Loss Sharing in Central Clearinghouses: Winners and Losers

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Abstract

Central clearing counterparties (CCPs) were established to mitigate default losses resulting from counterparty risk in derivatives markets. In a parsimonious model, we show that clearing benefits are distributed unevenly across market participants. Loss sharing rules determine who wins or loses from clearing. Current rules disproportionately benefit market participants with flat portfolios. Instead, those with directional portfolios are relatively worse off, consistent with their reluctance to voluntarily use central clearing. Alternative loss sharing rules can address cross-sectional disparities in clearing benefits. However, we show that CCPs may favor current rules to maximize fee income, with externalities on clearing participation.

Keywords: Central Clearing, Counterparty Risk, Loss Sharing, OTC markets, Derivatives.

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1 Introduction

Whether we choose to bolster the tools for CCP resilience, CCP recovery or CCP resolution, we will need to be aware of potential trade-offs in the way losses are allocated, and remember that there may be no ideal approach.[1]

Default losses occur when counterparties fail to fulfill their obligations, e.g., when they default. Counterparty risk, which refers to the risk of default losses, is one of the most important risks in over-the-counter (OTC) derivatives markets and has been identified as a significant factor in the 2007-08 financial crisis. To mitigate this risk, regulators worldwide have advocated for the central clearing of OTC derivative transactions through central clearing counterparties, known as *CCPs* (G20, 2009).[2] The main tasks of CCPs are to reduce the total amount of default losses through netting and margin requirements and to allocate the remaining losses to non-defaulted clearing members through loss sharing. Thus, loss sharing rules determine how the potential benefits of central clearing for counterparty risk are divided among clearing members. As a result, loss sharing rules may impact clearing participation, with important consequences for risk sharing. Motivated by the importance of counterparty risk and central clearing for financial stability, this paper provides an in-depth examination of the impact of loss sharing rules on counterparty risk in markets with heterogeneous market participants.

We investigate the role of loss sharing rules from the perspective of counterparty risk, which is measured by expected default losses. Default losses have significant economic implications. For instance, in September 2018, the default of a single trader at the Swedish clearinghouse *Nasdaq Clearing AB* resulted in EUR 107 million to be shared among surviving clearing members (Faruqui

[1]Remarks by Benoît Cœuré, then member of the Executive Board of the European Central Bank, at the Federal Reserve Bank of Chicago 2015 Symposium on Central Clearing. Available at https://www.ecb.europa.eu/press/key/date/2015/html/sp150411.en.html.

[2]OTC derivatives markets are very large, with a worldwide gross market value outstanding of $18 trillion in 2017 (*source: BIS OTC derivatives statistics 2022:H1*). Before the 2007-08 financial crisis, the derivatives market architecture was dominated by bilateral trades (FSB, 2017). The G20 initiative in 2009 was followed by the Dodd-Frank Wall Street Reform and Consumer Protection Act (DFA) in 2010 and the European Market Infrastructure Regulation (EMIR) in 2012, with the mandatory central clearing of certain OTC derivatives as a key element. More recently, a central clearing mandate has also been suggested for other asset classes, such as US treasuries (Duffie, 2020; Fleming and Keane, 2021).

et al., 2018). More generally, counterparty risk is an important determinant of clearing participation (FSB, 2018; Bellia et al., 2023; Vuillemey, 2020) and affects derivatives prices (Boissel et al., 2017; Cenedese et al., 2020). Using a parsimonious model, we compare both the aggregate and entity-specific counterparty risk between a market with central clearing and an uncleared market. Importantly, we focus on environments with heterogeneous market participants. For example, in core-periphery networks some entities (akin to dealers in practice) trade with many counterparties while maintaining flat portfolios, whereas other entities (akin to end-users in practice) exhibit a directional portfolio with a small number of counterparties. Our results shed light on disparities in clearing benefits between such market participants. We provide comparative statics for the impact of changes in the characteristics of the market, derivatives contracts, portfolios, and margin costs as well as in the loss sharing rule on clearing benefits. Finally, we endogenize the loss sharing rule in a model with a profit-maximizing CCP.

There are four main insights. First, the net-to-gross ratio, which represents portfolio directionality, is the key determinant of clearing benefits, both in aggregate as well as for individual entities. Second, the current market practice of proportionally sharing a CCP's default losses based on net portfolio risk favors entities with a less directional portfolio, resulting in greater clearing benefits for them. At the same time, entities with more directional portfolios are worse off and may even increase their counterparty risk compared to an uncleared market. Third, an alternative loss sharing rule, which is based on a weighted average of net and gross portfolio risk, balances clearing benefits across different market participants, ensuring that all entities benefit equally from clearing. Finally, even though this alternative loss sharing rule mitigates disincentives to use central clearing, it does not necessarily maximize the profit a CCP can generate from fees. Instead, we demonstrate that a CCP might prefer to attract only entities with a flat portfolio because these exhibit a higher willingness to pay for clearing.

Despite the increasing importance of central clearing in derivatives markets, with a gross market value of $8.2 trillion in interest rate and $104 billion in foreign exchange contract positions at

CCPs (*source: BIS OTC derivatives statistics 2022:H1*), research on loss sharing in clearinghouses is still scarce. We extend the existing literature by investigating heterogeneity in clearing benefits among market participants and the role of loss sharing rules. Our results have significant implications for policymakers. First, heterogeneity in expected default losses is crucial from a financial stability perspective because assigning default losses to systemically important entities can lead to amplification of initial losses. Second, loss sharing rules influence the incentives of market participants to use central clearing, with significant consequences for risk sharing in derivatives markets. Third, due to their impact on default losses, loss sharing rules may affect derivative prices, with a potential feedback effect on hedging costs for the real economy. Lastly, as clearing regulation continues to be refined, it is essential to understand the incentives of agents in the political process.

We commence our analysis by extending Duffie and Zhu (2011)'s model of counterparty risk exposure to incorporate heterogeneity in market participants' derivative portfolios. Counterparty risk exposures reflect the expected default losses in case all counterparties default. Hence, it serves as a measure for netting efficiency. Our results demonstrate that portfolio directionality, given by the net-to-gross ratio, is the primary driver of clearing benefits. The smaller the directionality, the larger is the scope for netting through the CCP (so-called *multilateral netting*) and, thus, the more beneficial is central clearing. Instead, entities with a sufficiently directional portfolio do not experience a reduction in counterparty risk exposure through central clearing: in exchange for fewer bilateral netting benefits (across derivative classes with individual counterparties) they receive limited multilateral netting benefits.

This intuition carries over to the impact of central clearing on expected aggregate default losses. Moreover, margin requirements are important. The stricter the margin requirement for cleared relative to uncleared positions, the greater the reduction in expected default losses through central clearing. We show that central clearing reduces expected aggregate default losses only when either it is accompanied by a stricter margin requirement or at least one market participant has multilateral netting opportunities.

We then shift our focus to individual market participants. This is where the loss sharing rule becomes relevant as it determines the allocation of the CCP's default losses that remain after multilateral netting and the use of a defaulted entity's collateral. We begin by considering loss sharing proportional to net portfolio risk, which closely aligns with current market practice. In this case, entities with lower portfolio directionality benefit more from central clearing. We conduct comparative statics and analyze the impact of central clearing on expected default losses in markets with homogeneous market participants as well as in core-periphery networks. Specifically, we demonstrate that central clearing can be beneficial in aggregate and for entities with a flat portfolio ("core entities") but, at the same time, *harmful* for entities with a directional portfolio ("peripheral entities").

Our results are consistent with the reluctance of end-users to participate in loss sharing in practice. End-users largely either do not use central clearing (if it is not mandatory) or minimize their exposure to loss sharing by using client clearing.[3] Based on anecdotal evidence from the industry and regulators, an important driver of this reluctance is loss sharing. In fact, end-users emphasize that they bear disproportionately large costs of loss sharing (Novick et al., 2018), which is confirmed by our model. Our results provide an explanation for the reluctance of end-users to use loss sharing and central clearing by showing that market participants with a directional portfolio, such as end-users, benefit the least from or might even be hurt by loss sharing rules used in practice. While there are also other determinants of end-users' clearing decision, such as fixed costs to satisfy membership requirements, our analysis reveals one important determinant of clearing costs and participation, which is important for understanding both dynamics in derivatives markets and the potential costs of clearing mandates.

Loss sharing rules are neither mandated by regulation to be net-based nor is a net-based loss sharing rule necessarily optimal. We investigate the design of the loss sharing rule and consider

[3] CCP membership requirements do not generally prohibit end-users from becoming clearing members. Instead, regulation requires non-discriminatory access to clearing. Nonetheless, only very few financial institutions other than banks and broker-dealers (e.g., insurers, investment or pension funds, or non-financial companies) are clearing members (BIS, 2018).

loss sharing to be proportional to a weighted average of net and gross portfolio risk. Gross portfolio risk reflects a market participant's total transaction volume and, for this reason, has been highlighted as an important dimension to take into account when allocating default losses (Cont, 2015).[4] We show that the impact of increasing the weight of gross relative to net risk in the loss sharing rule depends on entities' portfolio directionality. Whereas a larger weight of gross risk increases the clearing benefit of entities with directional portfolios, it reduces that of entities with flat portfolios. Thus, initial disparities in clearing benefits between market participants shrink. We show that in core-periphery networks there exists a unique weight of gross risk such that the associated loss sharing rule exactly balances clearing benefits between core and peripheral entities.

Importantly, changes in the loss sharing rule do not affect the aggregate clearing benefit, which is determined by netting and margin requirements, but only impact the *distribution* of remaining default losses among clearing members. Thus, our analysis illustrates and distinguishes three roles of multilateral netting. First, multilateral netting reduces exposure *to* the CCP and, thereby margin requirements. Second, it reduces overall default risk and, thus, determines the benefit of central clearing for aggregate default losses. Third, netting may impact how clearing benefits are split among individual clearing members, depending on the loss sharing rule.

Our results show that taking gross risk into account can remove differences in clearing benefits between different entities and, thereby, maximize clearing participation. Why are loss sharing rules, instead, based entirely on net risk in practice? We highlight the market power of for-profit CCPs as one potential explanation. Post-crisis financial regulation requires loss sharing but does not prescribe a specific loss sharing rule, which, instead, is chosen by the CCP. At the same time, the market for central clearing is highly concentrated and dominated by few for-profit clearinghouses.[5] Motivated by this observation, the CCP in our model maximizes its total fee income by setting both the loss sharing rule and fees. Fees are volume-based, whereas loss sharing rules

[4]Rules based on gross risk are not uncommon. For example, the Basel III leverage ratio is based on derivatives' gross notional amount (see https://www.bis.org/publ/bcbs270.htm).

[5]For instance, in the USD and EUR interest rate and credit risk derivatives markets, four clearinghouses (*LCH*, *CME*, *Eurex*, and *ICE*) account for nearly 100% of cleared transactions, and all of them are for-profit organizations.

discriminate depending on clearing members' portfolio risk, consistent with market practice and regulation. The optimal clearing rule must be consistent with clearing members' incentives to use central clearing in the sense that the fee does not exceed the benefit of central clearing.

Two possible optimal clearing rules emerge. Either the CCP maximizes clearing participation by attracting all entities in the market or it maximizes clearing members' willingness to pay by attracting only entities with a flat portfolio. In the first case, the optimal loss sharing rule offers the same clearing benefit to all entities by taking gross risk into account. In the second case and using a refinement based on small perturbations in clearing participation, the optimal loss sharing rule is proportional to net risk because it maximizes the willingness to pay for central clearing of entities with a flat portfolio and, thus, enables the CCP to request a larger fee. We show that the CCP prefers this second rule, curtailing clearing participation, if overall central clearing benefits are relatively small, e.g., if there are few opportunities for multilateral netting through the CCP. In this case, it is optimal for the CCP to not attract peripheral entities in order to be able to charge a larger fee from remaining clearing members. Hence, our analysis reveals the incentives for the CCP to use net-based loss sharing rules to maximize fee income from entities with a flat portfolio.

In addition, other considerations may shape the choice of loss sharing rules. On one hand, net-based loss sharing rewards low portfolio directionality and, thereby, may incentivize market participants to reduce directional risk and provide liquidity. On the other hand, penalizing portfolio directionality implies an increase in hedging costs for end-users. Trading off these effects provides an important avenue for future research.

Regulators aim to enlarge clearing participation (G20, 2009; FSB, 2018). Broad adoption of central clearing may be desirable to boost risk sharing and transparency in derivatives markets (Acharya and Bisin, 2014), and mitigate information frictions (Vuillemey, 2020) and counterparty risk (Bernstein et al., 2019). Our analysis suggests that CCPs' incentives may not fully align with this goal. Instead, it may be optimal for a CCP to *not* maximize clearing participation in order to extract larger fees from dealers. Therefore, an important avenue for future research is to investigate the implications of loss sharing rule choice on social welfare and the extent to which regulatory policies can mitigate its potential externalities.

In an extension of our model, we show that our baseline results are robust to including a small cost of collateral. In this case, the beneficial effect of collateral on default losses dominates. In contrast, if collateral is sufficiently costly, a larger margin requirement for cleared positions *reduces* clearing benefits.

Our analysis focuses on the risk of default losses and, therefore, does not incorporate other potential benefits or costs of central clearing, such as its impact on capital requirements, market transparency, or market liquidity. Throughout the paper, we consider expected default losses as a function of positions, which we treat as exogenous. We note that our results have potentially important implications for derivatives trading behavior, which suggests an interesting avenue of future research beyond the scope this paper. We discuss these implications and related equilibrium trade-offs and policy implications.

2 Literature Review

We contribute to a growing literature on central clearing and its role in derivatives markets. Previous studies have examined loss sharing and its interaction with CCP collateral and fee policies (Capponi et al., 2017; Capponi and Cheng, 2018; Huang, 2019) and with risk management incentives (Biais et al., 2012, 2016; Antinolfi et al., 2022; Wang et al., 2022).[6] In Kuong and Maurin (2022)'s model, the tension between loss sharing and risk management incentives motivates the CCP's ownership structure and default waterfall design. Wang et al. (2022) show that pre-funded default fund contributions are economically more efficient to align risk management incentives than initial margins if covering losses ex-post is costly. In these models, market participants typically trade one contract and differ only in the direction of trade, i.e., whether they are sellers or

[6] Huang and Zhu (2021) examine the design of default auctions. Menkveld (2017) and Huang et al. (2020) take a CCP's perspective and identify extreme price movements as well as portfolio concentration as important risks to CCP stability. Menkveld and Vuillemey (2021) provide an overview of the literature on central clearing.

buyers. We complement previous studies by focusing on heterogeneity in market participants' portfolio directionality. Thus, market participants may trade (partly) offsetting contracts, such as dealers in practice (Getmansky et al., 2016). We are, to the best of our knowledge, the first to investigate the distributional effects of loss sharing on market participants with different portfolio directionality. Moreover, we complement the previous literature by investigating the role of different loss sharing rules and a profit-maximizing CCP's incentives when choosing the loss sharing rule.

Duffie and Zhu (2011), Cont and Kokholm (2014), and Lewandowska (2015) study the impact of multilateral versus bilateral netting on counterparty risk exposure. Their main result is that a sufficiently large number of clearing members guarantees that central clearing reduces counterparty risk. Ghamami and Glasserman (2017) study the capital and collateral costs of central clearing and conclude that margin costs likely dominate potential clearing benefits in practice. Their result is contrasted by the FSB (2018)'s assessment that central clearing reforms create an overall incentive to clear.

Our framework builds on the model of Duffie and Zhu (2011) and considers mainly two important extensions. First, whereas Duffie and Zhu (2011) take an ex-ante perspective from which derivatives positions are random, we consider loss sharing as a function of a (fixed) set of derivatives portfolios. This allows us to explicitly distinguish between entities with different portfolios. We show that it is *not* a large number of counterparties per se but, instead, a low portfolio directionality that creates clearing benefits (which is more likely to realize when entities trade randomly with more counterparties in Duffie and Zhu (2011)'s model). Second, whereas Duffie and Zhu (2011) focus on the case that all counterparties—including the CCP—default, we more generally allow any number of market participants to default. Thus, a given entity is exposed to the risk of loss sharing contributions even when clearing members which are not the entity's counterparties default. We show that this is important to take into account in order to reveal the role of loss sharing and loss sharing rules.

Empirical evidence on the impact of central clearing on derivative markets has been growing only recently, fueled by the increasing availability of granular data. Recent examples are Loon and Zhong (2014), Duffie et al. (2015), Bellia et al. (2023), and Du et al. (2022) for single-name CDS, Menkveld et al. (2015) for equity, Mancini et al. (2016) and Boissel et al. (2017) for interbank repo, and Cenedese et al. (2020) and Dalla Fontana et al. (2019) for IRS markets. The results by Bellia et al. (2023) show that contracts with risky counterparties and large netting benefits are more likely to be cleared than uncleared, suggesting that counterparty risk and netting are indeed highly relevant for clearing participation. This result is consistent with the historical evidence documented by Vuillemey (2020), who shows that the global coffee crisis in 1880-81 motivated a group of coffee traders to create a CCP specifically to mitigate counterparty risk.

3 Model

In this section, we describe our model. Default losses result from replacement costs, which are changes in contract values during the settlement period, i.e., the time until liquidation or settlement after a counterparty's default (see Figure 1).[7] Without loss of generality, we consider a one-period model. At time $t = 0$, derivative contracts are written (or, equivalently, contracts are marked to market by the exchange of variation margin) and, subsequently, counterparties might default. At time $t = 1$, contracts are settled.

[Place Figure 1 about here]

Derivative positions are sorted into $K \geq 2$ derivative classes. This classification can result for different reasons, for example from grouping derivatives by contract type or underlying, such as interest rate, credit, commodities, or equities.

[7]The length of the settlement period depends on the liquidity of contracts and typically ranges from 2 to 5 days (Arnsdorf, 2012). For example, initial margins for OTC foreign exchange and IRS trades is based on a 5-day settlement period at CME (see their CPMI-IOSCO Quantitative Disclosures for 2019Q3).

There are $N \geq 3$ market participants (or, equivalently, *entities*), indexed $i = 1, ..., N$, which trade in all derivative classes K. The binary random variable D_i indicates the event that entity i defaults ($D_i = 1$) or survives ($D_i = 0$). The probability of default is equal to $\mathbb{P}(D_i = 1) = \pi \in (0, 1)$. Defaults are mutually independent. A defaulted entity does not honor any obligations arising from derivative contracts to its counterparties (including the CCP). However, liabilities from surviving entities (or the CCP) toward a defaulted entity must be paid.

We denote by $v_{ij} \in \mathbb{R}$ the position of entity i with j in class k.[8] We allow positions to differ across counterparties but not across derivative classes.[9]

The absolute size $|v_{ij}|$ is the trade volume and $sign(v_{ij})$ the direction. By symmetry, $v_{ij} = -v_{ji}$, and it is $v_{ii} = 0$. We define by $\mathcal{N}_i = \{j : v_{ij} \neq 0\}$ the set and by $N_i = |\mathcal{N}_i|$ the number of i's counterparties. By definition, $v_{ij} = 0$ if $j \notin \mathcal{N}_i$. Each entity trades at least with one other counterparty, $N_i > 0$.

During the settlement period, entity i's net profit with j in derivative class k is given by $X_{ij}^k = v_{ij} r^k$, where r^k is the return in class-k during the settlement period. r^k is the same for all entities, i.e., all entities trade the same class-k contract (or portfolio). Thus, profits across entities within each derivative class only differ by positions v_{ij}.[10]

Contract returns are normally distributed with zero mean, $\mathbb{E}[r^k] = 0$. Symmetry substantially reduces the dimension of our model and improves its tractability.[11] We consider a single-factor

[8]We treat positions as exogenous and focus on the impact of central clearing on default losses as a function of positions. We note that our results have implications for trading behavior, which suggests an interesting avenue of future research beyond the scope this paper.

[9]The assumption that networks are similar across derivative classes is broadly consistent with empirical evidence. For example, Abad et al. (2016) document that the network of gross notional links between counterparties in the European interest rate swap market resembles those of the European CDS and foreign exchange derivatives markets. Nonetheless, the specific positions of single entities may differ in practice across derivative classes. It is possible to extent our model to incorporate such heterogeneity in networks across positions, however, we do not expect that it would qualitatively affect our results.

[10]In a previous version of the paper, we have additionally considered risk that is idiosyncratic to entities, which does not qualitatively affect the results.

[11]Due to the small time horizon of the settlement period, the risk-free rate and risk premium are negligible. Individual contracts may exhibit skewed and fat-tailed distributions. However, the assumption of normally distributed returns may be appropriate for diversified portfolios. It allows us to work with closed-form analytical solutions and we do not expect that it affects the main results qualitatively.

model for contract returns:

$$r^k = \beta M + \sigma \varepsilon^k. \tag{1}$$

$\varepsilon^k \sim \mathcal{N}(0,1)$ is idiosyncratic risk, i.e., for $k \neq m$, ε^k and ε^m are independently distributed, and ε^k and M are independently distributed for all k. The systematic risk factor $M \sim \mathcal{N}(0, \sigma_M^2)$ serves as a latent variable that reflects macroeconomic conditions (e.g., the S&P 500 stock market index), and β is the systematic risk exposure of derivative contracts. For simplicity and tractability, we assume identical distributional properties across entities and derivative classes.

Remark 1 (Difference to Duffie and Zhu, 2011)**.** *Equation (1) implies that contract returns are correlated across entities within derivative classes, e.g., because all entities trade the same contract (portfolio). This nonzero correlation is an important difference to the model of Duffie and Zhu (2011), in which they assume uncorrelated contract returns, namely that* $cor(X_{ij}^k, X_{mn}^k) = 0$ *for* $\{i,j\} \neq \{m,n\}$*. The reason is a difference in perspectives. Duffie and Zhu (2011) consider positions to be unknown and, thus,* $cor(X_{ij}^k, X_{mn}^k) = 0$ *reflects that positions of entity pairs* (i,j) *and* (m,n) *are independently distributed. Instead, we consider deterministic positions* v_{ij} *to reveal how differences in positions affect the impact of central clearing.*

Market participants exchange collateral (i.e., initial margin) with each other and with the CCP. We assume that collateral is based on portfolio risk but not on default risk, which improves the model's tractability, is consistent with common market practice, and does not qualitatively affect heterogeneity in clearing benefits across market participants because we assume that market participants exhibit the same default risk.

For uncleared positions, we parametrize the collateral posted by i to j as a Value-at-Risk of i's bilateral portfolio profit, namely $C_{ij}^K = VaR_{\alpha_{uc}}\left(\sum_{k=1}^K X_{ij}^k\right)$, where $\alpha_{uc} \in [0.5, 1)$ is the confidence level.[12] $\alpha_{uc} = 0.5$ corresponds to an environment without collateral. The larger α_{uc}, the more

[12]Using a Value-at-Risk approach is common industry practice (ISDA, 2013) and consistent with regulation (BIS, 2019). For example, CME sets initial margins at the 99% VaR for futures and options and at the 99.7% VaR for interest rate swaps (see CME's CPMI-IOSCO Quantitative Disclosures 2019Q3).

protected is j against a default of i. Analogously, the collateral posted by i to the CCP is given by the Value-at-Risk of i's portfolio profit with the CCP, namely $C_i^{CCP} = VaR_{\alpha_{CCP}}\left(\sum_{j=1}^{N} X_{ij}^{K}\right)$, where $\alpha_{CCP} \in [0.5, 1)$ is the confidence level.

4 Counterparty Risk Exposure

We start our analysis by investigating netting efficiency through the lens of counterparty risk exposure before collateral (i.e., with $\alpha_{CCP} = \alpha_{uc} = 0.5$) in the spirit of Duffie and Zhu (2011), which reflects expected default losses conditional on the default of counterparties and the CCP. We first define portfolio directionality:

Definition 1. *The gross position of entity i in a given derivative class k is given by*

$$G_i = \sum_{j \in \mathcal{N}_i} \left| v_{ij} \right|. \tag{2}$$

The net-to-gross-ratio, defined by

$$\eta_i = \frac{\left| \sum_{j \in \mathcal{N}_i} v_{ij} \right|}{G_i}, \tag{3}$$

is a measure for the directionality of entity i's portfolio. η_i corresponds to the average net position per \$1 traded, and ranges from zero (flat) to one (directional). Both gross position and net-to-gross ratio are independent of trading direction, i.e., whether a portfolio is net long or short.

The following lemma decomposes portfolio risk into an entity's gross position, directionality, and contract volatility.

Lemma 1 (Portfolio risk)**.** *The standard deviation of entity i's portfolio in a given derivative class is given* *by*

$$\bar{\sigma}_i = G_i \eta_i \sqrt{\beta^2 \sigma_M^2 + \sigma^2}.$$

First, we consider an uncleared market. We assume that all entity pairs have bilateral (close-out) netting agreements with each other. Netting agreements aggregate outstanding positions into one single claim (Bergman et al., 2004) and are common market practice (Mengle, 2010). Bilateral netting offsets gains and losses of different derivative trades across different derivative classes (e.g., IRS and CDS) with a single counterparty. Thus, a counterparty j's default results in default losses for entity i only if i makes a *net* profit. If all derivative classes are uncleared, then the total counterparty risk exposure of entity i is given by

$$\mathbb{E}\left[E_i^K\right] = \mathbb{E}\left[\sum_{j \in \mathcal{N}_i} \max\left(\sum_{k=1}^{K} X_{ij}^k, 0\right)\right] = \varphi(0) G_i f(K), \tag{4}$$

where $f(K) = \sqrt{\beta^2 \sigma_M^2 K^2 + \sigma^2 K}$.

Second, we introduce central clearing. Following Duffie and Zhu (2011), we examine the case that one derivative class is centrally cleared while others remain uncleared. If derivative class K is centrally cleared by a CCP, all entities $i = 1, ..., N$ become clearing members at the CCP (we relax this assumption in Section 6) while the CCP is the single counterparty to all positions in this derivative class. Thus, there is netting across counterparties, which is called *multilateral netting*. For example, in Figure 2, A can reduce its total exposure from \$100 to \$40 with multilateral netting, as the exposure of \$100 to B is offset with a loss of \$60 to C.

[Place Figure 2 about here]

Due to multilateral netting, exposure to the CCP in derivative class K is determined by the net portfolio profit across counterparties:

$$\mathbb{E}\left[E_i^{CCP}\right] = \mathbb{E}\left[\max\left(\sum_{j\in\mathcal{N}_i} X_{ij}^K, 0\right)\right] = \eta_i \varphi(0) G_i f(1). \tag{5}$$

We examine the impact of centrally clearing derivative class K on entity i's counterparty risk exposure relative to an uncleared market, which we define by

$$\Delta E_i = \frac{\mathbb{E}[E_i^{K-1} + E_i^{CCP}] - \mathbb{E}[E_i^K]}{\mathbb{E}[E_i{}^K]}. \tag{6}$$

If $\Delta E_i < 0$, central clearing reduces counterparty risk exposure. Central clearing is *more beneficial* if ΔE_i is smaller, which means that it achieves a larger reduction (or, equivalently, smaller increase) of counterparty risk exposure.

In the following proposition, we provide an analytical expression for ΔE_i and comparative statics. ΔE_i is driven by two components: directionality and risk. The larger an entity i's portfolio directionality η_i, the less beneficial is central clearing, i.e., the larger is ΔE_i. The reason is that multilateral netting opportunities are decreasing with portfolio directionality.

In contrast, higher systematic risk exposure β increases clearing benefits, i.e., reduces ΔE_i. The intuition is that higher systematic risk leads to a larger correlation of contract returns across derivative classes, which impairs bilateral but not multilateral netting efficiency. Thus, central clearing, by facilitating multilateral netting, becomes relatively more beneficial.

Finally, we show that a sufficiently low portfolio directionality is a necessary and sufficient condition for central clearing to reduce counterparty risk exposure, i.e., for $\Delta E_i < 0$. Only in this case is multilateral netting efficiency sufficiently large to compress overall counterparty risk exposure relative to bilateral netting.

Proposition 1 (Impact of central clearing on counterparty risk exposure)**.** *The impact of central* *clearing on counterparty risk exposure is equal to*

$$\Delta E_i = \frac{f(K-1) + \eta_i f(1)}{f(K)} - 1, \tag{7}$$

where $f(K) = \sqrt{\beta^2\sigma_M^2 K^2 + \sigma^2 K}$. *The larger the portfolio directionality, the less beneficial is central clearing for counterparty risk exposure,* $\frac{\partial \Delta E_i}{\partial \eta_i} > 0$. *The larger derivatives' systematic risk exposure, the more beneficial is central clearing for counterparty risk exposure,* $\frac{\partial \Delta E_i}{\partial \beta} < 0$.

Central clearing reduces counterparty risk exposure if, and only if, $\eta_i < \bar{\eta}$, *i.e., if directionality is sufficiently low, with* $\bar{\eta} = \frac{f(K) - f(K-1)}{f(1)} \in (0, 1)$.

Remark 2 (The role of the number of counterparties)**.** *Proposition 1 shows that the impact of central clearing on counterparty risk exposure,* ΔE_i, *is independent of the number of counterparties* N. *This is in contrast with the result of Duffie and Zhu (2011) that a larger number of counterparties increases clearing benefits. Their result hinges on the assumption that profits in the same asset class exhibit entity-specific idiosyncratic risk (e.g., because positions are randomly distributed), which is compressed through multilateral netting. When removing such entity-specific risk, we show that a larger number of counterparties does not raise the benefit of central clearing. Instead, our results show that directionality is the key element that drives multilateral netting and, thus, clearing benefits.*

Corollary 1. *The larger the number of derivative classes* K, *the lower is the portfolio directionality required for central clearing to reduce counterparty risk exposure,* $\frac{\partial \bar{\eta}}{\partial K} < 0$. *Figure 3 illustrates this result.*

[Place Figure 3 about here]

5 Default Losses

Counterparty risk exposure examined in the previous section reflects expected default losses in case all counterparties and the CCP default. In the following, we extend the analysis to consider default losses more generally. Crucially, we also consider contributions to loss sharing at the CCP in case only *some* clearing members default. The amount of such contributions critically depends on how the CCP allocates losses among surviving clearing members, i.e., its *loss sharing rule*.

5.1 Aggregate Default Loss

We start by considering the expected aggregate default loss, which is the sum of expected default losses for cleared and uncleared positions across all market participants. Default losses for uncleared positions of entity i arise from a counterparty j's default if the bilateral portfolio profit exceeds the collateral C_{ji}^K posted by j to i. The CCP suffers default losses only in case at least one clearing member j defaults and the net liability of j toward the CCP exceeds the collateral C_j^{CCP} posted by j. In the following, we provide formal definitions of default losses.

Definition 2 (Default loss)**.** *The CCP's total default losses is defined as*

$$DL^{CCP} = \sum_{j=1}^{N} D_j \max\left(\sum_{g\in\mathcal{N}_j} X_{gj}^K - C_j^{CCP}, 0\right) \tag{8}$$

and the total uncleared default losses of entity i in derivative classes 1 *to K is defined as*

$$DL_i^K = \sum_{j\in\mathcal{N}_i} D_j \max\left(\sum_{k=1}^{K} X_{ij}^k - C_{ji}^K, 0\right). \tag{9}$$

Moreover, we define the function $\xi(\alpha)$, which reflects the distribution of losses in excess of collateral. The larger the confidence level of the collateral requirement α, the smaller is $\xi(\alpha)$ (see Lemma 3 in the Appendix).

Definition 3 (Collateral-weighted loss distribution)**.** *We define the function*
$\xi(\alpha) = (1-\alpha)\Phi^{-1}(1-\alpha) + \varphi(\Phi^{-1}(\alpha))$ *for* $\alpha \in [0.5, 1)$.

The following proposition provides an analytical formula for the expected default losses of uncleared derivative positions.

Proposition 2. *The expected default losses of entity i's uncleared positions in derivative classes 1 to K is equal to*

$$\mathbb{E}[DL_i^K] = \pi G_i \xi(\alpha_{uc}) \sqrt{\beta^2 \sigma_M^2 K^2 + \sigma^2 K}. \tag{10}$$

A measure for the aggregate counterparty risk when class-K derivatives are centrally cleared is given by the expected default losses aggregated across uncleared and cleared derivative classes, which is given by

$$ADL = \mathbb{E}\left[DL^{CCP} + \sum_{i=1}^{N} DL_i^{K-1} \right]. \tag{11}$$

Similarly to before, we examine the effect of centrally clearing derivative class K relative to an uncleared market. The following proposition provides analytical expressions for the expected aggregate default losses if class K is centrally cleared and for the impact of central clearing on the expected aggregate default loss. The latter is driven by the average net-to-gross ratio, η_{agg}, which is a measure for the average net position per $1 traded in class K. It reflects the average portfolio directionality. Intuitively, and as implied by Lemma 1, larger directionality lowers multilateral netting efficiency and, thereby, clearing benefits. Similarly, a lower collateral requirement for cleared relative to uncleared positions reduces clearing benefits. We illustrate these comparative statics in Figure 4. The proposition also provides a necessary condition for central clearing to be overall beneficial, i.e., to reduce the expected aggregate default loss, which is that the average portfolio directionality is sufficiently small.

[Place Figure 4 about here]

Proposition 3 (Impact of central clearing on aggregate default loss)**.** *The expected aggregate default* *losses with central clearing is equal to*

$$ADL = \pi \sum_{i=1}^{N} G_i \left(\xi(\alpha_{CCP}) \eta_i f(1) + \xi(\alpha_{uc}) f(K-1) \right), \tag{12}$$

where $f(K) = \sqrt{\beta^2 \sigma_M^2 K^2 + \sigma^2 K}$*. The impact of central clearing on the expected aggregate default losses is equal to*

$$\Delta ADL = \frac{ADL - \sum_{i=1}^{N} DL_i^K}{\sum_{i=1}^{N} DL_i^K} = \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \eta_{agg} + \frac{f(K-1)}{f(K)} - 1, \tag{13}$$

where $\eta_{agg} = \frac{\sum_{i=1}^{N} \left| \sum_{j \in \mathcal{N}_i} v_{ij} \right|}{\sum_{i=1}^{N} G_i}$ *is the average net-to-gross ratio.* $\Delta ADL < 0$ *holds only if*

$$\eta_{agg} < \frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})}. \tag{14}$$

Remark 3 (Comparison between ΔADL and ΔE_i)**.** *The impact of central clearing on the expected aggregate default loss,* ΔADL*, in Proposition 3 closely resembles that on counterparty risk exposure,* ΔE_i*, in Proposition 1.* ΔE_i *is driven entirely by derivative risk and multilateral netting efficiency (inversely related to portfolio directionality).* ΔADL *additionally takes differences in collateral requirements between cleared and uncleared positions into account. The larger the collateral requirement for cleared relative to uncleared positions, the larger is the benefit of central clearing, i.e., the smaller is* ΔADL*. If collateral requirements are identical* ($\alpha_{CCP} = \alpha_{uc}$), ΔADL *is equal to the impact of central clearing on the exposure of an entity with directionality* η_{agg}*.*

The following corollary further specifies the necessary condition for central clearing to be overall beneficial. We show that central clearing is overall beneficial only if there are either tighter collateral requirements for cleared than uncleared positions ($\alpha_{uc}, \alpha_{CCP}$) or at least one entity with an imperfectly directional portfolio ($\eta_i < 1$) or both. In other words, central clearing is *not* overall beneficial if there are no multilateral netting benefits and collateral requirements are tighter for uncleared than cleared positions.

Corollary 2. *Central clearing reduces the expected aggregate default loss, $\Delta ADL < 0$, only if at least one of the following conditions holds:*

- $\alpha_{uc} < \alpha_{CCP}$
- $\eta_{agg} < 1$.

The latter condition requires that $\min_{i \in \{1,\ldots,N\}} \eta_i < 1$.

5.2 Loss Sharing Based on Net Risk

The CCP's default loss, DL^{CCP}, is offset by *loss sharing contributions* made by surviving (i.e., non-defaulting) clearing members.[13] Contributions are made, first, out of the pre-funded default fund and, second, through cash calls or other recovery tools.[14] Because default funds must be replenished by clearing members within a short time window after defaults (typically within one month; see Appendix A for an example), from the perspective of our model, loss sharing through the default fund has a similar impact on realized default losses as cash calls. Therefore, we do not distinguish between different implementations of loss sharing but, instead, focus on the total amount of losses allocated to a specific clearing member, i.e., the sum of pre-funded contributions used and any additional contributions.

We start by assuming that losses are allocated proportionally to net portfolio risk and relax this assumption in Section 5.3. Since a clearing member's net portfolio risk is proportional to the collateral (initial margin) posted to the CCP in our model, this loss sharing rule is equivalent to allocating losses proportionally to initial margin, which resembles current market practice (see Appendix A for an example). With net-based loss sharing, a clearing member i's expected loss

[13]Before allocating losses to surviving members, default losses are (partly) absorbed by a share of the CCP's capital, its *skin-in-the-game* (SITG). Since CCPs' SITG is small in practice, typically below 20% of pre-funded default fund contributions (ESRB, 2021), we do not explicitly consider SITG in the model.

[14] Pre-funded default fund contributions are 4% of initial margin for cleared OTC IRS at LCH and 7% for cleared CDS at ICE Clear Credit in 2021 (Source: CPMI-IOSCO Quantitative Disclosures 2021Q1), which are the largest CCPs for USD- and Euro-denominated IRS and CDS, respectively. For a detailed discussion of CCPs' default waterfall see Elliott (2013), Cont (2015), Duffie (2015), or Armakolla and Laurent (2017).

sharing contribution equals the CCP's total default losses times the share of i's net risk relative to all survivors' net risk in case i survives, and zero otherwise. In states in which all surviving members have zero net risk, i.e., if $\sum_{g=1}^{N}(1-D_g)\bar{\sigma}_g = 0$, we make the technical assumption that losses are shared proportionally to the gross risk of an entity i's cleared portfolio, defined as

$$\bar{\Sigma}_i = \sum_{j \in \mathcal{N}_i} \sqrt{var\left(X_{ij}^K\right)} = G_i \sqrt{\beta^2 \sigma_M^2 + \sigma^2}, \tag{15}$$

whereas the impact of gross risk on loss sharing is infinitesimally small in other states. Then, a clearing member i's expected loss sharing contribution is equal to

$$\mathbb{E}[LSC_i] = \mathbb{P}(1-D_i)\mathbb{E}\left[\frac{\delta\bar{\Sigma}_i + \bar{\sigma}_i}{\sum_{g=1}^{N}(1-D_g)(\delta\bar{\Sigma}_g + \bar{\sigma}_g)} DL^{CCP} \mid D_i = 0\right], \tag{16}$$

where $\delta > 0$ is infinitesimally small. In the market environments we consider below, the limit when δ approaches 0 is well-defined, in which case we consider
$\lim_{\delta \searrow 0} \mathbb{P}(1-D_i)\mathbb{E}\left[\frac{\delta\bar{\Sigma}_i + \bar{\sigma}_i}{\sum_{g=1}^{N}(1-D_g)(\delta\bar{\Sigma}_g + \bar{\sigma}_g)} DL^{CCP} \mid D_i = 0\right]$ as the expected loss sharing contribution with loss sharing based on net risk.

Equation (16) illustrates that loss sharing contributions do not only depend on an entity's own portfolio risk but also on that of other clearing members. Because the share of losses borne by clearing member i is inversely proportional to the number of surviving clearing members, $\sum_{g=1}^{N}(1-D_g)$, there is, in general, no analytical expression for $\mathbb{E}[LSC_i]$. The following proposition simplifies Equation (16) by taking the expectation with respect to the CCP's default losses and clearing member i's default indicator.

Proposition 4. *With loss sharing proportional to net risk, clearing member i's expected loss sharing contribution is equal to*

$$\mathbb{E}[LSC_i] = (1-\pi)\xi(\alpha_{CCP})(\delta\bar{\Sigma}_i + \bar{\sigma}_i)\mathbb{E}\left[\frac{\sum_{j=1, j\neq i}^{N} D_j \bar{\sigma}_j}{\delta\bar{\Sigma}_i + \bar{\sigma}_i + \sum_{j=1, j\neq i}^{N}(1-D_j)(\delta\bar{\Sigma}_j + \bar{\sigma}_j)}\right]. \tag{17}$$

To assess the impact of central clearing on an entity i's expected default loss, we compute the change in expected default losses with central clearing of derivative class K relative to an uncleared market, as given by

$$\Delta DL_i = \frac{\mathbb{E}[(1-D_i)DL_i^{K-1} + LSC_i]}{\mathbb{E}[(1-D_i)DL_i^K]} - 1. \tag{18}$$

Analogously to loss sharing contributions, we assume that the uncleared default losses equals zero if i defaults because limited liability protects entity i in states with negative entity. If $\Delta DL_i < 0$, central clearing is beneficial since it reduces entity i's expected default losses compared to an uncleared market. In the following, we investigate the determinants of ΔDL_i. Our focus on ΔDL_i is motivated by the role of counterparty risk as a key determinant for clearing participation (Bellia et al., 2023; FSB, 2018; Vuillemey, 2020). In Section 6, we formalize this role in a model with endogenous clearing participation.

The following proposition characterizes ΔDL_i and derives several comparative statistics. We show that ΔDL_i is increasing with tighter collateral requirements for uncleared contracts, α_{uc}, and decreasing with tighter collateral requirements for cleared contracts, α_{CCP}. Intuitively, the *safer* central clearing is relative to uncleared contracts, the larger is the relative benefit of central clearing, i.e., the smaller is ΔDL_i. A larger number of derivative classes K has two effects. On one hand, it increases bilateral netting efficiency for uncleared contracts. On the other hand, it increases the total risk of uncleared contracts. If central clearing is relatively safe, i.e., if α_{CCP} is large, the former effect dominates and increasing bilateral netting efficiency makes central clearing relatively less beneficial, i.e., increases ΔDL_i. Instead, if central clearing associates with sufficiently large risk, the latter effect dominates and increasing the total risk of uncleared contracts makes central clearing relatively more beneficial, i.e., reduces ΔDL_i. Finally, we show that systematic risk exposure β increases central clearing benefits, i.e., reduces ΔDL_i. The reason is, analogously to Proposition 1, that higher systematic risk impairs bilateral but not multilateral netting efficiency.

Proposition 5 (Loss sharing based on net risk)**.** *The impact of central clearing on the expected default* *losses of entity i is equal to*

$$\Delta DL_i = \frac{f(K-1)}{f(K)} + (\delta+\eta_i)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}\mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^{N} D_j G_j \eta_j}{(\delta+\eta_i)G_i + \sum_{j=1,j\neq i}^{N}(1-D_j)(\delta+\eta_j)G_j}\right] - 1, \tag{19}$$

where $f(K) = \sqrt{\beta^2\sigma_M^2 K^2 + \sigma^2 K}$. ΔDL_i *is*

(a) *decreasing with the collateral requirement for cleared contracts,* $\frac{\partial \Delta DL_i}{\partial \alpha_{CCP}} < 0$*, and increasing with the collateral requirement for uncleared contracts,* $\frac{\partial \Delta DL_i}{\partial \alpha_{uc}} > 0$*,*

(b) *increasing with the number of derivative classes,* $\frac{\partial \Delta DL_i}{\partial K} > 0$*, if, and only if,* $\alpha_{CCP} > c$*, where* $c > 0$ *is a constant,*

(c) *decreasing with the systematic risk exposure,* $\frac{\partial \Delta DL_i}{\partial \beta} < 0$*.*

The impact of portfolio directionality on central clearing benefits is ex ante not obvious because a given entity's portfolio cannot be viewed in isolation. Instead, a change in entity i's portfolio directionality also implies a change in the portfolio of its counterparties. On one hand, lower directionality implies a smaller contribution to loss sharing, all else equal. On the other hand, it may increase the directionality of other clearing members and, thereby, the CCP's default loss. The following proposition shows that the first effect dominates if gross positions do not positively correlate with directionality. In this case, central clearing is more beneficial (i.e., ΔDL_i smaller) for entities with a marginally lower directionality.

Proposition 6 (Loss sharing based on net risk: directionality)**.** *Assume that at least three entities have a portfolio that is not perfectly flat. Consider two entities* $h, g \in \{1, ..., N\}, h \neq g$*, with* $G_h \geq G_g$*. Then there exists* $\varepsilon < 0$ *such that the following holds: if entity h exhibits a lower portfolio directionality than g,* $\eta_h < \eta_g$*, and either* $\eta_h = 0$ *or* $\eta_g < \eta_h + \varepsilon$*, then the impact of central clearing on expected default losses is smaller for h than for g,*

$$\Delta DL_h < \Delta DL_g. \tag{20}$$

Exploring other comparative statics, e.g., with respect to the probability of default, is challenging because a closed-form expression for ΔDL_i is, in general, not readily available. We address this challenge in the following by considering two specific classes of networks.

First, we study the class of homogeneous networks, which we define as networks in which all entities exhibit the same total gross position and portfolio directionality. This class is very broad. It includes markets with only one counterparty per entity as well as complete networks in which all entities trade with each other, e.g., an interdealer market (Getmansky et al., 2016).

Assumption 1 (Homogeneous network)**.** *In a homogeneous network, market participants have the same gross positions, $G_i \equiv G > 0$, and directionality, $\eta_i \equiv \eta > 0$, for all $i = 1, ..., N$.*

The following proposition derives a closed-form expression for the impact of central clearing in homogeneous networks. In such networks, all surviving clearing members bear the same share of the CCP's default losses and the expected loss sharing contribution is solely driven by netting efficiency and default dynamics. Portfolio directionality reduces the benefit of central clearing, i.e., increases ΔDL_i, due to a reduction in multilateral netting efficiency. Building on this result, we provide a complementary characterization of the role of K. A larger number of (uncleared) derivative classes K makes central clearing relatively less beneficial if directionality is sufficiently small since, in this case, larger bilateral netting efficiency undermines relative clearing benefits. Finally, we show that an increase in the probability of default π reduces clearing benefits. Intuitively, a larger probability of default increases the risk that fewer clearing members survive and, thereby, increases the share of losses an individual survivor has to bear.

Proposition 7 (Loss sharing based on net risk in homogeneous networks)**.** *Consider a homogeneous network as in Assumption 1. Then, the impact of central clearing with loss sharing based on net risk on the expected default losses of entity i with $\delta = 0$ is equal to*

$$\Delta DL_i = \frac{f(K-1)}{f(K)} + \eta \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \frac{1-\pi^{N-1}}{1-\pi} - 1, \tag{21}$$

where $f(K) = \sqrt{\beta^2\sigma_M^2 K^2 + \sigma^2 K}$. ΔDL_i *is*

(a) *increasing with directionality,* $\frac{\partial \Delta DL_i}{\partial \eta} > 0$,

(b) *increasing with the number of derivative classes,* $\frac{\partial \Delta DL_i}{\partial K} > 0$, *if, and only if,* $\eta < c$, *where* $c > 0$ *is a constant,*

(c) *increasing with the probability of default,* $\frac{\partial \Delta DL_i}{\partial \pi} > 0$.

The closed-form expression for ΔDL_i allows us to compare the impact of central clearing on expected default losses with that on counterparty risk exposure in Equation (7). Both coincide if margin requirements for cleared and uncleared positions are the same, $\alpha_{uc} = \alpha_{CCP}$, and there are exactly two market participants, $N = 2$. Unsurprisingly, since counterparty risk exposure does not take collateral into account, it does not reflect potential differences between α_{uc} and α_{CCP}. The more important insight is that the two perspectives on clearing benefits differ when there are more than two market participants. The reason is that counterparty risk exposure does not take loss sharing into account. A larger number of clearing members, on one hand, increases the expected number of defaulters, raising the CCP's expected default loss, but, on the other hand, also increases the expected number of survivors, reducing the expected share of losses borne by one clearing member. Equation (21) shows that the former effect dominates, as ΔDL_i increases with N.

Corollary 3 (Comparison with counterparty risk exposure in homogeneous networks)**.** *In a homogeneous network as in Assumption 1 the impact of central clearing on expected default losses is equal to the impact of central clearing on counterparty risk exposure if* $\alpha_{uc} = \alpha_{CCP}$ *and* $N = 2$. *The larger* N, *the larger* ΔDL_i.

Second, we add heterogeneity across clearing members. For this purpose, we consider core-periphery networks, which can be found in many OTC markets in practice (Getmansky et al., 2016; Di Maggio et al., 2017; Li and Schürhoff, 2019). The network's core can be interpreted as an interdealer market, where dealers trade with each other, whereas core-periphery links may reflect dealer intermediation between end-users.

Assumption 2 (Core-periphery network)**.** *A core-periphery network with $N \in \{3n : n \in \mathbb{N}$ uneven$\}$ exhibits the following properties:*

(1) $\mathcal{N}_{per} = \{1, ..., \frac{N}{3}, \frac{2N}{3} + 1, ..., N\}$ are peripheral entities and $\mathcal{N}_{core} = \{\frac{N}{3} + 1, ..., \frac{2N}{3}\}$ are core entities.

(2) Peripheral entities trade with only one entity in the core, such that, for all $i = 1, .., \frac{N}{3}$ and $j = \frac{2N}{3} - i + 1$, $v_{ij} = G_{per}$ if i is even, and $v_{ij} = -G_{per}$ if i is uneven; and for all $i = \frac{N}{3} + 1, ..., \frac{2N}{3}$ and $j = \frac{4N}{3} - i + 1$, $v_{ij} = G_{per}$ if i is uneven, and $v_{ij} = -G_{per}$ if i is even, $G_{per} \neq 0$.

(3) Each core entity trades with two peripheral entities with gross position G_{per} each and with all other entities in the core with unit gross position each, such that its portfolio is flat, $\eta_i = 0$ if $i \in \mathcal{N}_{core}$. Thus, a core entity's total gross position equals $G_{core} = \frac{N-3}{3} + 2G_{per}$.

(4) For all $j \geq i$, $v_{ij} = 0$ if not specified otherwise. For all i, j, it is $v_{ij} = -v_{ji}$.

To illustrate Assumption 2, we depict an exemplary core-periphery network with $N = 15$ and

$G = G_{per}$:

$$\begin{pmatrix}
 & & & & & & & & & -G & & & & & \\
 & & & & & & & & G & & & & & & \\
 & & & & & & & -G & & & & & & & \\
 & & & & & & G & & & & & & & & \\
 & & & & & -G & & & & & & & & & \\
 & & & & G & & 1 & -1 & 1 & -1 & & & & & -G \\
 & & & -G & & -1 & & 1 & -1 & 1 & & & & G & \\
 & & G & & & 1 & -1 & & 1 & -1 & & & -G & & \\
 & -G & & & & -1 & 1 & -1 & & 1 & & G & & & \\
G & & & & & 1 & -1 & 1 & -1 & & -G & & & & \\
 & & & & & & & & & G & & & & & \\
 & & & & & & & & -G & & & & & & \\
 & & & & & & & G & & & & & & & \\
 & & & & & & -G & & & & & & & & \\
 & & & & & G & & & & & & & & &
\end{pmatrix} \tag{22}$$

The core in Equation (22) is marked in gray, namely rows and columns 6-10, which correspond to entities in the core. The remaining rows (and columns) 1-5 and 11-15 correspond to entities in the periphery. No two entities in the periphery trade with each other, each entity in the periphery trades with one entity in the core, and all entities in the core trade with each other. Peripheral entities exhibit a purely directional and core entities a flat portfolio.

We first derive a closed-form expression for the impact of central clearing on the expected default losses of core and peripheral entities in Equations (23) and (24). Core entities' flat portfolio prevents them from contributing to loss sharing in states in which at least one peripheral entity survives. Thus, core entities' expected loss sharing contribution is driven by the probability that all peripheral entities default, $\pi^{2N/3}$, in which case each surviving core entity makes the same loss sharing contribution. In contrast, if peripheral entities survive, they bear all losses with an equal share and, thus, the impact of central clearing on their expected default losses in Equation (24) resembles that in a homogeneous network in Equation (21).

Second, we derive conditions under which peripheral entities are hurt by central clearing, i.e., $\Delta DL_g > 0$. This is the case if there is a sufficiently large number of clearing members, boosting the CCP's expected default loss, if there is a sufficiently large number of (uncleared) derivative classes, raising bilateral netting benefits, or if the collateral requirement for uncleared contracts is sufficiently large, boosting the safety of uncleared positions.

Finally, we compare peripheral to core entities. In contrast to peripheral entities, core entities *benefit* from central clearing if the number of clearing members is sufficiently large, reducing the likelihood that they need to contribute to loss sharing. Importantly, core entities unambiguously benefit *more* from central clearing than peripheral entities. The reason is that net-based loss sharing allocates more default losses to peripheral than to core entities relative to their respective expected uncleared default loss. The following proposition summarizes these findings.

Proposition 8 (Loss sharing based on net risk in core-periphery networks)**.** *Consider a core-periphery network as in Assumption 2. Then, the impact of central clearing with loss sharing based on net risk as δ approaches 0 on the expected default losses of a peripheral entity $g \in \mathcal{N}_{per}$ is equal to*

$$\Delta DL_g = \frac{f(K-1)}{f(K)} + \frac{1-\pi^{2N/3-1}}{1-\pi}\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)} - 1, \tag{23}$$

and for a core entity $h \in \mathcal{N}_{core}$ it is equal to

$$\Delta DL_h = \frac{f(K-1)}{f(K)} + \pi^{2N/3-1}\frac{6G_{per}}{(N-3)+6G_{per}}\frac{1-\pi^{N/3}}{1-\pi}\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)} - 1, \tag{24}$$

where $f(K) = \sqrt{\beta^2\sigma_M^2 K^2 + \sigma^2 K}$.

For peripheral entities, central clearing is not beneficial, i.e., $\Delta DL_g > 0$, if, and only if,

$$\frac{1-\pi^{2N/3-1}}{1-\pi} - \frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})}\frac{f(K)-f(K-1)}{f(1)} > 0, \tag{25}$$

which holds under the following conditions:

(a) If $\alpha_{CCP} \leq \alpha_{uc}$, there exists $\hat{N} < \infty$ such that $\Delta DL_g > 0$ for all $N > \hat{N}$.

(b) There exists $\hat{K} < \infty$ such that $\Delta DL_g > 0$ for all $K > \hat{K}$.

(c) There exists $\hat{\alpha}_{uc} < 1$ such that $\Delta DL_g > 0$ for all $\alpha_{uc} > \hat{\alpha}_{uc}$.

For core entities $h \in \mathcal{N}_{core}$, central clearing is

- *beneficial, i.e., $\Delta DL_h < 0$, if $N > \hat{N}$ for $\hat{N} < \infty$,*
- *and strictly more beneficial than for peripheral entities $g \in \mathcal{N}_{per}$, $\Delta DL_h < \Delta DL_g$.*

Importantly, central clearing can be beneficial overall ($\Delta ADL < 0$) and for core entities ($\Delta DL_h < 0$), but harmful for peripheral entities ($\Delta DL_g > 0$). We illustrate this insight and comparative statics in the following example.

Example 1. *Consider a core-periphery network. Central clearing with loss sharing based on net risk reduces expected default losses in aggregate but not that of peripheral entities for the following parameters: $G_{per} = 1$, $\pi = 0.05$, $N = 21$, $K = 10$, $\alpha_{uc} = \alpha_{CCP} = 0.99$, $\sigma = \sigma_M = 1$, $\beta = 0.3$.*

Figure 5 illustrates comparative statics varying either the number of market participants, N, or the systematic risk exposure, β, while holding all other parameters constant to those above. Figure 5 (a) shows that larger N reduces ΔADL. Intuitively, a larger market enables more risk sharing and, thus, central clearing reduces expected aggregate default losses by more. In other words, central clearing becomes more beneficial overall. However, the impact of central clearing on an individual entity's expected default losses *is largely unaffected by N. This is intuitive from the closed-form expressions in Proposition 8. A larger expected number of defaulters roughly balances a larger expected number of survivors.*

Figure 5 (b) shows that a larger systematic risk exposure β reduces ΔADL as well as each entity's ΔDL. This result is in line with Proposition 5, which shows that larger β reduces bilateral netting efficiency and, thereby, makes central clearing relatively more beneficial. This effect is particularly pronounced for peripheral entities because they make larger loss sharing contributions.

[Place Figure 5 about here]

We also compare the impact of central clearing on expected default losses in core-periphery networks with that on counterparty risk exposure, assuming that $\alpha_{uc} = \alpha_{CCP}$. Both perspectives on clearing benefits coincide only for peripheral entities and if the market consists of three entities. Otherwise, the impact on expected default losses is unambiguously larger than that on counterparty risk exposure. Similarly to Corollary 3, the reason is that counterparty risk exposure neglects loss sharing. For example, core entities exhibit zero counterparty risk exposure due to a flat portfolio, but they cannot fully escape loss sharing in case all peripheral entities default.

Corollary 4 (Comparison with counterparty risk exposure in core-periphery networks)**.** *Consider a core-periphery network as in Assumption 2 and assume that $\alpha_{uc} = \alpha_{CCP}$. Then, the impact of central clearing with loss sharing based on net risk on expected default losses is equal to the impact of central clearing on counterparty risk exposure for peripheral entities $i \in \mathcal{N}_{per}$ if $N = 3$, and strictly larger if $N > 3$ and for core entities $i \in \mathcal{N}_{core}$.*

5.3 Loss Sharing Based on Net and Gross Risk

In the following, we relax the assumption of net-based loss sharing. The natural alternative is loss sharing proportional to gross portfolio risk.[15] In this case, loss sharing does not reward multilateral netting. Instead, two clearing members with the same gross notional make the same

[15] In practice, loss sharing proportionally to gross risk may be implemented by aggregating the gross *flow* of cleared transactions instead of the existing (net) *stock* of outstanding exposures.

contribution to loss sharing even if one member's portfolio is directional and the other one's is perfectly hedged. Nonetheless, it is important to note that changing the loss sharing rule does neither change *aggregate* default losses, *aggregate* multilateral netting benefits, nor the amount of required collateral, holding clearing participation fixed. Instead, the loss sharing rule solely determines how realized losses are distributed among surviving clearing members.

We define loss sharing rules that are based on net and/or gross portfolio risk:

Definition 4 (Loss sharing rule)**.** *A loss sharing rule $w \in [0,1]^N$ determines the share of the CCP's default losses allocated to each clearing member, such that member i conditional on its survival contributes the share*

$$\frac{w_i}{\sum_{g=1}^{N}(1-D_g)w_g}. \tag{26}$$

We consider loss sharing rules of the following form:

$$w_i(\delta) = \delta\bar{\Sigma}_i + (1-\delta)\bar{\sigma}_i. \tag{27}$$

The larger δ, the larger is the weight of net relative to gross portfolio risk in loss sharing. It is $w_i(\delta) > 0$ for all $\delta > 0$.

In the following proposition, we first compute an entity's loss sharing contribution and the impact of central clearing on its expected default losses ΔDL_i for a general loss sharing rule $w(\delta)$. Second, we examine how changing the loss sharing rule affects ΔDL_i. A larger weight of gross risk in loss sharing, δ, reduces the benefit of central clearing, i.e., increases ΔDL_i, if, and only if, Inequality (30) holds. This condition suggests that entity i loses from a larger weight on gross risk if its net risk, $w_i(0)$, is sufficiently small compared to that of other entities, $w_j(0)$.

Third, we make this intuition more explicit. We show that if all entities exhibit the same portfolio directionality, then an individual entity's clearing benefit is independent of the loss sharing rule. In this case, all rules are equivalent to allocating losses proportionally to gross risk. Moreover, we zoom in on entities with the most extreme portfolio directionality. Entities with a flat portfolio, such as core entities in a core-periphery network, lose from a larger weight of gross risk in loss sharing; instead, entities with a directional portfolio, such as peripheral entities, benefit. Figure 6 illustrates this result for an exemplary core-periphery network. In this example, with loss sharing based on net risk (i.e., with $\delta = 0$) only core but not peripheral entities benefit from central clearing. Increasing the weight of gross risk in loss sharing aligns the impact of central clearing such that all entities strictly benefit from central clearing.

[Place Figure 6 about here]

Proposition 9 (Loss sharing based on net and gross risk)**.** *With the loss sharing rule $w(\delta)$, clearing member i's expected loss sharing contribution is equal to*

$$\mathbb{E}[LSC_i] = (1-\pi)\xi(\alpha_{CCP})w_i(\delta)\mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^{N} D_j\bar{\sigma}_j}{w_i(\delta)+\sum_{j=1,j\neq i}^{N}(1-D_j)w_j(\delta)}\right]. \tag{28}$$

The impact of central clearing on i's expected default losses is given by

$$\Delta DL_i = \frac{f(K-1)}{f(K)} + \frac{w_i(\delta)}{G_i f(K)}\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{1}{\pi}\mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^{N} D_j G_j \eta_j}{w_i(\delta)+\sum_{j=1,j\neq i}^{N}(1-D_j)w_j(\delta)}\right] - 1, \tag{29}$$

which is increasing with δ if, and only if,

$$\mathbb{E}\left[\tilde{H}\sum_{j=1,j\neq i}^{N}(1-D_j)(w_j(0)w_i(\delta)-w_j(\delta)w_i(0))\right] > 0, \tag{30}$$

where $\tilde{H}$ is a nonnegative random variable with $\mathbb{E}[\tilde{H}] > 0$.

(a) Assume that $\eta_j = \eta \in [0,1]$ for all $j = 1,...,N$. Then, $\frac{\partial \Delta DL_i}{\partial \delta} = 0$.

(b) Consider an entity with a flat portfolio, $\eta_i = 0$. Assume that there exist at least two fellow clearing

members a and b, $a \neq b$, with portfolio directionality $\eta_a > 0$ and $\eta_b > 0$. Then,

$$\frac{\partial \Delta DL_i}{\partial \delta} > 0.$$

(c) *Consider an entity with a fully directional portfolio, $\eta_i = 1$. Assume that there exist at least two fellow clearing members a and b, $a \neq b$, with portfolio directionality $\eta_a < 1$ and $\eta_b > 0$. Then,*

$$\frac{\partial \Delta DL_i}{\partial \delta} < 0.$$

When allocating losses exclusively based on gross risk ($\delta = 1$), there is no difference in the impact of central clearing between surviving clearing members. The reason is that gross-based loss sharing aligns loss sharing contributions with the expected default losses of uncleared positions, which is proportional to gross (but not net) risk. Nonetheless, expected default losses differ across clearing members when the CCP's risk depends on the identity of defaulting clearing members. Then, central clearing is more beneficial for riskier entities because their survival reduces the CCP's risk compared to the survival of less risky entities. Intuitively, riskier entities benefit more from risk pooling than less risky entities. The following proposition formalizes this intuition. In particular, we show that, if the probability of default is not too large, then the clearing benefit with gross-based loss sharing is larger for peripheral than core entities.

Proposition 10 (Loss sharing based on gross risk)**.** *Consider two entities g, h, $g \neq h$, and assume that loss sharing is proportional to gross portfolio risk, $\delta = 1$. Then, the difference in the impact of central clearing between the two entities is equal to*

$$\begin{aligned} &\Delta DL_g - \Delta DL_h \\ =&\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}\left(\mathbb{E}\left[\frac{\sum_{j=1}^{N} D_j G_j \eta_j}{\sum_{j=1}^{N}(1-D_j)G_j} \mid D_g = 0\right] - \mathbb{E}\left[\frac{\sum_{j=1}^{N} D_j G_j \eta_j}{\sum_{j=1}^{N}(1-D_j)G_j} \mid D_h = 0\right]\right). \end{aligned} \tag{31}$$

(a) *Conditional on $D_g = D_h$, the impact of central clearing is the same across entities:*

$$\Delta DL_{g|D_g=D_h} = \Delta DL_{h|D_g=D_h}. \tag{32}$$

(b) *If $\eta_g = \eta_h$, then*

$$G_h > G_g \;\Rightarrow\; \Delta DL_h < \Delta DL_g. \tag{33}$$

(c) *If $G_g = G_h$, then*

$$\eta_h > \eta_g \;\Leftrightarrow\; \Delta DL_h < \Delta DL_g. \tag{34}$$

(d) *If $h \in \mathcal{N}_{core}$ and $g \in \mathcal{N}_{per}$ in a core-periphery network, then there exists $\hat{\pi} > 0$ such that for all $\pi \in (0, \hat{\pi})$ it is*

$$\Delta DL_g < \Delta DL_h. \tag{35}$$

Result (e) implies that, if the probability of default is not too large, there exists a loss sharing rule $w(\hat{\delta})$ that perfectly smooths clearing benefits across core and peripheral entities:

Corollary 5. *Consider a core-periphery network and let $g \in \mathcal{N}_{per}$ and $h \in \mathcal{N}_{core}$. If π is sufficiently small, there exists $\hat{\delta} \in (0, 1)$ such that $\Delta DL_g = \Delta DL_h$ for the loss sharing rule $w(\hat{\delta})$ and that $\Delta DL_g > \Delta DL_h$ if, and only if, $\delta < \hat{\delta}$.*

5.4 Extension: Cost of Collateral

In our baseline model, collateral protects counterparties against losses but we abstract from the cost of posting collateral. In this section, we extent the model by including a cost of collateral. Specifically, we denote by $c > 0$ the marginal cost of collateral. Thus, the collateral cost for entity i

is cC_{ij}^K for uncleared positions with j and cC_i^{CCP} for cleared positions with the CCP. For consistency and without loss of generality, we assume that collateral costs arise only upon an entity's survival. Then, the impact of central clearing on expected default losses and collateral costs is given by

$$\Delta DLC_i = \frac{\mathbb{E}[(1-D_i)(DL_i^{K-1} + c\sum_{j\in\mathcal{N}_i} C_{ij}^{K-1} + cC_i^{CCP}) + LSC_i]}{\mathbb{E}[(1-D_i)DL_i^K + c\sum_{j\in\mathcal{N}_i} C_{ij}^K]}. \tag{36}$$

Whereas in the baseline model (with $c = 0$) a higher collateral requirement is unambiguously beneficial, with $c > 0$ it trades off against higher collateral costs. For entities' with a flat portfolio ($\eta_i = 0$), there is no collateral requirement due to zero net portfolio risk. Instead, for entities with $\eta_i > 0$, a higher collateral requirement for cleared positions, α_{CCP}, increases the benefit of central clearing (i.e., reduces ΔDLC_i) only if c is small. In this case, the beneficial impact of collateral on default risk dominates. If, instead, c is sufficiently large, the adverse impact on collateral costs undermines clearing benefits.

Proposition 11 (Costly collateral)**.** *Assume that at least two entities have a portfolio that is not perfectly flat. ΔDLC_i is equal to*

$$\Delta DLC_i = \frac{f(K-1)}{f(K)} + \frac{f(1)}{f(K)} \frac{\xi(\alpha_{CCP})\frac{w_i(\delta)}{G_i}\mathbb{E}[H] + c\eta_i\Phi^{-1}(\alpha_{CCP})}{\pi\xi(\alpha_{uc}) + c\Phi^{-1}(\alpha_{uc})} - 1, \tag{37}$$

where $H = \frac{\sum_{j=1,j\neq i}^N D_j G_j \eta_j}{w_i(\delta)+\sum_{j=1,j\neq i}^N (1-D_j)w_j(\delta)}$.

(1) If entity i has a flat portfolio, $\eta_i = 0$, then the impact of central clearing on expected default losses and collateral costs is decreasing with the CCP's margin requirement, $\frac{\partial \Delta DLC_i}{\partial \alpha_{CCP}} < 0$.

(2) If entity i's portfolio is not flat, $\eta_i > 0$, and $\alpha_{CCP} > 0$, there exists $0 < \hat{c} < \infty$ such that the impact of central clearing on expected default losses and collateral costs is decreasing with the CCP's margin requirement if, and only if, the marginal cost of collateral c is below $\hat{c}$,

$$\frac{\partial \Delta DLC_i}{\partial \alpha_{CCP}} < 0 \iff c < \hat{c}. \tag{38}$$

The effect of the marginal cost of collateral c on ΔDLC_i is not obvious ex ante because it affects both cleared and uncleared positions. The following proposition sheds light on the role of c in core-periphery networks when losses are shared based on net risk and collateral requirements are the same for cleared and uncleared positions. In core-periphery networks, expected loss sharing contributions per unit of cleared risk $f(1)$ are smaller than expected uncleared default losses per unit of uncleared risk $f(K)$ (see Proposition 8). A larger marginal cost of collateral c amplifies this difference between cleared and uncleared positions and, thereby, increases relative clearing benefits. This effect is particularly pronounced for core entities, which do not post collateral to the CCP due to their flat portfolio. In this case, a larger marginal collateral cost increases only the cost of uncleared but not of cleared positions, amplifying clearing benefits.

Proposition 12 (Costly collateral in core-periphery networks)**.** *Consider a core-periphery network and loss sharing based on net risk. Assume that $\alpha_{uc} = \alpha_{CCP}$. Then, for any entity $i \in \{1, ..., N\}$, the impact of central clearing on expected default losses and collateral costs is decreasing with the marginal cost of collateral,*

$$\frac{\partial \Delta DLC_i}{\partial c} < 0. \tag{39}$$

6 The CCP's Objective

In the previous section, we vary loss sharing rules along one important dimension, the degree to which they consider net relative to gross risk, and their impact on clearing benefits. What forces shape a CCP's decision to use a particular loss sharing rule? Answering this question is crucial for understanding the potential externalities associated with loss sharing rules. In the following, we provide one potential answer by exploring the incentives of a profit-maximizing CCP with market power, motivated by the observation that most CCPs are for-profit institutions and the market for central clearing is extremely concentrated in practice.[16]

We consider a CCP that sets a per-volume fee F and the loss sharing rule w before entities decide whether or not to clear. The CCP's objective is to maximize its total fee income. Providing multilateral netting benefits enables the CCP to charge positive fees from clearing members. Our analysis highlights an important difference between fees and loss sharing rules. Whereas loss sharing rules can be adjusted to discriminate between clearing members, the clearing fee F is paid per unit of notional cleared, as in Capponi and Cheng (2018) and Huang (2019).[17] For example, LCH SwapClear charges \$0.9 per-million notional for short-term interest rate swaps independently of clearing member characteristics (https://www.lch.com/services/swapclear/fees). Because the fee is uniform across clearing members, the optimal fee is determined by the clearing member with the lowest willingness to pay. Instead, regulation requires clearing members' loss sharing contributions to be "proportional to the exposures of each clearing member" (EMIR Article 42(2)), which allows for discrimination across clearing members. We show that the CCP optimally uses the loss sharing rule to maximize the minimum willingness to pay, which then determines the optimal fee.

We consider loss sharing rules as in Definition 4. Thus, choosing w is equivalent to choosing the weight of gross risk in loss sharing, δ. Without loss of generality, fees are paid upon a clearing member's survival. Entities use central clearing (i.e., become clearing members) if the sum of expected total fees and the expected default losses with central clearing does not exceed the expected default losses without central clearing. The optimal clearing rule (F^*, δ^*) maximizes the

[16] The CCPs *LCH*, *CME*, *Eurex*, and *ICE* jointly account for nearly 100% of cleared USD and EUR interest rate and credit risk derivatives (see https://www.clarusft.com/2021-ccp-volumes-and-market-share-in-ird/ and https://www.clarusft.com/2021-ccp-volumes-and-share-in-crd/) and are owned by publicly listed companies (Huang, 2019). The high concentration in the market for central clearing is consistent with the presence of significant network externalities (Menkveld and Vuillemey, 2021).

[17] F is uniform across clearing members since regulation restricts CCPs' ability to discriminate across clearing members (e.g., see EMIR Article 7(1) and Capponi and Cheng (2018) for an in-depth discussion). In our model, we focus on the trade-off between clearing fee, loss sharing rule, and clearing participation, and, for tractability, do not consider other dimensions that affect the optimal rule (e.g., a potential impact of fees on default risk).

CCP's expected total fee income subject to entities' participation constraints:

$$\max_{F,\delta} \quad \sum_{i\in\Omega} \mathbb{E}\left[(1-D_i)\sum_{j\in\mathcal{N}_i\cap\Omega}|v_{ij}|\,F\right] \tag{40}$$

$$\text{s.t.} \quad \mathbb{E}\left[(1-D_i)\sum_{j\in\mathcal{N}_i} DL_{ij}^K\right] \geq \mathbb{E}\left[(1-D_i)\sum_{j\in\mathcal{N}_i\cap\Omega}|v_{ij}|\,F\right] + \mathbb{E}[LSC_i(\delta,\Omega)]$$
$$+\,\mathbb{E}\left[(1-D_i)\left(\sum_{j\in\mathcal{N}_i\cap\Omega} DL_{ij}^{K-1} + \sum_{j\in\mathcal{N}_i\setminus\Omega} DL_{ij}^K\right)\right] \quad \forall i\in\Omega, \tag{41}$$

$$\mathbb{E}\left[(1-D_g)\sum_{j\in\mathcal{N}_g} DL_{gj}^K\right] < \mathbb{E}\left[(1-D_g)\sum_{j\in\mathcal{N}_g\cap\Omega}|v_{gj}|\,F\right] + \mathbb{E}[LSC_g(\delta,\Omega)]$$
$$+\,\mathbb{E}\left[(1-D_g)\left(\sum_{j\in\mathcal{N}_g\cap\Omega} DL_{gj}^{K-1} + \sum_{j\in\mathcal{N}_g\setminus\Omega} DL_{gj}^K\right)\right] \quad \forall g\notin\Omega, \tag{42}$$

where $DL_{ij}^K = D_j \max\left(\sum_{k=1}^K X_{ij}^k - C_{ji}^K, 0\right)$ are the uncleared default losses of i on positions with counterparty j in derivative classes 1 to K, analogously to Equation (9). $\Omega \subseteq \{1, ..., N\}$ is the set of clearing members implied by F, δ, and the participation constraints (41) and (42).[18] The participation constraint (41) ensures that clearing members (weakly) benefit from central clearing, and the participation constraint (42) ensures that non-clearing members do not benefit from central clearing. The expected loss sharing contribution depends on both Ω and δ and, analogously to Proposition 9, is equal to

$$\mathbb{E}[LSC_i(\delta,\Omega)] = (1-\pi)\xi(\alpha_{CCP})w_i(\delta)\mathbb{E}\left[\frac{\sum_{j\in\Omega\setminus\{i\}} D_j\bar{\sigma}_j}{w_i(\delta)+\sum_{j\in\Omega\setminus\{i\}}(1-D_j)w_j(\delta)}\right]. \tag{43}$$

Given the set of clearing members Ω and the loss sharing rule δ, the optimal fee set by the CCP equals the minimum willingness to pay across clearing members. The following lemma shows that it is determined by the minimum benefit of central clearing, $\min_i(-\Delta DL_i)$, within the cleared segment of the market.

[18] In general, given (δ, F), Ω is not necessarily unique. For the class of core-periphery networks that we consider below, we show that Ω is uniquely determined by (δ, F).

Lemma 2 (Optimal fee)**.** *For an optimal clearing rule* (F^*, δ^*)*, defined as the solution to (40) subject to (41) and (42), the optimal fee is equal to*

$$F^* = \pi f(K)\xi(\alpha_{uc}) \min_{i\in\Omega} \left(-\Delta DL_i(\delta^*, \Omega)\right), \tag{44}$$

where $\Delta DL_i(\delta, \Omega)$ *is the impact of central clearing on i's expected default losses considering only the set* Ω *of market participants, analogously to Equation (18),*

$$\Delta DL_i(\delta, \Omega) = \frac{\mathbb{E}\left[(1-D_i)\sum_{j\in\mathcal{N}_i\cap\Omega} DL_{ij}^{K-1} + LSC_i(\delta, \Omega)\right]}{\mathbb{E}\left[(1-D_i)\sum_{j\in\mathcal{N}_i\cap\Omega} DL_{ij}^{K}\right]} - 1. \tag{45}$$

For the remaining analysis, we focus on core-periphery networks. In such networks, there are two types of entities, core and peripheral entities, whereby peripheral entities trade only with core entities. Hence, the set of clearing members includes core entities.[19] The following proposition shows that the optimal clearing rules are such that either only core entities or all entities use central clearing, and it derives the associated optimal loss sharing rule and fee. Because the market participant with the lowest clearing benefit determines the optimal fee (Lemma 2), if all entities use central clearing, then the CCP seeks to balance clearing benefits across clearing members. In contrast, if only core entities use central clearing, the optimal fee dissuades peripheral entities from central clearing and, therefore, any loss sharing rule is optimal.

Proposition 13 (Optimal clearing rule)**.** *Consider a core-periphery network. Assume that* π *is sufficiently small, such that Corollary 5 applies. Then, the optimal clearing rule is one of the following:*

(A) All entities use central clearing, $\Omega = \{1, ..., N\}$*, the loss sharing rule balances the impact of central clearing across entities,* $\delta^* = \hat{\delta}$*, and the fee is equal to*

$$F_A^* = -\pi\xi(\alpha_{uc}) f(K)\Delta DL_1(\Omega). \tag{46}$$

[19] We assume that the parameters are such that $\min_{i\in\mathcal{N}_{core}}(-DL_i(0, \mathcal{N}_{core})) > 0$, which implies that central clearing is beneficial at least for core entities when only these use central clearing.

(B) *Only core entities use central clearing, $\Omega = \mathcal{N}_{core}$, the loss sharing rule is indeterminate, and the fee is equal to*

$$F_B^* = \pi\xi(\alpha_{uc})(f(K) - f(K-1)). \tag{47}$$

Clearing rule (A) in Proposition 13 maximizes clearing participation but associates with a smaller per-volume fee than rule (B). Thus, when setting the clearing rule, the CCP faces a trade-off between larger clearing volume and larger per-volume fee, which gives rise to inequality (48). The left hand side reflects entities' willingness to pay for multilateral netting, as in the fee F_B^* in Equation (47). The right hand side reflects the additional expected default losses when clearing peripheral entities' positions. If the latter exceeds the former, clearing benefits are relatively small and the CCP prefers to reduce the number of clearing members in exchange for a smaller expected default loss. In particular, if the bilateral netting efficiency is sufficiently high (large K) and, thus, multilateral netting through the CCP is relatively less beneficial, or if the collateral requirement for cleared contracts is sufficiently small relative to that for uncleared contracts (small α_{CCP} or large α_{uc}), or if balancing clearing benefits across core and peripheral entities requires a sufficiently large weight on gross risk (large $\hat{\delta}$), then it is less profitable for the CCP to attract peripheral entities. In these cases, the CCP maximizes its total fee income by dissuading peripheral entities from central clearing. In this case, clearing rule (B) is optimal.

Proposition 14 (Curtailing clearing participation)**.** *In the setting of Proposition 13, clearing rule (B) strictly dominates (A) if*

$$\left(f(K) - f(K-1)\right)\xi(\alpha_{uc}) < \max\left\{\frac{2N-3}{4N}, \frac{\hat{\delta}}{2}\right\} f(1)\xi(\alpha_{CCP}). \tag{48}$$

In this case, it is optimal for the CCP to dissuade peripheral entities from using central clearing. There exist $\hat{K} < \infty$ and $\hat{\alpha}_{uc} < 1$ such that condition (48) holds if $K > \hat{K}$ or $\alpha_{uc} > \hat{\alpha}_{uc}$.

The large fee in clearing rule (B) from Proposition 13 disincentivizes peripheral entities from using central clearing. In this case, since the remaining core entities share the same net and gross risk, any loss sharing rule will result in the same fee income to the CCP. Nonetheless, a net-based loss sharing rule is then more robust than other rules with respect to small perturbations in the clearing member base. If a small mass of peripheral entities use central clearing regardless of clearing rules (e.g., because they are forced to centrally clear their positions), only the net-based loss sharing rule maximizes core entities' willingness to pay and, thereby, the CCP's total fee income.[20] In this case, the CCP uses the loss sharing rule to allocate benefits of central clearing from peripheral to core entities. We following proposition formalizes this intuition. It shows that the CCP may strategically use net-based loss sharing to maximize its fee income.

Proposition 15 (Robust optimal clearing rule)**.** *If clearing rule (B) in Proposition 13 is strictly preferred over (A), then only a net-based loss sharing rule is robust to small perturbations in the following sense:*

There exists a sequence $(n_\ell)_{\ell \in \mathbb{N}}$ that converges to 0 and associates with the following sequence of core-periphery networks:

- *Each peripheral entity has the perturbed position $\tilde{G}^\ell_{per} = G_{per} + n_\ell$.*
- *Peripheral entities always centrally clear n_ℓ, independently of the clearing rule, and centrally clear G_{per} if, and only if, the participation constraint is satisfied.*
- *Core entities use central clearing if, and only if, the participation constraint is satisfied.*

Denote by $(F^{*,\ell}, \delta^{*,\ell})$ an optimal clearing rule for the ℓ-th perturbation. Then, (F^*, δ^*) is a robust optimal clearing rule for the original network if $F^{*,\ell} \to F^*$ and $\delta^{*,\ell} \to \delta^*$ for $\ell \to \infty$.

Remark 4 (The role of margin requirements)**.** *Our analysis focuses on fees and loss sharing rules as the key ingredients of clearing rules. In addition, CCPs in practice also choose (at least to some extent) margin requirements. Proposition 13 provides intuition about this choice in the absence of collateral costs. If rule*

[20]Selecting the optimal clearing rule based on small disturbances in agents' actions is reminiscent of well-known approaches to address equilibrium multiplicity in game theory, e.g., in Azevedo and Gottlieb (2017) or Selten (1975).

(A) is optimal, it is optimal for the CCP to maximize margin requirements to increase clearing benefits and, thereby, clearing members' willingness to pay. However, if collateral is costly, the optimal margin requirement trades off higher safety against higher collateral costs, analogously to Proposition 11.

Instead, if clearing rule (B) is optimal, multilateral netting entirely removes any default risk for the CCP because all clearing members exhibit a flat portfolio. As a result, clearing margins for cleared positions are equal to zero independently of the confidence level of margins.

7 Discussion

Counterparty risk is an important determinant of clearing and derivatives market equilibria (Boissel et al., 2017; Bellia et al., 2023; Bernstein et al., 2019; Cenedese et al., 2020; Vuillemey, 2020) and financial stability. Therefore, and in light of post-crisis regulation that mandates central clearing for certain derivatives, it is important to understand how and through which channels loss sharing affects the level and distribution of counterparty risk. Our results make several empirical predictions and have important policy implications, which we discuss in the following.

First, we observe that, in practice, market participants are on average reluctant to centrally clear derivatives in the absence of a clearing mandate. For instance, only 28% of CDS trades and less than 1% of foreign exchange derivatives were voluntarily cleared in December 2016 (Wooldridge, 2017). Considering expected default losses, we show that central clearing is indeed not necessarily beneficial for (all) market participants compared to an uncleared market. In contrast, loss sharing exposes market participants to risk which can disincentivize them from using central clearing. The comparative statics in our model provide guidance on how clearing benefits interact with market characteristics. We show that clearing benefits are larger when portfolio net-to-gross risk is small or returns are more exposed to systematic risk, or market participants are active in fewer derivative classes (in the presence of strict margin requirements for cleared contracts).

Second, clearing participation in practice varies significantly across different types of market participants. Clearing members are predominantly dealers and large banks in practice, while only a small number of end-users (such as investment funds and non-financial firms) participate in central clearing (BIS, 2018). For example, the European insurance company *Allianz* reports interest rate swap positions of more than EUR 2 billion notional outstanding at end-2020, while it is not a clearing member of any of the authorized European central clearinghouses for interest rate swaps.[21] The reason is not obvious. End-users are not prohibited from being clearing members.[22] Anecdotal evidence we collected from regulators and the industry suggests that expected loss sharing contributions are an important driver for end-users' reluctance to use central clearing. For example, large end-users, such as the asset manager *Blackrock*, emphasize that loss sharing "unfairly penalizes end-users, who in general hold directional positions, vs. CMs [clearing members] or dealers, who generally manage to a flat market position" (Novick et al., 2018). To mitigate exposure to loss sharing, entities might either abstain from central clearing (if it is optional) or choose to use client clearing. For instance, in the European interest rate swap market with mandatory central clearing, most non-G16 banks, insurance companies, and pension funds choose to use client clearing over being a direct clearing member (Fiedor et al., 2017).[23]

The reluctance of end-users to participate in loss sharing is consistent with our result that net-based loss sharing, as common in practice, disadvantages end-users relative to dealers. Being a clearing member may also have other disadvantages for end-users, such as fixed costs associated with the CCP's operational requirements and the degree to which dealers pass-through clearing

[21]Sources: https://www.allianz.com/en/investor_relations/results-reports/annual-reports.html and the membership lists of LCH, Eurex, Nasdaq, KDPW, and CME Clearing Europe as of April 2021.

[22]Instead, regulation forces CCPs to provide non-discriminatory access to clearing and to use membership requirements only to manage the CCP's risk (e.g., see EMIR Article 37). For example, the membership criteria of LCH include minimum levels of capital and experienced staff, but they do not restrict access for particular types of financial institutions (https://www.lch.com/membership/ltd-membership).

[23]Consistent with the rationale that client clearing is used to avoid participation in loss sharing in the presence of clearing mandates, in OTC derivatives markets without clearing mandates, client clearing is less common. For example, less than 5% of initial margin for OTC foreign exchange derivatives and less than 10% of initial margin for OTC CDS at the London-based clearinghouse LCH attributes to client clearing activities (Source: LCH LTD and SA CPMI-IOSCO Quantitative Disclosures 2020Q4). We do not explicitly incorporate client clearing in our model because its implementation varies across CCPs and jurisdictions (Braithwaite, 2016). Depending on dealers' market power and portfolio, they may charge clients for directional exposure that results from clients' trades.

fees. Our analysis suggests that expected loss sharing contributions can significantly add to such clearing costs for end-users. In contrast to end-users, dealers in our model receive the largest benefits from central clearing, which is consistent with the price discount they offer on centrally cleared relative to uncleared transactions (Cenedese et al., 2020).

Third, although we show that loss sharing that takes gross portfolio risk into account can lead to a more balanced distribution of clearing benefits across market participants, loss sharing is based on net risk in practice (see Appendix A).[24] Consistent with this observation, we show that a profit-maximizing CCP may have little incentive to deviate from net-based loss sharing because it allows to extract larger fees from dealers. In this case, the CCP's choice of the fee and loss sharing rule has externalities on clearing participation. Such externalities are important since large clearing participation may be socially desirable because it facilitates trade by increasing the scope for risk sharing and mitigating counterparty risk and financial frictions (Acharya and Bisin, 2014; Bernstein et al., 2019; Vuillemey, 2020).

There are other potential determinants of loss sharing rules that are beyond the scope of our model. First, default as a form of financial contagion may contribute to systemic risk. Because dealers are often systematically important (Billio et al., 2012), it can be socially optimal to over-proportionally reduce dealers' expected default losses using net-based loss sharing. Second, whereas we take derivative positions as exogenous, in practice loss sharing rules may impact trading. Because net-based loss sharing penalizes portfolio directionality, it may incentivize entities to hold less directional derivatives positions. Instead, gross-based loss sharing penalizes large derivatives portfolio size. The overall impact on social welfare is ambiguous. On one hand, penalizing high portfolio directionality may mitigate moral hazard incentives, reducing externalities on other clearing members and overall default losses, and facilitate liquidity provision. On the other hand, it may increase hedging costs for end-users and incentivize entities to build up very large

[24] It is important to note that, for the main part of the paper, we define clearing benefits based on expected default losses. In addition to this dimension, multilateral netting through central clearing also over-proportionately benefits entities with a flat portfolio by reducing their margin requirements. This effect is independent of loss sharing rules.

positions, which can create significant liquidity costs (Cont, 2015).[25] Whereas it is ultimately an empirical question which forces dominate, these trade-offs highlight that it is not ex-ante obvious that a net-based loss sharing rule maximizes welfare. Hence, an important avenue for future research is to investigate the implications of loss sharing rule choice on social welfare and the extent to which regulatory policies can mitigate potential externalities.

8 Conclusion

The recent global financial crisis 2007-08 exposed vulnerabilities in the derivatives market architecture, which was dominated by uncleared trades. The introduction of mandatory central clearing has clearly increased transparency in derivatives markets; however, was it successful in reducing counterparty risks in derivatives markets, as well, and, if so, have all market participants benefited?

To address these questions, we present a theoretical analysis of the impact of central clearing on default losses in derivatives markets. We focus on loss sharing in central clearinghouses, namely the allocation of losses caused by the default of some clearing members to surviving clearing members. We show that the effect of loss sharing on entities' expected default losses, relative to an uncleared market, can differ substantially across market participants and is highly sensitive toward the directionality in market participants' derivatives portfolios, loss sharing rules, and market characteristics.

In particular, our results show that market participants with flat portfolios, e.g., dealers, disproportionately benefit from loss sharing compared to an uncleared market—at the expense of entities with directional portfolios, e.g., end-users. Because clearing participation is affected by market participants' objective to reduce counterparty risk (FSB, 2018; Bellia et al., 2023; Vuillemey, 2020), our result is consistent with the reluctance of end-users to participate in loss sharing in

[25] In canonical models, end-users buy derivatives to protect themselves against risks outside of derivatives markets (Biais et al., 2012, 2016). In this case, they forego hedging benefits when choosing a portfolio that is less directional than the one that provides full insurance.

practice. The result emerges due to sharing of default losses among surviving clearing members proportionally to their *net* portfolio risk. While this is current standard practice, we contrast this rule with alternative loss sharing rules that take *gross* risk into account. We show that the latter can remove heterogeneity across market participants in clearing benefits.

Finally, we ask why, nevertheless, net-based loss sharing prevails in practice. We show that a profit-maximizing CCP might prefer dissuading end-users from central clearing in order to maximize the fee volume it can extract from dealers, rather than to maximize the number of clearing members. In this case, choosing a net-based loss sharing rule is optimal for the CCP. Our results emphasize loss sharing rules as a crucial determinant of clearing participation and, thereby, have important policy implications for the optimal design and regulation of central clearinghouses.

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Figures

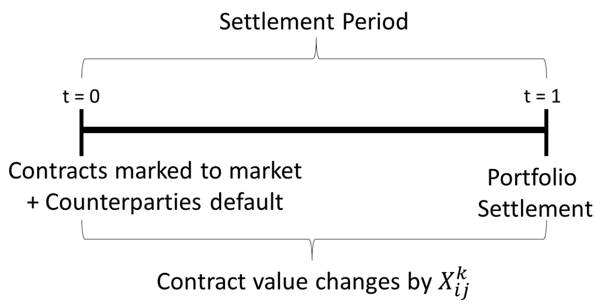
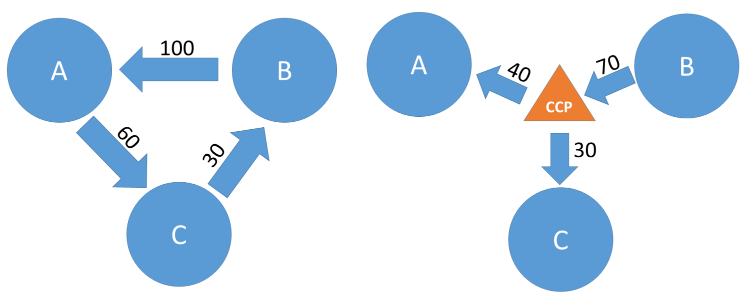


Figure 1: Timeline of the model.
Losses due to counterparty default occur between time $t = 0$, the most recent date where contracts have been marked to market and counterparties might default, and time $t = 1$, at which time the portfolio is settled.



(a) Bilateral netting. (b) Multilateral netting.

Figure 2: Illustration of bilateral and multilateral netting.
(a) Bilateral netting and (b) multilateral netting across counterparties. Arrows illustrate the flow of profits and losses (e.g., B owes \$100 to A).

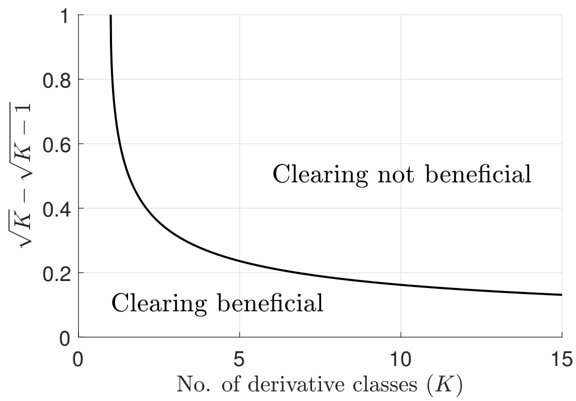


Figure 3: Maximum directionality for clearing to reduce counterparty risk exposure.
The figure depicts the function $\frac{f(K)-f(K-1)}{f(1)} = \sqrt{K} - \sqrt{K-1}$ for $\beta = 0$. If entity i's portfolio directionality η_i exceeds the function, central clearing does not reduce but increases counterparty risk exposure, i.e., is not beneficial. Instead, if η_i is below the function, central clearing reduces counterparty risk exposure, i.e., is beneficial.

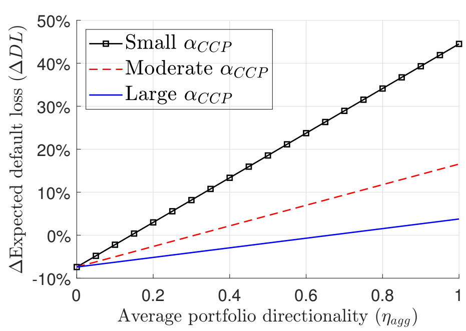
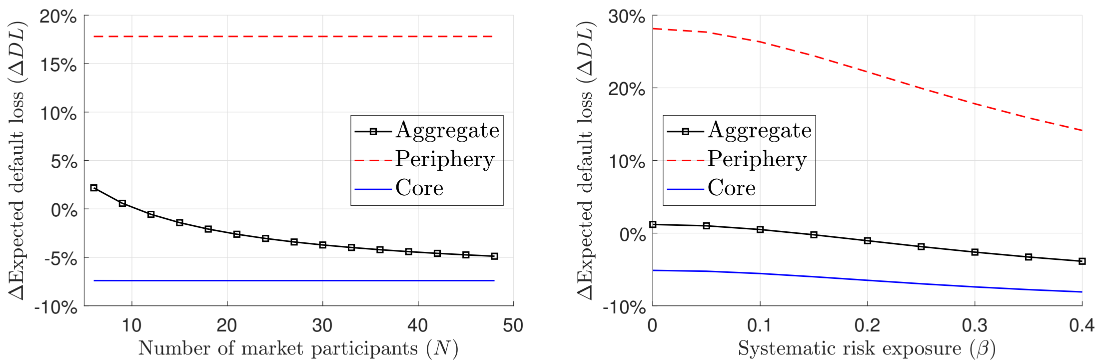


Figure 4: Impact of central clearing on the expected aggregate default loss.
The figure depicts the impact of central clearing on the expected aggregate default loss, as implied by Proposition 3. We fix the parameters to $K = 10$, $\alpha_{uc} = 0.99$, $\sigma = \sigma_M = 1$, $\beta = 0.3$, and vary η_{agg} on the x-axis for different values of α_{CCP}, namely small ($\alpha_{CCP} = 0.98$), moderate ($\alpha_{CCP} = 0.99$), and large ($\alpha_{CCP} = 0.995$).



(a) Varying N.

(b) Varying β.

Figure 5: Impact of central clearing on expected default losses in a core-periphery network.
The figures depict the impact of central clearing on the expected aggregate default loss, as implied by Proposition 3 as well as for peripheral and core entities as implied by Proposition 8. We fix the parameters to $G_{per} = 1$, $\pi = 0.05$, $N = 21$, $K = 10$, $\alpha_{uc} = \alpha_{CCP} = 0.99$, $\sigma = \sigma_M = 1$, $\beta = 0.3$, and vary N in figure (a) and β in figure (b).

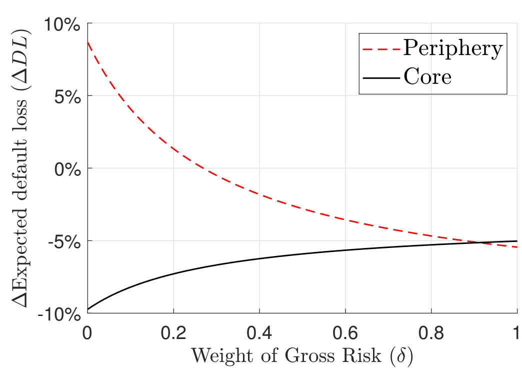


Figure 6: Impact of central clearing on expected default losses with varying loss sharing rules.
The figures depicts an exemplary core-periphery network as defined in Assumption 2. We vary the weight of gross risk δ in the loss sharing rule $w(\delta)$ on the x-axis. Larger δ corresponds to a larger weight of gross relative to net risk in loss sharing.

Internet Appendix for

Loss Sharing in Central Clearinghouses: Winners and Losers

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The views expressed in this paper are the authors' and do not necessarily reflect those of the European Central Bank or the Eurosystem.

A Loss Sharing Rules in Practice

We investigate the Default Rules of LCH Limited Rates Service, one of the largest clearinghouses worldwide, as of September 2022 (available at https://www.lch.com/resources/rulebooks/lch-limited). Using the terminology of default rules (we report the relevant excerpts of the rule book below), a clearing member i's default fund contribution is approximately equal to

$$\begin{aligned}
\text{Contribution}_i &\approx \text{Non-Tolerance Contribution}_i && (49)\\
&= \text{Non-Tolerance Amount} \times \text{Non-Tolerance Weight}_i && (50)\\
&= \text{Service Fund Amount} \times \frac{\text{Uncovered Stress Loss}_i}{\sum_j \text{Uncovered Stress Loss}_j} && (51)\\
&\approx \text{Total Uncovered Stress Loss} \times \frac{\text{Stress Loss}_i - \text{Margin}_i}{\sum_j \text{Stress Loss}_j - \text{Margin}_j} && (52)\\
&\approx \sum VaR_i \times \frac{VaR_i}{\sum VaR_i} && (53)\\
&= VaR_i = -\bar{\sigma}_i \Phi^{-1}(\alpha_{stress}), && (54)
\end{aligned}$$

where, in the first step, we ignore an additional ("tolerance") contribution that is related to temporary forbearance of initial margin.[IA.1] In the final two steps, we assume that the stress testing approach (which determines stress losses) resembles a Value-at-Risk approach with confidence level α_{stress} and is additive (as in the case of a Normal distribution), in which case the contribution is equal to entity i's portfolio Value-at-Risk.

According to default rule 21 (b), loss sharing contributions are proportional to default fund contributions, which implies that entity i's allocated share of default losses equals

$$\frac{\bar{\sigma}_i \Phi^{-1}(\alpha_{stress})}{\sum_j (1 - D_j)\bar{\sigma}_j \Phi^{-1}(\alpha_{stress})} = \frac{\bar{\sigma}_i}{\sum_j (1 - D_j)\bar{\sigma}_j}, \tag{55}$$

which is equivalent to loss sharing based on net portfolio risk.

Finally, Swapclear's Default Fund Supplement rule S1 (a) implies that the default fund must be replenished within 30 days after default events.

In the following, we provide the relevant excerpts from the LCH Limited Default Rules (as of September 2022):

[IA.1] Rule SC2 (i) on page 113 states: *The "SwapClear Tolerance" which shall be the aggregate amount of temporary initial margin forbearance provided by the Clearing House to SwapClear Clearing Members to enable registration of SwapClear Contracts.*

From Schedule 6 Rates Service Default Fund Supplement - Part A Rates Service Default Fund Supplement - Swapclear S1, p.127 ff.:

> *(b) the "SwapClear Tolerance Weight" of an SCM [...] shall be calculated by dividing (x) the average SwapClear Tolerance Utilisation of the relevant SCM during the 20 business day period preceding the relevant SwapClear Determination Date [...] by (y) the total of such average SwapClear Tolerance Utilisations of all Non-Defaulting SCMs [...]*
>
> *(c) the value of the "SwapClear Tolerance Contribution Amount" of: (x) an SCM [...] shall be calculated by multiplying the SwapClear Tolerance Amount by the SCM's SwapClear Tolerance Weight [...]*
>
> *(d) the "SwapClear Non-Tolerance Amount" shall be the value of that portion of the Rates Service Fund Amount - SwapClear after deducting the SwapClear Tolerance Amount*
>
> *(e) the value of the "SwapClear Non-Tolerance Contribution Amount" for a given SCM [...] shall be calculated by multiplying the SwapClear Non-Tolerance Amount by the SCM's SwapClear Non-Tolerance Weight*
>
> *(f) the "SwapClear Non-Tolerance Weight" of an SCM shall be calculated by dividing (i) the Uncovered Stress Loss [...] by (ii) the total Uncovered Stress Loss [...]. An SCM's "Uncovered Stress Loss," [...] shall be determined by the Clearing House [...] by, inter alia, deducting the amount of eligible margin held by the Clearing House with respect to the relevant SwapClear Contracts [...] from the stress loss [...]*
>
> *(g) the "SwapClear Contribution" of: (x) an SCM [...] shall be the sum of (i) that SCM's SwapClear Non-Tolerance Contribution Amount [...] and (ii) that SCM's Tolerance Contribution Amount [...]*

From Schedule 6 Rates Service Default Fund Supplement CS2, p.112 ff.:

> *(b) "The "Non-Tolerance Amount" which shall be the sum of: (1) the Combined Loss Value - Limb (1); plus (2) an amount equal to 10 per cent of the Combined Loss Value - Limb (1)"*

From the general default rules 21 (b) (p.21):

the amount due by a Non-Defaulting Clearing Member in respect of an Excess Loss shall [...] be the Non-Defaulting Clearing Member's pro rata share of such loss arising upon the relevant Default calculated as the proportion of such Non-Defaulting Clearing Member's relevant Contribution [...] relative to the aggregate relevant Contributions [...] of all Clearing Members engaged in the Relevant Business other than the relevant Defaulter at the time of the relevant Default.

From Schedule 6 Rates Service Default Fund Supplement - Part A Rates Service Default Fund Supplement - Swapclear S1 (a), p.127:

[...] following a Default, any determinations on a SwapClear Determination Date and any such SwapClear Determination Date which might otherwise have occurred under this Rule S1 shall be suspended for the duration of the period (the "SwapClear Default Period") commencing on the date of such Default and terminating on the later to occur of the following dates:

(i) the date which is the close of business on the day falling 30 calendar days after the Rates Service Default Management Process Completion Date in relation to such Default [...]; and

(ii) where, prior to the end of the period referred to in sub-paragraph (i) above [...] one or more subsequent Defaults (each a "Relevant Default") occur, the date which is the close of business on the day falling 30 calendar days after the Rates Service Default Management Process Completion Date in relation to a Relevant Default which falls latest in time [...].

B Additional Statements

In the following proofs, we will make extensive use of the following property of the Normal distribution: For $Y \sim \mathcal{N}(\mu, \sigma^2)$ the truncated expected value is given by $\mathbb{E}[Y \mid Y > 0] = \mu + \sigma \frac{\varphi(-\mu/\sigma)}{\Phi(\mu/\sigma)}$, and thus $\mathbb{E}[\max(Y, 0)] = \mathbb{E}[Y \mid Y > 0]\Phi(\mu/\sigma) = \mu\Phi(\mu/\sigma) + \sigma\varphi(-\mu/\sigma)$, where $\varphi(\cdot)$ and $\Phi(\cdot)$ denote the probability density function and the cumulative density function of the standard normal distribution, respectively. From this property, we derive the following lemma:

Lemma 3. *Let $Y \sim \mathcal{N}(0, \sigma^2)$ and $C = \sigma\Phi^{-1}(\alpha)$ with $\alpha \in (0, 1)$. Then,*

$$\mathbb{E}[\max(Y - C, 0)] = \sigma\xi(\alpha), \tag{56}$$

where $\xi(\alpha) = (1-\alpha)\Phi^{-1}(1-\alpha) + \varphi(\Phi^{-1}(\alpha))$ with $\xi(0.5) = \varphi(0)$, $\xi'(\alpha) < 0$, $0 < \xi(\alpha) < \varphi(0)$ for all $\alpha \in (0.5, 1)$, and $\xi(\alpha) \to 0$ for $\alpha \to 1$.

Proof.

$$\begin{aligned}
\mathbb{E}[\max(Y - C, 0)] &= (-C)\Phi((-C)/\sigma) + \sigma\varphi(C/\sigma) && (57)\\
&= (-\sigma\Phi^{-1}(\alpha))\Phi((-\sigma\Phi^{-1}(\alpha))/\sigma) + \sigma\varphi(\sigma\Phi^{-1}(\alpha)/\sigma) && (58)\\
&= \sigma\left[(-\Phi^{-1}(\alpha))\Phi\left(-\Phi^{-1}(\alpha)\right) + \varphi(\Phi^{-1}(\alpha))\right] && (59)\\
&= \sigma\left[(-\Phi^{-1}(\alpha))\Phi\left(\Phi^{-1}(1-\alpha)\right) + \varphi(\Phi^{-1}(\alpha))\right] && (60)\\
&= \sigma\xi(\alpha) && (61)
\end{aligned}$$

with $\xi(\alpha) = (1-\alpha)\Phi^{-1}(1-\alpha) + \varphi(\Phi^{-1}(\alpha))$, where we use that $-\Phi^{-1}(\alpha) = \Phi^{-1}(1-\alpha)$. If $\alpha = 0.5$, then it is $\xi(\alpha) = 0.5\Phi^{-1}(0.5) + \varphi(\Phi^{-1}(0.5)) = \varphi(0)$. Using that $\varphi'(x) = (-x)\varphi(x)$ and the inverse function rule, the first derivative of ξ is equal to

$$\begin{aligned}
\xi'(\alpha) &= (-1)\Phi^{-1}(1-\alpha) + (1-\alpha)\frac{(-1)}{\Phi'(\Phi^{-1}(1-\alpha))} + (-\Phi^{-1}(\alpha))\varphi(\Phi^{-1}(\alpha))\frac{1}{\Phi'(\Phi^{-1}(\alpha))}\\
&= (-1)\Phi^{-1}(1-\alpha) + (1-\alpha)\frac{(-1)}{\varphi(\Phi^{-1}(1-\alpha))} + (-\Phi^{-1}(\alpha))\\
&= (-1)\Phi^{-1}(1-\alpha) - \frac{1-\alpha}{\varphi(\Phi^{-1}(1-\alpha))} + \Phi^{-1}(1-\alpha) = -\frac{1-\alpha}{\varphi(\Phi^{-1}(1-\alpha))} < 0. && (62)
\end{aligned}$$

Moreover, it is

$$\lim_{\alpha\to 1}(1-\alpha)\Phi^{-1}(1-\alpha)+\lim_{\alpha\to 1}\varphi(\underbrace{\Phi^{-1}(\alpha)}_{\to\infty}) \tag{63}$$

$$=\lim_{\alpha\to 1}\frac{1-\alpha}{1/\Phi^{-1}(1-\alpha)}+0 \tag{64}$$

$$=\lim_{\alpha\to 1}\frac{-1}{(-1)\times(\Phi^{-1}(1-\alpha))^{-2}\times\frac{1}{\Phi'(\Phi^{-1}(1-\alpha))}\times(-1)} \tag{65}$$

$$=\lim_{\alpha\to 1}(-1)\times(\Phi^{-1}(1-\alpha))^{2}\times\varphi(\Phi^{-1}(1-\alpha)) \tag{66}$$

$$=\lim_{\alpha\to 1}(-1)\times\frac{(\Phi^{-1}(1-\alpha))^{2}}{\frac{1}{\varphi(\Phi^{-1}(1-\alpha))}} \tag{67}$$

$$=\lim_{\alpha\to 1}(-1)\times\frac{2\times\Phi^{-1}(1-\alpha)\times\frac{(-1)}{\varphi(\Phi^{-1}(1-\alpha))}}{(-1)\times(\varphi(\Phi^{-1}(1-\alpha)))^{-2}\times\varphi'(\Phi^{-1}(1-\alpha))\times\frac{-1}{\varphi(\Phi^{-1}(1-\alpha))}} \tag{68}$$

$$=\lim_{\alpha\to 1}(-1)\times\frac{2\times\Phi^{-1}(1-\alpha)\times\frac{(-1)}{\varphi(\Phi^{-1}(1-\alpha))}}{(-1)\times(\varphi(\Phi^{-1}(1-\alpha)))^{-2}\times(-\Phi^{-1}(1-\alpha))\times\varphi(\Phi^{-1}(1-\alpha))\times\frac{-1}{\varphi(\Phi^{-1}(1-\alpha))}}$$

$$=\lim_{\alpha\to 1}-\frac{2\times\Phi^{-1}(1-\alpha)\times(\varphi(\Phi^{-1}(1-\alpha)))^{2}}{\Phi^{-1}(1-\alpha)\times\varphi(\Phi^{-1}(1-\alpha))}=\lim_{\alpha\to 1}(-2)\times\varphi(\Phi^{-1}(1-\alpha))=0, \tag{69}$$

using L'Hôpital's rule and the inverse function rule. Together with $\xi'(\alpha)<0$, this implies $0<\xi(\alpha)<\varphi(0)$ for all $\alpha\in(0.5,1)$. From the above, it follows that $\xi(\alpha)\to 0$ for $\alpha\to 1$. □

Another result will be useful:

Lemma 4. *Define $f:(0,\infty)\to(0,\infty)$ by $f(K)=\sqrt{\beta^2\sigma_M^2K^2+\sigma^2K}$ with $\sigma,\beta,\sigma_M>0$. Then, $f'(K)>0$, $f''(K)<0$, and for all $K>1$ it is*

$$\frac{\partial}{\partial K}[f(K)-f(K-1)]<0.$$

Moreover, it is $\frac{\partial f}{\partial\beta}=\frac{\beta\sigma_M^2K^2}{f(K)}$, and $\frac{\partial}{\partial\beta}\frac{f(K_1)}{f(K_2)}<0$ for all K_1,K_2 with $0<K_1<K_2$ and $\beta>0$.

Proof. Rewrite $f(K)=\sqrt{X(K)}$ with $X(K)=\beta^2\sigma_M^2K^2+\sigma^2K$. It is $f'(K)=\frac{2\beta^2\sigma_M^2K+\sigma^2}{2\sqrt{X(K)}}>0$ and

$$f''(K)=\frac{2\beta^2\sigma_M^2 2\sqrt{X(K)}-\frac{2\beta^2\sigma_M^2K+\sigma^2}{\sqrt{X(K)}}(2\beta^2\sigma_M^2K+\sigma^2)}{4X(K)},$$

which is negative, if and only if,

$$
\begin{aligned}
& 4\beta^2\sigma_M^2 X - (2\beta^2\sigma_M^2 K + \sigma^2)(2\beta^2\sigma_M^2 K + \sigma^2) < 0 \\
\Leftrightarrow\ & 2\beta^2\sigma_M^2(2X - K(2\beta^2\sigma_M^2 K + \sigma^2)) - \sigma^2(2\beta^2\sigma_M^2 K + \sigma^2) < 0 \\
\Leftrightarrow\ & 4\beta^2\sigma_M^2(X - \underbrace{(\beta^2\sigma_M^2 K^2 + \sigma^2 K)}_{=X}) - \sigma^4 < 0 \\
\Leftrightarrow\ & -\sigma^4 < 0,
\end{aligned}
$$

which holds by the assumption that $\sigma > 0$. Thus, $f'(K) < f'(K-1)$ and, therefore, $\frac{\partial}{\partial K}[f(K) - f(K-1)] = f'(K) - f'(K-1) < 0$. The derivative with respect to β is straightforward to calculate. Because $f(K) > 0$ for all $K > 0$, for $K_1, K_2 > 0$ it is

$$
\frac{\partial}{\partial \beta}\frac{f(K_1)}{f(K_2)} < 0 \ \Leftrightarrow\ \frac{\partial}{\partial \beta}\frac{X(K_1)}{X(K_2)} < 0, \tag{70}
$$

which, if $\beta > 0$, is equivalent to

$$
\begin{aligned}
& \frac{\partial}{\partial \beta}\frac{\beta^2\sigma_M^2 K_1^2 + \sigma^2 K_1}{\beta^2\sigma_M^2 K_2^2 + \sigma^2 K_2} < 0 & (71)\\
\Leftrightarrow\ & 2\beta\sigma_M^2 K_1^2(\beta^2\sigma_M^2 K_2^2 + \sigma^2 K_2) - 2\beta\sigma_M^2 K_2^2(\beta^2\sigma_M^2 K_1^2 + \sigma^2 K_1) < 0 & (72)\\
\Leftrightarrow\ & \sigma^2(K_1^2 K_2 - K_2^2 K_1) + \beta^2\sigma_M^2(K_2^2 K_1^2 - K_2^2 K_1^2) < 0 & (73)\\
\Leftrightarrow\ & \sigma^2(K_1 - K_2) < 0 \ \Leftrightarrow\ K_1 < K2. & (74)
\end{aligned}
$$

□

C Proofs for Section 4 (Counterparty Risk Exposure)

Lemma 1 (Portfolio risk)**.** *The standard deviation of entity i's portfolio in a given derivative class is given by*

$$\bar{\sigma}_i = G_i \eta_i \sqrt{\beta^2 \sigma_M^2 + \sigma^2}.$$

Proof. The standard deviation of the portfolio in derivative class k is given by

$$\begin{aligned}\bar{\sigma}_i =& \sqrt{\operatorname{var}\left(\sum_{j \in \mathcal{N}_i} X_{ij}^k\right)} = \sqrt{\operatorname{var}\left((\beta M + \varepsilon^K) \sum_{j \in \mathcal{N}_i} v_{ij}^k\right)} = \sqrt{(\beta^2 \sigma_M^2 + \sigma^2)} \left|\sum_{j \in \mathcal{N}_i} v_{ij}^k\right| \\ =& G_i \eta_i \sqrt{\beta^2 \sigma_M^2 + \sigma^2}.\end{aligned}$$

□

Proposition 1 (Impact of central clearing on counterparty risk exposure)**.** *The impact of central clearing on counterparty risk exposure is equal to*

$$\Delta E_i = \frac{f(K-1) + \eta_i f(1)}{f(K)} - 1, \tag{7}$$

where $f(K) = \sqrt{\beta^2 \sigma_M^2 K^2 + \sigma^2 K}$*. The larger the portfolio directionality, the less beneficial is central clearing for counterparty risk exposure,* $\frac{\partial \Delta E_i}{\partial \eta_i} > 0$*. The larger derivatives' systematic risk exposure, the more beneficial is central clearing for counterparty risk exposure,* $\frac{\partial \Delta E_i}{\partial \beta} < 0$*.*

Central clearing reduces counterparty risk exposure if, and only if, $\eta_i < \bar{\eta}$*, i.e., if directionality is sufficiently low, with* $\bar{\eta} = \frac{f(K) - f(K-1)}{f(1)} \in (0, 1)$*.*

Proof. The impact of central clearing is equal to

$$\Delta E_i = \frac{G_i f(K-1) + G_i \eta_i f(1)}{G_i f(K)} - 1 = \frac{f(K-1) + \eta_i f(1)}{f(K)} - 1, \tag{75}$$

where $f(K) = \sqrt{\beta^2 \sigma_M^2 K^2 + \sigma^2 K}$. Therefore, it is

$$\Delta E_i < 0 \Leftrightarrow \eta_i < \frac{f(K) - f(K-1)}{f(1)}. \tag{76}$$

Hence, $\bar{\eta} = \frac{f(K)-f(K-1)}{f(1)}$. Since it is $\frac{f(K)-f(K-1)}{f(1)} = 1$ for $K = 1$ and $f(K) - f(K-1)$ is strictly decreasing with K (see Lemma 4), $\bar{\eta} < 1$ for all $K > 1$.

Moreover, it is

$$\frac{\partial \Delta E_i}{\partial \beta} = \frac{\partial}{\partial \beta} \frac{f(K-1)}{f(K)} + \eta_i \frac{\partial}{\partial \beta} \frac{f(1)}{f(K)} < 0, \tag{77}$$

using Lemma 4. □

Corollary 1. *The larger the number of derivative classes K, the lower is the portfolio directionality required for central clearing to reduce counterparty risk exposure, $\frac{\partial \bar{\eta}}{\partial K} < 0$. Figure 3 illustrates this result.*

Proof. The result follows from Proposition 1 and from

$$\frac{\partial \bar{\eta}}{\partial K} = \frac{\partial}{\partial K} \frac{f(K) - f(K-1)}{f(1)} < 0, \tag{78}$$

using Lemma 4. □

D Proofs for Section 5 (Default Losses)

Proposition 2. *The expected default losses of entity i's uncleared positions in derivative classes 1 to K is equal to*

$$\mathbb{E}[DL_i^K] = \pi G_i \xi(\alpha_{uc}) \sqrt{\beta^2 \sigma_M^2 K^2 + \sigma^2 K}. \tag{10}$$

Proof. Entity i's expected default losses of uncleared positions in classes 1 to K is given by

$$\begin{aligned}
\mathbb{E}\left[DL_i^K\right] &= \sum_{j \in \mathcal{N}_i} \mathbb{E}\left[D_j \max\left(\sum_{k=1}^{K} X_{ij}^k - C_{ji}^K, 0\right)\right] && (79)\\
&= \pi \sum_{j \in \mathcal{N}_i} \mathbb{E}\left[\max\left(\sum_{k=1}^{K} v_{ij}(\beta M + \sigma \varepsilon^k) - C_{ji}^K, 0\right)\right] && (80)\\
&= \pi \sum_{j \in \mathcal{N}_i} \sqrt{\beta^2 \sigma_M^2 K^2 v_{ij}^2 + K \sigma^2 v_{ij}^2}\, \xi(\alpha_{uc}) && (81)\\
&= \pi G_i \xi(\alpha_{uc}) \sqrt{\beta^2 \sigma_M^2 K^2 + \sigma^2 K}, && (82)
\end{aligned}$$

where we use that defaults D_j are distributed independently of profits X_{ij}^k, that

$$C_{ji}^K = VaR_{\alpha_{uc}}\left(\sum_{k=1}^{K} X_{ji}^k\right) \tag{83}$$

$$= -\sqrt{var\left(\sum_{k=1}^{K} X_{ji}^k\right)}\Phi^{-1}(1-\alpha_{uc}) \tag{84}$$

$$= \sqrt{var\left(-\sum_{k=1}^{K} X_{ij}^k\right)}\Phi^{-1}(\alpha_{uc}) \tag{85}$$

$$= \sqrt{var\left(\sum_{k=1}^{K} X_{ij}^k\right)}\Phi^{-1}(\alpha_{uc}), \tag{86}$$

and Lemma 3. □

Proposition 3 (Impact of central clearing on aggregate default loss)**.** *The expected aggregate default losses with central clearing is equal to*

$$ADL = \pi \sum_{i=1}^{N} G_i \left(\xi(\alpha_{CCP})\eta_i f(1) + \xi(\alpha_{uc}) f(K-1)\right), \tag{12}$$

where $f(K) = \sqrt{\beta^2\sigma_M^2 K^2 + \sigma^2 K}$*. The impact of central clearing on the expected aggregate default losses is equal to*

$$\Delta ADL = \frac{ADL - \sum_{i=1}^{N} DL_i^K}{\sum_{i=1}^{N} DL_i^K} = \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\eta_{agg} + \frac{f(K-1)}{f(K)} - 1, \tag{13}$$

where $\eta_{agg} = \frac{\sum_{i=1}^{N}\left|\sum_{j\in\mathcal{N}_i} v_{ij}\right|}{\sum_{i=1}^{N} G_i}$ *is the average net-to-gross ratio.* $\Delta ADL < 0$ *holds only if*

$$\eta_{agg} < \frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})}. \tag{14}$$

Proof. The CCP's expected total default losses is given by

$$\mathbb{E}\left[DL^{CCP}\right] = \sum_{j=1}^{N} \mathbb{E}\left[D_j \max\left(\sum_{g\in\mathcal{N}_j} X_{gj}^K - C_j^{CCP}, 0\right)\right] \tag{87}$$

$$= \pi \sum_{j=1}^{N} \mathbb{E}\left[\max\left(\sum_{g\in\mathcal{N}_j} v_{gj}^K(\beta M + \sigma\varepsilon^K) - C_j^{CCP}, 0\right)\right] \tag{88}$$

$$= \pi \sum_{j=1}^{N} \sqrt{var(\sum_{g\in\mathcal{N}_j} v_{jg}^K(\beta M + \sigma\varepsilon^K))}\xi(\alpha_{CCP}) \tag{89}$$

$$= \pi\xi(\alpha_{CCP}) \sum_{j=1}^{N} \bar{\sigma}_j^K \tag{90}$$

$$= \pi\xi(\alpha_{CCP}) f(1) \sum_{j=1}^{N} G_j \eta_j, \tag{91}$$

with $f(K) = \sqrt{\beta^2\sigma_M^2 K^2 + \sigma^2 K}$, where we use that

$$C_j^{CCP} = VaR_{\alpha_{CCP}}\left(\sum_{g=1}^{N} X_{jg}^K\right) \tag{92}$$

$$= -\sqrt{var\left(\sum_{g=1}^{N} X_{jg}^K\right)}\Phi^{-1}(1-\alpha_{CCP}) \tag{93}$$

$$= \sqrt{var\left(-\sum_{k=1}^{K} X_{gj}^k\right)}\Phi^{-1}(\alpha_{CCP}) \tag{94}$$

$$= \sqrt{var\left(\sum_{k=1}^{K} X_{gj}^k\right)}\Phi^{-1}(\alpha_{CCP}), \tag{95}$$

and Lemma 3. Together with Proposition 2, the expected aggregate default losses with central clearing is thus equal to

$$\mathbb{E}\left[DL^{CCP} + \sum_{i=1}^{N} DL_i^{K-1}\right] \tag{96}$$

$$= \pi\xi(\alpha_{CCP})f(1)\sum_{i=1}^{N} G_i\eta_i + \sum_{i=1}^{N} \pi G_i\xi(\alpha_{uc})f(K-1) \tag{97}$$

$$= \pi\sum_{i=1}^{N} G_i\left(\xi(\alpha_{CCP})\eta_i f(1) + \xi(\alpha_{uc})f(K-1)\right) \tag{98}$$

and without central clearing it is equal to

$$\mathbb{E}\left[\sum_{i=1}^{N} DL_i^{K}\right] = \pi\xi(\alpha_{uc})\sum_{i=1}^{N} G_i f(K). \tag{99}$$

The derivation of ΔADL is straightforward. $\Delta ADL < 0$ is equivalent to

$$\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\eta_{agg} + \frac{f(K-1)}{f(K)} < 1 \tag{100}$$

$$\Leftrightarrow \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\eta_{agg} < 1 - \frac{f(K-1)}{f(K)} \tag{101}$$

$$\Leftrightarrow \eta_{agg} < \frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})f(1)}[f(K) - f(K-1)]. \tag{102}$$

The statement follows from

$$\frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})f(1)}[f(K) - f(K-1)] \leq \frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})f(1)}[f(1) - f(0)] = \frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})} \tag{103}$$

using that $f(K) - f(K-1)$ is strictly decreasing in K for all $K > 1$ (Lemma 4) and $f(0) = 0$. □

Corollary 2. *Central clearing reduces the expected aggregate default loss, $\Delta ADL < 0$, only if at least one of the following conditions holds:*

- $\alpha_{uc} < \alpha_{CCP}$
- $\eta_{agg} < 1$.

The latter condition requires that $\min_{i\in\{1,\ldots,N\}} \eta_i < 1$.

Proof. From Lemma 3, $\alpha_{uc} \geq \alpha_{CCP}$ implies that $\xi(\alpha_{uc}) \leq \xi(\alpha_{CCP})$ and, thus, $\frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})} \leq 1$. Together with Proposition 3 the first statement follows. For the second statement, note that the aggregate net-to-gross ratio is a weighted average of individual entities' net-to-gross ratio,

$$\eta_{agg} = \frac{\sum_{i=1}^{N} G_i \eta_i}{\sum_{i=1}^{N} G_i}, \tag{104}$$

and, thus, $\eta_{agg} < 1$ requires that there exists at least one entity with $\eta_i < 1$. □

Proposition 4. *With loss sharing proportional to net risk, clearing member i's expected loss sharing contribution is equal to*

$$\mathbb{E}[LSC_i] = (1-\pi)\xi(\alpha_{CCP})(\delta\bar{\Sigma}_i + \bar{\sigma}_i)\mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^{N} D_j \bar{\sigma}_j}{\delta\bar{\Sigma}_i + \bar{\sigma}_i + \sum_{j=1,j\neq i}^{N}(1-D_j)(\delta\bar{\Sigma}_j + \bar{\sigma}_j)}\right]. \tag{17}$$

Proof. $\mathbb{E}[LSC_i]$ is equal to

$$\begin{aligned}
&\mathbb{P}(D_i = 0)\mathbb{E}\left[\frac{\delta\bar{\Sigma}_i + \bar{\sigma}_i}{\sum_{g=1}^{N}(1-D_g)(\delta\bar{\Sigma}_g + \bar{\sigma}_g)} DL^{CCP} \mid D_i = 0\right] \\
=&\mathbb{P}(D_i = 0)\mathbb{E}\left[\frac{\delta\bar{\Sigma}_i + \bar{\sigma}_i}{\sum_{g=1}^{N}(1-D_g)(\delta\bar{\Sigma}_g + \bar{\sigma}_g)} \sum_{j=1}^{N} D_j \max\left(\sum_{g\in\mathcal{N}_g} X_{gj}^K - C_j^{CCP}, 0\right) \mid D_i = 0\right] \\
=&(1-\pi)\mathbb{E}\left[\mathbb{E}\left[\frac{\delta\bar{\Sigma}_i + \bar{\sigma}_i}{\sum_{g=1}^{N}(1-D_g)(\delta\bar{\Sigma}_g + \bar{\sigma}_g)} \sum_{j=1}^{N} D_j \max\left(\sum_{g\in\mathcal{N}_g} X_{gj}^K - C_j^{CCP}, 0\right) \mid D_1, ..., D_N\right] \mid D_i = 0\right] \\
=&(1-\pi)\mathbb{E}\left[\mathbb{E}\left[\frac{\delta\bar{\Sigma}_i + \bar{\sigma}_i}{\sum_{g=1}^{N}(1-D_g)(\delta\bar{\Sigma}_g + \bar{\sigma}_g)} \sum_{j=1}^{N} D_j \xi(\alpha_{CCP})\bar{\sigma}_j \mid D_1, ..., D_N\right] \mid D_i = 0\right] \\
=&(1-\pi)\xi(\alpha_{CCP})(\delta\bar{\Sigma}_i + \bar{\sigma}_i)\mathbb{E}\left[\frac{\sum_{j=1}^{N} D_j \bar{\sigma}_j}{\sum_{j=1}^{N}(1-D_j)(\delta\bar{\Sigma}_j + \bar{\sigma}_j)} \mid D_i = 0\right] \\
=&(1-\pi)\xi(\alpha_{CCP})(\delta\bar{\Sigma}_i + \bar{\sigma}_i)\mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^{N} D_j \bar{\sigma}_j}{\delta\bar{\Sigma}_i + \bar{\sigma}_i + \sum_{j=1,j\neq i}^{N}(1-D_j)(\delta\bar{\Sigma}_j + \bar{\sigma}_j)}\right],
\end{aligned}$$

using the definition of DL^{CCP} and the law of total expectation. □

Proposition 5 (Loss sharing based on net risk)**.** *The impact of central clearing on the expected default losses of entity i is equal to*

$$\Delta DL_i = \frac{f(K-1)}{f(K)} + (\delta + \eta_i)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}\mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^{N} D_j G_j \eta_j}{(\delta + \eta_i)G_i + \sum_{j=1,j\neq i}^{N}(1-D_j)(\delta + \eta_j)G_j}\right] - 1, \tag{19}$$

where $f(K) = \sqrt{\beta^2\sigma_M^2 K^2 + \sigma^2 K}$. ΔDL_i *is*

(a) *decreasing with the collateral requirement for cleared contracts,* $\frac{\partial \Delta DL_i}{\partial \alpha_{CCP}} < 0$*, and increasing with the collateral requirement for uncleared contracts,* $\frac{\partial \Delta DL_i}{\partial \alpha_{uc}} > 0$*,*

(b) *increasing with the number of derivative classes,* $\frac{\partial \Delta DL_i}{\partial K} > 0$*, if, and only if,* $\alpha_{CCP} > c$*, where* $c > 0$ *is a constant,*

(c) *decreasing with the systematic risk exposure,* $\frac{\partial \Delta DL_i}{\partial \beta} < 0$*.*

Proof. Using Propositions 2 and 4, the impact of central clearing for entity i is given by

$$\Delta DL_i = \frac{(1-\pi)\pi G_i \xi(\alpha_{uc}) f(K-1) + (1-\pi)\xi(\alpha_{CCP})(\delta\bar{\Sigma}_i + \bar{\sigma}_i)\mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^N D_j \bar{\sigma}_j}{\delta\bar{\Sigma}_i + \bar{\sigma}_i + \sum_{j=1,j\neq i}^N (1-D_j)(\delta\bar{\Sigma}_j + \bar{\sigma}_j)}\right]}{(1-\pi)\pi G_i \xi(\alpha_{uc}) f(K)} - 1$$

$$= \frac{(1-\pi)\pi G_i \xi(\alpha_{uc}) f(K-1) + (1-\pi)\xi(\alpha_{CCP})(\delta+\eta_i) G_i f(1)\mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^N D_j G_j \eta_j f(1)}{(\delta+\eta_i)G_i f(1) + \sum_{j=1,j\neq i}^N (1-D_j)(\delta+\eta_j) G_j f(1)}\right]}{(1-\pi)\pi G_i \xi(\alpha_{uc}) f(K)} - 1$$

$$= \frac{f(K-1)}{f(K)} + (\delta+\eta_i)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}\mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^N D_j G_j \eta_j}{(\delta+\eta_i)G_i + \sum_{j=1,j\neq i}^N (1-D_j)(\delta+\eta_j)G_j}\right] - 1,$$

where $f(K) = \sqrt{\beta^2\sigma_M^2 K^2 + \sigma^2 K}$, using that D_i and D_j are independently distributed for $i \neq j$. Denote

$$H = \frac{1}{\pi}\mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^N D_j G_j \eta_j}{G_i(\delta+\eta_i) + \sum_{j=1,j\neq i}^N (1-D_j) G_j (\delta+\eta_j)}\right].$$

It is $H > 0$.

(a) The derivative of ΔDL_i with respect to α_{CCP} is equal to

$$\frac{\partial \Delta DL_i}{\partial \alpha_{CCP}} = \frac{\xi'(\alpha_{CCP})}{\xi(\alpha_{uc})}(\delta+\eta_i)\frac{f(1)}{f(K)} H < 0 \tag{105}$$

and the derivative with respect to α_{uc} is equal to

$$\frac{\partial \Delta DL_i}{\partial \alpha_{uc}} = -\frac{\xi'(\alpha_{uc})\xi(\alpha_{CCP})}{\xi(\alpha_{uc})^2}(\delta+\eta_i)\frac{f(1)}{f(K)} H > 0, \tag{106}$$

using in both cases that $\xi'(\alpha) < 0$ from Lemma 3.

(b) The derivative of ΔDL_i with respect to K is equal to

$$\frac{\partial \Delta DL_i}{\partial K} = \frac{f'(K-1)f(K) - f'(K)f(K-1)}{f^2(K)} - f'(K)\frac{f(1)}{f^2(K)}(\delta + \eta_i)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}H \tag{107}$$

$$= \frac{f'(K-1)f(K) - f'(K)\left[f(K-1) + f(1)(\delta+\eta_i)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}H\right]}{f^2(K)}, \tag{108}$$

which is positive if, and only if,

$$f'(K-1)f(K) > f'(K)\left[f(K-1) + f(1)(\delta+\eta_i)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}H\right] \tag{109}$$

$$\Leftrightarrow \frac{f'(K-1)f(K) - f'(K)f(K-1)}{f'(K)f(1)}\frac{1}{(\delta+\eta_i)H} > \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \tag{110}$$

$$\Leftrightarrow \xi^{-1}\left(\frac{f'(K-1)f(K) - f'(K)f(K-1)}{f'(K)f(1)}\frac{1}{(\delta+\eta_i)H}\xi(\alpha_{uc})\right) < \alpha_{CCP}. \tag{111}$$

(c) The derivative with respect to β is equal to

$$\frac{\partial \Delta DL_i}{\partial \beta} = \frac{\partial}{\partial \beta}\frac{f(K-1)}{f(K)} + (\delta+\eta_i)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}H\frac{\partial}{\partial \beta}\frac{f(1)}{f(K)} < 0, \tag{112}$$

using Lemma 4.

□

Proposition 6 (Loss sharing based on net risk: directionality). *Assume that at least three entities have a portfolio that is not perfectly flat. Consider two entities $h, g \in \{1, ..., N\}, h \neq g$, with $G_h \geq G_g$. Then there exists $\varepsilon < 0$ such that the following holds: if entity h exhibits a lower portfolio directionality than g, $\eta_h < \eta_g$, and either $\eta_h = 0$ or $\eta_g < \eta_h + \varepsilon$, then the impact of central clearing on expected default losses is smaller for h than for g,*

$$\Delta DL_h < \Delta DL_g. \tag{20}$$

Proof. Consider two different entities $h, g \in \{1, ..., N\}, h \neq g$. By assumption, there exist at least one other entity with positive net risk, $w \notin \{h, g\}$ with $G_w \eta_w > 0$. For $i \in \{h, g\}$, define

$$
\begin{aligned}
H_i =& \mathbb{E}\left[\frac{\sum_{j=1, j\neq i}^{N} D_j G_j \eta_j}{(\delta+\eta_i)G_i + \sum_{j=1,j\neq i}^{N}(1-D_j)(\delta+\eta_j)G_j}\right] \\
=& \mathbb{E}\left[\frac{1_{\{i=h\}} D_g G_g \eta_g + 1_{\{i=g\}} D_h G_h \eta_h + \sum_{j=1, j\notin\{h,g\}}^{N} D_j G_j \eta_j}{(1-1_{\{i=h\}}D_g)(\delta+\eta_g)G_g + (1-1_{\{i=g\}}D_h)(\delta+\eta_h)G_h + \sum_{j=1,j\notin\{h,g\}}^{N}(1-D_j)(\delta+\eta_j)G_j}\right] \\
=& \mathbb{E}\left[\frac{\tilde{D}(1_{\{i=h\}} G_g \eta_g + 1_{\{i=g\}} G_h \eta_h) + A}{(1-1_{\{i=h\}}\tilde{D})(\delta+\eta_g)G_g + (1-1_{\{i=g\}}\tilde{D})(\delta+\eta_h)G_h + B}\right],
\end{aligned}
$$

where we define by $\tilde{D} \sim Bern(\pi)$ a Bernoulli distributed random variable with success probability π that is independent from D_j for all $j \in \{1, ..., N\}\backslash\{h, g\}$, $A = \sum_{j=1,j\notin\{h,g\}}^{N} D_j G_j \eta_j$, and $B = \sum_{j=1,j\notin\{h,g\}}^{N}(1-D_j)(\delta+\eta_j)G_j$. Using Proposition 5, $\Delta DL_h < \Delta DL_g$ is equivalent to

$$
\begin{aligned}
& \frac{f(K-1)}{f(K)} + (\delta+\eta_h)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}H_h - 1 < \frac{f(K-1)}{f(K)} + (\delta+\eta_g)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}H_g - 1 \\
\Leftrightarrow\ & (\delta+\eta_h)H_h < (\delta+\eta_g)H_g \\
\Leftrightarrow\ & (\delta+\eta_h)\mathbb{E}\left[\frac{\tilde{D}G_g\eta_g + A}{(\delta+\eta_h)G_h + (1-\tilde{D})(\delta+\eta_g)G_g + B}\right] \\
& \qquad < (\delta+\eta_g)\mathbb{E}\left[\frac{\tilde{D}G_h\eta_h + A}{(\delta+\eta_g)G_g + (1-\tilde{D})(\delta+\eta_h)G_h + B}\right] \\
\Leftrightarrow\ & \mathbb{E}\left[\frac{(\delta+\eta_h)(\tilde{D}G_g\eta_g + A)}{(\delta+\eta_h)G_h + (1-\tilde{D})(\delta+\eta_g)G_g + B} - \frac{(\delta+\eta_g)(\tilde{D}G_h\eta_h + A)}{(\delta+\eta_g)G_g + (1-\tilde{D})(\delta+\eta_h)G_h + B}\right] < 0 \\
\Leftrightarrow\ & \mathbb{E}\left[\frac{\substack{(\delta+\eta_h)(\tilde{D}G_g\eta_g + A)((\delta+\eta_g)G_g + (1-\tilde{D})(\delta+\eta_h)G_h + B) \\ -(\delta+\eta_g)(\tilde{D}G_h\eta_h + A)((\delta+\eta_h)G_h + (1-\tilde{D})(\delta+\eta_g)G_g + B)}}{\underbrace{((\delta+\eta_g)G_g + (1-\tilde{D})(\delta+\eta_h)G_h + B)((\delta+\eta_h)G_h + (1-\tilde{D})(\delta+\eta_g)G_g + B)}_{=C}}\right] < 0.
\end{aligned}
$$

The denominator is almost surely strictly positive since $\delta > 0$, $\eta_j \geq 0$, and $G_j > 0$ for all j. Assume that $\eta_h < \eta_g$ and $G_h \geq G_g$. Then, if $\delta = 0$, for the nominator it holds that

$$
\begin{aligned}
&\eta_h(\tilde{D}G_g\eta_g + A)(\eta_g G_g + (1-\tilde{D})\eta_h G_h + B) - \eta_g(\tilde{D}G_h\eta_h + A)(\eta_h G_h + (1-\tilde{D})\eta_g G_g + B)\\
=&A\left[\eta_h\left(\eta_g G_g + (1-\tilde{D})\eta_h G_h + B\right) - \eta_g\left(\eta_h G_h + (1-\tilde{D})\eta_g G_g + B\right)\right]\\
&+\tilde{D}\left[\eta_h G_g \eta_g\left(\eta_g G_g + (1-\tilde{D})\eta_h G_h + B\right) - \eta_g G_h \eta_h\left(\eta_h G_h + (1-\tilde{D})\eta_g G_g + B\right)\right]\\
=&A\left[\eta_h\left(\eta_g G_g + (1-\tilde{D})\eta_h G_h + B\right) - \eta_g\left(\eta_h G_h + (1-\tilde{D})\eta_g G_g + B\right)\right]\\
&+\eta_h\eta_g\tilde{D}\left[B\left(G_g - G_h\right) + (1-\tilde{D})G_h G_g\left(\eta_h - \eta_g\right) + \eta_g G_g^2 - \eta_h G_h^2\right]\\
\leq&A\left[B(\eta_h - \eta_g) + \eta_h\left(\eta_g G_g + (1-\tilde{D})\eta_h G_h\right) - \eta_g\left(\eta_h G_h + (1-\tilde{D})\eta_g G_g\right)\right]\\
&+\eta_h\eta_g\tilde{D}\left[B\left(G_g - G_h\right) + G_h^2(\eta_g - \eta_h)\right]\\
\leq&A\left[B(\eta_h - \eta_g) + \eta_h\eta_g\left(G_g - G_h\right) + (1-\tilde{D})\left((\eta_h)^2 G_h - (\eta_g)^2 G_g\right)\right] + \tilde{D}\eta_h\eta_g G_h^2(\eta_g - \eta_h)\\
\leq&A\left[(\eta_h)^2 G_h + \eta_h\eta_g\left(G_g - G_h\right) - (\eta_g)^2 G_g\right] + \tilde{D}G_h^2\eta_h\eta_g(\eta_g - \eta_h),
\end{aligned}
\tag{113}
$$

using that $\tilde{D} \in \{0, 1\}$ implies that $\tilde{D}(1-\tilde{D}) = 0$. Because for $x > 0$ it is

$$
\begin{aligned}
&x^2 G_h + x\eta_g\left(G_g - G_h\right) - (\eta_g)^2 G_g < 0 &(114)\\
\Leftrightarrow\ & x < \frac{-\eta_g\left(G_g - G_h\right) + \sqrt{(\eta_g)^2\left(G_g - G_h\right)^2 + 4G_h(\eta_g)^2 G_g}}{2G_h} &(115)\\
\Leftrightarrow\ & x < \eta_g\frac{G_h - G_g + \sqrt{\left(G_h - G_g\right)^2 + 4G_h G_g}}{2G_h} &(116)\\
\Leftrightarrow\ & x < \eta_g\frac{G_h - G_g + \sqrt{\left(G_h + G_g\right)^2}}{2G_h} &(117)\\
\Leftrightarrow\ & x < \eta_g\frac{G_h - G_g + G_h + G_g}{2G_h} = \eta_g, &(118)
\end{aligned}
$$

if $A > 0$, then it holds that

$$
A\left[(\eta_h)^2 G_h + \eta_h\eta_g\left(G_g - G_h\right) - (\eta_g)^2 G_g\right] < 0. \tag{119}
$$

Therefore, there exists $\varepsilon_1 > 0$ such that Expression (113) is strictly negative if $A > 0$ and $\eta_h\eta_g(\eta_g - \eta_h) < \varepsilon_1$. Because the nominator of C is continuous in δ, there exists $\tilde{\delta}$ such that the nominator of C is strictly negative if $A > 0$, $\eta_h\eta_g(\eta_g - \eta_h) < \varepsilon_1$, and $\delta < \tilde{\delta}$. Let $\delta \in (0, \tilde{\delta})$. From the definition of A, $\pi > 0$, and the existence of an entity $w \notin \{h, g\}$ with $G_w\eta_w > 0$, it is $\mathbb{P}(A > 0) > \pi > 0$ and $\mathbb{P}(A < 0) = 0$. Therefore, there exists $0 < \varepsilon$ such that if either $\eta_h = 0$ or $\eta_g - \eta_h < \varepsilon$, then it holds that

$$\begin{aligned}\mathbb{E}[C] &= \mathbb{P}(A=0)\mathbb{E}[C \mid A=0] + \mathbb{P}(A>0)\mathbb{E}[C \mid A>0] \\ &\leq \mathbb{P}(A=0)\pi\mathbb{E}\left[\frac{G_h^2\eta_h\eta_g(\eta_g-\eta_h)}{((\delta+\eta_g)G_g+B)((\delta+\eta_h)G_h+B)}\right] + \underbrace{\underbrace{\mathbb{P}(A>0)}_{>0}\underbrace{\mathbb{E}[C \mid A>0]}_{<0}}_{<0} < 0,\end{aligned}$$

and, thus, $\Delta DL_h < \Delta DL_g$. □

Proposition 7 (Loss sharing based on net risk in homogeneous networks)**.** *Consider a homogeneous network as in Assumption 1. Then, the impact of central clearing with loss sharing based on net risk on the expected default losses of entity i with $\delta = 0$ is equal to*

$$\Delta DL_i = \frac{f(K-1)}{f(K)} + \eta\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1-\pi^{N-1}}{1-\pi} - 1, \tag{21}$$

where $f(K) = \sqrt{\beta^2\sigma_M^2K^2 + \sigma^2K}$. ΔDL_i is

(a) increasing with directionality, $\frac{\partial \Delta DL_i}{\partial \eta} > 0$,

(b) increasing with the number of derivative classes, $\frac{\partial \Delta DL_i}{\partial K} > 0$, if, and only if, $\eta < c$, where $c > 0$ is a constant,

(c) increasing with the probability of default, $\frac{\partial \Delta DL_i}{\partial \pi} > 0$.

Proof. Under Assumption 1, it is $G_i \equiv G > 0$ and $\eta_i \equiv \eta > 0$ for all $i = 1, ..., N,$. Then, the following identity holds:

$$\mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^{N} D_j G_j \eta_j}{G_i(\delta+\eta_i)+\sum_{j=1,j\neq i}^{N}(1-D_j)G_j(\delta+\eta_j)}\right] \tag{120}$$

$$=\mathbb{E}\left[\frac{G\eta\sum_{j=1,j\neq i}^{N} D_j}{G(\delta+\eta)+\sum_{j=1,j\neq i}^{N}(1-D_j)G(\delta+\eta)}\right] \tag{121}$$

$$=\frac{\eta}{\delta+\eta}\mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^{N} D_j-\sum_{j=1,j\neq i}^{N}(1-D_j)+\sum_{j=1,j\neq i}^{N}(1-D_j)}{1+\sum_{j=1,j\neq i}^{N}(1-D_j)}\right] \tag{122}$$

$$=\frac{\eta}{\delta+\eta}\mathbb{E}\left[\frac{N-1-\sum_{j=1,j\neq i}^{N}(1-D_j)}{1+\sum_{j=1,j\neq i}^{N}(1-D_j)}\right] \tag{123}$$

$$=\frac{\eta}{\delta+\eta}\mathbb{E}\left[\frac{N}{1+\sum_{j=1,j\neq i}^{N}(1-D_j)}-1\right] \tag{124}$$

$$=\frac{N\eta}{\delta+\eta}\left(\mathbb{E}\left[\frac{1}{1+Y}\right]-\frac{1}{N}\right), \tag{125}$$

where $Y \sim Bin(N-1, 1-\pi)$. Using the properties of the Binomial distribution, it is

$$\mathbb{E}\left[\frac{1}{1+Y}\right]=\frac{1-\pi^N}{N(1-\pi)}.$$

Plugging into the formula in Proposition 5 yields

$$\Delta DL_i = \frac{f(K-1)}{f(K)}+(\delta+\eta_i)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}\frac{N\eta}{\delta+\eta}\left(\mathbb{E}\left[\frac{1}{1+Y}\right]-\frac{1}{N}\right)-1 \tag{126}$$

$$= \frac{f(K-1)}{f(K)}+(\delta+\eta)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}\frac{N\eta}{\delta+\eta}\left(\frac{1-\pi^N}{N(1-\pi)}-\frac{1}{N}\right)-1 \tag{127}$$

$$= \frac{f(K-1)}{f(K)}+(\delta+\eta)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}\frac{\eta}{\delta+\eta}\frac{1-\pi^N-1+\pi}{1-\pi}-1, \tag{128}$$

$$= \frac{f(K-1)}{f(K)}+\eta\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1-\pi^{N-1}}{1-\pi}-1, \tag{129}$$

where in the last step we set $\delta = 0$.

(a) The derivative with respect to portfolio directionality η is equal to

$$\frac{\partial \Delta DL_i}{\partial \eta}=\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1-\pi^{N-1}}{1-\pi}>0. \tag{130}$$

(b) The derivative with respect to the number of derivative classes K is:

$$\frac{\partial \Delta DL_i}{\partial K} = \frac{f'(K-1)f(K) - f'(K)f(K-1)}{f(K)^2} - \eta \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)f'(K)}{f(K)^2} \frac{1-\pi^{N-1}}{1-\pi}, \tag{131}$$

which is positive if, and only if,

$$\eta \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)f'(K)}{f(K)^2} \frac{1-\pi^{N-1}}{1-\pi} < \frac{f'(K-1)f(K) - f'(K)f(K-1)}{f(K)^2} \tag{132}$$

$$\Leftrightarrow \eta < \frac{f'(K-1)f(K) - f'(K)f(K-1)}{f(1)f'(K)} \frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})} \frac{1-\pi}{1-\pi^{N-1}}, \tag{133}$$

where the right hand side is strictly positive because $f'(\cdot) > 0$ and $f''(\cdot) < 0$ (see Lemma 4) imply that $f'(K-1)f(K) > f'(K)f(K-1)$.

(c) The derivative with respect to π is equal to

$$\frac{\partial \Delta DL_i}{\partial \pi} = \eta \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \frac{-(N-1)\pi^{N-2}(1-\pi) - (-1)(1-\pi^{N-1})}{(1-\pi)^2} \tag{134}$$

$$= \eta \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \frac{1-\pi^{N-1} - \pi^{N-2}(N-1) + \pi^{N-1}(N-1)}{(1-\pi)^2} \tag{135}$$

$$= \eta \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \frac{1 + \pi\pi^{N-2}(N-2) - \pi^{N-2}(N-1)}{(1-\pi)^2} \tag{136}$$

$$= \eta \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \frac{1 + \pi^{N-2}(\pi(N-1) - \pi - (N-1))}{(1-\pi)^2} \tag{137}$$

$$= \eta \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \frac{1 - \pi^{N-2}((N-1)(1-\pi) + \pi)}{(1-\pi)^2}. \tag{138}$$

Note that $g(N) = 1 - \pi^{N-2}((N-1)(1-\pi)+\pi)$ equals zero for $N = 2$, $g(2) = 1 - \pi^0(1 - \pi + \pi) = 1 - 1 = 0$, and that

$$g'(N) = -\log(\pi)\pi^{N-2}((N-1)(1-\pi)+\pi) - \pi^{N-2}(1-\pi) \tag{139}$$

$$= \pi^{N-2}(-\log(\pi)((N-1)(1-\pi)+\pi) - (1-\pi)), \tag{140}$$

which is strictly positive if, and only if,

$$-\log(\pi)((N-1)(1-\pi)+\pi) - (1-\pi) > 0 \tag{141}$$

$$\Leftrightarrow N - 1 > \frac{1}{-\log(\pi)} - \frac{\pi}{1-\pi}. \tag{142}$$

It is $\frac{1}{-\log(\pi)} - \frac{\pi}{1-\pi} < 1 \;\Leftrightarrow\; \log(\pi) < \pi - 1$, which holds for all $\pi \in (0,1)$. Therefore,

$$\frac{1}{-\log(\pi)} - \frac{\pi}{1-\pi} < 1 \leq N - 1$$

using that $N > 2$. Thus, $g'(N) > 0$, which, together with $g(2) = 0$, implies that $g(N) > 0$ for all $N \geq 2$. Therefore,

$$\frac{\partial \Delta DL_i}{\partial \pi} = \eta \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \frac{g(N)}{(1-\pi)^2} > 0. \tag{143}$$

□

Proposition 8 (Loss sharing based on net risk in core-periphery networks)**.** *Consider a core-periphery network as in Assumption 2. Then, the impact of central clearing with loss sharing based on net risk as δ approaches 0 on the expected default losses of a peripheral entity $g \in \mathcal{N}_{per}$ is equal to*

$$\Delta DL_g = \frac{f(K-1)}{f(K)} + \frac{1-\pi^{2N/3-1}}{1-\pi} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} - 1, \tag{23}$$

and for a core entity $h \in \mathcal{N}_{core}$ it is equal to

$$\Delta DL_h = \frac{f(K-1)}{f(K)} + \pi^{2N/3-1} \frac{6G_{per}}{(N-3)+6G_{per}} \frac{1-\pi^{N/3}}{1-\pi} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} - 1, \tag{24}$$

where $f(K) = \sqrt{\beta^2\sigma_M^2 K^2 + \sigma^2 K}$.

For peripheral entities, central clearing is not beneficial, i.e., $\Delta DL_g > 0$, if, and only if,

$$\frac{1-\pi^{2N/3-1}}{1-\pi} - \frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})} \frac{f(K)-f(K-1)}{f(1)} > 0, \tag{25}$$

which holds under the following conditions:

(a) If $\alpha_{CCP} \leq \alpha_{uc}$, there exists $\hat{N} < \infty$ such that $\Delta DL_g > 0$ for all $N > \hat{N}$.

(b) There exists $\hat{K} < \infty$ such that $\Delta DL_g > 0$ for all $K > \hat{K}$.

(c) There exists $\hat{\alpha}_{uc} < 1$ such that $\Delta DL_g > 0$ for all $\alpha_{uc} > \hat{\alpha}_{uc}$.

For core entities $h \in \mathcal{N}_{core}$, central clearing is

- *beneficial, i.e.,* $\Delta DL_h < 0$*, if* $N > \hat{N}$ *for* $\hat{N} < \infty$*,*
- *and strictly more beneficial than for peripheral entities* $g \in \mathcal{N}_{per}$*,* $\Delta DL_h < \Delta DL_g$*.*

Proof. In the core-periphery network, the CCP's expected default losses per loss allocation unit is equal to

$$
\begin{aligned}
H_i =& \mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^{N} D_j G_j \eta_j}{(\delta+\eta_i)G_i + \sum_{j=1,j\neq i}^{N}(1-D_j)(\delta+\eta_j)G_j}\right] \\
=& \mathbb{E}\left[\frac{\sum_{j\in\mathcal{N}_{per},j\neq i} D_j G_j \eta_j + \sum_{j\in\mathcal{N}_{core},j\neq i} D_j G_j \eta_j}{G_i(\delta+\eta_i) + \sum_{j\in\mathcal{N}_{per},j\neq i}(1-D_j)G_j(\delta+\eta_j) + \sum_{j\in\mathcal{N}_{core},j\neq i}(1-D_j)G_j(\delta+\eta_j)}\right] \\
=& \mathbb{E}\left[\frac{G_{per}\sum_{j\in\mathcal{N}_{per},j\neq i} D_j}{G_i(\delta+\eta_i) + G_{per}\sum_{j\in\mathcal{N}_{per},j\neq i}(1-D_j)(\delta+1) + \delta G_{core}\sum_{j\in\mathcal{N}_{core},j\neq i}(1-D_j)}\right],
\end{aligned}
\tag{144}
$$

using that $\eta_j = 1$ if $j \in \mathcal{N}_{per}$ and $\eta_j = 0$ if $j \in \mathcal{N}_{core}$ by Assumption 2.
If $i \in \mathcal{N}_{per}$, then

$$
H_i = \mathbb{E}\left[\frac{G_{per}\sum_{j\in\mathcal{N}_{per},j\neq i} D_j}{G_{per}(1+\delta) + G_{per}(1+\delta)\sum_{j\in\mathcal{N}_{per},j\neq i}(1-D_j) + \delta G_{core}\sum_{j\in\mathcal{N}_{core}}(1-D_j)}\right]. \tag{145}
$$

For $\delta = 0$ and $i \in \mathcal{N}_{per}$, H_i is equal to (note that the expectation is well-defined since $G_{per} > 0$)

$$
\begin{aligned}
H_i|_{\delta=0} =&\mathbb{E}\left[\frac{G_{per}\sum_{j\in\mathcal{N}_{per},j\neq i} D_j}{G_{per} + G_{per}\sum_{j\in\mathcal{N}_{per},j\neq i}(1-D_j)}\right] && (146)\\
=&\mathbb{E}\left[\frac{\sum_{j\in\mathcal{N}_{per},j\neq i} D_j}{1+\sum_{j\in\mathcal{N}_{per},j\neq i}(1-D_j)}\right] && (147)\\
=&\mathbb{E}\left[\frac{\sum_{j\in\mathcal{N}_{per},j\neq i} D_j + 1 + \sum_{j\in\mathcal{N}_{per},j\neq i}(1-D_j)}{1+\sum_{j\in\mathcal{N}_{per},j\neq i}(1-D_j)} - 1\right] && (148)\\
=&\mathbb{E}\left[\frac{\sum_{j\in\mathcal{N}_{per},j\neq i} 1 + 1}{1+\sum_{j\in\mathcal{N}_{per},j\neq i}(1-D_j)} - 1\right] && (149)\\
=&\mathbb{E}\left[\frac{|\mathcal{N}_{per}|}{1+\sum_{j\in\mathcal{N}_{per},j\neq i}(1-D_j)} - 1\right] && (150)\\
=&|\mathcal{N}_{per}|\mathbb{E}\left[\frac{1}{1+\sum_{j\in\mathcal{N}_{per},j\neq i}(1-D_j)}\right] - 1 && (151)\\
=&|\mathcal{N}_{per}|\frac{1-\pi^{|\mathcal{N}_{per}|}}{|\mathcal{N}_{per}|(1-\pi)} - 1 = \frac{1-\pi^{|\mathcal{N}_{per}|}}{1-\pi} - 1, && (152)
\end{aligned}
$$

where in the last step we use the properties of the Binomial distribution. Using that $|\mathcal{N}_{per}| = \frac{2N}{3}$ is the number of entities in the periphery, applying the dominated convergence theorem, and plugging into the formula in Proposition 5 it is thus

$$
\begin{aligned}
\lim_{\delta\to 0}\Delta DL_i &= \frac{f(K-1)}{f(K)} + \lim_{\delta\to 0}(\delta+\eta_i)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}H_i - 1 && (153)\\
&= \frac{f(K-1)}{f(K)} + \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}\left(\frac{1-\pi^{2N/3}}{1-\pi} - 1\right) - 1 && (154)\\
&= \frac{f(K-1)}{f(K)} + \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}\frac{1-\pi^{2N/3}-1+\pi}{1-\pi} - 1 && (155)\\
&= \frac{f(K-1)}{f(K)} + \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1-\pi^{2N/3-1}}{1-\pi} - 1. && (156)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\lim_{\delta\to 0} \Delta DL_i > 0 && (157)\\
\Leftrightarrow\ & \frac{f(K-1)}{f(K)} + \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1-\pi^{2N/3-1}}{1-\pi} - 1 > 0 && (158)\\
\Leftrightarrow\ & \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1-\pi^{2N/3-1}}{1-\pi} - \frac{f(K)-f(K-1)}{f(K)} > 0 && (159)\\
\Leftrightarrow\ & \underbrace{\frac{1-\pi^{2N/3-1}}{1-\pi} - \frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})}\frac{f(K)-f(K-1)}{f(1)}}_{A} > 0. && (160)
\end{aligned}
$$

(a) A is increasing with N since $\frac{\partial A}{\partial N} = (-\log(\pi))\frac{\frac{2}{3}\pi^{2N/3-1}}{1-\pi} > 0$, and it is

$$\lim_{N\to\infty} A = \frac{1}{1-\pi} - \frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})}\frac{f(K)-f(K-1)}{f(1)},$$

which is positive if, and only if,

$$
\begin{aligned}
& \frac{1}{1-\pi} > \frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})}\frac{f(K)-f(K-1)}{f(1)} && (161)\\
\Leftrightarrow\ & \pi > 1 - \underbrace{\frac{f(1)}{f(K)-f(K-1)}}_{>1}\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}. && (162)
\end{aligned}
$$

Note that $\frac{f(1)}{f(K)-f(K-1)} = 1$ for $K = 1$ and $\frac{f(1)}{f(K)-f(K-1)} > 1$ for all $K > 1$ since $f(K) - f(K-1)$ is decreasing with K (see Lemma 4). Since $\xi(\alpha)$ is decreasing with α (see Lemma 3), if $\alpha_{CCP} \leq \alpha_{uc}$, it is $\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \geq 1$ and $1 - \frac{f(1)}{f(K)-f(K-1)}\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} < 0$. In this case, $\lim_{N\to\infty} A > 0$. Therefore, if $\alpha_{CCP} \leq \alpha_{uc}$, there exists $\hat{N} < \infty$ such that $\lim_{\delta\to 0} \Delta DL_i > 0$ for all $N > \hat{N}$.

(b) A is increasing with K and it is

$$\lim_{K\to\infty} A = \frac{1-\pi^{2N/3-1}}{1-\pi} > 0,$$

since $\frac{2N}{3} > 1$. Thus, there exists $\hat{K} < \infty$ such that $\lim_{\delta\to 0} \Delta DL_i > 0$ for all $K > \hat{K}$.

(c) Since $\xi(\alpha)$ is decreasing with α and $\lim_{\alpha\to 1} \xi(\alpha) = 0$ and $\xi(0.5) = \varphi(0)$, it is

$$\lim_{\alpha_{uc}\to 1} A = \frac{1-\pi^{2N/3-1}}{1-\pi} > 0,$$

and, thus, there exists $\hat{\alpha}_{uc} < 1$ such that $\lim_{\delta \to 0} \Delta DL_i > 0$ for all $\alpha_{uc} > \hat{\alpha}_{uc}$.

If $i \in \mathcal{N}_{core}$, then

$$\begin{aligned}
\delta H_i = & \mathbb{E}\left[\frac{\delta G_{per} \sum_{j \in \mathcal{N}_{per}} D_j}{G_{core}\delta + G_{per} \sum_{j \in \mathcal{N}_{per}} (1-D_j)(1+\delta) + G_{core} \sum_{j \in \mathcal{N}_{core}, j \neq i} (1-D_j)\delta}\right] \\
= & \mathbb{E}\left[\frac{\delta G_{per} \sum_{j \in \mathcal{N}_{per}} D_j}{G_{core}\delta + G_{per} \sum_{j \in \mathcal{N}_{per}} (1-D_j)(1+\delta) + \delta G_{core} \sum_{j \in \mathcal{N}_{core}, j \neq i} (1-D_j)}\right] \\
= & \mathbb{P}(\mathcal{D}_{per}) \mathbb{E}\left[\frac{\delta G_{per} \sum_{j \in \mathcal{N}_{per}} D_j}{G_{core}\delta + G_{per} \sum_{j \in \mathcal{N}_{per}} (1-D_j)(1+\delta) + \delta G_{core} \sum_{j \in \mathcal{N}_{core}, j \neq i} (1-D_j)} \mid \mathcal{D}_{per}\right] \\
& + (1-\mathbb{P}(\mathcal{D}_{per})) \mathbb{E}\left[\frac{\delta G_{per} \sum_{j \in \mathcal{N}_{per}} D_j}{G_{core}\delta + G_{per} \sum_{j \in \mathcal{N}_{per}} (1-D_j)(1+\delta) + \delta G_{core} \sum_{j \in \mathcal{N}_{core}, j \neq i} (1-D_j)} \mid \overline{\mathcal{D}_{per}}\right] \\
= & \mathbb{P}(\mathcal{D}_{per}) \underbrace{\mathbb{E}\left[\frac{\delta G_{per} \sum_{j \in \mathcal{N}_{per}} 1}{G_{core}\delta + \delta G_{core} \sum_{j \in \mathcal{N}_{core}, j \neq i} (1-D_j)}\right]}_{=A_1} \\
& + (1-\mathbb{P}(\mathcal{D}_{per})) \underbrace{\mathbb{E}\left[\frac{\delta G_{per} \sum_{j \in \mathcal{N}_{per}} D_j}{G_{core}\delta + G_{per} \sum_{j \in \mathcal{N}_{per}} (1-D_j)(1+\delta) + \delta G_{core} \sum_{j \in \mathcal{N}_{core}, j \neq i} (1-D_j)} \mid \overline{\mathcal{D}_{per}}\right]}_{=A_2},
\end{aligned}$$

using that D_n and D_m are independently distributed for $n \neq m$, where $\mathcal{D}_{per} = \{D \in \{0,1\}^N : D_j = 1 \ \forall j \in \mathcal{N}_{per}\}$ is the set of states in which all peripheral entities default and $\overline{\mathcal{D}_{per}}$ its complement. Since conditional on $\overline{\mathcal{D}_{per}}$, there exists $j \in \mathcal{N}_{per}$ such that $(1-D_j)(1+\delta) = 1+\delta > 0$, A_2 almost surely has a strictly positive denominator and is, thus, well-defined for $\delta = 0$, which implies that (using the dominated convergence theorem)

$$\lim_{\delta \to 0} A_2 = 0.$$

Moreover, for all $\delta > 0$, it is

$$\begin{aligned}
A_1 = & \frac{|\mathcal{N}_{per}| G_{per}}{G_{core}} \mathbb{E}\left[\frac{1}{1+\sum_{j \in \mathcal{N}_{core}, j \neq i} (1-D_j)}\right] = \frac{|\mathcal{N}_{per}| G_{per}}{G_{core}} \frac{1-\pi^{|\mathcal{N}_{core}|}}{|\mathcal{N}_{core}|(1-\pi)} \\
= & \frac{\frac{2N}{3} G_{per}}{\frac{N-3}{3} + 2G_{per}} \frac{1-\pi^{N/3}}{N/3(1-\pi)} = \frac{6G_{per}}{(N-3)+6G_{per}} \frac{1-\pi^{N/3}}{1-\pi},
\end{aligned}$$

using that $N_{core} = \frac{N-3}{3} + 2G_{per}$, $|\mathcal{N}_{per}| = \frac{2N}{3}$, and $|\mathcal{N}_{core}| = \frac{N}{3}$ and the properties of the Binomial distribution. Therefore,

$$\begin{aligned}\lim_{\delta\to 0}\delta H_i &= \mathbb{P}(\mathcal{D}_{per})\lim_{\delta\to 0}A_1 + (1-\mathbb{P}(\mathcal{D}_{per}))\lim_{\delta\to 0}A_2\\ &= \pi^{2N/3}\frac{6G_{per}}{(N-3)+6G_{per}}\frac{1-\pi^{N/3}}{1-\pi}\end{aligned}$$

and

$$\begin{aligned}\lim_{\delta\to 0}\Delta DL_i &= \frac{f(K-1)}{f(K)} + \lim_{\delta\to 0}\delta H_i\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi} - 1\\ &= \frac{f(K-1)}{f(K)} + \pi^{2N/3}\frac{6G_{per}}{(N-3)+6G_{per}}\frac{1-\pi^{N/3}}{1-\pi}\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi} - 1\\ &= \frac{f(K-1)}{f(K)} + \pi^{2N/3}\frac{6G_{per}}{(N-3)+6G_{per}}\frac{1-\pi^{N/3}}{\pi(1-\pi)}\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)} - 1.\end{aligned}$$

Consequently, $\lim_{N\to\infty}\lim_{\delta\to 0}\Delta DL_i = \frac{f(K-1)}{f(K)} - 1 < 0$. Therefore, there exists $\hat{N}$ such that $\lim_{\delta\to 0}\Delta DL_i < 0$ for all $N > \hat{N}$, i.e., such that entities in the core benefit from central clearing.

For $g \in \mathcal{N}_{per}$ and $h \in \mathcal{N}_{core}$ it is

$$\lim_{\delta\to 0}\Delta DL_g > \lim_{\delta\to 0}\Delta DL_h \tag{163}$$

$$\Leftrightarrow \frac{1-\pi^{2N/3-1}}{1-\pi}\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)} > \pi^{2N/3-1}\frac{6G_{per}}{(N-3)+6G_{per}}\frac{1-\pi^{N/3}}{1-\pi}\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)} \tag{164}$$

$$\Leftrightarrow \frac{1-\pi^{2N/3-1}}{1-\pi} > \pi^{2N/3-1}\frac{6G_{per}}{(N-3)+6G_{per}}\frac{1-\pi^{N/3}}{1-\pi} \tag{165}$$

which holds because

$$\pi^{2N/3-1}\underbrace{\frac{6G_{per}}{(N-3)+6G_{per}}}_{\leq 1}\frac{1-\pi^{N/3}}{1-\pi} \leq \pi^{N/3-1}\frac{1-\pi^{N/3}}{1-\pi} \tag{166}$$

$$= \frac{\pi^{N/3-1} - \pi^{N/3-1}\pi^{N/3}}{1-\pi} < \frac{1-\pi^{2N/3-1}}{1-\pi}. \tag{167}$$

□

Example 1. *Consider a core-periphery network. Central clearing with loss sharing based on net risk reduces expected default losses in aggregate but not that of peripheral entities for the following parameters:*

$G_{per} = 1$, $\pi = 0.05$, $N = 21$, $K = 10$, $\alpha_{uc} = \alpha_{CCP} = 0.99$, $\sigma = \sigma_M = 1$, $\beta = 0.3$.

Figure 5 illustrates comparative statics varying either the number of market participants, N, or the systematic risk exposure, β, while holding all other parameters constant to those above. Figure 5 (a) shows that larger N reduces ΔADL. Intuitively, a larger market enables more risk sharing and, thus, central clearing reduces expected aggregate default losses by more. In other words, central clearing becomes more beneficial overall. However, the impact of central clearing on an individual entity's expected default losses is largely unaffected by N. This is intuitive from the closed-form expressions in Proposition 8. A larger expected number of defaulters roughly balances a larger expected number of survivors.

Figure 5 (b) shows that a larger systematic risk exposure β reduces ΔADL as well as each entity's ΔDL. This result is in line with Proposition 5, which shows that larger β reduces bilateral netting efficiency and, thereby, makes central clearing relatively more beneficial. This effect is particularly pronounced for peripheral entities because they make larger loss sharing contributions.

Proof. From Proposition 3, the impact of clearing on expected aggregate default losses is equal to

$$\Delta ADL = \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \eta_{agg} + \frac{f(K-1)}{f(K)} - 1, \tag{168}$$

where

$$\eta_{agg} = \frac{\sum_{i=1}^{N} \left| \sum_{j \in \mathcal{N}_i} v_{ij} \right|}{\sum_{i=1}^{N} G_i} = \frac{\frac{2N}{3} G_{per} + \frac{N}{3} \cdot 0}{\frac{2N}{3} G_{per} + \frac{N}{3} \frac{N-3+6G_{per}}{3}} \tag{169}$$

$$= \frac{6G_{per}}{6G_{per} + N - 3 + 6G_{per}} = \frac{6G_{per}}{12G_{per} + N - 3} \tag{170}$$

in the case of a core-periphery network. The statement follows from setting the variables equal to the parameters. □

Corollary 4 (Comparison with counterparty risk exposure in core-periphery networks)**.** *Consider a core-periphery network as in Assumption 2 and assume that $\alpha_{uc} = \alpha_{CCP}$. Then, the impact of central clearing with loss sharing based on net risk on expected default losses is equal to the impact of central clearing on counterparty risk exposure for peripheral entities $i \in \mathcal{N}_{per}$ if $N = 3$, and strictly larger if $N > 3$ and for core entities $i \in \mathcal{N}_{core}$.*

Proof. From Proposition 1, the impact of central clearing on counterparty risk exposure is equal to

$$\Delta E_i = \begin{cases} \frac{f(K-1)}{f(K)} + \frac{f(1)}{f(K)} - 1, \text{ if } i \in \mathcal{N}_{per}, \\ \frac{f(K-1)}{f(K)} - 1, \text{ if } i \in \mathcal{N}_{core}. \end{cases} \tag{171}$$

Therefore, if $\alpha_{uc} = \alpha_{CCP}$, $N = 3$, and $i \in \mathcal{N}_{per}$, then

$$\Delta E_i = \frac{f(K-1)}{f(K)} + \frac{f(1)}{f(K)} - 1 = \lim_{\delta \to 0} \Delta DL_i, \tag{172}$$

and if $N > 3$, then

$$\Delta E_i = \frac{f(K-1)}{f(K)} + \frac{f(1)}{f(K)} - 1 < \lim_{\delta \to 0} \Delta DL_i. \tag{173}$$

Since $\pi^{2N/3-1} \frac{6G_{per}}{(N-3)+6G_{per}} \frac{1-\pi^{N/3}}{1-\pi} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)}$ is strictly positive, for $i \in \mathcal{N}_{core}$ it holds that

$$\lim_{\delta \to 0} \Delta DL_i = \frac{f(K-1)}{f(K)} + \pi^{2N/3-1} \frac{6G_{per}}{(N-3)+6G_{per}} \frac{1-\pi^{N/3}}{1-\pi} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} - 1, \tag{174}$$

$$> \frac{f(K-1)}{f(K)} - 1 = \Delta E_i. \tag{175}$$

□

Proposition 9 (Loss sharing based on net and gross risk)**.** *With the loss sharing rule $w(\delta)$, clearing member i's expected loss sharing contribution is equal to*

$$\mathbb{E}[LSC_i] = (1-\pi)\xi(\alpha_{CCP}) w_i(\delta) \mathbb{E}\left[\frac{\sum_{j=1, j\neq i}^N D_j \bar{\sigma}_j}{w_i(\delta) + \sum_{j=1,j\neq i}^N (1-D_j) w_j(\delta)}\right]. \tag{28}$$

The impact of central clearing on i's expected default losses is given by

$$\Delta DL_i = \frac{f(K-1)}{f(K)} + \frac{w_i(\delta)}{G_i f(K)} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{1}{\pi} \mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^N D_j G_j \eta_j}{w_i(\delta) + \sum_{j=1,j\neq i}^N (1-D_j) w_j(\delta)}\right] - 1, \tag{29}$$

which is increasing with δ if, and only if,

$$\mathbb{E}\left[\tilde{H} \sum_{j=1,j\neq i}^N (1-D_j)(w_j(0) w_i(\delta) - w_j(\delta) w_i(0))\right] > 0, \tag{30}$$

where $\tilde{H}$ is a nonnegative random variable with $\mathbb{E}[\tilde{H}] > 0$.

(a) Assume that $\eta_j = \eta \in [0,1]$ for all $j = 1, ..., N$. Then, $\frac{\partial \Delta DL_i}{\partial \delta} = 0$.

(b) Consider an entity with a flat portfolio, $\eta_i = 0$. Assume that there exist at least two fellow clearing members a and b, $a \neq b$, with portfolio directionality $\eta_a > 0$ and $\eta_b > 0$. Then,

$$\frac{\partial \Delta DL_i}{\partial \delta} > 0.$$

(c) Consider an entity with a fully directional portfolio, $\eta_i = 1$. Assume that there exist at least two fellow clearing members a and b, $a \neq b$, with portfolio directionality $\eta_a < 1$ and $\eta_b > 0$. Then,

$$\frac{\partial \Delta DL_i}{\partial \delta} < 0.$$

Proof. From Definition 4 and Proposition 1, it is

$$w_i(\delta) = \delta G_i f(1) + (1-\delta)\eta_i G_i f(1) = (\delta + (1-\delta)\eta_i) G_i f(1). \tag{176}$$

The expected loss sharing contribution of entity i with loss sharing rule $w(\delta)$ is given by

$$\begin{aligned}
\mathbb{E}[LSC_i] &= \mathbb{P}(D_i = 0)\mathbb{E}\left[\frac{w_i(\delta)}{\sum_{g=1}^{N}(1-D_g)w_g(\delta)} DL^{CCP} \mid D_i = 0\right] \\
&= (1-\pi)\xi(\alpha_{CCP}) w_i(\delta) \mathbb{E}\left[\frac{\sum_{j=1, j\neq i}^{N} D_j \bar{\sigma}_j}{w_i(\delta) + \sum_{j=1, j\neq i}^{N}(1-D_j)w_j(\delta)}\right],
\end{aligned}$$

following the same steps as in the proof of Proposition 4.

Using Proposition 2, the impact of central clearing for entity i is then given by

$$\begin{aligned}
\Delta DL_i &= \frac{(1-\pi)\pi G_i \xi(\alpha_{uc}) f(K-1) + (1-\pi)\xi(\alpha_{CCP}) w_i(\delta)\mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^{N} D_j \bar{\sigma}_j}{w_i(\delta) + \sum_{j=1,j\neq i}^{N}(1-D_j)w_j(\delta)}\right]}{(1-\pi)\pi G_i \xi(\alpha_{uc}) f(K)} - 1 \\
&= \frac{f(K-1)}{f(K)} + \frac{w_i(\delta)}{G_i}\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{1}{\pi f(K)}\mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^{N} D_j G_j \eta_j}{w_i(\delta) + \sum_{j=1,j\neq i}^{N}(1-D_j)w_j(\delta)}\right] - 1.
\end{aligned}$$

The derivative of $w_i(\delta)$ with respect to δ is equal to

$$\frac{\partial w_i}{\partial \delta} = (1-\eta_i) G_i f(1). \tag{177}$$

Define by $H = \frac{\sum_{j=1,j\neq i}^{N} D_j G_j \eta_j}{w_i(\delta)+\sum_{j=1,j\neq i}^{N}(1-D_j)w_j(\delta)}$ the CCP's default losses per unit of loss sharing weight. The derivative of ΔDL_i with respect to δ is equal to

$$
\begin{aligned}
\frac{\partial \Delta DL_i}{\partial \delta} &= \frac{1}{G_i \pi f(K)} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{\partial}{\partial \delta} w_i(\delta) \mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^{N} D_j G_j \eta_j}{w_i(\delta)+\sum_{j=1,j\neq i}^{N}(1-D_j)w_j(\delta)}\right] \\
&= \frac{1}{G_i \pi f(K)} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \Bigg((1-\eta_i) G_i f(1) \mathbb{E}[H] \\
&\qquad - w_i(\delta)\mathbb{E}\left[H \frac{(1-\eta_i)G_i f(1) + \sum_{j=1,j\neq i}^{N}(1-D_j)(1-\eta_j)G_j f(1)}{G_i f(1)(\delta+(1-\delta)\eta_i) + \sum_{j=1,j\neq i}^{N}(1-D_j)w_j(\delta)}\right] \Bigg) \\
&= \frac{1}{\pi f(K)} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \Bigg((1-\eta_i) f(1) \mathbb{E}[H] \\
&\qquad - f(1)(\delta+(1-\delta)\eta_i)\mathbb{E}\left[H \frac{(1-\eta_i)G_i + \sum_{j=1,j\neq i}^{N}(1-D_j)(1-\eta_j)G_j}{G_i(\delta+(1-\delta)\eta_i) + \sum_{j=1,j\neq i}^{N}(1-D_j)(\delta+(1-\delta)\eta_j)G_j}\right] \Bigg),
\end{aligned}
$$

which is positive if, and only if,

$$
\begin{aligned}
& \frac{1-\eta_i}{\delta+(1-\delta)\eta_i}\mathbb{E}[H] > \mathbb{E}\left[H \frac{(1-\eta_i)G_i + \sum_{j=1,j\neq i}^{N}(1-D_j)(1-\eta_j)G_j}{G_i(\delta+(1-\delta)\eta_i) + \sum_{j=1,j\neq i}^{N}(1-D_j)(\delta+(1-\delta)\eta_j)G_j}\right] \\
\Leftrightarrow\ & \frac{1-\eta_i}{\delta+(1-\delta)\eta_i}\mathbb{E}[H] > \mathbb{E}\left[H \frac{1}{\delta}\left(1 - \frac{w_i(0)+\sum_{j=1,j\neq i}^{N}(1-D_j)w_j(0)}{w_i(\delta)+\sum_{j=1,j\neq i}^{N}(1-D_j)w_j(\delta)}\right)\right] \\
\Leftrightarrow\ & \delta \frac{1-\eta_i}{\delta+(1-\delta)\eta_i}\mathbb{E}[H] > \mathbb{E}[H] - \mathbb{E}\left[H \frac{w_i(0)+\sum_{j=1,j\neq i}^{N}(1-D_j)w_j(0)}{w_i(\delta)+\sum_{j=1,j\neq i}^{N}(1-D_j)w_j(\delta)}\right] \\
\Leftrightarrow\ & \mathbb{E}\left[H \frac{w_i(0)+\sum_{j=1,j\neq i}^{N}(1-D_j)w_j(0)}{w_i(\delta)+\sum_{j=1,j\neq i}^{N}(1-D_j)w_j(\delta)}\right] > \mathbb{E}[H] - \delta \frac{1-\eta_i}{\delta+(1-\delta)\eta_i}\mathbb{E}[H] \\
\Leftrightarrow\ & \mathbb{E}\left[H \frac{w_i(0)+\sum_{j=1,j\neq i}^{N}(1-D_j)w_j(0)}{w_i(\delta)+\sum_{j=1,j\neq i}^{N}(1-D_j)w_j(\delta)}\right] > \mathbb{E}\left[H \frac{w_i(0)}{w_i(\delta)}\right] \\
\Leftrightarrow\ & \mathbb{E}\left[H \left(\frac{w_i(0)+\sum_{j=1,j\neq i}^{N}(1-D_j)w_j(0)}{w_i(\delta)+\sum_{j=1,j\neq i}^{N}(1-D_j)w_j(\delta)} - \frac{w_i(0)}{w_i(\delta)}\right)\right] > 0 \\
\Leftrightarrow\ & \mathbb{E}\left[H \frac{\sum_{j=1,j\neq i}^{N}(1-D_j)(w_j(0)w_i(\delta) - w_j(\delta)w_i(0))}{w_i(\delta)\left(w_i(\delta)+\sum_{j=1,j\neq i}^{N}(1-D_j)w_j(\delta)\right)}\right] > 0 \\
\Leftrightarrow\ & \mathbb{E}\left[\tilde{H} \sum_{j=1,j\neq i}^{N}(1-D_j)(w_j(0)w_i(\delta) - w_j(\delta)w_i(0))\right] > 0, \qquad (178)
\end{aligned}
$$

where we define $\tilde{H} = \frac{h}{w_i(\delta)\left(w_i(\delta)+\sum_{j=1,j\neq i}^{N}(1-D_j)w_j(\delta)\right)}$, which is nonnegative with probability one.

From Equation (178) it follows that:

(a) $\frac{\partial \Delta DL_i}{\partial \delta} = 0$ if $\eta_j \equiv \eta \in [0,1]$ for all $j = 1, ..., N$, since in this case

$$\begin{aligned}
&\mathbb{E}\left[\tilde{H} \sum_{j=1, j\neq i}^{N} (1-D_j)(w_j(0)w_i(\delta) - w_j(\delta)w_i(0))\right] \\
=&f(1)\mathbb{E}\left[\tilde{H} \sum_{j=1, j\neq i}^{N} (1-D_j)(\eta G_j w_i(\delta) - w_j(\delta)\eta G_i)\right] \\
=&f(1)\eta\mathbb{E}\left[\tilde{H} \sum_{j=1, j\neq i}^{N} (1-D_j)(G_j(\delta+(1-\delta)\eta)G_i f(1) - (\delta+(1-\delta)\eta)G_j f(1) G_i)\right] \\
=&f(1)^2\eta(\delta+(1-\delta)\eta)G_i\mathbb{E}\left[\tilde{H} \sum_{j=1, j\neq i}^{N} (1-D_j)(G_j - G_j)\right] = 0.
\end{aligned}$$

(b) $\frac{\partial \Delta DL_i}{\partial \delta} > 0$ if $\eta_i = 0$ since in this case $w_i(0) = \eta_i f(1) G_i = 0$ and, thus,

$$\begin{aligned}
&\mathbb{E}\left[\tilde{H} \sum_{j=1, j\neq i}^{N} (1-D_j)(w_j(0)w_i(\delta) - w_j(\delta)w_i(0))\right] \\
=&\mathbb{E}\left[\tilde{H} \sum_{j=1, j\neq i}^{N} (1-D_j)(\eta_j f(1) G_j w_i(\delta))\right] \\
\geq& w_i(\delta) f(1)\mathbb{E}\left[\tilde{H}\left((1-D_a)\eta_a G_a + (1-D_b)\eta_b G_b\right)\right] > 0,
\end{aligned}$$

where we use that by assumption there exist $a, b \in \{1, ..., N\}\backslash\{i\}, a \neq b$, with $\eta_a > 0$ and $\eta_b > 0$ such that $\mathbb{P}(D_a = 1, D_b = 0) + \mathbb{P}(D_a = 0, D_b = 1) > 0$ implies that $\mathbb{P}(\tilde{h} > 0, (1 - D_a)\eta_a G_a + (1 - D_b)\eta_b G_b > 0) > 0$.

(c) $\frac{\partial \Delta DL_i}{\partial \delta} < 0$ if $\eta_i = 1$ since in this case $w_i(\delta) = (\delta + (1-\delta)) f(1) G_i \equiv f(1) G_i$ and, thus,

$$\begin{aligned}
&\mathbb{E}\left[\tilde{H} \sum_{j=1, j\neq i}^{N} (1-D_j)(w_j(0)w_i(\delta) - w_j(\delta)w_i(0))\right] \\
=&f(1)G_i\mathbb{E}\left[\tilde{H} \sum_{j=1, j\neq i}^{N} (1-D_j)(w_j(0) - w_j(\delta))\right] \\
\leq& f(1)G_i\mathbb{E}\left[\tilde{H}(1-D_a)(w_a(0) - w_a(\delta))\right] < 0,
\end{aligned}$$

where we use that by assumption there exist $a, b \in \{1, ..., N\}\backslash\{i\}, a \neq b$, with $\eta_a < 1$ and $\eta_b > 0$ such that $w_a(0) - w_a(\delta) = (\eta_a - (\delta + (1-\delta)\eta_a)) f(1) G_a = -\delta(1-\eta_a) f(1) G_a < 0$ for all

$\delta > 0$ and that $P(D_b\eta_b G_b > 0, D_a = 0) > 0$, implying that $\mathbb{P}(\tilde{h} > 0, (1-D_a)(w_a(0) - w_a(\delta)) < 0) > 0$ and $\mathbb{P}(\tilde{h} < 0, (1-D_a)(w_a(0) - w_a(\delta)) < 0) = 0$.

□

Proposition 10 (Loss sharing based on gross risk)**.** *Consider two entities g, h, $g \neq h$, and assume that loss sharing is proportional to gross portfolio risk, $\delta = 1$. Then, the difference in the impact of central clearing between the two entities is equal to*

$$\begin{aligned}&\Delta DL_g - \Delta DL_h \\ =&\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}\left(\mathbb{E}\left[\frac{\sum_{j=1}^N D_j G_j \eta_j}{\sum_{j=1}^N (1-D_j)G_j} \mid D_g = 0\right] - \mathbb{E}\left[\frac{\sum_{j=1}^N D_j G_j \eta_j}{\sum_{j=1}^N (1-D_j)G_j} \mid D_h = 0\right]\right).\end{aligned} \tag{31}$$

(a) Conditional on $D_g = D_h$, the impact of central clearing is the same across entities:

$$\Delta DL_{g|D_g=D_h} = \Delta DL_{h|D_g=D_h}. \tag{32}$$

(b) If $\eta_g = \eta_h$, then

$$G_h > G_g \;\Rightarrow\; \Delta DL_h < \Delta DL_g. \tag{33}$$

(c) If $G_g = G_h$, then

$$\eta_h > \eta_g \;\Leftrightarrow\; \Delta DL_h < \Delta DL_g. \tag{34}$$

(d) If $h \in \mathcal{N}_{core}$ and $g \in \mathcal{N}_{per}$ in a core-periphery network, then there exists $\hat{\pi} > 0$ such that for all $\pi \in (0, \hat{\pi})$ it is

$$\Delta DL_g < \Delta DL_h. \tag{35}$$

Proof. When $\delta = 1$, loss sharing weights are equal to $w_i = G_i f(1)$. Using Proposition 9, the impact of central clearing on i's expected default losses is then given by

$$\Delta DL_i = \frac{f(K-1)}{f(K)} + \frac{w_i(\delta)}{G_i}\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}\mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^{N} D_j G_j \eta_j}{w_i(\delta)+\sum_{j=1,j\neq i}^{N}(1-D_j)w_j(\delta)}\right] - 1 \tag{179}$$

$$= \frac{f(K-1)}{f(K)} + \frac{G_i f(1)}{G_i}\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}\mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^{N} D_j G_j \eta_j}{G_i f(1)+\sum_{j=1,j\neq i}^{N}(1-D_j)G_j f(1)}\right] - 1 \tag{180}$$

$$= \frac{f(K-1)}{f(K)} + \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}\mathbb{E}\left[\frac{\sum_{j=1}^{N} D_j G_j \eta_j}{\sum_{j=1}^{N}(1-D_j)G_j} \mid D_i = 0\right] - 1. \tag{181}$$

Consider two entities $g, h \in \{1, ..., N\}, g \neq h$. Then, the difference in the impact of central clearing is equal to

$$\Delta DL_g - \Delta DL_h \tag{182}$$

$$= \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}\left(\mathbb{E}\left[\frac{\sum_{j=1}^{N} D_j G_j \eta_j}{\sum_{j=1}^{N}(1-D_j)G_j} \mid D_g = 0\right] - \mathbb{E}\left[\frac{\sum_{j=1}^{N} D_j G_j \eta_j}{\sum_{j=1}^{N}(1-D_j)G_j} \mid D_h = 0\right]\right).$$

Define by $\tilde{D}$ a Bernoulli distributed random variable with success probability π such that $\tilde{D}$ and D_j are independently distributed for all $j \notin \{g, h\}$. With $A = \sum_{j=1,j\notin\{g,h\}}^{N} D_j G_j \eta_j \geq 0$ and $B = \sum_{j=1,j\notin\{g,h\}}^{N}(1-D_j)G_j \geq 0$

$$\frac{\Delta DL_g - \Delta DL_h}{\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}} \tag{183}$$

$$= \mathbb{E}\left[\frac{\tilde{D}G_h\eta_h + A}{G_g + (1-\tilde{D})G_h + B} - \frac{\tilde{D}G_g\eta_g + A}{G_h + (1-\tilde{D})G_g + B}\right] \tag{184}$$

$$= \mathbb{E}\left[\frac{(\tilde{D}G_h\eta_h + A)(G_h + (1-\tilde{D})G_g + B) - (\tilde{D}G_g\eta_g + A)(G_g + (1-\tilde{D})G_h + B)}{(G_g + (1-\tilde{D})G_h + B)(G_h + (1-\tilde{D})G_g + B)}\right] \tag{185}$$

$$= \mathbb{E}\left[\frac{A\tilde{D}[G_h - G_g] + \tilde{D}\left[G_h^2\eta_h - G_g^2\eta_g + G_g G_h(1-\tilde{D})(\eta_h - \eta_g) + B(G_h\eta_h - G_g\eta_g)\right]}{(G_g + (1-\tilde{D})G_h + B)(G_h + (1-\tilde{D})G_g + B)}\right] \tag{186}$$

$$= \mathbb{E}\left[\tilde{D}\frac{A(G_h - G_g) + G_h^2\eta_h - G_g^2\eta_g + B(G_h\eta_h - G_g\eta_g)}{(G_g + (1-\tilde{D})G_h + B)(G_h + (1-\tilde{D})G_g + B)}\right] \tag{187}$$

$$= \pi\mathbb{E}\left[\frac{A(G_h - G_g) + G_h^2\eta_h - G_g^2\eta_g + B(G_h\eta_h - G_g\eta_g)}{(G_g + B)(G_h + B)}\right], \tag{188}$$

using that

$$\tilde{D}(1-\tilde{D}) = \begin{cases} 0 \times (1-0) = 0, \text{ if } \tilde{D} = 0 \\ 1 \times (1-1) = 0, \text{ if } \tilde{D} = 1. \end{cases}$$

(a) If $D_g = 0$ and $D_h = 0$, then Equation (182) implies that the impact of central clearing is the same for entities h and g. Moreover, if $D_g = 1$ and $D_h = 1$, cleared and uncleared default losses are zero and the impact of central clearing coincides, as well. Therefore, conditional on $D_g = D_h$, the impact of central clearing is the same across entities, $\Delta DL_{g|D_g=D_h} = \Delta DL_{h|D_g=D_h}$.

(b) If $\eta_g = \eta_h$, then Expression (188) is equal to

$$\pi\mathbb{E}\left[\frac{A(G_h - G_g) + \eta_g\left[G_h^2 - G_g^2 + B(G_h - G_g)\right]}{(G_g + B)(G_h + B)}\right] \tag{189}$$

$$=\pi\mathbb{E}\left[\frac{(G_h - G_g)(A + \eta_g B) + \eta_g(G_h^2 - G_g^2)}{(G_g + B)(G_h + B)}\right], \tag{190}$$

which is positive if $G_h > G_g$. Thus, $\Delta DL_g - \Delta DL_h > 0$ if $G_h > G_g$.

(c) If $G_g = G_h$, then Expression (188) is equal to

$$\pi\mathbb{E}\left[\frac{G_h^2(\eta_h - \eta_g) + BG_h(\eta_h - \eta_g)}{(G_h + B)(G_h + B)}\right] = \pi(\eta_h - \eta_g)G_h\mathbb{E}\left[\frac{G_h + B}{(G_h + B)^2}\right], \tag{191}$$

which is positive if, and only if, $G_h > G_g$. Thus, $\Delta DL_g - \Delta DL_h > 0$ if, and only if, $\eta_h > \eta_g$.

(d) In a core-periphery network as in Assumption 2 with $h \in \mathcal{N}_{core}$ and $g \in \mathcal{N}_{per}$, it is $G_h = \frac{N-3}{3} + 2G_{per}$, $G_g = G_{per}$, $\eta_h = 0$, and $\eta_g = 1$, and, thus, Expression (188) is equal to

$$\pi\mathbb{E}\left[\frac{A(\frac{N-3}{3} + 2G_{per} - G_{per}) - G_{per}^2 - BG_{per}}{(G_{per} + B)(\frac{N-3}{3} + 2G_{per} + B)}\right] \tag{192}$$

$$=\pi\mathbb{E}\left[\frac{A\frac{N-3+3G_{per}}{3} - (G_{per} + B)G_{per}}{(G_{per} + B)(\frac{N-3+6G_{per}}{3} + B)}\right]. \tag{193}$$

Moreover, it is

$$A = \sum_{j=1, j\notin\{g,h\}}^{N} D_j G_j \eta_j = G_{per} \sum_{j\in\mathcal{N}_{per}\setminus\{g\}} D_j$$

$$B = \sum_{j=1, j\notin\{g,h\}}^{N} (1 - D_j)G_j = \frac{N - 3 + 6G_{per}}{3} \sum_{j\in\mathcal{N}_{core}\setminus\{h\}} (1 - D_j) + G_{per} \sum_{j\in\mathcal{N}_{per}\setminus\{g\}} (1 - D_j),$$

which implies that the nominator in the expectation in Expression (193) is equal to

$$\begin{aligned}
\tilde{A} =& A\frac{N-3+3G_{per}}{3} - (G_{per}+B)G_{per} \\
=& G_{per}\frac{N-3+3G_{per}}{3}\sum_{j\in\mathcal{N}_{per}\setminus\{g\}} D_j - G_{per}^2 \\
& - G_{per}\left(\frac{N-3+6G_{per}}{3}\sum_{j\in\mathcal{N}_{core}\setminus\{h\}}(1-D_j) + G_{per}\sum_{j\in\mathcal{N}_{per}\setminus\{g\}}(1-D_j)\right) \\
=& G_{per}\Bigg(\sum_{j\in\mathcal{N}_{per}\setminus\{g\}}\left(D_j\frac{N-3+3G_{per}}{3} - (1-D_j)G_{per}\right) \\
& - \frac{N-3+6G_{per}}{3}\sum_{j\in\mathcal{N}_{core}\setminus\{h\}}(1-D_j) - G_{per}\Bigg) \\
=& G_{per}\left(\sum_{j\in\mathcal{N}_{per}\setminus\{g\}}\left(D_j\frac{N-3+6G_{per}}{3} - G_{per}\right) - \frac{N-3+6G_{per}}{3}\sum_{j\in\mathcal{N}_{core}\setminus\{h\}}(1-D_j) - G_{per}\right) \\
=& G_{per}\Bigg(\sum_{j\in\mathcal{N}_{per}\setminus\{g\}}\left(D_j\frac{N-3+6G_{per}}{3}\right) - G_{per}\frac{2N-3}{3} - \frac{N-3+6G_{per}}{3}\frac{N-3}{3} \\
& + \frac{N-3+6G_{per}}{3}\sum_{j\in\mathcal{N}_{core}\setminus\{h\}} D_j - G_{per}\Bigg) \\
=& G_{per}\left(\frac{N-3+6G_{per}}{3}\sum_{j\in\{1,\dots,N\}\setminus\{g,h\}} D_j - G_{per}\frac{2N-3}{3} - \frac{N-3+6G_{per}}{3}\frac{N-3}{3} - G_{per}\right) \\
=& \tilde{a}\tilde{D} - \tilde{b},
\end{aligned}$$

with

$$\tilde{a} = G_{per}\frac{N-3+6G_{per}}{3} > 0, \tag{194}$$

$$\tilde{b} = G_{per}\left(G_{per}\frac{2N-3}{3} + \frac{N-3+6G_{per}}{3}\frac{N-3}{3} + G_{per}\right) > 0, \tag{195}$$

$$\tilde{D} = \sum_{j\in\{1,\dots,N\}\setminus\{g,h\}} D_j \sim Bin(N-2,\pi). \tag{196}$$

We define $\hat{d} = \tilde{b}/\tilde{a} > 0$. Then,

$$\tilde{A} \geq 0 \;\Leftrightarrow\; \tilde{a}\tilde{D} - \tilde{b} \geq 0 \;\Leftrightarrow\; \tilde{D} \geq \hat{d}. \tag{197}$$

We consider the following two cases:

$\underline{\tilde{D} \geq \hat{d}}$: In this case, $\tilde{A} \geq 0$. Then, using that $B \geq 0$, it is

$$\frac{\tilde{A}}{(G_{per} + B)(\frac{N-3+6G_{per}}{3} + B)} \leq \frac{\tilde{A}}{G_{per}\frac{N-3+6G_{per}}{3}}. \tag{198}$$

$\underline{\tilde{D} < \hat{d}}$: In this case, $\tilde{A} < 0$. Then, using that

$$B \leq \frac{N-3+6G_{per}}{3}(|\mathcal{N}_{core}| - 1) + G_{per}(|\mathcal{N}_{per}| - 1) = \overline{b} > 0, \tag{199}$$

it is

$$\frac{\tilde{A}}{(G_{per} + B)(\frac{N-3+6G_{per}}{3} + B)} \leq \frac{\tilde{A}}{(G_{per} + \overline{b})(\frac{N-3+6G_{per}}{3} + \overline{b})}. \tag{200}$$

Combining both cases, Expression (193) is equal to

$$\pi\mathbb{E}\left[\frac{A\frac{N-3+3G_{per}}{3}-(G_{per}+B)G_{per}}{(G_{per}+B)(\frac{N-3+6G_{per}}{3}+B)}\right] \tag{201}$$

$$=\pi\Bigg(\mathbb{P}(\tilde{D}\geq\hat{d})\mathbb{E}\left[\frac{\tilde{A}}{(G_{per}+B)(\frac{N-3+6G_{per}}{3}+B)}\mid\tilde{D}\geq\hat{d}\right]$$
$$+\mathbb{P}(\tilde{D}<\hat{d})\mathbb{E}\left[\frac{\tilde{A}}{(G_{per}+B)(\frac{N-3+6G_{per}}{3}+B)}\mid\tilde{D}<\hat{d}\right]\Bigg) \tag{202}$$

$$\leq\pi\Bigg(\mathbb{P}(\tilde{D}\geq\hat{d})\frac{\mathbb{E}\left[\tilde{A}\mid\tilde{D}\geq\hat{d}\right]}{G_{per}\frac{N-3+6G_{per}}{3}}+\mathbb{P}(\tilde{D}<\hat{d})\frac{\mathbb{E}\left[\tilde{A}\mid\tilde{D}<\hat{d}\right]}{(G_{per}+\overline{b})(\frac{N-3+6G_{per}}{3}+\overline{b})}\Bigg) \tag{203}$$

$$=\pi\Bigg(\frac{\mathbb{E}\left[\tilde{A}\right]}{(G_{per}+\overline{b})(\frac{N-3+6G_{per}}{3}+\overline{b})}$$
$$-\mathbb{P}(\tilde{D}\geq\hat{d})\frac{\mathbb{E}\left[\tilde{A}\mid\tilde{D}\geq\hat{d}\right]}{(G_{per}+\overline{b})(\frac{N-3+6G_{per}}{3}+\overline{b})}+\mathbb{P}(\tilde{D}\geq\hat{d})\frac{\mathbb{E}\left[\tilde{A}\mid\tilde{D}\geq\hat{d}\right]}{G_{per}\frac{N-3+6G_{per}}{3}}\Bigg) \tag{204}$$

$$=\pi\Bigg(\frac{\mathbb{E}\left[\tilde{A}\right]}{(G_{per}+\overline{b})(\frac{N-3+6G_{per}}{3}+\overline{b})}$$
$$+\mathbb{P}(\tilde{D}\geq\hat{d})\mathbb{E}\left[\tilde{A}\mid\tilde{D}\geq\hat{d}\right]\Bigg(\frac{1}{G_{per}\frac{N-3+6G_{per}}{3}}-\frac{1}{(G_{per}+\overline{b})(\frac{N-3+6G_{per}}{3}+\overline{b})}\Bigg)\Bigg) \tag{205}$$

$$=\pi\Bigg(\frac{\mathbb{E}\left[\tilde{A}\right]}{(G_{per}+\overline{b})(\frac{N-3+6G_{per}}{3}+\overline{b})}$$
$$+\mathbb{P}(\tilde{D}\geq\hat{d})\mathbb{E}\left[\tilde{A}\mid\tilde{D}\geq\hat{d}\right]\frac{(G_{per}+\overline{b})(\frac{N-3+6G_{per}}{3}+\overline{b})-G_{per}\frac{N-3+6G_{per}}{3}}{G_{per}\frac{N-3+6G_{per}}{3}(G_{per}+\overline{b})(\frac{N-3+6G_{per}}{3}+\overline{b})}\Bigg) \tag{206}$$

$$=\pi\Bigg(\frac{\tilde{a}(N-2)\pi-\tilde{b}}{(G_{per}+\overline{b})(\frac{N-3+6G_{per}}{3}+\overline{b})}+\mathbb{P}(\tilde{D}\geq\hat{d})\mathbb{E}\left[\tilde{A}\mid\tilde{D}\geq\hat{d}\right]\tilde{g}\Bigg) \tag{207}$$

with $\tilde{g}=\frac{(G_{per}+\overline{b})(\frac{N-3+6G_{per}}{3}+\overline{b})-G_{per}\frac{N-3+6G_{per}}{3}}{G_{per}\frac{N-3+6G_{per}}{3}(G_{per}+\overline{b})(\frac{N-3+6G_{per}}{3}+\overline{b})}>0$. Using Markov's inequality (note that $\hat{d}>0$), it is

$$\mathbb{P}(\tilde{D}\geq\hat{d})\leq\frac{\mathbb{E}[\tilde{D}]}{\hat{d}}=\frac{(N-2)\pi}{\hat{d}}. \tag{208}$$

Moreover, it is $\mathbb{E}\left[\tilde{A} \mid \tilde{D} \geq \hat{d}\right] \leq \tilde{a}(N-2) - \tilde{b}$. Using this in Expression (207) yields that

$$\pi\left(\frac{\tilde{a}(N-2)\pi - \tilde{b}}{(G_{per}+\overline{b})(\frac{N-3+6G_{per}}{3}+\overline{b})} + \mathbb{P}(\tilde{D} \geq \hat{d})\mathbb{E}\left[\tilde{A} \mid \tilde{D} \geq \hat{d}\right]\tilde{g}\right) \tag{209}$$

$$\leq \pi\Bigg(\underbrace{\frac{\tilde{a}(N-2)\pi - \tilde{b}}{(G_{per}+\overline{b})(\frac{N-3+6G_{per}}{3}+\overline{b})} + \frac{(N-2)\pi}{\hat{d}}(\tilde{a}(N-2)-\tilde{b})\tilde{g}}_{=\tilde{C}}\Bigg). \tag{210}$$

When π approaches zero, the term inside the parentheses becomes negative:

$$\tilde{C} \to -\frac{\tilde{b}}{(G_{per}+\overline{b})(\frac{N-3+6G_{per}}{3}+\overline{b})} < 0 \quad \text{for} \quad \pi \to 0. \tag{211}$$

Due to continuity, there exists $\hat{\pi} > 0$ such that for all $\pi \in (0, \hat{\pi})$ it holds that $\pi\tilde{C} < 0$. Using Equality (188), for $\pi \in (0, \hat{\pi})$ it is, thus, $\Delta DL_g - \Delta DL_h < 0 \Leftrightarrow \Delta DL_g < \Delta DL_h$.

□

Corollary 5. *Consider a core-periphery network and let $g \in \mathcal{N}_{per}$ and $h \in \mathcal{N}_{core}$. If π is sufficiently small, there exists $\hat{\delta} \in (0,1)$ such that $\Delta DL_g = \Delta DL_h$ for the loss sharing rule $w(\hat{\delta})$ and that $\Delta DL_g > \Delta DL_h$ if, and only if, $\delta < \hat{\delta}$.*

Proof. From Proposition 8, it is $\Delta DL_g > \Delta DL_h$ if loss sharing is based on net risk, i.e., when δ approaches zero. From Proposition 10 (d), it is $\Delta DL_g < \Delta DL_h$ if loss sharing is based on gross risk ($\delta = 1$) and π is sufficiently small. From Proposition 9, it is $\frac{\partial \Delta DL_g}{\partial \delta} < 0$ and $\frac{\partial \Delta DL_h}{\partial \delta} > 0$, which implies monotonicity of the differential impact of central clearing in δ, i.e.,

$$\frac{\partial(\Delta DL_g - \Delta DL_h)}{\partial \delta} < 0.$$

Together with continuity, the statement follows. □

Proposition 11 (Costly collateral)**.** *Assume that at least two entities have a portfolio that is not perfectly flat. ΔDLC_i is equal to*

$$\Delta DLC_i = \frac{f(K-1)}{f(K)} + \frac{f(1)}{f(K)} \frac{\xi(\alpha_{CCP})\frac{w_i(\delta)}{G_i}\mathbb{E}\left[H\right] + c\eta_i\Phi^{-1}(\alpha_{CCP})}{\pi\xi(\alpha_{uc}) + c\Phi^{-1}(\alpha_{uc})} - 1, \tag{37}$$

where $H = \frac{\sum_{j=1,j\neq i}^{N} D_j G_j \eta_j}{w_i(\delta)+\sum_{j=1,j\neq i}^{N}(1-D_j)w_j(\delta)}$.

(1) If entity i has a flat portfolio, $\eta_i = 0$, then the impact of central clearing on expected default losses and collateral costs is decreasing with the CCP's margin requirement, $\frac{\partial \Delta DLC_i}{\partial \alpha_{CCP}} < 0$.

(2) If entity i's portfolio is not flat, $\eta_i > 0$, and $\alpha_{CCP} > 0$, there exists $0 < \hat{c} < \infty$ such that the impact of central clearing on expected default losses and collateral costs is decreasing with the CCP's margin requirement if, and only if, the marginal cost of collateral c is below $\hat{c}$,

$$\frac{\partial \Delta DLC_i}{\partial \alpha_{CCP}} < 0 \;\Leftrightarrow\; c < \hat{c}. \tag{38}$$

Proof. Using Lemma 1, the collateral posted by entity i to the CCP is equal to

$$C_i^{CCP} = \bar{\sigma}_i \Phi^{-1}(\alpha_{CCP}) = \eta_i G_i f(1) \Phi^{-1}(\alpha_{CCP}). \tag{212}$$

The total collateral posted by entity i to its bilateral counterparties in uncleared derivative classes $1, ..., K$ is equal to

$$\sum_{j \in \mathcal{N}_i} C_{ij}^K = \sum_{j \in \mathcal{N}_i} |v_{ij}| f(K) \Phi^{-1}(\alpha_{uc}) = G_i f(K) \Phi^{-1}(\alpha_{uc}). \tag{213}$$

Then, ΔDLC_i is equal to

$$\begin{aligned}
\Delta DLC_i &= \frac{\mathbb{E}[(1-D_i)(DL_i^{K-1} + c\sum_{j \in \mathcal{N}_i} C_{ij}^{K-1} + cC_i^{CCP}) + LSC_i]}{\mathbb{E}[(1-D_i)(DL_i^K + c\sum_{j\in\mathcal{N}_i} C_{ij}^K)]} \\
&= \frac{\mathbb{E}[(1-D_i)(DL_i^{K-1} + cG_i(f(K-1)\Phi^{-1}(\alpha_{uc}) + \eta_i f(1)\Phi^{-1}(\alpha_{CCP}))) + LSC_i]}{\mathbb{E}[(1-D_i)(DL_i^K + cG_i f(K)\Phi^{-1}(\alpha_{uc}))]}.
\end{aligned}$$

Using Propositions 2 and 9 and following the steps in previous proofs, the impact of central clearing on the expected default losses and collateral cost of entity i is then given by

$$\begin{aligned}
\Delta DLC_i &= \frac{(1-\pi)\left(\pi G_i \xi(\alpha_{uc}) f(K-1) + \xi(\alpha_{CCP}) w_i(\delta) \mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^N D_j \bar{\sigma}_j}{w_i(\delta) + \sum_{j=1,j\neq i}^N (1-D_j) w_j(\delta)}\right]\right)}{(1-\pi)\left[\pi G_i \xi(\alpha_{uc}) f(K) + cG_i f(K)\Phi^{-1}(\alpha_{uc})\right]} \\
&\quad + \frac{(1-\pi)\left(cG_i(f(K-1)\Phi^{-1}(\alpha_{uc}) + \eta_i f(1)\Phi^{-1}(\alpha_{CCP}))\right)}{(1-\pi)\left[\pi G_i \xi(\alpha_{uc}) f(K) + cG_i f(K)\Phi^{-1}(\alpha_{uc})\right]} - 1 \\
&= \frac{f(K-1)}{f(K)} + \frac{\xi(\alpha_{CCP}) \frac{w_i(\delta)}{G_i} \mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^N D_j G_j \eta_j f(1)}{w_i(\delta) + \sum_{j=1,j\neq i}^N (1-D_j) w_j(\delta)}\right] + c\eta_i f(1)\Phi^{-1}(\alpha_{CCP})}{\pi \xi(\alpha_{uc}) f(K) + cf(K)\Phi^{-1}(\alpha_{uc})} - 1 \\
&= \frac{f(K-1)}{f(K)} + \frac{f(1)}{f(K)} \frac{\xi(\alpha_{CCP}) \frac{w_i(\delta)}{G_i} \mathbb{E}[H] + c\eta_i \Phi^{-1}(\alpha_{CCP})}{\pi \xi(\alpha_{uc}) + c\Phi^{-1}(\alpha_{uc})} - 1,
\end{aligned}$$

where $H = \frac{\sum_{j=1,j\neq i}^{N} D_j G_j \eta_j}{w_i(\delta)+\sum_{j=1,j\neq i}^{N}(1-D_j)w_j(\delta)}$.

The derivative of ΔDLC_i with respect to α_{CCP} is equal to

$$\frac{\partial \Delta DLC_i}{\partial \alpha_{CCP}} = \frac{f(1)}{f(K)} \frac{\xi'(\alpha_{CCP})\frac{w_i(\delta)}{G_i}\mathbb{E}\left[H\right] + c\eta_i \frac{1}{\varphi(\Phi^{-1}(1-\alpha_{CCP}))}}{\pi\xi(\alpha_{uc}) + c\Phi^{-1}(\alpha_{uc})}$$
$$= \frac{f(1)}{f(K)} \frac{-\frac{1-\alpha_{CCP}}{\varphi(\Phi^{-1}(1-\alpha_{CCP}))}\frac{w_i(\delta)}{G_i}\mathbb{E}\left[H\right] + c\eta_i \frac{1}{\varphi(\Phi^{-1}(1-\alpha_{CCP}))}}{\pi\xi(\alpha_{uc}) + c\Phi^{-1}(\alpha_{uc})}$$
$$= \frac{f(1)}{f(K)} \frac{1}{\varphi(\Phi^{-1}(1-\alpha_{CCP}))(\pi\xi(\alpha_{uc}) + c\Phi^{-1}(\alpha_{uc}))} \left(c\eta_i - (1-\alpha_{CCP})\frac{w_i(\delta)}{G_i}\mathbb{E}\left[H\right]\right),$$

using Lemma 4 and that the inverse function rule and the properties of the Normal distribution imply that

$$\frac{\partial \Phi^{-1}(\alpha_{CCP})}{\partial \alpha_{CCP}} = \frac{1}{\Phi'(\Phi^{-1}(\alpha_{CCP}))}$$
$$= \frac{1}{\varphi(\Phi^{-1}(\alpha_{CCP}))} = \frac{1}{\varphi(-\Phi^{-1}(1-\alpha_{CCP}))} = \frac{1}{\varphi(\Phi^{-1}(1-\alpha_{CCP}))}.$$

By assumption, $\alpha_{CCP} \in [0.5, 1)$ and, using that at least two entities have a non-flat portfolio and $\pi > 0$, $\mathbb{E}[H] > 0$.

(1) Clearly, if $\eta_i = 0$, then $\frac{\partial \Delta DLC_i}{\partial \alpha_{CCP}} < 0$.

(2) If $\eta_i > 0$, then
$$\frac{\partial \Delta DLC_i}{\partial \alpha_{CCP}} < 0 \;\Leftrightarrow\; c < (1-\alpha_{CCP})\frac{w_i(\delta)}{G_i}\mathbb{E}\left[H\right] > 0.$$

□

Proposition 12 (Costly collateral in core-periphery networks)**.** *Consider a core-periphery network and loss sharing based on net risk. Assume that $\alpha_{uc} = \alpha_{CCP}$. Then, for any entity $i \in \{1, ..., N\}$, the impact of central clearing on expected default losses and collateral costs is decreasing with the marginal cost of collateral,*

$$\frac{\partial \Delta DLC_i}{\partial c} < 0. \tag{39}$$

Proof. Let $g \in \mathcal{N}_{per}$ and $\delta = 0$. Using Proposition 4, the proof of Proposition 8, and that $\eta_g = 1$, it is

$$\mathbb{E}[LSC_g] = (1-\pi)\xi(\alpha_{CCP})\bar{\sigma}_g \mathbb{E}\left[\frac{\sum_{j=1,j\neq g}^{N} D_j \bar{\sigma}_j}{\bar{\sigma}_g + \sum_{j=1,j\neq g}^{N}(1-D_j)\bar{\sigma}_j}\right] \tag{214}$$

$$= (1-\pi)\xi(\alpha_{CCP})\eta_g G_g f(1)\mathbb{E}\left[\frac{\sum_{j=1,j\neq g}^{N} D_j \eta_j G_j f(1)}{\eta_g G_g f(1) + \sum_{j=1,j\neq g}^{N}(1-D_j)\eta_j G_j f(1)}\right] \tag{215}$$

$$= (1-\pi)\xi(\alpha_{CCP})\eta_g G_g f(1)\mathbb{E}\left[\frac{\sum_{j=1,j\neq g}^{N} D_j \eta_j G_j}{\eta_g G_g + \sum_{j=1,j\neq g}^{N}(1-D_j)\eta_j G_j}\right] \tag{216}$$

$$= (1-\pi)\xi(\alpha_{CCP})\eta_g G_g f(1)\frac{1-\pi^{2N/3}-1+\pi}{1-\pi} \tag{217}$$

$$= G_g(1-\pi)\xi(\alpha_{CCP})f(1)\frac{\pi-\pi^{2N/3}}{1-\pi} \tag{218}$$

and, therefore,

$$\Delta DLC_g = \frac{\mathbb{E}[(1-D_g)(DL_g^{K-1} + c\sum_{j\in\mathcal{N}_g} C_{ij}^{K-1} + cC_g^{CCP}) + LSC_g]}{\mathbb{E}[(1-D_g)DL_g^K + c\sum_{j\in\mathcal{N}_g} C_{ij}^K]} - 1 \tag{219}$$

$$= \frac{(1-\pi)\left(\pi G_g \xi(\alpha_{uc})f(K-1) + G_g\xi(\alpha_{CCP})f(1)\frac{\pi-\pi^{2N/3}}{1-\pi}\right)}{(1-\pi)\left[\pi G_g\xi(\alpha_{uc})f(K) + cG_g f(K)\Phi^{-1}(\alpha_{uc})\right]} \tag{220}$$

$$+ \frac{(1-\pi)\left(cG_g(f(K-1)\Phi^{-1}(\alpha_{uc}) + f(1)\Phi^{-1}(\alpha_{CCP}))\right)}{(1-\pi)\left[\pi G_g\xi(\alpha_{uc})f(K) + cG_g f(K)\Phi^{-1}(\alpha_{uc})\right]} - 1 \tag{221}$$

$$= \frac{f(K-1)}{f(K)} + \frac{f(1)}{f(K)}\frac{\pi\xi(\alpha_{CCP})\frac{1-\pi^{2N/3-1}}{1-\pi} + c\Phi^{-1}(\alpha_{CCP})}{\pi\xi(\alpha_{uc}) + c\Phi^{-1}(\alpha_{uc})} - 1. \tag{222}$$

The derivative of ΔDLC_g with respect to c is equal to

$$\frac{\partial \Delta DLC_g}{\partial c} = \pi\frac{f(1)}{f(K)}\frac{\Phi^{-1}(\alpha_{CCP})\xi(\alpha_{uc}) - \Phi^{-1}(\alpha_{uc})\xi(\alpha_{CCP})\frac{1-\pi^{2N/3-1}}{1-\pi}}{(\pi\xi(\alpha_{uc}) + c\Phi^{-1}(\alpha_{uc}))^2}. \tag{223}$$

If $\alpha_{uc} = \alpha_{CCP}$, then $\frac{\partial \Delta DLC_g}{\partial c} < 0$ if, and only if,

$$1-\pi < 1-\pi^{2N/3-1} \tag{224}$$

$$\Leftrightarrow \pi > \pi^{2N/3-1}, \tag{225}$$

which holds since $2N/3-1>1 \;\Leftrightarrow\; N>3$ and $\pi<1$, which hold by assumption.

If $h \in \mathcal{N}_{core}$ and for $\lim \delta \searrow 0$, using Proposition 4 and (the notation from) the proof of Proposition 8 it is

$$\lim_{\delta\searrow 0}\delta H_h = \mathbb{P}(\mathcal{D}_{per})\lim_{\delta\to 0}A_1 + (1-\mathbb{P}(\mathcal{D}_{per}))\lim_{\delta\to 0}A_2 \tag{226}$$

$$= \pi^{2N/3}\frac{6G_{per}}{(N-3)+6G_{per}}\frac{1-\pi^{N/3}}{1-\pi} \tag{227}$$

and

$$\lim_{\delta\searrow 0}\mathbb{E}[LSC_h] = \lim_{\delta\searrow 0}(1-\pi)\xi(\alpha_{CCP})(\delta\bar{\Sigma}_h+\bar{\sigma}_h)\mathbb{E}\left[\frac{\sum_{j=1,j\neq h}^{N}D_j\bar{\sigma}_j}{\delta\bar{\Sigma}_h+\bar{\sigma}_h+\sum_{j=1,j\neq h}^{N}(1-D_j)(\delta\bar{\Sigma}_j+\bar{\sigma}_j)}\right]$$

$$= (1-\pi)\xi(\alpha_{CCP})\bar{\Sigma}_h\lim_{\delta\searrow 0}\delta H_h \tag{228}$$

$$= (1-\pi)\xi(\alpha_{CCP})G_h f(1)\pi^{2N/3}\frac{6G_{per}}{(N-3)+6G_{per}}\frac{1-\pi^{N/3}}{1-\pi}, \tag{229}$$

and, therefore, using that $\eta_h = 0$ and for $\lim_{\delta\searrow 0}$,

$$\begin{aligned}
\Delta DLC_h &= \frac{\mathbb{E}[(1-D_h)(DL_h^{K-1}+c\sum_{j\in\mathcal{N}_h}C_{ij}^{K-1}+cC_h^{CCP})+LSC_h]}{\mathbb{E}[(1-D_h)DL_h^K+c\sum_{j\in\mathcal{N}_h}C_{ij}^K]}-1\\
&= \frac{(1-\pi)\left(\pi G_h\xi(\alpha_{uc})f(K-1)+\xi(\alpha_{CCP})G_h f(1)\pi^{2N/3}\frac{6G_{per}}{(N-3)+6G_{per}}\frac{1-\pi^{N/3}}{1-\pi}\right)}{(1-\pi)\left[\pi G_h\xi(\alpha_{uc})f(K)+cG_h f(K)\Phi^{-1}(\alpha_{uc})\right]}\\
&\quad+\frac{(1-\pi)\left(cG_h(f(K-1)\Phi^{-1}(\alpha_{uc})+f(1)\eta_h\Phi^{-1}(\alpha_{CCP}))\right)}{(1-\pi)\left[\pi G_h\xi(\alpha_{uc})f(K)+cG_h f(K)\Phi^{-1}(\alpha_{uc})\right]}-1\\
&= \frac{f(K-1)}{f(K)}+\frac{f(1)}{f(K)}\frac{\xi(\alpha_{CCP})\pi^{2N/3}\frac{6G_{per}}{(N-3)+6G_{per}}\frac{1-\pi^{N/3}}{1-\pi}}{\xi(\alpha_{uc})\pi+c\Phi^{-1}(\alpha_{uc})}-1,
\end{aligned}$$

which is decreasing with c. □

E Proofs for Section 6 (The CCP's Objective)

Lemma 2 (Optimal fee)**.** *For an optimal clearing rule (F^*, δ^*), defined as the solution to (40) subject to (41) and (42), the optimal fee is equal to*

$$F^* = \pi f(K)\xi(\alpha_{uc})\min_{i\in\Omega}\left(-\Delta DL_i(\delta^*,\Omega)\right), \tag{44}$$

where $\Delta DL_i(\delta,\Omega)$ is the impact of central clearing on i's expected default losses considering only the set Ω of market participants, analogously to Equation (18),

$$\Delta DL_i(\delta,\Omega) = \frac{\mathbb{E}\left[(1-D_i)\sum_{j\in\mathcal{N}_i\cap\Omega} DL_{ij}^{K-1} + LSC_i(\delta,\Omega)\right]}{\mathbb{E}\left[(1-D_i)\sum_{j\in\mathcal{N}_i\cap\Omega} DL_{ij}^{K}\right]} - 1. \tag{45}$$

Proof. The participation constraint (41) is equivalent to

$$\begin{aligned}
&(1-\pi)F\sum_{j\in\mathcal{N}_i\cap\Omega}|v_{ij}| \le (1-\pi)\left(\mathbb{E}\left[\sum_{j\in\mathcal{N}_i} DL_{ij}^K - \sum_{j\in\mathcal{N}_i\cap\Omega} DL_{ij}^{K-1} - \sum_{j\in\mathcal{N}_i\setminus\Omega} DL_{ij}^K\right]\right) \\
&\qquad\qquad - \mathbb{E}[LSC_i(\delta,\Omega)] \\
\Leftrightarrow &(1-\pi)F\sum_{j\in\mathcal{N}_i\cap\Omega}|v_{ij}| \le (1-\pi)\left(\mathbb{E}\left[\sum_{j\in\mathcal{N}_i\cap\Omega} DL_{ij}^K - \sum_{j\in\mathcal{N}_i\cap\Omega} DL_{ij}^{K-1}\right]\right) - \mathbb{E}[LSC_i(\delta,\Omega)] \\
\Leftrightarrow &(1-\pi)FG_i(\Omega) \le (1-\pi)\left(\mathbb{E}\left[DL_i^K(\Omega) - DL_i^{K-1}(\Omega)\right]\right) - \mathbb{E}[LSC_i(\delta,\Omega)] \\
\Leftrightarrow &\frac{(1-\pi)FG_i(\Omega)}{(1-\pi)\mathbb{E}[DL_i^K(\Omega)]} \le -\left(\frac{(1-\pi)\mathbb{E}[DL_i^{K-1}(\Omega)] + \mathbb{E}[LSC_i(\delta,\Omega)]}{(1-\pi)\mathbb{E}[DL_i^K(\Omega)]} - 1\right) \\
\Leftrightarrow &\frac{FG_i(\Omega)}{\mathbb{E}[DL_i^K(\Omega)]} \le -\Delta DL_i(\delta,\Omega) \\
\Leftrightarrow &F \le -\pi f(K)\xi(\alpha_{uc})\Delta DL_i(\delta,\Omega),
\end{aligned}$$

where $G_i(\Omega)$, $DL_i^K(\Omega)$, and $\Delta DL_i(\delta,\Omega)$ are the gross position, uncleared default loss, and impact of central clearing on the default losses of entity i considering only the set Ω of market participants.

Because the participation constraint must hold for all $i\in\Omega$, it is

$$F^* \le \min_{i\in\Omega} -\pi f(K)\xi(\alpha_{uc})\Delta DL_i(\delta^*,\Omega) = \pi f(K)\xi(\alpha_{uc})\min_{i\in\Omega}(-\Delta DL_i(\delta^*,\Omega)). \tag{230}$$

Since the objective (40) is increasing in F, the optimal clearing fee maximizes F with respect to the participation constraints, which implies that

$$F^* = \pi f(K)\xi(\alpha_{uc})\min_{i\in\Omega}(-\Delta DL_i(\delta,\Omega)). \tag{231}$$

□

Proposition 13 (Optimal clearing rule)**.** *Consider a core-periphery network. Assume that π is sufficiently small, such that Corollary 5 applies. Then, the optimal clearing rule is one of the following:*

(A) All entities use central clearing, $\Omega = \{1,...,N\}$, the loss sharing rule balances the impact of central

clearing across entities, $\delta^* = \hat{\delta}$, and the fee is equal to

$$F_A^* = -\pi\xi(\alpha_{uc}) f(K) \Delta DL_1(\Omega). \tag{46}$$

(B) *Only core entities use central clearing, $\Omega = \mathcal{N}_{core}$, the loss sharing rule is indeterminate, and the fee is equal to*

$$F_B^* = \pi\xi(\alpha_{uc})(f(K) - f(K-1)). \tag{47}$$

Proof. Entities only differ in whether they are in the core or periphery of the network, but otherwise face the same participation constraints. Let $g \in \mathcal{N}_{per}$ and $h \in \mathcal{N}_{core}$. Let $\hat{\delta} \in (0,1)$ such that $\Delta DL_g(\hat{\delta}, \{1, ..., N\}) = \Delta DL_h(\hat{\delta}, \{1, ..., N\})$, which exists due to Corollary 5. We rewrite the objective function (40) as

$$\mathcal{O} = \sum_{i \in \Omega} \mathbb{E}\left[(1 - D_i) \sum_{j \in \mathcal{N}_i \cap \Omega} \left|v_{ij}\right| F\right] = (1 - \pi) F G(\Omega), \tag{232}$$

where $G(\Omega) = \sum_{i \in \Omega} \sum_{j \in \mathcal{N}_i \cap \Omega} \left|v_{ij}\right|$ is the total gross volume cleared.

Because each peripheral entity trades only with a core entity, it is not feasible that only peripheral entities use central clearing. Therefore, $\mathcal{N}_{core} \subseteq \Omega$. Thus, there are two possible sets of clearing members Ω:[IA.2]

(A) Assume that $\Omega = \{1, ..., N\}$. In this case, all entities use central clearing. Assume that $\delta^* \leq \hat{\delta}$. Then, using Corollary 5, it is $\Delta DL_h(\delta^*, \Omega) \leq \Delta DL_g(\delta^*, \Omega)$, and, thus, using Lemma 2, the optimal fee is equal to

$$F_A^* = \pi f(K) \xi(\alpha_{uc}) \min_{i \in \Omega}(-\Delta DL_i(\delta^*, \Omega)) = -\pi f(K) \xi(\alpha_{uc}) \Delta DL_g(\delta^*, \Omega).$$

From Proposition 9, it is $\frac{\partial \Delta DL_g}{\partial \delta} < 0$, and, thus, for all $\delta^* < \hat{\delta}$,

$$\frac{\partial \mathcal{O}(\delta^*)}{\partial \delta} = (1 - \pi) G(\Omega) \frac{\partial F_A^*}{\partial \delta} = -(1 - \pi)\ G(\Omega) \pi f(K) \xi(\alpha_{uc}) \frac{\partial \Delta DL_g(\delta^*, \Omega)}{\partial \delta} > 0.$$

Therefore, $\delta^* < \hat{\delta}$ is not optimal.

[IA.2] Ω is nonempty by the assumption in Footnote 19.

Assume that $\delta^* > \hat{\delta}$. Then, $\Delta DL_h(\delta^*, \Omega) > \Delta DL_g(\delta^*, \Omega)$, and, thus, using Lemma 2 it is

$$F_A^* = \pi f(K)\xi(\alpha_{uc}) \min_{i\in\Omega}(-\Delta DL_i(\delta^*, \Omega)) = -\pi f(K)\xi(\alpha_{uc})\Delta DL_h(\delta^*, \Omega).$$

From Proposition 9, it is $\frac{\partial \Delta DL_h}{\partial \delta} > 0$, and, thus, for all $\delta > \hat{\delta}$,

$$\frac{\partial \mathcal{O}(\delta^*)}{\partial \delta} = (1-\pi)G(\Omega)\frac{\partial F_A^*}{\partial \delta} = -(1-\pi)\pi f(K)\xi(\alpha_{uc})G(\Omega)\frac{\partial \Delta DL_h(\delta^*, \Omega)}{\partial \delta} < 0.$$

Therefore, $\delta > \hat{\delta}$ is not optimal, and $\delta^* = \hat{\delta}$ is a maximum. Thus, $\delta^* = \hat{\delta}$ maximizes the CCP's profit.

(B) Assume that $\Omega = \mathcal{N}_{core}$. In this case, only core entities use central clearing. Because core entities have zero net risk, $\bar{\sigma}_j = 0$ for all $j \in \mathcal{N}_{core}$, using Proposition 9, the expected loss sharing contribution is equal to

$$\mathbb{E}[LSC_i] = (1-\pi)\xi(\alpha_{CCP})w_i(\delta)\mathbb{E}\left[\frac{\sum_{j\in\mathcal{N}_{core}, j\neq i} D_j \bar{\sigma}_j}{w_i(\delta) + \sum_{j\in\mathcal{N}_{core}, j\neq i}(1-D_j)w_j(\delta)}\right] = 0. \tag{233}$$

Therefore, for all $i \in \mathcal{N}_{core}$ the impact of central clearing on the expected default losses is equal to

$$\Delta DL_i(\delta, \mathcal{N}_{core}) = \frac{f(K-1) - f(K)}{f(K)}, \tag{234}$$

independently of the loss sharing rule δ. Therefore, using Lemma 2, the optimal fee is equal to

$$F_B^* = -\pi f(K)\xi(\alpha_{uc})\frac{f(K-1) - f(K)}{f(K)} = \pi\xi(\alpha_{uc})(f(K) - f(K-1)).$$

Assume that the loss sharing rule is $\delta^* \in [0, 1]$. If any peripheral entity $g \in \mathcal{N}_{per}$ joins the CCP, the CCP's expected default losses become strictly positive. Thus,

$$\Delta DL_g(\delta^*, \mathcal{N}_{core} \cup \{g\}) > \frac{f(K-1) - f(K)}{f(K)}.$$

From the proof of Lemma 2, entity g prefers not to use central clearing if, and only if,

$$\begin{aligned}
&-\pi f(K)\xi(\alpha_{uc})\Delta DL_g(\delta^*, \mathcal{N}_{core} \cup \{g\}) < F_B^* \\
\Leftrightarrow &-\pi f(K)\xi(\alpha_{uc})\Delta DL_g(\delta^*, \mathcal{N}_{core} \cup \{g\}) < \pi\xi(\alpha_{uc})(f(K) - f(K-1)) \\
\Leftrightarrow &-\Delta DL_g(\delta^*, \mathcal{N}_{core} \cup \{g\}) < \frac{f(K) - f(K-1)}{f(K)} \\
\Leftrightarrow &\Delta DL_g(\delta^*, \mathcal{N}_{core} \cup \{g\}) > \frac{f(K-1) - f(K)}{f(K)}.
\end{aligned}$$

Therefore, constraint (42) holds for all $g \in \mathcal{N}_{per}$.

□

Proposition 14 (Curtailing clearing participation). *In the setting of Proposition 13, clearing rule (B) strictly dominates (A) if*

$$(f(K) - f(K-1))\,\xi(\alpha_{uc}) < \max\left\{\frac{2N-3}{4N}, \frac{\hat{\delta}}{2}\right\} f(1)\xi(\alpha_{CCP}). \tag{48}$$

In this case, it is optimal for the CCP to dissuade peripheral entities from using central clearing. There exist $\hat{K} < \infty$ and $\hat{\alpha}_{uc} < 1$ such that condition (48) holds if $K > \hat{K}$ or $\alpha_{uc} > \hat{\alpha}_{uc}$.

Proof. Let $k \in \{1, ..., N\}$. Clearing rule (B) results in a strictly larger fee income to the CCP than (A) if, and only if,

$$\begin{aligned}
&F_B^* G(\mathcal{N}_{core}) > F_A^* G(\{1, ..., N\}) \\
\Leftrightarrow\ &\pi\xi(\alpha_{uc})(f(K) - f(K-1))G(\mathcal{N}_{core}) > -\pi\xi(\alpha_{uc})f(K)\Delta DL_k(\hat{\delta}, \{1, .., N\})G(\{1, ..., N\}) \qquad (235) \\
\Leftrightarrow\ &\frac{f(K) - f(K-1)}{f(K)}G(\mathcal{N}_{core}) > -\Delta DL_k(\hat{\delta}, \{1, .., N\})G(\{1, ..., N\}) \\
\Leftrightarrow\ &\frac{f(K) - f(K-1)}{f(K)}G(\mathcal{N}_{core}) > G(\{1, ..., N\})\left[\frac{f(K) - f(K-1)}{f(K)} - \frac{w_k(\hat{\delta})f(1)}{G_k f(K)}\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{1}{\pi}H\right] \\
\Leftrightarrow\ &\frac{f(K) - f(K-1)}{f(K)}(G(\mathcal{N}_{core}) - G(\{1, ..., N\})) > -G(\{1, ..., N\})\frac{w_k(\hat{\delta})f(1)}{G_k f(K)}\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{1}{\pi}H \\
\Leftrightarrow\ &\frac{f(K) - f(K-1)}{f(K)}(G(\{1, ..., N\}) - G(\mathcal{N}_{core})) < G(\{1, ..., N\})\frac{w_k(\hat{\delta})f(1)}{G_k f(K)}\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{1}{\pi}H \qquad (236)
\end{aligned}$$

where $H = \mathbb{E}\left[\frac{\sum_{j=1, j\neq k}^N D_j G_j \eta_j}{w_k(\hat{\delta}) + \sum_{j=1, j\neq k}^N (1-D_j)w_j(\hat{\delta})}\right]$. In the following, we use that

$$w_k(\hat{\delta}) = \hat{\delta}G_k f(1) + (1-\hat{\delta})G_k\eta_k f(1) \leq G_k f(1).$$

(1) Let $k \in \mathcal{N}_{core}$. Then, using the properties of core-periphery networks,

$$H = \mathbb{E}\left[\frac{\sum_{j=1,j\neq k}^{N} D_j G_j \eta_j}{w_k(\hat{\delta}) + \sum_{j=1,j\neq k}^{N}(1-D_j)w_j(\hat{\delta})}\right] = \mathbb{E}\left[\frac{\sum_{j\in\mathcal{N}_{per}} D_j G_{per}}{w_k(\hat{\delta}) + \sum_{j=1,j\neq g}^{N}(1-D_j)w_j(\hat{\delta})}\right]$$
$$\geq \mathbb{E}\left[\frac{\sum_{j\in\mathcal{N}_{per}} D_j G_{per}}{f(1)\sum_{j=1}^{N} G_j(\{1,...,N\})}\right] = \frac{\frac{2N}{3}\pi G_{per}}{G(\{1,...,N\})f(1)}. \tag{237}$$

Because $k \in \mathcal{N}_{core}$, it is $w_k(\hat{\delta}) = \hat{\delta} G_{core} f(1)$. Therefore, inequality (236) holds if

$$\frac{f(K)-f(K-1)}{f(K)}(G(\{1,...,N\}) - G(\mathcal{N}_{core})) < \frac{\hat{\delta} G_{core} f(1) f(1)}{G_{core} f(K)} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{1}{\pi} \frac{\frac{2N}{3}\pi G_{per} G(\{1,...,N\})}{G(\{1,...,N\})f(1)}$$
$$\Leftrightarrow \frac{f(K)-f(K-1)}{f(K)}(G(\{1,...,N\}) - G(\mathcal{N}_{core})) < \frac{\hat{\delta} f(1)}{f(K)} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{2N}{3} G_{per}$$
$$\Leftrightarrow \frac{f(K)-f(K-1)}{f(K)}\left(\frac{2N}{3}G_{per} + \frac{N}{3}\frac{N-3+6G_{per}}{3} - \frac{N}{3}\frac{N-3}{3}\right) < \frac{\hat{\delta} f(1)}{f(K)} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{2N}{3} G_{per}$$
$$\Leftrightarrow \frac{f(K)-f(K-1)}{f(K)} 4G_{per} < \frac{\hat{\delta} f(1)}{f(K)} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} 2G_{per}$$
$$\Leftrightarrow 2\frac{f(K)-f(K-1)}{f(K)} < \hat{\delta}\frac{f(1)}{f(K)} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}$$
$$\Leftrightarrow \frac{f(K)-f(K-1)}{f(1)} \frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})} < \frac{\hat{\delta}}{2}.$$

(2) Let $k \in \mathcal{N}_{per}$. Then,

$$H = \mathbb{E}\left[\frac{\sum_{j=1,j\neq k}^{N} D_j G_j \eta_j}{w_k(\hat{\delta}) + \sum_{j=1,j\neq k}^{N}(1-D_j)w_j(\hat{\delta})}\right] = \mathbb{E}\left[\frac{\sum_{j\in\mathcal{N}_{per}\setminus\{k\}} D_j G_{per}}{w_k(\hat{\delta}) + \sum_{j=1,j\neq g}^{N}(1-D_j)w_k(\hat{\delta})}\right]$$
$$\geq \mathbb{E}\left[\frac{\sum_{j\in\mathcal{N}_{per}\setminus\{k\}} D_j G_{per}}{f(1)\sum_{j=1}^{N} G_j(\{1,...,N\}}\right] = \frac{\frac{2N-3}{3}\pi G_{per}}{G(\{1,...,N\})f(1)}. \tag{238}$$

Because $k \in \mathcal{N}_{per}$, it is $w_k(\hat{\delta}) = G_{per} f(1)$. Therefore, it is sufficient for inequality (236) to hold if

$$
\begin{aligned}
&\frac{f(K)-f(K-1)}{f(K)}(G(\{1,...,N\})-G(\mathcal{N}_{core})) < \frac{G_{per}f(1)f(1)}{G_{per}f(K)}\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{1}{\pi}\frac{\frac{2N-3}{3}\pi G_{per}G(\{1,...,N\})}{G(\{1,...,N\})f(1)}\\
\Leftrightarrow\ &\frac{f(K)-f(K-1)}{f(K)}(G(\{1,...,N\})-G(\mathcal{N}_{core})) < \frac{f(1)}{f(K)}\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{2N-3}{3}G_{per}\\
\Leftrightarrow\ &\frac{f(K)-f(K-1)}{f(K)}\frac{N}{3}\frac{12G_{per}}{3} < \frac{f(1)}{f(K)}\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{2N-3}{3}G_{per}\\
\Leftrightarrow\ &\frac{f(K)-f(K-1)}{f(1)}\frac{4N}{2N-3} < \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\\
\Leftrightarrow\ &\frac{f(K)-f(K-1)}{f(1)}\frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})} < \frac{2N-3}{4N}.
\end{aligned}
$$

Therefore, the CCP strictly prefers rule (B) over (A) if

$$
(f(K)-f(K-1))\,\xi(\alpha_{uc}) < \max\left\{\frac{2N-3}{4N}, \frac{\hat{\delta}}{2}\right\} f(1)\xi(\alpha_{CCP}).
$$

The LHS converges to zero for $K \to \infty$ (using Lemma 4) and for $\alpha_{uc} \to 1$ (Lemma 3). Therefore, there exist $\hat{K} < \infty$ and $\hat{\alpha}_{uc} < 1$ such that the CCP strictly prefers rule (B) over (A) if either $K > \hat{K}$ or $\alpha_{uc} > \hat{\alpha}_{uc}$ or both. □

Proposition 15 (Robust optimal clearing rule). *If clearing rule (B) in Proposition 13 is strictly preferred over (A), then only a net-based loss sharing rule is robust to small perturbations in the following sense:*

There exists a sequence $(n_\ell)_{\ell\in\mathbb{N}}$ that converges to 0 and associates with the following sequence of core-periphery networks:

- *Each peripheral entity has the perturbed position $\tilde{G}^{\ell}_{per} = G_{per} + n_\ell$.*
- *Peripheral entities always centrally clear n_ℓ, independently of the clearing rule, and centrally clear G_{per} if, and only if, the participation constraint is satisfied.*
- *Core entities use central clearing if, and only if, the participation constraint is satisfied.*

Denote by $(F^{*,\ell}, \delta^{*,\ell})$ an optimal clearing rule for the ℓ-th perturbation. Then, (F^*, δ^*) is a robust optimal clearing rule for the original network if $F^{*,\ell} \to F^*$ and $\delta^{*,\ell} \to \delta^*$ for $\ell \to \infty$.

Proof. Consider clearing rule (B) associated with clearing members $\Omega = \mathcal{N}_{core}$ and fee F^*_B. The constraint (42) implies for the original network that peripheral entities strictly prefer not to become clearing members. By continuity, there exists $\bar{\ell} > 0$ such that constraint (42) holds for all perturbed networks with $\ell < \bar{\ell}$.

Let $\ell < \bar{\ell}$ and consider the ℓ-th perturbed network. Note that peripheral entities centrally clear n_ℓ but not G_{per}. Lemma 2 implies that the optimal fee is

$$F^{*,\ell} = -\pi f(K)\xi(\alpha_{uc})\Delta DL_h(\delta^{*,\ell}, \Omega^{*,\ell}),$$

where $h \in \mathcal{N}_{core}$. Proposition 9 (b) implies that the impact of central clearing on a core entity's expected default loss, ΔDL_h, is increasing with δ. Because the CCP's profit is increasing with the fee $F^{*,\ell}$, it is optimal to maximize $F^{*,\ell}$ by minimizing δ. Thus, $\delta^{*,\ell} = 0$ and, using Proposition 8,

$$\Delta DL_h(0, \Omega^{*,\ell}) = \frac{f(K-1)}{f(K)} + \pi^{2N/3-1}\frac{6n_\ell}{(N-3)+6n_\ell}\frac{1-\pi^{N/3}}{1-\pi}\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)} - 1. \tag{239}$$

Therefore,

$$\lim_{\ell\to\infty} F^{*,\ell} = -\pi(1-\pi)\xi(\alpha_{uc})f(K)\left[\frac{f(K-1)}{f(K)} - 1\right] = F_B^*.$$

Therefore, $(F_B^*, 0)$ is a robust optimal clearing rule for the original network.

$\square$