CENTER FOR
ECONOMIC
BEHAVIOR &
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Consumption-Saving Models

An Introduction to Dynamic Programming

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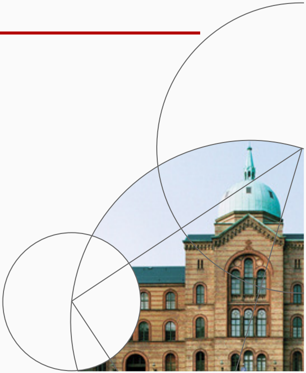


Introduction

Introduction

- **Why are consumption-saving models important?**
  1. Important topic in itself (70 percent of GDP)
  2. **Central aspect of many other decisions**
     a) Labor supply, retirement, family and choices
     b) Portfolio choices and asset pricing
     c) Housing and location choices
  3. Households are the **cornerstone of general equilibrium models**
- **Dynamic programming** essential for recent advances
  1. Idiosyncratic and aggregate uncertainty
  2. Ex ante and ex post heterogeneity
  3. Internal and external optimization frictions
     (bounded rationality, adjustment costs etc.)

Introduction

- **Part of mini-course on dynamic programming:**
  ConsumptionSavingNotebooks/DynamicProgramming
  before: general introduction to dynamic programming
  after: estimation + general equilibrium

- **Focus in most of these slides:**
  Carroll (2020, QE), *Theoretical foundations of buffer stock saving*
  (partial equilibrium, PE)

General references

- **Dynamic programming and computational methods in general:** Stokey and Lucas (1989), Judd (1998), Adda and Cooper (2003), Ljungqvist and Sargent (2004), Puterman (2009), Powell (2011), Bertsekas (2012), Schmedders and Judd (2013)
- **Surveys of consumption-saving litteratures:** Browning and Lusardi (1996), Browning and Crossley (2001), Heathcote et al. (2009), Krusell and Smith (2006), Krueger et al. (2016), Pistaferri (2017), Kaplan and Violante (2018)
- **End-of-slides:** Many more references

Plan

1. Introduction
2. PIH
3. Buffer-stock
4. Details
5. Life-cycle
6. EGM
7. Further perspectives
8. Estimation
9. GE
10. Summary

PIH

- **Household problem**

$$
\begin{aligned}
V_0(M_0, P_0) &= \max_{\{C_t\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t \frac{C_t^{1-\rho}}{1-\rho}, \;\; \beta < 1,\, \rho \geq 1 \\
&\text{s.t.} \\
A_t &= M_t - C_t \\
B_{t+1} &= R \cdot A_t, \;\; R > 0 \\
M_{t+1} &= B_{t+1} + P_{t+1} \\
P_{t+1} &= G \cdot P_t, \;\; G > 0 \\
A_T &\geq 0
\end{aligned}
$$

- **Well-defined analytical solution, also for $T \to \infty$ if**
  1. Return impatience (RI): $(\beta R)^{1/\rho}/R < 1$
  2. Finite human wealth (FHW): $G/R < 1$
- **What do you think is missing?**

The Intertemporal Budget Constraint (IBC)

- **Substitution** implies

$$\begin{aligned} A_T &= M_T - C_T = (RA_{T-1} + P_T) - C_T \\ &= R(M_{T-1} - C_{T-1}) + P_T - C_T \\ &= R^2 A_{T-2} + RP_{T-1} - RC_{T-1} + P_T - C_T \\ &= R^{T+1} A_{-1} + \sum_{t=0}^{T} R^{T-t}(P_t - C_t) \end{aligned}$$

- Use **terminal condition** (why equality?)

$$\begin{aligned} A_T = 0 \Leftrightarrow R^{-T} A_T = 0 \Leftrightarrow RA_{-1} + \sum_{t=0}^{T} R^{-t}(P_t - C_t) &= 0 \Leftrightarrow \\ B_0 + H_0 &= \sum_{t=0}^{T} R^{-t} C_t \end{aligned}$$

where $H_0 \equiv \sum_{t=0}^{T} (G/R)^t P_0 = \frac{1-(G/R)^{T+1}}{1-G/R} P_0$

Static problem → Lagrangian

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \frac{C_t^{1-\rho}}{1-\rho} + \lambda \left[\sum_{t=0}^{T} R^{-t} C_t - (B_0 + H_0) \right]$$

- **First order conditions**

$$\forall t:\ 0 = \beta^t C_t^{-\rho} - \lambda R^{-t}$$

- ***Short-run* Euler** equation**:** $\frac{C_{t+1}}{C_t} = (\beta R)^{1/\rho}$
- ***Long-run* Euler** equation**:** $\frac{C_t}{C_0} = (\beta R)^{t/\rho}$

Consumption function

- Insert **Euler** into **IBC**

$$\begin{aligned} \sum_{t=0}^{T} R^{-t}(\beta R)^{t/\rho} C_0 &= B_0 + H_0 \Leftrightarrow \\ C_0 \sum_{t=0}^{T} ((\beta R)^{1/\rho}/R)^t &= B_0 + H_0 \end{aligned}$$

- **Solve** for C_0

$$C_0 = \frac{1-(\beta R)^{1/\rho}/R}{1-((\beta R)^{1/\rho}/R)^{T+1}}(B_0 + H_0)$$

  - **MPC:** $\frac{\partial C_0}{\partial B_0} \approx 1 - [(\beta R)^{1/\rho}/R] \approx 1 - R^{-1} \approx r$, where $R = 1 + r$
  - **MPCP:** $\frac{\partial C_0}{\partial P_0} \approx 1 - [(\beta R)^{1/\rho}/R] \frac{\partial H_0}{\partial P_0} \approx \frac{1-1/R}{1-G/R} \approx 1$

Side-note: Value function

- **Analytical expression** for the value function

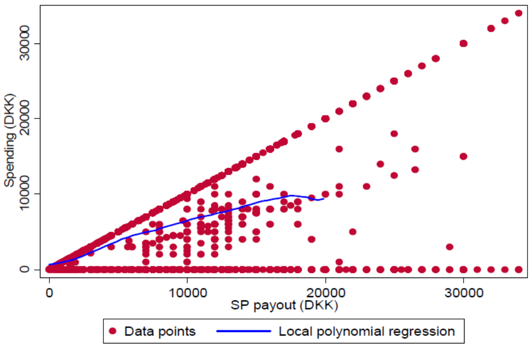
$$
\begin{aligned}
V_0(M_0, P_0) &= \sum_{t=0}^{T} \beta^t u((\beta R)^{t/\rho} C_0) \\
&= \sum_{t=0}^{T} \beta^t (\beta R)^{(1-\rho)t/\rho} \frac{C_0^{1-\rho}}{1-\rho} \\
&= \sum_{t=0}^{T} ((\beta R)^{1/\rho}/R)^t \frac{C_0^{1-\rho}}{1-\rho} \\
&= \frac{1-((\beta R)^{1/\rho}/R)^{T+1}}{1-(\beta R)^{1/\rho}/R} \frac{C_0^{1-\rho}}{1-\rho}
\end{aligned}
$$

Empirical evidence

- **Pro**
  1. Micro-founded consumption-saving
     - Theoretically appealing (humans are intentional)
     - Empirically appealing (testable implications on micro-data)
  2. Larger responses to permanent than to transitory shocks
  3. Consumption smoothing - save for retirement (future low income)
- **Con**
  1. Households seems to have a high MPC in the range 0.20-0.40
     - Survey studies (Kreiner et al., 2019)
     - Tax rebates studies (Johnson et al., 2006; Parker et al., 2013)
     - Lottery studies (Fagereng et al., 2019)
     - ARM payments studies (Druedahl et al., 2019; Di Maggio et al., 2017)
  2. Consumption responds to anticipated income changes
  3. Households with more volatile income have larger savings
  4. Consumption tracks income over the life-cycle
  5. (Households are only boundedly rational)

High MPC: Danish SP payout

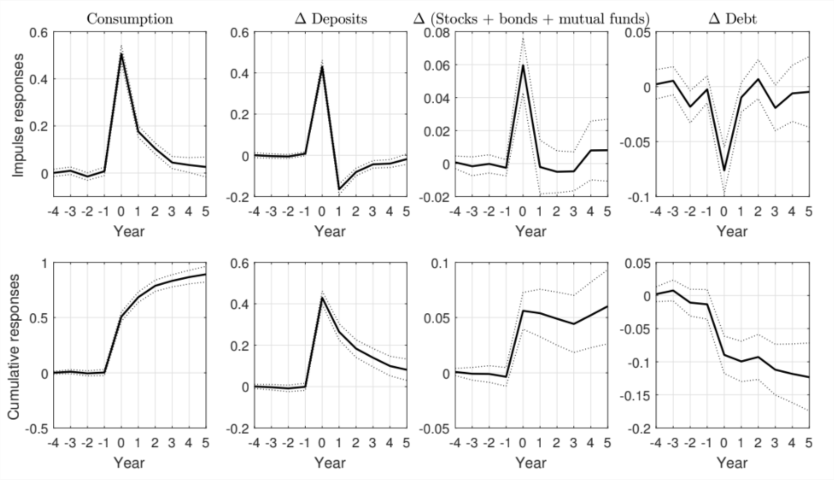
Figure 4: Spending and the size of the SP payout



NOTE: 5055 observations.

Source: Kreiner, Lassen og Leth-Petersen (AEJ:Pol, 2019)

High MPC: Norwegian lottery winners



Source: Fagereng, Holm, Natvik (WP, 2019)

Buffer-stock

Buffer-stock model (Deaton-Carroll)

+ borrowing constraints
+ income uncertainty

$\Rightarrow$

$$
\begin{aligned}
V_0(M_0, P_0) &= \max_{\{C_t\}_{t=0}^{T}} \mathbb{E}_0 \sum_{t=0}^{T} \beta^t \frac{C_t^{1-\rho}}{1-\rho} \\
&\text{s.t.} \\
A_t &= M_t - C_t \\
M_{t+1} &= RA_t + Y_{t+1} \\
Y_{t+1} &= \xi_{t+1} P_{t+1} \\
\xi_{t+1} &= \begin{cases} \mu & \text{with prob. } \pi \\ (\epsilon_{t+1} - \pi\mu)/(1-\pi) & \text{else} \end{cases} \\
& \quad \epsilon_t \sim \exp \mathcal{N}(-0.5\sigma_\xi^2, \sigma_\xi^2) \\
P_{t+1} &= GP_t\psi_{t+1}, \;\; \psi_t \sim \exp \mathcal{N}(-0.5\sigma_\psi^2, \sigma_\psi^2) \\
A_t &\geq -\lambda P_t \\
A_T &\geq 0
\end{aligned}
$$

Note: Later analytical results hold only for $\mu = 0$ and $\pi > 0$

How to solve the model?

- **Borrowing constraints** $\rightarrow$ inequalities $\rightarrow$ high-dimensional **Kuhn-Tucker problem**
- **Uncertainty** $\rightarrow$ fully dynamic problem $\rightarrow$ no simple Lagrangian
- **No analytical solution with CRRA preferences**
  - Quadratic or CARA utility, which give some analytical results, have implausible properties

$$\text{CRRA: } u(c) = \frac{c^{1-\rho}}{1-\rho} \rightarrow \text{RRA} = \rho$$

$$\text{Qudratic: } u(c) = ac - \frac{b}{2}c^2 \rightarrow \text{ RRA} = \frac{b}{a-bc}c$$

$$\text{CARA: } u(c) = \frac{1}{\alpha}e^{-\alpha c} \rightarrow \text{ RRA} = \alpha c$$

  where RRA = relative risk aversion $= \frac{-u''(c)}{u'(c)}c$

- **Solution:** Bellman equation $\rightarrow$ numerical dynamic programming

Bellman equation

$$
\begin{aligned}
V_t(M_t, P_t) &= \max_{C_t} \frac{C_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[V_{t+1}(M_{t+1}, P_{t+1}) \right] \\
&\text{s.t.} \\
A_t &= M_t - C_t \\
M_{t+1} &= RA_t + Y_{t+1} \\
Y_{t+1} &= \xi_{t+1} P_{t+1} \\
\xi_{t+1} &= \begin{cases} \mu & \text{with prob. } \pi \\ (\epsilon_{t+1} - \pi\mu)/(1-\pi) & \text{else} \end{cases} \\
P_{t+1} &= G P_t \psi_{t+1} \\
A_t &\geq -\lambda P_t \\
A_T &\geq 0
\end{aligned}
$$

Normalization I

- **Defining** $c_t \equiv C_t/P_t$, $m_t \equiv M_t/P_t$ etc. implies

$$
\begin{aligned}
A_t = M_t - C_t &\Leftrightarrow A_t/P_t = M_t/P_t - C_t/P_t \\
&\Leftrightarrow a_t = m_t - c_t
\end{aligned}
$$

$$
\begin{aligned}
M_{t+1} = RA_t + Y_{t+1} &\Leftrightarrow M_{t+1}/P_{t+1} = RA_t/P_{t+1} + Y_{t+1}/P_{t+1} \\
&\Leftrightarrow m_{t+1} = Ra_t P_t/P_{t+1} + \xi_{t+1} \\
&\Leftrightarrow m_{t+1} = \frac{R}{G\psi_{t+1}} a_t + \xi_{t+1}
\end{aligned}
$$

The **adjustment factor** $\frac{1}{G\psi_{t+1}}$ is due to changes in permanent income

Normalization II

- **Defining** $v_t(m_t) = V_t(M_t, P_t)/P_t^{1-\rho}$ finally implies

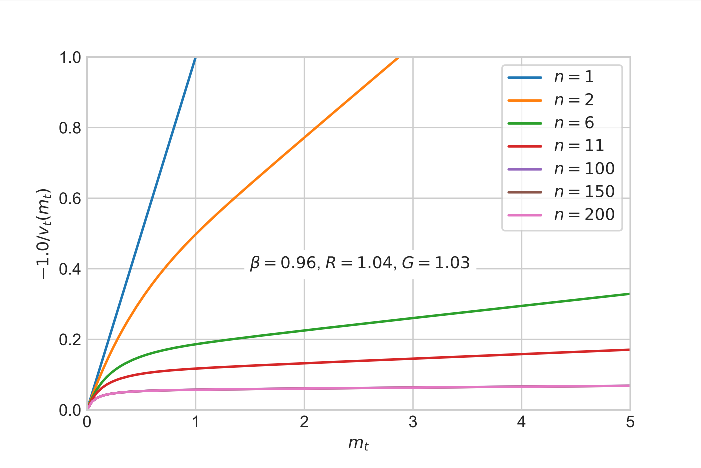
$$
\begin{aligned}
V_t(M_t, P_t) &= \max_{C_t} \frac{C_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[V_{t+1}(M_{t+1}, P_{t+1}) \right] \\
&= \max_{c_t} \frac{(c_t P_t)^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[V_{t+1}(M_{t+1}, P_{t+1}) \right] \Leftrightarrow \\
V_t(M_t, P_t)/P_t^{1-\rho} &= \max_{c_t} \frac{(c_t P_t)^{1-\rho}/P_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[V_{t+1}(M_{t+1}, P_{t+1})/P_t^{1-\rho} \right] \Leftrightarrow \\
v_t(m_t) &= \max_{c_t} \frac{c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[V_{t+1}(M_{t+1}, P_{t+1})/P_{t+1}^{1-\rho} \cdot P_{t+1}^{1-\rho}/P_t^{1-\rho} \right] \\
&= \max_{c_t} \frac{c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[(G\psi_{t+1})^{1-\rho} v_{t+1}(m_{t+1}) \right]
\end{aligned}
$$

Bellman equation in ratio form

$$
\begin{aligned}
v_t(m_t) &= \max_{c_t} \frac{c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[(G\psi_{t+1})^{1-\rho} v_{t+1}(m_{t+1}) \right] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
m_{t+1} &= \frac{1}{G\psi_{t+1}} R a_t + \xi_{t+1} \\
\xi_{t+1} &= \begin{cases} \mu & \text{with prob. } \pi \\ (\epsilon_{t+1} - \pi\mu)/(1-\pi) & \text{else} \end{cases} \\
a_t &\geq -\lambda \\
a_T &\geq 0
\end{aligned}
$$

- **Benefit:** Dimensionality of state space reduced
  Can this always be done?
- Easy to solve by **VFI**

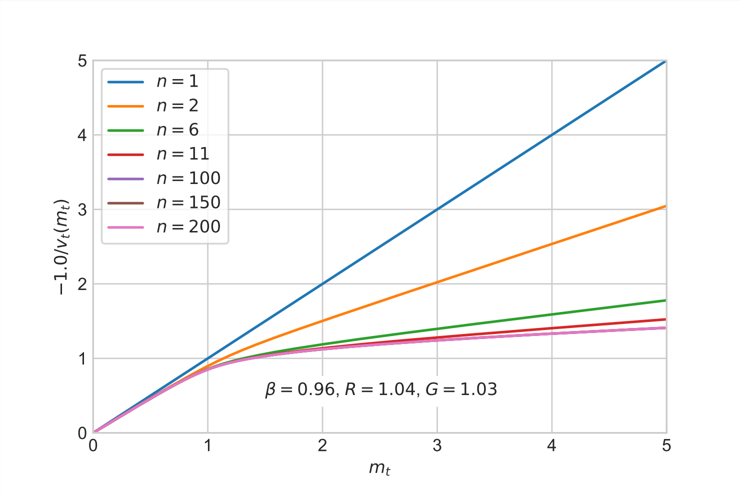
$T \to \infty$; Convergence of $-1.0/v_t(m_t) \to -1.0/v^\star(m_t)$



Other parameters: $\rho = 2$, $\pi = 0.005$, $\mu = 0.0$, $\sigma_\psi = \sigma_\xi = 0.10$

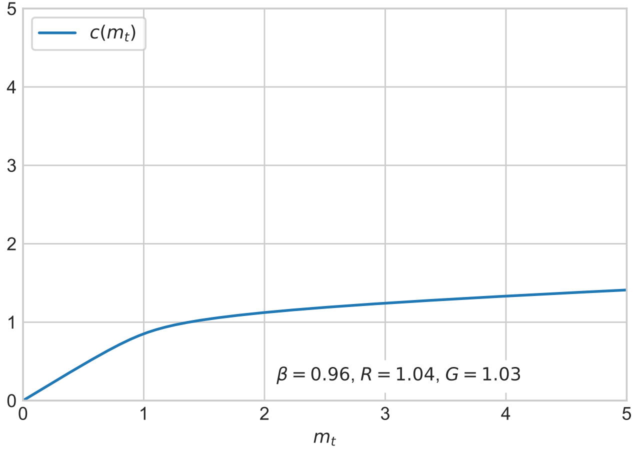
Note: $-1.0/v_t(m_t)$ is a numerically more stable object than $v_t(m_t)$

$T \to \infty$: Convergence of $c_t(m_t) \to c^{\star}(m_t)$

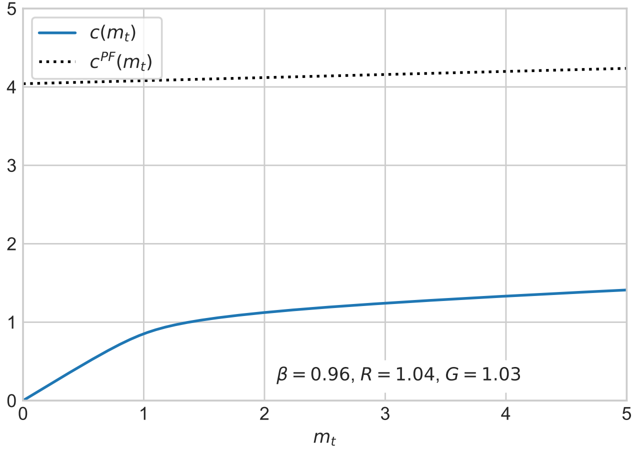


- What is the MPC?

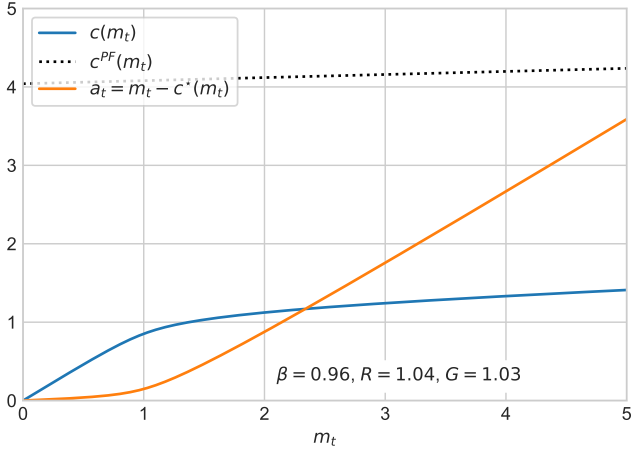
$T \to \infty$: The buffer-stock target



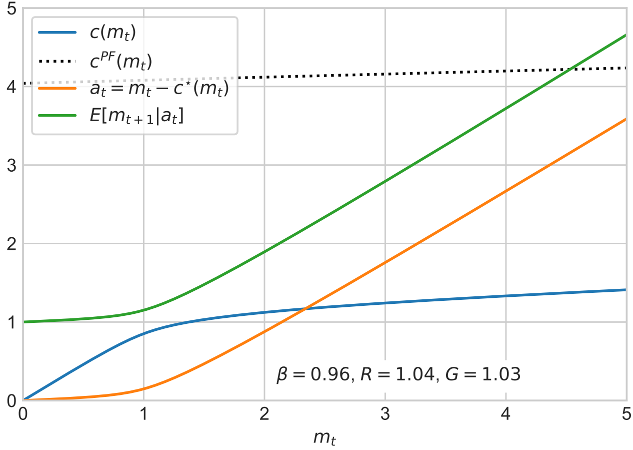
$T \to \infty$: The buffer-stock target



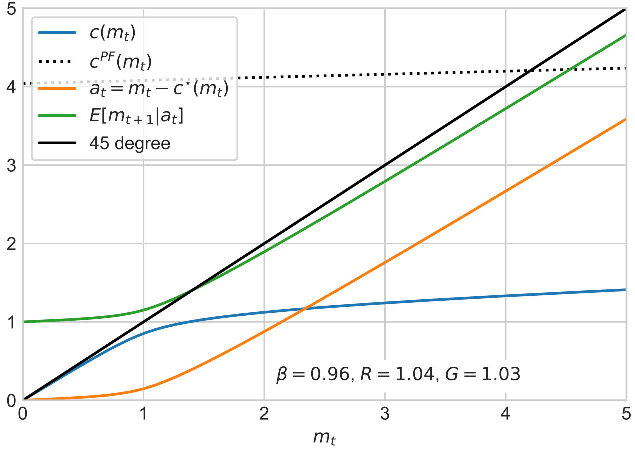
$T \to \infty$: The buffer-stock target



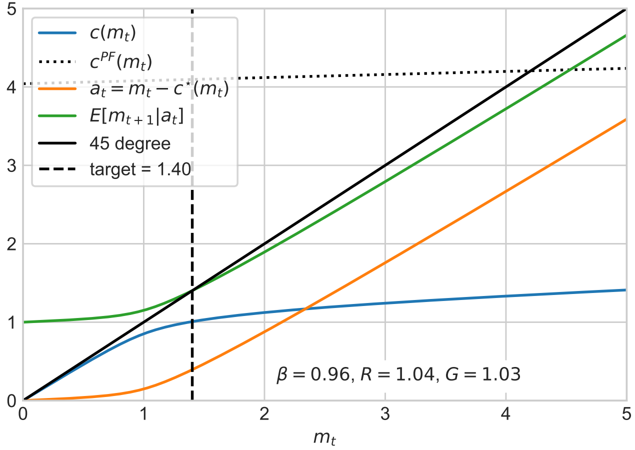
$T \to \infty$: The buffer-stock target



$T \to \infty$: The buffer-stock target



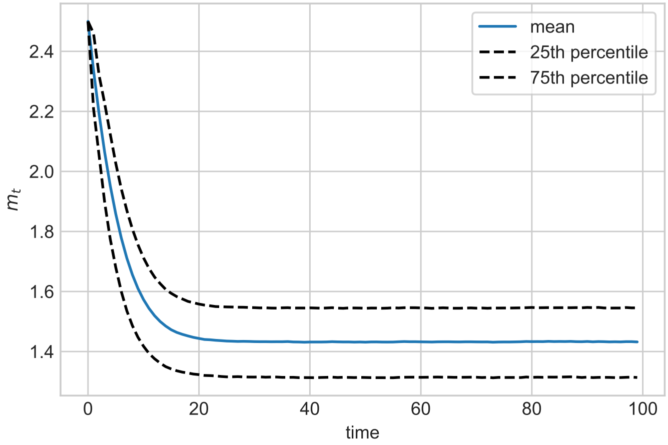
$T \to \infty$: The buffer-stock target



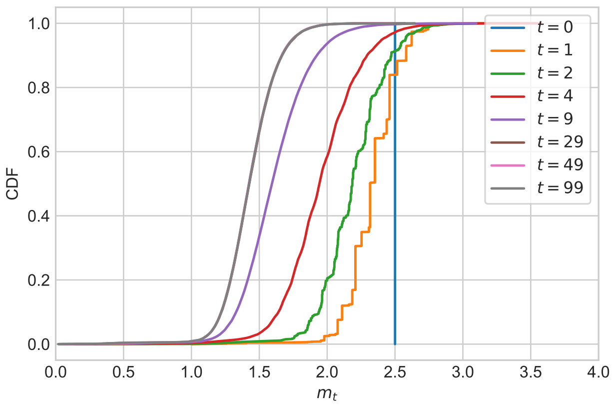
Simulation for $t \in \{0, 1, \ldots, T-1\}$

1. Choose m_0 and set $t = 0$
2. Calculate $c_t = c^\star(m_t)$
3. Calculate $a_t = m_t - c_t$
4. Draw (pseudo-)random numbers
$$\begin{aligned} \epsilon_{t+1} &\sim \exp \mathcal{N}(-0.5\sigma_\xi^2, \sigma_\xi^2) \\ \psi_{t+1} &\sim \exp \mathcal{N}(-0.5\sigma_\psi^2, \sigma_\psi^2) \\ \eta_{t+1} &\sim \mathcal{U}(0,1) \end{aligned}$$
5. Calculate $\xi_{t+1} = \begin{cases} \mu & \text{if } \eta_{t+1} < \pi \\ (\epsilon_{t+1} - \pi\mu)/(1-\pi) & \text{else} \end{cases}$
6. Calculate $m_{t+1} = \frac{R}{G\psi_{t+1}} a_t + \xi_{t+1}$
7. Set $t = t + 1$
8. Stop if $t \geq T$ else go to step 2

Simulation: Avg. cash-on-hand



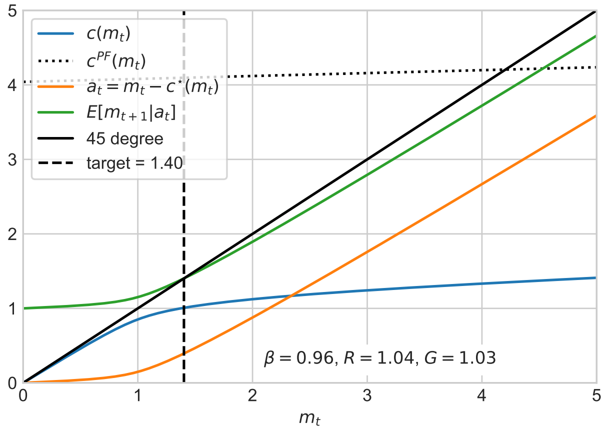
Simulation: Distribution of cash-on-hand



- **Proof of convergence:** Szeidl (2006)

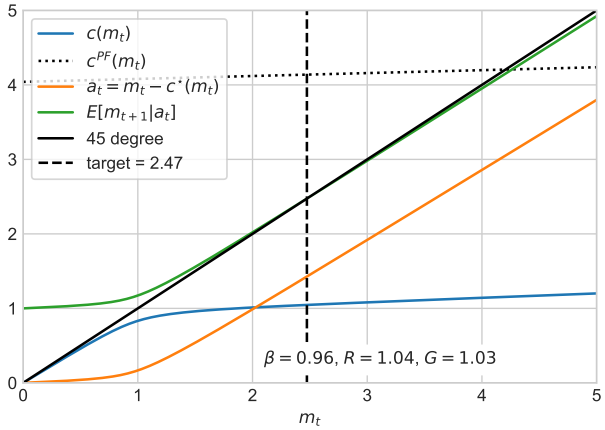
Details

$\sigma_\psi = 0.10$



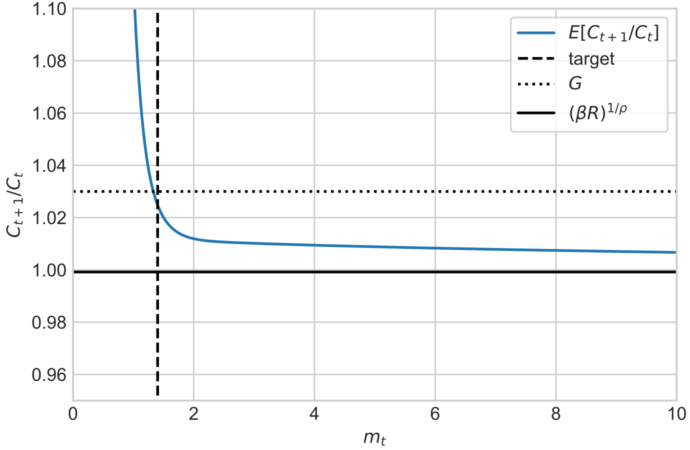
Target with standard risk: 1.40

$\sigma_\psi = 0.15$



Target with high risk: 2.47

Consumption growth I



Consumption growth II

- Remember **Euler-equation**

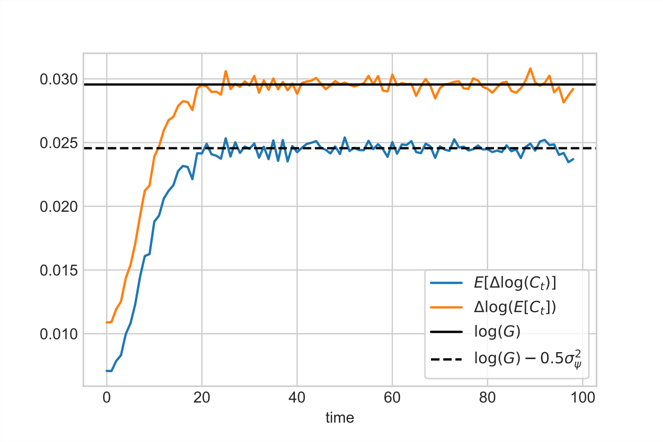
$$C_t^{-\rho} = \beta R \mathbb{E}_t \left[C_{t+1}^{-\rho} \right] \text{ if no uncertainty} \Rightarrow C_{t+1}/C_t = (\beta R)^{1/\rho}$$

- **Results**
  1. C_{t+1}/C_t is declining in m_t
  2. $\lim_{m_t \to \infty} C_{t+1}/C_t = (\beta R)^{1/\rho} = \text{RI}$
  3. $\lim_{m_t \to 0} C_{t+1}/C_t = \infty$
  4. $C_{t+1}/C_t < G$ at buffer-stock target
- **Intuition** for $C_{t+1}/C_t > (\beta R)^{1/\rho}$
  1. Uncertainty $\Rightarrow$ expected marginal utility $\uparrow$ [$C_{t+1}^{-\rho}$ is convex function]
  2. Consumer must be lowered today, $C_t \downarrow$
  3. Consumption growth will increase, $C_{t+1}/C_t \uparrow$

Further: *The above arguments are stronger for lower cash-on-hand relative to permanent income*

Consumption growth III

1. Growth of average consumption $= G$
2. Average consumption growth $= G - 0.5\sigma_{\psi}^2$

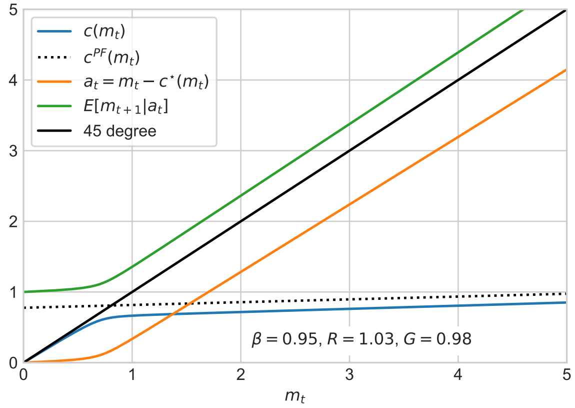


Always a buffer-stock target? I

1. **Utility impatience** (UI): $\beta < 1$
2. **Return impatience** (RI): $(\beta R)^{1/\rho}/R < 1$
3. **Weak return impatience** (WRI): $\pi^{1/\rho}(\beta R)^{1/\rho}/R < 1$
4. **Growth impatience** (GI): $(\beta R)^{1/\rho}\mathbb{E}_t[\psi_{t+1}^{-1}]/G < 1$
5. **Absolute impatience** (AI): $(\beta R)^{1/\rho} < 1$
6. **Finite value of autarky** (FVA): $\beta\mathbb{E}_t[(G\psi_{t+1})^{1-\rho}] < 1$

Always a buffer-stock target? II

- **GI ensures buffer-stock target**
- If not *GI* then inifinite accumulation is possible like:



Existence of solution

- **Existence of solution:** WRI + FVA
  - **Proof:** Use *Boyds weighted contraction mapping theorem*
  - **Standard assumptions:** FHW, RI, GI
- The **consumption function** is twice continuously differentiable**,** **increasing** and **concave**

The borrowing constraint

- Assume **perfect foresight** ($\sigma_\psi = \sigma_\epsilon = \pi = 0$), but **no borrowing**, $\lambda = 0$.
- **Solution:** RI + FHW is still *sufficient* (with $\lambda = \infty$ they are *necessary*)
- **Standard solutions:** RI + FHW
  1. **GI** $\Rightarrow$ *constraint will eventually be binding*

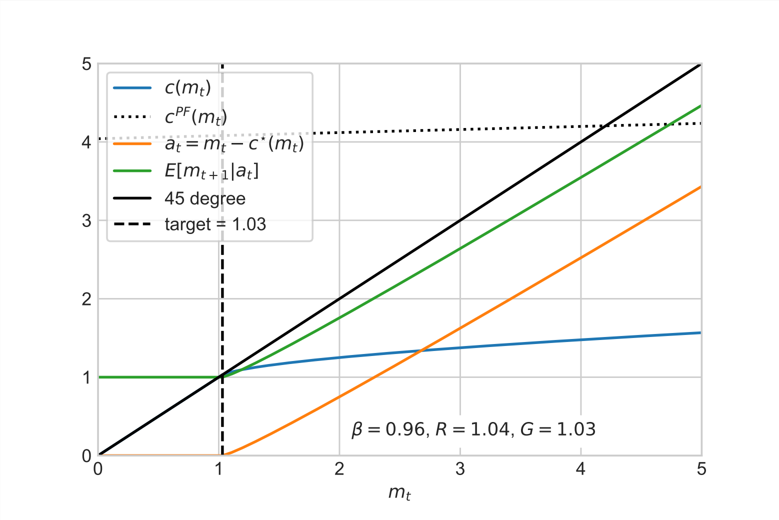
     $$c^{\star}(m_t) \text{ converge to } c^{PF}(m_t) \text{ from below as } m_t \rightarrow \infty$$

  2. **Not GI** $\Rightarrow$ *constraint is never reached*

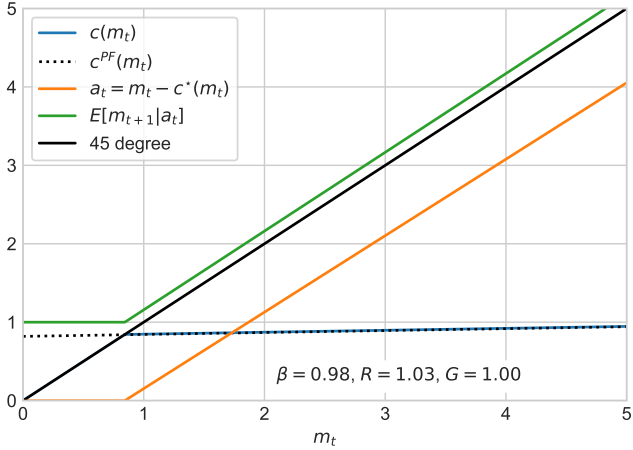
     $$c^{\star}(m_t) = c^{PF}(m_t) \text{ for } m_t \geq 1$$

- **Exotic solutions without FHW** exists (GI necessary)

Perfect foresight with $\lambda = 0$ and GI



Perfect foresight with $\lambda = 0$, but not GI



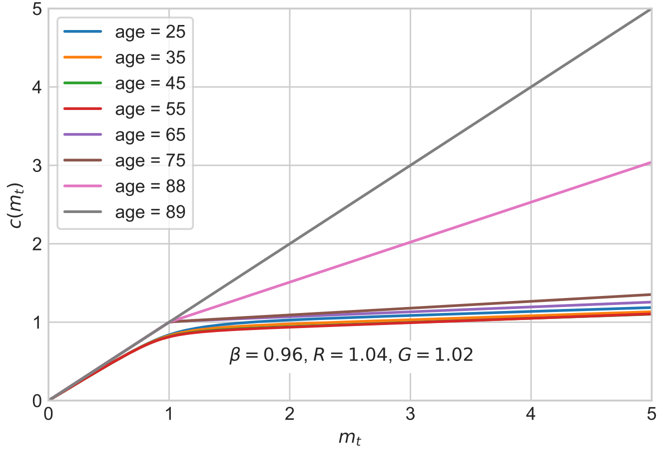
Life-cycle

Adding a life-cycle (normalized)

$$
\begin{aligned}
v_t(m_t, z_t) &= \max_{c_t} \frac{v(z_t) c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[(GL_{t+1}\psi_{t+1})^{1-\rho} v_{t+1}(\bullet) \right] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
m_{t+1} &= \frac{1}{GL_t \psi_{t+1}} Ra_t + \xi_{t+1} \\
\xi_{t+1} &= \begin{cases} \mu & \text{with prob. } \pi \\ (\epsilon_{t+1} - \pi\mu)/(1-\pi) & \text{else} \end{cases} \\
a_t &\geq \lambda_t = \begin{cases} -\lambda & \text{if } t < T_R \\ 0 & \text{if } t \geq T_R \end{cases}
\end{aligned}
$$

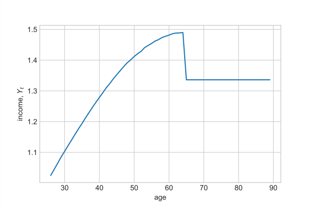
- **Demographics**: z_t (exogenous)
- **Income profile:** $P_{t+1} = GL_t P_t \psi_{t+1}$
- **No shocks in retirement:** $\psi_t = \xi_t = 1$ if $t > T_R$
- **Euler equation:** $C_t^{-\rho} = \beta R \mathbb{E}_t [\frac{v(z_{t+1})}{v(z_t)} C_{t+1}^{-\rho}]$

Consumption functions ($v(z_t) = 1$)

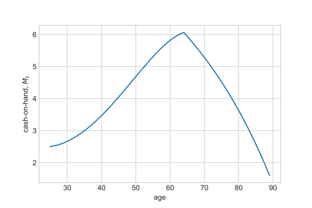


Simulation: LIfe-cycle profiles ($v(z_t) = 1$)

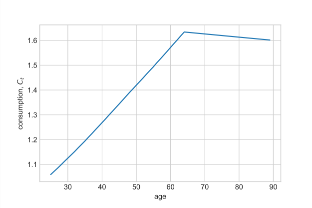
Income, Y_t (implied by G and L_t)



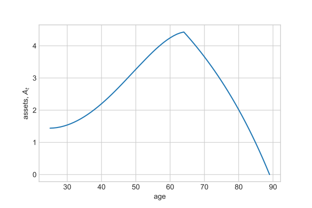
Cash-on-hand, M_t



Consumption, C_t



End-of-period assets, A_t



EGM

Euler-equation

- Reference: Carroll (2006)
- Assume for simplicity **no borrowing**: $\lambda = 0$
- All optimal **interior choices** must satisfy

$$
\begin{aligned}
C_t^{-\rho} &= \beta R \mathbb{E}_t \left[C_{t+1}^{-\rho} \right] \Leftrightarrow \\
c_t^{-\rho} &= \beta R \mathbb{E}_t \left[(G \psi_{t+1} c_{t+1})^{-\rho} \right]
\end{aligned}
$$

- Else optimal choice is **constrained**

$$
\begin{aligned}
C_t^{-\rho} &\geq \beta R \mathbb{E}_t \left[C_{t+1}^{-\rho} \right] \Leftrightarrow \\
C_t &= M_t \Leftrightarrow \\
c_t &= m_t
\end{aligned}
$$

Endogenous grid method: Intuition

- **Obs.:** Given $C^{\star}_{t+1}(M_{t+1}, P_{t+1})$ and A_t and P_t we have

$$\begin{aligned}
C_t^{-\rho} &= \beta R \mathbb{E}_t\left[\left(C^{\star}_{t+1}(M_{t+1}, P_{t+1})\right)^{-\rho}\right] \Leftrightarrow \\
C_t &= \mathbb{E}_t\left[\beta R\left(C^{\star}_{t+1}(M_{t+1}, P_{t+1})\right)^{-\rho}\right]^{-\frac{1}{\rho}} \\
&= \mathbb{E}_t\left[\beta R\left(C^{\star}_{t+1}(RA_t + Y_{t+1}, P_{t+1})\right)^{-\rho}\right]^{-\frac{1}{\rho}} \\
&= \mathbb{E}_t\left[\beta R\left(C^{\star}_{t+1}(RA_t + P_t\psi_{t+1}\xi_{t+1}, P_t\psi_{t+1})\right)^{-\rho}\right]^{-\frac{1}{\rho}} \\
&\equiv F(A_t, P_t)
\end{aligned}$$

- **Endogenous grid:** $A_t = M_t - C_t \Leftrightarrow M_t = C_t + A_t$
- **Conclusion:** $(M_t,\ P_t,\ C_t)$ is a solution to the Bellman equation because it satisfies the Euler equation
- **Perspectives:** Varying A_t (and P_t) we can map out the consumption function without using any numerical solver!
- **Borrowing constraint:** Binding below lowest generated M_t

... in ratio form

- **Prerequisites**:
  1. Next-period **consumption function**: $c^{\star}_{t+1}(m_{t+1})$
  2. **Asset grid:** $\mathcal{G}_a = \{a_1, a_2, \dots, a_\#\}$ with $a_1 = 10^{-6}$
- **Algorithm:** For each $a_i \in \mathcal{G}_a$
  1. Find consumption using Euler equation

$$c_i = \mathbb{E}_t \left[\beta R \left(G\psi_{t+1} c^{\star}_{t+1} \left(\frac{R}{G\psi_{t+1}} a_i + \xi_{t+1} \right) \right)^{-\rho} \right]^{-\frac{1}{\rho}}$$

  2. Find endogenous state: $a_i = m_i - c_i \Leftrightarrow m_i = a_i + c_i$
- The **consumption function**, $c_t(m_t)$, is given by

$$\{0, c_1, c_2, \dots, c_\#\} \text{ for } \{\underline{a}_t, m_1, m_2, \dots, m_\#\}$$

- *We can find all consumption functions in this way!*

Addendum: The natural borrowing constraint ($\lambda > 0$)

- The **optimal end-of-period asset choice satisfies** the backwards recursion

$$a_t \geq \underline{a}_t = \begin{cases} 0 & \text{if } t \geq T_R \\ -\min\left\{\Lambda_t, \lambda_t\right\} GL_t \underline{\psi} & \text{if } t < T_R \end{cases}$$

where

$$\mathit{\Lambda}_t \quad \equiv \quad \begin{cases} R^{-1} GL_t \underline{\psi}\,\underline{\xi} & \text{if } t = T_R - 1 \\ R^{-1}\left[\min\left\{\mathit{\Lambda}_{t+1}, \lambda_t\right\} + \underline{\xi}\right] GL_t \underline{\psi} & \text{if } t < T - 1 \end{cases}$$

and $\underline{\psi}$ and $\underline{\xi}$ are the minimum realizations of ψ_{t+1} and ξ_{t+1}

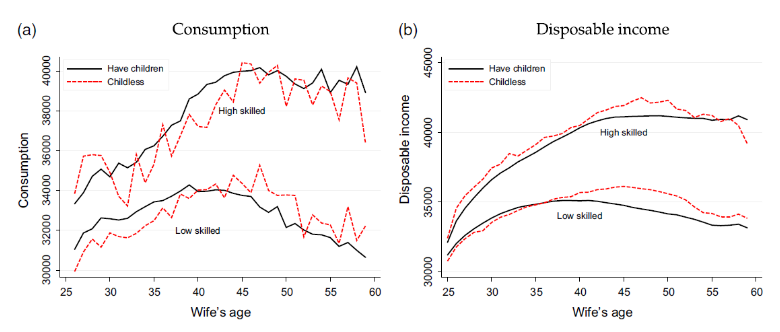
- **Proof:** Can be shown as a consequence of the household wanting to avoid $c_t = 0$ at *any cost* because $\lim_{c_t \to 0} u'(c_t) = \infty$.

Further perspectives

Three generations of models

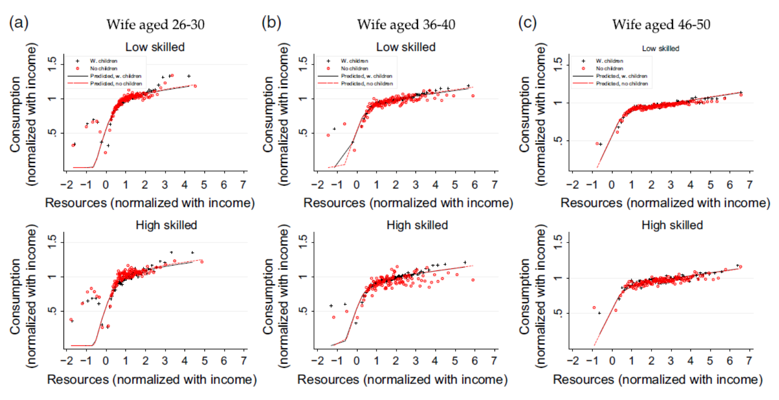
- **1st:** *Permanent income hypothesis* (Friedman, 1957) or *life-cycle model* (Modigliani and Brumburg, 1954)
- **2nd:** *Buffer-stock consumption model* (Deaton, 1991, 1992; Carroll, 1992, 1997, 2020)
- **3nd:** *Multiple-asset buffer-stock consumption models* (e.g. Kaplan and Violante (2014))

Denmark: Life-cycle profiles fit



Source: Jørgensen (2017)

Denmark: Consumption function fit



Source: Jørgensen (2017)

Level of wealth and MPC

- Consumption-saving models a few years ago **could not endogenously fit** both
    1. The level of wealth observed
    2. The high MPCs found in quasi experiments
- **Three solutions:**
    1. Exogenous **hands-too-mouth households** (Campbell and Mankiw, 1990)
    2. **Preference heterogeneity**
    3. **Wealthy hands-to-mouth** (Kaplan and Violante, 2014)
       Many households hold mostly illiquid assets with a high return
       $\rightarrow$ *consumption adjust in response to small income shock*

Kaplan-Violante model (two-asset model)

$$V_t(M_t, N_t, P_t) = \max\left\{v_t^{keep}(M_t, N_t, P_t), v_t^{adj.}(M_t + N_t - \lambda, P_t)\right\}$$

$$\begin{aligned}
v_t^{keep}(M_t, N_t, P_t) &= \max_{C_t} u(C_t, B_t) + \beta W_t(A_t, B_t, P_t) \text{ s.t.} \\
A_t &= M_t - C_t \\
B_t &= N_t \\
A_t &\geq -\omega P_t.
\end{aligned}$$

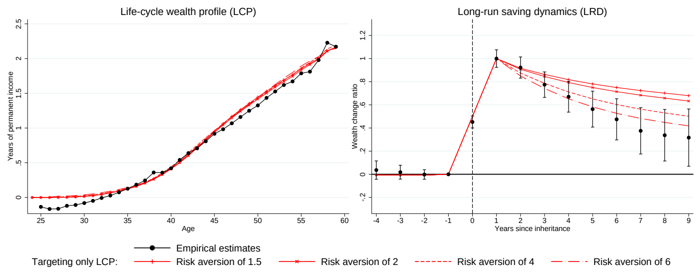
$$\begin{aligned}
\tilde{v}_t^{adj.}(X_t, P_t) &= \max_{B_t, C_t} u(C_t, B_t) + \beta W_t(A_t, B_t, P_t) \text{ s.t.} \\
M_t &= X_t - B_t \\
A_t &= M_t - C_t \\
A_t &\geq -\omega P_t.
\end{aligned}$$

$$W_t(A_t, B_t, P_t) = \mathbb{E}_t[V_t(RA_t + P_t\psi_{t+1}\xi_{t+1}, R_b B_t, P_t\psi_{t+1})]$$

Level of wealth and long-run dynamics I

- **Best test of a life-cycle consumption-saving model:**
  A *sudden*, *sizable* and *salient* shock to wealth
  + long panel to observe how the extra wealth is spend
- **My own research:** Druedahl and Martinello (2018)
  Compare individuals in the Danish register data who
  1. Receive a similar inheritance, but at different points in time
  2. From parents dying due to heart attacks or car crashes

Level of wealth and long-run dynamics II



- **Net worth:** Good fit for different levels of risk-aversion (ρ) when re-calibrating patience (β)
- **Also dynamics:** Good fit only if risk-aversion (ρ) is high

Frontier topics - curated papers

- **Durable consumption**: Berger and Vavra (2015), Harmenberg and Oberg (2017)
- **Labor supply**, **retirement** and **family formation**: Low et al. (2010), French and Jones (2011), Keane and Wasi (2016), Adda et al. (2016), Blundell et al. (2016)
- **Non-Gaussian income uncertainty**: De Nardi et al. (2018), Guvenen et al. (2019), Druedahl and Munk-Nielsen (2019)
- **Housing**: Landvoigt (2017), Kaplan et al. (2019)
- **Imperfect information and bounded rationality**: Pagel (2017). Carroll et al. (2019), Moran and Kovacs (2019), Druedahl and Jørgensen (2020)
- **Level and dynamics of inequality** – circumstances or behavior? De Nardi and Fella (2017), Hubmer et al. (2019)

Frontier solution methods - curated papers

- **EGM in non-convex multi-dimensional models:** Druedahl and Jørgensen (2017) and Druedahl (2020)
- **Sparse grids:** Judd et al. (2014), Brumm and Scheidegger (2017)
- **Machine learning:** Azinovic et al. (2019), Maliar et al. (2019)

Estimation

Reduced form estimation

- **Code:** »04. Structural estimation«
- Critic of structural estimation: **Requires many assumptions**
- **But:** To turn reduced form parameter estimates into policy advice *a lot of assumptions are often implicitely required*

  > *»All econometric work relies heavily on a priori assumptions. The main difference between structural and experimental (or "atheoretic") approaches is not in the number of assumptions but the extent to which they are made explicit.« (Keane, 2012)*

- **The beauty of models:**
  1. Ensure *consistent* world view
  2. Allow us to combine *heterogenous facts* and extrapolate from a myriad of past experiences
  3. Better models are clearly defined – even if we never find *the* true model we can make *progress*
- **Frontier:** Combine the two and use exogenous variation to estimate structural model (Nakamura and Steinsson, 2018)

The Lucas critique

- **The Lucas critique:** *Behavioral rules change with policy*
  $\Rightarrow$ policy advice can not rely on estimated behavioral rules
  $\Rightarrow$ we need to estimate *structural parameters*

  »Invariance of parameters in an economic model is not, of course, a property which can be assured in advance, but it seems reasonable to hope that neither tastes nor technology vary systematically with variations in counter-cyclical policies.« (Lucas, 1977)

- **Other stuff might be approximately invariant**
- **Rigourous microfoundations:**
  1. **Mathematically:** Based on (boundedly) rational behavior derived as a solution to a formal optimization problem
  2. **Economically:** The assumptions are realistic

Estimation

1. **Focus:** Closely related estimators ***indirectly*** using **micro-data**

   Simulated Method of Moments (**SMM**) (McFadden, 1989)
   Simulated Minimium Distance (**SMD**) (Duffie and Singleton, 1990)
   Indirect Inference (**II**) (Gouriéroux and Monfort, 1997)

   Main alternative:
   Simulated Maximum Likelihood (**SML**) ***directly*** using **micro-data**
   (see e.g. Adda and Cooper (2003) or Druedahl et al. (2018))

2. **Examples:** Gourinchas and Parker (2002), Cagetti (2003), Guvenen and Smith (2014), Druedahl and Jørgensen (2020)
3. **Extended toolbox:** Jørgensen (2020) and Honore et al. (2020)

GE

Heterogenous Agent (HA) models

1. **Stationary equilibrium:**
   Deterministic steady state and transition path
   Foundational papers: Bewley (1986), Imrohoroğlu (1989), Huggett (1993), Aiyagari (1994)
   A few policy examples: Aiyagari and McGrattan (1998), Conesa et al. (2009), Heathcote et al. (2014),
2. **Dynamic/recursive/sequential equilibrium:**
   Aggregate shocks and stochastic dynamics
   Foundational papers: Krusell and Smith (1997, 1998), Carroll (2000), Carroll et al. (2015)
3. **Reviews:** Heathcote et al. (2009), Krusell and Smith (2006), Krueger et al. (2016)

Heterogenous Agent New Keynesian (HANK) models

1. **Frontier:** Kaplan et al. (2018), Bayer et al. (2019), Luetticke (2019), Alves et al. (2019), Hagedorn et al. (2019), Auclert et al. (2020), Bayer et al. (2020), Fernandez-Villaverde et al. (2020)
2. **Analytical:** Bilbiie (2008, 2019a,b), Werning (2015), Challe et al. (2017), Acharya and Dogra (2018), Bilbiie et al. (2020), Debortoli and Galí (2018), Auclert et al. (2018), Broer et al. (2020), Ravn et al. (2020), Auclert and Rognlie (2020)
3. **Others:** Oh and Reis (2012), Gornemann et al. (2016), McKay and Reis (2016), McKay et al. (2016), Guerrieri and Lorenzoni (2017),Den Haan et al. (2017), Ravn and Sterk (2017)
4. **Empirical:** Cloyne et al. (2020), Slacalek et al. (2020), Holm and Paul (2020), Wolf (2020)
5. **Reviews:** Kaplan and Violante (2018)

Computational methods

- **Early reviews:** Den Haan et al. (2010), Schmedders and Judd (2013)
- **Continous time:** Achdou et al. (2020) (code), Ahn et al. (2018) (code)
- **Local aggregate solution:**
  1. State space: Bayer and Luetticke (2019) (MATLAB, Python)
  2. Sequence space: Boppart et al. (2018), Auclert et al. (2020) (code)
- **Global aggregate solution:** Kubler and Scheidegger (2018), Azinovic et al. (2019), Scheidegger and Bilionis (2019), Pröhl (2019) (code), Maliar et al. (2019) (code, video), Fernandez-Villaverde et al. (2020) (code)

Simple general equilibrium model

- **Code:** »05. General Equilibrium.ipynb«
- **Population:** Continuum of measure 1
  1. Owns stocks, a_{t-1}
  2. Supplies labor with productivity e_t (exogenous and stochastic, mean one)
  3. Consumes, c_t
- **Capital:** Depreciation rate δ
- **Firms:** Rent capital and hire labor to produce
- **Prices** are taken as given by households and firms
  1. r_t^k, rental rate
  2. $r_t = r_t^k - \delta$, interest rate
  3. w_t, wage rate

Households

- **Perfect foresight:** Price sequence known, $\{r_t, w_t\}_{t\geq 0}$
- **Households solve:**

$$
\begin{aligned}
v_t(e_t, a_{t-1}) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{t+1}(e_{t+1}, a_t) \right] \\
&\text{s.t.} \\
a_t + c_t &= (1+r_t)a_{t-1} + w_t e_t \\
a_t &\geq 0
\end{aligned}
$$

- **Optimal saving:** $a^\star(e_t, a_{t-1})$
- **Optimal consumption:** $c^\star(e_t, a_{t-1})$
- **Distribution:** D_t over e_t and a_{t-1}

Firms

- **Production function:** $Y_t = Z_t K_{t-1}^{\alpha} L_t^{1-\alpha}$
- **Define** $k_{t-1} \equiv K_{t-1}/L_t$
- Standard **pricing equations:**

$$
\begin{aligned}
r_t^k &= \alpha Z_t k_{t-1}^{\alpha-1} \\
w_t &= (1-\alpha) Z_t k_{t-1}^{\alpha}
\end{aligned}
$$

- Useful **implications:**

$$
\begin{aligned}
k_{t-1} &= \left(\frac{r_t+\delta}{\alpha Z_t}\right)^{\frac{1}{\alpha-1}} \equiv k(r_t, Z_t) \\
r_t &= \alpha Z_t k_{t-1}^{\alpha-1} \equiv r(k_{t-1}, Z_t) \\
w_t &= (1-\alpha) Z_t \left(\frac{r_t+\delta}{\alpha Z_t}\right)^{\frac{\alpha}{\alpha-1}} \equiv w(r_t, Z_t)
\end{aligned}
$$

Definition: Stationary equilibrium

A **stationary equilibrium** for a given Z_{ss} is a

1. A set of quantities K_{ss} and L_{ss} and prices r_{ss} and w_{ss}
2. A distribution D_{ss} over e_t and a_{t-1}
3. A policy functions $a^*_{ss}(e_t, a_{t-1})$ and $c^*_{ss}(e_t, a_{t-1})$

such that

1. $a^*_{ss}(\bullet)$ and $c^*_{ss}(\bullet)$ solves the household problem
2. D_{ss} is the invariant distribution implied by the household problem
3. Firms maximizes profit, $r_{ss} = r(K_{ss}/L_{ss}, Z_{ss})$ and $w_{ss} = w(r_{ss}, Z_{ss})$
4. The labor market clears, i.e. $L_{ss} = \int e_t dD_{ss} = 1$
5. The capital market clears, i.e. $K_{ss} = \int a^*_{ss}(e_t, a_{t-1})dD_{ss}$
6. The goods market clears, i.e. $Y_{ss} - \delta K_{ss} = \int c^*_{ss}(e_t, a_{t-1})dD_{ss}$

Find stationary equilibrium

1. Guess on r_{ss}
2. Calculate $w_{ss} = w(r_{ss}, Z_{ss})$
3. Solve the infinite horizon household problem
4. Simulate until convergence of D_{ss}
5. Calculate supply $k_{ss}^s = \int a_{ss}^*(e_t, a_{t-1})dD_{ss}$
6. Calculate demand $k_{ss}^d = k(r_{ss})L_{ss}$
7. If for some tolerance ϵ

$$\left|k_{ss}^s - k_{ss}^d\right| < \epsilon$$

   then stop, otherwise update r_{ss} appropriately and return to step 2

$\Rightarrow$ this is just a **root-finding problem**

Solve household problem with EGM

- **Grids:**
  1. $e_t \in \{e^1, e^2, \dots, e^{\#_e}\}$(using Tauchen and Hussey (1991))
  2. $a_t \in \{a^1, a^2, \dots, a^{\#_a}\}$
- **Guess:** $v_{a,t+1}(e^i, a^j)$, $\forall i, j$
- **Time iteration:**
  1. Calcualte: $q_t(e^i, a^j) = \sum_{k=1}^{\#_e} \Pr[e^k | e^i] v_{a,t+1}(e^i, a^j)$
  2. Calculate $\tilde{c}^{ij} = q_t(e^i, a^j)^{-\sigma}$ and $\tilde{m}^{ij} = \tilde{c}^{ij} + a^j$
  3. Interpolate $\{\tilde{m}^{ij}, a^j\}_{j=1}^{\#_a}$ at $m^j = (1 + r_t)a^j + w_t e^i$ to find $a^\star(e^i, a^j)$
  4. Calculate $c^\star(e^i, a^j) = m_t - a^\star(e^i, a^j)$
  5. Calculate $v_{a,t+1}(e^i, a^j) = (1 + r)c^\star(e^i, a^j)^{-\sigma}$
     (use of the envelope theorem)

Simulate household behavior on grid

- **Initial distribution:** $D_0(e^i, a^j) = \frac{\Pr[e^i]}{\#_a}$ (*ergodic* in e, *uniform* in a)
- **Update:**

$$D_{t+1}(e^k, a^l) = \sum_{i=1}^{\#_e} \Pr[e^i | e^l] \sum_{j=1}^{\#_a} D_t(e^i, a^j) \omega(a^j, a^{\max\{l-1,0\}}, a^l, a^{\min\{l+1,l\}})$$
$$+ \sum_{i=1}^{\#_e} \Pr[e^i | e^l] \sum_{j=1}^{\#_a} D_t(e^i, a^j) \omega(a^j, a^l, a^{l+1})$$

where

$$\omega(a, \underline{a}, \not{a}, \overline{a}) = 1\{a \in [\underline{a}, \overline{a}]\} \begin{cases} \frac{\overline{a}-a}{\overline{a}-\not{a}} & \text{if } a \geq \not{a} \\ \frac{a-\underline{a}}{\not{a}-\underline{a}} & \text{if } a < \not{a} \end{cases}$$

Definition: Transition path

A **transition path** for $t \in \{0, 1, 2, \dots\}$ given an initial distribution D_0 and a path of Z_t, is paths of quantities K_t and L_t, prices r_t and w_t, policy functions $a_t^*(\bullet)$ and $c_t^*(\bullet)$, distributions D_t, such that for all t

1. $a_t^*(\bullet)$ and $c_t^*(\bullet)$ solve the household problem given price paths
2. D_t are implied by the household problem given price paths and D_0
3. Firms maximizes profit, $r_t = r(K_{t-1}/L_t, Z_t)$ and $w_t = w(r_t, Z_t)$
4. The labor market clears, i.e. $L_t = \int e_t d\kappa_t = 1$
5. The capital market clears, i.e. $K_t = \int a_t^*(\bullet) dD_t$
6. The goods market clears, i.e. $Y_t = \int c_t^*(\bullet) dD_t + \delta K_{t-1}$

Find transition path

1. Chose truncation horizon $\mathcal{T}$
2. Guess on $\{r_t\}_{t=0}^{\mathcal{T}} = \{r_{ss}\}_{t=0}^{\mathcal{T}}$ (or something else)
3. Calculate $\{w_t\}_{t=0}^{\mathcal{T}} = \{w(r_{ss}, Z_t)\}_{t=0}^{\mathcal{T}}$
4. Solve the household problem backwards along the transition path
5. Simulate households forward along the transition path
6. Calculate $\{k_t\}_{t=0}^{\mathcal{T}} = \{\int a_t^*(\bullet)dD_t\}_{t=0}^{\mathcal{T}}$
7. Calculate $\{r_t'\}_{t=0}^{\mathcal{T}} = \{r(k_{t-1}, Z_t)\}_{t=0}^{\mathcal{T}}$
8. Stop if for some tolerance ϵ

$$\max_{t \in \{0,1,2,\ldots,\mathcal{T}\}} |r_t - r_t'| < \epsilon$$

   otherwise return to step 2 with $\{r_t\}_{t=0}^{\mathcal{T}} = \{\nu r_t + (1-\nu) r_t'\}_{t=0}^{\mathcal{T}}$

Note: Typically the relaxation parameter is $\nu = 0.90$ (Kirkby, 2017)

Aggregate shocks: Households

- **Aggregate shocks:** Assume Z_t is a stochastic process
- **Root problem:** There is no longer perfect foresight wrt. r_t and w_t
- **Extended problem:**

$$\begin{aligned}
v(e_t, a_{t-1}, Z_t, D_t) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v(e_{t+1}, a_t, Z_{t+1}, D_{t+1}) \right] \\
\text{s.t.} & \\
a_t + c_t &= (1+r_t)a_{t-1} + w_t e_t \\
k_t &= \int a^*(e_t, a_{t-1}) dD_t \\
r_t &= r(k_t, Z_t) \\
w_t &= w(r_t, Z_t) \\
a_t &\geq 0
\end{aligned}$$

- **Ultimate problem:** D_t is not easy to discretize...

Aggregate shocks: Households

- **Krussell-Smith idea:** Approximate D_t with some selected moments, e.g. just the mean
- **Approximate problem:**

$$
\begin{aligned}
v(e_t, a_{t-1}, Z_t, k_t) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v(e_{t+1}, a_t, Z_{t+1}, k_{t+1}) \right] \\
&\text{s.t.} \\
a_t + c_t &= (1+r_t)a_{t-1} + w_t e_t \\
r_t &= r(k_t, Z_t) \\
w_t &= w(r_t, Z_t) \\
k_{t+1} &= \text{PLM}(k_t, Z_t, Z_{t+1}) \\
a_t &\geq 0
\end{aligned}
$$

where $\text{PLM}(k_t, Z_t, Z_{t+1})$ is **perceived law of motion**

Definition: Dynamic equilibrium

An **approximate dynamic equilibrium** is a PLM, policy functions $a_t^*(\bullet)$ and $c_t^*(\bullet)$, and paths of quantities K_t and L_t, prices r_t and w_t, distributions D_t such that for all t

1. $a_t^*(\bullet)$ and $c_t^*(\bullet)$ solve the household problem given the PLM
2. D_t are implied by the household problem
3. Firm profit maximize $r_t = r(K_{t-1}/L_t, Z_t)$ and $w_t = w(r_t, Z_t)$
4. The labor market clears, i.e. $L_t = \int e_t d\kappa_t = 1$
5. The capital market clears, i.e. $K_t = \int a_t^*(\bullet) dD_t$
6. The goods market clears, i.e. $Y_t = \int c_t^*(\bullet) dD_t + \delta K_{t-1}$
7. $\text{PLM}(k_t, Z_t, Z_{t+1})$ does not imply systematic expectations errors

Find dynamic equilibrium

1. Guess on the $\mathrm{PLM}(k_t, Z_t, Z_{t+1})$
2. Solve the household problem
3. Simulate a path of Z_t and D_t and thus k_t
4. Compare simulated behavior with the $\mathrm{PLM}(k_t, Z_t, Z_{t+1})$
   Stop if »good enough«
   otherwise update $\mathrm{PLM}(k_t, Z_t, Z_{t+1})$ and return to step

Terminology:

1. The Krussell-Smith method is a **global method**
2. The newest **local methods** rely on linearization of the aggregate dynamics, but solve for the full non-linear stationary equilibrium

Summary

Summary

- **Dynamic programming** is needed to solve **empirically realistic consumption-saving models**
- The **buffer-stock consumption model**, and it's two asset cousin, can fit central stylized facts
  1. High MPC
  2. Responses to expected windfalls
  3. Households with more volatile income save more
  4. Consumption tracks income over the life-cycle
- Advances in micro-data, numerical methods and computational power are leading to **new discoveries**
- **EGM is a powerful solution method** (and can be generalized)
- Realistic consumption-saving behavior can be included in **general equilibrium models** $\rightarrow$ welfare analysis with full distributional effects

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