



# Consumption-Saving Models

## An Introduction to Dynamic Programming

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# Introduction

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- **Why are consumption-saving models important?**
  1. Important topic in itself (70 percent of GDP)
  2. **Central aspect of many other decisions**
    - a) Labor supply, retirement, and fertility choices
    - b) Portfolio choices and asset pricing
    - c) Housing and location choices
  3. Households are the **cornerstone of general equilibrium models** designed to study the cause and effect of inequality
- **Dynamic programming** essential for recent advances
  1. Idiosyncratic and aggregate uncertainty
  2. Ex ante and ex post heterogeneity
  3. Internal and external optimization frictions  
(bounded rationality, adjustment costs etc.)

# Introduction

- **Part of mini-course on dynamic programming:**  
[ConsumptionSavingNotebooks/DynamicProgramming](#)
- **Focus in the partial equilibrium (PE) part of these slides:**  
Carroll (2020, QE), *Theoretical foundations of buffer stock saving*
- **Acknowledgments:** Christoffer Jessen Weissert, Emil Holst Partsch, Anders Yding, previous students in Dynamic Programming

# General references

- **Dynamic programming and computational methods in general:** Stokey and Lucas (1989), Judd (1998), Adda and Cooper (2003), Ljungqvist and Sargent (2004), Puterman (2009), Powell (2011), Bertsekas (2012), Schmedders and Judd (2013)
- **Surveys of consumption-saving literatures:** Browning and Lusardi (1996), Browning and Crossley (2001), Heathcote et al. (2009), Krusell and Smith (2006), Krueger et al. (2016), Pistaferri (2017), Kaplan and Violante (2018)
- **End-of-slides:** Many more references

1. Introduction
2. PIH
3. Buffer-stock
4. EGM
5. Further perspectives
6. Estimation
7. GE
8. Summary

**PIH**



# Permanent Income Hypothesis (PIH)

- Household problem

$$V_0(M_0, P_0) = \max_{\{C_t\}_{t=0}^T} \sum_{t=0}^T \beta^t \frac{C_t^{1-\rho}}{1-\rho}, \quad \beta < 1, \rho \geq 1$$

s.t.

$$A_t = M_t - C_t$$

$$B_{t+1} = R \cdot A_t, \quad R > 0$$

$$M_{t+1} = B_{t+1} + P_{t+1}$$

$$P_{t+1} = G \cdot P_t, \quad G > 0$$

$$A_T \geq 0$$

- Well-defined analytical solution, also for  $T \rightarrow \infty$  if

1. Return impatience (RI):  $(\beta R)^{1/\rho} / R < 1$
2. Finite human wealth (FWH):  $G/R < 1$

- What do you think is missing?



# The Intertemporal Budget Constraint (IBC)

- **Substitution** implies

$$\begin{aligned}A_T &= M_T - C_T = (RA_{T-1} + P_T) - C_T \\&= R(M_{T-1} - C_{T-1}) + P_T - C_T \\&= R^2 A_{T-2} + RP_{T-1} - RC_{T-1} + P_T - C_T \\&= R^{T+1} A_{-1} + \sum_{t=0}^T R^{T-t} (P_t - C_t)\end{aligned}$$

- Use **terminal condition**  $A_T = 0$  (why equality?)

$$R^{-T} A_T = 0 \Leftrightarrow B_0 + H_0 = \sum_{t=0}^T R^{-t} C_t$$

$$\text{where } H_0 \equiv \sum_{t=0}^T (G/R)^t P_0 = \frac{1 - (G/R)^{T+1}}{1 - G/R} P_0$$

# Static problem → Lagrangian

$$\mathcal{L} = \sum_{t=0}^T \beta^t \frac{C_t^{1-\rho}}{1-\rho} + \lambda \left[ \sum_{t=0}^T R^{-t} C_t - (B_0 + H_0) \right]$$

- **First order conditions**

$$\forall t : 0 = \beta^t C_t^{-\rho} - \lambda R^{-t}$$

- **Short-run Euler** equation:  $\frac{C_{t+1}}{C_t} = (\beta R)^{1/\rho}$
- **Long-run Euler** equation:  $\frac{C_t}{C_0} = (\beta R)^{t/\rho}$

# Consumption function

- Insert **Euler** into **IBC**

$$\sum_{t=0}^T R^{-t} (\beta R)^{t/\rho} C_0 = B_0 + H_0 \Leftrightarrow$$
$$C_0 \sum_{t=0}^T ((\beta R)^{1/\rho} / R)^t = B_0 + H_0$$

- **Solve** for  $C_0$

$$C_0 = \frac{1 - (\beta R)^{1/\rho} / R}{1 - ((\beta R)^{1/\rho} / R)^{T+1}} (B_0 + H_0)$$

- **MPC:**  $\frac{\partial C_0}{\partial B_0} \approx 1 - [(\beta R)^{1/\rho} / R] \approx 1 - R^{-1} \approx r$ , where  $R = 1 + r$
- **MPCP:**  $\frac{\partial C_0}{\partial P_0} \approx 1 - [(\beta R)^{1/\rho} / R] \frac{\partial H_0}{\partial P_0} \approx \frac{1 - 1/R}{1 - G/R} \approx 1$

- **Analytical expression** for the value function

$$\begin{aligned} V_0(M_0, P_0) &= \sum_{t=0}^T \beta^t u((\beta R)^{t/\rho} C_0) \\ &= \sum_{t=0}^T \beta^t (\beta R)^{(1-\rho)t/\rho} \frac{C_0^{1-\rho}}{1-\rho} \\ &= \sum_{t=0}^T ((\beta R)^{1/\rho}/R)^t \frac{C_0^{1-\rho}}{1-\rho} \\ &= \frac{1 - ((\beta R)^{1/\rho}/R)^{T+1}}{1 - (\beta R)^{1/\rho}/R} \frac{C_0^{1-\rho}}{1-\rho} \end{aligned}$$

- **Pro**

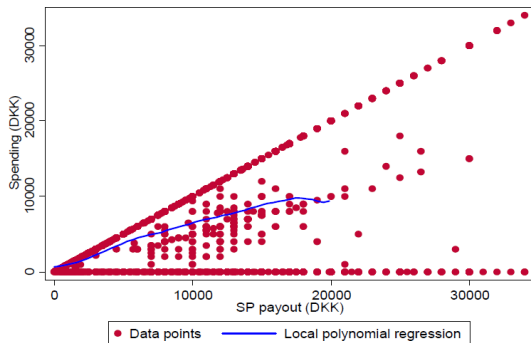
1. Micro-founded consumption-saving
  - Theoretically appealing (humans are intentional)
  - Empirically appealing (testable implications on micro-data)
2. Larger responses to permanent than to transitory shocks
3. Consumption smoothing - save for retirement (future low income)

- **Con**

1. Households seems to have a high MPC in the range 0.20-0.40
  - Survey studies (Kreiner et al., 2019)
  - Tax rebates studies (Johnson et al., 2006; Parker et al., 2013)
  - Lottery studies (Fagereng et al., 2020)
  - ARM payments studies (Di Maggio et al., 2017; Druedahl et al., 2020b)
2. Consumption responds to anticipated income changes
3. Households with more volatile income have larger savings
4. Consumption tracks income over the life-cycle
5. (Households are only boundedly rational)

# High MPC: Danish SP payout

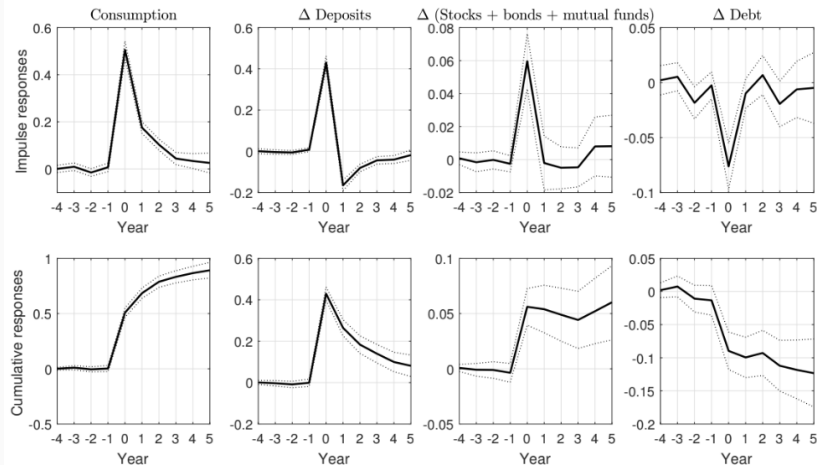
Figure 4: Spending and the size of the SP payout



NOTE: 5055 observations.

**Source:** Kreiner, Lassen og Leth-Petersen (AEJ:Pol, 2019)

# High MPC: Norwegian lottery winners



**Source:** Fagereng, Holm, Natvik (AEJ:Macro, 2020)

# Buffer-stock

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# Buffer-stock model (Deaton-Carroll)

+ borrowing constraints

+ income uncertainty

$$\begin{aligned}\Rightarrow \quad V_0(M_0, P_0) &= \max_{\{C_t\}_{t=0}^T} \mathbb{E}_0 \sum_{t=0}^T \beta^t \frac{C_t^{1-\rho}}{1-\rho} \\ &\text{s.t.} \\ A_t &= M_t - C_t \\ M_{t+1} &= RA_t + Y_{t+1} \\ Y_{t+1} &= \xi_{t+1} P_{t+1} \\ \xi_{t+1} &= \begin{cases} \mu & \text{with prob. } \pi \\ (\epsilon_{t+1} - \pi\mu)/(1-\pi) & \text{else} \end{cases} \\ &\epsilon_t \sim \exp \mathcal{N}(-0.5\sigma_\xi^2, \sigma_\xi^2) \\ P_{t+1} &= GP_t \psi_{t+1}, \quad \psi_t \sim \exp \mathcal{N}(-0.5\sigma_\psi^2, \sigma_\psi^2) \\ A_t &\geq -\lambda P_t \\ A_T &\geq 0\end{aligned}$$

**Note:** Later analytical results hold only for  $\mu = 0$  and  $\pi > 0$

# How to solve the model?

- **Borrowing constraints** → inequalities → high-dimensional **Kuhn-Tucker problem**
- **Uncertainty** → fully dynamic problem → no simple Lagrangian
- **No analytical solution with CRRA preferences**
  - Quadratic or CARA utility, which give some analytical results, have implausible properties

$$\text{CRRA: } u(c) = \frac{c^{1-\rho}}{1-\rho} \rightarrow \text{RRA} = \rho$$

$$\text{Quadratic: } u(c) = ac - \frac{b}{2}c^2 \rightarrow \text{RRA} = \frac{b}{a-bc}c$$

$$\text{CARA: } u(c) = \frac{1}{\alpha}e^{-\alpha c} \rightarrow \text{RRA} = \alpha c$$

where  $\text{RRA} = \text{relative risk aversion} = \frac{-u''(c)}{u'(c)}c$

- **Solution:** Bellman equation → numerical dynamic programming

# Bellman equation

$$V_t(M_t, P_t) = \max_{C_t} \frac{C_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t [V_{t+1}(M_{t+1}, P_{t+1})]$$

s.t.

$$A_t = M_t - C_t$$

$$M_{t+1} = RA_t + Y_{t+1}$$

$$Y_{t+1} = \xi_{t+1} P_{t+1}$$

$$\xi_{t+1} = \begin{cases} \mu & \text{with prob. } \pi \\ (\epsilon_{t+1} - \pi\mu)/(1 - \pi) & \text{else} \end{cases}$$

$$P_{t+1} = GP_t \psi_{t+1}$$

$$A_t \geq -\lambda P_t$$

$$A_T \geq 0$$

- Defining  $c_t \equiv C_t/P_t$ ,  $m_t \equiv M_t/P_t$  etc. implies

$$A_t = M_t - C_t \Leftrightarrow A_t/P_t = M_t/P_t - C_t/P_t$$

$$\Leftrightarrow a_t = m_t - c_t$$

$$M_{t+1} = RA_t + Y_{t+1} \Leftrightarrow M_{t+1}/P_{t+1} = RA_t/P_{t+1} + Y_{t+1}/P_{t+1}$$

$$\Leftrightarrow m_{t+1} = Ra_t P_t/P_{t+1} + \xi_{t+1}$$

$$\Leftrightarrow m_{t+1} = \frac{R}{G\psi_{t+1}} a_t + \xi_{t+1}$$

The **adjustment factor**  $\frac{1}{G\psi_{t+1}}$  is due to changes in permanent income

- Defining  $v_t(m_t) = V_t(M_t, P_t)/P_t^{1-\rho}$  finally implies

$$\begin{aligned} V_t(M_t, P_t) &= \max_{C_t} \frac{C_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t [V_{t+1}(M_{t+1}, P_{t+1})] \\ &= \max_{c_t} \frac{(c_t P_t)^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t [V_{t+1}(M_{t+1}, P_{t+1})] \Leftrightarrow \\ V_t(M_t, P_t)/P_t^{1-\rho} &= \max_{c_t} \frac{(c_t P_t)^{1-\rho}/P_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t [V_{t+1}(M_{t+1}, P_{t+1})/P_t^{1-\rho}] \Leftrightarrow \\ v_t(m_t) &= \max_{c_t} \frac{c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t [V_{t+1}(M_{t+1}, P_{t+1})/P_{t+1}^{1-\rho} \cdot P_{t+1}^{1-\rho}/P_t^{1-\rho}] \\ &= \max_{c_t} \frac{c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t [(G\psi_{t+1})^{1-\rho} v_{t+1}(m_{t+1})] \end{aligned}$$

# Bellman equation in ratio form

$$v_t(m_t) = \max_{c_t} \frac{c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t [(G\psi_{t+1})^{1-\rho} v_{t+1}(m_{t+1})]$$

s.t.

$$a_t = m_t - c_t$$

$$m_{t+1} = \frac{1}{G\psi_{t+1}} Ra_t + \xi_{t+1}$$

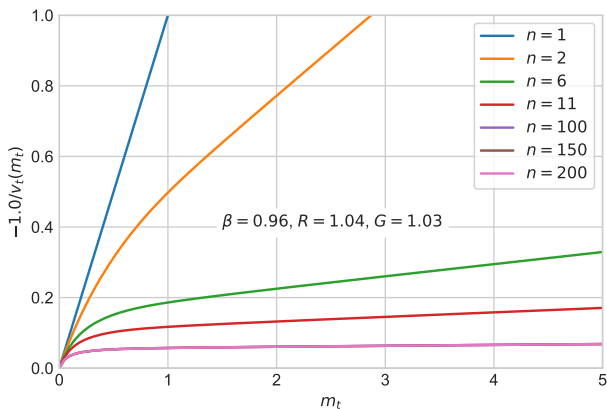
$$\xi_{t+1} = \begin{cases} \mu & \text{with prob. } \pi \\ (\epsilon_{t+1} - \pi\mu)/(1-\pi) & \text{else} \end{cases}$$

$$a_t \geq -\lambda$$

$$a_T \geq 0$$

- **Benefit:** Dimensionality of state space reduced  
**Can this always be done?**
- Easy to solve by **VFI**

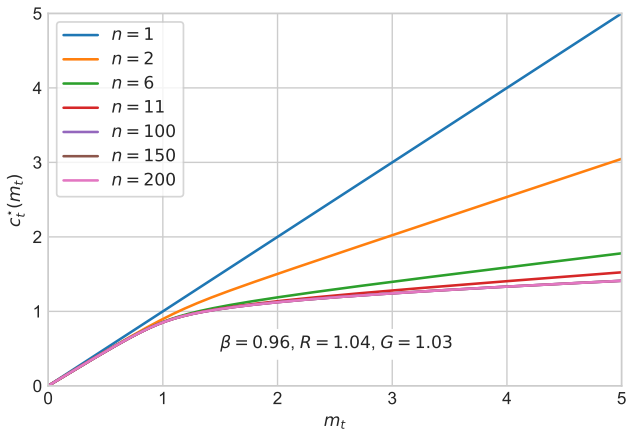
$T \rightarrow \infty$ ; **Convergence of  $-1.0/v_t(m_t) \rightarrow -1.0/v^*(m_t)$**



**Other parameters:**  $\rho = 2, \pi = 0.005, \mu = 0.0, \sigma_\psi = \sigma_\xi = 0.10$

**Note:**  $-1.0/v_t(m_t)$  is a numerically more stable object than  $v_t(m_t)$

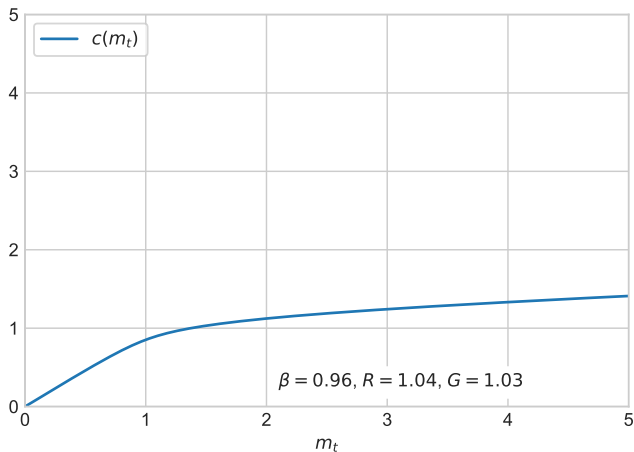
# $T \rightarrow \infty$ : Convergence of $c_t(m_t) \rightarrow c^*(m_t)$



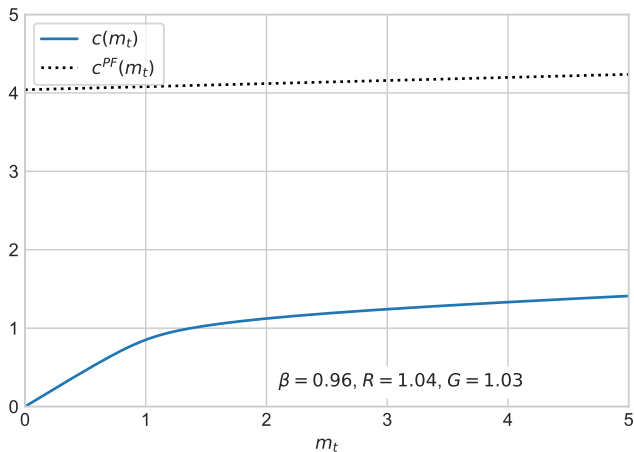
- What is the MPC?



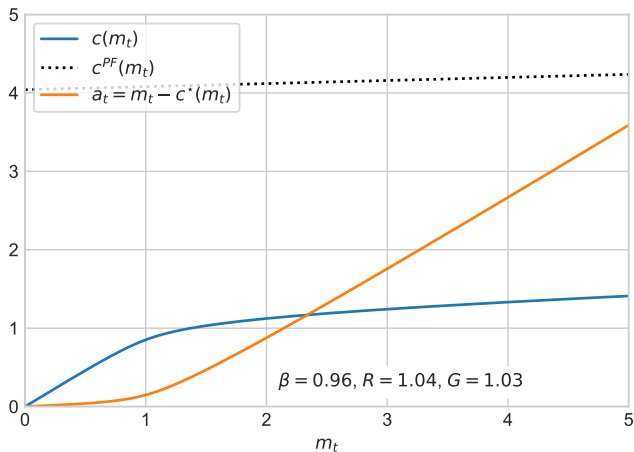
## $T \rightarrow \infty$ : The buffer-stock target



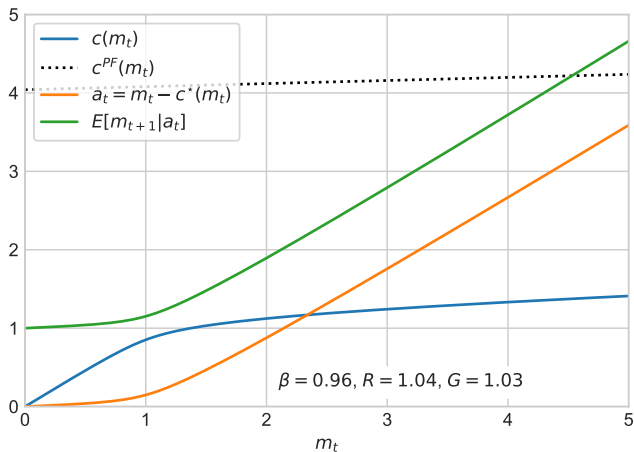
## $T \rightarrow \infty$ : The buffer-stock target



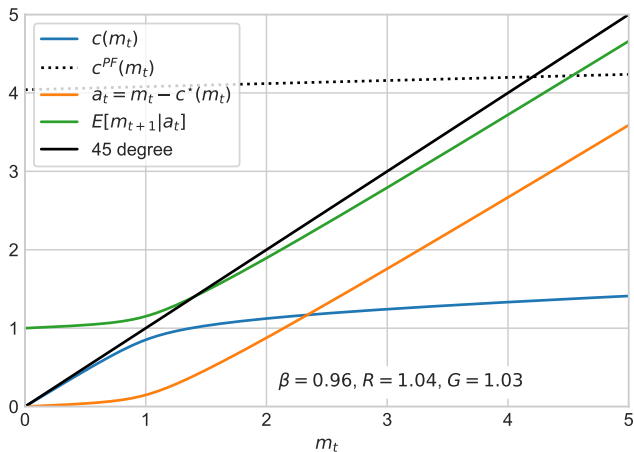
## $T \rightarrow \infty$ : The buffer-stock target



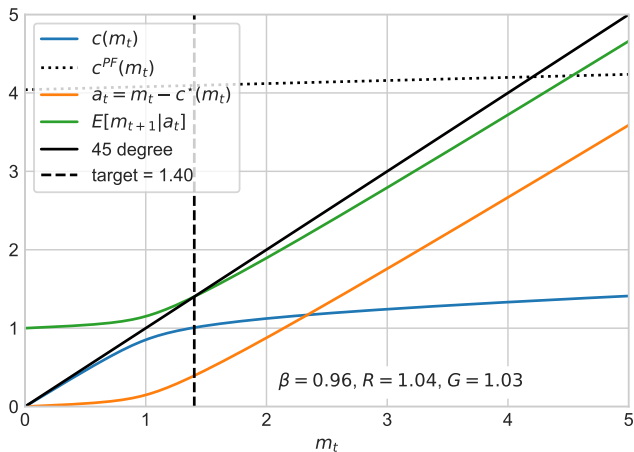
## $T \rightarrow \infty$ : The buffer-stock target



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# $T \rightarrow \infty$ : The buffer-stock target



## Simulation for $t \in \{0, 1, \dots, T - 1\}$

1. Choose  $m_0$  and set  $t = 0$
2. Calculate  $c_t = c^*(m_t)$
3. Calculate  $a_t = m_t - c_t$
4. Draw (pseudo-)random numbers

$$\epsilon_{t+1} \sim \exp \mathcal{N}(-0.5\sigma_\xi^2, \sigma_\xi^2)$$

$$\psi_{t+1} \sim \exp \mathcal{N}(-0.5\sigma_\psi^2, \sigma_\psi^2)$$

$$\eta_{t+1} \sim \mathcal{U}(0, 1)$$

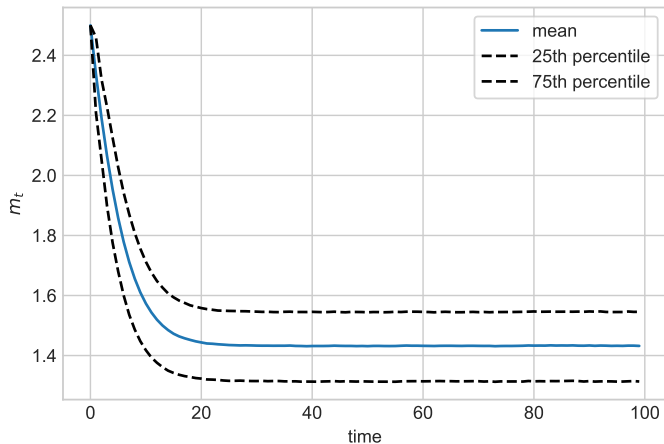
5. Calculate  $\xi_{t+1} = \begin{cases} \mu & \text{if } \eta_{t+1} < \pi \\ (\epsilon_{t+1} - \pi\mu)/(1 - \pi) & \text{else} \end{cases}$

6. Calculate  $m_{t+1} = \frac{R}{G\psi_{t+1}} a_t + \xi_{t+1}$

7. Set  $t = t + 1$

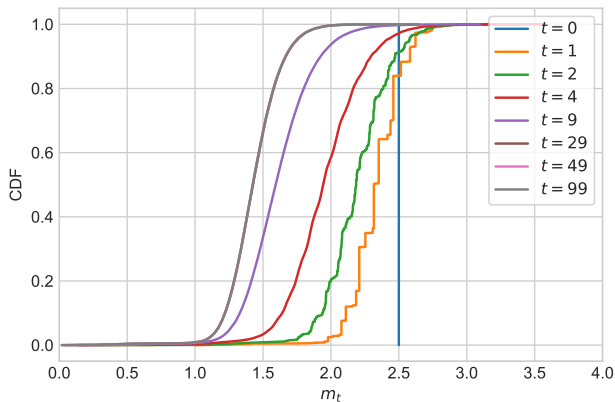
8. Stop if  $t \geq T$  else go to step 2

# Simulation: Avg. cash-on-hand





# Simulation: Distribution of cash-on-hand



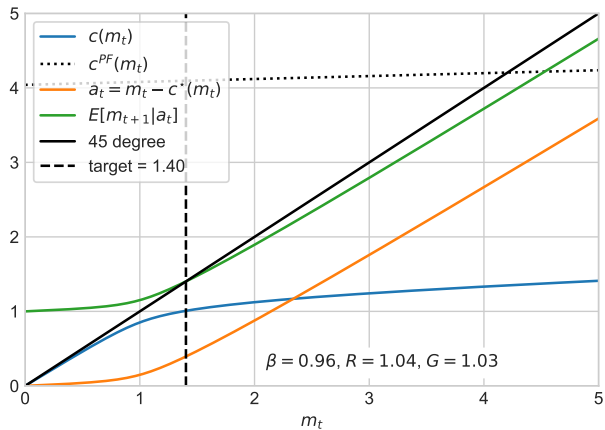
- **Proof of convergence:** Szeidl (2006)

# Buffer-stock

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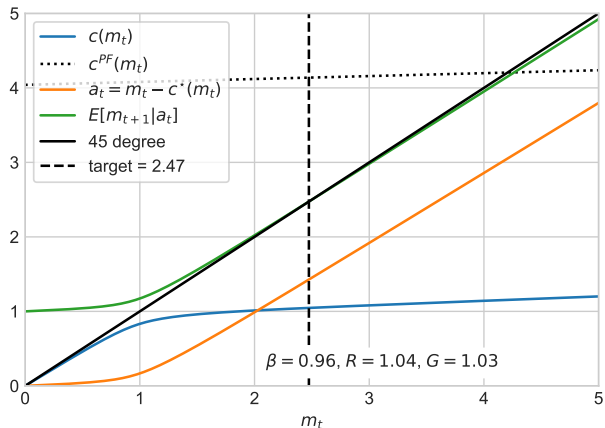
## Details

## Precautionary saving: $\sigma_\psi = 0.10$



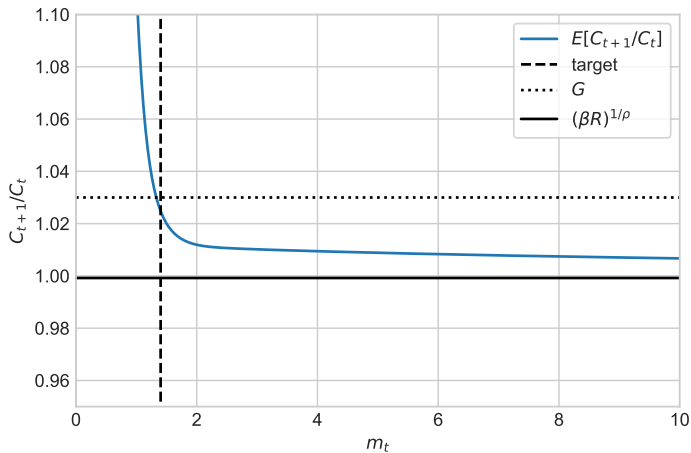
Target with baseline risk: 1.40

## Precautionary saving: $\sigma_\psi = 0.15$



Target with high risk: 2.47

# Consumption growth I



- Remember **Euler-equation**

$$C_t^{-\rho} = \beta R \mathbb{E}_t [C_{t+1}^{-\rho}] \text{ if no uncertainty } \Rightarrow C_{t+1}/C_t = (\beta R)^{1/\rho}$$

- Results**

1.  $C_{t+1}/C_t$  is declining in  $m_t$
2.  $\lim_{m_t \rightarrow \infty} C_{t+1}/C_t = (\beta R)^{1/\rho} = \text{RI}$
3.  $\lim_{m_t \rightarrow 0} C_{t+1}/C_t = \infty$
4.  $C_{t+1}/C_t < G$  at buffer-stock target

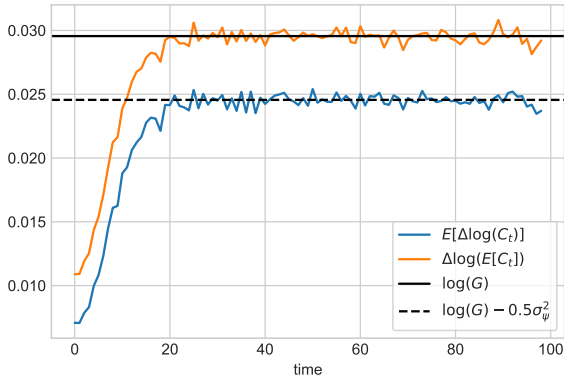
- Intuition** for  $C_{t+1}/C_t > (\beta R)^{1/\rho}$

1. Uncertainty  $\Rightarrow$  expected marginal utility  $\uparrow$  [ $C_{t+1}^{-\rho}$  is convex function]
2. Consumer must be lowered today,  $C_t \downarrow$
3. Consumption growth will increase,  $C_{t+1}/C_t \uparrow$

**Further:** *The above arguments are stronger for lower cash-on-hand relative to permanent income*

# Consumption growth III

1. Growth of average consumption =  $G$
2. Average consumption growth =  $G - 0.5\sigma_\psi^2$



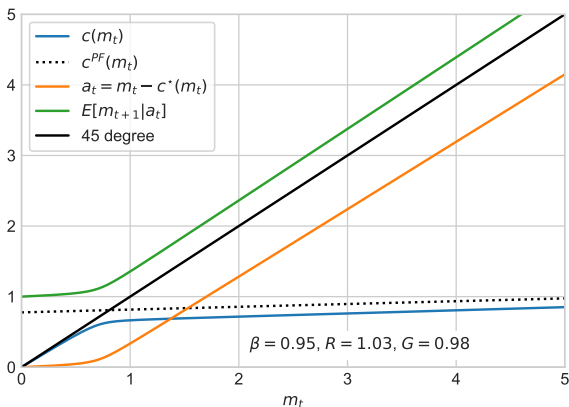
# Always a buffer-stock target? I

1. **Utility impatience (UI):**  $\beta < 1$
2. **Return impatience (RI):**  $(\beta R)^{1/\rho} / R < 1$
3. **Weak return impatience (WRI):**  $\pi^{1/\rho} (\beta R)^{1/\rho} / R < 1$
4. **Growth impatience (GI):**  $(\beta R)^{1/\rho} \mathbb{E}_t[\psi_{t+1}^{-1}] / G < 1$
5. **Absolute impatience (AI):**  $(\beta R)^{1/\rho} < 1$
6. **Finite value of autarky (FVA):**  $\beta \mathbb{E}_t[(G\psi_{t+1})^{1-\rho}] < 1$



# Always a buffer-stock target? II

- **GI ensures buffer-stock target**
- If not  $G/$  then infinite accumulation is possible like:



# Existence of solution

- **Existence of solution:** WRI + FVA
  - **Proof:** Use *Boyd's weighted contraction mapping theorem*
  - **Standard assumptions:** FHW, RI, GI
- The **consumption function** is twice continuously differentiable, **increasing** and **concave**

# The borrowing constraint

- Assume **perfect foresight** ( $\sigma_\psi = \sigma_\epsilon = \pi = 0$ ), but **no borrowing**,  $\lambda = 0$ .

- **Solution:** RI + FHW is still *sufficient* (with  $\lambda = \infty$  they are *necessary*)

- **Standard solutions:** RI + FHW

1. **GI**  $\Rightarrow$  *constraint will eventually be binding*

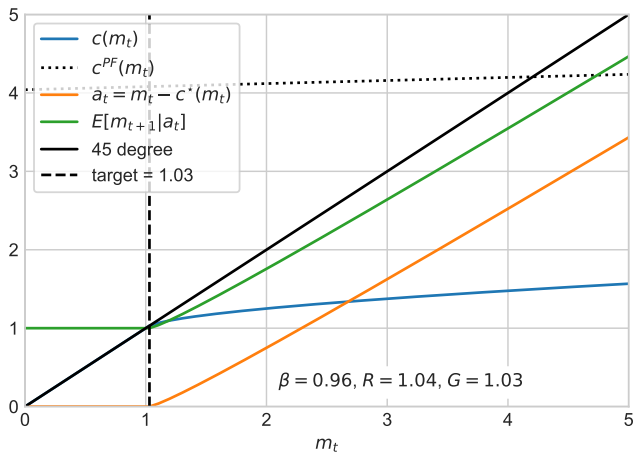
$c^*(m_t)$  converge to  $c^{PF}(m_t)$  from below as  $m_t \rightarrow \infty$

2. **Not GI**  $\Rightarrow$  *constraint is never reached*

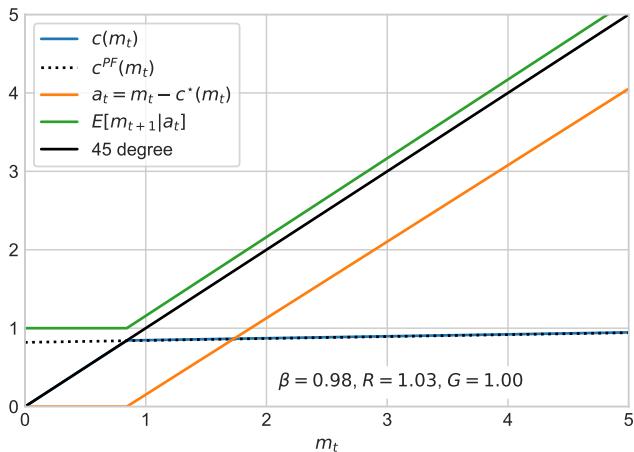
$$c^*(m_t) = c^{PF}(m_t) \text{ for } m_t \geq 1$$

- **Exotic solutions without FHW** exists (GI necessary)

# Perfect foresight with $\lambda = 0$ and GI



# Perfect foresight with $\lambda = 0$ , but not GI



# Buffer-stock

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## Life-cycle

# Adding a life-cycle (normalized)

$$v_t(m_t, z_t) = \max_{c_t} \frac{v(z_t)c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t [(GL_{t+1}\psi_{t+1})^{1-\rho} v_{t+1}(\bullet)]$$

s.t.

$$a_t = m_t - c_t$$

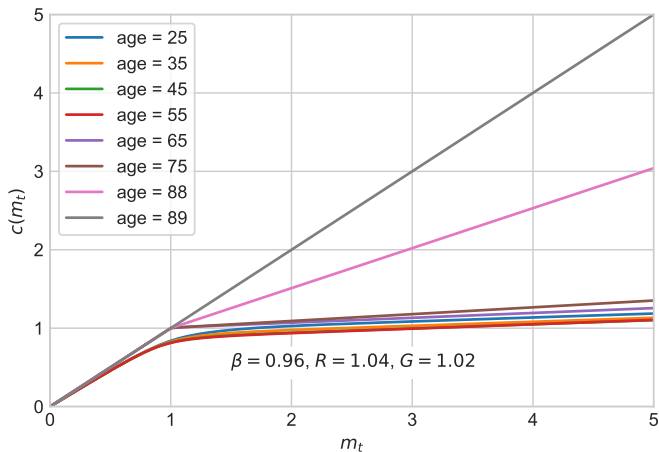
$$m_{t+1} = \frac{1}{GL_t\psi_{t+1}} Ra_t + \xi_{t+1}$$

$$\xi_{t+1} = \begin{cases} \mu & \text{with prob. } \pi \\ (\epsilon_{t+1} - \pi\mu)/(1-\pi) & \text{else} \end{cases}$$

$$a_t \geq \lambda_t = \begin{cases} -\lambda & \text{if } t < T_R \\ 0 & \text{if } t \geq T_R \end{cases}$$

- **Demographics:**  $z_t$  (exogenous). What could it be specifically?
- **Income profile:**  $P_{t+1} = GL_t P_t \psi_{t+1}$
- **No shocks in retirement:**  $\psi_t = \xi_t = 1$  if  $t > T_R$
- **Euler equation:**  $C_t^{-\rho} = \beta R \mathbb{E}_t [\frac{v(z_{t+1})}{v(z_t)} C_{t+1}^{-\rho}]$

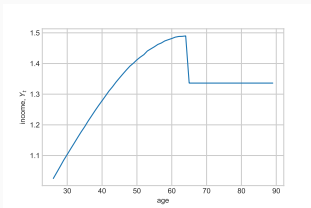
# Consumption functions ( $v(z_t) = 1$ )



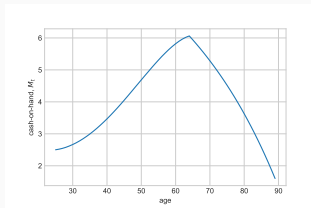


# Simulation: Life-cycle profiles ( $v(z_t) = 1$ )

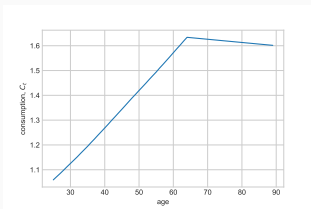
Income,  $Y_t$  (implied by  $G$  and  $L_t$ )



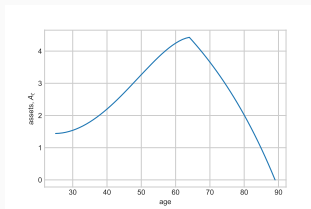
Cash-on-hand,  $M_t$



Consumption,  $C_t$



End-of-period assets,  $A_t$



What is the most unrealistic here?

**EGM**



# Euler-equation

- **Reference:** Carroll (2006)
- Assume for simplicity **no borrowing**:  $\lambda = 0$
- All optimal **interior choices** must satisfy

$$\begin{aligned}C_t^{-\rho} &= \beta R \mathbb{E}_t [C_{t+1}^{-\rho}] \Leftrightarrow \\c_t^{-\rho} &= \beta R \mathbb{E}_t [(G\psi_{t+1}c_{t+1})^{-\rho}]\end{aligned}$$

- Else optimal choice is **constrained**

$$\begin{aligned}C_t^{-\rho} &\geq \beta R \mathbb{E}_t [C_{t+1}^{-\rho}] \Leftrightarrow \\C_t &= M_t \Leftrightarrow \\c_t &= m_t\end{aligned}$$

# Endogenous grid method: Intuition

- **Obs.:** Given  $C_{t+1}^*(M_{t+1}, P_{t+1})$  and  $A_t$  and  $P_t$  we have

$$\begin{aligned}C_t^{-\rho} &= \beta R \mathbb{E}_t [(C_{t+1}^*(M_{t+1}, P_{t+1}))^{-\rho}] \Leftrightarrow \\C_t &= \mathbb{E}_t [\beta R (C_{t+1}^*(M_{t+1}, P_{t+1}))^{-\rho}]^{-\frac{1}{\rho}} \\&= \mathbb{E}_t [\beta R (C_{t+1}^*(RA_t + Y_{t+1}, P_{t+1}))^{-\rho}]^{-\frac{1}{\rho}} \\&= \mathbb{E}_t [\beta R (C_{t+1}^*(RA_t + P_t \psi_{t+1} \xi_{t+1}, P_t \psi_{t+1}))^{-\rho}]^{-\frac{1}{\rho}} \\&\equiv F(A_t, P_t)\end{aligned}$$

- **Endogenous grid:**  $A_t = M_t - C_t \Leftrightarrow M_t = C_t + A_t$
- **Conclusion:**  $(M_t, P_t, C_t)$  is a solution to the Bellman equation because it satisfies the Euler equation
- **Perspectives:** Varying  $A_t$  (and  $P_t$ ) we can map out the consumption function without using any numerical solver!
- **Borrowing constraint:** Binding below lowest generated  $M_t$

- **Prerequisites:**

1. Next-period **consumption function**:  $c_{t+1}^*(m_{t+1})$
2. **Asset grid**:  $\mathcal{G}_a = \{a_1, a_2, \dots, a_{\#}\}$  with  $a_1 = 10^{-6}$

- **Algorithm:** For each  $a_i \in \mathcal{G}_a$

1. Find consumption using Euler equation

$$c_i = \mathbb{E}_t \left[ \beta R \left( G\psi_{t+1} c_{t+1}^* \left( \frac{R}{G\psi_{t+1}} a_i + \xi_{t+1} \right) \right)^{-\rho} \right]^{-\frac{1}{\rho}}$$

2. Find endogenous state:  $a_i = m_i - c_i \Leftrightarrow m_i = a_i + c_i$

- The **consumption function**,  $c_t(m_t)$ , is given by interpolating

$$\{0, c_1, c_2, \dots, c_{\#}\} \text{ for } \{\underline{a}_t, m_1, m_2, \dots, m_{\#}\}$$

- *We can find all consumption functions in this way!*

## Addendum: The natural borrowing constraint ( $\lambda > 0$ )

- The **optimal end-of-period asset choice satisfies** the backwards recursion

$$a_t \geq \underline{a}_t = \begin{cases} 0 & \text{if } t \geq T_R \\ -\min\{\Lambda_t, \lambda_t\} GL_t \underline{\psi} & \text{if } t < T_R \end{cases}$$

where

$$\Lambda_t \equiv \begin{cases} R^{-1} GL_t \underline{\psi} \underline{\xi} & \text{if } t = T_R - 1 \\ R^{-1} [\min\{\Lambda_{t+1}, \lambda_t\} + \underline{\xi}] GL_t \underline{\psi} & \text{if } t < T - 1 \end{cases}$$

and  $\underline{\psi}$  and  $\underline{\xi}$  are the minimum realizations of  $\psi_{t+1}$  and  $\xi_{t+1}$

- **Proof:** Can be shown as a consequence of the household wanting to avoid  $c_t = 0$  at *any cost* because  $\lim_{c_t \rightarrow 0} u'(c_t) = \infty$ .

## Further perspectives

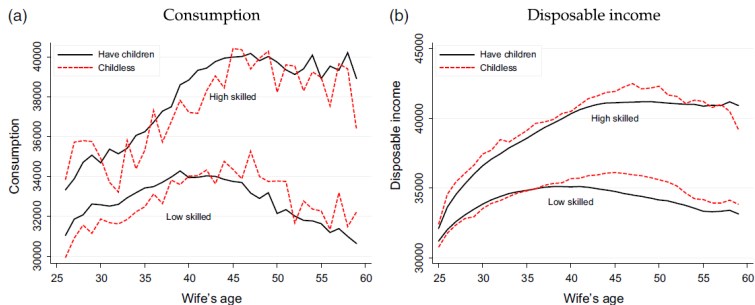
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# Three generations of models

- **1st:** *Permanent income hypothesis* (Friedman, 1957)  
or *life-cycle model* (Modigliani and Brumberg, 1954)
- **2nd:** *Buffer-stock consumption model*  
(Deaton, 1991, 1992; Carroll, 1992, 1997, 2020)
- **3rd:** *Multiple-asset buffer-stock consumption models*  
(e.g. Kaplan and Violante (2014))

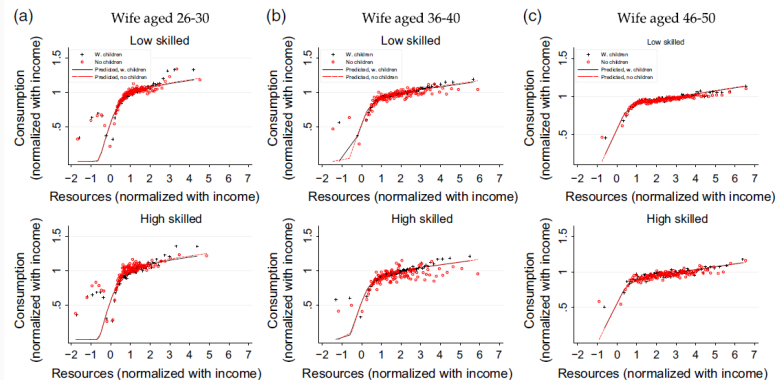


# Denmark: Life-cycle profiles fit



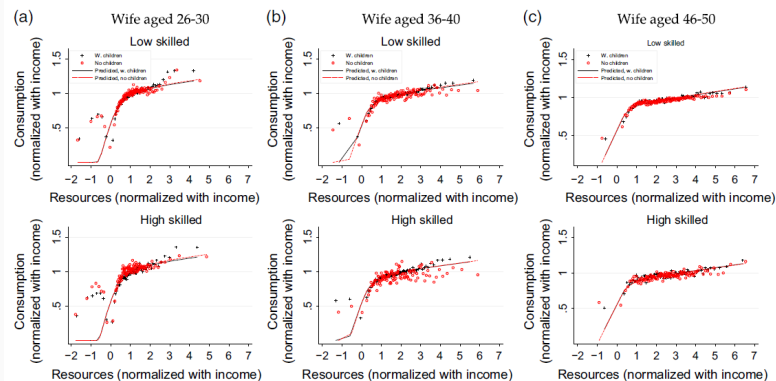
Source: Jørgensen (2017)

# Denmark: Consumption function fit



Source: Jørgensen (2017)

# Denmark: Consumption function fit



Source: Jørgensen (2017)

# Level of wealth and long-run dynamics I

- **Best test of a life-cycle consumption-saving model:**

*A sudden, sizable and salient shock to wealth*

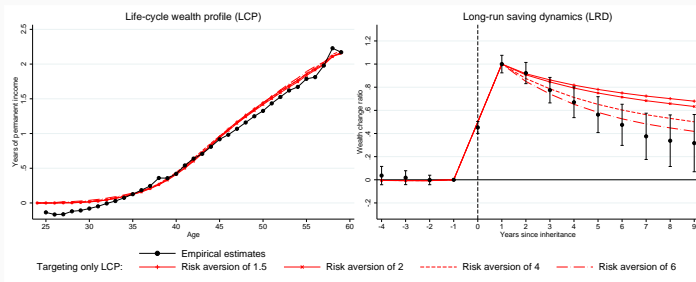
+ long panel to observe how the extra wealth is spend

- **My own research:** Druedahl and Martinello (2018)

Compare individuals in the Danish register data who

1. Receive a similar inheritance, but at different points in time
2. From parents dying due to heart attacks or car crashes

# Level of wealth and long-run dynamics II



- **Net worth:** Good fit for different levels of risk-aversion ( $\rho$ ) when re-calibrating patience ( $\beta$ )
- **Also dynamics:** Good fit only if risk-aversion ( $\rho$ ) is high

- **Central property of buffer-stock model:**
  1. **Non-constrained:** Increase consumption today when obtaining information of higher income tomorrow
  2. **Constrained:** Increase consumption when cash flow arrives
- **How to test test this?**
- **A paper of mine:** The Intertemporal Marginal Propensity to Consume out of Future Persistent Cash-Flows
  1. **Data:** Account data for all Nykredit customers
  2. **Experiment:** Letter with bank's expectations for interest rate in next mortgage auction 2-3 months ahead
  3. **Result:** Observed behavior can be rationalized in a simple buffer-stock consumption model
  4. **Code:** [GitHub](#)

# Level of wealth and MPC

- Consumption-saving models a few years ago **could not endogenously fit** both
  1. The level of wealth observed
  2. The high MPCs found in quasi experiments
- **Three solutions:**
  1. Exogenous **hands-too-mouth households**  
(Campbell and Mankiw, 1990)
  2. **Preference heterogeneity**
  3. **Wealthy hands-to-mouth** (Kaplan and Violante, 2014)  
*Many households hold mostly illiquid assets with a high return*  
→ *consumption adjust in response to small income shock*

# Kaplan-Violante model (two-asset model)

$$V_t(M_t, N_t, P_t) = \max_{B_t, C_t} u(C_t, B_t) + \beta \mathbb{E}_t[V_{t+1}(M_{t+1}, N_{t+1}, P_{t+1})]$$

s.t.

$$A_t = M_t - C_t + (N_t - B_t) - 1\{N_t \neq B_t\}\omega$$

$$M_{t+1} = R + P_{t+1}\xi_{t+1}$$

$$N_{t+1} = R_b B_t$$

$$P_{t+1} = P_t \psi_{t+1}$$

$$A_t \geq -\lambda P_t.$$

- **Cost of liquidation:**  $\omega$
- **Illiquid assets give higher return:**  $R_b > R$  (+ potentially utility)



# Kaplan-Violante model (two-asset model)

$$V_t(M_t, N_t, P_t) = \max \left\{ v_t^{keep}(M_t, N_t, P_t), v_t^{adj.}(M_t + N_t - \lambda, P_t) \right\}$$

$$v_t^{keep}(M_t, N_t, P_t) = \max_{C_t} u(C_t, B_t) + \beta W_t(A_t, B_t, P_t) \text{ s.t.}$$

$$A_t = M_t - C_t$$

$$B_t = N_t$$

$$A_t \geq -\omega P_t.$$

$$\tilde{v}_t^{adj.}(X_t, P_t) = \max_{B_t, C_t} u(C_t, B_t) + \beta W_t(A_t, B_t, P_t) \text{ s.t.}$$

$$M_t = X_t - B_t$$

$$A_t = M_t - C_t$$

$$A_t \geq -\omega P_t.$$

$$W_t(A_t, B_t, P_t) = \mathbb{E}_t[V_t(RA_t + P_t\psi_{t+1}\xi_{t+1}, R_b B_t, P_t\psi_{t+1})]$$

- **Durable consumption:** Berger and Vavra (2015), Harmenberg and Öberg (2020)
- **Labor supply, retirement and family formation:** Low et al. (2010), French and Jones (2011), Keane and Wasi (2016), Adda et al. (2016), Blundell et al. (2016)
- **Non-Gaussian income uncertainty:** Guvenen et al. (2019), De Nardi et al. (2020), Druedahl and Munk-Nielsen (2020)
- **Housing:** Landvoigt (2017), Kaplan et al. (2019)
- **Imperfect information and bounded rationality:** Pagel (2017). Carroll et al. (2019), Moran and Kovacs (2019), Druedahl and Jørgensen (2020)
- **Level and dynamics of inequality** – circumstances or behavior? De Nardi and Fella (2017), Hubmer et al. (2020)

- **EGM in non-convex multi-dimensional models:** Druedahl and Jørgensen (2017) and Druedahl (2020)
- **Sparse grids:** Judd et al. (2014), Brumm and Scheidegger (2017)
- **Machine learning:** Azinovic et al. (2019), Maliar et al. (2019)

# Estimation

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- Critic of structural estimation: **Requires many assumptions**
- **But:** To turn reduced form parameter estimates into policy advice *a lot of assumptions are often implicitly required*

*»All econometric work relies heavily on a priori assumptions. The main difference between structural and experimental (or “atheoretic”) approaches is not in the number of assumptions but the extent to which they are made explicit.« (Keane, 2012)*

- **The beauty of models:**
  1. Ensure *consistent* world view
  2. Allow us to combine *heterogenous facts* and extrapolate from a myriad of past experiences
  3. Better models are clearly defined – even if we never find *the* true model we can make *progress*
- **Frontier:** Combine the two and use exogenous variation to estimate structural model (Nakamura and Steinsson, 2018)

- **The Lucas critique:** *Behavioral rules change with policy*
  - ⇒ policy advice can not rely on estimated behavioral rules
  - ⇒ we need to estimate *structural parameters*

*»Invariance of parameters in an economic model is not, of course, a property which can be assured in advance, but it seems reasonable to hope that neither tastes nor technology vary systematically with variations in counter-cyclical policies.« (Lucas, 1977)*

- **Other stuff might be approximately invariant**
- **Rigorous microfoundations:**
  1. **Mathematically:** Based on (boundedly) rational behavior derived as a solution to a formal optimization problem
  2. **Economically:** The assumptions are realistic

1. **Focus:** Closely related estimators *indirectly* using **micro-data**

Simulated Method of Moments (**SMM**) (McFadden, 1989)

Simulated Minimum Distance (**SMD**) (Duffie and Singleton, 1990)

Indirect Inference (**II**) (Gouriéroux and Monfort, 1997)

## **Main alternative:**

Simulated Maximum Likelihood (**SML**) *directly* using **micro-data**  
(see e.g. Adda and Cooper (2003) or Druedahl et al. (2018))

2. **Examples:** Gourinchas and Parker (2002), Cagetti (2003), Guvenen and Smith (2014), Druedahl and Jørgensen (2020)
3. **Extended toolbox:** Jørgensen (2020) and Honore et al. (2020)

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# Heterogenous Agent (HA) models

## 1. **Stationary equilibrium:**

*Deterministic steady state and transition path*

**Foundational papers:** Bewley (1986), Imrohoroglu (1989), Huggett (1993), Aiyagari (1994)

**A few policy examples:** Aiyagari and McGrattan (1998), Conesa et al. (2009), Heathcote et al. (2014)

## 2. **Dynamic/recursive/sequential equilibrium:**

*Aggregate shocks and stochastic dynamics*

**Foundational papers:** Krusell and Smith (1997, 1998), Carroll (2000), Carroll et al. (2015)

## 3. **Reviews:** Heathcote et al. (2009), Krusell and Smith (2006), Krueger et al. (2016)

# Heterogenous Agent New Keynesian (HANK) models

1. **Frontier:** Kaplan et al. (2018), Bayer et al. (2019), Hagedorn et al. (2019b), Alves et al. (2020), Auclert et al. (2020c), Fernandez-Villaverde et al. (2020)
2. **Analytical:** Bilbiie (2008, 2019a,b), Werning (2015), Challe et al. (2017), Acharya and Dogra (2020), Bilbiie et al. (2020), Debortoli and Galí (2018), Auclert et al. (2018), Broer et al. (2020), Ravn and Sterk (2020), Auclert and Rognlie (2020)
3. **Others:** Oh and Reis (2012), Gornemann et al. (2016), McKay and Reis (2016), McKay et al. (2016), Guerrieri and Lorenzoni (2017), Ravn and Sterk (2017), Den Haan et al. (2018), Luetticke (2020)
4. **Empirical:** Cloyne et al. (2020), Slacalek et al. (2020), Holm and Paul (2020), Wolf (2020)
5. **Reviews:** Kaplan and Violante (2018)

- **Early reviews:** Den Haan et al. (2010), Schmedders and Judd (2013)
- **Continuous time:** Achdou et al. (2020) ([code](#)), Ahn et al. (2018) ([code](#))
- **Local aggregate solution:**
  1. *State space:* Bayer and Luetticke (2020) ([MATLAB](#), [Python](#))
  2. *Sequence space:* Boppart et al. (2018), Auclert et al. (2020c) ([code](#))
- **Global aggregate solution:** Kubler and Scheidegger (2018), Azinovic et al. (2019), Scheidegger and Bilonis (2019), Pröhl (2019) ([code](#)), Maliar et al. (2019) ([code](#), [video](#)), Fernandez-Villaverde et al. (2020) ([code](#))

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**Stationary equilibrium**

# The Aiyagari model

- **Households:** Continuum of measure 1 who
  1. Own stocks,  $a_{t-1}$  (end-of-period)
  2. Supply labor with productivity  $e_t$   
(exogenous and stochastic, mean one)
  3. Consume,  $c_t$
- **Firms:** Rent capital and hire labor to produce
- **Capital:**
  1. Predetermined:  $Y_t = F(Z_t, K_{t-1}, L_t)$ , where  
 $Z_t$  is technology,  $K_{t-1}$  is capital, and  $L_t$  is labor
  2. Depreciates with rate  $\delta$
- **Prices** are taken as given by households and firms
  1.  $r_t^k$ , rental rate
  2.  $r_t = r_t^k - \delta$ , interest rate
  3.  $w_t$ , wage rate

- **Production function:**  $Y_t = Z_t K_{t-1}^\alpha L_t^{1-\alpha}$
- **Define**  $k_{t-1} \equiv K_{t-1}/L_t$
- **Standard pricing equations:**

$$r_t^k = \alpha Z_t k_{t-1}^{\alpha-1}$$
$$w_t = (1 - \alpha) Z_t k_{t-1}^\alpha$$

- **Useful implications:**

$$k_{t-1} = \left( \frac{r_t + \delta}{\alpha Z_t} \right)^{\frac{1}{\alpha-1}} \equiv k(r_t, Z_t)$$
$$r_t = \alpha Z_t k_{t-1}^{\alpha-1} \equiv r(k_{t-1}, Z_t)$$
$$w_t = (1 - \alpha) Z_t \left( \frac{r_t + \delta}{\alpha Z_t} \right)^{\frac{\alpha}{\alpha-1}} \equiv w(r_t, Z_t)$$

- **Perfect foresight:** Price sequence known,  $\{r_t, w_t\}_{t \geq 0}$
- **Households solve:**

$$\begin{aligned}v_t(e_t, a_{t-1}) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t [v_{t+1}(e_{t+1}, a_t)] \\ \text{s.t.} \\ a_t + c_t &= (1 + r_t)a_{t-1} + w_t e_t \\ a_t &\geq 0\end{aligned}$$

- **Value function:** We could write

$$v_t(e_t, a_{t-1}) = \mathcal{V}(e_t, a_{t-1}, \{r_k, w_k\}_{k \geq t})$$

where  $\mathcal{V}$  does not have a time subscript, but where the price sequences are state variables

- **Optimal saving and consumption:**  $a_t^*(e_t, a_{t-1})$  and  $c_t^*(e_t, a_{t-1})$

# Supply of capital

- **Distribution:**  $D_t$  over  $e_t$  and  $a_{t-1}$
- **Supply of capital:**  $\mathcal{K}_t = \int a_t^*(e_t, a_{t-1}) dD_t = \int a_t dD_{t+1}$
- **Details:**
  - **Formulation I:**  $\int a_t^*(e_t, a_{t-1}) dD_t$  is an integral over  $e_t$  and  $a_{t-1}$  applying the optimal saving function in period  $t$ , i.e.  $a_t^*(e_t, a_{t-1})$ , thus summing up savings at the end-of-period  $t$
  - **Formulation II:**  $\int a_t dD_{t+1}$  is an integral over  $e_{t+1}$  and  $a_t$  directly summing up savings at the end-of-period  $t$
  - **Equivalence:** The two formulations gives the same result because  $D_{t+1}$  is generated from  $D_t$  assuming saving according to  $a_t^*(e_t, a_{t-1})$  (and the exogenous process for  $e_t$ )



- **Market clearing** requires

$$\text{Capital: } K_t = \mathcal{K}_t = \int a_t dD_{t+1} = \int a_t^*(e_t, a_{t-1}) dD_t$$

$$\text{Labour: } L_t = \int e_t dD_t = 1$$

$$\text{Goods: } Y_t = \int c_t^*(e_t, a_{t-1}) dD_t + \delta K_{t-1}$$

- The **labor market clears trivially**, while we can leave out the **goods market** due to **Walras's Law**

# Solve household problem by EGM

- **Grids:**

1.  $e_t \in \{e^1, \dots, e^{\#_e}\}$  (discretized with Tauchen and Hussey (1991))
2.  $a_t \in \{a^1, \dots, a^{\#_a}\}$

- **Guess:**  $v_{a,t+1}(e^i, a^j), \forall i, j$

- **Time iteration:**

1. Calculate:  $q_t(e^i, a^j) = \sum_{k=1}^{\#_e} \Pr[e^k | e^i] v_{a,t+1}(e^i, a^j)$
2. Calculate  $\tilde{c}^{ij} = q_t(e^i, a^j)^{-\frac{1}{\sigma}}$  and  $\tilde{m}^{ij} = \tilde{c}^{ij} + a^j$
3. Interpolate  $\{\tilde{m}^{ij}, a^j\}_{j=1}^{\#_a}$  at  $m^j = (1 + r_t)a^j + w_t e^i$  to find  $a^*(e^i, a^j)$
4. Calculate  $c^*(e^i, a^j) = m_t - a^*(e^i, a^j)$
5. Calculate  $v_{a,t+1}(e^i, a^j) = (1 + r)c^*(e^i, a^j)^{-\sigma}$   
(use of the envelope theorem)

- **Note:** Any other *solution* method could have been used

# Simulate household behavior on grid

- **Initial distribution:**  $D_0(e^i, a^j) = \frac{\Pr[e^i]}{\#_a}$  (*ergodic in  $e$ , uniform in  $a$* )
- **Idea:** Re-distribute mass to grid points, where optimal decisions lie
- **Update:** Calculate  $D_{t+1}(e^k, a^l)$  as

$$\sum_{i=1}^{\#_e} \Pr[e^k | e^i] \sum_{j=1}^{\#_a} D_t(e^i, a^j) \omega(a^*(e^i, a^j), a^{\max\{l-1, 1\}}, a^l, a^{\min\{l+1, \#_a\}})$$

where

$$\omega(a, \underline{a}, \tilde{a}, \bar{a}) = 1\{a \in [\underline{a}, \bar{a}]\} \begin{cases} \frac{\bar{a}-a}{\bar{a}-\tilde{a}} & \text{if } a \geq \tilde{a} \\ \frac{a-\underline{a}}{\tilde{a}-\underline{a}} & \text{if } a < \tilde{a} \end{cases}$$

- **Weight:**  $\omega$  is a weight on each grid point calculated using linear interpolation
- **Note:** Any other *simulation* method could have been used

## Definition: Stationary equilibrium

A **stationary equilibrium** for a given  $Z_{ss}$  is one where

1. Quantities  $K_{ss}$  and  $L_{ss}$ ,
2. prices  $r_{ss}$  and  $w_{ss}$ ,
3. a distribution  $D_{ss}$  over  $e_t$  and  $a_{t-1}$
4. and policy functions  $a_{ss}^*(e_t, a_{t-1})$  and  $c_{ss}^*(e_t, a_{t-1})$

are such that

1.  $a_{ss}^*(\bullet)$  and  $c_{ss}^*(\bullet)$  solves the household problem with  $\{r_{ss}, w_{ss}\}_{k \geq t}$
2.  $D_{ss}$  is the invariant distribution implied by the household problem
3. Firms maximize profits,  $r_{ss} = r(K_{ss}/L_{ss}, Z_{ss})$  and  $w_{ss} = w(r_{ss}, Z_{ss})$
4. The labor market clears, i.e.  $L_{ss} = \int e_t dD_{ss} = 1$
5. The capital market clears, i.e.  $K_{ss} = \int a_{ss}^*(e_t, a_{t-1}) dD_{ss}$
6. The goods market clears, i.e.  $Y_{ss} = \int c_{ss}^*(e_t, a_{t-1}) dD_{ss} + \delta K_{ss}$

# Finding stationary equilibrium

1. Guess on  $r_{ss}$
2. Calculate  $w_{ss} = w(r_{ss}, Z_{ss})$
3. Solve the infinite horizon household problem
4. Simulate until convergence of  $D_{ss}$
5. Calculate supply  $\mathcal{K}_{ss} = \int a_{ss}^*(e_t, a_{t-1}) dD_{ss}$
6. Calculate demand  $K_{ss} = k(r_{ss}, Z_{ss})L_{ss}$
7. If for some tolerance  $\epsilon$

$$|\mathcal{K}_{ss} - K_{ss}| < \epsilon$$

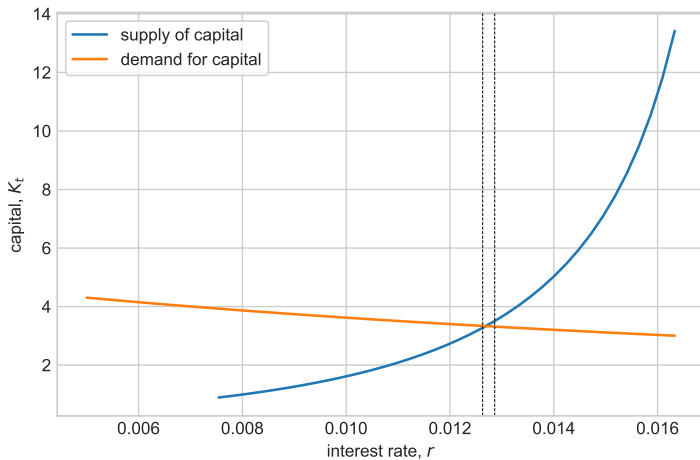
then stop, otherwise update  $r_{ss}$  appropriately and return to step 2

This is just a **root-finding problem**

# Equilibrium interest rate

**Step 1:** Perform grid search

**Step 2:** Use standard root finder



# Precautionary saving and the interest rate

- **Baseline:**  $\sigma_e = 0.1$ 
  1. Interest rate:  $r^* = 0.0127$
  2. Capital-output ratio: 2.92
- **Higher risk:**  $\sigma_e = 0.2$ 
  1. Interest rate:  $r^* = 0.0029$
  2. Capital-output ratio: 3.95
- **Intuition:** Saving motive  $\uparrow$ , marginal product of capital  $\downarrow$
- **Implication:** Important for the »natural« interest rate!  
**Example:** On Secular Stagnation in the Industrialized World

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**Transition path**



## Definition: Transition path (to MIT shock)

A **transition path** for  $t \in \{0, 1, 2, \dots\}$ , given an initial distribution  $D_0$  and a path of  $Z_t$ , is paths of quantities  $K_t$  and  $L_t$ , prices  $r_t$  and  $w_t$ , policy functions  $a_t^*(\bullet)$  and  $c_t^*(\bullet)$ , distributions  $D_t$ , such that for all  $t$

1.  $a_t^*(\bullet)$  and  $c_t^*(\bullet)$  solve the household problem given price paths
2.  $D_t$  are implied by the household problem given price paths and  $D_0$
3. Firms maximizes profit,  $r_t = r(K_{t-1}/L_t, Z_t)$  and  $w_t = w(r_t, Z_t)$
4. The labor market clears, i.e.  $L_t = \int e_t dD_t = 1$
5. The capital market clears, i.e.  $K_{t-1} = \int a_{t-1} dD_t$
6. The goods market clears, i.e.  $Y_t = \int c_t^*(\bullet) dD_t + \delta K_{t-1}$

**MIT shock**  $\equiv$  »Shock in a world without shocks«

# Solving the house problem along the transition path

1. **Assumption:** Back at stationary equilibrium after  $\mathcal{T}$  periods
2. **Consequence:**  $v_{\mathcal{T}+1}(e_t, a_{t-1}) = v_{ss}(e_t, a_{t-1})$
3. **Perfect foresight price paths:**

$$\{r_k, w_k\}_{k \geq \mathcal{T}} = \{r_{\mathcal{T}}, w_{\mathcal{T}}, r_{ss}, w_{ss} \dots\}$$

$$\{r_k, w_k\}_{k \geq \mathcal{T}-1} = \{r_{\mathcal{T}-1}, w_{\mathcal{T}-1}, r_{\mathcal{T}}, w_{\mathcal{T}}, r_{ss}, w_{ss} \dots\}$$

$\vdots$

$$\{r_k, w_k\}_{k \geq 0} = \{r_0, w_0, \dots, r_{\mathcal{T}}, w_{\mathcal{T}}, r_{ss}, w_{ss} \dots\}$$

# Relaxation-method

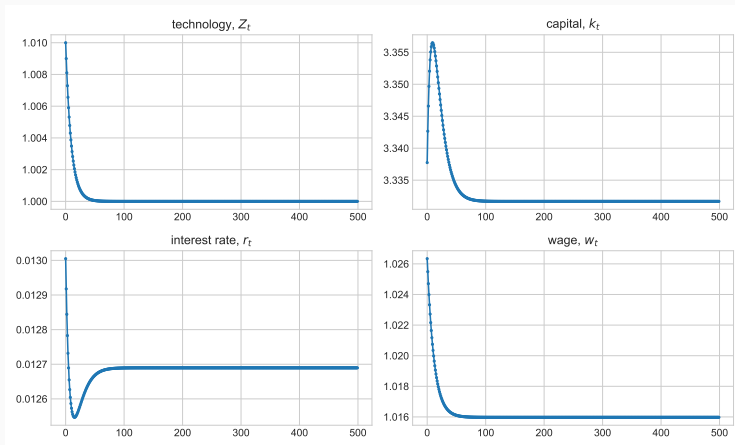
1. Guess on  $\{r_t\}_{t=0}^T = \{r_{ss}\}_{t=0}^T$  (or something else)
2. Calculate  $\{w_t\}_{t=0}^T = \{w(r_t, Z_t)\}_{t=0}^T$
3. Solve the household problem backwards along the transition path
4. Simulate households forward along the transition path
5. Calculate  $\{k_t\}_{t=0}^T = \{\int a_t^*(\bullet) dD_{t-1}\}_{t=0}^T$
6. Calculate  $\{r'_t\}_{t=0}^T = \{r(k_{t-1}, Z_t)\}_{t=0}^T$
7. Stop if for some tolerance  $\epsilon$

$$\max_{t \in \{0, 1, 2, \dots, T\}} |r_t - r'_t| < \epsilon$$

otherwise return to step 2 with  $\{r_t\}_{t=0}^T = \{\nu r_t + (1 - \nu) r'_t\}_{t=0}^T$

**Note:** Typically the relaxation parameter is  $\nu = 0.90$  (Kirkby, 2017)

# Transition path



**Remember:** We also have a transition path for  $D_t$   
→ we can calculate distribution of utility effects!

- **Reformulation:**  $K_t$  given by initial distribution and price sequence

$$\begin{aligned} K_t &= \mathcal{K}_t(\{r_s, w_s\}_{s \geq 0}, D_0)) \\ &= \int a_t^*(e_t, a_{t-1}) dD_t \end{aligned}$$

- **Define:**  $\mathbf{K} = (K_0, K_1, \dots)$  and  $\mathbf{Z} = (Z_0, Z_1, \dots)$
- **Transition path** is, for given  $\mathbf{Z}$ , solution for  $t = 0, 1, \dots$ , to

$$H_t(\mathbf{K}, \mathbf{Z}, D_0) \equiv \mathcal{K}_t(\{r(Z_s, K_{s-1}), w(Z_s, K_{s-1})\}_{s \geq 0}, D_0) - K_t = 0$$

or in **time-stacked form:**

$$\mathbf{H}(\mathbf{K}, \mathbf{Z}, D_0) = \mathbf{0}$$

- **Standard:**  $D_0 = D_{ss}$

**Total differentiation** implies

$$\mathbf{H}_K d\mathbf{K} + \mathbf{H}_Z d\mathbf{Z} = 0 \Leftrightarrow d\mathbf{K} = -\mathbf{H}_K^{-1} \mathbf{H}_Z d\mathbf{Z}$$

where

$$\mathbf{H}_K = \begin{bmatrix} \frac{\partial H_0}{\partial K_0} & \frac{\partial H_0}{\partial K_1} & \dots \\ \frac{\partial H_1}{\partial K_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}, \mathbf{H}_Z = \begin{bmatrix} \frac{\partial H_0}{\partial Z_0} & \frac{\partial H_0}{\partial Z_1} & \dots \\ \frac{\partial H_1}{\partial Z_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}$$

**Decomposition:**  $H_t = \mathcal{K}_t(\{r(Z_s, K_{s-1}), w(Z_s, K_{s-1})\}_{s \geq 0}, D_0) - K_t$

$$\mathbf{H}_K = \mathcal{J}^{\mathcal{K},r} \mathcal{J}^{r,K} + \mathcal{J}^{\mathcal{K},w} \mathcal{J}^{w,K} - \mathbf{I}$$

$$\mathbf{H}_Z = \mathcal{J}^{\mathcal{K},r} \mathcal{J}^{r,Z} + \mathcal{J}^{\mathcal{K},w} \mathcal{J}^{w,Z}$$

where generically

$$\mathcal{J}^{x,y} = \begin{bmatrix} \frac{\partial x_0}{\partial y_0} & \frac{\partial x_0}{\partial y_1} & \dots \\ \frac{\partial x_1}{\partial y_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}$$

# Calculating Jacobians

- Analytical:**

$$\mathcal{J}^{r,K} = \alpha(\alpha - 1)Z_t K_t^{\alpha-2} \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & 1 & 0 & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & 1 \end{bmatrix}, \mathcal{J}^{r,Z} = \alpha K_t^{\alpha-1} I$$

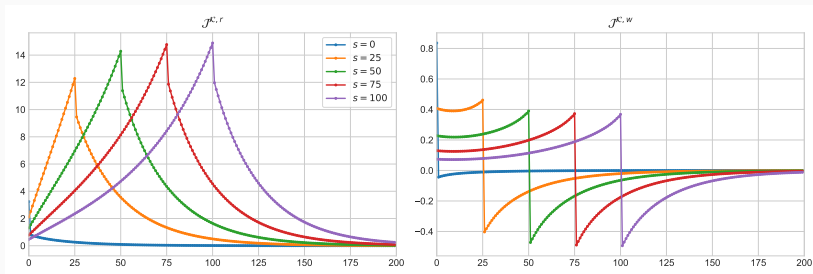
$$\mathcal{J}^{w,K} = (1 - \alpha)\alpha Z_t K_t^{\alpha-1} \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & 1 & 0 & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & 1 \end{bmatrix}, \mathcal{J}^{w,Z} = (1 - \alpha)K_t^{\alpha} I$$

**Note**  $\forall t \neq s - 1 : \frac{\partial r_t}{\partial K_s} = \frac{\partial w_t}{\partial K_s} = 0$  and  $\forall s \neq t : \frac{\partial r_t}{\partial Z_s} = \frac{\partial w_s}{\partial Z_s} = 0$

- Numerical:**  $\mathcal{J}^{K,r}$  and  $\mathcal{J}^{K,w}$
- Truncation:** The matrices are truncated such that they are  $\mathcal{T} \times \mathcal{T}$

# Interpreting Jacobians

1. **Along a row:** The effect on a variable in a given period of a shock in an arbitrary period
2. **Along a column:** The effect of a shock in a given period on a variable in an arbitrary period





# Transition path after MIT shock

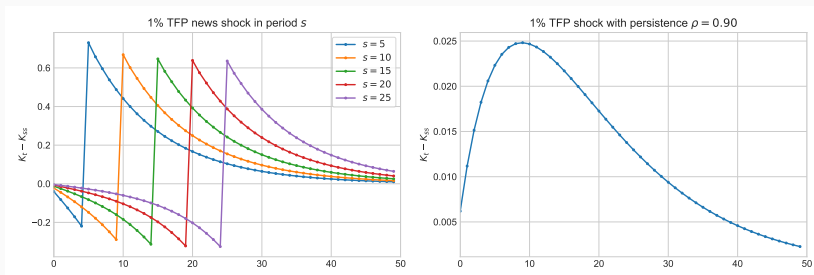
1. Assume we **start in the stationary equilibrium**
2. **MIT-shock:** Known path of future  $\mathbf{Z}$ , i.e. in terms of changes from steady state  $d\mathbf{Z} = \mathbf{Z} - \mathbf{Z}_{ss}$
3. **Question:** What happens to aggregate capital?
4. **Answer:** To a first order we have

$$\mathbf{G}^{K,Z} \equiv \frac{d\mathbf{K}}{d\mathbf{Z}} = -\mathbf{H}_K^{-1} \mathbf{H}_Z$$

where all derivatives are **evaluated at the stationary equilibrium**  
**Additionally:**

$$\begin{aligned}\mathbf{G}^{r,Z} &\equiv \frac{d\mathbf{r}}{d\mathbf{Z}} = \mathcal{J}^{r,Z} + \mathcal{J}^{r,K} \mathbf{G}^{K,Z}, & \mathbf{G}^{w,Z} \frac{d\mathbf{w}}{d\mathbf{Z}} &= \mathcal{J}^{w,Z} + \mathcal{J}^{w,K} \mathbf{G}^{K,Z} \\ \mathbf{G}^{C,Z} &\equiv \frac{d\mathbf{C}}{d\mathbf{Z}} = \mathcal{J}^{C,r} \mathbf{G}^{r,Z} + \mathcal{J}^{C,w} \mathbf{G}^{w,Z}, & \mathbf{G}^{Y,Z} &\equiv \frac{d\mathbf{Y}}{d\mathbf{Z}} = \mathcal{J}^{Y,Z} + \mathcal{J}^{Y,K} \mathbf{G}^{K,Z}\end{aligned}$$

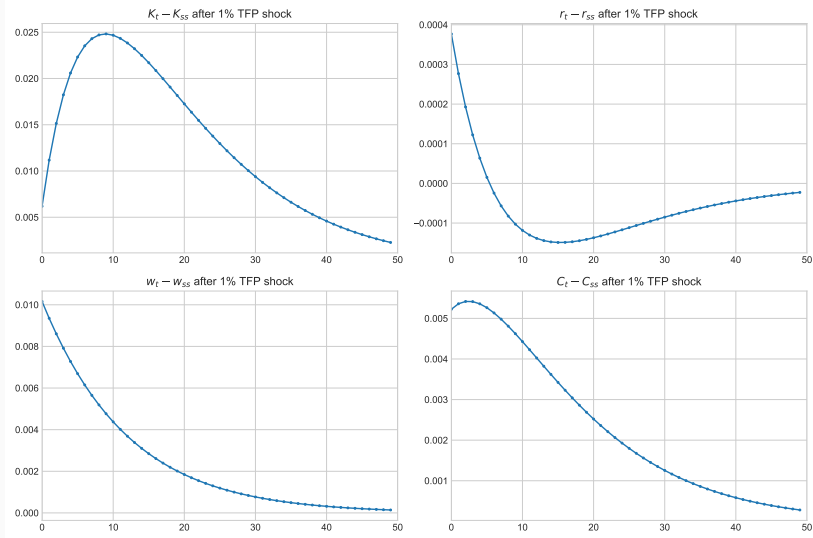
# News shock vs. persistent shock



**Persistent shock:**  $Z_t = (1 - \rho_Z)Z_{ss} + \rho_Z Z_t$  and  $Z_0 = (1 + \sigma_Z)Z_{ss}$

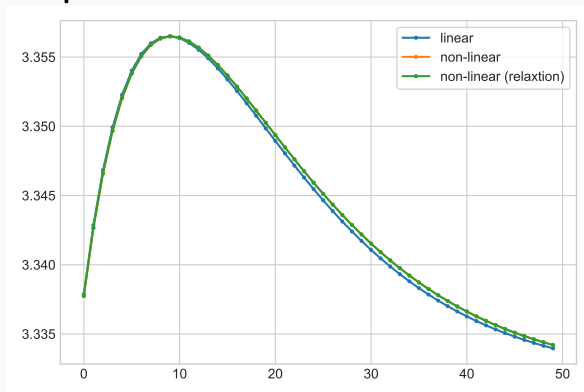
**Note:** One set of Jacobians, infinite many impulse-responses!

# Impulse-responses



# Linear vs. non-linear

- **Beyond first order approximation?** *Solve full non-linear equation system using the Jacobian at the stationary equilibrium as an approximation of the true Jacobians along the transition path*
- **Comparison:**



**GE**

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**Dynamic equilibrium**

# Household problem with aggregate shocks

- **Aggregate shocks:** Assume  $Z_t$  is a stochastic process
- **Root problem:** There is no longer perfect foresight wrt.  $r_t$  and  $w_t$
- **Extended problem:**

$$v(e_t, a_{t-1}, Z_t, D_t) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t [v(e_{t+1}, a_t, Z_{t+1}, D_{t+1})]$$

s.t.

$$a_t + c_t = (1 + r_t)a_{t-1} + w_t e_t$$

$$k_{t-1} = \int a_{t-1} dD_t$$

$$r_t = r(k_{t-1}, Z_t)$$

$$w_t = w(r_t, Z_t)$$

$$a_t \geq 0$$

- **Ultimate problem:**  $D_t$  is not easy to discretize...

# Approximate household problem

- **Krusell-Smith idea:** Approximate  $D_t$  with some selected moments, e.g. just the mean
- **Approximate problem:**

$$\begin{aligned} v(e_t, a_{t-1}, Z_t, k_{t-1}) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t [v(e_{t+1}, a_t, Z_{t+1}, k_t)] \\ &\text{s.t.} \end{aligned}$$

$$a_t + c_t = (1 + r_t)a_{t-1} + w_t e_t$$

$$r_t = r(k_{t-1}, Z_t)$$

$$w_t = w(r_t, Z_t)$$

$$k_t = \text{PLM}(k_{t-1}, Z_t)$$

$$a_t \geq 0$$

where  $\text{PLM}(k_{t-1}, Z_t)$  is the **perceived law of motion**

**For example:**  $\text{PLM}(k_{t-1}, Z_t) = \beta_{Z_t} + \alpha_{Z_t} \log k_{t-1}$

## Definition: Dynamic equilibrium

An **(approximate) dynamic equilibrium** is a PLM, policy functions  $a^*(\bullet)$  and  $c^*(\bullet)$ , and paths of quantities  $K_t$  and  $L_t$ , prices  $r_t$  and  $w_t$ , distributions  $D_t$  such that for all  $t$

1.  $a^*(\bullet)$  and  $c^*(\bullet)$  solve the household problem given the PLM
2.  $D_t$  is implied by the household problem
3. Firms profit maximize  $r_t = r(K_{t-1}/L_t, Z_t)$  and  $w_t = w(r_t, Z_t)$
4. The labor market clears, i.e.  $L_t = \int e_t dD_t = 1$
5. The capital market clears, i.e.  $K_t = \int a^*(e_t, a_{t-1}) dD_t$
6. The goods market clears, i.e.  $Y_t = \int c^*(e_t, a_{t-1}) dD_t + \delta K_{t-1}$
7.  $\text{PLM}(k_{t-1}, Z_t)$  does not imply systematic expectations errors

**Note:** When  $Z_t = Z_{ss} \forall Z_t$  the dynamic equilibrium does *not* converge to the stationary equilibrium unless the households know  $Z_t$  is actually not stochastic.



# Finding dynamic equilibrium

1. Guess on the PLM( $k_{t-1}, Z_t$ )
2. Solve the household problem
3. Simulate a path of  $Z_t$  and  $D_t$  and thus  $k_t$
4. Compare simulated behavior with the PLM( $k_{t-1}, Z_t$ )  
Stop if »good enough«  
otherwise update PLM( $k_{t-1}, Z_t$ ) and return to step 2

## Terminology:

1. The Krusell-Smith method is a **global solution method**
2. The newest **local solution methods** rely on linearization of the aggregate dynamics, but solve for the full non-linear stationary equilibrium

# Connection to sequence space

- **Insights in** Auclert et al. (2020b):
  1. In the limit where the shock variance disappears the dynamic equilibrium path converge to the stationary equilibrium.
  2. In the limit where the shock variance disappears the transition path to the MIT shock around the stationary equilibrium is the same as the impulse-response in the dynamic equilibrium.
- **Implications:** From the transition paths first order approximations of variances and co-variances of all variables can be calculated.
- **Estimation:** Parameters affecting
  1. The stationary equilibrium are computationally *very costly*
  2. Only the Jacobians are computationally *rather costly*
  3. Only affecting e.g. the shock process are computationally *cheap*

# Summary

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# Summary

- **Dynamic programming** is needed to solve **empirically realistic consumption-saving models**
- The **buffer-stock consumption model**, and its two asset cousin, can fit central stylized facts
  1. High MPC
  2. Responses to expected windfalls
  3. Households with more volatile income save more
  4. Consumption tracks income over the life-cycle
- Advances in micro-data, numerical methods and computational power are leading to **new discoveries**
- **EGM is a powerful solution method** (and can be generalized)
- Realistic consumption-saving behavior can be included in **general equilibrium models** → welfare analysis with full distributional effects

## References

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- Acharya, S. and Dogra, K. (2020). Understanding HANK: Insights from a PRANK. forthcoming in *Econometrica*.
- Achdou, Y., Han, J., Lasry, J.-M., Lions, P.-L., and Moll, B. (2020). Income and wealth distribution in macroeconomics: A continuous-time approach. Working Paper.
- Adda, J. and Cooper, R. W. (2003). *Dynamic Economics: Quantitative Methods and Applications*. MIT Press.
- Adda, J., Dustmann, C., and Stevens, K. (2016). The Career Costs of Children. *Journal of Political Economy*, forthcoming.
- Ahn, S., Kaplan, G., Moll, B., Winberry, T., and Wolf, C. (2018). When Inequality Matters for Macro and Macro Matters for Inequality. *NBER Macroeconomics Annual*, 32:1–75.
- Aiyagari, S. R. (1994). Uninsured Idiosyncratic Risk and Aggregate Saving. *The Quarterly Journal of Economics*, 109(3):659–684.
- Aiyagari, S. R. and McGrattan, E. R. (1998). The optimum quantity of debt. *Journal of Monetary Economics*, 42(3):447–469.
- Alves, F., Kaplan, G., Moll, B., and Violante, G. L. (2020). A Further Look at the Propagation of Monetary Policy Shocks in HANK. *Journal of Money, Credit and Banking*, 52(S2):521–559.
- Angeletos, G.-M. and Lian, C. (2017). Dampening General Equilibrium: From Micro to Macro. NBER Working Paper 23379.
- Angeletos, G.-M. and Lian, C. (2018). Forward Guidance without Common Knowledge. *American Economic Review*, 108(9):2477–2512.
- Angeletos, G.-M. and Lian, C. (2019). Confidence and the Propagation of Demand Shocks. Working Paper.
- Attanasio, O. P. (1999). Consumption. In Taylor, J. B. and Woodford, M., editors, *Handbook of Macroeconomics, Volume 1*, volume 1, pages 741–812. Elsevier B.V.

- Attanasio, O. P. and Browning, M. (1995). Consumption over the Life Cycle and over the Business Cycle. *The American Economic Review*, 85:1118–1137.
- Auclert, A. (2019). Monetary Policy and the Redistribution Channel. *American Economic Review*, 109(6):2333–2367.
- Auclert, A., Bardóczy, B., and Rognlie, M. (2020a). MPCs, MPEs and Multipliers: A Trilemma for New Keynesian Models. Working Paper 27486, National Bureau of Economic Research.
- Auclert, A., Bardóczy, B., Rognlie, M., and Straub, L. (2020b). Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models. Technical Report w26123, National Bureau of Economic Research, Cambridge, MA.
- Auclert, A. and Rognlie, M. (2017). A note on multipliers in NK models with GHH preferences.
- Auclert, A. and Rognlie, M. (2020). Inequality and Aggregate Demand. Working Paper.
- Auclert, A., Rognlie, M., and Straub, L. (2018). The Intertemporal Keynesian Cross. NBER Working Paper 25020, National Bureau of Economic Research.
- Auclert, A., Rognlie, M., and Straub, L. (2020c). Micro Jumps, Macro Humps: Monetary Policy and Business Cycles in an Estimated HANK Model. NBER Working Paper 26647.
- Azinovic, M., Gaegauf, L., and Scheidegger, S. (2019). Deep Equilibrium Nets. Working Paper.
- Baker, S. R. (2018). Debt and the Response to Household Income Shocks: Validation and Application of Linked Financial Account Data. *Journal of Political Economy*, 126(4):1504–1557.
- Bayer, C., Born, B., and Luetticke, R. (2020). Shocks, Frictions, and Inequality. Working Paper.

- Bayer, C. and Luetticke, R. (2020). Solving discrete time heterogeneous agent models with aggregate risk and many idiosyncratic states by perturbation. *Quantitative Economics*, 11(4):1253–1288–1288.
- Bayer, C., Luetticke, R., Pham-Dao, L., and Tjaden, V. (2019). Precautionary Savings, Illiquid Assets, and the Aggregate Consequences of Shocks to Household Income Risk. *Econometrica*, 87(1):255–290.
- Beaudry, P., Galizia, D., and Portier, F. (2020). Putting the Cycle Back into Business Cycle Analysis. *American Economic Review*, 110(1):1–47.
- Berger, D., Bocola, L., and Dovis, A. (2019). Imperfect Risk-Sharing and the Business Cycle. Technical Report w26032, National Bureau of Economic Research, Cambridge, MA.
- Berger, D. and Vavra, J. (2015). Consumption Dynamics During Recessions. *Econometrica*, 83(1):101–154.
- Bertsekas, D. P. (2012). *Dynamic Programming and Optimal Control: Approximate dynamic programming*. Athena Scientific.
- Bewley, T. (1986). Stationary Monetary Equilibrium with a Continuum of Independently Fluctuating Consumers. In Hildenbrand, W. and Mas-Collel, A., editors, *Contributions to Mathematical Economics in Honor of Gerard Debreu*. North-Holland, Amsterdam.
- Bilbiie, F. O. (2008). Limited asset markets participation, monetary policy and (inverted) aggregate demand logic. *Journal of Economic Theory*, 140(1):162–196.
- Bilbiie, F. O. (2019a). Monetary Policy and Heterogeneity: An Analytical Framework. Working Paper.
- Bilbiie, F. O. (2019b). The New Keynesian cross. *Journal of Monetary Economics*, forthcoming.
- Bilbiie, F. O. (2020). Everything you always wanted to know about T HANK (but were too afraid to ask). page 60.



- Bilbiie, F. O., Känzig, D. R., and Surico, P. (2020). Capital, Income Inequality, and Consumption: the Missing Link. Working Paper.
- Blundell, R., Pistaferri, L., and Saporta-Eksten, I. (2016). Consumption Inequality and Family Labor Supply. *American Economic Review*, 106(2):387–435.
- Boppart, T., Krusell, P., and Mitman, K. (2018). Exploiting MIT shocks in heterogeneous-agent economies: the impulse response as a numerical derivative. *Journal of Economic Dynamics and Control*, 89:68–92.
- Broer, T., Harbo Hansen, N.-J., Krusell, P., and Öberg, E. (2020). The New Keynesian Transmission Mechanism: A Heterogeneous-Agent Perspective. *The Review of Economic Studies*, 87(1):77–101.
- Broer, T. and Krusell, P. (2020). Fiscal Multipliers: A Heterogenous-Agent Perspective. Working Paper.
- Browning, M. and Crossley, T. F. (2001). The life-cycle model of consumption and saving. *The Journal of Economic Perspectives*, 15(3):3–22.
- Browning, M. and Lusardi, A. (1996). Household Saving: Micro Theories and Micro Facts. *Journal of Economic Literature*, 34(4):1797–1855.
- Brumm, J. and Scheidegger, S. (2017). Using Adaptive Sparse Grids to Solve High-Dimensional Dynamic Models. *Econometrica*, 85(5):1575–1612.
- Caballero, R. J. and Farhi, E. (2018). The Safety Trap. *The Review of Economic Studies*, 85(1):223–274.
- Cagetti, M. (2003). Wealth Accumulation Over the Life Cycle and Precautionary Savings. *Journal of Business & Economic Statistics*, 21(3):339–353.
- Campbell, J. Y. and Mankiw, N. G. (1990). Permanent Income, Current Income, and Consumption. *Journal of Business & Economic Statistics*, 8(3):265–279.

- Carroll, C., Crawley, E., Slacalek, J., Tokuoka, K., and White, M. (2019). Sticky Expectations and Consumption Dynamics. Working Paper.
- Carroll, C., Slacalek, J., Tokuoka, K., and White, M. N. (2017). The distribution of wealth and the marginal propensity to consume. *Quantitative Economics*, 8(3):977–1020.
- Carroll, C. D. (1992). The buffer-stock theory of saving: Some macroeconomic evidence. *Brookings Papers on Economic Activity*, 2:61–156.
- Carroll, C. D. (1997). Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis. *The Quarterly Journal of Economics*, 112(1):1–55.
- Carroll, C. D. (2000). Requiem for the representative consumer? Aggregate implications of microeconomic consumption behavior. *The American Economic Review: Papers and Proceedings of the One Hundred Twelfth Annual Meeting of the American economic Association*, 90(2):110–115.
- Carroll, C. D. (2001). A Theory of the Consumption Function, with and without Liquidity Constraints. *The Journal of Economic Perspectives*, 15(3):23–45.
- Carroll, C. D. (2006). The Method of Endogenous Gridpoints for Solving Dynamic Stochastic Optimization Problems. *Economics Letters*, 91(3):312–320.
- Carroll, C. D. (2020). Theoretical Foundations of Buffer Stock Saving. forthcoming in *Quantitative Economics*.
- Carroll, C. D., Slacalek, J., and Tokuoka, K. (2015). Buffer-stock saving in a Krusell–Smith world. *Economics Letters*, 132:97–100.
- Challe, E. (2020). Uninsured Unemployment Risk and Optimal Monetary Policy in a Zero-Liquidity Economy. *American Economic Journal: Macroeconomics*, 12(2):241–283.
- Challe, E., Matheron, J., Ragot, X., and Rubio-Ramirez, J. F. (2017). Precautionary saving and aggregate demand. *Quantitative Economics*, 8(2):435–478.

- Challe, E. and Ragot, X. (2015). Precautionary saving over the business cycle. *The Economic Journal*.
- Cloyne, J., Ferreira, C., and Surico, P. (2020). Monetary Policy when Households have Debt: New Evidence on the Transmission Mechanism. *The Review of Economic Studies*, 87(1):102–129.
- Coibion, O., Gorodnichenko, Y., Kueng, L., and Silvia, J. (2017). Innocent Bystanders? Monetary policy and inequality. *Journal of Monetary Economics*, 88:70–89.
- Conesa, J. C., Kitao, S., and Krueger, D. (2009). Taxing Capital? Not a Bad Idea after All! *American Economic Review*, 99(1):25–48.
- de Ferra, S., Mitman, K., and Romei, F. (2020). Household heterogeneity and the transmission of foreign shocks. *Journal of International Economics*, page 103303.
- De Nardi, M. and Fella, G. (2017). Saving and wealth inequality. *Review of Economic Dynamics*, 26(Supplement C):280–300.
- De Nardi, M., Fella, G., and Paz-Pardo, G. (2020). Nonlinear Household Earnings Dynamics, Self-Insurance, and Welfare. *Journal of the European Economic Association*, 18(2):890–926.
- Deaton, A. (1991). Saving and liquidity constraints. *Econometrica*, 59(5):1221–1248.
- Deaton, A. (1992). *Understanding Consumption*. Oxford University Press.
- Debortoli, D. and Galí, J. (2018). Monetary Policy with Heterogeneous Agents: Insights from TANK models. Working Paper.
- Den Haan, W. J., Judd, K. L., and Juillard, M. (2010). Computational suite of models with heterogeneous agents: Incomplete markets and aggregate uncertainty. *Journal of Economic Dynamics and Control*, 34(1):1–3.
- Den Haan, W. J., Rendahl, P., and Riegler, M. (2018). Unemployment (Fears) and Deflationary Spirals. *Journal of the European Economic Association*, 16(5):1281–1349.

- Di Maggio, M., Kermani, A., Keys, B. J., Piskorski, T., Ramcharan, R., Seru, A., and Yao, V. (2017). Interest Rate Pass-Through: Mortgage Rates, Household Consumption, and Voluntary Deleveraging. *American Economic Review*, 107(11):3550–3588.
- Drue Dahl, J. (2020). A Guide On Solving Non-Convex Consumption-Saving Models. forthcoming in *Computational Economics*.
- Drue Dahl, J., Graber, M., and Jørgensen, T. H. (2020a). High Frequency Income Dynamics. Working Paper.
- Drue Dahl, J., Jensen, E. B., and Leth-Petersen, S. (2020b). The Intertemporal Marginal Propensity to Consume out of Future Persistent Cash-Flows:. Working Paper.
- Drue Dahl, J. and Jørgensen, T. H. (2017). A general endogenous grid method for multi-dimensional models with non-convexities and constraints. *Journal of Economic Dynamics and Control*, 74:87–107.
- Drue Dahl, J. and Jørgensen, T. H. (2020). Can Consumers Distinguish Persistent from Transitory Income Shocks? *The Economic Journal*, 130(632):2410–2437.
- Drue Dahl, J., Kristensen, D., and Jørgensen, T. H. (2018). Estimating Dynamic Economic Models with Unobserved Heterogeneity. Working Paper.
- Drue Dahl, J. and Martinello, A. (2018). Long-Run Saving Dynamics: Evidence from Unexpected Inheritances. forthcoming in *Review of Economics and Statistics*.
- Drue Dahl, J. and Munk-Nielsen, A. (2020). Higher-order income dynamics with linked regression trees. *The Econometrics Journal*, 23(3):S25–S58.
- Duffie, D. and Singleton, K. J. (1990). Simulated moments estimation of Markov models of asset prices. NBER Working Paper 83.
- Dupraz, S., Nakamura, E., and Steinsson, J. (2020). A Plucking Model of Business Cycles. Technical report.

- Egger, D., Haushofer, J., Miguel, E., Niehaus, P., and Walker, M. W. (2019). General Equilibrium Effects of Cash Transfers: Experimental Evidence from Kenya. Working Paper 26600, National Bureau of Economic Research.
- Fagereng, A., Holm, M. B., and Natvik, G. J. (2020). MPC Heterogeneity and Household Balance Sheets. forthcoming in *American Economic Journal: Macroeconomics*.
- Farhi, E. and Werning, I. (2019). Monetary Policy, Bounded Rationality, and Incomplete Markets. *American Economic Review*, 109(11):3887–3928.
- Farmer, R. (2019). The Importance of Beliefs in Shaping Macroeconomic Outcomes. NBER Working Paper 26557, National Bureau of Economic Research.
- Fernandez-Villaverde, J., Hurtado, S., and Nuno, G. (2020). Financial Frictions and the Wealth Distribution. Working Paper.
- Fernández-Villaverde, J. and Valencia, D. Z. (2018). A Practical Guide to Parallelization in Economics. NBER Working Paper 24561.
- Flodén, M., Kilström, M., Sigurdsson, J., and Vestman, R. (2020). Household Debt and Monetary Policy: Revealing the Cash-Flow Channel. forthcoming in *Economic Journal*.
- French, E. and Jones, J. B. (2011). The Effects of Health Insurance and Self-Insurance on Retirement Behavior. *Econometrica*, 79(3):693–732.
- Friedman, M. (1957). *A theory of the consumption function*. Princeton university Press for NBER.
- Gabaix, X. (2020). A Behavioral New Keynesian Model. *American Economic Review*, 110(8):2271–2327.
- Gelman, M., Kariv, S., Shapiro, M. D., Silverman, D., and Tadelis, S. (2014). Harnessing Naturally Occurring Data to Measure the Response of Spending to Income. *Science*, 345(6193):212–215.

- Gornemann, N., Kuester, K., and Nakajima, M. (2016). Doves for the Rich, Hawks for the Poor? Distributional Consequences of Monetary Policy. Technical report.
- Gouriéroux, C. and Monfort, A. (1997). *Simulation-based Econometric Methods*. Oxford University Press, New York, NY.
- Gourinchas, P.-O. and Parker, J. A. (2002). Consumption over the life cycle. *Econometrica*, 70(1):47–89.
- Guerrieri, V. and Lorenzoni, G. (2017). Credit Crises, Precautionary Savings, and the Liquidity Trap. *The Quarterly Journal of Economics*, 132(3):1427–1467.
- Guerrieri, V., Lorenzoni, G., Werning, I., and Straub, L. (2020). Macroeconomic Implications of COVID-19. Working Paper.
- Guren, A., McKay, A., Nakamura, E., and Steinsson, J. (2020). What Do We Learn From Cross-Regional Empirical Estimates in Macroeconomics? NBER Working Paper 26881.
- Guvenen, F. (2011). Macroeconomics With Heterogeneity: A Practical Guide. NBER Working Paper 17622.
- Guvenen, F., Karahan, F., Ozkan, S., and Song, J. (2019). What Do Data on Millions of U.S. Workers Reveal about Life-Cycle Earnings Dynamics? forthcoming in *Econometrica*.
- Guvenen, F. and Smith, A. A. (2014). Inferring labor income risk and partial insurance from economic choices. *Econometrica*, 82(6):2085–2129.
- Hagedorn, M., Luo, J., Manovskii, I., and Mitman, K. (2019a). Forward guidance. *Journal of Monetary Economics*, 102:1–23.
- Hagedorn, M., Manovskii, I., and Mitman, K. (2019b). The Fiscal Multiplier. NBER Working Paper 25571, National Bureau of Economic Research.
- Harmenberg, K. and Öberg, E. (2020). Consumption dynamics under time-varying unemployment risk. *Journal of Monetary Economics*.

- Heathcote, J., Storesletten, K., and Violante, G. L. (2009). Quantitative Macroeconomics with Heterogeneous Households. *Annual Review of Economics*, 1(1):319–354.
- Heathcote, J., Storesletten, K., and Violante, G. L. (2014). Consumption and Labor Supply with Partial Insurance: An Analytical Framework. *The American Economic Review*, 104(7):2075–2126.
- Holm, M. B. (2018). Monetary Policy Transmission with Income Risk. Technical report.
- Holm, M. B. and Paul, P. (2020). The Transmission of Monetary Policy under the Microscope a. forthcoming in *Journal of Political Economy*.
- Honore, B., Jorgensen, T., and de Paula, A. (2020). The Informativeness of Estimation Moments. arXiv: 1907.02101 version: 2.
- Hubmer, J., Krusell, P., and Smith, A. A. (2020). Sources of U.S. Wealth Inequality: Past, Present, and Future. forthcoming in *NBER Macroeconomics Annual 2020*.
- Huggett, M. (1993). The risk-free rate in heterogeneous-agent incomplete-insurance economies. *Journal of Economic Dynamics and Control*, 17(5-6):953–969.
- Imrohoroglu, A. (1989). Cost of Business Cycles with Indivisibilities and Liquidity Constraints. *Journal of Political Economy*, 97(6):1364–1383.
- Johnson, D. S., Parker, J. A., and Souleles, N. S. (2006). Household Expenditure and the Income Tax Rebates of 2001. *The American Economic Review*, 96(5):1589–1610.
- Jørgensen, T. H. (2017). Life-Cycle Consumption and Children: Evidence from a Structural Estimation. *Oxford Bulletin of Economics and Statistics*, 79(5):717–746.
- Jørgensen, T. H. (2020). Sensitivity to Calibrated Parameters.
- Judd, K. L. (1998). *Numerical Methods in Economics*. MIT Press.
- Judd, K. L., Maliar, L., and Maliar, S. (2017). How to Solve Dynamic Stochastic Models Computing Expectations Just Once. *Quantitative Economics*, 8(3).

- Judd, K. L., Maliar, L., Maliar, S., and Valero, R. (2014). Smolyak method for solving dynamic economic models: Lagrange interpolation, anisotropic grid and adaptive domain. *Journal of Economic Dynamics and Control*, 44:92–123.
- Kaas, L. (2020). Block-Recursive Equilibria in Heterogeneous-Agent Models. Technical report.
- Kaplan, G., Mitman, K., and Violante, G. L. (2019). The Housing Boom and Bust: Model Meets Evidence. *Journal of Political Economy*, page 89.
- Kaplan, G., Moll, B., and Violante, G. L. (2018). Monetary Policy According to HANK. *American Economic Review*, 108(3):697–743.
- Kaplan, G., Violante, G., and Weidner, J. (2014). The Wealthy Hand-to-Mouth. *Brookings Papers on Economic Activity*, pages 77–138.
- Kaplan, G. and Violante, G. L. (2014). A Model of the Consumption Response to Fiscal Stimulus Payments. *Econometrica*, 82(4):1199–1239.
- Kaplan, G. and Violante, G. L. (2018). Microeconomic Heterogeneity and Macroeconomic Shocks. *Journal of Economic Perspectives*, 32(3):167–194.
- Keane, M. P. and Wasi, N. (2016). Labour Supply: The Roles of Human Capital and The Extensive Margin. *The Economic Journal*, 126(592):578–617.
- Kekre, R. and Lenel, M. (2020). Monetary Policy, Redistribution, and Risk Premia. *SSRN Electronic Journal*.
- Kirkby, R. (2017). Transition paths for Bewley-Huggett-Aiyagari models: Comparison of some solution algorithms.
- Kreiner, C. T., Dreyer Lassen, D., and Leth-Petersen, S. (2019). Liquidity Constraint Tightness and Consumer Responses to Fiscal Stimulus Policy. *American Economic Journal: Economic Policy*, 11(1):351–379.
- Krueger, D., Mitman, K., and Perri, F. (2016). Chapter 11 - Macroeconomics and Household Heterogeneity. In Taylor, J. B. and Uhlig, H., editors, *Handbook of Macroeconomics*, volume 2, pages 843–921. Elsevier.



- Krusell, P., Mukoyama, T., and Smith, A. A. (2011). Asset prices in a Huggett economy. *Journal of Economic Theory*, 146(3):812–844.
- Krusell, P. and Smith, A. A. (1997). Income and wealth heterogeneity, portfolio choice, and equilibrium asset returns. *Macroeconomic Dynamics*, 1(02):387–422.
- Krusell, P. and Smith, A. A. (1998). Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economy*, 106(5):867–896.
- Krusell, P. and Smith, A. A. (2006). Quantitative macroeconomic models with heterogeneous agents. In Blundell, R., editor, *Advanced in Economics and Econometrics: Theory and Applications*, pages 298–340. Cambridge University Press.
- Kubler, F. and Scheidegger, S. (2018). Self-justified equilibria: Existence and computation. Working Paper.
- Kueng, L. (2018). Excess Sensitivity of High-Income Consumers. *The Quarterly Journal of Economics*, 133(4):1693–1751.
- La Cava, G., Hughson, H., and Kaplan, G. (2016). The Household Cash Flow Channel of Monetary Policy. RBA Research Discussion Papers, Reserve Bank of Australia.
- Landvoigt, T. (2017). Housing Demand During the Boom: The Role of Expectations and Credit Constraints. *The Review of Financial Studies*, 30(6):1865–1902.
- Lewis, D. J., Melcangi, D., and Piososop, L. (2019). Latent Heterogeneity in the Marginal Propensity to Consume. Working Paper.
- Ljungqvist, L. and Sargent, T. J. (2004). *Recursive Macroeconomic Theory*. MIT Press.
- Low, H., Meghir, C., and Pistaferri, L. (2010). Wage Risk and Employment Risk over the Life Cycle. *American Economic Review*, 100(4):1432–1467.
- Luetticke, R. (2020). Transmission of Monetary Policy with Heterogeneity in Household Portfolios. forthcoming in *American Economic Journal: Macroeconomics*.

- Maliar, L., Maliar, S., and Winant, P. (2019). Will Artificial Intelligence Replace Computational Economists Any Time Soon? Working Paper.
- McFadden, D. (1989). A Method of Simulated Moments for Estimation of Discrete Response Models Without Numerical Integration. *Econometrica*, 57(5):995–1026.
- McKay, A., Nakamura, E., and Steinsson, J. (2016). The Power of Forward Guidance Revisited. *American Economic Review*, 106(10):3133–3158.
- McKay, A., Nakamura, E., and Steinsson, J. (2017). The Discounted Euler Equation: A Note. *Economica*, 84(336):820–831.
- McKay, A. and Reis, R. (2016). The Role of Automatic Stabilizers in the U.S. Business Cycle. *Econometrica*, 84(1):141–194.
- McKay, A. and Reis, R. (2020). Optimal Automatic Stabilizers. Working Paper.
- McKay, A. and Wieland, J. (2020). Lumpy Durable Consumption Demand and the Limited Ammunition of Monetary Policy. Working Paper.
- Mian, A., Rao, K., and Sufi, A. (2013). Household Balance Sheets, Consumption, and the Economic Slump. *The Quarterly Journal of Economics*, 128(4):1687–1726.
- Mian, A. R., Straub, L., and Sufi, A. (2021). Indebted Demand. forthcoming in *Quarterly Journal of Economics*.
- Michaillat, P. and Saez, E. (2018). Resolving New Keynesian Anomalies with Wealth in the Utility Function. NBER Working Paper 24971, National Bureau of Economic Research.
- Misra, K. and Surico, P. (2014). Consumption, Income Changes, and Heterogeneity: Evidence from Two Fiscal Stimulus Programs. *American Economic Journal: Macroeconomics*, 6(4):84–106.
- Modigliani, F. and Brumberg, R. (1954). Utility Analysis and the Consumption Function: An Interpretation of Cross-Section Data. In Kurihara, K. and Brunswick, N., editors, *Post-Keynesian Economics*, pages 338–436. Rutgers University Press.

- Moran, P. and Kovacs, A. (2019). Temptation and commitment: understanding the demand for illiquidity. NBER Working Paper.
- Nakamura, E. and Steinsson, J. (2018). Identification in Macroeconomics. *Journal of Economic Perspectives*, 32(3):59–86.
- Oh, H. and Reis, R. (2012). Targeted Transfers and the Fiscal Response to the Great Recession. *Journal of Monetary Economics*, 59:S50–S64.
- Ottobello, P. and Winberry, T. (2020). Financial Heterogeneity and the Investment Channel of Monetary Policy. *Econometrica*, 88(6):2473–2502.
- Pagel, M. (2017). Expectations-Based Reference-Dependent Life-Cycle Consumption. *The Review of Economic Studies*, 84(2):885–934.
- Parker, J. A., Souleles, N. S., Johnson, D. S., and McClelland, R. (2013). Consumer Spending and the Economic Stimulus Payments of 2008. *The American Economic Review*, 103(6):2530–2553.
- Pistaferri, L. (2017). *The Economics of Consumption*. Oxford University Press.
- Powell, W. B. (2011). *Approximate Dynamic Programming: Solving the Curses of Dimensionality*. John Wiley & Sons.
- Pröhl, E. (2019). Approximating Equilibria with Ex-Post Heterogeneity and Aggregate Risk. Working Paper.
- Puterman, M. L. (2009). *Markov Decision Processes: Discrete Stochastic Dynamic Programming*. John Wiley & Sons.
- Ravn, M. O. and Sterk, V. (2017). Job uncertainty and deep recessions. *Journal of Monetary Economics*, 90:125–141.
- Ravn, M. O. and Sterk, V. (2020). Macroeconomic Fluctuations with HANK&SAM: an Analytical Approach. forthcoming in Journal of the European Economic Association.

- Reiter, M. (2009). Solving heterogeneous-agent models by projection and perturbation. *Journal of Economic Dynamics and Control*, 33(3):649–665.
- Reiter, M. (2010). Solving the incomplete markets model with aggregate uncertainty by backward induction. *Journal of Economic Dynamics and Control*, 34(1):28–35.
- Scheidegger, S. and Bilionis, I. (2019). Machine learning for high-dimensional dynamic stochastic economies. *Journal of Computational Science*, 33:68–82.
- Schmedders, K. and Judd, K. L. (2013). *Handbook of Computational Economics Vol. 3*. Newnes. Google-Books-ID: xDhO6L\_Psp8C.
- Slacalek, J., Tristani, O., and Violante, G. L. (2020). Household Balance Sheet Channels of Monetary Policy: A Back of the Envelope Calculation for the Euro Area. page 44.
- Stokey, N. L. and Lucas, R. E. (1989). *Recursive methods in economic dynamics*. Harvard University Press.
- Tauchen, G. (1986). Finite state markov-chain approximations to univariate and vector autoregressions. *Economics Letters*, 20(2):177–181.
- Tauchen, G. and Hussey, R. (1991). Quadrature-Based Methods for Obtaining Approximate Solutions to Nonlinear Asset Pricing Models. *Econometrica*, 59(2):371–396.
- Werning, I. (2015). Incomplete Markets and Aggregate Demand. NBER Working Paper 21448.
- Winberry, T. (2018). A method for solving and estimating heterogeneous agent macro models. *Quantitative Economics*, 9(3):1123–1151–1151.
- Wolf, C. K. (2020). The Missing Intercept: A Demand Equivalence Approach. Working Paper.
- Zeldes, S. P. (1989a). Consumption and Liquidity Constraints: An Empirical Investigation. *Journal of Political Economy*, 97(2):305–346.
- Zeldes, S. P. (1989b). Optimal Consumption with Stochastic Income: Deviations from Certainty Equivalence. *The Quarterly Journal of Economics*, 104(2):275–298.