} Vy_{\tau-1} + \sum_{\tau=t'+1}^{t} \overline{V}(1 - y_{\tau-1})$, which implies that $x_t - x_{t'} \leq \overline{V} + (t-t'-1)V$. Because $y_{t'} = 0$, by (2d), $x_{t'} = 0$. By (2c), $-x_{t-k} \leq -\underline{C}y_{t-k}$. Hence, $x_t - x_{t-k} \leq \overline{V} + (t-t'-1)V - \underline{C}y_{t-k}$. Thus, the left-hand side of (21) is less than the right-hand side.

Case 3.3: $y_\tau = 1$ for all $\tau \in [t-k, t+m+1]_{\mathbb{Z}}$. In this case, the right-hand side of (21) is at least kV. By (2e), $\sum_{\tau=t-k+1}^{t}(x_\tau - x_{\tau-1}) \leq \sum_{\tau=t-k+1}^{t} Vy_{\tau-1} + \sum_{\tau=t-k+1}^{t} \overline{V}(1-y_{\tau-1})$, which implies that $x_t - x_{t-k} \leq kV$. Thus, the left-hand side of (21) is less than or equal to the right-hand side.

In Cases 3.1–3.3, $(\mathbf{x}, \mathbf{y})$ satisfies (21). Summarizing Cases 1–3, we conclude that (21) is valid for $\text{conv}(\mathcal{P})$.

Consider any $t \in [k+1, T-m-1]_{\mathbb{Z}}$. To prove that inequality (21) is facet-defining for $\text{conv}(\mathcal{P})$, it suffices to show that there exist $2T$ affinely independent points in $\text{conv}(\mathcal{P})$ that satisfy (21) at equality. Because $\mathbf{0} \in \text{conv}(\mathcal{P})$ and $\mathbf{0}$ satisfies (21) at equality, it suffices to create the remaining $2T-1$ nonzero linearly independent points. We denote these $2T-1$ points as $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ for $r \in [1, T]_{\mathbb{Z}} \setminus \{t-k\}$ and $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ for $r \in [1, T]_{\mathbb{Z}}$, and denote the qth component of $\bar{\mathbf{x}}^r$, $\bar{\mathbf{y}}^r$, $\hat{\mathbf{x}}^r$, and $\hat{\mathbf{y}}^r$ as $\bar{x}_q^r$, $\bar{y}_q^r$, $\hat{x}_q^r$, and $\hat{y}_q^r$, respectively. Let $\epsilon = \min\{\overline{V} - \underline{C}, \overline{C} - \underline{C} - kV\} > 0$.

We divide these $2T-1$ points into the following eight groups.

(A1) For each $r \in [1, t-1]_{\mathbb{Z}} \setminus \{t-k\}$, we create a point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as follows:

$$\bar{x}_q^r = \begin{cases} \underline{C}, & \text{for } q \in [1, t-1]_{\mathbb{Z}} \setminus \{r\}; \\ \underline{C} + \epsilon, & \text{for } q = r; \\ \overline{V}, & \text{for } q = t; \\ 0, & \text{for } q \in [t+1, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\bar{y}_q^r = \begin{cases} 1, \text{ for } q \in [1,t]_{\mathbb{Z}}; \\ 0, \text{ for } q \in [t+1,T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (2a)–(2f). Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \operatorname{conv}(\mathcal{P})$. Note that $\bar{x}_t^r = \overline{V}$, $\bar{x}_{t-k}^r = \underline{C}$, $\bar{y}_t^r = \bar{y}_{t-k}^r = 1$, $\bar{y}_{t+m+1}^r = 0$, $\sum_{i=1}^m \bar{y}_{t+i}^r = 0$, and $\bar{y}_{t-s}^r - \bar{y}_{t-s-1}^r = 0$ for all $s \in \mathcal{S}$. Hence, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (21) at equality.

(A2) We create the same point $(\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t)$ as in group (A3) in the proof of Proposition 9. Thus, $(\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t) \in \operatorname{conv}(\mathcal{P})$. Note that $\bar{x}_t^t = \underline{C} + kV + \epsilon$, $\bar{x}_{t-k}^t = \underline{C} + \epsilon$, $\bar{y}_t^t = \bar{y}_{t-k}^t = \bar{y}_{t+m+1}^t = 1$, $\sum_{i=1}^m \bar{y}_{t+i}^t = m$, and $\bar{y}_{t-s}^t - \bar{y}_{t-s-1}^t = 0$ for all $s \in \mathcal{S}$. Hence, $(\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t)$ satisfies (21) at equality.

(A3) For each $r \in [t+1,T]_{\mathbb{Z}}$, we create a point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as follows:

$$\bar{x}_q^r = \begin{cases} \underline{C}, & \text{for } q \in [1,t-k-1]_{\mathbb{Z}}; \\ \underline{C} + (q-t+k)V, & \text{for } q \in [t-k,t]_{\mathbb{Z}}; \\ \underline{C} + kV, & \text{for } q \in [t+1,T]_{\mathbb{Z}} \setminus \{r\}; \\ \underline{C} + kV + \epsilon, & \text{for } q = r; \end{cases}$$

and $\bar{y}_q^r = 1$ for all $q \in [1,T]_{\mathbb{Z}}$. It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (2a)–(2d). Note that $\bar{x}_q^r - \bar{x}_{q-1}^r = 0$ when $q \in [2,t-k]_{\mathbb{Z}}$, $\bar{x}_q^r - \bar{x}_{q-1}^r = V$ when $q \in [t-k+1,t]_{\mathbb{Z}}$, and $-\epsilon \le \bar{x}_q^r - \bar{x}_{q-1}^r \le \epsilon$ when $q \in [t+1,T]_{\mathbb{Z}}$. Thus, $-V\bar{y}_q^r - \overline{V}(1-\bar{y}_q^r) \le \bar{x}_q^r - \bar{x}_{q-1}^r \le V\bar{y}_{q-1}^r + \overline{V}(1-\bar{y}_{q-1}^r)$ for all $q \in [2,T]_{\mathbb{Z}}$. Hence, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (2e) and (2f). Therefore, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \operatorname{conv}(\mathcal{P})$. Note that $\bar{x}_t^r = \underline{C} + kV$, $\bar{x}_{t-k}^r = \underline{C}$, $\bar{y}_t^r = \bar{y}_{t-k}^r = \bar{y}_{t+m+1}^r = 1$, $\sum_{i=1}^m \bar{y}_{t+i}^r = m$, and $\bar{y}_{t-s}^r - \bar{y}_{t-s-1}^r = 0$ for all $s \in \mathcal{S}$. Hence, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (21) at equality.

(A4) For each $r \in [1,t-1]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (A2) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \operatorname{conv}(\mathcal{P})$. To show that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (21) at equality, we first consider the case where $t-r-1 \notin \mathcal{S}$. In this case, $\hat{x}_q^r = \hat{y}_q^r = 0$ for all $q \in [t,t+m+1]_{\mathbb{Z}}$. Because $t-k \le t-s_{\max}-1 \le r$, we have $\hat{x}_{t-k}^r = \underline{C}$, and $\hat{y}_{t-k}^r = 1$. Because $t-s-1 \ne r$ for all $s \in \mathcal{S}$, we have $\hat{y}_{t-s}^r - \hat{y}_{t-s-1}^r = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (21) at equality. Next, we consider the case where $t-r-1 \in \mathcal{S}$. In this case, $\hat{x}_t^r = \overline{V} + (t-r-1)V$ and $\hat{y}_q^r = 1$ for all $q \in [t,t+m+1]_{\mathbb{Z}}$. Because $t-k \le t-s_{\max}-1 \le r$, we have $\hat{x}_{t-k}^r = \hat{y}_{t-k}^r = 0$. In addition, $\hat{y}_{t-s}^r - \hat{y}_{t-s-1}^r = 1$ when $s = t-r-1$, and $\hat{y}_{t-s}^r - \hat{y}_{t-s-1}^r = 0$ when $s \ne t-r-1$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (21) at equality.

(A5) For each $r \in [t,t+m]_{\mathbb{Z}}$, we create a point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as follows:

$$\hat{x}_q^r = \begin{cases} \underline{C}, & \text{for } q \in [1,2t-r-1]_{\mathbb{Z}}; \\ \overline{V} + (q+r-2t)V, & \text{for } q \in [2t-r,t-1]_{\mathbb{Z}}; \\ \overline{V} + (r-q)V, & \text{for } q \in [t,r]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [r+1,T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^r = \begin{cases} 1, \text{ for } q \in [1,r]_{\mathbb{Z}}; \\ 0, \text{ for } q \in [r+1,T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2a)–(2d). Note that $-V \le \hat{x}^r_q - \hat{x}^r_{q-1} \le V$ when $q \in [2, r]_{\mathbb{Z}}$, $\hat{x}^r_q - \hat{x}^r_{q-1} = -\overline{V}$ when $q = r+1$, and $\hat{x}^r_q - \hat{x}^r_{q-1} = 0$ when $q \in [r+2, T]_{\mathbb{Z}}$. Thus, $-V\hat{y}^r_q - \overline{V}(1-\hat{y}^r_q) \le \hat{x}^r_q - \hat{x}^r_{q-1} \le V\hat{y}^r_{q-1} + \overline{V}(1-\hat{y}^r_{q-1})$ for all $q \in [2, T]_{\mathbb{Z}}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2e) and (2f). Therefore, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \operatorname{conv}(\mathcal{P})$. Note that $\hat{x}^r_t = \overline{V} + (r-t)V$ and $\hat{y}^r_t = 1$. Note also that $\hat{y}^r_{t+m+1} = 0$ and $V\sum_{i=1}^m y^r_{t+i} = (r-t)V$. Because $t-k \le t-m-1 \le 2t-r-1$, we have $\hat{x}^r_{t-k} = \underline{C}$ and $\hat{y}^r_{t-k} = 1$. For any $s \in \mathcal{S}$, because $t-s \le t \le r$, we have $\hat{y}^r_{t-s} - \hat{y}^r_{t-s-1} = 0$. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (21) at equality.

(A6) We create a point $(\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1})$ as follows:

$$\hat{\mathbf{x}}^{t+m+1}_q = \begin{cases} \underline{C}, & \text{for } q \in [1, t-k-1]_{\mathbb{Z}}; \\ \underline{C} + (q-t+k)V, & \text{for } q \in [t-k, t]_{\mathbb{Z}}; \\ \underline{C} + kV, & \text{for } q \in [t+1, T]_{\mathbb{Z}}; \end{cases}$$

and $\hat{y}^{t+m+1}_q = 1$ for all $q \in [1, T]_{\mathbb{Z}}$. It is easy to verify that $(\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1})$ satisfies (2a)–(2d). Note that $\hat{x}^{t+m+1}_q - \hat{x}^{t+m+1}_{q-1} = 0$ when $q \in [2, t-k]_{\mathbb{Z}}$, $\hat{x}^{t+m+1}_q - \hat{x}^{t+m+1}_{q-1} = V$ when $q \in [t-k+1, t]_{\mathbb{Z}}$, and $\hat{x}^{t+m+1}_q - \hat{x}^{t+m+1}_{q-1} = 0$ when $q \in [t+1, T]_{\mathbb{Z}}$. Thus, $-V\hat{y}^{t+m+1}_q - \overline{V}(1-\hat{y}^{t+m+1}_q) \le \hat{x}^{t+m+1}_q - \hat{x}^{t+m+1}_{q-1} \le V\hat{y}^{t+m+1}_{q-1} + \overline{V}(1-\hat{y}^{t+m+1}_{q-1})$ for all $q \in [2, T]_{\mathbb{Z}}$. Hence, $(\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1})$ satisfies (2e) and (2f). Therefore, $(\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1}) \in \operatorname{conv}(\mathcal{P})$. Note that $\hat{x}^{t+m+1}_t = \underline{C} + kV$, $\hat{x}^{t+m+1}_{t-k} = \underline{C}$, $\hat{y}^{t+m+1}_{t+m+1} = \hat{y}^{t+m+1}_t = \hat{y}^{t+m+1}_{t-k} = 1$, $\sum_{i=1}^m y^{t+m+1}_{t+i} = m$, and $\hat{y}^{t+m+1}_{t-s} - \hat{y}^{t+m+1}_{t-s-1} = 0$ for all $s \in \mathcal{S}$, Thus, $(\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1})$ satisfies (21) at equality.

(A7) For each $r \in [t+m+2, T]_{\mathbb{Z}}$, we create a point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as follows:

$$\hat{x}^r_q = \begin{cases} 0, & \text{for } q \in [1, r-1]_{\mathbb{Z}}; \\ \underline{C}, & \text{for } q \in [r, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}^r_q = \begin{cases} 0, & \text{for } q \in [1, r-1]_{\mathbb{Z}}; \\ 1, & \text{for } q \in [r, T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2a)–(2f). Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \operatorname{conv}(\mathcal{P})$. Note that $\hat{x}^r_t = \hat{x}^r_{t-k} = 0$, $\hat{y}^r_t = \hat{y}^r_{t-k} = \hat{y}^r_{t+m+1} = 0$, $\sum_{i=1}^m y^r_{t+i} = 0$, and $\hat{y}^r_{t-s} - \hat{y}^r_{t-s-1} = 0$ for all $s \in \mathcal{S}$, Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (21) at equality.

Table EC.9 shows a matrix with $2T-1$ rows, where each row represents a point created by this process. This matrix can be transformed into the matrix in Table EC.10 via the following Gaussian elimination process:

(i) For each $r \in [1, t-1]_{\mathbb{Z}} \setminus \{t-k\}$, the point with index r in group (B1), denoted $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point with index r in group (A1), and $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point with index t in group (A5).

(ii) The point in group (B2), denoted $(\underline{\bar{\mathbf{x}}}^t, \underline{\bar{\mathbf{y}}}^t)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^t, \underline{\bar{\mathbf{y}}}^t) = (\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t) - (\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1})$. Here, $(\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t)$ is the point in group (A2), and $(\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1})$ is the point in group (A6).

Table EC.9 A matrix with the rows representing $2T-1$ linearly independent points in $\text{conv}(\mathcal{P})$ satisfying inequality (21) at equality

Group	Point	Index r	$\mathbf{x}$																$\mathbf{y}$										
			1	$\cdots$	$t-k-1$	$t-k$	$t-k+1$	$\cdots$	$t-1$	t	$t+1$	$\cdots$	$t+m$	$t+m+1$	$t+m+2$	$\cdots$	T	1	$\cdots$	$t-1$	t	$t+1$	$\cdots$	$t+m$	$t+m+1$	$t+m+2$	$\cdots$	T
(A1)		1	$\underline{C}+\epsilon$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	$\overline{V}$	0	$\cdots$	0	0	0	$\cdots$	0	1	$\cdots$	1	1	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-k-1$	$\underline{C}$	$\cdots$	$\underline{C}+\epsilon$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	$\overline{V}$	0	$\cdots$	0	0	0	$\cdots$	0	1	$\cdots$	1	1	0	$\cdots$	0	0	0	$\cdots$	0
		$t-k+1$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}+\epsilon$	$\cdots$	$\underline{C}$	$\overline{V}$	0	$\cdots$	0	0	0	$\cdots$	0	1	$\cdots$	1	1	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-1$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}+\epsilon$	$\overline{V}$	0	$\cdots$	0	0	0	$\cdots$	0	1	$\cdots$	1	1	0	$\cdots$	0	0	0	$\cdots$	0
(A2)	$(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$	t	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}+\epsilon$	$\underline{C}+V+\epsilon$	$\cdots$	$\underline{C}+(k-1)V+\epsilon$	$\underline{C}+kV+\epsilon$	$\underline{C}+kV$	$\cdots$	$\underline{C}+kV$	$\underline{C}+kV$	$\underline{C}+kV$	$\cdots$	$\underline{C}+kV$	1	$\cdots$	1	1	1	$\cdots$	1	1	1	$\cdots$	1
(A3)		$t+1$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}+V$	$\cdots$	$\underline{C}+(k-1)V$	$\underline{C}+kV$	$\underline{C}+kV+\epsilon$	$\cdots$	$\underline{C}+kV$	$\underline{C}+kV$	$\underline{C}+kV$	$\cdots$	$\underline{C}+kV$	1	$\cdots$	1	1	1	$\cdots$	1	1	1	$\cdots$	1
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t+m$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}+V$	$\cdots$	$\underline{C}+(k-1)V$	$\underline{C}+kV$	$\underline{C}+kV$	$\cdots$	$\underline{C}+kV+\epsilon$	$\underline{C}+kV$	$\underline{C}+kV$	$\cdots$	$\underline{C}+kV$	1	$\cdots$	1	1	1	$\cdots$	1	1	1	$\cdots$	1
		$t+m+1$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}+V$	$\cdots$	$\underline{C}+(k-1)V$	$\underline{C}+kV$	$\underline{C}+kV$	$\cdots$	$\underline{C}+kV$	$\underline{C}+kV+\epsilon$	$\underline{C}+kV$	$\cdots$	$\underline{C}+kV$	1	$\cdots$	1	1	1	$\cdots$	1	1	1	$\cdots$	1
		$t+m+2$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}+V$	$\cdots$	$\underline{C}+(k-1)V$	$\underline{C}+kV$	$\underline{C}+kV$	$\cdots$	$\underline{C}+kV$	$\underline{C}+kV$	$\underline{C}+kV+\epsilon$	$\cdots$	$\underline{C}+kV$	1	$\cdots$	1	1	1	$\cdots$	1	1	1	$\cdots$	1
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		T	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}+V$	$\cdots$	$\underline{C}+(k-1)V$	$\underline{C}+kV$	$\underline{C}+kV$	$\cdots$	$\underline{C}+kV$	$\underline{C}+kV$	$\underline{C}+kV$	$\cdots$	$\underline{C}+kV+\epsilon$	1	$\cdots$	1	1	1	$\cdots$	1	1	1	$\cdots$	1
(A4)		1	(See Note EC.9-1)															(See Note EC.9-1)										
		$\vdots$																										
		$t-1$																										
(A5)	$(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$	t	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	$\overline{V}$	0	$\cdots$	0	0	0	$\cdots$	0	1	$\cdots$	1	1	0	$\cdots$	0	0	0	$\cdots$	0
		$t+1$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}$	$\cdots$	$\overline{V}$	$\overline{V}+V$	$\overline{V}$	$\cdots$	0	0	0	$\cdots$	0	1	$\cdots$	1	1	1	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t+m$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	(See Note EC.9-2)	$\cdots$	$\overline{V}+(m-1)V$	$\overline{V}+mV$	$\overline{V}+(m-1)V$	$\cdots$	$\overline{V}$	0	0	$\cdots$	0	1	$\cdots$	1	1	1	$\cdots$	1	0	0	$\cdots$	0
(A6)		$t+m+1$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}+V$	$\cdots$	$\underline{C}+(k-1)V$	$\underline{C}+kV$	$\underline{C}+kV$	$\cdots$	$\underline{C}+kV$	$\underline{C}+kV$	$\underline{C}+kV$	$\cdots$	$\underline{C}+kV$	1	$\cdots$	1	1	1	$\cdots$	1	1	1	$\cdots$	1
(A7)		$t+m+2$	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\underline{C}$	$\cdots$	$\underline{C}$	0	$\cdots$	0	0	0	$\cdots$	0	0	1	$\cdots$	1
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	$\underline{C}$	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	1

Note EC.9-1: For $r \in [1, t-1]_{\mathbb{Z}}$, the $\mathbf{x}$ and $\mathbf{y}$ vectors in group (A4) are given as follows:

$\hat{\mathbf{x}}^r = (\underbrace{\underline{C}, \ldots, \underline{C}}_{r \text{ terms}}, \underbrace{0, \ldots, 0}_{T-r \text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{1, \ldots, 1}_{r \text{ terms}}, \underbrace{0, \ldots, 0}_{T-r \text{ terms}})$ if $t-r-1 \notin \mathcal{S}$;

$\hat{\mathbf{x}}^r = (\underbrace{0, \ldots, 0}_{r \text{ terms}}, \underbrace{\overline{V}, \overline{V}+V, \overline{V}+2V, \ldots, \overline{V}(r-t-1)V}_{t-r \text{ terms}}, \underbrace{\overline{V}+(t-r-1)V, \ldots, \overline{V}+(t-r-1)V}_{T-t \text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{0, \ldots, 0}_{r \text{ terms}}, \underbrace{1, \ldots, 1}_{T-r \text{ terms}})$ if $t-r-1 \in \mathcal{S}$.

Note EC.9-2: In group (A5), $\hat{x}^{t+m}_{t-k+1} = \underline{C}$ if $m < k-1$, and $\hat{x}^{t+m}_{t-k+1} = \overline{V}$ if $m = k-1$.

Table EC.10 Lower triangular matrix obtained from Table EC.9 via Gaussian elimination

Group	Point	Index r	$\mathbf{x}$															$\mathbf{y}$										
			1	$\cdots$	$t-k-1$	$t-k$	$t-k+1$	$\cdots$	$t-1$	t	$t+1$	$\cdots$	$t+m$	$t+m+1$	$t+m+2$	$\cdots$	T	1	$\cdots$	$t-1$	t	$t+1$	$\cdots$	$t+m$	$t+m+1$	$t+m+2$	$\cdots$	T
(B1)		1	ϵ	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-k-1$	0	$\cdots$	ϵ	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0
		$t-k+1$	0	$\cdots$	0	0	ϵ	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-1$	0	$\cdots$	0	0	0	$\cdots$	ϵ	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0
(B2)	$(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$	t	0	0	$\cdots$	ϵ	ϵ	$\cdots$	ϵ	ϵ	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0
(B3)		$t+1$	0	0	$\cdots$	0	0	$\cdots$	0	0	ϵ	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t+m$	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	ϵ	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0
		$t+m+1$	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	ϵ	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0
		$t+m+2$	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	ϵ	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		T	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	ϵ	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0
(B4)		1																1	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	(Omitted)															$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-1$																1	$\cdots$	1	0	0	$\cdots$	0	0	0	$\cdots$	0
(B5)		t																1	$\cdots$	1	1	0	$\cdots$	0	0	0	$\cdots$	0
		$t+1$																1	$\cdots$	1	1	1	$\cdots$	0	0	0	$\cdots$	0
	$(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$	$\vdots$	(Omitted)															$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t+m$																1	$\cdots$	1	1	1	$\cdots$	1	0	0	$\cdots$	0
(B6)		$t+m+1$	(Omitted)															1	$\cdots$	1	1	1	$\cdots$	1	1	0	$\cdots$	0
(B7)		$t+m+2$																0	$\cdots$	0	0	0	$\cdots$	0	0	1	$\cdots$	0
		$\vdots$	(Omitted)															$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T																0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	1

(iii) For each $r \in [t+1, T]_{\mathbb{Z}}$, the point with index r in group (B3), denoted $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1})$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point with index r in group (A3), and $(\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1})$ is the point in group (A6).

(iv) For each $r \in [1, t-1]_{\mathbb{Z}}$, the point with index r in group (B4), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $t-r-1 \notin \mathcal{S}$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1}) - (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $t-r-1 \in \mathcal{S}$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (A4), and $(\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1})$ is the point in group (A6).

(v) For each $r \in [t, t+m]_{\mathbb{Z}}$, the point with index r in group (B5), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (A5).

(vi) The point in group (B6), denoted $(\underline{\hat{\mathbf{x}}}^{t+m+1}, \underline{\hat{\mathbf{y}}}^{t+m+1})$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^{t+m+1}, \underline{\hat{\mathbf{y}}}^{t+m+1}) = (\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1}) - (\hat{\mathbf{x}}^{t+m+2}, \hat{\mathbf{y}}^{t+m+2})$. Here, $(\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1})$ is the point in group (A6), and $(\hat{\mathbf{x}}^{t+m+2}, \hat{\mathbf{y}}^{t+m+2})$ is the point with index $t+m+2$ in group (A7).

(vii) For each $r \in [t+m+2, T]_{\mathbb{Z}}$, the point with index r in group (B7), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ if $r \neq T$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $r = T$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ and $(\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ are the points with indices r and $r+1$, respectively, in group (A7).

The matrix shown in Table EC.10 is lower triangular; that is, the position of the last nonzero component of a row of the matrix is greater than the position of the last nonzero component of the previous row. This implies that the $2T-1$ points in groups (A1)–(A7) are linearly independent. Therefore, inequality (21) is facet-defining for $\text{conv}(\mathcal{P})$.

Next, we show that inequality (22) is valid and facet-defining for $\text{conv}(\mathcal{P})$. Denote $x'_t = x_{T-t+1}$ and $y'_t = y_{T-t+1}$ for $t \in [1, T]_{\mathbb{Z}}$. Because inequality (21) is valid and facet-defining for $\text{conv}(\mathcal{P})$ for any $t \in [k+1, T-m-1]_{\mathbb{Z}}$, the inequality

$$x'_{T-t+1} - x'_{T-t+k+1} \leq (\underline{C} + (k-m)V - \overline{V})y'_{T-t-m} + V\sum_{i=1}^{m} y'_{T-t-i+1} + \overline{V}y'_{T-t+1} - \underline{C}y'_{T-t+k+1} - \sum_{s \in \mathcal{S}}(\underline{C} + (k-s)V - \overline{V})(y'_{T-t+s+1} - y'_{T-t+s+2})$$

is valid and facet-defining for $\text{conv}(\mathcal{P}')$ for any $t \in [k+1, T-m-1]_{\mathbb{Z}}$. Let $t' = T-t+1$. Then, the inequality

$$x'_{t'} - x'_{t'+k} \leq (\underline{C} + (k-m)V - \overline{V})y'_{t'-m-1} + V\sum_{i=1}^{m} y'_{t'-i} + \overline{V}y'_{t'} - \underline{C}y'_{t'+k} - \sum_{s \in \mathcal{S}}(\underline{C} + (k-s)V - \overline{V})(y'_{t'+s} - y'_{t'+s+1})$$

is valid and facet-defining for $\text{conv}(\mathcal{P}')$ for any $t' \in [m+2, T-k]_{\mathbb{Z}}$. Hence, by Lemma 2, inequality (22) is valid and facet-defining for $\text{conv}(\mathcal{P})$ for any $t \in [m+2, T-k]_{\mathbb{Z}}$. $\square$

A.14. Proof of Proposition 12

Proposition 12. *For any given point* $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{2T}$*, the most violated inequalities* (21) *and* (22) *can be determined in* $O(T^3)$ *time if such violated inequalities exist.*

Proof. We first consider inequality (21). Consider any given $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{2T}$. For notational convenience, denote $\hat{k} = \max\{k \in [1, T-1]_{\mathbb{Z}} : \overline{C} - \underline{C} - kV > 0\}$, and denote $\hat{s}_{km} = \min\{k-1, L-m-2\}$ for any $k \in [1, \hat{k}]_{\mathbb{Z}}$ and $m \in [0, k-1]_{\mathbb{Z}}$. For any $t \in [1, T]_{\mathbb{Z}}$, let

$$\theta(t) = \sum_{\tau=2}^{t} \max\{y_\tau - y_{\tau-1}, 0\}.$$

Then, for any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [k+1, T-m-1]_{\mathbb{Z}}$,

$$\sum_{s=1}^{\hat{s}_{km}} \max\{y_{t-s} - y_{t-s-1}, 0\} = \sum_{\tau=t-\hat{s}_{km}}^{t-1} \max\{y_\tau - y_{\tau-1}, 0\} = \theta(t-1) - \theta(t - \hat{s}_{km} - 1). \tag{EC.24}$$

For any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, $t \in [k+1, T-m-1]_{\mathbb{Z}}$, and $\mathcal{S} \subseteq [0, \hat{s}_{km}]_{\mathbb{Z}}$, let

$$\begin{aligned}\tilde{v}_{km}(\mathcal{S}, t) = x_t - x_{t-k} - (\underline{C} + (k-m)V - \overline{V})y_{t+m+1} - V\sum_{i=1}^{m} y_{t+i} - \overline{V}y_t + \underline{C}y_{t-k} \\ + \sum_{s \in \mathcal{S}} (\underline{C} + (k-s)V - \overline{V})(y_{t-s} - y_{t-s-1}).\end{aligned}$$

If $\tilde{v}_{km}(\mathcal{S}, t) > 0$, then $\tilde{v}_{km}(\mathcal{S}, t)$ is the amount of violation of inequality (21). If $\tilde{v}_{km}(\mathcal{S}, t) \le 0$, there is no violation of inequality (21). For any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [k+1, T-m-1]_{\mathbb{Z}}$, let

$$v_{km}(t) = \max_{\mathcal{S} \subseteq [0, \hat{s}_{km}]_{\mathbb{Z}}} \{\tilde{v}_{km}(\mathcal{S}, t)\}.$$

If $v_{km}(t) > 0$, then $v_{km}(t)$ is the largest possible violation of inequality (21) for this combination of k, m, and t. If $v_{km}(t) \le 0$, the largest possible violation of inequality (21) is zero for this combination of k, m, and t. Because $\underline{C} + V > \overline{V}$, we have $\underline{C} + (k-s)V - \overline{V} > 0$ for all $k \in [1, \hat{k}]_{\mathbb{Z}}$, $s \in [0, \hat{s}_{km}]_{\mathbb{Z}}$, and $m \in [0, k-1]_{\mathbb{Z}}$. Thus, for any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [k+1, T-m-1]_{\mathbb{Z}}$, $\tilde{v}_{km}(\mathcal{S}, t)$ is maximized when $\mathcal{S}$ contains all $s \in [0, \hat{s}_{km}]_{\mathbb{Z}}$ such that $y_{t-s} - y_{t-s-1} > 0$ (if any). If it does not exist any $s \in [0, \hat{s}]_{\mathbb{Z}}$ such that $y_{t-s} - y_{t-s-1} > 0$, then $\tilde{v}_{km}(\mathcal{S}, t)$ is maximized when $\mathcal{S} = \emptyset$, and $v_{km}(t) = x_t - x_{t-k} - (\underline{C} + (k-m)V - \overline{V})y_{t+m+1} - V\sum_{i=1}^{m} y_{t+i} - \overline{V}y_t + \underline{C}y_{t-k}$. Hence, for any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [k+1, T-m-1]_{\mathbb{Z}}$,

$$\begin{aligned}v_{km}(t) = x_t - x_{t-k} - (\underline{C} + (k-m)V - \overline{V})y_{t+m+1} - V\sum_{i=1}^{m} y_{t+i} - \overline{V}y_t + \underline{C}y_{t-k} \\ + \sum_{s=0}^{\hat{s}_{km}} (\underline{C} + (k-s)V - \overline{V}) \max\{y_{t-s} - y_{t-s-1}, 0\}.\end{aligned}$$

Determining $\theta(t)$ for all $t \in [1, T]_{\mathbb{Z}}$ can be done recursively in $O(T)$ time by setting $\theta(1) = 0$ and setting $\theta(t) = \theta(t-1) + \max\{y_t - y_{t-1}, 0\}$ for $t = 2, \ldots, T$. Clearly, for each $k \in [1, \hat{k}]_{\mathbb{Z}}$ and each $m \in [0, k-1]_{\mathbb{Z}}$, the value of $v_{km}(k+1)$ can be determined in $O(T)$ time. For any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [k+2, T-m-1]_{\mathbb{Z}}$,

$$\begin{aligned}
v_{km}(t) - v_{km}(t-1) = {} & (x_t - x_{t-1}) - (x_{t-k} - x_{t-k-1}) - (\underline{C} + (k-m)V - \overline{V})(y_{t+m+1} - y_{t+m}) \\
& - V\left[\sum_{i=1}^{m} y_{t+i} - \sum_{i=1}^{m} y_{t+i-1}\right] - \overline{V}(y_t - y_{t-1}) + \underline{C}(y_{t-k} - y_{t-k-1}) \\
& + (\underline{C} + kV - \overline{V})\left[\sum_{s=0}^{\hat{s}_{km}} \max\{y_{t-s} - y_{t-s-1}, 0\} - \sum_{s=0}^{\hat{s}_{km}} \max\{y_{t-s-1} - y_{t-s-2}, 0\}\right] \\
& - V\left[\sum_{s=0}^{\hat{s}_{km}} s \max\{y_{t-s} - y_{t-s-1}, 0\} - \sum_{s=0}^{\hat{s}_{km}} s \max\{y_{t-s-1} - y_{t-s-2}, 0\}\right] \\
= {} & (x_t - x_{t-1}) - (x_{t-k} - x_{t-k-1}) - (\underline{C} + (k-m)V - \overline{V})(y_{t+m+1} - y_{t+m}) \\
& - V(y_{t+m} - y_t) - \overline{V}(y_t - y_{t-1}) + \underline{C}(y_{t-k} - y_{t-k-1}) \\
& + (\underline{C} + kV - \overline{V})\left[\max\{y_t - y_{t-1}, 0\} - \max\{y_{t-\hat{s}_{km}-1} - y_{t-\hat{s}_{km}-2}, 0\}\right] \\
& - V\left[\sum_{s=1}^{\hat{s}_{km}} \max\{y_{t-s} - y_{t-s-1}, 0\} - \hat{s}_{km} \max\{y_{t-\hat{s}_{km}-1} - y_{t-\hat{s}_{km}-2}, 0\}\right].
\end{aligned}$$

This, together with (EC.24), implies that

$$\begin{aligned}
v_{km}(t) = {} & v_{km}(t-1) + (x_t - x_{t-1}) - (x_{t-k} - x_{t-k-1}) - (\underline{C} + (k-m)V - \overline{V})(y_{t+m+1} - y_{t+m}) \\
& - V(y_{t+m} - y_t) - \overline{V}(y_t - y_{t-1}) + \underline{C}(y_{t-k} - y_{t-k-1}) \\
& + (\underline{C} + kV - \overline{V})\left[\max\{y_t - y_{t-1}, 0\} - \max\{y_{t-\hat{s}_{km}-1} - y_{t-\hat{s}_{km}-2}, 0\}\right] \\
& - V\left[\theta(t-1) - \theta(t - \hat{s}_{km} - 1) - \hat{s}_{km} \max\{y_{t-\hat{s}_{km}-1} - y_{t-\hat{s}_{km}-2}, 0\}\right].
\end{aligned}$$

Thus, for each $k \in [1, \hat{k}]_{\mathbb{Z}}$ and $m \in [0, k-1]_{\mathbb{Z}}$, the values of $v_{km}(k+1), v_{km}(k+2), \ldots, v_{km}(T-m-1)$ can be determined recursively in $O(T)$ time. Hence, the values of k, m, t and the set $\mathcal{S}$ corresponding to the largest possible violation of inequality (21) can be obtained in $O(T^3)$ time.

Next, we consider inequality (22). Consider any given $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{2T}$. Let $x'_t = x_{T-t+1}$ and $y'_t = y_{T-t+1}$ for $t \in [1, T]_{\mathbb{Z}}$. Inequality (22) becomes

$$\begin{aligned}
x'_{T-t+1} - x'_{T-t-k+1} \le {} & (\underline{C} + (k-m)V - \overline{V})y'_{T-t+m+2} + V\sum_{i=1}^{m} y'_{T-t+i+1} + \overline{V}y'_{T-t+1} - \underline{C}y'_{T-t-k+1} \\
& - \sum_{s \in \mathcal{S}} (\underline{C} + (k-s)V - \overline{V})(y'_{T-t-s+1} - y'_{T-t-s}).
\end{aligned}$$

Letting $t' = T - t + 1$, this inequality becomes

$$\begin{aligned}
x'_{t'} - x'_{t'-k} \le {} & (\underline{C} + (k-m)V - \overline{V})y'_{t'+m+1} + V\sum_{i=1}^{m} y'_{t'+i} + \overline{V}y'_{t'} - \underline{C}y'_{t'-k} \\
& - \sum_{s \in \mathcal{S}} (\underline{C} + (k-s)V - \overline{V})(y'_{t'-s} - y'_{t'-s-1}). \qquad \text{(EC.25)}
\end{aligned}$$

Because the values of k, m, t and the set $\mathcal{S}$ corresponding to the largest possible violation of inequality (21) can be obtained in $O(T^3)$ time, the values of k, m, t' and the set $\mathcal{S}$ corresponding to the largest possible violation of inequality (EC.25) can be obtained in $O(T^3)$ time. Hence, the values of k, m, t and the set $\mathcal{S}$ corresponding to the largest possible violation of inequality (22) can be obtained in $O(T^3)$ time. $\square$

Appendix B: Additional Computational Results

B.1. Results of the Third Experiment Under Different Demand Settings

For the third experiment, we consider two additional cases, namely a less congested demand setting and a more congested demand setting. For the more congested demand setting, we obtain the new demand $\bar{d}_t^b$ at bus $b \in \mathcal{B}$ for each period $t \in [1, T]_{\mathbb{Z}}$ by increasing the corresponding demand d_t^b in the third experiment by 10%, while ensuring that $\bar{d}_t^b$ will not exceed the total generation capacity of these 54 thermal generators at the same time. Thus, we set $\bar{d}_t^b = \min\{1.1 d_t^b, \sum_{g \in \mathcal{G}} \overline{C}^g\}$. For the less congested demand setting, we obtain $\bar{d}_t^b$ by decreasing the demand d_t^b by 10%; that is, we set $\bar{d}_t^b = 0.9 d_t^b$.

Tables EC.12 and EC.13 present the computational results for the less congested demand setting. Under this setting, formulations F1$^+$ and F1$^+$-X exhibit similar overall performance. Both formulations successfully solved 19 instances within the time limit. The integrality gaps for F1$^+$ and F1$^+$-X are nearly identical across all instances. This can be attributed to the decrease in demand for each time period, resulting in a smaller number of generators being started up, as well as a reduction in the number of ramp-ups and ramp-downs. Therefore, few valid inequalities are applied in each solution process. Consequently, the reduction in the integrality gap is small from F1$^+$ to

Table EC.12 Performance of MIP Formulations in the Third Experiment Under a Less Congested Demand Setting

Instance	# var	# bin var	# cstr		CPU time [TGap]		# nodes		# user cuts
			F1$^+$	F1$^+$-X	F1$^+$	F1$^+$-X	F1$^+$	F1$^+$-X	F1$^+$-X
1	6372	1296	36124	43576	307.8	212.0	19395	12293	88
2					174.7	233.8	13023	10430	95
3					166.4	522.7	11248	13803	96
4					588.9	342.1	36443	17313	98
5					62.0	106.6	6775	2699	70
6					**[0.11%]	**[0.07%]	115728	103776	229
7					1116.6	848.6	43524	36756	126
8					1247.9	1985.2	63266	59530	217
9					821.9	1774.3	41451	56742	198
10					1881.1	2230.6	60109	69543	188
11					745.1	2793.1	40789	72982	182
12					87.8	226.9	9905	9968	93
13					35.5	66.7	2370	2702	45
14					1187.4	1043.2	49699	43317	143
15					272.5	818.3	21934	39631	173
16					306.6	206.8	21323	9596	99
17					200.1	614.2	18638	27580	128
18					82.3	543.2	9904	18321	88
19					70.6	119.5	6875	3618	73
20					81.8	148.5	10301	10550	66

Table EC.13 The Strength of LP Relaxations of MIP Formulations in the Third Experiment Under a Less Congested Demand Setting

IGap	Instance	1	2	3	4	5	6	7	8	9	10
	$F1^+$	0.57%	0.59%	0.50%	0.63%	0.53%	0.58%	0.58%	0.64%	0.53%	0.58%
	$F1^+$-X	0.57%	0.59%	0.50%	0.63%	0.53%	0.58%	0.58%	0.63%	0.52%	0.57%
	Instance	11	12	13	14	15	16	17	18	19	20
	$F1^+$	0.60%	0.57%	0.47%	0.58%	0.58%	0.59%	0.60%	0.59%	0.47%	0.46%
	$F1^+$-X	0.60%	0.57%	0.47%	0.58%	0.58%	0.59%	0.60%	0.59%	0.47%	0.46%
Pct. reduction	Instance	1	2	3	4	5	6	7	8	9	10
	$F1^+$-X	0.14%	0%	0%	0%	0%	0%	0%	1.36%	1.82%	1.17%
	Instance	11	12	13	14	15	16	17	18	19	20
	$F1^+$-X	0%	0.17%	0%	0.04%	0.09%	0.13%	0%	0%	0.65%	0.11%

$F1^+$-X. For most instances, the solution time of $F1^+$ is smaller than $F1^+$-X, because valid inequalities (5)–(10) are added as constraints in $F1^+$-X, making the subproblems at each node of the branching process more difficult to solve.

Tables EC.14 and EC.15 present the computational results for the more congested setting. Under this setting, formulation $F1^+$-X outperforms formulation $F1^+$ in all instances. The integrality gaps of the $F1^+$-X formulation are much smaller than those of $F1^+$, and the "Pct. reduction" is approximately 90% for each instance. Eight instances are solved to optimality by formulation $F1^+$-X, and the TGaps for the remaining twelve instances are all within 0.05%. On the other hand, none of these instances are solved successfully by formulation $F1^+$, and the TGaps are all above 0.26%. For most instances, formulation $F1^+$-X explores less nodes than formulation $F1^+$. This is because as the demand for each time period increases, more generators are needed, and the number of ramp-ups and ramp-downs also increases. As a result, more user cuts are used during the solution process.

Table EC.14 Computational Results of the Third Experiment Under a More Congested Demand Setting

Instance	# var	# bin var	# cstr		CPU time [TGap]		# nodes		# user cuts
			F1$^+$	F1$^+$-X	F1$^+$	F1$^+$-X	F1$^+$	F1$^+$-X	F1$^+$-X
1	6372	1296	36124	43576	**[0.27%]	616.8	409894	124616	809
2					**[0.30%]	503.2	893931	62034	847
3					**[0.42%]	471.4	49184	587395	581
4					**[0.28%]	928.7	579540	87807	652
5					**[0.33%]	**[0.04%]	516315	860515	684
6					**[0.31%]	**[0.03%]	588982	1021155	764
7					**[0.40%]	1195.5	602518	223840	669
8					**[0.53%]	**[0.05%]	634564	523103	1087
9					**[0.51%]	**[0.05%]	543776	289863	819
10					**[0.53%]	**[0.02%]	589285	434974	925
11					**[0.39%]	**[0.04%]	589812	432163	1088
12					**[0.45%]	**[0.03%]	641832	609698	932
13					**[0.43%]	1714.3	610207	251633	897
14					**[0.45%]	**[0.03%]	603593	286153	827
15					**[0.46%]	**[0.04%]	834771	506532	797
16					**[0.35%]	**[0.04%]	617076	432809	734
17					**[0.27%]	**[0.02%]	838514	462097	778
18					**[0.27%]	3221.3	603741	562787	892
19					**[0.35%]	880.5	603631	310824	792
20					**[0.58%]	**[0.02%]	639278	614575	962

Table EC.15 The Strength of LP Relaxations of MIP Formulations in the Third Experiment Under a More Congested Demand Setting

IGap	Instance	1	2	3	4	5	6	7	8	9	10
	F1$^+$	2.33%	2.61%	2.53%	2.24%	2.43%	2.40%	2.48%	2.60%	2.56%	2.48%
	F1$^+$-X	0.21%	0.21%	0.26%	0.20%	0.21%	0.17%	0.23%	0.32%	0.29%	0.26%
	Instance	11	12	13	14	15	16	17	18	19	20
	F1$^+$	2.59%	2.76%	2.61%	2.41%	2.56%	2.56%	2.35%	2.25%	2.41%	2.48%
	F1$^+$-X	0.24%	0.22%	0.23%	0.30%	0.26%	0.22%	0.24%	0.26%	0.25%	0.25%
Pct. reduction	Instance	1	2	3	4	5	6	7	8	9	10
	F1$^+$-X	90.81%	91.93%	89.87%	91.24%	91.36%	93.04%	90.54%	87.50%	88.79%	89.52%
	Instance	11	12	13	14	15	16	17	18	19	20
	F1$^+$-X	90.62%	91.88%	91.33%	87.41%	89.93%	91.36%	89.80%	88.40%	89.72%	89.76%

B.2. Results of the Three Experiments by Disabling the Smart Features of the Solver

In this section, we run the three experiments in Section 5 by disabling the smart features of CPLEX, including the presolve, heuristics, and default cut generation, to examine the strength of our proposed strong valid inequalities to help solve UC formulations. As the numbers of variables and constraints remain unchanged for all tested formulations in these three experiments, they are not reported in the following tables.

The results of the first experiment are summarized in Tables EC.16 and EC.17. Using formulation F1, CPLEX is unable to solve any of these 20 instances to optimality within the one-hour time limit. In Table EC.16, the "—" rows denote the instances for which a feasible solution cannot be found by CPELX within the time limit using formulation F1. Using formulation F1-X, CPLEX can solve 2 instances to optimality within the time limit. The number of nodes explored by F1-X is much smaller than that of formulation F1. The number of user cuts added by F1-X in the solution process is smaller compared with the total number of constraints in formulations F1 and F1-X. For the instances that cannot be solved to optimality using formulation F1-X, the terminating gaps are all within 0.05% except for instances 2 and 7, and are much smaller than those using formulation F1. The integrality gaps generated by formulation F1-X are substantially smaller than those of F1, with reductions of at least 55%. These results demonstrate that the proposed strong valid inequalities significantly tighten the single-binary formulation, thereby enhancing the efficiency of the solution process.

Tables EC.18 and EC.19 present the results for the second experiment. Using formulation F2, no instance can be solved to optimality by CPLEX within the time limit. In contrast, using formulation F2-X, the two-binary formulation with our proposed valid inequalities, CPLEX can solve 5 instances to optimality within the time limit. Using formulation F2-Y, the two-binary formulation with valid inequalities from Pan and Guan (2016), CPLEX is able to solve 2 instances to optimality within the time limit. Using formulation F2-Z, the two-binary formulation with our and Pan and Guan's inequalities, CPLEX can solve only 1 instance to optimality within the time limit. This can be attributed to the increase in the size of the LP relaxation at each node during the branching process, resulting from the addition of a large number of user cuts, which adversely affects the solution time. Fewer nodes are explored by the three strong formulations F2-X, F2-Y, and F2-Z, compared to formulation F2. The number of user cuts added during the solution process by these three strong formulations is smaller compared to the total number of constraints in formulations F2, F2-X, F2-Y, and F2-Z. For nearly all of the unsolved instances, the terminating gaps of the three strong formulations are within 0.05%. The integrality gaps of these three strong formulations are also much smaller than those using formulation F2. Furthermore, formulations

Table EC.16 Performance of MIP Formulations in the First Experiment by Disabling Smart Features

Instance	CPU time [TGap]		# nodes		# user cuts
	F1	F1-X	F1	F1-X	F1-X
1	**[0.25%]	**[0.04%]	3243082	1331163	428
2	**[0.33%]	**[0.10%]	1687199	906171	768
3	**[0.35%]	**[0.05%]	1398467	403775	937
4	**[0.30%]	**[0.01%]	1217117	1592032	490
5	**[0.48%]	**[0.03%]	1218680	494229	823
6	**[0.53%]	**[0.02%]	2492793	837237	823
7	**[0.31%]	**[0.06%]	1647817	576308	472
8	**[0.53%]	**[0.05%]	1431881	547503	732
9	**[0.44%]	**[0.04%]	1062217	481563	899
10	**[0.62%]	**[0.03%]	1232270	552396	790
11	**[0.47%]	671.2	397815	17587	1050
12	**[0.66%]	**[0.03%]	345767	132849	2423
13	**[0.71%]	**[0.02%]	300637	50590	1907
14	—	**[0.03%]	—	70987	3195
15	—	**[0.03%]	—	17541	2050
16	**[0.70%]	935.7	262321	15358	1347
17	—	**[0.02%]	—	68966	2168
18	**[1.63%]	**[0.01%]	155156	106438	2116
19	—	**[0.02%]	—	61927	2338
20	—	**[0.02%]	—	66462	1810

Table EC.17 The Strength of LP Relaxations of MIP Formulations in the First Experiment by Disabling Smart Features

IGap	Instance	1	2	3	4	5	6	7	8	9	10
	F1	0.45%	0.39%	0.41%	0.36%	0.51%	0.62%	0.34%	0.55%	0.51%	0.64%
	F1-X	0.20%	0.14%	0.08%	0.06%	0.05%	0.05%	0.07%	0.06%	0.05%	0.05%
	Instance	11	12	13	14	15	16	17	18	19	20
	F1	0.32%	0.32%	0.40%	0.39%	0.52%	0.36%	0.45%	0.42%	0.37%	0.44%
	F1-X	0.05%	0.02%	0.02%	0.03%	0.03%	0.08%	0.01%	0.01%	0.02%	0.02%
Pct. reduction	Instance	1	2	3	4	5	6	7	8	9	10
	F1-X	55.7%	63.9%	81.6%	84.5%	90.2%	92.4%	78.7%	89.5%	89.5%	92.8%
	Instance	11	12	13	14	15	16	17	18	19	20
	F1-X	83.1%	94.1%	95.1%	92.3%	94.0%	77.1%	96.0%	96.6%	94.2%	94.4%

F1-X and F2-Y have similar performance, suggesting that the single-binary formulation with our proposed valid inequalities exhibits comparable results to the two-binary formulation with strong valid inequalities. In addition, formulation F2-X outperforms both F2-Y and F2-Z, demonstrating that the effectiveness of our proposed valid inequalities in tightening the two-binary formulation is more significant than that of valid inequalities from Pan and Guan (2016).

Table EC.18 Performance of MIP Formulations in the Second Experiment by Disabling Smart Features

Instance	CPU time [TGap]				# nodes				# user cuts		
	F2	F2-X	F2-Y	F2-Z	F2	F2-X	F2-Y	F2-Z	F2-X	F2-Y	F2-Z
1	**[0.26%]	2429.2	184.4	**[0.03%]	3749277	4217040	266731	1948319	461	211	571
2	** [0.29%]	**[0.05%]	**[0.09%]	**[0.05%]	3496986	2166004	1841074	1683486	521	304	576
3	**[0.34%]	**[0.06%]	**[0.05%]	**[0.05%]	1936293	1603280	1303901	520019	730	293	426
4	** [0.26%]	**[0.02%]	**[0.02%]	**[0.02%]	2231992	2161428	1690409	1537977	396	171	338
5	**[0.46%]	**[0.04%]	**[0.04%]	**[0.03%]	1786842	1790680	1151283	715773	780	238	535
6	**[0.51%]	3241.8	**[0.01%]	**[0.01%]	4239872	3331564	1556754	1175628	782	298	563
7	**[0.27%]	**[0.05%]	332.8	**[0.06%]	1824764	1492391	161419	915783	504	177	450
8	** [0.46%]	**[0.05%]	**[0.05%]	**[0.05%]	1955325	1620740	703513	631110	542	220	444
9	**[0.43%]	**[0.03%]	**[0.05%]	**[0.05%]	2294145	1603961	661026	403170	1007	351	608
10	**[0.60%]	**[0.02%]	**[0.04%]	**[0.03%]	2024929	1983320	875849	950878	869	299	720
11	**[0.42%]	**[0.05%]	**[0.05%]	**[0.05%]	647225	479026	448558	184850	1066	643	1392
12	**[0.38%]	**[0.02%]	**[0.06%]	**[0.03%]	558586	410723	66363	42267	1983	639	1077
13	**[0.41%]	**[0.02%]	**[0.02%]	**[0.02%]	562676	437129	148136	148136	1420	1650	1483
14	**[0.57%]	**[0.04%]	**[0.03%]	**[0.03%]	478871	317574	412241	164262	2285	3048	1140
15	**[0.82%]	797.1	**[0.02%]	**[0.03%]	617261	70838	181056	41661	1827	658	1739
16	**[0.48%]	1812.6	**[0.03%]	**[0,02%]	521416	283080	236032	107088	1576	721	1372
17	**[0.80%]	788.0	**[0.02%]	2570.1	314340	85493	204695	34590	1446	563	1239
18	**[0.81%]	**[0.01%]	**[0.01%]	**[0,02%]	482334	304661	195106	125472	1987	679	1501
19	**[0.69%]	**[0.02%]	**[0.02%]	**[0.02%]	403289	370957	173920	94781	2124	600	1238
20	**[0.81%]	**[0.01%]	**[0.02%]	**[0.04%]	452740	290555	149267	22281	1764	496	1315

Th results for the third experiment are presented in Tables EC.20 and EC.21. Using formulation F1$^+$, CPLEX cannot solve any of these 20 instances to optimality within the time limit while using formulation F1$^+$-X, CPLEX can solve 1 instance to optimality. For the instance that cannot be solved to optimality within the time limit, the terminating gaps of formulation F1$^+$-X are much smaller than those of formulation F1$^+$. The number of nodes explored by formulation F1$^+$-X is smaller than that of formulation F1$^+$. The number of user cuts added in the solution process is also smaller than the total number of constraints in formulations F1$^+$ and F1$^+$-X. The integrality gaps of formulation F1$^+$-X are also much smaller than those using formulation F1$^+$, which indicates that our proposed valid inequalities can tighten the single-binary formulation significantly.

Table EC.19 The Strength of LP Relaxations of MIP Formulations in the Second Experiment by Disabling Smart Features

	Instance	1	2	3	4	5	6	7	8	9	10
IGap	F2	0.37%	0.31%	0.25%	0.20%	0.26%	0.25%	0.25%	0.29%	0.27%	0.36%
	F2-X	0.20%	0.14%	0.08%	0.06%	0.05%	0.04%	0.07%	0.06%	0.06%	0.05%
	F2-Y	0.20%	0.14%	0.08%	0.05%	0.05%	0.05%	0.08%	0.06%	0.06%	0.05%
	F2-Z	0.20%	0.14%	0.08%	0.05%	0.05%	0.04%	0.07%	0.06%	0.06%	0.05%
	Instance	11	12	13	14	15	16	17	18	19	20
	F2	0.23%	0.19%	0.20%	0.20%	0.26%	0.20%	0.22%	0.19%	0.17%	0.22%
	F2-X	0.05%	0.02%	0.02%	0.03%	0.03%	0.02%	0.03%	0.02%	0.02%	0.04%
	F2-Y	0.06%	0.03%	0.02%	0.03%	0.03%	0.02%	0.02%	0.02%	0.02%	0.03%
	F2-Z	0.05%	0.02%	0.02%	0.03%	0.03%	0.02%	0.02%	0.02%	0.02%	0.03%
Pct. reduction	Instance	1	2	3	4	5	6	7	8	9	10
	F2-X	45.5%	54.8%	69.2%	69.6%	80.0%	83.4%	72.1%	78.0%	77.4%	86.9%
	F2-Y	45.5%	54.7%	67.9%	73.2%	81.8%	80.6%	69.6%	78.8%	78.6%	85.9%
	F2-Z	45.5%	54.8%	69.4%	73.2%	81.8%	84.1%	72.1%	80.3%	78.6%	86.9%
	Instance	11	12	13	14	15	16	17	18	19	20
	F2-X	78.2%	87.5%	89.3%	86.9%	88.6%	91.4%	88.2%	89.7%	85.6%	83.2%
	F2-Y	76.2%	84.9%	89.0%	86.0%	89.3%	87.8%	89.3%	91.8%	88.1%	84.3%
	F2-Z	78.1%	87.7%	89.5%	87.0%	89.4%	91.4%	89.3%	91.8%	88.1%	84.4%

Table EC.20 Performance of MIP Formulations in the Third Experiment by Disabling Smart Features

Instance	CPU time [TGap]		# nodes		# user cuts
	$F1^{+}$	$F1^{+}$-X	$F1^{+}$	$F1^{+}$-X	$F1^{+}$-X
1	**[1.07%]	**[0.31%]	678548	267276	1636
2	**[1.14%]	**[0.33%]	812095	204492	1470
3	**[0.85%]	**[0.23%]	755989	276532	1500
4	**[1.04%]	**[0.45%]	828645	157520	1421
5	**[1.32%]	3398.7	601218	298525	1943
6	**[1.23%]	**[0.17%]	511022	324346	1594
7	**[1.42%]	**[0.23%]	962694	491768	1712
8	**[1.20%]	**[0.37%]	747792	294910	1424
9	**[1.10%]	**[0.27%]	880796	215853	1512
10	**[0.91%]	**[0.26%]	1046090	285761	1862
11	**[1.11%]	**[0.31%]	870860	382853	1457
12	**[1.59%]	**[0.26%]	765177	275495	1955
13	**[0.96%]	**[0.22%]	902408	206672	1464
14	**[1.09%]	**[0.47%]	902171	285107	1594
15	**[1.27%]	**[0.35%]	704154	267693	1580
16	**[1.23%]	**[0.30%]	850453	101352	1338
17	**[1.17%]	**[0.38%]	569100	276125	1412
18	**[0.96%]	**[0.27%]	710371	282015	1551
19	**[1.23%]	**[0.27%]	639002	274153	1569
20	**[1.20%]	**[0.25%]	608163	414109	1420

Table EC.21 The Strength of LP Relaxations of MIP Formulations in the Third Experiment by Disabling Smart Features

IGap	Instance	1	2	3	4	5	6	7	8	9	10
	F1$^+$	0.94%	1.10%	0.81%	1.04%	0.86%	0.89%	1.15%	1.08%	1.03%	0.88%
	F1$^+$-X	0.45%	0.46%	0.43%	0.63%	0.32%	0.37%	0.46%	0.58%	0.48%	0.46%
	Instance	11	12	13	14	15	16	17	18	19	20
	F1$^+$	0.97%	1.23%	0.94%	1.07%	1.08%	0.93%	0.96%	0.89%	1.04%	0.90%
	F1$^+$-X	0.47%	0.53%	0.44%	0.59%	0.49%	0.42%	0.49%	0.48%	0.48%	0.40%
Pct. reduction	Instance	1	2	3	4	5	6	7	8	9	10
	F1$^+$-X	52.2%	58.4%	46.6%	39.7%	62.3%	59.1%	59.7%	46.4%	52.8%	47.4%
	Instance	11	12	13	14	15	16	17	18	19	20
	F1$^+$-X	51.6%	57.1%	53.5%	44.8%	54.5%	55.4%	48.9%	46.3%	54.0%	56.1%