A Note on Solving Heterogenous Agent General Equilibrium Models

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Abstract

This note provides a brief introduction to equilibrium concepts and solution methods for heterogenous agent general equilibrium models.

Code: github.com/NumEconCopenhagen/ConsumptionSavingNotebooks/

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Overview

1. **Section 1:** This section provides an overview of the literature.
2. **Section 2:** This section introduces the model, which will be used as an example throughout this note, and the three fundamental equilibrium concepts:
   (a) Stationary equilibrium
   (b) Transition path
   (c) Dynamic equilibrium
3. **Section 3:** This section explains how to use the sequence-space method proposed in Auclert et al. (2019).

1 Literature

1.1 Heterogenous Agent (HA) models

1. **Stationary equilibrium:**
   Deterministic steady state and transition path
   Foundational papers: Bewley (1986), Imrohoroğlu (1989), Huggett (1993), Aiyagari (1994), Carroll (2000)

   A few policy examples: Aiyagari and McGrattan (1998), Conesa et al. (2009), Heathcote et al. (2014)

2. **Dynamic/recursive/sequential equilibrium:**
   Aggregate shocks and stochastic dynamics
   Foundational papers: Krusell and Smith (1997, 1998)

3. **Reviews:** Heathcote et al. (2009), Krusell and Smith (2006), Guvenen (2011), Krueger et al. (2016)

1.2 Heterogenous Agent New Keynesian (HANK) models

1. **Frontier:** Kaplan et al. (2018), Bayer et al. (2019), Luetticke (2019), Alves et al. (2019), Hagedorn et al. (2019), Auclert et al. (2020), Bayer et al. (2020), Fernandez-Villaverde et al. (2020)

2. **Analytical:** Bilbiie (2008, 2019a,b), Werning (2015), Challe et al. (2017), Bilbiie et al. (2020), Debortoli and Galí (2018), Auclert et al. (2018), Acharya and Dogra (2020), Broer et al. (2020), Ravn et al. (2020), Auclert and Rognlie (2020)

3. **Others:** Oh and Reis (2012), Gornemann et al. (2016), McKay and Reis (2016), McKay et al. (2016), Guerrieri and Lorenzoni (2017),Den Haan et al. (2017), Ravn and Sterk (2017)

4. **Empirical:** Cloyne et al. (2020), Slacalek et al. (2020), Holm and Paul (2020), Wolf (2020), Guren et al. (2020)

5. **Reviews:** Kaplan and Violante (2018)

1.3 Computational methods

- **Early reviews:** Den Haan et al. (2010), Schmedders and Judd (2013)
- **Continuous time:** Achdou et al. (2020) (code)
- **Local aggregate solution in *state space*:** Solve for the non-linear stationary equilibrium, and then linearize (first order perturbation) wrt. to the aggregate shocks. Parametric approximation of value and policy must provide dimension reduction to avoid a too large equation system.

  1. ***Ex ante* dimension reduction *before* finding stationary equilibrium:** Reiter (2009), Winberry (2018) (code I, code II) and in continuous time: Ahn et al. (2018) (code)
  2. ***Ex post* dimension reduction *after* finding stationary equilibrium:** Bayer and Luetticke (2019) (MATLAB, Python) - first attempt at a second-order perturbation

- **Local aggregate solution in *sequence space*:** Boppart et al. (2018), Auclert et al. (2020) (code)
- **Global aggregate solution:** Kubler and Scheidegger (2018), Azinovic et al. (2019), Scheidegger and Bilionis (2019), Pröhl (2019) (code), Maliar et al. (2019) (code, video), Fernandez-Villaverde et al. (2020) (code)

2 Equilibrium concepts

Throughout this note, we consider a simple one-asset economy without nominal frictions.

1. **Households:** Continuum of measure 1 who
   (a) Own stocks, a_{t-1} (end-of-period)
   (b) Supply labor with productivity e_t
   (exogenous and stochastic, mean one)
   (c) Consume, c_t
2. **Firms:** Rent capital and hire labor to produce
3. **Capital:**
   (a) Predetermined: $Y_t = F(K_{t-1}, L_t)$, where K_{t-1} is capital and L_t is labor supply
   (b) Depreciates with rate δ
4. **Prices** are taken as given by households and firms
   (a) r_t^k, rental rate
   (b) $r_t = r_t^k - \delta$, interest rate
   (c) w_t, wage rate

We consider **two versions of the model**, where the **technology level**, Z_t, is either:

1. **Deterministic -** no aggregate uncertainty
2. **Stochastic** - aggregate uncertainty

2.1 Firms

- **Production function:** $Y_t = Z_t K_{t-1}^{\alpha} L_t^{1-\alpha}$
- **Define** $k_{t-1} \equiv K_{t-1}/L_t$
- Standard **pricing equations:**

$$
\begin{aligned}
r_t^k &= \alpha Z_t k_{t-1}^{\alpha-1} \\
w_t &= (1-\alpha) Z_t k_{t-1}^{\alpha}
\end{aligned}
$$

- Useful **implications:**

$$
\begin{aligned}
k_{t-1} &= \left(\frac{r_t+\delta}{\alpha Z_t}\right)^{\frac{1}{\alpha-1}} \equiv k(r_t, Z_t) \\
r_t &= \alpha Z_t k_{t-1}^{\alpha-1} \equiv r(k_{t-1}, Z_t) \\
w_t &= (1-\alpha) Z_t \left(\frac{r_t+\delta}{\alpha Z_t}\right)^{\frac{\alpha}{\alpha-1}} \equiv w(r_t, Z_t)
\end{aligned}
$$

2.2 Households - no aggregate uncertainty

- **Perfect foresight:** Price sequence known, $\{r_t, w_t\}_{t\geq 0}$
- **Households solve:**

$$
\begin{aligned}
v_t(e_t, a_{t-1}) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{t+1}(e_{t+1}, a_t) \right] \\
&\text{s.t.} \\
a_t + c_t &= (1+r_t) a_{t-1} + w_t e_t \\
a_t &\geq 0
\end{aligned}
$$

  - **Value function:** We could write $v_t(e_t, a_{t-1}) = \mathcal{V}(e_t, a_{t-1}, \{r_k, w_k\}_{k\geq t})$, where $\mathcal{V}$ does not have a time subscript, but where the price sequences are state variables.
  - **Optimal saving:** $a_t^*(e_t, a_{t-1})$
  - **Optimal consumption:** $c_t^*(e_t, a_{t-1})$
  - **Distribution:** D_t over e_t and a_{t-1}

- **Supply of capital:** $\mathcal{K}_t = \int a_t^*(e_t, a_{t-1}) dD_t = \int a_t dD_{t+1}$
  * **Note I:** $\int a_t^*(e_t, a_{t-1}) dD_t$ is an integral over e_t and a_{t-1} applying the optimal saving function in period t, i.e. $a_t^*(e_t, a_{t-1})$, thus summing up savings at the end-of-period t.
  * **Note II:** $\int a_t dD_{t+1}$ is an integral over e_{t+1} and a_t directly summing up savings at the end-of-period t.
  * **Note III:** The two formulations gives the same result because D_{t+1} is generated from D_t assuming saving according to $a_t^*(e_t, a_{t-1})$ (and the exogenous process for e_t).

2.3 Market clearing

Market clearing requires

$$
\begin{aligned}
\text{Capital: } K_t &= \mathcal{K}_t = \int a_t dD_{t+1} = \int a_t^*(e_t, a_{t-1}) dD_t \\
\text{Labour: } L_t &= \int e_t dD_t = 1 \\
\text{Goods: } Y_t &= \int c_t^*(e_t, a_{t-1}) dD_t + \delta K_{t-1}
\end{aligned}
$$

The **labor market clears trivially**, while we can leave out the **goods market** due to **Walras's Law**.

2.4 Solve household problem with EGM

- **Grids:**
  1. $e_t \in \{e^1, e^2, \ldots, e^{\#_e}\}$ (discretized using Tauchen and Hussey (1991))
  2. $a_t \in \{a^1, a^2, \ldots, a^{\#_a}\}$
- **Guess:** $v_{a,t+1}(e^i, a^j)$, $\forall i, j$
- **Time iteration:**
  1. Calculate: $q_t(e^i, a^j) = \sum_{k=1}^{\#_e} \Pr[e^k | e^i] v_{a,t+1}(e^i, a^j)$
  2. Calculate $\tilde{c}^{ij} = q_t(e^i, a^j)^{-\sigma}$ and $\tilde{m}^{ij} = \tilde{c}^{ij} + a^j$

3. Interpolate $\{\tilde{m}^{ij}, a^j\}_{j=1}^{\#_a}$ at $m^j = (1+r_t)a^j + w_t e^i$ to find $a^*(e^i, a^j)$
4. Calculate $c^*(e^i, a^j) = m_t - a^*(e^i, a^j)$
5. Calculate $v_{a,t+1}(e^i, a^j) = (1+r)c^*(e^i, a^j)^{-\sigma}$
   (use of the envelope theorem)

Note: Any other *solution* method could have been used.

2.5 Simulate household behavior on grid

- **Initial distribution:** $D_0(e^i, a^j) = \frac{\Pr[e^i]}{\#_a}$ (*ergodic* in e, *uniform* in a)
- **Update:**

$$D_{t+1}(e^k, a^l) = \sum_{i=1}^{\#_e} \Pr[e^k | e^i] \sum_{j=1}^{\#_a} D_t(e^i, a^j) \omega(a^\star(e^i, a^j), a^{\max\{l-1,1\}}, a^l, a^{\min\{l+1,\#_a\}})$$

  where

$$\omega(a, \underline{a}, \tilde{a}, \overline{a}) = 1\{a \in [\underline{a}, \overline{a}]\} \begin{cases} \frac{\overline{a}-a}{\overline{a}-\tilde{a}} & \text{if } a \geq \tilde{a} \\ \frac{a-\underline{a}}{\tilde{a}-\underline{a}} & \text{if } a < \tilde{a} \end{cases}$$

- **Note I:** If only on grid choices where possible then

$$D_{t+1}(e^k, a^l) = \sum_{i=1}^{\#_e} \Pr[e^k | e^i] \sum_{j=1}^{\#_a} D_t(e^i, a^j) 1\{a^\star(e^i, a^j) = a^l\}$$

- **Note II:** ω is a weight on each grid point calculated using linear interpolation

$$\begin{aligned} \omega(\underline{a}, \underline{a}, \tilde{a}, \overline{a}) &= \omega(\overline{a}, \underline{a}, \tilde{a}, \overline{a}) = 1 \\ \omega(\tilde{a}, \underline{a}, \tilde{a}, \overline{a}) &= 1 \end{aligned}$$

Note: Any other *simulation* method could have been used.

2.6 Definition: Stationary equilibrium

A **stationary equilibrium** for a given Z_{ss} is one where

1. Quantities K_{ss} and L_{ss},
2. prices r_{ss} and w_{ss},
3. a distribution D_{ss} over e_t and a_{t-1}
4. and policy functions $a^*_{ss}(e_t, a_{t-1})$ and $c^*_{ss}(e_t, a_{t-1})$

are such that

1. $a^*_{ss}(\bullet)$ and $c^*_{ss}(\bullet)$ solves the household problem
2. D_{ss} is the invariant distribution implied by the household problem
3. Firms maximize profits, $r_{ss} = r(K_{ss}/L_{ss}, Z_{ss})$ and $w_{ss} = w(r_{ss}, Z_{ss})$
4. The labor market clears, i.e. $L_{ss} = \int e_t dD_{ss} = 1$
5. The capital market clears, i.e. $K_{ss} = \int a^*_{ss}(e_t, a_{t-1}) dD_{ss}$
6. The goods market clears, i.e. $Y_{ss} = \int c^*_{ss}(e_t, a_{t-1}) dD_{ss} + \delta K_{ss}$

2.7 Find stationary equilibrium

1. Guess on r_{ss}
2. Calculate $w_{ss} = w(r_{ss}, Z_{ss})$
3. Solve the infinite horizon household problem
4. Simulate until convergence of D_{ss}
5. Calculate supply $\mathcal{K}_{ss} = \int a^*_{ss}(e_t, a_{t-1}) dD_{ss}$
6. Calculate demand $K_{ss} = k(r_{ss}, Z_{ss}) L_{ss}$
7. If for some tolerance ϵ
$$|\mathcal{K}_{ss} - K_{ss}| < \epsilon$$
then stop, otherwise update r_{ss} appropriately and return to step 2

This is just a **root-finding problem**

2.8 Definition: Transition path

A **transition path** for $t \in \{0, 1, 2, \dots\}$, given an initial distribution D_0 and a path of Z_t, is paths of quantities K_t and L_t, prices r_t and w_t, policy functions $a_t^*(\bullet)$ and $c_t^*(\bullet)$, distributions D_t, such that for all t

1. $a_t^*(\bullet)$ and $c_t^*(\bullet)$ solve the household problem given price paths
2. D_t are implied by the household problem given price paths and D_0
3. Firms maximizes profit, $r_t = r(K_{t-1}/L_t, Z_t)$ and $w_t = w(r_t, Z_t)$
4. The labor market clears, i.e. $L_t = \int e_t dD_t = 1$
5. The capital market clears, i.e. $K_{t-1} = \int a_{t-1} dD_t$
6. The goods market clears, i.e. $Y_t = \int c_t^*(\bullet) dD_t + \delta K_{t-1}$

2.9 Find transition path

1. Choose truncation horizon $\mathcal{T}$
2. Guess on $\{r_t\}_{t=0}^{\mathcal{T}} = \{r_{ss}\}_{t=0}^{\mathcal{T}}$ (or something else)
3. Calculate $\{w_t\}_{t=0}^{\mathcal{T}} = \{w(r_{ss}, Z_t)\}_{t=0}^{\mathcal{T}}$
4. Solve the household problem backwards along the transition path
5. Simulate households forward along the transition path
6. Calculate $\{k_t\}_{t=0}^{\mathcal{T}} = \{\int a_t^*(\bullet) dD_t\}_{t=0}^{\mathcal{T}}$
7. Calculate $\{r_t'\}_{t=0}^{\mathcal{T}} = \{r(k_{t-1}, Z_t)\}_{t=0}^{\mathcal{T}}$
8. Stop if for some tolerance ϵ

$$\max_{t \in \{0,1,2,\dots,\mathcal{T}\}} \left| r_t - r_t' \right| < \epsilon$$

   otherwise return to step 2 with $\{r_t\}_{t=0}^{\mathcal{T}} = \{\nu r_t + (1-\nu) r_t'\}_{t=0}^{\mathcal{T}}$

Note: Typically the relaxation parameter is $\nu = 0.90$ (Kirkby, 2017)

2.10 Households - with aggregate shocks

- **Aggregate shocks:** Assume Z_t is a stochastic process
- **Root problem:** There is no longer perfect foresight wrt. r_t and w_t
- **Extended problem:**

$$
\begin{aligned}
v(e_t, a_{t-1}, Z_t, D_t) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v(e_{t+1}, a_t, Z_{t+1}, D_{t+1}) \right] \\
&\text{s.t.} \\
a_t + c_t &= (1 + r_t) a_{t-1} + w_t e_t \\
k_{t-1} &= \int a_{t-1} dD_t \\
r_t &= r(k_{t-1}, Z_t) \\
w_t &= w(r_t, Z_t) \\
a_t &\geq 0
\end{aligned}
$$

- **Ultimate problem:** D_t is not easy to discretize...
- **Krusell-Smith idea:** Approximate D_t with some selected moments, e.g. just the mean
- **Approximate problem:**

$$
\begin{aligned}
v(e_t, a_{t-1}, Z_t, k_{t-1}) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v(e_{t+1}, a_t, Z_{t+1}, k_t) \right] \\
&\text{s.t.} \\
a_t + c_t &= (1 + r_t) a_{t-1} + w_t e_t \\
r_t &= r(k_{t-1}, Z_t) \\
w_t &= w(r_t, Z_t) \\
k_t &= \text{PLM}(k_{t-1}, Z_t) \\
a_t &\geq 0
\end{aligned}
$$

  where $\text{PLM}(k_{t-1}, Z_t)$ is the **perceived law of motion**

2.11 Definition: Dynamic equilibrium

An **(approximate) dynamic equilibrium** is a PLM, policy functions $a^*(\bullet)$ and $c^*(\bullet)$, and paths of quantities K_t and L_t, prices r_t and w_t, distributions D_t such that for all t

1. $a^*(\bullet)$ and $c^*(\bullet)$ solve the household problem given the PLM
2. D_t is implied by the household problem
3. Firms profit maximize $r_t = r(K_{t-1}/L_t, Z_t)$ and $w_t = w(r_t, Z_t)$
4. The labor market clears, i.e. $L_t = \int e_t dD_t = 1$
5. The capital market clears, i.e. $K_t = \int a^*(e_t, a_{t-1}) dD_t$
6. The goods market clears, i.e. $Y_t = \int c^*(e_t, a_{t-1}) dD_t + \delta K_{t-1}$
7. $\text{PLM}(k_{t-1}, Z_t)$ does not imply systematic expectations errors

Note: When $Z_t = Z_{ss} \forall Z_t$ the dynamic equilibrium does *not* converge to the stationary equilibrium unless the households know Z_t is actually not stochastic.

2.12 Find dynamic equilibrium

1. Guess on the $\text{PLM}(k_{t-1}, Z_t)$
2. Solve the household problem
3. Simulate a path of Z_t and D_t and thus k_t
4. Compare simulated behavior with the $\text{PLM}(k_{t-1}, Z_t)$

   Stop if »good enough«
   otherwise update $\text{PLM}(k_{t-1}, Z_t)$ and return to step 2

Terminology:

1. The Krusell-Smith method is a **global solution method**
2. The newest **local solution methods** rely on linearization of the aggregate dynamics, but solve for the full non-linear stationary equilibrium

3 Sequence Space Method

Main idea: We are interested in the dynamic equilibrium, but want to avoid solving the model with aggregate shocks.

Starting point: Consider the model in **sequence-space,** such that

$$\begin{aligned} K_t &= \int a_t^*(e_t, a_{t-1}) dD_t \\ &\equiv \mathcal{K}_t\{\{r_s, w_s\}_{s\geq 0}, D_0)\} \end{aligned} \tag{1}$$

where the second line is just a re-formulation of the model in **sequence space** conditional on the distribution in period 0. Let $\boldsymbol{K} = (K_0, K_1, \ldots)$ and $\boldsymbol{Z} = (Z_0, Z_1, \ldots)$. For given $\boldsymbol{Z}$ the model is now the solution to the following equation

$$H_t(\boldsymbol{K}, \boldsymbol{Z}, D_0) \equiv \mathcal{K}_t(\{r(Z_s, K_{s-1}), w(Z_s, K_{s-1})\}_{s\geq 0}, D_0) - K_t = 0, \qquad t = 0, 1, \ldots \tag{2}$$

Or in **time-stacked form**

$$\boldsymbol{H}(\boldsymbol{K}, \boldsymbol{Z}, D_0) = \boldsymbol{0} \tag{3}$$

3.1 Linearization

Total differentiation implies

$$\begin{aligned} \boldsymbol{H_K} d\boldsymbol{K} + \boldsymbol{H_Z} d\boldsymbol{Z} = 0 \Leftrightarrow& \\ d\boldsymbol{K} &= -\boldsymbol{H_K}^{-1} \boldsymbol{H_Z} d\boldsymbol{Z} \end{aligned} \tag{4}$$

where

$$\boldsymbol{H_K} = \begin{bmatrix} \frac{\partial H_0}{\partial K_0} & \frac{\partial H_0}{\partial K_1} & \cdots \\ \frac{\partial H_1}{\partial K_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}, \boldsymbol{H_Z} = \begin{bmatrix} \frac{\partial H_0}{\partial Z_0} & \frac{\partial H_0}{\partial Z_1} & \cdots \\ \frac{\partial H_1}{\partial Z_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}$$

A central insight is that $\mathcal{K}_t$ **only depends on the prices** and not the aggregate quantities in themselves. Consequently, applying the chain rule, we have

$$\boldsymbol{H_K} = \mathcal{J}^{\mathcal{K},r}\mathcal{J}^{r,K} + \mathcal{J}^{\mathcal{K},w}\mathcal{J}^{w,K} - \boldsymbol{I} \tag{5}$$

$$\boldsymbol{H}_Z = \mathcal{J}^{\mathcal{K},r}\mathcal{J}^{r,Z} + \mathcal{J}^{\mathcal{K},w}\mathcal{J}^{w,Z} \tag{6}$$

where generically

$$\mathcal{J}^{x,y} = \begin{bmatrix} \frac{\partial x_0}{\partial y_0} & \frac{\partial x_0}{\partial y_1} & \cdots \\ \frac{\partial x_1}{\partial y_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}$$

and specifically

$$\mathcal{J}^{\mathcal{K},r} = \begin{bmatrix} \frac{\partial \mathcal{K}_0}{\partial r_0} & \frac{\partial \mathcal{K}_0}{\partial r_1} & \cdots \\ \frac{\partial \mathcal{K}_1}{\partial r_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}, \mathcal{J}^{r,K} = \alpha(\alpha-1)Z_t K_t^{\alpha-2} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & 1 & 0 & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \mathcal{J}^{r,Z} = \alpha K_t^{\alpha-1}\boldsymbol{I}$$

and

$$\mathcal{J}^{\mathcal{K},w} = \begin{bmatrix} \frac{\partial \mathcal{K}_0}{\partial w_0} & \frac{\partial \mathcal{K}_0}{\partial w_1} & \cdots \\ \frac{\partial \mathcal{K}_1}{\partial w_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}, \mathcal{J}^{w,K} = (1-\alpha)\alpha Z_t K_t^{\alpha-1} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & 1 & 0 & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \mathcal{J}^{w,Z} = (1-\alpha)K_t^{\alpha}\boldsymbol{I}$$

Note that $\forall t \neq s-1 : \frac{\partial r_t}{\partial K_s} = \frac{\partial w_t}{\partial K_s} = 0$ and $\forall s \neq t : \frac{\partial r_t}{\partial Z_s} = \frac{\partial w_s}{\partial Z_s} = 0$.

Truncation: In practice the matrices are truncated, such that they are $T \times T$.

1. **Along a row:** The effect on a variable in a given period of a shock in an arbitrary period.
2. **Along a column:** The effect of a shock in a given period on a variable in an arbitrary period.

3.2 Intermezzo: No uncertainty

In the case where there is no income risk, the household problem reduces to a **permanent income hypothesis model** with the solution

$$
\begin{aligned}
C_0(\{r_s, w_s\}_{s\geq 0}) &= \frac{(1+r_0)a_{-1} + \sum_{t=0}^{\infty} \frac{1}{\mathcal{R}_t} w_t}{\sum_{t=0}^{\infty} \beta^{t/\rho} \mathcal{R}_t^{\frac{1-\rho}{\rho}}} \qquad (7) \\
C_t(\{r_s, w_s\}_{s\geq 0}) &= \beta^{1/\rho}(1+r_t)^{1/\rho} C_{t-1}(\{r_s, w_s\}_{s\geq 0}) \\
&= \beta^{t/\rho} \mathcal{R}_t^{1/\rho} C_0(\{r_s, w_s\}_{s\geq 0})
\end{aligned}
$$

where the **compound interest rate factor** is

$$
\mathcal{R}_t = \begin{cases} 1 & \text{if } t = 0, \\ (1+r_t)\mathcal{R}_{t-1} & \text{else} \end{cases} \qquad (8)
$$

The partial derivatives of a price change at time s can all be written in terms of the effect on **period 0 consumption**

$$
\frac{\partial C_t}{\partial w_s} = \beta^{1/\rho}(1+r_t)^{1/\rho} \frac{\partial C_{t-1}}{\partial w_s} = \beta^{t/\rho} \mathcal{R}_t^{1/\rho} \frac{\partial C_0}{\partial w_s}, \qquad (9)
$$

$$
\frac{\partial C_t}{\partial r_s} = \beta^{t/\rho} \mathcal{R}_t^{1/\rho} \frac{\partial C_0}{\partial r_s} + \begin{cases} 0 & \text{if } t < s, \\ \beta^{t/\rho} (\mathcal{R}_t)^{1/\rho} \frac{C_0}{\rho(1+r_s)} & \text{else.} \end{cases} \qquad (10)
$$

Capital accumulation follows the recursion

$$
\begin{aligned}
\mathcal{K}_0 &= (1+r_0)K_{-1} + w_0 - C_0(\{r_s, w_s\}_{s\geq 0}) \\
\mathcal{K}_1 &= (1+r_1)\mathcal{K}_0 + w_1 - C_1(\{r_s, w_s\}_{s\geq 0}) \\
&\vdots \\
\mathcal{K}_t &= (1+r_t)\mathcal{K}_{t-1} + w_t - C_t(\{r_s, w_s\}_{s\geq 0}) \\
&\vdots
\end{aligned}
$$

The Jacobian $\mathcal{J}^{\mathcal{K},r}$ can now be constructed from the recursion

$$\begin{aligned}
\frac{\partial \mathcal{K}_0}{\partial r_s} &= 1\{s=0\}K_{-1} + \frac{\partial C_0}{\partial r_s} \\
\frac{\partial \mathcal{K}_1}{\partial r_s} &= 1\{s=1\}\mathcal{K}_0 + (1+r_1)\frac{\partial \mathcal{K}_0}{\partial r_s} + \frac{\partial C_1}{\partial r_s} \\
&\vdots \\
\frac{\partial \mathcal{K}_t}{\partial r_s} &= 1\{s=t\}\mathcal{K}_s + (1+r_t)\frac{\partial \mathcal{K}_{t-1}}{\partial r_s} + \frac{\partial C_t}{\partial r_s}
\end{aligned}$$

and similar for$\mathcal{J}^{\mathcal{K},w}$

$$\begin{aligned}
\frac{\partial \mathcal{K}_0}{\partial r_s} &= \frac{\partial C_0}{\partial w_s} \\
\frac{\partial \mathcal{K}_1}{\partial r_s} &= (1+r_1)\frac{\partial \mathcal{K}_0}{\partial w_s} + \frac{\partial C_1}{\partial w_s} \\
&\vdots \\
\frac{\partial \mathcal{K}_t}{\partial r_s} &= (1+r_t)\frac{\partial \mathcal{K}_{t-1}}{\partial w_s} + \frac{\partial C_t}{\partial w_s}
\end{aligned}$$

3.3 Transition path after MIT shock

1. Assume we **start in the stationary equilibrium**

2. **MIT-shock:** Known path of future $\boldsymbol{Z}$, i.e. in terms of changes from steady state $d\boldsymbol{Z} = \boldsymbol{Z} - Z_{ss}$

3. **Question:** What happens to aggregate capital?

4. **Answer:** To a first order using equation (4) we have

$$\boldsymbol{G}^{K,Z} \equiv \frac{d\boldsymbol{K}}{d\boldsymbol{Z}} = -\boldsymbol{H}_{\boldsymbol{K}}^{-1}\boldsymbol{H}_{\boldsymbol{Z}}$$

   where all derivatives are **evaluated at the stationary equilibrium**.

   Additional responses are easily calculated:

$$\boldsymbol{G}^{r,Z} \equiv \frac{d\boldsymbol{r}}{d\boldsymbol{Z}} = \mathcal{J}^{r,Z} + \mathcal{J}^{r,K}\boldsymbol{G}^{K,Z}$$

$$G^{w,Z} \equiv \frac{d\boldsymbol{w}}{d\boldsymbol{Z}} = \mathcal{J}^{w,Z} + \mathcal{J}^{w,K}\boldsymbol{G}^{K,Z}$$

$$G^{C,Z} \equiv \frac{d\boldsymbol{C}}{d\boldsymbol{Z}} = \mathcal{J}^{C,r}\boldsymbol{G}^{r,Z} + \mathcal{J}^{C,w}\boldsymbol{G}^{w,Z}$$

$$G^{Y,Z} \equiv \frac{d\boldsymbol{Y}}{d\boldsymbol{Z}} = \mathcal{J}^{Y,Z} + \mathcal{J}^{Y,K}\boldsymbol{G}^{K,Z}$$

Transition path for variable o is then $d\boldsymbol{X}^o = \boldsymbol{G}^{o,Z}d\boldsymbol{Z}$.

3.4 Impulse-responses and and covariances with aggregate risk

Let us now return to the **model with aggregate risk** where $\tilde{Z}_t$ is a stochastic variable as indicated by the tilde.

1. In the limit where the shock variance disappears the dynamic equilibrium path converge to the stationary equilibrium.
2. In the limit where the shock variance disappears the transition path to the MIT shock around the stationary equilibrium is the same as the impulse-response in the dynamic equilibrium.

Remark: This is formally proved in Auclert et al. (2019).

Think as follows:

1. **MA representation:**
   Assume that the TFP process has a $MA(\infty)$ representation given by

   $$d\tilde{Z}_t = \sum_{s=0}^{\infty} m_s^Z \epsilon_{t-s},$$

   where $d\tilde{Z}_t \equiv \tilde{Z}_t - Z_{ss}$ is deviations from stationary equilibrium, ϵ_t's are mutually iid **standard normally** distributed innovations and m_s^Z is the MA coefficients.

   After a **first order linearization** any *stochastic* output variable $\tilde{X}_t^o$ also have an $MA(\infty)$ representation

$$d\tilde{X}_t^o = \sum_{s=0}^{\infty} m_s^{o,Z} \epsilon_{t-s}^Z.$$

2. **Equivalence result:** Think of the following two experiments.

   (a) **In the stochastic model:** Assume that there is a shock in period t, and no shocks in any other past or future period. This implies

$$\mathbb{E}_t[d\tilde{X}_{t+s}|\epsilon_t = 1, \epsilon_k = 0 \forall k \neq t] = m_s^{o,Z}$$
$$\mathbb{E}_t[d\tilde{Z}_{t+s}|\epsilon_t = 1, \epsilon_k = 0 \forall k \neq t] = m_s^Z$$

   and likewise

$$\mathbb{E}_t[d\tilde{\boldsymbol{X}}_t|\epsilon_t = 1, \epsilon_k = 0 \forall k \neq t] = \boldsymbol{m}^{o,Z} = (m_0^{o,Z}, m_1^{o,Z}, \dots)^T$$
$$\mathbb{E}_t[d\tilde{\boldsymbol{Z}}_t|\epsilon_t = 1, \epsilon_k = 0 \forall k \neq t] = \boldsymbol{m}^Z = (m_0^Z, m_1^Z, \dots)^T$$

   (b) **In the deterministic model:** Think of an MIT-shock where $d\boldsymbol{Z} = \boldsymbol{m}^Z$. The path of the output variable then is

$$d\boldsymbol{X}^o = \boldsymbol{G}^{o,Z} d\boldsymbol{Z} = \boldsymbol{G}^{o,Z} \boldsymbol{m}^Z$$

   (c) **Certainty equivalence:** Because of certainty equivalence these two experiments yield the same result and therefore

$$\begin{aligned} d\boldsymbol{X}^o &= \mathbb{E}_t[d\tilde{\boldsymbol{X}}_t|\epsilon_t = 1, \epsilon_k = 0 \forall k \neq t] \Leftrightarrow \\ \boldsymbol{m}^{o,Z} &= \boldsymbol{G}^{o,Z} \boldsymbol{m}^Z \end{aligned}$$

3. The **covariance** between any two output variables is given by

$$\begin{aligned} \mathrm{Cov}(d\tilde{X}_t^o, d\tilde{X}_{t+k}^o) =& \mathbb{E}_t\left[d\tilde{X}_t^o \left(d\tilde{X}_{t+k}^o\right)^T\right] \\ =& \mathbb{E}_t\left[\left(\boldsymbol{m}^{o,Z}\right)^T \boldsymbol{\epsilon}_t^Z \left(\boldsymbol{\epsilon}_{t+k}^Z\right)^T \boldsymbol{m}^{o,Z}\right] \\ =& \sum_{s=0}^{\infty} m_s^{o,Z} m_{s+k}^{o,Z} \end{aligned}$$

where we use that

$$\mathbb{E}_t\left[\epsilon_t^Z\left(\epsilon_{t+k}^Z\right)^T\right]=\mathbb{E}_t\begin{bmatrix}\epsilon_t^Z\epsilon_{t+k}^Z & \epsilon_t^Z\epsilon_{t+k-1}^Z & \cdots & \epsilon_t^Z\epsilon_t^Z & \cdots & \cdots & \cdots\\ \epsilon_{t-1}^Z\epsilon_{t+k}^Z & \epsilon_{t-1}^Z\epsilon_{t+k-1}^Z & \cdots & \epsilon_{t-1}^Z\epsilon_t^Z & \epsilon_t^Z\epsilon_{t-1}^Z & \cdots & \cdots\\ \epsilon_{t-2}^Z\epsilon_{t+k}^Z & \epsilon_{t-2}^Z\epsilon_{t+k-1}^Z & \cdots & \epsilon_{t-2}^Z\epsilon_t^Z & \epsilon_t^Z\epsilon_{t-1}^Z & \epsilon_{t-2}^Z\epsilon_{t-2}^Z & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & \cdots & 1 & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & 0 & 1 & \cdots & \cdots\\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots\end{bmatrix}.$$

Truncation: Since the Jacobians are only computed for some truncated horizon T, we have that

$$\mathrm{Cov}(d\tilde{X}_t, d\tilde{X}_{t+k})=\sum_{s=0}^{T-k} m_s^{o,Z} m_{s+k}^{o,Z}$$

4. **Example:** Let TFP follow the $AR(1)$ process

$$\tilde{Z}_t = 1-\rho+\rho\tilde{Z}_{t-1}+\sigma\epsilon_t.$$
$$\Rightarrow d\tilde{Z}_t=\sigma\sum_{s=0}^{\infty}\rho^s\epsilon_{t-s},$$

with $m_s^Z=\sigma\rho^s$. Given $\boldsymbol{G}^{o,Z}$, we have that

$$\mathrm{Cov}(d\tilde{X}_t^o, dX_{t+k}^{\tilde{o}})=\sigma^2\sum_{s=0}^{T-k}\rho^{2s+k}\boldsymbol{G}_{t,s}^{o,Z}\boldsymbol{G}_{t+k,s}^{o,Z}.$$

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3.5 Computational efficiency

Much of Auclert et al. (2019) is about **computational efficiency**. How to:

1. Compute the jacobians of the heterogenous agent block efficiently.
2. Multiply the (sparse) jacobians efficiently.
3. Compute the covariance matrix efficiently.

Additionally, the paper shows:

1. How to check for determinancy.
2. How to use the computed jacobians to speed-up the solution of the fully non-linear transition path.

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