CENTER FOR
ECONOMIC
BEHAVIOR &
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Consumption-Saving Models

An Introduction to Dynamic Programming

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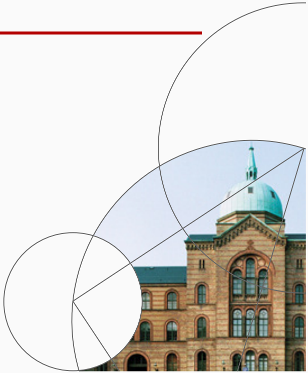


Introduction

Introduction

- **Why are consumption-saving models important?**
  1. Important topic in itself (70 percent of GDP)
  2. **Central aspect of many other decisions**
     a) Labor supply, retirement, and fertility choices
     b) Portfolio choices and asset pricing
     c) Housing and location choices
  3. Households are the **cornerstone of general equilibrium models** designed to study the cause and effect of inequality
- **Dynamic programming** essential for recent advances
  1. Idiosyncratic and aggregate uncertainty
  2. Ex ante and ex post heterogeneity
  3. Internal and external optimization frictions (bounded rationality, adjustment costs etc.)

Introduction

- **Part of mini-course on dynamic programming:** ConsumptionSavingNotebooks/DynamicProgramming
- **Focus in the partial equilibrium (PE) part of these slides:** Carroll (2020, QE), *Theoretical foundations of buffer stock saving*
- **Acknowledgments:** Christoffer Jessen Weissert, Emil Holst Partsch, Anders Yding, previous students in Dynamic Programming

General references

- **Dynamic programming and computational methods in general:** Stokey and Lucas (1989), Judd (1998), Adda and Cooper (2003), Ljungqvist and Sargent (2004), Puterman (2009), Powell (2011), Bertsekas (2012), Schmedders and Judd (2013)
- **Surveys of consumption-saving litteratures:** Browning and Lusardi (1996), Browning and Crossley (2001), Heathcote et al. (2009), Krusell and Smith (2006), Krueger et al. (2016), Pistaferri (2017), Kaplan and Violante (2018)
- **End-of-slides:** Many more references

Plan

1. Introduction
2. PIH
3. Buffer-stock
4. EGM
5. Further perspectives
6. Estimation
7. GE
8. Summary

PIH

Permanent Income Hypothesis (PIH)

- **Household problem**

$$
\begin{aligned}
V_0(M_0, P_0) &= \max_{\{C_t\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t \frac{C_t^{1-\rho}}{1-\rho}, \;\; \beta < 1, \, \rho \geq 1 \\
& \text{s.t.} \\
A_t &= M_t - C_t \\
B_{t+1} &= R \cdot A_t, \;\; R > 0 \\
M_{t+1} &= B_{t+1} + P_{t+1} \\
P_{t+1} &= G \cdot P_t, \;\; G > 0 \\
A_T &\geq 0
\end{aligned}
$$

- **Well-defined analytical solution, also for $T \to \infty$ if**
    1. Return impatience (RI): $(\beta R)^{1/\rho}/R < 1$
    2. Finite human wealth (FHW): $G/R < 1$
- **What do you think is missing?**

The Intertemporal Budget Constraint (IBC)

- **Substitution** implies

$$
\begin{aligned}
A_T &= M_T - C_T = (RA_{T-1} + P_T) - C_T \\
&= R(M_{T-1} - C_{T-1}) + P_T - C_T \\
&= R^2 A_{T-2} + RP_{T-1} - RC_{T-1} + P_T - C_T \\
&= R^{T+1} A_{-1} + \sum_{t=0}^{T} R^{T-t}(P_t - C_t)
\end{aligned}
$$

- Use **terminal condition** $A_T = 0$ (why equality?)

$$
R^{-T} A_T = 0 \quad \Leftrightarrow \quad B_0 + H_0 = \sum_{t=0}^{T} R^{-t} C_t
$$

where $H_0 \equiv \sum_{t=0}^{T} (G/R)^t P_0 = \frac{1-(G/R)^{T+1}}{1-G/R} P_0$

Static problem $\rightarrow$ Lagrangian

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \frac{C_t^{1-\rho}}{1-\rho} + \lambda \left[\sum_{t=0}^{T} R^{-t} C_t - (B_0 + H_0) \right]$$

- **First order conditions**

$$\forall t : \; 0 = \beta^t C_t^{-\rho} - \lambda R^{-t}$$

- ***Short-run*** **Euler** equation**:** $\frac{C_{t+1}}{C_t} = (\beta R)^{1/\rho}$
- ***Long-run*** **Euler** equation**:** $\frac{C_t}{C_0} = (\beta R)^{t/\rho}$

Consumption function

- Insert **Euler** into **IBC**

$$
\begin{aligned}
\sum_{t=0}^{T} R^{-t}(\beta R)^{t/\rho} C_0 &= B_0 + H_0 \Leftrightarrow \\
C_0 \sum_{t=0}^{T} ((\beta R)^{1/\rho}/R)^t &= B_0 + H_0
\end{aligned}
$$

- **Solve** for C_0

$$
C_0 = \frac{1-(\beta R)^{1/\rho}/R}{1-((\beta R)^{1/\rho}/R)^{T+1}}(B_0 + H_0)
$$

  - **MPC:** $\frac{\partial C_0}{\partial B_0} \approx 1 - [(\beta R)^{1/\rho}/R] \approx 1 - R^{-1} \approx r$, where $R = 1 + r$
  - **MPCP:** $\frac{\partial C_0}{\partial P_0} \approx 1 - [(\beta R)^{1/\rho}/R] \frac{\partial H_0}{\partial P_0} \approx \frac{1-1/R}{1-G/R} \approx 1$

Side-note: Value function

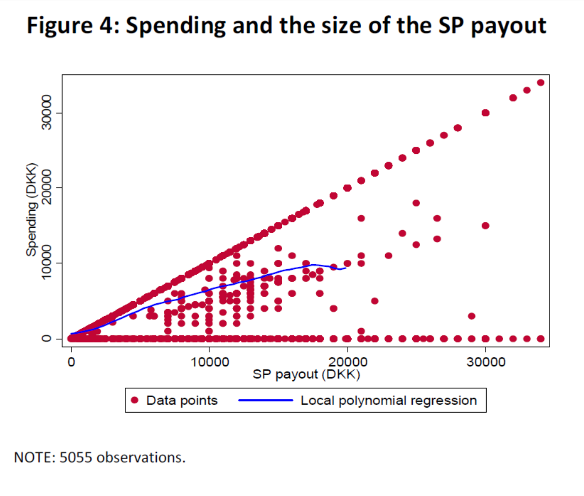
- **Analytical expression** for the value function

$$
\begin{aligned}
V_0(M_0, P_0) &= \sum_{t=0}^{T} \beta^t u((\beta R)^{t/\rho} C_0) \\
&= \sum_{t=0}^{T} \beta^t (\beta R)^{(1-\rho)t/\rho} \frac{C_0^{1-\rho}}{1-\rho} \\
&= \sum_{t=0}^{T} ((\beta R)^{1/\rho}/R)^t \frac{C_0^{1-\rho}}{1-\rho} \\
&= \frac{1-((\beta R)^{1/\rho}/R)^{T+1}}{1-(\beta R)^{1/\rho}/R} \frac{C_0^{1-\rho}}{1-\rho}
\end{aligned}
$$

Empirical evidence

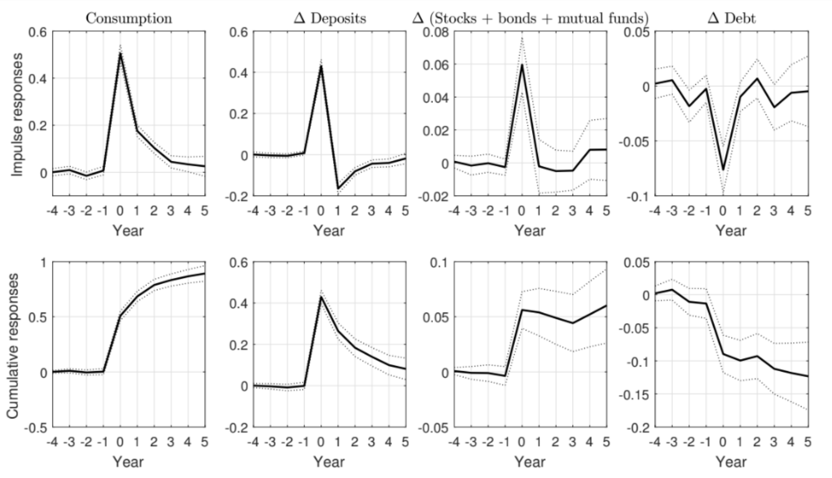
- **Pro**
  1. Micro-founded consumption-saving
     - Theoretically appealing (humans are intentional)
     - Empirically appealing (testable implications on micro-data)
  2. Larger responses to permanent than to transitory shocks
  3. Consumption smoothing - save for retirement (future low income)
- **Con**
  1. Households seems to have a high MPC in the range 0.20-0.40
     - Survey studies (Kreiner et al., 2019)
     - Tax rebates studies (Johnson et al., 2006; Parker et al., 2013)
     - Lottery studies (Fagereng et al., 2020)
     - ARM payments studies (Di Maggio et al., 2017; Druedahl et al., 2020b)
  2. Consumption responds to anticipated income changes
  3. Households with more volatile income have larger savings
  4. Consumption tracks income over the life-cycle
  5. (Households are only boundedly rational)

High MPC: Danish SP payout



Source: Kreiner, Lassen og Leth-Petersen (AEJ:Pol, 2019)

High MPC: Norwegian lottery winners



Source: Fagereng, Holm, Natvik (AEJ:Macro, 2020)

Buffer-stock

Buffer-stock model (Deaton-Carroll)

+ borrowing constraints

+ income uncertainty

$\Rightarrow$

$$
\begin{aligned}
V_0(M_0, P_0) &= \max_{\{C_t\}_{t=0}^{T}} \mathbb{E}_0 \sum_{t=0}^{T} \beta^t \frac{C_t^{1-\rho}}{1-\rho} \\
&\text{s.t.} \\
A_t &= M_t - C_t \\
M_{t+1} &= RA_t + Y_{t+1} \\
Y_{t+1} &= \xi_{t+1} P_{t+1} \\
\xi_{t+1} &= \begin{cases} \mu & \text{with prob. } \pi \\ (\epsilon_{t+1} - \pi\mu)/(1-\pi) & \text{else} \end{cases} \\
& \quad \epsilon_t \sim \exp \mathcal{N}(-0.5\sigma_\xi^2, \sigma_\xi^2) \\
P_{t+1} &= G P_t \psi_{t+1}, \;\; \psi_t \sim \exp \mathcal{N}(-0.5\sigma_\psi^2, \sigma_\psi^2) \\
A_t &\geq -\lambda P_t \\
A_T &\geq 0
\end{aligned}
$$

Note: Later analytical results hold only for $\mu = 0$ and $\pi > 0$

How to solve the model?

- **Borrowing constraints** $\rightarrow$ inequalities $\rightarrow$ high-dimensional **Kuhn-Tucker problem**
- **Uncertainty** $\rightarrow$ fully dynamic problem $\rightarrow$ no simple Lagrangian
- **No analytical solution with CRRA preferences**
  - Quadratic or CARA utility, which give some analytical results, have implausible properties

$$\text{CRRA: } u(c) = \frac{c^{1-\rho}}{1-\rho} \rightarrow \text{RRA} = \rho$$

$$\text{Qudratic: } u(c) = ac - \frac{b}{2}c^2 \rightarrow \text{ RRA} = \frac{b}{a-bc}c$$

$$\text{CARA: } u(c) = \frac{1}{\alpha}e^{-\alpha c} \rightarrow \text{ RRA} = \alpha c$$

  where RRA = relative risk aversion = $\frac{-u''(c)}{u'(c)}c$

- **Solution:** Bellman equation $\rightarrow$ numerical dynamic programming

Bellman equation

$$
\begin{aligned}
V_t(M_t, P_t) &= \max_{C_t} \frac{C_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[V_{t+1}(M_{t+1}, P_{t+1}) \right] \\
& \text{s.t.} \\
A_t &= M_t - C_t \\
M_{t+1} &= RA_t + Y_{t+1} \\
Y_{t+1} &= \xi_{t+1} P_{t+1} \\
\xi_{t+1} &= \begin{cases} \mu & \text{with prob. } \pi \\ (\epsilon_{t+1} - \pi\mu)/(1-\pi) & \text{else} \end{cases} \\
P_{t+1} &= G P_t \psi_{t+1} \\
A_t &\geq -\lambda P_t \\
A_T &\geq 0
\end{aligned}
$$

Normalization I

- **Defining** $c_t \equiv C_t/P_t$, $m_t \equiv M_t/P_t$ etc. implies

$$\begin{aligned} A_t = M_t - C_t &\Leftrightarrow A_t/P_t = M_t/P_t - C_t/P_t \\ &\Leftrightarrow a_t = m_t - c_t \end{aligned}$$

$$\begin{aligned} M_{t+1} = RA_t + Y_{t+1} &\Leftrightarrow M_{t+1}/P_{t+1} = RA_t/P_{t+1} + Y_{t+1}/P_{t+1} \\ &\Leftrightarrow m_{t+1} = Ra_t P_t/P_{t+1} + \xi_{t+1} \\ &\Leftrightarrow m_{t+1} = \frac{R}{G\psi_{t+1}} a_t + \xi_{t+1} \end{aligned}$$

The **adjustment factor** $\frac{1}{G\psi_{t+1}}$ is due to changes in permanent income

Normalization II

- **Defining** $v_t(m_t) = V_t(M_t, P_t)/P_t^{1-\rho}$ finally implies

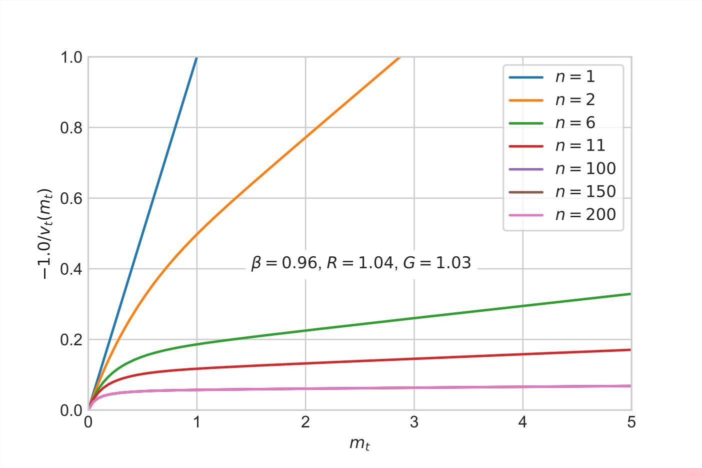
$$
\begin{aligned}
V_t(M_t, P_t) &= \max_{C_t} \frac{C_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[V_{t+1}(M_{t+1}, P_{t+1}) \right] \\
&= \max_{c_t} \frac{(c_t P_t)^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[V_{t+1}(M_{t+1}, P_{t+1}) \right] \Leftrightarrow \\
V_t(M_t, P_t)/P_t^{1-\rho} &= \max_{c_t} \frac{(c_t P_t)^{1-\rho}/P_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[V_{t+1}(M_{t+1}, P_{t+1})/P_t^{1-\rho} \right] \Leftrightarrow \\
v_t(m_t) &= \max_{c_t} \frac{c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[V_{t+1}(M_{t+1}, P_{t+1})/P_{t+1}^{1-\rho} \cdot P_{t+1}^{1-\rho}/P_t^{1-\rho} \right] \\
&= \max_{c_t} \frac{c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[(G\psi_{t+1})^{1-\rho} v_{t+1}(m_{t+1}) \right]
\end{aligned}
$$

Bellman equation in ratio form

$$
\begin{aligned}
v_t(m_t) &= \max_{c_t} \frac{c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[(G\psi_{t+1})^{1-\rho} v_{t+1}(m_{t+1}) \right] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
m_{t+1} &= \frac{1}{G\psi_{t+1}} R a_t + \xi_{t+1} \\
\xi_{t+1} &= \begin{cases} \mu & \text{with prob. } \pi \\ (\epsilon_{t+1} - \pi\mu)/(1-\pi) & \text{else} \end{cases} \\
a_t &\geq -\lambda \\
a_T &\geq 0
\end{aligned}
$$

- **Benefit:** Dimensionality of state space reduced
  Can this always be done?
- Easy to solve by **VFI**

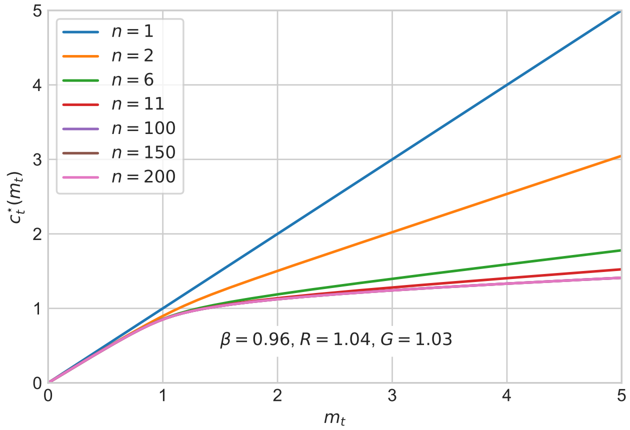
$T \to \infty$; Convergence of $-1.0/v_t(m_t) \to -1.0/v^\star(m_t)$



Other parameters: $\rho = 2$, $\pi = 0.005$, $\mu = 0.0$, $\sigma_\psi = \sigma_\xi = 0.10$

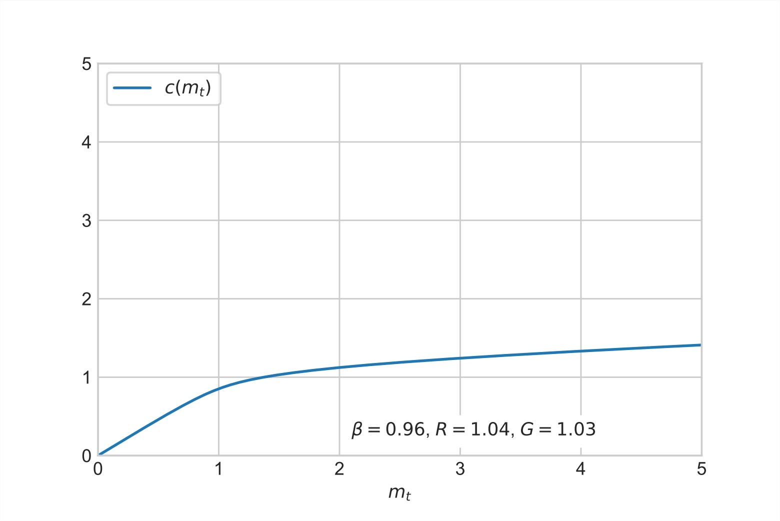
Note: $-1.0/v_t(m_t)$ is a numerically more stable object than $v_t(m_t)$

$T \to \infty$: Convergence of $c_t(m_t) \to c^\star(m_t)$

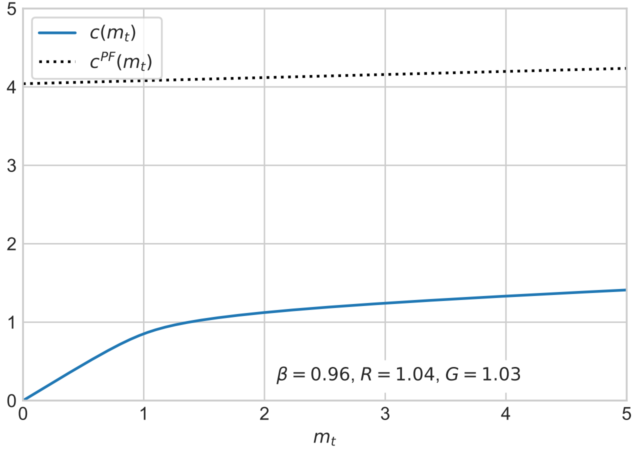


- What is the MPC?

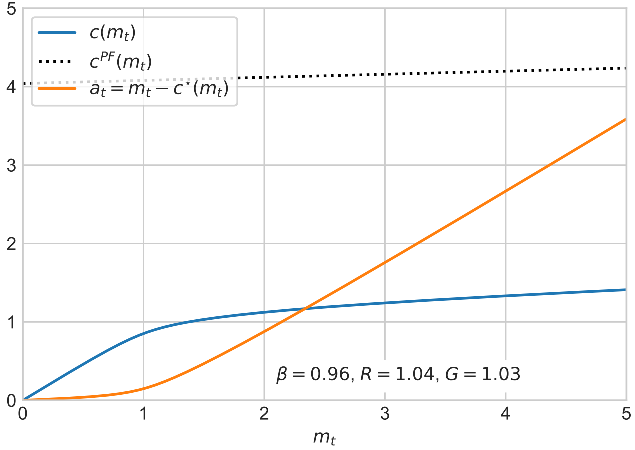
$T \to \infty$: The buffer-stock target



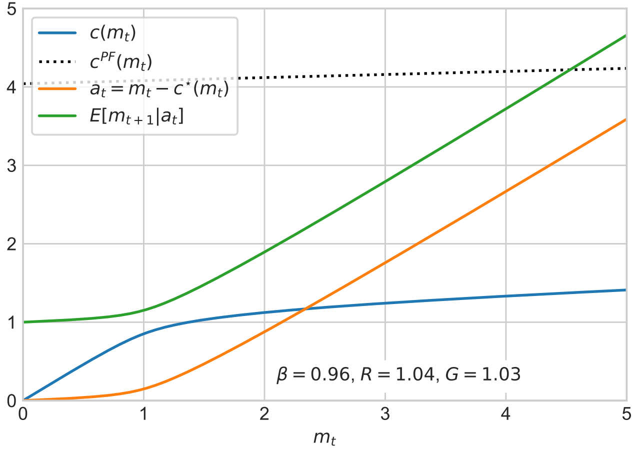
$T \to \infty$: The buffer-stock target



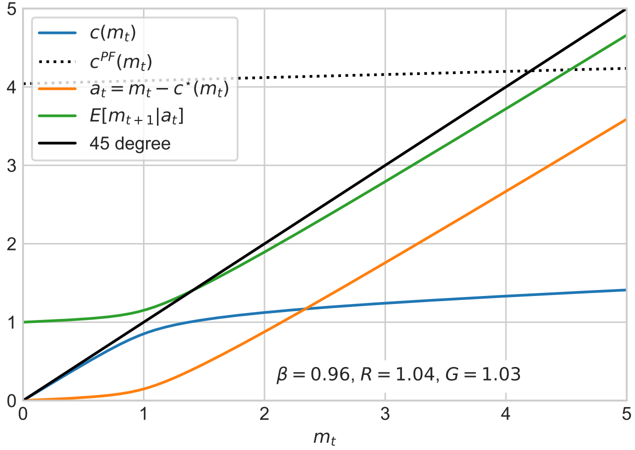
$T \to \infty$: The buffer-stock target



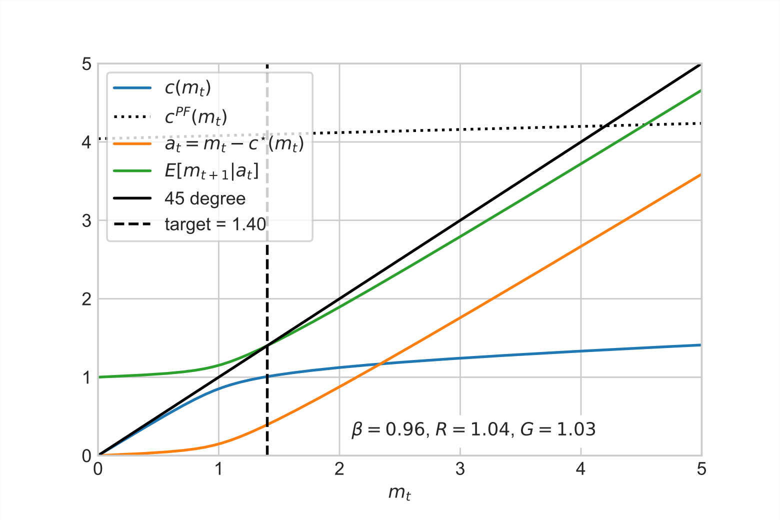
$T \to \infty$: The buffer-stock target



$T \to \infty$: The buffer-stock target



$T \to \infty$: The buffer-stock target



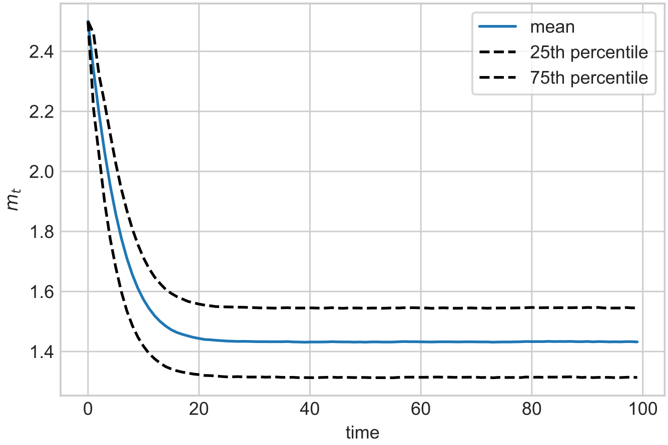
Simulation for $t \in \{0, 1, \dots, T-1\}$

1. Choose m_0 and set $t = 0$
2. Calculate $c_t = c^\star(m_t)$
3. Calculate $a_t = m_t - c_t$
4. Draw (pseudo-)random numbers

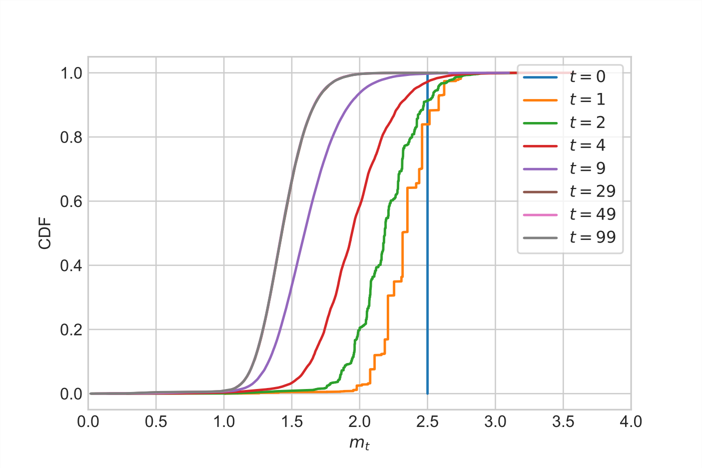
$$\begin{aligned} \epsilon_{t+1} &\sim \exp\mathcal{N}(-0.5\sigma_\xi^2, \sigma_\xi^2) \\ \psi_{t+1} &\sim \exp\mathcal{N}(-0.5\sigma_\psi^2, \sigma_\psi^2) \\ \eta_{t+1} &\sim \mathcal{U}(0,1) \end{aligned}$$

5. Calculate $\xi_{t+1} = \begin{cases} \mu & \text{if } \eta_{t+1} < \pi \\ (\epsilon_{t+1} - \pi\mu)/(1-\pi) & \text{else} \end{cases}$
6. Calculate $m_{t+1} = \frac{R}{G\psi_{t+1}} a_t + \xi_{t+1}$
7. Set $t = t + 1$
8. Stop if $t \geq T$ else go to step 2

Simulation: Avg. cash-on-hand



Simulation: Distribution of cash-on-hand

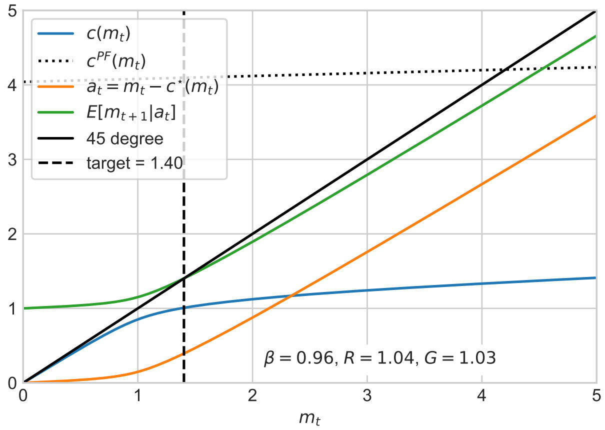


- **Proof of convergence:** Szeidl (2006)

Buffer-stock

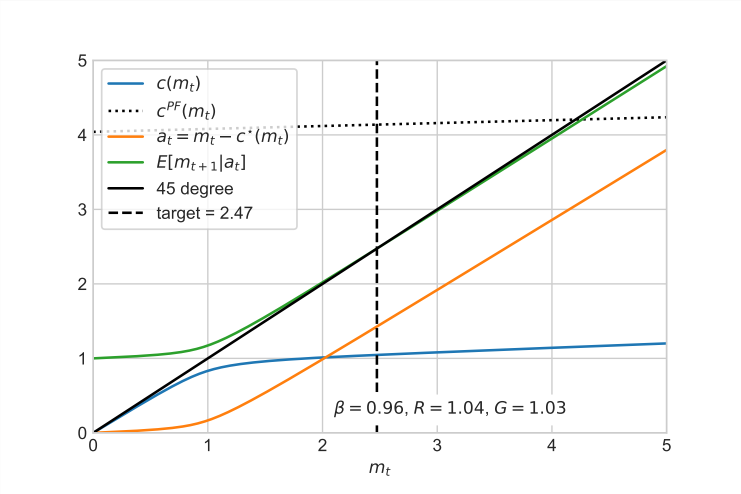
Details

Precautionary saving: $\sigma_\psi = 0.10$



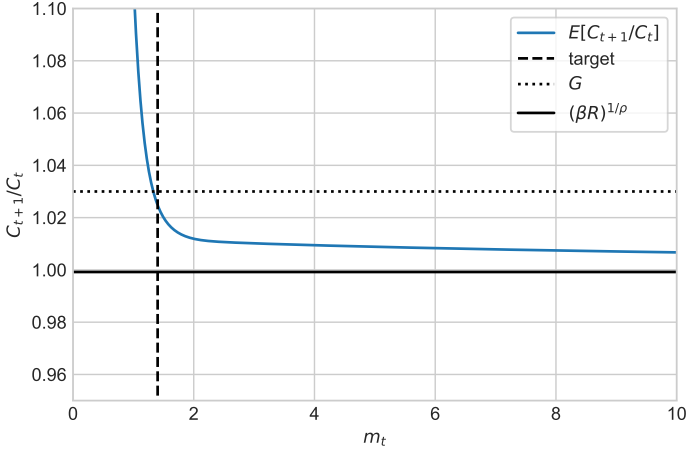
Target with baseline risk: 1.40

Precautionary saving: $\sigma_\psi = 0.15$



Target with high risk: 2.47

Consumption growth I



Consumption growth II

- Remember **Euler-equation**

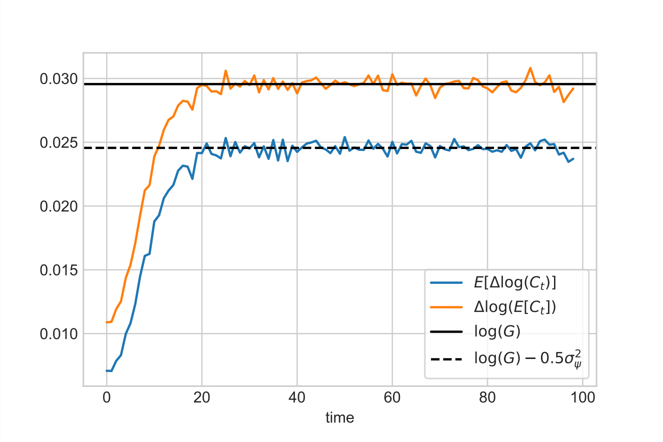
$$C_t^{-\rho} = \beta R \mathbb{E}_t \left[C_{t+1}^{-\rho} \right] \text{ if no uncertainty} \Rightarrow C_{t+1}/C_t = (\beta R)^{1/\rho}$$

- **Results**
  1. C_{t+1}/C_t is declining in m_t
  2. $\lim_{m_t \to \infty} C_{t+1}/C_t = (\beta R)^{1/\rho} = \text{RI}$
  3. $\lim_{m_t \to 0} C_{t+1}/C_t = \infty$
  4. $C_{t+1}/C_t < G$ at buffer-stock target
- **Intuition** for $C_{t+1}/C_t > (\beta R)^{1/\rho}$
  1. Uncertainty $\Rightarrow$ expected marginal utility $\uparrow$ [$C_{t+1}^{-\rho}$ is convex function]
  2. Consumer must be lowered today, $C_t \downarrow$
  3. Consumption growth will increase, $C_{t+1}/C_t \uparrow$

Further: *The above arguments are stronger for lower cash-on-hand relative to permanent income*

Consumption growth III

1. Growth of average consumption $= G$
2. Average consumption growth $= G - 0.5\sigma_{\psi}^2$

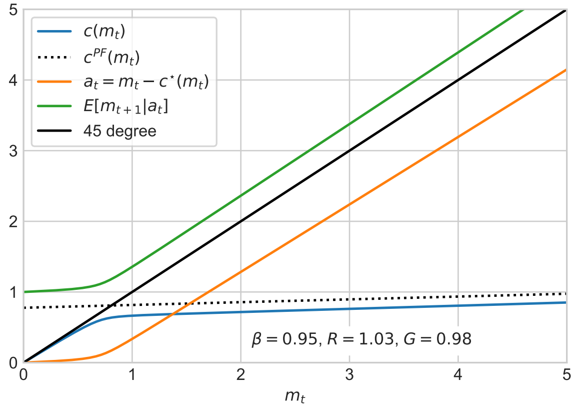


Always a buffer-stock target? I

1. **Utility impatience** (UI): $\beta < 1$
2. **Return impatience** (RI): $(\beta R)^{1/\rho}/R < 1$
3. **Weak return impatience** (WRI): $\pi^{1/\rho}(\beta R)^{1/\rho}/R < 1$
4. **Growth impatience** (GI): $(\beta R)^{1/\rho}\mathbb{E}_t[\psi_{t+1}^{-1}]/G < 1$
5. **Absolute impatience** (AI): $(\beta R)^{1/\rho} < 1$
6. **Finite value of autarky** (FVA): $\beta\mathbb{E}_t[(G\psi_{t+1})^{1-\rho}] < 1$

Always a buffer-stock target? II

- **GI ensures buffer-stock target**
- If not *GI* then inifinite accumulation is possible like:



Existence of solution

- **Existence of solution:** WRI + FVA
  - **Proof:** Use *Boyds weighted contraction mapping theorem*
  - **Standard assumptions:** FHW, RI, GI
- The **consumption function** is twice continuously differentiable**, increasing** and **concave**

The borrowing constraint

- Assume **perfect foresight** ($\sigma_\psi = \sigma_\epsilon = \pi = 0$), but **no borrowing**, $\lambda = 0$.
- **Solution:** RI + FHW is still *sufficient* (with $\lambda = \infty$ they are *necessary*)
- **Standard solutions:** RI + FHW
  1. **GI** $\Rightarrow$ *constraint will eventually be binding*

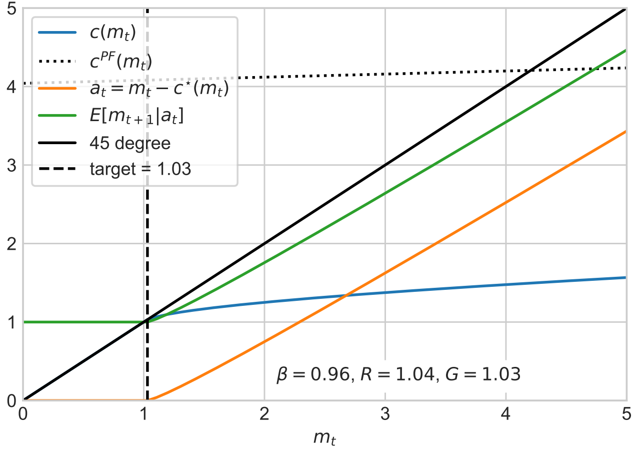
     $$c^\star(m_t) \text{ converge to } c^{PF}(m_t) \text{ from below as } m_t \to \infty$$

  2. **Not GI** $\Rightarrow$ *constraint is never reached*

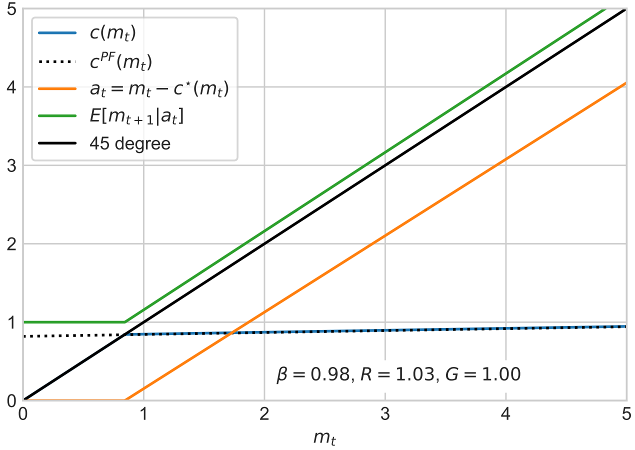
     $$c^\star(m_t) = c^{PF}(m_t) \text{ for } m_t \geq 1$$

- **Exotic solutions without FHW** exists (GI necessary)

Perfect foresight with $\lambda = 0$ and GI



Perfect foresight with $\lambda = 0$, but not GI



Buffer-stock

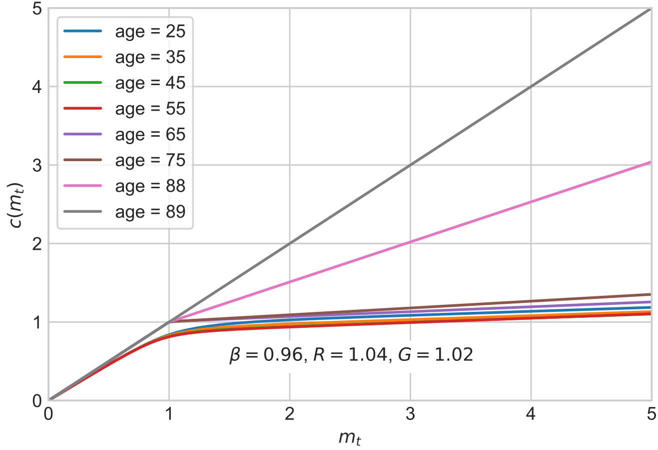
Life-cycle

Adding a life-cycle (normalized)

$$
\begin{aligned}
v_t(m_t, z_t) &= \max_{c_t} \frac{v(z_t)c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t\left[(GL_{t+1}\psi_{t+1})^{1-\rho} v_{t+1}(\bullet)\right] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
m_{t+1} &= \frac{1}{GL_t\psi_{t+1}} Ra_t + \xi_{t+1} \\
\xi_{t+1} &= \begin{cases} \mu & \text{with prob. } \pi \\ (\epsilon_{t+1} - \pi\mu)/(1-\pi) & \text{else} \end{cases} \\
a_t &\geq \lambda_t = \begin{cases} -\lambda & \text{if } t < T_R \\ 0 & \text{if } t \geq T_R \end{cases}
\end{aligned}
$$

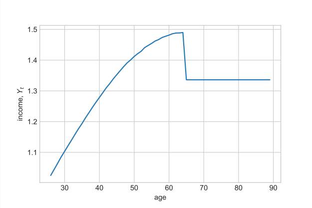
- **Demographics**: z_t (exogenous). What could it be specifically?
- **Income profile:** $P_{t+1} = GL_t P_t \psi_{t+1}$
- **No shocks in retirement:** $\psi_t = \xi_t = 1$ if $t > T_R$
- **Euler equation:** $C_t^{-\rho} = \beta R \mathbb{E}_t[\frac{v(z_{t+1})}{v(z_t)} C_{t+1}^{-\rho}]$

Consumption functions ($v(z_t) = 1$)

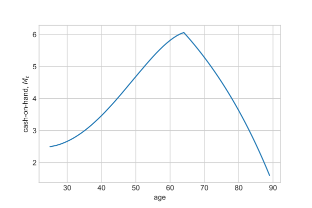


Simulation: LIfe-cycle profiles ($v(z_t) = 1$)

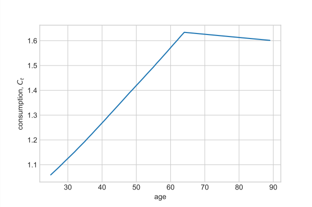
Income, Y_t (implied by G and L_t)



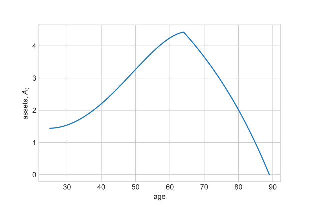
Cash-on-hand, M_t



Consumption, C_t



End-of-period assets, A_t



What is the most unrealistic here?

EGM

Euler-equation

- **Reference:** Carroll (2006)
- Assume for simplicity **no borrowing**: $\lambda = 0$
- All optimal **interior choices** must satisfy

$$\begin{aligned} C_t^{-\rho} &= \beta R \mathbb{E}_t \left[C_{t+1}^{-\rho} \right] \Leftrightarrow \\ c_t^{-\rho} &= \beta R \mathbb{E}_t \left[(G \psi_{t+1} c_{t+1})^{-\rho} \right] \end{aligned}$$

- Else optimal choice is **constrained**

$$\begin{aligned} C_t^{-\rho} &\geq \beta R \mathbb{E}_t \left[C_{t+1}^{-\rho} \right] \Leftrightarrow \\ C_t &= M_t \Leftrightarrow \\ c_t &= m_t \end{aligned}$$

Endogenous grid method: Intuition

- **Obs.:** Given $C^{\star}_{t+1}(M_{t+1}, P_{t+1})$ and A_t and P_t we have

$$
\begin{aligned}
C_t^{-\rho} &= \beta R \mathbb{E}_t \left[\left(C^{\star}_{t+1}(M_{t+1}, P_{t+1}) \right)^{-\rho} \right] \Leftrightarrow \\
C_t &= \mathbb{E}_t \left[\beta R \left(C^{\star}_{t+1}(M_{t+1}, P_{t+1}) \right)^{-\rho} \right]^{-\frac{1}{\rho}} \\
&= \mathbb{E}_t \left[\beta R \left(C^{\star}_{t+1}(RA_t + Y_{t+1}, P_{t+1}) \right)^{-\rho} \right]^{-\frac{1}{\rho}} \\
&= \mathbb{E}_t \left[\beta R \left(C^{\star}_{t+1}(RA_t + P_t \psi_{t+1} \xi_{t+1}, P_t \psi_{t+1}) \right)^{-\rho} \right]^{-\frac{1}{\rho}} \\
&\equiv F(A_t, P_t)
\end{aligned}
$$

- **Endogenous grid:** $A_t = M_t - C_t \Leftrightarrow M_t = C_t + A_t$
- **Conclusion:** $(M_t,\ P_t,\ C_t)$ is a solution to the Bellman equation because it satisfies the Euler equation
- **Perspectives:** Varying A_t (and P_t) we can map out the consumption function without using any numerical solver!
- **Borrowing constraint:** Binding below lowest generated M_t

… in ratio form

- **Prerequisites**:
  1. Next-period **consumption function**: $c^{\star}_{t+1}(m_{t+1})$
  2. **Asset grid:** $\mathcal{G}_a = \{a_1, a_2, \ldots, a_{\#}\}$ with $a_1 = 10^{-6}$

- **Algorithm:** For each $a_i \in \mathcal{G}_a$
  1. Find consumption using Euler equation

$$c_i = \mathbb{E}_t \left[\beta R \left(G\psi_{t+1} c^{\star}_{t+1} \left(\frac{R}{G\psi_{t+1}} a_i + \xi_{t+1} \right) \right)^{-\rho} \right]^{-\frac{1}{\rho}}$$

  2. Find endogenous state: $a_i = m_i - c_i \Leftrightarrow m_i = a_i + c_i$

- The **consumption function**, $c_t(m_t)$, is given by interpolating

$$\{0, c_1, c_2, \ldots, c_{\#}\} \text{ for } \{\underline{a}_t, m_1, m_2, \ldots, m_{\#}\}$$

- *We can find all consumption functions in this way!*

Addendum: The natural borrowing constraint ($\lambda > 0$)

- The **optimal end-of-period asset choice satisfies** the backwards recursion

$$a_t \geq \underline{a}_t = \begin{cases} 0 & \text{if } t \geq T_R \\ -\min\left\{\Lambda_t, \lambda_t\right\} GL_t \underline{\psi} & \text{if } t < T_R \end{cases}$$

where

$$\Lambda_t \equiv \begin{cases} R^{-1} GL_t \underline{\psi}\, \underline{\xi} & \text{if } t = T_R - 1 \\ R^{-1}\left[\min\left\{\Lambda_{t+1}, \lambda_t\right\} + \underline{\xi}\right] GL_t \underline{\psi} & \text{if } t < T - 1 \end{cases}$$

and $\underline{\psi}$ and $\underline{\xi}$ are the minimum realizations of ψ_{t+1} and ξ_{t+1}

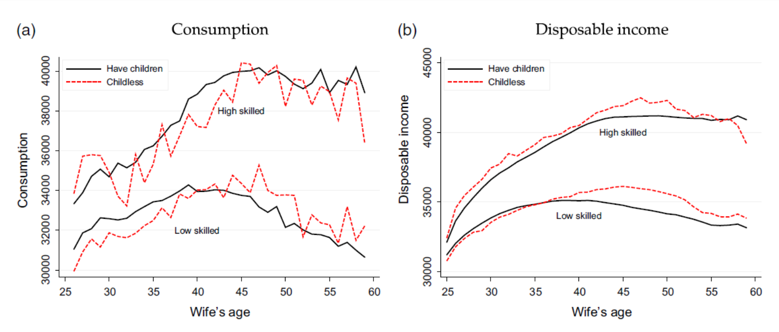
- **Proof:** Can be shown as a consequence of the household wanting to avoid $c_t = 0$ at *any cost* because $\lim_{c_t \to 0} u'(c_t) = \infty$.

Further perspectives

Three generations of models

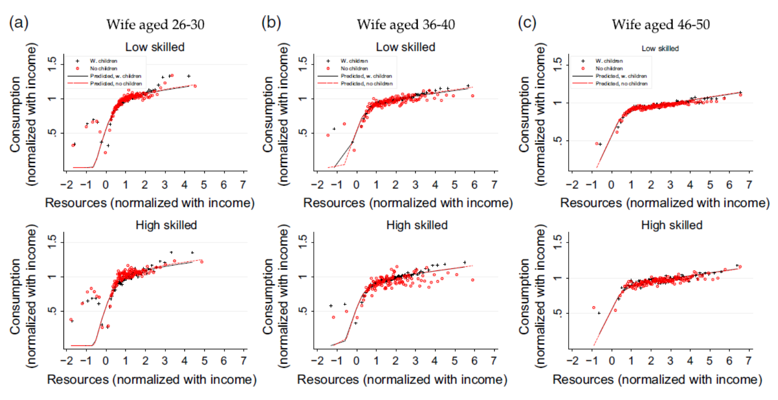
- **1st:** *Permanent income hypothesis* (Friedman, 1957) or *life-cycle model* (Modigliani and Brumburg, 1954)
- **2nd:** *Buffer-stock consumption model* (Deaton, 1991, 1992; Carroll, 1992, 1997, 2020)
- **3nd:** *Multiple-asset buffer-stock consumption models* (e.g. Kaplan and Violante (2014))

Denmark: Life-cycle profiles fit



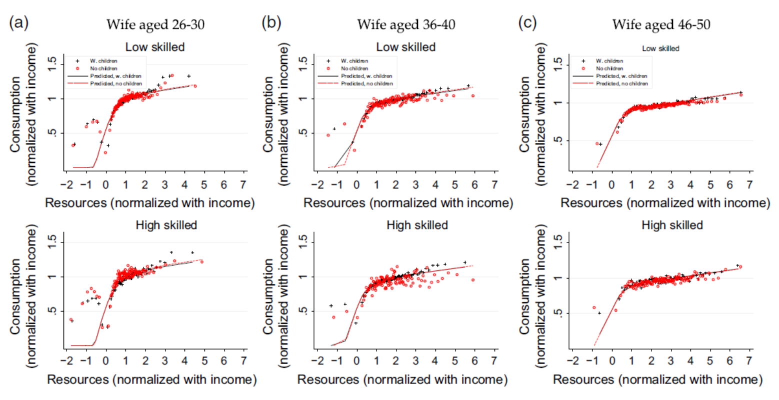
Source: Jørgensen (2017)

Denmark: Consumption function fit



Source: Jørgensen (2017)

Denmark: Consumption function fit

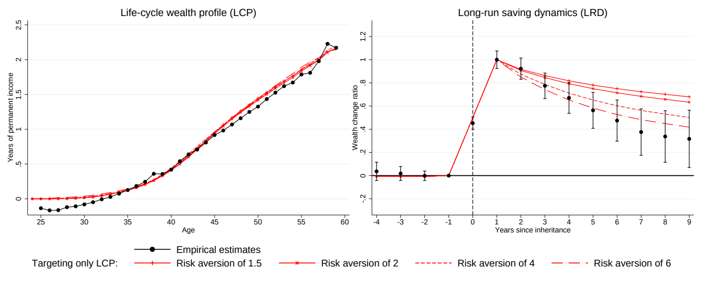


Source: Jørgensen (2017)

Level of wealth and long-run dynamics I

- **Best test of a life-cycle consumption-saving model:**
  A *sudden*, *sizable* and *salient* shock to wealth
  $+$ long panel to observe how the extra wealth is spend
- **My own research:** Druedahl and Martinello (2018)
  Compare individuals in the Danish register data who
  1. Receive a similar inheritance, but at different points in time
  2. From parents dying due to heart attacks or car crashes

Level of wealth and long-run dynamics II



- **Net worth:** Good fit for different levels of risk-aversion (ρ) when re-calibrating patience (β)
- **Also dynamics:** Good fit only if risk-aversion (ρ) is high

MPC out of future shocks

- **Central property of buffer-stock model:**
  1. **Non-constrained:** Increase consumption today when obtaining information of higher income tomorrow
  2. **Constrained:** Increase consumption when cash flow arrives
- **How to test test this?**
- **A paper of mine:** The Intertemporal Marginal Propensity to Consume out of Future Persistent Cash-Flows
  1. **Data:** Account data for all Nykredit customers
  2. **Experiment:** Letter with bank's expectations for interest rate in next mortgage auction 2-3 months ahead
  3. **Result:** Observed behavior can be rationalized in a simple buffer-stock consumption model
  4. **Code:** GitHub

Level of wealth and MPC

- Consumption-saving models a few years ago **could not endogenously fit** both
  1. The level of wealth observed
  2. The high MPCs found in quasi experiments
- **Three solutions:**
  1. Exogenous **hands-too-mouth households** (Campbell and Mankiw, 1990)
  2. **Preference heterogeneity**
  3. **Wealthy hands-to-mouth** (Kaplan and Violante, 2014)
     Many households hold mostly illiquid assets with a high return
     $\rightarrow$ *consumption adjust in response to small income shock*

Kaplan-Violante model (two-asset model)

$$V_t(M_t, N_t, P_t) = \max_{B_t, C_t} u(C_t, B_t) + \beta \mathbb{E}_t[V_{t+1}(M_{t+1}, N_{t+1}, P_{t+1})]$$

$$\begin{aligned} & \text{s.t.} \\ A_t &= M_t - C_t + (N_t - B_t) - 1\{N_t \neq B_t\}\omega \\ M_{t+1} &= R + P_{t+1}\xi_{t+1} \\ N_{t+1} &= R_b B_t \\ P_{t+1} &= P_t \psi_{t+1} \\ A_t &\geq -\lambda P_t. \end{aligned}$$

- **Cost of liquidation:** ω
- **Illiquid assets give higher return:** $R_b > R$ (+ potentially utility)

Kaplan-Violante model (two-asset model)

$$V_t(M_t, N_t, P_t) = \max\left\{v_t^{keep}(M_t, N_t, P_t), v_t^{adj.}(M_t + N_t - \lambda, P_t)\right\}$$

$$\begin{aligned}
v_t^{keep}(M_t, N_t, P_t) &= \max_{C_t} u(C_t, B_t) + \beta W_t(A_t, B_t, P_t) \text{ s.t.} \\
A_t &= M_t - C_t \\
B_t &= N_t \\
A_t &\geq -\omega P_t.
\end{aligned}$$

$$\begin{aligned}
\tilde{v}_t^{adj.}(X_t, P_t) &= \max_{B_t, C_t} u(C_t, B_t) + \beta W_t(A_t, B_t, P_t) \text{ s.t.} \\
M_t &= X_t - B_t \\
A_t &= M_t - C_t \\
A_t &\geq -\omega P_t.
\end{aligned}$$

$$W_t(A_t, B_t, P_t) = \mathbb{E}_t[V_t(RA_t + P_t\psi_{t+1}\xi_{t+1}, R_b B_t, P_t\psi_{t+1})]$$

Frontier topics - curated papers

- **Durable consumption**: Berger and Vavra (2015), Harmenberg and Öberg (2020)
- **Labor supply**, **retirement** and **family formation**: Low et al. (2010), French and Jones (2011), Keane and Wasi (2016), Adda et al. (2016), Blundell et al. (2016)
- **Non-Gaussian income uncertainty**: Guvenen et al. (2019), De Nardi et al. (2020), Druedahl and Munk-Nielsen (2020)
- **Housing**: Landvoigt (2017), Kaplan et al. (2019)
- **Imperfect information and bounded rationality**: Pagel (2017). Carroll et al. (2019), Moran and Kovacs (2019), Druedahl and Jørgensen (2020)
- **Level and dynamics of inequality** – circumstances or behavior? De Nardi and Fella (2017), Hubmer et al. (2020)

Frontier solution methods - curated papers

- **EGM in non-convex multi-dimensional models:** Druedahl and Jørgensen (2017) and Druedahl (2020)
- **Sparse grids:** Judd et al. (2014), Brumm and Scheidegger (2017)
- **Machine learning:** Azinovic et al. (2019), Maliar et al. (2019)

Estimation

Reduced form estimation

- Critic of structural estimation: **Requires many assumptions**
- **But:** To turn reduced form parameter estimates into policy advice *a lot of assumptions are often implicitely required*

  »All econometric work relies heavily on a priori assumptions. The main difference between structural and experimental (or "atheoretic") approaches is not in the number of assumptions but the extent to which they are made explicit.« (Keane, 2012)

- **The beauty of models:**
  1. Ensure *consistent* world view
  2. Allow us to combine *heterogenous facts* and extrapolate from a myriad of past experiences
  3. Better models are clearly defined – even if we never find *the* true model we can make *progress*
- **Frontier:** Combine the two and use exogenous variation to estimate structural model (Nakamura and Steinsson, 2018)

The Lucas critique

- **The Lucas critique:** *Behavioral rules change with policy*
  $\Rightarrow$ policy advice can not rely on estimated behavioral rules
  $\Rightarrow$ we need to estimate *structural parameters*

  »Invariance of parameters in an economic model is not, of course, a property which can be assured in advance, but it seems reasonable to hope that neither tastes nor technology vary systematically with variations in counter-cyclical policies.« (Lucas, 1977)

- **Other stuff might be approximately invariant**
- **Rigourous microfoundations:**
  1. **Mathematically:** Based on (boundedly) rational behavior derived as a solution to a formal optimization problem
  2. **Economically:** The assumptions are realistic

Estimation

1. **Focus:** Closely related estimators ***indirectly*** using **micro-data**

   Simulated Method of Moments (**SMM**) (McFadden, 1989)
   Simulated Minimium Distance (**SMD**) (Duffie and Singleton, 1990)
   Indirect Inference (**II**) (Gouriéroux and Monfort, 1997)

   Main alternative:
   Simulated Maximum Likelihood (**SML**) ***directly*** using **micro-data** (see e.g. Adda and Cooper (2003) or Druedahl et al. (2018))

2. **Examples:** Gourinchas and Parker (2002), Cagetti (2003), Guvenen and Smith (2014), Druedahl and Jørgensen (2020)
3. **Extended toolbox:** Jørgensen (2020) and Honore et al. (2020)

GE

Heterogenous Agent (HA) models

1. **Stationary equilibrium:**
   Deterministic steady state and transition path
   Foundational papers: Bewley (1986), Imrohoroğlu (1989), Huggett (1993), Aiyagari (1994)
   A few policy examples: Aiyagari and McGrattan (1998), Conesa et al. (2009), Heathcote et al. (2014)
2. **Dynamic/recursive/sequential equilibrium:**
   Aggregate shocks and stochastic dynamics
   Foundational papers: Krusell and Smith (1997, 1998), Carroll (2000), Carroll et al. (2015)
3. **Reviews:** Heathcote et al. (2009), Krusell and Smith (2006), Krueger et al. (2016)

Heterogenous Agent New Keynesian (HANK) models

1. **Frontier:** Kaplan et al. (2018), Bayer et al. (2019), Hagedorn et al. (2019b), Alves et al. (2020), Auclert et al. (2020c), Fernandez-Villaverde et al. (2020)
2. **Analytical:** Bilbiie (2008, 2019a,b), Werning (2015), Challe et al. (2017), Acharya and Dogra (2020), Bilbiie et al. (2020), Debortoli and Galí (2018), Auclert et al. (2018), Broer et al. (2020), Ravn and Sterk (2020), Auclert and Rognlie (2020)
3. **Others:** Oh and Reis (2012), Gornemann et al. (2016), McKay and Reis (2016), McKay et al. (2016), Guerrieri and Lorenzoni (2017), Ravn and Sterk (2017), Den Haan et al. (2018), Luetticke (2020)
4. **Empirical:** Cloyne et al. (2020), Slacalek et al. (2020), Holm and Paul (2020), Wolf (2020)
5. **Reviews:** Kaplan and Violante (2018)

Computational methods

- **Early reviews:** Den Haan et al. (2010), Schmedders and Judd (2013)
- **Continuous time:** Achdou et al. (2020) (code), Ahn et al. (2018) (code)
- **Local aggregate solution:**
  1. *State space:* Bayer and Luetticke (2020) (MATLAB, Python)
  2. *Sequence space:* Boppart et al. (2018), Auclert et al. (2020c) (code)
- **Global aggregate solution:** Kubler and Scheidegger (2018), Azinovic et al. (2019), Scheidegger and Bilionis (2019), Pröhl (2019) (code), Maliar et al. (2019) (code, video), Fernandez-Villaverde et al. (2020) (code)

GE

Stationary equilibrium

The Aiyagari model

- **Households:** Continuum of measure 1 who
  1. Own stocks, a_{t-1} (measured end-of-period)
  2. Supply labor with productivity e_t (exogenous and stochastic, mean one)
  3. Consume, c_t
- **Firms:** Rent capital and hire labor to produce
- **Capital:**
  1. Predetermined: $Y_t = F(Z_t, K_{t-1}, L_t)$, where Z_t is technology, K_{t-1} is capital, and L_t is labor
  2. Depreciates with rate δ
- **Prices** are taken as given by households and firms
  1. r_t^k, rental rate
  2. $r_t = r_t^k - \delta$, interest rate
  3. w_t, wage rate

Firms

- **Production function:** $Y_t = Z_t K_{t-1}^{\alpha} L_t^{1-\alpha}$
- **Define** $k_{t-1} \equiv K_{t-1}/L_t$
- Standard **pricing equations:**

$$
\begin{aligned}
r_t^k &= \alpha Z_t k_{t-1}^{\alpha-1} \\
w_t &= (1-\alpha) Z_t k_{t-1}^{\alpha}
\end{aligned}
$$

- Useful **implications:**

$$
\begin{aligned}
k_{t-1} &= \left(\frac{r_t + \delta}{\alpha Z_t}\right)^{\frac{1}{\alpha-1}} \equiv k(r_t, Z_t) \\
r_t &= \alpha Z_t k_{t-1}^{\alpha-1} \equiv r(k_{t-1}, Z_t) \\
w_t &= (1-\alpha) Z_t \left(\frac{r_t + \delta}{\alpha Z_t}\right)^{\frac{\alpha}{\alpha-1}} \equiv w(r_t, Z_t)
\end{aligned}
$$

Households

- **Perfect foresight:** Price sequence known, $\{r_t, w_t\}_{t\geq 0}$
- **Households solve:**

$$\begin{aligned} v_t(e_t, a_{t-1}) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{t+1}(e_{t+1}, a_t) \right] \\ \text{s.t.} & \\ a_t + c_t &= (1+r_t)a_{t-1} + w_t e_t \\ a_t &\geq 0 \end{aligned}$$

- **Alternatively:** $v_t(e_t, a_{t-1}) = \mathcal{V}(e_t, a_{t-1}, \{r_k, w_k\}_{k\geq t})$, where $\mathcal{V}$ have no time subscript, but the price sequences are state variables.
- **FOC:** $c_t^{-\sigma} = \beta \mathbb{E}_t[v_{a,t+1}]$
- **Envelope:** $v_{a,t} = (1+r_t)c_t^{-\sigma}$
- **Optimal saving and consumption:** $a_t^*(e_t, a_{t-1})$ and $c_t^*(e_t, a_{t-1})$

Supply of capital

- **Distribution:** D_t over e_t and a_{t-1}
- **Supply of capital:** $\mathcal{K}_t = \int a_t^*(e_t, a_{t-1}) dD_t = \int a_t dD_{t+1}$
- **Details:**
  - **Formulation I:** $\int a_t^*(e_t, a_{t-1}) dD_t$ is an integral over e_t and a_{t-1} applying the optimal saving function in period t, i.e. $a_t^*(e_t, a_{t-1})$, thus summing up savings at the end-of-period t
  - **Formulation II:** $\int a_t dD_{t+1}$ is an integral over e_{t+1} and a_t directly summing up savings at the end-of-period t
  - **Equivalence:** The two formulations gives the same result because D_{t+1} is generated from D_t assuming saving according to $a_t^*(e_t, a_{t-1})$ (and the exogenous process for e_t)

Market clearing

- **Market clearing** requires

$$\text{Capital: } K_t = \mathcal{K}_t = \int a_t dD_{t+1} = \int a_t^*(e_t, a_{t-1}) dD_t$$

$$\text{Labour: } L_t = \int e_t dD_t = 1$$

$$\text{Goods: } Y_t = \int c_t^*(e_t, a_{t-1}) dD_t + \delta K_{t-1}$$

- The **labor market clears trivially**, while we can leave out the **goods market** due to **Walras's Law**

Solve household problem by EGM

- **Grids:**
  1. $e_t \in \{e^0, \dots, e^{\#_e - 1}\}$ (discretized with Tauchen and Hussey (1991))
  2. $a_t \in \{a^0, \dots, a^{\#_a - 1}\}$
- **Guess:** $v_{a,t+1}(e^i, a^j)$, $\forall i, j$
- **Time iteration:**
  1. Calculate: $q_t(e^i, a^j) = \sum_{k=0}^{\#_e - 1} \Pr[e^k | e^i] v_{a,t+1}(e^i, a^j)$
  2. Calculate $\tilde{c}^{ij} = q_t(e^i, a^j)^{-\frac{1}{\sigma}}$ and $\tilde{m}^{ij} = \tilde{c}^{ij} + a^j$ (*use FOC*)
  3. Interpolate $\{\tilde{m}^{ij}, a^j\}_{j=0}^{\#_a - 1}$ at $m^j = (1 + r_t)a^j + w_t e^i$ to find $a_t^*(e^i, a^j)$
  4. Calculate $c^*(e^i, a^j) = m_t - a_t^*(e^i, a^j)$
  5. Calculate $v_{a,t}(e^i, a^j) = (1 + r_t) c_t^*(e^i, a^j)^{-\sigma}$
     (*use envelope theorem*)
- **Note:** Any other *solution* method could have been used.

Simulate household behavior on grid

- **Initial distribution:** $D_0(e^i, a^j) = \frac{\Pr[e^i]}{\#_a}$ (*ergodic* in e, *uniform* in a)
- **Idea:** Re-distribute mass to grid points based on optimal decisions
- **Update:** Calculate $D_{t+1}(e^k, a^l)$ as

$$\sum_{i=0}^{\#_e-1} \Pr[e^k|e^i] \sum_{j=0}^{\#_a-1} D_t(e^i, a^j)\omega(a_t^*(e^i, a^j), a^{\max\{l-1,0\}}, a^l, a^{\min\{l+1,\#_a-1\}})$$

where ω is a weight calculated using linear interpolation

$$\omega(a, \underline{a}, \tilde{a}, \overline{a}) = 1\{a \in [\underline{a}, \overline{a}]\} \begin{cases} \frac{\overline{a}-a}{\overline{a}-\tilde{a}} & \text{if } a \geq \tilde{a} \\ \frac{a-\underline{a}}{\tilde{a}-\underline{a}} & \text{if } a < \tilde{a} \end{cases}$$

- **Note:** Any other *simulation* method could have been used.

Definition: Stationary equilibrium

A **stationary equilibrium** for a given Z_{ss} is one where

1. Quantities K_{ss} and L_{ss},
2. prices r_{ss} and w_{ss},
3. a distribution D_{ss} over e_t and a_{t-1}
4. and policy functions $a^*_{ss}(e_t, a_{t-1})$ and $c^*_{ss}(e_t, a_{t-1})$

are such that

1. $a^*_{ss}(\bullet)$ and $c^*_{ss}(\bullet)$ solves the household problem with $\{r_{ss}, w_{ss}\}_{k \geq t}$
2. D_{ss} is the invariant distribution implied by the household problem
3. Firms maximize profits, $r_{ss} = r(K_{ss}/L_{ss}, Z_{ss})$ and $w_{ss} = w(r_{ss}, Z_{ss})$
4. The labor market clears, i.e. $L_{ss} = \int e_t dD_{ss} = 1$
5. The capital market clears, i.e. $K_{ss} = \int a^*_{ss}(e_t, a_{t-1}) dD_{ss}$
6. The goods market clears, i.e. $Y_{ss} = \int c^*_{ss}(e_t, a_{t-1}) dD_{ss} + \delta K_{ss}$

Finding stationary equilibrium

1. Guess on r_{ss}
2. Calculate $w_{ss} = w(r_{ss}, Z_{ss})$
3. Solve the infinite horizon household problem
4. Simulate until convergence of D_{ss}
5. Calculate supply $\mathcal{K}_{ss} = \int a^*_{ss}(e_t, a_{t-1}) dD_{ss}$
6. Calculate demand $K_{ss} = k(r_{ss}, Z_{ss}) L_{ss}$
7. If for some tolerance ϵ

$$|\mathcal{K}_{ss} - K_{ss}| < \epsilon$$

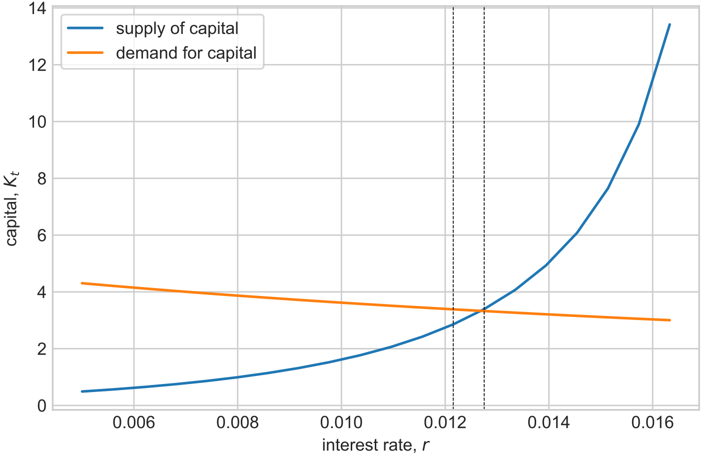
   then stop, otherwise update r_{ss} appropriately and return to step 2

This is just a **root-finding problem**

Equilibrium interest rate

Step 1: Perform grid search

Step 2: Use standard root finder



Precautionary saving and the interest rate

- **Baseline:** $\sigma_e = 0.1$
  1. Interest rate: $r^* = 0.0127$
  2. Capital-output ratio: 2.92
- **Higher risk:** $\sigma_e = 0.2$
  1. Interest rate: $r^* = 0.0029$
  2. Capital-output ratio: 3.95
- **Intuition:** Saving motive $\uparrow$, marginal product of capital $\downarrow$
- **Implication:** Important for the »natural« interest rate!
  Example: On Secular Stagnation in the Industrialized World
  (by Lukasz Racehl and Lawrance Summers)

GE

Transition path

Definition: Transition path (to MIT shock)

A **transition path** for $t \in \{0, 1, 2, \dots\}$, given an initial distribution D_0 and a path of Z_t, is paths of quantities K_t and L_t, prices r_t and w_t, policy functions $a_t^*(\bullet)$ and $c_t^*(\bullet)$, distributions D_t, such that for all t

1. $a_t^*(\bullet)$ and $c_t^*(\bullet)$ solve the household problem given price paths
2. D_t are implied by the household problem given price paths and D_0
3. Firms maximizes profit, $r_t = r(K_{t-1}/L_t, Z_t)$ and $w_t = w(r_t, Z_t)$
4. The labor market clears, i.e. $L_t = \int e_t dD_t = 1$
5. The capital market clears, i.e. $K_{t-1} = \int a_{t-1} dD_t$
6. The goods market clears, i.e. $Y_t = \int c_t^*(\bullet) dD_t + \delta K_{t-1}$

MIT shock $\equiv$ »Shock in a world without shocks«

Solving the house problem along the transition path

1. **Assumption:** Back at stationary equilibrium after $\mathcal{T}$ periods
2. **Consequence:** $v_{\mathcal{T}}(e_t, a_{t-1}) = v_{ss}(e_t, a_{t-1})$
3. **Perfect foresight price paths:**

$$\{r_k, w_k\}_{k \geq \mathcal{T}-1} = \{r_{\mathcal{T}-1}, w_{\mathcal{T}-1}, r_{ss}, w_{ss} \dots\}$$
$$\{r_k, w_k\}_{k \geq \mathcal{T}-2} = \{r_{\mathcal{T}-2}, w_{\mathcal{T}-2}, r_{\mathcal{T}-1}, w_{\mathcal{T}-1}, r_{ss}, w_{ss} \dots\}$$
$$\vdots$$
$$\{r_k, w_k\}_{k \geq 0} = \{r_0, w_0, \dots, r_{\mathcal{T}-1}, w_{\mathcal{T}-1}, r_{ss}, w_{ss} \dots\}$$

Relaxation-method

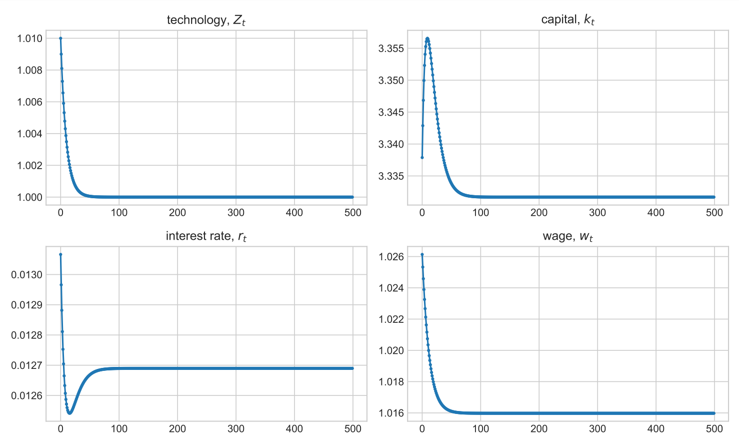
1. Guess on $\{r_t\}_{t=0}^{\mathcal{T}-1} = \{r_{ss}\}_{t=0}^{\mathcal{T}-1}$ (or something else)
2. Calculate $\{w_t\}_{t=0}^{\mathcal{T}-1} = \{w(r_t, Z_t)\}_{t=0}^{\mathcal{T}-1}$
3. Solve the household problem backwards along the transition path
4. Simulate households forward along the transition path
5. Calculate $\{k_{t-1}\}_{t=0}^{\mathcal{T}-1} = \{\int a_{t-1} dD_t\}_{t=0}^{\mathcal{T}-1}$
6. Calculate $\{r'_t\}_{t=0}^{\mathcal{T}} = \{r(k_{t-1}, Z_t)\}_{t=0}^{\mathcal{T}-1}$
7. Stop if for some tolerance ϵ

$$\max_{t \in \{0,1,2,\dots,\mathcal{T}\}} |r_t - r'_t| < \epsilon$$

otherwise return to step 2 with $\{r_t\}_{t=0}^{\mathcal{T}} = \{\nu r_t + (1-\nu) r'_t\}_{t=0}^{\mathcal{T}}$

Note: Typically the relaxation parameter is $\nu = 0.90$ (Kirkby, 2017)

Transition path



Remember: We also have a transition path for D_t

$\rightarrow$ we can calculate distribution of utility effects!

Sequence space method

- **Reformulation**: K_t given by initial distribution and price sequence

$$
\begin{aligned}
K_t &= \mathcal{K}_t(\{r_s, w_s\}_{s\geq 0}, D_0)) \\
&= \int a_t^*(e_t, a_{t-1}) dD_t
\end{aligned}
$$

- **Define:** $\boldsymbol{K} = (K_0, K_1, \dots)$ and $\boldsymbol{Z} = (Z_0, Z_1, \dots)$
- **Transition path** is, for given $\boldsymbol{Z}$, solution for $t = 0, 1, \dots,$ to

$$
H_t(\boldsymbol{K}, \boldsymbol{Z}, D_0) \equiv \mathcal{K}_t(\{r(Z_s, K_{s-1}), w(Z_s, K_{s-1})\}_{s\geq 0}, D_0) - K_t = 0
$$

or in **time-stacked form:**

$$
\boldsymbol{H}(\boldsymbol{K}, \boldsymbol{Z}, D_0) = \boldsymbol{0}
$$

- **In practice often:** $D_0 = D_{ss}$

Time-stacked form

Total differentiation implies

$$\boldsymbol{H_K} d\boldsymbol{K} + \boldsymbol{H_Z} d\boldsymbol{Z} = 0 \Leftrightarrow d\boldsymbol{K} = -\boldsymbol{H_K}^{-1}\boldsymbol{H_Z} d\boldsymbol{Z}$$

where

$$\boldsymbol{H_K} = \begin{bmatrix} \frac{\partial H_0}{\partial K_0} & \frac{\partial H_0}{\partial K_1} & \cdots \\ \frac{\partial H_1}{\partial K_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}, \boldsymbol{H_Z} = \begin{bmatrix} \frac{\partial H_0}{\partial Z_0} & \frac{\partial H_0}{\partial Z_1} & \cdots \\ \frac{\partial H_1}{\partial Z_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}$$

Decomposition: $H_t = \mathcal{K}_t(\{r(Z_s, K_{s-1}), w(Z_s, K_{s-1})\}_{s\geq 0}, D_0) - K_t$

$$\begin{aligned} \boldsymbol{H_K} &= \mathcal{J}^{\mathcal{K},r}\mathcal{J}^{r,K} + \mathcal{J}^{\mathcal{K},w}\mathcal{J}^{w,K} - \boldsymbol{I} \\ \boldsymbol{H}_Z &= \mathcal{J}^{\mathcal{K},r}\mathcal{J}^{r,Z} + \mathcal{J}^{\mathcal{K},w}\mathcal{J}^{w,Z} \end{aligned}$$

where generically

$$\mathcal{J}^{x,y} = \begin{bmatrix} \frac{\partial x_0}{\partial y_0} & \frac{\partial x_0}{\partial y_1} & \cdots \\ \frac{\partial x_1}{\partial y_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}$$

Calculating Jacobians at steady state

- **Analytical:** Subscript $[t, s]$ denotes the t'th row and s'th column

  Immediate impact of Z_t:

$$\mathcal{J}^{r,Z}_{[t,s]} = \begin{cases} (1-\alpha) Z_{ss} K_{ss}^{\alpha-1} & \text{if } t = s \\ 0 & \text{else} \end{cases}, \mathcal{J}^{w,Z}_{[t,s]} = \begin{cases} (1-\alpha) Z_{ss} K_{ss}^{\alpha} & \text{if } t = s \\ 0 & \text{else} \end{cases}$$

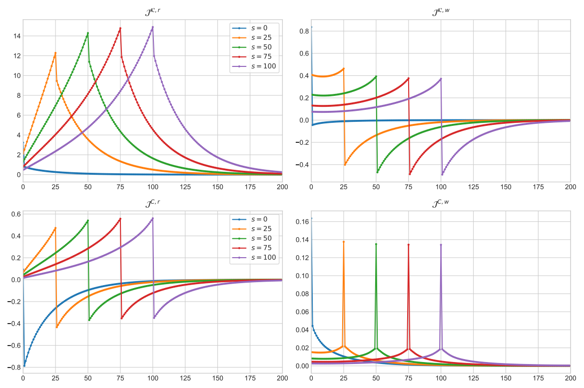
  Lagged impact of K_t:

$$\mathcal{J}^{r,K}_{[t,s]} = \begin{cases} \alpha(\alpha-1) Z_{ss} K_{ss}^{\alpha-2} & \text{if } t = s+1 \\ 0 & \text{else} \end{cases}, \mathcal{J}^{w,K}_{[t,s]} = \begin{cases} \alpha(1-\alpha) Z_{ss} K_{ss}^{\alpha-1} & \text{if } t = s+1 \\ 0 & \text{else} \end{cases}$$

- **Numerical:** Find $\mathcal{J}^{\mathcal{K},r}$ and $\mathcal{J}^{\mathcal{K},w}$ by solving backwards and simulating forwards with r or w changed from steady state value in single period.
- **Truncation:** The matrices are truncated such that they are $\mathcal{T} \times \mathcal{T}$

Interpreting Jacobians

1. **Along a row:** The effect on a variable in a given period of a shock in an arbitrary period
2. **Along a column:** The effect of a shock in a given period on a variable in an arbitrary period



Transition path after MIT shock

1. Assume we **start in the stationary equilibrium**
2. **MIT-shock:** Known path of future $\boldsymbol{Z}$, i.e. in terms of changes from steady state $d\boldsymbol{Z} = \boldsymbol{Z} - Z_{ss}$
3. **Question:** What happens to aggregate capital?
4. **Answer:** To a first order we have

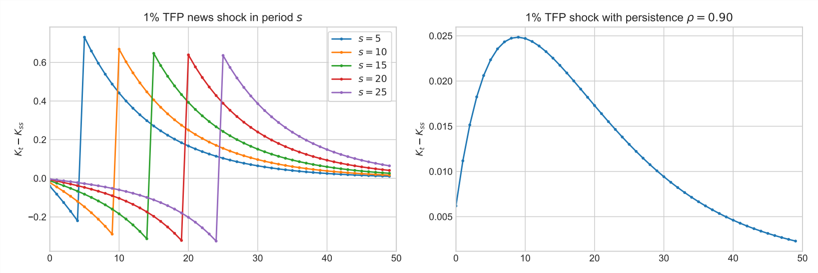
$$\boldsymbol{G}^{K,Z} \equiv \frac{d\boldsymbol{K}}{d\boldsymbol{Z}} = -\boldsymbol{H}_{\boldsymbol{K}}^{-1}\boldsymbol{H}_{\boldsymbol{Z}}$$

   where all derivatives are **evaluated at the stationary equilibrium**
   Additionally:

$$\boldsymbol{G}^{r,Z} \equiv \frac{d\boldsymbol{r}}{d\boldsymbol{Z}} = \mathcal{J}^{r,Z} + \mathcal{J}^{r,K}\boldsymbol{G}^{K,Z}, \;\; \boldsymbol{G}^{w,Z}\frac{d\boldsymbol{w}}{d\boldsymbol{Z}} = \mathcal{J}^{w,Z} + \mathcal{J}^{w,K}\boldsymbol{G}^{K,Z}$$

$$\boldsymbol{G}^{C,Z} \equiv \frac{d\boldsymbol{C}}{d\boldsymbol{Z}} = \mathcal{J}^{C,r}\boldsymbol{G}^{r,Z} + \mathcal{J}^{C,w}\boldsymbol{G}^{w,Z}, \;\; \boldsymbol{G}^{Y,Z} \equiv \frac{d\boldsymbol{Y}}{d\boldsymbol{Z}} = \mathcal{J}^{Y,Z} + \mathcal{J}^{Y,K}\boldsymbol{G}^{K,Z}$$

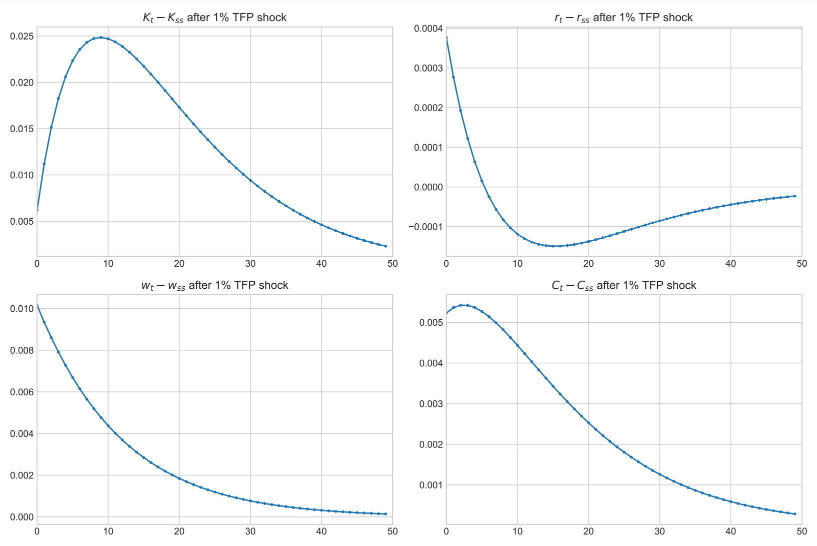
News shock vs. persistent shock



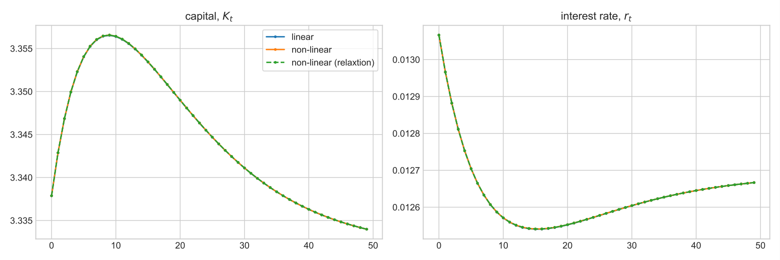
Persistent shock: $Z_t = (1 - \rho_Z) Z_{ss} + \rho Z_t$ and $Z_0 = (1 + \sigma_Z) Z_{ss}$

Note: One set of Jacobians, infinitely many impulse-respones!

Impulse-responses



- **Beyond first order approximation?** *Solve full non-linear equation system using the Jacobian at the stationary equilibrium as an approximation of the true Jacobians along the transition path*
- **Comparison:**



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Fast Jacobians

Problem formulation

- **Problem:** Calculating the Jacobians $\mathcal{J}^{\mathcal{K},r}$ and $\mathcal{J}^{\mathcal{K},w}$ using direct numerical derivatives is very costly
- **Central result in Auclert et. al. (2020):** A much simpler algorithm can be constructed. See their paper for the proof and some intuition.

Reformulation with linear interpolation

1. **Productivity:** e_t, indexed by i, lives on $\mathcal{G}_e = \{e^0, e^1, \dots, e^{\#_e - 1}\}$ with transition matrix Π^e with elements
$$\pi^e_{[i,i+]} = \Pr[e_{t+1} = e^{i+1} | e_t = e^i]$$
2. **Assets:** a_t, indexed by j, lives on $\mathcal{G}_a = \{a^0, a^1, \dots, a^{\#_a - 1}\}$.
3. **Value and policy functions:** $\boldsymbol{v}$, $\boldsymbol{a}^*$ and $\boldsymbol{c}^*$ lives on $\mathcal{G}_e \times \mathcal{G}_a$ with
$$\boldsymbol{v}_{[i,j]} = u(\boldsymbol{c}^*_{[i,j]}) + \sum_{j_+=0}^{\#_a-1} \boldsymbol{Q}^i_{[j,j_+]} \beta \sum_{k=0}^{\#_e-1} \pi^e_{[i,i_+]} v_{[i_+ j_+]}$$
where $\boldsymbol{c}^*_{[i,j]} = c^*(e_i, a_j)$ and $\boldsymbol{Q}^i_{[j,k]}$ are the weights implied by linear interpolation of $a^*(e_t, a_{t-1})$ at $\boldsymbol{a}^*_{[i,j]} = a^*(e_i, a_j)$ given by
$$\boldsymbol{Q}^i_{[j,k]} = \begin{cases} \frac{a^*_{ij} - a^{j_+ - 1}}{a^{j_+} - a^{j_+ - 1}} & \text{if } j_+ > 0, \text{and } a^*_{ij} \in [a^{j_+ - 1}, a^{j_+}] \\ \frac{a^*_{ij} - a^{j_+}}{a^{j_+ + 1} - a^{j_+}} & \text{if } j_+ < \#_a - 1, \text{and } a^*_{ij} \in [a^{j_+}, a^{j_+ + 1}] \\ 0 & \text{else} \end{cases}$$

Reformulation in matrix form

- **Definition:** $\overrightarrow{x}$ is the row-stacked version of the matrix x
- **Bellman equation** can be written

$$\overrightarrow{\boldsymbol{v}}_{\boldsymbol{t}} = u(\overrightarrow{\boldsymbol{c}}^*_t) + \beta \boldsymbol{Q}_t \tilde{\Pi}^e \overrightarrow{\boldsymbol{v}}_{t+1}$$

where $\tilde{\Pi} = \Pi \otimes \boldsymbol{I}_{\#_a \times \#_a}$ and Q_t is the policy matrix given by

$$\boldsymbol{Q}_t = \begin{bmatrix} \boldsymbol{Q}_t^0 & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \ddots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{Q}_t^{\#_e - 1} \end{bmatrix}, \boldsymbol{Q}_t^i = \begin{bmatrix} \ddots & \vdots & ⋰ \\ \cdots & q^i_{[j,j_+]} & \cdots \\ ⋰ & \vdots & \ddots \end{bmatrix}$$

- **Simulation** is now the inverse operation:

$$\overrightarrow{D}_{t+1} = \tilde{\Pi}^{e\prime} \boldsymbol{Q}_t' \overrightarrow{D}_t$$

where $\prime$ denoted transpose

Numerically: The sparsity of $\boldsymbol{Q}_t$ should be used

Important result in Auclert et. al. (2020)

Step 1: Solve backwards $T-1$ periods from a shock Δ_x to price x.

$\boldsymbol{a}_s^{*,x}$ is the optimal saving policy with s periods until shock arrival

$\boldsymbol{Q}_s^x$ is the associated policy matrix

Step 2: *Numerical derivatives,*

$$\Delta_{D,x}^s = \frac{\tilde{\Pi}^{e\prime} Q_s^{x\prime} \overrightarrow{D}_{ss} - \overrightarrow{D}_{ss}}{\Delta_x}, \Delta_{a,x}^s = \frac{\overrightarrow{\boldsymbol{a}}_s^{*,x\prime} \overrightarrow{D}_{ss} - \overrightarrow{\boldsymbol{a}_{ss}}^{*,\prime} \overrightarrow{D}_{ss}}{\Delta_x}$$

Step 3: *Expectation factors,* $\mathcal{E}_t = \begin{cases} \boldsymbol{a}_{ss}^* & \text{if } t=0 \\ \boldsymbol{Q}_{ss} \tilde{\Pi}^e \mathcal{E}_t & \text{else} \end{cases}$

Step 4: *Fake news matrix,* $\mathcal{F}_{[t,s]}^{\mathcal{K}} = \begin{cases} \Delta_{a,x}^s & \text{if } t=0 \\ \overrightarrow{\mathcal{E}_t} \Delta_{D,x}^s & \text{else} \end{cases}$

Step 5: *Jacobian* $\mathcal{J}_{[t,s]}^{\mathcal{K},x} = \begin{cases} \mathcal{F}_{[t,s]} & \text{if } t=0 \vee s=0 \\ \sum_{k=0}^{\min\{t,s\}} \mathcal{F}_{[t-k,s-k]} & \text{else} \end{cases}$

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Dynamic equilibrium

Household problem with aggregate shocks

- **Aggregate shocks:** Assume Z_t is a stochastic process
- **Root problem:** There is no longer perfect foresight wrt. r_t and w_t
- **Extended problem:**

$$
\begin{aligned}
v(e_t, a_{t-1}, Z_t, D_t) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v(e_{t+1}, a_t, Z_{t+1}, D_{t+1}) \right] \\
&\text{s.t.} \\
a_t + c_t &= (1+r_t)a_{t-1} + w_t e_t \\
k_{t-1} &= \int a_{t-1} dD_t \\
r_t &= r(k_{t-1}, Z_t) \\
w_t &= w(r_t, Z_t) \\
a_t &\geq 0
\end{aligned}
$$

- **Ultimate problem:** D_t is not easy to discretize...

Approximate household problem

- **Krusell-Smith idea:** Approximate D_t with some selected moments, e.g. just the mean
- **Approximate problem:**

$$
\begin{aligned}
v(e_t, a_{t-1}, Z_t, k_{t-1}) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v(e_{t+1}, a_t, Z_{t+1}, k_t) \right] \\
& \text{s.t.} \\
a_t + c_t &= (1+r_t)a_{t-1} + w_t e_t \\
r_t &= r(k_{t-1}, Z_t) \\
w_t &= w(r_t, Z_t) \\
k_t &= \text{PLM}(k_{t-1}, Z_t) \\
a_t &\geq 0
\end{aligned}
$$

where $\text{PLM}(k_{t-1}, Z_t)$ is the **perceived law of motion**
For example: $\text{PLM}(k_{t-1}, Z_t) = \beta_{Z_t} + \alpha_{Z_t} \log k_{t-1}$

Definition: Dynamic equilibrium

An **(approximate) dynamic equilibrium** is a PLM, policy functions $a^*(\bullet)$ and $c^*(\bullet)$, and paths of quantities K_t and L_t, prices r_t and w_t, distributions D_t such that for all t

1. $a^*(\bullet)$ and $c^*(\bullet)$ solve the household problem given the PLM
2. D_t is implied by the household problem
3. Firms profit maximize $r_t = r(K_{t-1}/L_t, Z_t)$ and $w_t = w(r_t, Z_t)$
4. The labor market clears, i.e. $L_t = \int e_t dD_t = 1$
5. The capital market clears, i.e. $K_t = \int a^*(e_t, a_{t-1}) dD_t$
6. The goods market clears, i.e. $Y_t = \int c^*(e_t, a_{t-1}) dD_t + \delta K_{t-1}$
7. PLM(k_{t-1}, Z_t) does not imply systematic expectations errors

Note: When $Z_t = Z_{ss} \forall Z_t$ the dynamic equilibrium does *not* converge to the stationary equilibrium unless the households know Z_t is actually not stochastic.

Finding dynamic equilibrium

1. Guess on the $\mathrm{PLM}(k_{t-1}, Z_t)$
2. Solve the household problem
3. Simulate a path of Z_t and D_t and thus k_t
4. Compare simulated behavior with the $\mathrm{PLM}(k_{t-1}, Z_t)$
   Stop if »good enough«
   otherwise update $\mathrm{PLM}(k_{t-1}, Z_t)$ and return to step 2

Terminology:

1. The Krusell-Smith method is a **global solution method**
2. The newest **local solution methods** rely on linearization of the aggregate dynamics, but solve for the full non-linear stationary equilibrium

Connection to sequence space

- **Insights in** Auclert et al. (2020b)**:**
  1. In the limit where the shock variance disappears the dynamic equilibrium path converge to the stationary equilibrium.
  2. In the limit where the shock variance disappears the transition path to the MIT shock around the stationary equilibrium is the same as the impulse-response in the dynamic equilibrium.
- **Implications:** From the transition paths first order approximations of variances and co-variances of all variables can be calculated.
- **Estimation:** Parameters affecting
  1. The stationary equilibrium are computationally *very costly*
  2. Only the Jacobians are computationally *rather costly*
  3. Only affecting e.g. the shock process are computationally *cheap*

Summary

Summary

- **Dynamic programming** is needed to solve **empirically realistic consumption-saving models**
- The **buffer-stock consumption model**, and it's two asset cousin, can fit central stylized facts
    1. High MPC
    2. Responses to expected windfalls
    3. Households with more volatile income save more
    4. Consumption tracks income over the life-cycle
- Advances in micro-data, numerical methods and computational power are leading to **new discoveries**
- **EGM is a powerful solution method** (and can be generalized)
- Realistic consumption-saving behavior can be included in **general equilibrium models** $\rightarrow$ welfare analysis with full distributional effects

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