CS 520: Homework 9
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(1) Problem 1

(a) Base case: $n = 1$, $T(1) = 3(2^1) - (-1)^1$
Base case: $n = 2$, $T(2) = 3(2^2) - (-1)^2$
Inductive case: suppose $T(x) = 3(2^x) - (-1)^x$ for all $x \leq n$

$$\begin{aligned} T(n+1) &= T(n) + 2T(n-1) \\ &= 3(2^n) - (-1)^n + 2(3(2^{n-1}) - (-1)^{n-1}) \\ &= 3(2^n) - (-1)^n + 3(2^n) + 2(-1)^n \\ &= 2(3(2^n)) + (-1)^n \\ &= 3(2^{n+1}) - (-1)^{n+1} \end{aligned}$$

(b) Let $f(n) = 2^n$, $T(n) = O(f(n))$
$\exists c \exists x \forall (n \geq x)(T(n) \leq cf(n))$, let $c = 6$, $x = 3$
$3(2^n) - (-1)^n \leq 6(2^n)$ for all $n > 3$
$3(2^n) - (-1)^n \leq 3(2^n) + 1 \leq 6(2^n)$, since $1 < 3(2^n)$ for all $n \geq 3$

$T(n) = \Omega(f(n)) \leftrightarrow f(n) = O(T(n))$
Let $c = 3$, $x = 3$, then $2^n \leq 9(2^n) - 3(-1)^n$
$2^n \leq 9(2^n) - 3 \leq 9(2^n) - 3(-1)^n$, since $3 < 8(2^n)$ for all $n \geq 3$

(2) Problem 2

(a) Complexity of $T_c(n)$: Recursion tree: at depth $i = 0$ to k
  (i) Problem size is n/b^i
  (ii) Number of nodes is a^i
  (iii) Work at depth is $a^i cf(n/b^i)$

Complexity of $T_c(n) = \sum_{i=0}^{k} a^i cf(n/b^i)$
For notational simplicity, let $\sum$ denote a sum from $i = 0$ to k, and let $g(n) = a^i f(n/b^i)$

$T_c(n) = O(T_1(n))$, $T_c(n) \leq xT_1(n)$ for $n \geq y$
$\sum cg(n) \leq x \sum g(n)$ for $n \geq y$, let $x = kc + 1$, $y = 1$
$kc \sum g(n) \leq (kc+1) \sum g(n)$ for $n \geq y$

$T_1(n) = O(T_c(n))$, $T_1(n) \leq xT_c(n)$ for $n \geq y$
$\sum g(n) \leq x \sum cg(n)$ for $n \geq y$, let $x = (1/(kc)) + 1$, $y = 1$
$\sum g(n) \leq ((kc+1)/(kc)) \sum g(n)$ for $n \geq y$

(b) $U(n) = T(n) + d = aT(bn) + c + d$

$$\begin{aligned} U(n) &= aU(bn) \\ aT(bn) + c + d &= a^2T(b^n) + ac + ad \\ a^2T(b^2n) + 2c + d &= a^2T(b^2n) + ac + ad \\ 2c + d &= ac + ad \\ 2c - ac &= ad - d = d(a-1) \end{aligned}$$

$d = (2c - ac)/(a-1)$

(3) Problem 3

(a) $P(T_x > x) = O(c^t)$, so $P(T_x > t) \leq Cc^t$ for some C, $c < 1$, for large t (i.e. $t >> n$)
For any segment of n time, the probability of moving n steps in the same direction is $(1/2)^n$
This gives the following recurrence: $P(T_x > 0) = 1$
$P(T_x > kn) \leq (1 - (1/2)^n)P(T_x > (k-1)n)$
Let $t > kn$ for some k, then $P(T_x > t) \leq (1 - (1/2)^n)P(T_x > t - n)$
Since we can allow C to be arbitrarily large, let $c = (1 - (1/2)^n)$

$$\begin{aligned} E(T_x) &= \sum_{i=0}^{\infty} P(T_x = i)i \\ &\leq \sum_{i=0}^{\infty} P(T_x > i)i \\ &\leq \sum_{i=0}^{\infty} Cic^t, \text{ a convergent series (by ratio test)} \end{aligned}$$

(b) Since it takes 1 time to move, and $x+1$ and $x-1$ are equally likely from x

$$\begin{aligned} E(T_x) &= (1/2)E(T_{x-1}) + (1/2)E(T_{x+1}) + 1 \\ 2E(T_x) &= E(T_{x-1}) + E(T_{x+1}) + 2 \\ E(T_{x+1}) &= 2E(T_x) - E(T_{x-1}) - 2 \\ E(T_x) &= 2E(T_{x-1}) - E(T_{x-2}) - 2 \end{aligned}$$

(c) For notational simplicity, let $T(x) = E(T_x)$
Base cases: $T(0) = 0$ and $T(n) = 0$
Recursive case: $T(x) = 2T(x-1) - T(x-2) - 2$ for $x \neq 0, n$
Let $U(x) = T(x) - T(x-1)$, $U(x) - U(x-1) = (T(x) - T(x-1)) - (T(x-1) - T(x-2)) = -2$

$$\begin{aligned} U(x) &= U(x-1) - 2 \\ &= U(x-2) - 2(2) \\ &= U(x-3) - 3(2) \\ &= U(x-k) - 2k, \text{ let } k = x \\ U(x) &= U(0) - 2x \\ T(x) &= T(x-1) - 2x + U(0) \\ &= T(x-2) - 2(x-0) - 2(x-1) - 2x + 2U(0) \\ &= T(x-3) - 2(x-0) - 2(x-1) - 2(x-2) - 2x + 3U(0) \\ &= T(x-k) - (2\sum_{i=0}^{k-1}(x-i)) + kU(0), \text{ let } k = x \\ &= T(0) + (x-1)(x-2) - 2x^2 + xU(0) \\ T(x) &= x^2 + 3x - 2 - 2x^2 + xU(0) = -x^2 + (3 + U(0))x - 2 \end{aligned}$$

The general solution is of form: $T(x) = ax^2 + bx + c$, where $a = -1$
i.e. general solution is of form $T(x) = -x^2 + bx + c$ for constants b, c

(d) Considering the quadratic: $-x^2+bx+c$ Let $x = 0$, $T(0) = c = 0$, now let $x = n$, $T(n) = -n^2+bn = 0$, so $b = n$
Thus, the solution to the recurrence, given the previous quadratic, is $-x^2 + nx$

$T(n/2) = \Theta(n^2)$, $T(n/2) = -(n/2)^2 + n^2 = (3/4)n^2$
This result is trivial to prove (i.e. polynomials of the same degree are Θ of each other)

(4) Problem 4
Based on the induction argument given in class, we can suppose that $T(n) = O(n)$
This means that for some d and x, $T(n) \leq Cn$ for all $n \geq x$

$$\begin{aligned} C(an) + C(bn) + cn &\leq Cn \\ Ca + Cb + c &\leq C \\ C(a+b) + c &\leq C \\ c &\leq C(1-(a+b)) \end{aligned}$$

$C \geq c/(1-(a+b))$