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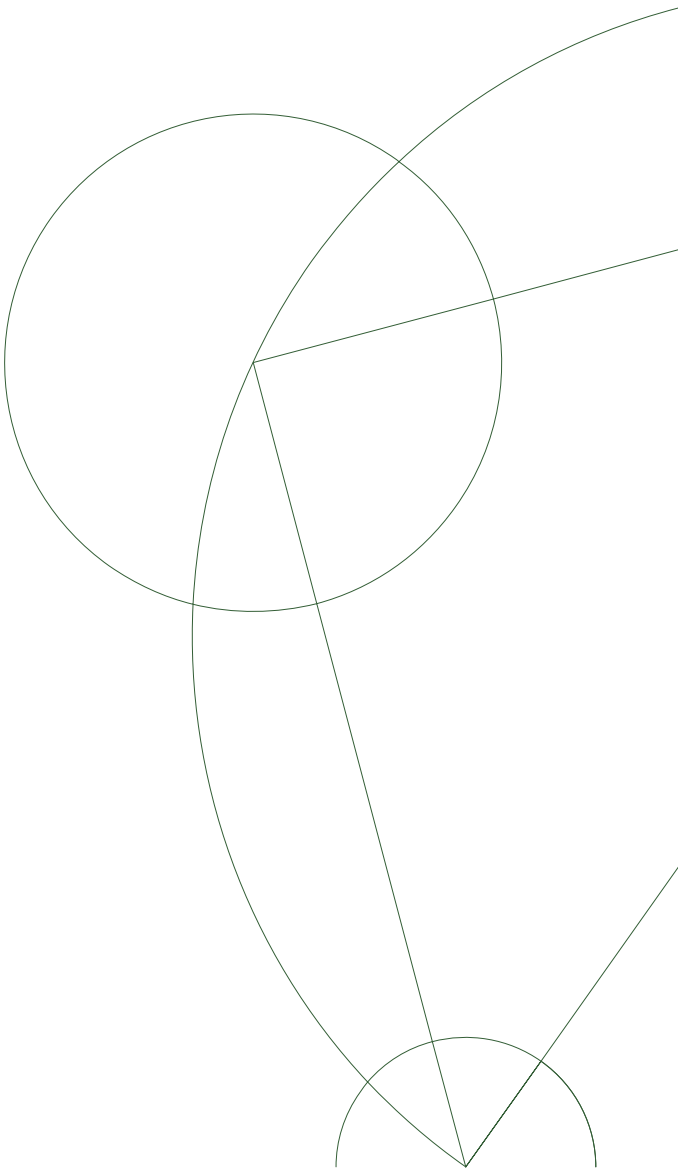
Integer Programming Models in Graph Coloring



Bachelor thesis in mathematics

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Abstract

The aim is to present and compare a few formulations of the graph colouring problem in the language of mixed-integer optimization. The student will learn and describe the principles of linear and integer optimization, formulate and implement a few known integer models of the graph colouring problem, and compare their performance on various benchmark graphs.

Contents

1. Linear Programming

In this chapter we introduce the concept of Linear programming. Most proofs will be omitted but proofs and more in depth explanations can be found in {some book}

1.1 The structure of a linear program

In a linear program we want to maximise or minimize a given linear function $z : \mathbb{R}^n \to \mathbb{R}$ subject to a number of linear inequalities or equalities $g_i : \mathbb{R}^n \to \mathbb{R}$ with $g_i(x_1, ..., x_n) \geq b_i$,$g_i(x_1, ..., x_n) \leq b_i$ or $g_i(x_1, ..., x_n) = b_i$. The function, z, that we want to optimise is called the **Objective** and the set of inequalities are called the **Constraints**. A general linear problem is written:

$$\begin{array}{ll} \min/\max & z(x_1, ..., x_n) \\ \text{s.t.} & g_1(x_1, ..., x_n) \textbf{ ordRel } b_1, \\ & \vdots \\ & g_m(x_1, ..., x_n) \textbf{ ordRel } b_m \\ & \text{where each } \textbf{ordRel} \text{ can be } \leq, \geq \text{ or } = \end{array} \tag{1.1}$$

Example 1.2
A maths student wants to save money on his diet while still remaining healthy. To stay healthy his diet must contain at least $b_1 = 5$ units of protein, $b_2 = 10$ units of carbs, $b_3 = 5$ units of fat and $b_4 = 10$ units of vitamins.
He consider buying 3 food products with different nutritional values and prices:

x_1 is a take away meal costing 5 and containing 3 units of protein, 12 units of carbs, 7 units of fat and 3 unit of vitamins.

x_2 is a vegetable costing 1 and containing 2 unit of protein, 2 units of carbs, 0 units of fat and 5 units of vitamins.

x_3 is a type of bread costing 2 and containing 1 unit of protein, 4 units of carbs, 2 units of fat and 2 units of vitamins.

He then define an optimization problem minimizing the cost of food subject to getting the right nutrition

$$\begin{aligned} \min \quad & 5x_1 + x_2 + 2x_3 \\ \text{s.t.} \quad & 3x_1 + 2x_2 + x_3 \geq 5, \\ & 12x_1 + 2x_2 + 4x_3 \geq 10, \\ & 7x_1 + 2x_3 \geq 5, \\ & 3x_1 + 5x_2 + 2x_3 \geq 10, \end{aligned} \tag{1.2}$$

1.3 feasible and optimal solutions

feasibility

Given a Linear problem on form 1.1 any point of $\mathbb{R}^n$ such that all m constraints are true is called a feasible solution. The set of all these points is called the feasible set. In the case Example 1.2 the feasible set is the set of all combinations of amounts of the different foods such that the nutritional requirements are met. If a problem has no feasible solutions, that problem is said to be infeasible. A feasibility problem is a special case in linear programming where our object function is constant and thus if any feasible solution exists, that solution is optimal.

optimal solutions

A feasible solution $\mathbf{x}_0 \in \mathbb{R}^n$ is said to be optimal if $z(\mathbf{x}_0) \geq z(\mathbf{x}_0)$ (when maximizing) or $z(\mathbf{x}_0) \leq z(\mathbf{x}_0)$ (when minimizing) for all feasible solutions $\mathbf{x} \in \mathbb{R}^n$.

- something about unbounded feasible sets.

- something about the result where an optimal solution is always in an extreme point of the polyhedron, leading us into the convexity and simplex part.

1.4 Convexity

Definition 1.5. *A set $X \in \mathbb{R}^n$ is said to be convex if for any two points $a, b \in X$ the straight line segment connecting a and b is entirely within X.*

Theorem 1.6. *A feasible region of a linear program is convex*

Proof.
easy short proof □

1.7 The geometric/graphical intuition

Section with pretty 3d polyhedron and a objective plane of Example 1.2 once i figure out how to use TikZ.

learn TikZ

1.8 Matrix representation of an LP and Slack variables

Since the objective function and the constraints of a linear program are all linear functions we can also write a linear program using matrix-vector products. The object can be written as the product of the vector of variables to be determined $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{x}$, and the transposed vector of coefficients $[c_1, c_2, ..., c_n] = \mathbf{c}^T$ of each x_i in the objective.
Likewise the constraints can be written as the matrix-vector products of $n \times m$ constraint matrices,A where m is the number of a constraints with a certain order relation, and $\mathbf{x}$ with some order relation ($\leq, \geq$ *or* $=$) to a $1 \times m$ vector $\mathbf{b}$.
Rewriting any $\leq$ constraint to $\geq$ in case of minimization or any $\geq$ to $\leq$ in case of maximization (the equality constraints can be written with two inequalities) we can write the linear program on it's *canonical form*:

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} \geq \mathbf{b}, \\ & \mathbf{x} \geq 0, \end{aligned} \tag{1.3}$$

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} \geq 0, \end{aligned} \tag{1.4}$$

in case of Example 1.2 we write it on it's canonical form:

$$\begin{aligned} \min \quad & [5,1,2]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ \text{s.t.} \quad & \begin{bmatrix} 3 & 2 & 1 \\ 12 & 2 & 4 \\ 7 & 0 & 2 \\ 3 & 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 5 \\ 10 \\ 5 \\ 10 \end{bmatrix}, \\ & \mathbf{x} \geq 0, \end{aligned} \tag{1.5}$$

Much like how we made an LP on canonical form by manipulating the inequalities, we can also write any LP on it's *standard form*

$$\begin{aligned} \max/\min \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq 0, \end{aligned} \tag{1.6}$$

by introducing a $1 \times m$ vector, $\mathbf{s}$, with one slack variable for each constraint and thus creating a new LP on standard with the same solutions as the old one.

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} + \mathbf{s} = \mathbf{b}, \\ & \mathbf{x},\mathbf{s} \geq 0, \end{aligned} \tag{1.7}$$

These forms and the concept of slack variables will come in handy later (I hope)

1.9 Duality

Another important result in Linear programming is the concept of duality.

Definition 1.10. *Given a maximization linear program on canonical form 1.4 We define it's dual problem as the minimization problem:*

$$\begin{aligned} \textit{min} \quad & \boldsymbol{b}^T\boldsymbol{y} \\ \textit{s.t.} \quad & A^T\boldsymbol{y} \geq \boldsymbol{c}, \\ & \boldsymbol{y} \geq 0, \end{aligned} \tag{1.8}$$

Theorem 1.11. *An optimal solution to a linear program on it's primal form is also optimal in the dual form*

Proof.
proof □

Weak and strong duality

1.12 Worst case runtime of an LP

This might be relevant since linear programming is in P and integer programming is NP-hard.

1.13 The simplex method

Result showing that Linear programming is in P.

2. Integer Programming and Graphs

In this chapter we introduce the concept of Integer programming. We will approach the subject with examples and results from graph theory, but apart from fundamental definitions we will omit most of the theory from this field of mathematics.

2.1 Some Graph notation

Definition 2.2. *A **graph** G is an ordered set (V, E) where V is a set of vertices and E is a set of edges and each edge in E is a subset of V containing two vertices.*

Definition 2.3. *let $G = (V, E)$ be a graph. Two vertices $v, u \in V$ are said to be **adjacent** if the edge (v, u) (or (u, v)) is in E.*

Definition 2.4. *let $G = (V, E)$ be a graph. A subset S of V form an **independent set** if no two vertices in S are adjacent in G*

Definition 2.5. *let $G = (V, E)$ be a graph. A subset S of V form a **clique** if all pairs of two vertices in S are adjacent in G*

Definition 2.6. *let $G = (V, E)$ be a graph. A **maximal independent set** or a **maximal clique** is an independent set or a clique respectively where if any other vertex $v \in V$ is added to the set, it will lose the property of being an independent set or clique respectively*

Definition 2.7. *A **maximum independent set** or a **maximum clique** in a graph G is an independent set or a clique respectively where no other independent set or clique in G respectively have more vertices.*

Definition 2.8. *the **clique number** $\omega(G)$ of a graph G is the number of vertices in a maximal clique in G.*

2.9 Integer and Binary Constraints

Example 2.10
Maximal clique problem
Let $G = (V, E)$ be a graph and let $H = (V, E')$ be it's complement graph.

$$\begin{array}{ll} \max & \sum_{v \in V} x_v \\ \text{s.t.} & x_u + x_v \leq 1, \forall (u, v) \in E' \\ & x_v \in \{0, 1\}, \ \forall v \in V \end{array} \tag{2.1}$$

2.11 The Convex Hull

2.12 Mixed Integer Programming - MIP

2.13 Relaxation

2.14 Integer Programming is NP-hard

3. Graph colouring

In this chapter we introduce main concept of graph vertex-colouring and some integer formulations for colouring graphs.

3.1 Fundamental colouring terms and results

Definition 3.2. *A **vertex-colouring** of a graph $G = (V, E)$ is an assignment of colors $C = 1, ..., k$ such that each $v \in V$ is assigned one color and no two adjacent vertices are assigned the same color.*

in this paper a colouring will always refer to a vertex-colouring.

Definition 3.3. *A k-**colouring** of a graph G is a vertex-colouring using k colours.*

Definition 3.4. *An **optimal colouring** of a graph, G, is a colouring of G using the minimum amount of colours possible.*

Definition 3.5. *The **chromatic number** $\chi(G)$ of a graph, G is the number number of colours in an optimal colouring of G.*

Theorem 3.6. *Vertices of one color form an independent set*

Proof.
Let $G = (V, E)$ be a graph and suppose that a set of vertices S are assigned the same color but does not form an independent set. Then there exists some edge in E connecting two vertices $s_1, s_2 \in S$. But that implies that s_1 and s_2 are assigned different colours which gives us contradiction □

Theorem 3.7. *Given a graph G*
$\chi(G) \geq \omega(G)$

Proof.
Let $G = (V, E)$ be a graph and suppose that a set S of $k = \omega(G)$ vertices form

a maximal clique in G. Then each $s_i \in S$ has edges in E to all other vertices of S and they must each have distinct colours by 3.2. And since G_S is a sub graph of G, we get $\omega(G) = \chi(G_S) \leq \chi(G)$. □

3.8 Integer programming formulations

Here we formulate some integer problems to calculate an optimal colouring. The same formulations can be used with minor simplifications to calculate if feasibility problems of weather graphs are k-colourable.

The standard formulation

In the standard formulation we want to introduce $k|V|$ binary variables, $x_{v,c}$ for a suitable $k \geq \mathcal{X}(G)$. where

$$x_{v,c} = \begin{cases} 1 & \text{if vertex } v \text{ is assigned colour } c \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

and k binary variables y_c indicating if color c is used in the colouring.

Proposition 3.9. *The following formulation assigns an optimal colouring to the graph and gives the chromatic number:*

$$\begin{array}{lll} \textit{min} & \sum_{c\in\{1\ldots k\}} y_c & \\ \textit{s.t.} & \sum_{c\in\{1\ldots k\}} x_{v,c} = 1, & \forall v \in V \\ & x_{v,c} + x_{u,c} \leq 1, & \forall (u,v) \in E, \forall c \in \{1\ldots k\} \\ & x_{v,c} - y_c \leq 0, & \forall v \in V, \forall c \in \{1\ldots k\} \\ & x_{v,c} \in \{0,1\}, & \forall v \in V, \forall c \in \{1\ldots k\} \\ & y_c \geq 0, & \forall c \in \{1\ldots k\} \end{array} \tag{3.2}$$

Proof.
Given a graph $G = (V, E)$:

1. An optimal colouring is feasible and optimal in 3.2:
   Given an optimal colouring of G and assigning each $x_{v,c}$ the values specified in 3.1, we see that:
   a) the first condition is satisfied since each vertex only has one colour.
   b) the second condition is satisfied since no two adjacent vertices has the same colour.

c) the third condition assign values to each y_c such that $y_{c_i} \geq 1$ if colour c_i is used to colour some vertex $\in V$ and otherwise greater than zero.

thus the solution is feasible, and since the object is to minimize the sum of all y_c's each y_c will be minimized to $y_c = \begin{cases} 1 & \text{if colour } c \text{ is used} \\ 0 & \text{otherwise} \end{cases}$ and since the colouring was optimal and used the fewest colours possible, their sum will be optimal and equal $\chi(G)$

2. An optimal solution to 3.2 returns $\chi(G)$ and assign the vertices colours such that the colouring is optimal:

□

A scheduling formulation

Another Integer formulation of the graph colouring problem for a graph $G = (V, E)$ is the scheduling formulation. Here we introduce V integer variables $\{X_1, \cdots X_v\}$ with the desired result: $X_v = c$ if vertex v is assigned colour c. And another E binary variables $x_{u,v} \forall (u, v) \in E$ with the desired result:

$$x_{u,v} = \begin{cases} 1 & \text{if vertecies } u, v \text{ are assigned colours } c_u, c_v \text{ respectively with } c_u < c_v \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

Furthermore we introduce the variable c that has to be greater or equal to the amount of colours used.

Proposition 3.10. *The following formulation assigns an optimal colouring to the graph and gives the chromatic number:*

$$\begin{array}{lll} \textit{min} & c & \\ \textit{s.t.} & X_u - X_v + kx_{u,v} \leq k-1, & \forall (u,v) \in E \\ & X_v - X_u - kx_{u,v} \leq -1, & \forall (u,v) \in E \ \textit{for } k \geq c \\ & X_v - c \leq 0, & \forall v \in V \\ & x_{u,v} \in \{0,1\}, & \forall (u,v) \in E \end{array} \tag{3.4}$$

Proof.

1. An optimal colouring is feasible and optimal in 3.4:

   a) Since $X_u \neq X_v$ for all adjacent $u, v \in V$ we have the option of $X_u < X_v$ and $X_u > X_v$. If $X_u < X_v$ then $x_{u,v}$ must equal 1 in order for the first two constraint to be feasible. If $X_u > X_v$ then $x_{u,v}$ must be 0. But since $x_{u,v}$ is not inherited from the colouring, we can assign these the desired values and the constraints will be satisfied.

   b) Under the assumption that the colours assigned to G are natural numbers $\leq \chi(G)$, the highest value of X_v is exactly the chromatic number of G. The last constraint then ensures that c is at least $\chi(G)$ and when minimized it will become the chromatic number and optimal.

2. An optimal solution to 3.4 returns $\chi(G)$ and assign the vertices colours such that the colouring is optimal:
   Let ζ be an optimal solution to the problem.

   a) The third constraint ensures that no value of X_v is greater than ζ.

   b) The first and second constraints ensures that no two adjacent vertices have the same colour. Thus it is a proper colouring and ζ must be at least $\chi(G)$ for a solution to be feasible. And since ζ is optimal, we can conclude that $\zeta = \chi(G)$

□

The binary encoding formulation

The last formulation of the graph colouring problem for a graph $G = (V, E)$, that this thesis will focus on is the binary encoding formulation. Here we introduce $V \lceil log_2(k) \rceil$ integer variables $x_{v,b}$ with the desired result:

$$x_{v,b} = \begin{cases} 1 & \text{if vertex } v \text{ has the } b\text{'th bit in it's assigned colour set to 1} \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

And another $2E \lceil log_2(k) \rceil$ binary variables $z_{u,v,b}$ and $z_{u,v,b}$ to help create the equality $z_{u,v,b} = |x_{u,b} - x_{v,b}|, \forall (u, v) \in E$. Furthermore we introduce the variable c that has to be greater or equal to the greatest value of any colour used.

Proposition 3.11. *The following formulation assigns an optimal colouring to the graph and gives the chromatic number:*

$$
\begin{array}{lllll}
min & c & & & \\
s.t. & c - \sum_{b=0}^{B} 2^b \cdot x_{v,b} \leq -1, & \forall v \in V & & \\
 & z_{v,u,b} - 2t_{v,u,b} + x_{v,b} - x_{u,b} = 0, & \forall (u,v) \in E \forall b \in [0, \cdots, B] & & \\
 & \sum_{b=0}^{B} z_{v,u,b} \geq 1, & \forall (u,v) \in E & for\ k \geq c, B = \lceil log_2(k) \rceil & \\
 & x_{v,b} \in \{0,1\}, & \forall v \in V \forall b \in [0, \cdots, B] & & \\
 & z_{u,v,b} \in \{0,1\}, & \forall (u,v) \in E \forall b \in [0, \cdots, B] & & \\
 & x_{u,v,b} \in \{0,1\}, & \forall (u,v) \in E \forall b \in [0, \cdots, B] & &
\end{array}
\tag{3.6}
$$

Proof.

1. An optimal colouring is feasible and optimal in 3.6:

   a) Given an optimal solution with values of $x_{v,b}$ defined as our desired results 3.5, we see that at, since adjacent vertices have different colours, at least one bit in the colours are different for such two vertices.
   For such a bit, $x_{v,b} - x_{u,b}$ from the second constraint function is either 1 or -1, and $z_{v,u,b}$ and $t_{v,u,b}$ must be either $z_{v,u,b} = 1$ and $t_{v,u,b} = 0$ or $z_{v,u,b} = 1$ and $t_{v,u,b} = 1$ respectively for the constraint to be satisfied. But since these values are not inherited from the colouring, they can take the appropriate values.

   b) Sine $z_{v,u,b} = 1$ for some bit of two adjacent vertices, the third constraint is also satisfied.

   c) since the first constraint states that c is at least one higher than the value of the highest value colour used (since a vertex can be assigned colour 0 by this formulation), and if we assume the colouring has assigned colours values as low as possible, c will minimize to the chromatic number of the graph.

2. An optimal solution to 3.6 returns $\chi(G)$ and assign the vertices colours such that the colouring is optimal:

a) since all $x_{v,b}$ has a value, all vertices have colours trivially.

b) As established before, a $z_{v,u,b}$ value of 1 indicates that the b'th bit of vertex u and v are different. Likewise we see that since $x_{v,b} - x_{u,b} \in \{-1, 0, 1\}$ and $2t_u, v, b \in \{0, 2\}$ a $z_{v,u,b}$ value of 0 will only be feasible if the b'th bit of u and v are the same.
By that observation we see that the third constraint ensures that at least one bit of the colours of two adjacent vertices are different, making the colours different. Thus the formulation will assign a proper colouring to a graph

c) the first constraint states that c is at least the value of the highest value colour assigned to any vertex. When minimizing c the formulation will therefore assign the lowest amount of colours possible with the lowest possible values where the solution is still feasible. Thus it will assign an optimal colouring to a graph and return the chromatic number.

□

A. Test appendix