2.1 Science, Mathematics and Mathematical Models

By John D. Donaldson
University of Tasmania (Australia), KAIST (South Korea), University of Aarhus (Denmark), University of Copenhagen (Denmark).

Contents

wo fold: dobbelt

1 What is Science?

In our experience the real world is a kaleidoscope of patterns. Happenings or phenomena repeat themselves over and over. Patterns are apparent in all disciplines from the study of the frequency of word use by authors, the usual and unusual psychological behaviour of humans and animals, population growth in biology, crystal growth in chemistry, the rise and fall of the tides to the study of planetary motion.

pparent: delig, enbar

2 The Study of Patterns – Quantitative and Qualitative

Science is the study of patterns in all matters concerned with our universe. They range from numerical patterns which are termed as quantitative problems to patterns of shape and form which are termed as qualitative problems. For example the fact that the planets move about the sun in particular elliptical orbits is quantitative while the fact that concentrations of heat or populations have a tendency to dissipate is qualitative.

ssipate: rede sig

3 History

The history of scientific methods goes back to the early astronomers who realised that there were patterns associated with the daily rising of the sun, the seasons of the year and the motion of the stars. These early observations gave rise to the early developments of seasonal planting in agriculture, while observation of the stars and constellations formed the foundations of navigation.

On a more sinister side the early priests used the eclipses to their advantage, frightening the general population with their power to have the sun disappear and appear again.

sinister: uhyggelig, dyster
eclipse: formørkelse

Having observed patterns, mankind has wished to catalogue and understand them.

4 Understanding the Patterns

John Barrow in his book "Theories of Everything" propounds that *"It seems to be a principal human drive to demonstrate one's insight, wisdom and inside knowledge about the hidden causes of everything."*

propound: foreslå, fremlægge

It is part of human thinking but we are even more rational. We wish to use that knowledge to better our planet and maybe our own lives.

Figure 1

Our aim would be to
- describe the pattern,
- explain the underlying causes,
- predict the future development or find new related patterns and
- control the pattern.

5 The Language of Science – Mathematics

ramework: amme

The natural language to provide a framework for this process is mathematics. Galileo wrote *"the book of nature is written in mathematical language and its characters are triangles, circles and other geometric figures."*

Isadore Singer, the joint recipient with Michael Atiyah of the 2004 Abel Prize for Mathematics, describes mathematics as the study of patterns and their relationships. The usefulness of mathematics especially in physics cannot be ignored but it has beauty within itself.

6 The Two Fold Purpose and Examples

Mankind's desire to understand his environment serves two main purposes, firstly to explain, perhaps satisfying man's curiosity, and then to use the knowledge to advantage.

A major driving force has been the attempt to explain the formation of the galaxies. It would appear not to have a bearing on our lives over the next million years or so but millions of dollars are being spent every year on trying to establish the Big Bang theory of the origins of the Universe. Three years ago, NASA launched a new telescopic satellite into orbit, which will measure characteristics of microwave light in the universe. If the measurements made are as expected, it will give us information about where we came from and the ultimate fate of the Universe.

ver: ftestang, egtstang

It is not difficult to come up with observations, which were developed to advantage. The lever indicated how more weight can be lifted by using the leverage principle.

Figure 2

It is easily seen that the longer the lever the more advantage there is. However one cannot keep on using longer and longer levers because of their weight and strength. One is usually faced with choosing an optimal length, as in the case of using a garden spade.

The idea of optimisation goes back several centuries. Queen Dido of Carthage is supposed to have used the idea of optimisation. She chose a circular boundary as the boundary of fixed length, which enclosed the maximum area when selecting the site for her city.

optimisation: optimering

Decade after decade, scientists are finding new ways of improving our lives.

7 Quantitative Information

Up until the last century many of the problems were solved in terms of quantitative information. The interest in the universe was concerned with the actual orbits of the planets and the actual number of people in a country.

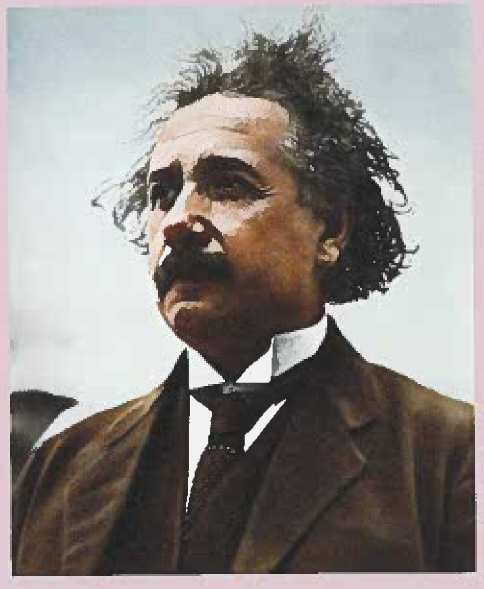

Figure 3
Albert Einstein (1879-1955).

The constants in Newton's equations were made more and more precise and Einstein's relativity went further in explaining the small differences in the orbit of Mercury. The concept was very important in putting satellites into orbit, satellites which would make measurements and provide data on the earth's environment of profound importance to man.

profound: dybtgående

The number of people the earth could support was also of major concern and the mathematical models were quite definitive. The parameters involved were obtained using statistics but the objective was quantitative.

8 Qualitative Information

With the introduction of computers scientists were able to handle much more complex problems and ask more general questions. As was noted earlier there is interest in how the galaxies were formed not in the number of galaxies. Questions can be asked about whether the hot house effect will happen or under what conditions it will happen. Questions about the stability of populations are asked rather than how many there are in the population.

We study general weather patterns to use in our predictions and we can attempt to predict whether the financial stock market will rise or fall rather than predict the price of a particular stock.

9 Mathematical Modelling

Throughout our world, many people have a fear or a dislike of mathematics, mainly because it is considered difficult and not very useful. To overcome this attitude, applications of mathematics in an area of interest to an individual or group of individuals could be explained. There is a wide range of applications at varying levels of sophistication that can be used to illustrate the beauty and power of mathematics. There are examples such as those in the areas of art, architecture, beauty, music, nature, astrology, ethics, economics, politics, literature, poetry, religion and even in the flow of traffic. The objective is to engage the individual with introductory level mathematical applications in their areas of interest, which are designed to capture their imagination. This could be followed by illustrations of the power of mathematics to give insight into more complex forms of the problem.

Effectively this process requires the development of a mathematical model, a description in the language of mathematics.

In forming a mathematical model, the steps taken are usually well defined. The following is an outline of the general process:

- identify the pattern or physical situation;
- choose the state variables – the variables for measuring the pattern;
- obtain relationships connecting the variables, such as equations from the laws of physics;
- obtain mathematical solutions – this may require new mathematics;
- compare the solutions with the physical situation.

ite:
tand

If the comparison is not valid, a return is made to one of the previous steps and the model is redeveloped.

If the comparison is valid, the model is used for:

- further understanding the physical situation – discovering new patterns,
- predicting what can happen,
- controlling the physical situation and extending the problem to a more complex physical situation.

Obtaining mathematical solutions are not necessarily straightforward and numerical or computational solutions may be the only ones available. The numerical methods must be checked; too often computer output is accepted without question.

The numerical output should compare well with the pattern. It should be compared with known solutions.

Experimenting with the parameters, the numerical output may reveal trends and ideas about the physical situation allowing for a closer look at mathematical solutions.

10 Example

Let us consider mathematics on the sporting field.

Throwing the javelin, putting the shot and the long jump have some obvious similar patterns. We identify that the objects in motion, the javelin, the shot or the jumper, follow curved paths from take-off point to landing. The shape of the curved path looks similar.

javelin: kastespyd

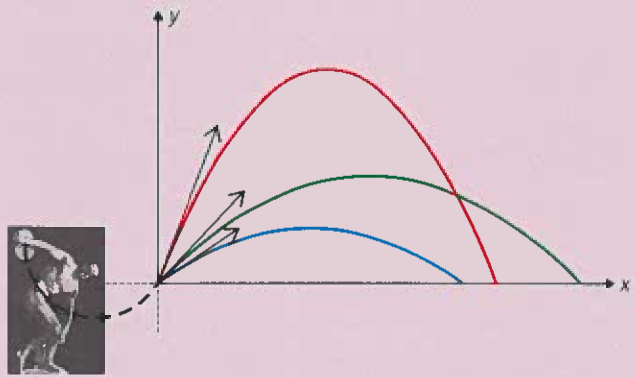

Figure 4

We would like to have more knowledge of the path and what causes it to change. Can we predict the landing point? What control can we exert over the path? How can we use the information to achieve the objective of maximizing the distance travelled by the object?

Suppose the motion takes place in a vertical plane whose origin is at the take-off point and the x-axis is horizontal in the direction of the throw and the y-axis is vertical. We will in the first instance suppose that there is no air resistance. The state variables which describe the path are given by the coordinates (x, y). The coordinates will change with time during the motion so we can write $x = x(t), y = y(t)$.

Our first objective is to find the relation between x and y which will determine the shape of the path. To do this we need to know how they change with t in accordance with the physical laws that control them.

Consider the x-coordinate. If there is no air resistance there is nothing to alter the speed, v_x or rate of change of x with respect to t, in that direction. So we have $x'(t) = \frac{dx}{dt} = v_{0x}$ for some constant, v_{0x}. This means that

$$x = \int v_{0x} dt = v_{0x} t + k,$$

where k is a constant.

In the motion in the direction of the y-coordinate or vertical direction, there is obviously some factor, which is causing the speed to slow down as the object gains height and increase as it falls back. We take the simplest case of the rate of slowing down or speeding up to be constant, the quantity, g, the value of the gravitational acceleration at sea level. In this case the rate of change of the speed, v_y, in the y-direction, satisfies $v_y'(t) = \frac{dv_y}{dt} = -g$. This gives us

$$v_y = \int -g\, dt = -gt + v_{0y}$$

and since $y'(t) = \frac{dy}{dt} = v_y$,

$$v_y = \int (-gt + v_{0y}) dt = -\frac{1}{2}gt^2 + v_{0y}t + c\,, \ (1)$$

where v_{0y} and c are constants to be determined.

We have

$$x = v_{0x}t + k,\ y = -\frac{1}{2}gt^2 + v_{0y}t + c\,. \ (2)$$

It is clear that there are significant differences in the paths of the objects and this will be reflected in the value of the constants, v_{0x}, v_{0y}, k and c.

If we choose the take-off point to be the origin at time $t = 0$, we see putting $t = 0$ in the equations (1) and (2), gives us $k = c = 0$. Since $\frac{dx}{dt} = v_{0x}$ and $\frac{dy}{dt} = v_{0y}$ at time $t = 0$, then v_{0x} and v_{0y} are the components of the velocity given to the object at take-off.

velocity: hastighed

Now we find on eliminating t, from equations (1) and (2) that

$$y = -\frac{g}{2v_{0x}^2} \cdot x^2 + \frac{v_{0y}}{v_{0x}} \cdot x\,.$$

This is the equation of a parabola and we can examine the parabola to give us further information and in particular the distance travelled in the horizontal direction. This will happen when the parabola once again intersects with the x-axis or when $y = 0$. Since the solution $x = 0$ is the starting point the other solution

$$x_{max} = \frac{2v_{0x}v_{0y}}{g} \ (3)$$

is the one of interest.

Alternatively we can find the distance in the x-direction by observing that the object will be at ground level when $y = 0$. This happens at time $t = 0$, when it leaves the take-off point, or at time $t_{max} = \frac{2v_{0y}}{g}$, when it returns to the ground level. So the distance in the horizontal direction is, as in equation (3),

$$x_{max} = v_{0x} \cdot t_{max} = v_{0x} \cdot \frac{2v_{0y}}{g}\,.$$

We have in fact predicted the length of the throw or jump.

Now we can turn our attention to controlling the motion to our advantage, in this case to maximize the distance x.

Firstly we notice that the bigger the values of v_{0x} and v_{0y} the further the distance. But, this is limited by the natural attributes (speed or strength) of the athlete, which govern the magnitude of the velocity at take-off.

However notice that if the effort is all put into the y-direction, $v_{0x} = 0$ and so $x = 0$, while if all the effort is put into the x-direction, $v_{0y} = 0$ and again $x = 0$. So we expect that there will be a way of splitting the effort into both directions, which will be better.

Suppose that this direction makes an angle, α, with the x-axis. Then, if the initial speed has magnitude, v_0, in that direction, the components in the x- and y-directions will be $v_{0x} = v_0\cos(\alpha)$ and $v_{0y} = v_0\sin(\alpha)$ respectively.

The distance travelled is

$$x_{max} = \frac{2v_0 \cos(\alpha) \cdot v_0 \sin(\alpha)}{g} = \frac{v_0^2 \sin(2\alpha)}{g} \quad *)$$

and is dependent only on the angle α.

The maximum distance will be when $\sin(2\alpha)$ is maximum. Now the maximum value of the sine function is 1 and this occurs when $2\alpha = 90°$ or $\alpha = 45°$.

The athlete can now try to reach this optimum value of take-off direction to increase the distance.

However, there are physical limits. It is found that long jump athletes cannot achieve take-off angles more than about 22°. Nevertheless the better jumpers take-off at angles closer to the optimum.

11 Conclusion: The Importance of Science and Mathematical Models

Science has created many problems for man such as the spread of disease by increase in the development of transport, the terror of atomic weapons and the possible permanent changes in our weather patterns. But the advantages of science have benefited mankind in so many ways as well as satisfied our curiosity with the nature of the patterns in our universe. The role, played by mathematical modellers in explaining and understanding these patterns and controlling and optimising the benefits, has considerable scope.

Experimentation through trials and tests is becoming more and more expensive and is being subjected to more ethical practices. Pharmaceutical drugs are readily designed for specific problems but to gain acceptance they require testing for efficacy. Animal experiments take several years and have high costs as well as being subject to strict controls and dissent from the general public. Sheer size also limits the range of experiments that can be performed – for example, the cost of testing differing shapes of ships' hulls is prohibitive. Experimentally replicating all of the conditions undergone in a particular situation is inconceivable.

Using mathematical models and computer simulations of these physical situations allows a wide range of trials at minimal cost. Mathematics transcribes patterns into a language that can be organised and reorganised logically to describe, explain, predict, and, if possible or if desired, suggest controls of those patterns to the advantage of humankind.

ɔpe:
ɪfang
ficacy:
knings-
ldhed
ssent:
enings-
rskel,
igende
ening
ill:
rog
ohibitive:
werkom-
elig
plication:
ntagelse
forsøg
conceiv-
ble:
ænkelig
anscribe:
nskrive

Opgaver til John D. Donaldsons artikel findes som opgave 2001-2008 i arbejdsbogen.

*) Her benyttes formlen $\sin(2x) = 2 \cdot \sin(x) \cdot \cos(x)$.

2.2 Modelbegrebet

Geometriske figurer ved bestemmelse af højden af en flagstang eller rumfanget af en cylinderformet dåse er eksempler på *matematiske modeller*. Andre eksempler på modeller er variabelsammenhænge. Nogle af disse eksempler tager vi her op til fornyet og udvidet behandling. Derudover behandler vi især modeller, hvor differentialregning har betydning.

Først eksemplet med Danny Taxa fra Grundforløbsbogen:
Ligningen $p = 20 \cdot s + 42$ viser sammenhængen mellem den kørte strækning s (km) og prisen p (kr.) for en taxatur med Danny Taxa. Denne ligning indeholder elementer, der er typiske for modeller:

- En sammenhæng (her: lineær) mellem nogle *variable* størrelser fra virkeligheden (her: pris p og kørt strækning s).
- Nogle talværdier, som kaldes modellens *parametre* (her: kilometerprisen er 20 kr., og startgebyret er 42 kr.).

Modellen $p = 20 \cdot s + 42$ gør det fx muligt at forudberegne, hvad det koster at køre en bestemt strækning. Modellen er en matematisk oversættelse af taxafirmaets takstreglement. Læg mærke til, at det er muligt at oversætte begge veje. Takstreglementet kan oversættes til ligningen $p = 20 \cdot s + 42$, men ud fra ligningen kan man også aflæse takstreglementet.

En model har som regel nogle begrænsninger. Vi kan næppe regne med, at en tur fra København til Rom med taxa går efter de samme regler. Vi kan godt sætte $s = 2200$ km ind i formlen. Derved fremkommer prisen $p = 44042$ kr., men resultatet har sikkert ikke noget med virkeligheden at gøre. Hvor stort denne models *gyldighedsområde* er, kan fremgå af takstreglementet.

Følgende er typisk for de matematiske modeller, vi behandler:

DEN MATEMATISKE MODEL

- beskriver en situation fra virkeligheden
- angiver sammenhænge mellem variable størrelser fra virkeligheden (tid, pris, temperatur, hastighed, befolkningstal, ...)
- indeholder parametre (kilometerpris, startgebyr, begyndelsestemperatur, årlig rente i procent, ...), der er karakteristiske for den situation fra virkeligheden, der skal beskrives
- kan have et begrænset gyldighedsområde
- kan bruges til at give større indsigt i og overblik over den situation fra virkeligheden, der skal beskrives
- kan fx anvendes til prognoser og andre beregninger.