A Note on Solving Heterogenous Agent General Equilibrium Models

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Abstract

This note provides a brief introduction to equilibrium concepts and solution methods for heterogenous agent general equilibrium models.

Code: github.com/NumEconCopenhagen/ConsumptionSavingNotebooks/

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1 Overview

1. **Section 1:** This section provides an overview of the note and the literature.

2. **Section 2:** This section introduces the model, which will be used as an example throughout this note, and the three fundamental equilibrium concepts:

   (a) Stationary equilibrium

   (b) Transition path

   (c) Dynamic equilibrium

3. **Section 3:** This section explains how to use the sequence-space method proposed in Auclert et al. (2019).

1.1 Heterogenous Agent (HA) models

1. **Stationary equilibrium:**
   Deterministic steady state and transition path
   Foundational papers: Bewley (1986), Imrohoroğlu (1989), Huggett (1993), Aiyagari (1994), Carroll (2000)

   A few policy examples: Aiyagari and McGrattan (1998), Conesa et al. (2009), Heathcote et al. (2014)

2. **Dynamic/recursive/sequential equilibrium:**
   Aggregate shocks and stochastic dynamics
   Foundational papers: Krusell and Smith (1997, 1998)

3. **Reviews:** Heathcote et al. (2009), Krusell and Smith (2006), Krueger et al. (2016)

1.2 Heterogenous Agent New Keynesian (HANK) models

1. **Frontier:** Kaplan et al. (2018), Bayer et al. (2019), Luetticke (2019), Alves et al. (2019), Hagedorn et al. (2019), Auclert et al. (2020), Bayer et al. (2020), Fernandez-Villaverde et al. (2020)

2. **Analytical:** Bilbiie (2008, 2019a,b), Werning (2015), Challe et al. (2017), **?**, Bilbiie et al. (2020), Debortoli and Galí (2018), Auclert et al. (2018), Broer et al. (2020), Ravn et al. (2020), Auclert and Rognlie (2020)

3. **Others:** Oh and Reis (2012), Gornemann et al. (2016), McKay and Reis (2016), McKay et al. (2016), Guerrieri and Lorenzoni (2017),Den Haan et al. (2017), Ravn and Sterk (2017)

4. **Empirical:** Cloyne et al. (2020), Slacalek et al. (2020), Holm and Paul (2020), Wolf (2020)

5. **Reviews:** Kaplan and Violante (2018)

1.3 Computational methods

- **Early reviews:** Den Haan et al. (2010),Schmedders and Judd (2013)

- **Continuous time:** Achdou et al. (2020) (code)
- **Full local solution:** Winberry (2018)
- **Local aggregate solution:**
  1. State space: Bayer and Luetticke (2019) (MATLAB, Python)
     Continuous time: Ahn et al. (2018) (code)
  2. Sequence space: Boppart et al. (2018), Auclert et al. (2020) (code)
- **Global aggregate solution:** Kubler and Scheidegger (2018), Azinovic et al. (2019), Scheidegger and Bilionis (2019), Pröhl (2019) (code), Maliar et al. (2019) (code, video), Fernandez-Villaverde et al. (2020) (code)

2 Equilibrium concepts

Throughout this note, we consider a simple one-asset economy without nominal frictions.

1. **Households:** Continuum of measure 1 who
   (a) Own stocks, a_{t-1}
   (b) Supply labor with productivity e_t
   (exogenous and stochastic, mean one)
   (c) Consume, c_t
2. **Firms:** Rent capital and hire labor to produce
3. **Capital:** Depreciates with rate δ
4. **Prices** are taken as given by households and firms
   (a) r_t^k, rental rate
   (b) $r_t = r_t^k - \delta$, interest rate
   (c) w_t, wage rate

We consider **two versions of the model**, where the **technology level**, Z_t, is either:

1. **Deterministic -** no aggregate uncertainty
2. **Stochastic** - aggregate uncertainty

2.1 Firms

- **Production function:** $Y_t = Z_t K_{t-1}^{\alpha} L_t^{1-\alpha}$
- **Define** $k_{t-1} \equiv K_{t-1}/L_t$
- Standard **pricing equations:**

$$
\begin{aligned}
r_t^k &= \alpha Z_t k_{t-1}^{\alpha-1} \\
w_t &= (1-\alpha) Z_t k_{t-1}^{\alpha}
\end{aligned}
$$

- Useful **implications:**

$$
\begin{aligned}
k_{t-1} &= \left(\frac{r_t+\delta}{\alpha Z_t}\right)^{\frac{1}{\alpha-1}} \equiv k(r_t, Z_t) \\
r_t &= \alpha Z_t k_{t-1}^{\alpha-1} \equiv r(k_{t-1}, Z_t) \\
w_t &= (1-\alpha) Z_t \left(\frac{r_t+\delta}{\alpha Z_t}\right)^{\frac{\alpha}{\alpha-1}} \equiv w(r_t, Z_t)
\end{aligned}
$$

2.2 Households - no aggregate uncertainty

- **Perfect foresight:** Price sequence known, $\{r_t, w_t\}_{t\geq 0}$
- **Households solve:**

$$
\begin{aligned}
v_t(e_t, a_{t-1}) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{t+1}(e_{t+1}, a_t) \right] \\
&\text{s.t.} \\
a_t + c_t &= (1+r_t) a_{t-1} + w_t e_t \\
a_t &\geq 0
\end{aligned}
$$

  - **Optimal saving:** $a^*(e_t, a_{t-1})$
  - **Optimal consumption:** $c^*(e_t, a_{t-1})$
  - **Distribution:** D_t over e_t and a_{t-1}
  - **Supply of capital:** $\mathcal{K}_t = \int a_t^*(e_t, a_{t-1}) dD_t$

2.3 Market clearing

Market clearing requires

$$
\begin{aligned}
\text{Capital: } K_t &= \mathcal{K}_t = \int a_t^*(e_t, a_{t-1}) dD_t \\
\text{Labour: } L_t &= \int e_t dD_t = 1 \\
\text{Goods: } Y_t &= \int c^*(e_t, a_{t-1}) dD_t + \delta K_{t-1}
\end{aligned}
$$

The **labor market clears trivially**, while we can leave out the **goods market** due to **Walras's Law**.

2.4 Solve household problem with EGM

- **Grids:**
  1. $e_t \in \{e^1, e^2, \dots, e^{\#_e}\}$ (discretized using Tauchen and Hussey (1991))
  2. $a_t \in \{a^1, a^2, \dots, a^{\#_a}\}$
- **Guess:** $v_{a,t+1}(e^i, a^j)$, $\forall i, j$
- **Time iteration:**
  1. Calculate: $q_t(e^i, a^j) = \sum_{k=1}^{\#_e} \Pr[e^k|e^i] v_{a,t+1}(e^i, a^j)$
  2. Calculate $\tilde{c}^{ij} = q_t(e^i, a^j)^{-\sigma}$ and $\tilde{m}^{ij} = \tilde{c}^{ij} + a^j$
  3. Interpolate $\{\tilde{m}^{ij}, a^j\}_{j=1}^{\#_a}$ at $m^j = (1 + r_t)a^j + w_t e^i$ to find $a^*(e^i, a^j)$
  4. Calculate $c^*(e^i, a^j) = m_t - a^*(e^i, a^j)$
  5. Calculate $v_{a,t+1}(e^i, a^j) = (1 + r)c^*(e^i, a^j)^{-\sigma}$
     (use of the envelope theorem)

Note: Any other *solution* method could have been used.

2.5 Simulate household behavior on grid

- **Initial distribution:** $D_0(e^i, a^j) = \frac{\Pr[e^i]}{\#_a}$ (*ergodic* in e, *uniform* in a)
- **Update:**

$$D_{t+1}(e^k, a^l) = \sum_{i=1}^{\#_e} \Pr[e^k|e^i] \sum_{j=1}^{\#_a} D_t(e^i, a^j) \omega(a^\star(e^i, a^j), a^{\max\{l-1,1\}}, a^l, a^{\min\{l+1,\#_a\}})$$

  where

$$\omega(a, \underline{a}, \tilde{a}, \overline{a}) = 1\{a \in [\underline{a}, \overline{a}]\} \begin{cases} \frac{\overline{a}-a}{\overline{a}-\tilde{a}} & \text{if } a \geq \tilde{a} \\ \frac{a-\underline{a}}{\tilde{a}-\underline{a}} & \text{if } a < \tilde{a} \end{cases}$$

- **Note I:** If only on grid choices where possible then

$$D_{t+1}(e^k, a^l) = \sum_{i=1}^{\#_e} \Pr[e^k|e^i] \sum_{j=1}^{\#_a} D_t(e^i, a^j) 1\{a^\star(e^i, a^j) = a^l\}$$

- **Note II:** ω is a weight on each grid point calculated using linear interpolation

$$
\begin{aligned}
&\omega(\underline{a}, \underline{a}, \tilde{a}, \overline{a}) = \omega(\overline{a}, \underline{a}, \tilde{a}, \overline{a}) = 1 \\
&\omega(\tilde{a}, \underline{a}, \tilde{a}, \overline{a}) = 1
\end{aligned}
$$

Note: Any other *simulation* method could have been used.

2.6 Definition: Stationary equilibrium

A **stationary equilibrium** for a given Z_{ss} is one where

1. Quantities K_{ss} and L_{ss},
2. prices r_{ss} and w_{ss},
3. a distribution D_{ss} over e_t and a_{t-1}
4. and policy functions $a^*_{ss}(e_t, a_{t-1})$ and $c^*_{ss}(e_t, a_{t-1})$

are such that

1. $a^*_{ss}(\bullet)$ and $c^*_{ss}(\bullet)$ solves the household problem
2. D_{ss} is the invariant distribution implied by the household problem
3. Firms maximize profits, $r_{ss} = r(K_{ss}/L_{ss}, Z_{ss})$ and $w_{ss} = w(r_{ss}, Z_{ss})$
4. The labor market clears, i.e. $L_{ss} = \int e_t dD_{ss} = 1$
5. The capital market clears, i.e. $K_{ss} = \int a^*_{ss}(e_t, a_{t-1}) dD_{ss}$
6. The goods market clears, i.e. $Y_{ss} = \int c^*_{ss}(e_t, a_{t-1}) dD_{ss} + \delta K_{ss}$

2.7 Find stationary equilibrium

1. Guess on r_{ss}
2. Calculate $w_{ss} = w(r_{ss}, Z_{ss})$
3. Solve the infinite horizon household problem

4. Simulate until convergence of D_{ss}
5. Calculate supply $\mathcal{K}_{ss} = \int a^*_{ss}(e_t, a_{t-1}) dD_{ss}$
6. Calculate demand $K_{ss} = k(r_{ss}, Z_{ss}) L_{ss}$
7. If for some tolerance ϵ
$$|\mathcal{K}_{ss} - K_{ss}| < \epsilon$$
then stop, otherwise update r_{ss} appropriately and return to step 2

This is just a **root-finding problem**

2.8 Definition: Transition path

A **transition path** for $t \in \{0, 1, 2, \dots\}$, given an initial distribution D_0 and a path of Z_t, is paths of quantities K_t and L_t, prices r_t and w_t, policy functions $a^*_t(\bullet)$ and $c^*_t(\bullet)$, distributions D_t, such that for all t

1. $a^*_t(\bullet)$ and $c^*_t(\bullet)$ solve the household problem given price paths
2. D_t are implied by the household problem given price paths and D_0
3. Firms maximizes profit, $r_t = r(K_{t-1}/L_t, Z_t)$ and $w_t = w(r_t, Z_t)$
4. The labor market clears, i.e. $L_t = \int e_t dD_t = 1$
5. The capital market clears, i.e. $K_t = \int a^*_t(\bullet) dD_t$
6. The goods market clears, i.e. $Y_t = \int c^*_t(\bullet) dD_t + \delta K_{t-1}$

2.9 Find transition path

1. Choose truncation horizon $\mathcal{T}$
2. Guess on $\{r_t\}_{t=0}^{\mathcal{T}} = \{r_{ss}\}_{t=0}^{\mathcal{T}}$ (or something else)
3. Calculate $\{w_t\}_{t=0}^{\mathcal{T}} = \{w(r_{ss}, Z_t)\}_{t=0}^{\mathcal{T}}$
4. Solve the household problem backwards along the transition path

5. Simulate households forward along the transition path

6. Calculate $\{k_t\}_{t=0}^{\mathcal{T}} = \{\int a_t^*(\bullet)dD_t\}_{t=0}^{\mathcal{T}}$

7. Calculate $\{r_t'\}_{t=0}^{\mathcal{T}} = \{r(k_{t-1}, Z_t)\}_{t=0}^{\mathcal{T}}$

8. Stop if for some tolerance ϵ

$$\max_{t\in\{0,1,2,\dots,\mathcal{T}\}} \left|r_t - r_t'\right| < \epsilon$$

otherwise return to step 2 with $\{r_t\}_{t=0}^{\mathcal{T}} = \{\nu r_t + (1-\nu)r_t'\}_{t=0}^{\mathcal{T}}$

Note: Typically the relaxation parameter is $\nu = 0.90$ (Kirkby, 2017)

2.10 Households - with aggregate shocks

- **Aggregate shocks:** Assume Z_t is a stochastic process
- **Root problem:** There is no longer perfect foresight wrt. r_t and w_t
- **Extended problem:**

$$\begin{aligned}
v(e_t, a_{t-1}, Z_t, D_t) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta\mathbb{E}_t\left[v(e_{t+1}, a_t, Z_{t+1}, D_{t+1})\right] \\
&\text{s.t.} \\
a_t + c_t &= (1+r_t)a_{t-1} + w_t e_t \\
k_t &= \int a^*(e_t, a_{t-1}, Z_t, D_t)dD_t \\
r_t &= r(k_t, Z_t) \\
w_t &= w(r_t, Z_t) \\
a_t &\geq 0
\end{aligned}$$

- **Ultimate problem:** D_t is not easy to discretize...
- **Krusell-Smith idea:** Approximate D_t with some selected moments, e.g. just the mean

- **Approximate problem:**

$$
\begin{aligned}
v(e_t, a_{t-1}, Z_t, k_t) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v(e_{t+1}, a_t, Z_{t+1}, k_{t+1}) \right] \\
& \text{s.t.} \\
a_t + c_t &= (1 + r_t) a_{t-1} + w_t e_t \\
r_t &= r(k_t, Z_t) \\
w_t &= w(r_t, Z_t) \\
k_{t+1} &= \text{PLM}(k_t, Z_t, Z_{t+1}) \\
a_t &\geq 0
\end{aligned}
$$

where $\text{PLM}(k_t, Z_t, Z_{t+1})$ is the **perceived law of motion**

2.11 Definition: Dynamic equilibrium

An **(approximate) dynamic equilibrium** is a PLM, policy functions $a_t^*(\bullet)$ and $c_t^*(\bullet)$, and paths of quantities K_t and L_t, prices r_t and w_t, distributions D_t such that for all t

1. $a_t^*(\bullet)$ and $c_t^*(\bullet)$ solve the household problem given the PLM
2. D_t is implied by the household problem
3. Firms profit maximize $r_t = r(K_{t-1}/L_t, Z_t)$ and $w_t = w(r_t, Z_t)$
4. The labor market clears, i.e. $L_t = \int e_t dD_t = 1$
5. The capital market clears, i.e. $K_t = \int a_t^*(\bullet) dD_t$
6. The goods market clears, i.e. $Y_t = \int c_t^*(\bullet) dD_t + \delta K_{t-1}$
7. $\text{PLM}(k_t, Z_t, Z_{t+1})$ does not imply systematic expectations errors

Note: When $Z_t = Z_{ss} \forall Z_t$ the dynamic equilibrium does *not* converge to the stationary equilibrium unless the households know Z_t is actually not stochastic.

2.12 Find dynamic equilibrium

1. Guess on the $\text{PLM}(k_t, Z_t, Z_{t+1})$

2. Solve the household problem
3. Simulate a path of Z_t and D_t and thus k_t
4. Compare simulated behavior with the PLM(k_t, Z_t, Z_{t+1})

   Stop if »good enough«
   otherwise update PLM(k_t, Z_t, Z_{t+1}) and return to step 2

Terminology:

1. The Krusell-Smith method is a **global solution method**
2. The newest **local solution methods** rely on linearization of the aggregate dynamics, but solve for the full non-linear stationary equilibrium
3. **Older solution methods** linearized both the **idiosyncratic** and the **aggregate dynamics**

3 Sequence-Space Method

Main idea: We are interested in the dynamic equilibrium, but want to avoid solving the model with aggregate shocks.
Starting point: Consider the model in **sequence-space,** such that

$$\begin{aligned} K_t^s &= \int a^*(e_t, a_{t-1}) dD_t \\ &\equiv \mathcal{K}_t\{\{r_s, w_s\}_{s\geq 0}, D_0)\} \end{aligned} \tag{1}$$

where the second line is just a re-formulation of the model in **sequence space** conditional on the distribution in period 0. Let $\boldsymbol{K} = (K_0, K_1, \dots)$ and $\boldsymbol{Z} = (Z_0, Z_1, \dots)$. For given $\boldsymbol{Z}$ the model is now the solution to the following equation

$$H_t(\boldsymbol{K}, \boldsymbol{Z}, D_0) \equiv \mathcal{K}_t(\{r(Z_s, K_{s-1}), w(Z_s, K_{s-1})\}_{s\geq 0}, D_0) - K_t = 0, \qquad t = 0, 1, \dots \tag{2}$$

Or in **time-stacked form**

$$\boldsymbol{H}(\boldsymbol{K}, \boldsymbol{Z}, D_0) = \boldsymbol{0} \tag{3}$$

3.1 Linearization

Total differentiation implies

$$
\begin{aligned}
\boldsymbol{H_K}\boldsymbol{H_Z}d\boldsymbol{K} + \boldsymbol{H_Z}d\boldsymbol{Z} = 0 \Leftrightarrow \\
d\boldsymbol{K} = -\boldsymbol{H_K}^{-1}\boldsymbol{H_Z}d\boldsymbol{Z}
\end{aligned} \tag{4}
$$

where

$$
\boldsymbol{H_K} = \begin{bmatrix} \frac{\partial H_0}{\partial K_0} & \frac{\partial H_0}{\partial K_1} & \cdots \\ \frac{\partial H_1}{\partial K_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}, \boldsymbol{H_Z} = \begin{bmatrix} \frac{\partial H_0}{\partial Z_0} & \frac{\partial H_0}{\partial Z_1} & \cdots \\ \frac{\partial H_1}{\partial Z_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}
$$

A central insight is that $\mathcal{K}_t$ **only depends on the prices** and not the aggregate quantities in themselves. Consequently, applying the chain rule, we have

$$
\boldsymbol{H_K} = \mathcal{J}^{\mathcal{K},r}\mathcal{J}^{r,K} + \mathcal{J}^{\mathcal{K},w}\mathcal{J}^{w,K} - \boldsymbol{I} \tag{5}
$$

$$
\boldsymbol{H}_Z = \mathcal{J}^{\mathcal{K},r}\mathcal{J}^{r,Z} + \mathcal{J}^{\mathcal{K},w}\mathcal{J}^{w,Z} \tag{6}
$$

where generically

$$
\mathcal{J}^{x,y} = \begin{bmatrix} \frac{\partial x_0}{\partial y_0} & \frac{\partial x_0}{\partial y_1} & \cdots \\ \frac{\partial x_1}{\partial y_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}
$$

and specifically

$$
\mathcal{J}^{\mathcal{K},r} = \begin{bmatrix} \frac{\partial \mathcal{K}_0}{\partial r_0} & \frac{\partial \mathcal{K}_0}{\partial r_1} & \cdots \\ \frac{\partial \mathcal{K}_1}{\partial r_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}, \mathcal{J}^{r,K} = \alpha(\alpha-1)Z_tK_t^{\alpha-2}\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & 1 & 0 & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \mathcal{J}^{r,Z} = \alpha K_t^{\alpha-1}\boldsymbol{I}
$$

and

$$\mathcal{J}^{\mathcal{K},w} = \begin{bmatrix} \frac{\partial \mathcal{K}_0}{\partial w_0} & \frac{\partial \mathcal{K}_0}{\partial w_1} & \cdots \\ \frac{\partial \mathcal{K}_1}{\partial w_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}, \mathcal{J}^{w,K} = (1-\alpha)\alpha Z_t K_t^{\alpha-1} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & 1 & 0 & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \mathcal{J}^{w,Z} = (1-\alpha)K_t^{\alpha}\boldsymbol{I}$$

Note that $\forall t \neq s-1 : \frac{\partial r_t}{\partial K_s} = \frac{\partial w_t}{\partial K_s} = 0$ and $\forall s \neq t : \frac{\partial r_t}{\partial Z_s} = \frac{\partial w_s}{\partial Z_s} = 0$.

Truncation: In practice the matrices are truncated, such that they are $T \times T$.

1. **Along a row:** The effect on a variable in a given period of a shock in an arbitrary period.
2. **Along a column:** The effect of a shock in a given period on a variable in an arbitrary period.

3.2 Intermezzo: No uncertainty

In the case where there is no income risk, the household problem reduces to a **permanent income hypothesis model** with the solution

$$\begin{aligned} C_0(\{r_s, w_s\}_{s\geq 0}) &= \frac{(1+r_0)a_{-1} + \sum_{t=0}^{\infty} \frac{1}{\mathcal{R}_t} w_t}{\sum_{t=0}^{\infty} \beta^{t/\rho} \mathcal{R}_t^{\frac{1-\rho}{\rho}}} \\ C_t(\{r_s, w_s\}_{s\geq 0}) &= \beta^{1/\rho}(1+r_t)^{1/\rho} C_{t-1}(\{r_s, w_s\}_{s\geq 0}) \\ &= \beta^{t/\rho}\mathcal{R}_t^{1/\rho} C_0(\{r_s, w_s\}_{s\geq 0}) \end{aligned} \tag{7}$$

where the **compound interest rate factor** is

$$\mathcal{R}_t = \begin{cases} 1 & \text{if } t = 0, \\ (1+r_t)\mathcal{R}_{t-1} & \text{else} \end{cases} \tag{8}$$

The partial derivatives of a price change at time s can all be written in terms of the effect on **period 0 consumption**

$$\frac{\partial C_t}{\partial w_s} = \beta^{1/\rho}(1+r_t)^{1/\rho}\frac{\partial C_{t-1}}{\partial w_s} = \beta^{t/\rho}\mathcal{R}_t^{1/\rho}\frac{\partial C_0}{\partial w_s}, \tag{9}$$

$$\frac{\partial C_t}{\partial r_s} = \beta^{t/\rho}\mathcal{R}_t^{1/\rho}\frac{\partial C_0}{\partial r_s} + \begin{cases} 0 & \text{if } t < s, \\ \beta^{t/\rho}(\mathcal{R}_t)^{1/\rho}\frac{C_0}{\rho(1+r_s)} & \text{else.} \end{cases} \tag{10}$$

Capital accumulation follows the recursion

$$\begin{aligned}
\mathcal{K}_0 &= (1+r_0)K_{-1} + w_0 - C_0(\{r_s, w_s\}_{s\geq 0}) \\
\mathcal{K}_1 &= (1+r_1)\mathcal{K}_0 + w_1 - C_1(\{r_s, w_s\}_{s\geq 0}) \\
&\vdots \\
\mathcal{K}_t &= (1+r_t)\mathcal{K}_{t-1} + w_t - C_t(\{r_s, w_s\}_{s\geq 0}) \\
&\vdots
\end{aligned}$$

The Jacobian $\mathcal{J}^{\mathcal{K},r}$ can now be constructed from the recursion

$$\begin{aligned}
\frac{\partial \mathcal{K}_0}{\partial r_s} &= 1\{s=0\}K_{-1} + \frac{\partial C_0}{\partial r_s} \\
\frac{\partial \mathcal{K}_1}{\partial r_s} &= 1\{s=1\}\mathcal{K}_0 + (1+r_1)\frac{\partial \mathcal{K}_0}{\partial r_s} + \frac{\partial C_1}{\partial r_s} \\
&\vdots \\
\frac{\partial \mathcal{K}_t}{\partial r_s} &= 1\{s=t\}\mathcal{K}_s + (1+r_t)\frac{\partial \mathcal{K}_{t-1}}{\partial r_s} + \frac{\partial C_t}{\partial r_s}
\end{aligned}$$

and similar for$\mathcal{J}^{\mathcal{K},w}$

$$\begin{aligned}
\frac{\partial \mathcal{K}_0}{\partial r_s} &= \frac{\partial C_0}{\partial w_s} \\
\frac{\partial \mathcal{K}_1}{\partial r_s} &= (1+r_1)\frac{\partial \mathcal{K}_0}{\partial w_s} + \frac{\partial C_1}{\partial w_s} \\
&\vdots \\
\frac{\partial \mathcal{K}_t}{\partial r_s} &= (1+r_t)\frac{\partial \mathcal{K}_{t-1}}{\partial w_s} + \frac{\partial C_t}{\partial w_s}
\end{aligned}$$

3.3 Transition path after MIT shock

1. Assume we **start in the stationary equilibrium**

2. **MIT-shock:** Known path of future $\boldsymbol{Z}$, i.e. in terms of changes from steady state $d\boldsymbol{Z} = \boldsymbol{Z} - Z_{ss}$

3. **Question:** What happens to aggregate capital?

4. **Answer:** To a first order using equation (4) we have

$$\boldsymbol{G}^{K,Z} \equiv \frac{d\boldsymbol{K}}{d\boldsymbol{Z}} = -\boldsymbol{H}_{\boldsymbol{K}}^{-1}\boldsymbol{H}_{\boldsymbol{Z}}$$

where all derivatives are **evaluated at the stationary equilibrium**.

Additional responses are easily calculated:

$$\boldsymbol{G}^{r,Z} \equiv \frac{d\boldsymbol{r}}{d\boldsymbol{Z}} = \mathcal{J}^{r,Z} + \mathcal{J}^{r,K}\frac{d\boldsymbol{K}}{d\boldsymbol{Z}}$$

$$\boldsymbol{G}^{w,Z} \equiv \frac{d\boldsymbol{w}}{d\boldsymbol{Z}} = \mathcal{J}^{w,Z} + \mathcal{J}^{w,K}\frac{d\boldsymbol{K}}{d\boldsymbol{Z}}$$

$$\boldsymbol{G}^{C,Z} \equiv \frac{d\boldsymbol{C}}{d\boldsymbol{Z}} = \mathcal{J}^{C,r}\frac{d\boldsymbol{r}}{d\boldsymbol{Z}} + \mathcal{J}^{C,w}\frac{d\boldsymbol{w}}{d\boldsymbol{Z}}$$

$$\boldsymbol{G}^{Y,Z} \equiv \frac{d\boldsymbol{Y}}{d\boldsymbol{Z}} = \mathcal{J}^{Y,Z} + \mathcal{J}^{Y,K}\frac{d\boldsymbol{K}}{d\boldsymbol{Z}}$$

Transition path for variable o is then $d\boldsymbol{X}^{o} = \boldsymbol{G}^{o,Z}d\boldsymbol{Z}$.

3.4 Impulse-responses and and covariances with aggregate risk

Let us now return to the **model with aggregate risk** where $\tilde{Z}_t$ is a stochastic variable as indicated by the tilde.

1. In the limit where the shock variance disappears the dynamic equilibrium path converge to the stationary equilibrium.

2. In the limit where the shock variance disappears the transition path to the MIT shock around the stationary equilibrium is the same as the impulse-response in the dynamic equilibrium.

Remark: This is formally proved in Auclert et al. (2019).

Think as follows:

1. **MA representation:**
   Assume that the TFP process has a $MA(\infty)$ representation given by

$$d\tilde{Z}_t = \sum_{s=0}^{\infty} m_s^Z \epsilon_{t-s},$$

   where $d\tilde{Z}_t \equiv \tilde{Z}_t - Z_{ss}$ is deviations from stationary equilibrium, ϵ_t's are mutually iid **standard normally** distributed innovations and m_s^Z is the MA coefficients.

   After a **first order linearization** any *stochastic* output variable $\tilde{X}_t^o$ also have an $MA(\infty)$ representation

$$d\tilde{X}_t^o = \sum_{s=0}^{\infty} m_s^{o,Z} \epsilon_{t-s}^Z.$$

2. **Equivalence result:** Think of the following two experiments.

   (a) **In the stochastic model:** Assume that there is a shock in period t, and no shocks in any other past or future period. This implies

$$\begin{aligned} \mathbb{E}_t[d\tilde{X}_{t+s}|\epsilon_t = 1, \epsilon_k = 0 \forall k \neq t] &= m_s^{o,Z} \\ \mathbb{E}_t[d\tilde{Z}_{t+s}|\epsilon_t = 1, \epsilon_k = 0 \forall k \neq t] &= m_s^Z \end{aligned}$$

   and likewise

$$\begin{aligned} \mathbb{E}_t[d\tilde{\boldsymbol{X}}_t|\epsilon_t = 1, \epsilon_k = 0 \forall k \neq t] &= \boldsymbol{m}^{o,Z} = (m_0^{o,Z}, m_1^{o,Z}, \dots)^T \\ \mathbb{E}_t[d\tilde{\boldsymbol{Z}}_t|\epsilon_t = 1, \epsilon_k = 0 \forall k \neq t] &= \boldsymbol{m}^Z = (m_0^Z, m_1^Z, \dots)^T \end{aligned}$$

   (b) **In the deterministic model:** Think of an MIT-shock where $d\boldsymbol{Z} = \boldsymbol{m}^Z$. The path of the output variable then is

$$d\boldsymbol{X}^o = \boldsymbol{G}^{o,Z} d\boldsymbol{Z} = \boldsymbol{G}^{o,Z} \boldsymbol{m}^Z$$

(c) **Certainty equivalence:** Because of certainty equivalence these two experiments yield the same result and therefore

$$\begin{aligned} d\boldsymbol{X}^o &= \mathbb{E}_t[d\tilde{\boldsymbol{X}}_t | \epsilon_t = 1, \epsilon_k = 0 \forall k \neq t] \Leftrightarrow \\ \boldsymbol{m}^{o,Z} &= \boldsymbol{G}^{o,Z} \boldsymbol{m}^Z \end{aligned}$$

3. The **covariance** between any two output variables is given by

$$\begin{aligned} \text{Cov}(d\tilde{X}^o_t, d\tilde{X}^o_{t+k}) =& \mathbb{E}_t\left[d\tilde{X}^o_t \left(d\tilde{X}^o_{t+k} \right)^T \right] \\ =& \mathbb{E}_t\left[\left(\boldsymbol{m}^{o,Z} \right)^T \boldsymbol{\epsilon}^Z_t \left(\boldsymbol{\epsilon}^Z_{t+k} \right)^T \boldsymbol{m}^{o,Z} \right] \\ =& \sum_{s=0}^{\infty} m^{o,Z}_s m^{o,Z}_{s+k} \end{aligned}$$

where we use that

$$\mathbb{E}_t\left[\boldsymbol{\epsilon}^Z_t \left(\boldsymbol{\epsilon}^Z_{t+k} \right)^T \right] = \mathbb{E}_t \begin{bmatrix} \epsilon^Z_t \epsilon^Z_{t+k} & \epsilon^Z_t \epsilon^Z_{t+k-1} & \cdots & \epsilon^Z_t \epsilon^Z_t & \cdots & \cdots & \cdots \\ \epsilon^Z_{t-1} \epsilon^Z_{t+k} & \epsilon^Z_{t-1} \epsilon^Z_{t+k-1} & \cdots & \epsilon^Z_{t-1} \epsilon^Z_t & \epsilon^Z_t \epsilon^Z_{t-1} & \cdots & \cdots \\ \epsilon^Z_{t-2} \epsilon^Z_{t+k} & \epsilon^Z_{t-2} \epsilon^Z_{t+k-1} & \cdots & \epsilon^Z_{t-2} \epsilon^Z_t & \epsilon^Z_t \epsilon^Z_{t-1} & \epsilon^Z_{t-2} \epsilon^Z_{t-2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 1 & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Truncation: Since the Jacobians are only computed for some truncated horizon T, we have that

$$\text{Cov}(d\tilde{X}_t, d\tilde{X}_{t+k}) = \sum_{s=0}^{T-k} m^{o,Z}_s m^{o,Z}_{s+k}$$

4. **Example:** Let TFP follow the $AR(1)$ process

$$\tilde{Z}_t = 1 - \rho + \rho \tilde{Z}_{t-1} + \sigma \epsilon_t.$$
$$\Rightarrow d\tilde{Z}_t = \sigma \sum_{s=0}^{\infty} \rho^s \epsilon_{t-s},$$

with $m_s^Z = \sigma \rho^s$. Given $\boldsymbol{G}^{o,Z}$, we have that

$$\mathrm{Cov}(d\tilde{X}_t^o, dX_{t+k}^{\tilde{o}}) = \sigma^2 \sum_{s=0}^{T-k} \rho^{2s+k} \boldsymbol{G}_{t,s}^{o,Z} \boldsymbol{G}_{t+k,s}^{o,Z}.$$

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3.5 Computational efficiency

Much of Auclert et al. (2019) is about **computational efficiency**. How to:

1. Compute the jacobians of the heterogenous agent block efficiently.
2. Multiply the (sparse) jacobians efficiently.
3. Compute the covariance matrix efficiently.

Additionally, the paper shows:

1. How to check for determinancy.
2. How to use the computed jacobians to speed-up the solution of the fully non-linear transition path.

References

Achdou, Y., Han, J., Lasry, J.-M., Lions, P.-L., and Moll, B. (2020). Income and wealth distribution in macroeconomics: A continuous-time approach. Working Paper.

Ahn, S., Kaplan, G., Moll, B., Winberry, T., and Wolf, C. (2018). When Inequality Matters for Macro and Macro Matters for Inequality. *NBER Macroeconomics Annual*, 32:1–75.

Aiyagari, S. R. (1994). Uninsured Idiosyncratic Risk and Aggregate Saving. *The Quarterly Journal of Economics*, 109(3):659–684.

Aiyagari, S. R. and McGrattan, E. R. (1998). The optimum quantity of debt. *Journal of Monetary Economics*, 42(3):447–469.

Alves, F., Kaplan, G., Moll, B., and Violante, G. L. (2019). A Further Look at the Propagation Mechanism of Monetary Policy Shocks in. Technical report.

Auclert, A., Bardóczy, B., Rognlie, M., and Straub, L. (2019). Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models. NBER Working Paper 26123, National Bureau of Economic Research.

Auclert, A. and Rognlie, M. (2020). Inequality and Aggregate Demand. Technical report.

Auclert, A., Rognlie, M., and Straub, L. (2018). The Intertemporal Keynesian Cross. NBER Working Paper 25020.

Auclert, A., Rognlie, M., and Straub, L. (2020). Micro Jumps, Macro Humps: Monetary Policy and Business Cycles in an Estimated HANK Model. NBER Working Paper 26647.

Azinovic, M., Gaegauf, L., and Scheidegger, S. (2019). Deep Equilibrium Nets. Technical report.

Bayer, C., Born, B., and Luetticke, R. (2020). Shocks, Frictions, and Inequality. Technical report.

Bayer, C. and Luetticke, R. (2019). Solving discrete time heterogeneous agent models with aggregate risk and many idiosyncratic states by perturbation. Working Paper.

Bayer, C., Luetticke, R., Pham-Dao, L., and Tjaden, V. (2019). Precautionary Savings, Illiquid Assets, and the Aggregate Consequences of Shocks to Household Income Risk. *Econometrica*, 87(1):255–290.

Bewley, T. (1986). Stationary Monetary Equilibrium with a Continuum of Independently Fluctuating Consumers. In Hildenbrand, W. and Mas-Collel, A., editors, *Contributions to Mathematical Economics in Honor of Gerad Debreu*. North-Holland, Amsterdam.

Bilbiie, F. O. (2008). Limited asset markets participation, monetary policy and (inverted) aggregate demand logic. *Journal of Economic Theory*, 140(1):162–196.

Bilbiie, F. O. (2019a). Monetary Policy and Heterogeneity: An Analytical Framework. Technical report.

Bilbiie, F. O. (2019b). The New Keynesian cross. *Journal of Monetary Economics*, forthcoming.

Bilbiie, F. O., Känzig, D. R., and Surico, P. (2020). Capital, Income Inequality, and Consumption: the Missing Link. Technical report.

Boppart, T., Krusell, P., and Mitman, K. (2018). Exploiting MIT shocks in heterogeneous-agent economies: the impulse response as a numerical derivative. *Journal of Economic Dynamics and Control*, 89:68–92.

Broer, T., Harbo Hansen, N.-J., Krusell, P., and Öberg, E. (2020). The New Keynesian Transmission Mechanism: A Heterogeneous-Agent Perspective. *The Review of Economic Studies*, 87(1):77–101.

Carroll, C. D. (2000). Requiem for the representative consumer? Aggregate implications of microeconomic consumption behavior. *The American Economic Review: Papers and Proceedings of the One Hundred Twelfth Annual Meeting of the American economic Association*, 90(2):110–115.

Challe, E., Matheron, J., Ragot, X., and Rubio-Ramirez, J. F. (2017). Precautionary saving and aggregate demand. *Quantitative Economics*, 8(2):435–478.

Cloyne, J., Ferreira, C., and Surico, P. (2020). Monetary Policy when Households have Debt: New Evidence on the Transmission Mechanism. *The Review of Economic Studies*, 87(1):102–129.

Conesa, J. C., Kitao, S., and Krueger, D. (2009). Taxing Capital? Not a Bad Idea after All! *American Economic Review*, 99(1):25–48.

Debortoli, D. and Galí, J. (2018). Monetary Policy with Heterogeneous Agents: Insights from TANK models. Technical report.

Den Haan, W. J., Judd, K. L., and Juillard, M. (2010). Computational suite of models with heterogeneous agents: Incomplete markets and aggregate uncertainty. *Journal of Economic Dynamics and Control*, 34(1):1–3.

Den Haan, W. J., Rendahl, P., Riegler, M., and Riegler, M. (2017). Unemployment (Fears) and Deflationary Spirals. *Journal of the European Economic Association*.

Fernandez-Villaverde, J., Hurtado, S., and Nuno, G. (2020). Financial Frictions and the Wealth Distribution. Technical report.

Gornemann, N., Kuester, K., and Nakajima, M. (2016). Doves for the Rich, Hawks for the Poor? Distributional Consequences of Monetary Policy. Technical report.

Guerrieri, V. and Lorenzoni, G. (2017). Credit Crises, Precautionary Savings, and the Liquidity Trap. *The Quarterly Journal of Economics*, 132(3):1427–1467.

Hagedorn, M., Manovskii, I., and Mitman, K. (2019). The Fiscal Multiplier. NBER Working Paper 25571.

Heathcote, J., Storesletten, K., and Violante, G. L. (2009). Quantitative Macroeconomics with Heterogeneous Households. *Annual Review of Economics*, 1(1):319–354.

Heathcote, J., Storesletten, K., and Violante, G. L. (2014). Consumption and Labor Supply with Partial Insurance: An Analytical Framework. *The American Economic Review*, 104(7):2075–2126.

Holm, M. B. and Paul, P. (2020). The Transmission of Monetary Policy under the Microscope a. Technical report.

Huggett, M. (1993). The risk-free rate in heterogeneous-agent incomplete-insurance economies. *Journal of Economic Dynamics and Control*, 17(5-6):953–969.

Imrohoroğlu, A. (1989). Cost of Business Cycles with Indivisibilities and Liquidity Constraints. *Journal of Political Economy*, 97(6):1364–1383.

Kaplan, G., Moll, B., and Violante, G. L. (2018). Monetary Policy According to HANK. *American Economic Review*, 108(3):697–743.

Kaplan, G. and Violante, G. L. (2018). Microeconomic Heterogeneity and Macroeconomic Shocks. *Journal of Economic Perspectives*, 32(3):167–194.

Kirkby, R. (2017). Transition paths for Bewley-Huggett-Aiyagari models: Comparison of some solution algorithms.

Krueger, D., Mitman, K., and Perri, F. (2016). Chapter 11 - Macroeconomics and Household Heterogeneity. In Taylor, J. B. and Uhlig, H., editors, *Handbook of Macroeconomics*, volume 2, pages 843–921. Elsevier.

Krusell, P. and Smith, A. A. (1997). Incoem and wealth heterogeneity, portfolio choice, and equilibrium asset returns. *Macroeconomic Dynamics*, 1(02):387–422.

Krusell, P. and Smith, A. A. (1998). Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economy*, 106(5):867–896.

Krusell, P. and Smith, A. A. (2006). Quantitative macroeconomic models with heterogeneous agents. In Blundell, R., editor, *Advanced in Economics and Econometrics: Theory and Applications*, pages 298–340. Cambridge University Press.

Kubler, F. and Scheidegger, S. (2018). Self-justied equilibria: Existence and computation. Technical report.

Luetticke, R. (2019). Transmission of monetary policy with heterogeneity in household portfolios. Working Paper.

Maliar, L., Maliar, S., and Winant, P. (2019). Will Artificial Intelligence Replace Computational Economists Any Time Soon? Technical report.

McKay, A., Nakamura, E., and Steinsson, J. (2016). The Power of Forward Guidance Revisited. *American Economic Review*, 106(10):3133–3158.

McKay, A. and Reis, R. (2016). The Role of Automatic Stabilizers in the U.S. Business Cycle. *Econometrica*, 84(1):141–194.

Oh, H. and Reis, R. (2012). Targeted transfers and the fiscal response to the great recession. *Journal of Monetary Economics*, 59:S50–S64.

Pröhl, E. (2019). Approximating Equilibria with Ex-Post Heterogeneity and Aggregate Risk. Technical report.

Ravn, M. O. and Sterk, V. (2017). Job uncertainty and deep recessions. *Journal of Monetary Economics*, 90:125–141.

Ravn, M. O., Sterk, V., and others (2020). Macroeconomic Fluctuations with HANK & SAM: An Analytical Approach. Working Paper.

Scheidegger, S. and Bilionis, I. (2019). Machine learning for high-dimensional dynamic stochastic economies. *Journal of Computational Science*, 33:68–82.

Schmedders, K. and Judd, K. L. (2013). *Handbook of Computational Economics Vol. 3*. Newnes. Google-Books-ID: xDhO6L_Psp8C.

Slacalek, J., Tristani, O., and Violante, G. (2020). Household Balance Sheet Channels of Monetary Policy: A Back of the Envelope Calculation for the Euro Area. Technical report.

Tauchen, G. and Hussey, R. (1991). Quadrature-Based Methods for Obtaining Approximate Solutions to Nonlinear Asset Pricing Models. *Econometrica*, 59(2):371–396.

Werning, I. (2015). Incomplete Markets and Aggregate Demand. NBER Working Paper 21448.

Winberry, T. (2018). A method for solving and estimating heterogeneous agent macro models. *Quantitative Economics*, 9(3):1123–1151–1151.

Wolf, C. K. (2020). The Missing Intercept: A Demand Equivalence Approach. Technical report.