A COMPUTATIONAL APPROACH FOR PERSISTENT RELATIVE HOMOLOGY

Applied Topology Beyond Persistence Diagrams

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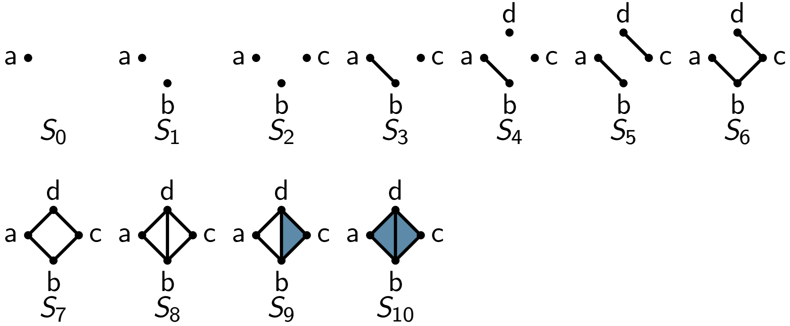
INTRODUCTION



a
a b
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S_0
S_1
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S_3
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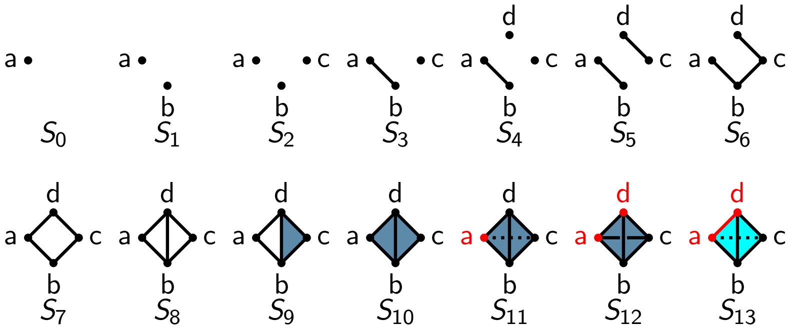
S_7
S_8
S_9
S_{10}


a b c d
S_0 S_1 S_2 S_3 S_4 S_5 S_6
S_7 S_8 S_9 S_{10} S_{11} S_{12} S_{13}

PERSISTENT RELATIVE HOMOLOGY

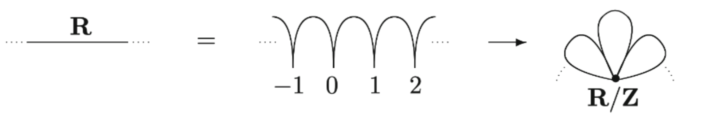
Definition

Let X be a topological space with $A \subseteq X$. We define the **Quotient Space** to be

$$X/A = (X \setminus A) \sqcup *$$

Where $*$ is a single point

Example (The Infinite Bouquet)



- What does the homology of a quotient space tell us?

RELATIVE HOMOLOGY

- Let K and K_0 be simplicial complexes such that $K_0 \subset K$
- **Relative Homology** is the homology of the quotient space K/K_0.

Definition

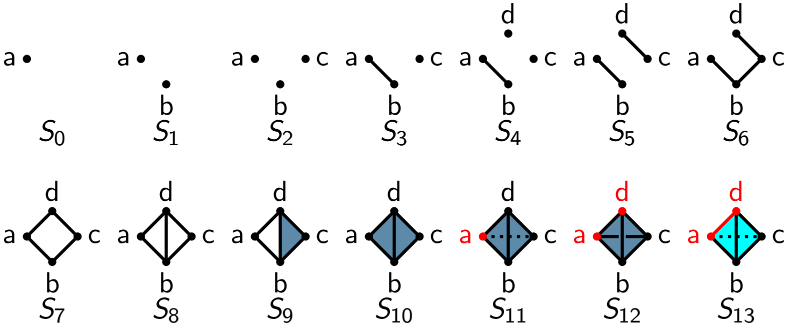
We define a **Relative n-Cycle** to be any n-chain $\alpha \in C_n(K)$ such that $\partial_n(\alpha) \in C_{n-1}(K_0)$.

Definition

We define a **Relative n-Boundary** to be any relative n-cycle $\alpha = \partial_{n+1}(\beta) + \gamma$ for some $\beta \in C_{n+1}(K)$ and $\gamma \in C_n(K_0)$.

PERSISTENT RELATIVE HOMOLOGY

- Goal: Track the relative homology through a pair of filtered spaces, K and K_0, such that $K_0 \subseteq K$ at each time step:



Note: The filtration on K_0 does not necessary follow that of K.

COMPUTATIONAL APPROACH FOR PRH

- Our work extends recently presented computational techniques for PH to a method for PRH.
- Here we will cover:
  - Modifying U-Match decomposition [1] for PRH
  - Matching bases
  - Implementation and the Open Applied Topology (*OAT*) project

[1] Hang, Haibin, et al. "U-match factorization: sparse homological algebra, lazy cycle representatives, and dualities in persistent (co) homology." arXiv preprint arXiv:2108.08831 (2021).

U-MATCH DECOMPOSITION

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- A tuple of matrices (T, M, D, S) which satisfy the following three conditions:
  - $TM = DS$
  - M is a **matching matrix**
  - T and S are both upper triangular and invertible

U-MATCH DECOMPOSITION

- A tuple of matrices (T, M, D, S) which satisfy the following three conditions:
  - $TM = DS$
  - M is a **matching matrix**
  - T and S are both upper triangular and invertible
- Assume D is the **block boundary matrix** of a chain complex, so D is square and $D^2 = 0$.

$$D = \begin{pmatrix} 0 & \partial_1 & & & \\ & 0 & \partial_2 & & \\ & & \ddots & \ddots & \\ & & & 0 & \partial_N \\ & & & & 0 \end{pmatrix}$$

- Reduce D bottom to top and left to right. T^{-1} records row operations, and S records column operations.

$$\begin{pmatrix} D & I_n \\ I_m & 0 \end{pmatrix} \mapsto \begin{pmatrix} M & T \\ S & 0 \end{pmatrix}$$

U-MATCH THEOREM 1

- Let $TM = DS$ be a U-match decomposition, where D is the block boundary matrix of a chain complex. Let $r_\bullet$ and $c_\bullet$ denote, respectively, the set of indices of **nonzero rows** and **columns** of the matching matrix M.

Theorem 1 [Hang, Haibin, et al.]

The set of indices $r_\bullet$ and $c_\bullet$ are disjoint. Hence, $r_\bullet \subseteq \overline{c_\bullet}$.

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Outline of proof.

- $TM = DS \Rightarrow S^{-1}TM = S^{-1}DS$.
- $(S^{-1}DS)^2 = S^{-1}D^2S = 0$
- $(S^{-1}TM)^2 = 0$ implies that indices of nonzero rows and columns of $S^{-1}TM$ are disjoint.

THE MATRIX J

- Construct a matrix, J, from the matrix S with the substitution

$$COL_{r_j}(S) \mapsto COL_{c_j}(TM)$$

- Nice properties of J:
  - The columns that we remove from and insert into S to form J are members of $Im(D)$. (1)
  - J is invertibe and upper-triangular. (2)
  - $i \in \overline{c_\bullet} \Rightarrow COL_i(DJ) = 0$ (3)
- This construction is helpful in showing how U-Match can be used for PH.

- Let $TM = DS$ be a U-match decomposition, where D is the **block boundary matrix** of a chain complex. Let $r_\bullet$ and $c_\bullet$ denote, respectively, the set of indices of **nonzero rows** and **columns** of the matching matrix M.

Theorem 2 [Hang, Haibin, et al.]

Columns of S indexed by the set $\overline{c_\bullet}$ contain a basis for $Ker(D)$, which are the cycles.

U-MATCH THEOREM 2

- Let $TM = DS$ be a U-match decomposition, where D is the **block boundary matrix** of a chain complex. Let $r_\bullet$ and $c_\bullet$ denote, respectively, the set of indices of **nonzero rows** and **columns** of the matching matrix M.

Theorem 2 [Hang, Haibin, et al.]

Columns of S indexed by the set $\overline{c_\bullet}$ contain a basis for $Ker(D)$, which are the cycles.

Outline of Proof:

- Construct the matrix J from S.
- Recall: $i \in \overline{c_\bullet} \Rightarrow COL_i(DJ) = 0$ (3), implying that $D \times COL_i(J) = 0$ for each $i \in \overline{c_\bullet}$. In other words, each of these columns has no boundary.

U-MATCH THEOREM 3

- Let $TM = DS$ be a U-match decomposition, where D is the **block boundary matrix** of a chain complex. Let $r_\bullet$ and $c_\bullet$ denote, respectively, the set of indices of **nonzero rows** and **columns** of the matching matrix M.

Theorem 3 [Hang, Haibin, et al.]

Columns of T indexed by the set $r_\bullet$ give a basis for $Im(D)$, which are the boundaries.

U-MATCH THEOREM 3

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Theorem 3 [Hang, Haibin, et al.]

Columns of T indexed by the set $r_\bullet$ give a basis for $Im(D)$, which are the boundaries.

Outline of Proof:

- $TM = DS \Rightarrow Im(TM) = Im(DS)$
- But $Im(DS) = Im(D)$ since S is invertible. So $Im(TM) = Im(D)$.
- M is row equivalent to the identity, so $Im(T) = Im(D)$.

MATCHED BASES

- U-Match allows us to compute **matched bases** for cycles and boundaries. This means a set of basis vectors for $Im(D)$ is a subset of a set of basis vectors for $Ker(D)$.
- How?
  - By construction, columns of J contain a basis for both $Im(D)$ and $Ker(D)$.
    - $COL_{\overline{c_\bullet}}(J) = Ker(D)$
    - $COL_{r_\bullet}(J) = Im(D)$
  - Recall that **Theorem 1** implies that $r_\bullet \subseteq \overline{c_\bullet}$.

U-MATCH FOR PRH

OVERVIEW

A (very high-level) overview:

1. Construct a boundary matrix, D.
2. Permute rows of D.
3. Perform a U-Match on D to get, $TM = DS$.
4. Permute columns of T and S.
5. Perform another U-Match.

COMPUTING RELATIVE BASES

- Compute (unmatched) bases for relative cycles and relative boundaries with a modified U-Match:
  1. Construct a boundary matrix, D, for a filtered simplicial complex K.
  2. Permute rows of D (top to bottom) to respect filtration of some subspace K_0.
  3. Perform U-Match Decomposition of D to get $TM = DS$.

COMPUTING RELATIVE BASES

- Compute (unmatched) bases for relative cycles and relative boundaries with a modified U-Match:
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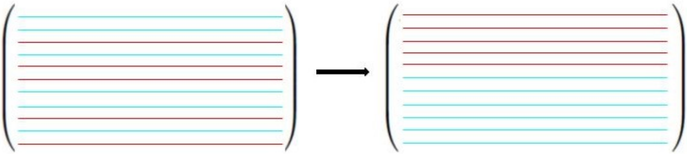
  2. Permute rows of D (top to bottom) to respect filtration of some subspace K_0.
  3. Perform U-Match Decomposition of D to get $TM = DS$.
- Why step 2?
  - Allows for simpler extraction of homological generators.
  - In the relative homology, we consider an n-chain to belong to one of the cosets $c + C_n(K_0)$ where $c \in C_n(K)$. Reordering the rows of D ensures our reduction records these *relative chains*.



- Suppose a filtered quotient space K/K_0, where the total number of simplices in K_0 is i. Given the modified U-Match process, we have the following two results:

Theorem 4 [Henselman-Petrusek, L, X, Ziegelmeier]

Define the set $I = \{c \in COL_{\overline{r_\bullet}}(S) : D(c) \in K_0, D(S) \neq 0\}$. Then $COL_I(S) \cup COL_{\overline{c_\bullet}}(S)$ gives a basis for $\overline{Ker}(D)$, which are the relative cycles.

Theorem 5 [Henselman-Petrusek, L, X, Ziegelmeier]

Let I be the set of indices corresponding to the first i columns of T. The subset of the columns of T given by $COL_{r_\bullet}(T) \cup COL_I(T)$ give a basis for $\overline{Im}(D)$, which are the relative boundaries.

ONE MORE U-MATCH THEOREM

- Suppose that:
  - A is a square, invertible matrix of size $m \times m$.
  - B is a (not necessarily square) matrix of size $m \times n$.
  - $F_\bullet$ is a filtration on a vector space $\mathbb{K}^m$ such that $F_i\mathbb{K}^m$ describes the span of the first i columns of A.
  - Similarly, define $G_\bullet$ to be a filtration on the columns of B.
  - If the columns of B do not span the columns of A, let $G_{n+1} = \mathbb{K}^m$ to ensure $G_\bullet$ terminates.

ONE MORE U-MATCH THEOREM

- Suppose that:
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Theorem 6 **[Henselman-Petrusek, L, X, Ziegelmeier]**

Assume the above conditions hold. It follows that, given the U-Match $TM = (A^{-1}B)S$, then the columns of AT contain a basis for each F_i and G_j for $i, j \in \{1, ..., m\}$.

MATCHING BASES

- Given Theorem 6, we can match bases for the relative cycles and boundaries.
- Suppose a U-Match $TM = DS$ where T and S, respectively, contain bases for relative boundaries and cycles of the filtered quotient space K/K_0. To match the bases:
  - Permute columns of T and S (left to right) according to the birth of their boundary with respect to the subspace filtration.
  - Let $A = T$ and $B = S$.
  - Perform the U-Match $\mathcal{TM} = (A^{-1}B)\mathcal{S}$.
- Suppose $dim(\overline{Ker}(D)) = i$ and $dim(\overline{Im}(D)) = j$.
- By Theorem 6:

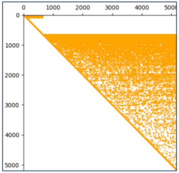
Theorem 7 [Henselman-Petrusek, L, X, Ziegelmeier]

The first j columns of $A\mathcal{T}$ contain a basis for $\overline{Im}(D)$, and the first i columns of $A\mathcal{T}\mathcal{M}$ contain a basis for $\overline{Ker}(D)$.

FINAL NOTES

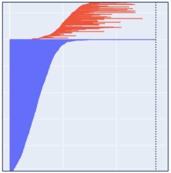
Open Applied Topology (OAT)

Fast, user-friendly homological algebra



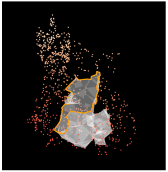
Sparse Matrices

Factorization
Multiplication
Inversion
Addition
Back-substitution



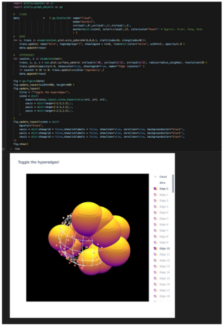
Homology

Persistence
Zigzag
Generators
Optimization
Duality



Topological Spaces

Simplicial
Cubical
Filtered
CW
Hypergraph



Languages

Python
Rust
Jupyter

Highlights

Documentation
Accessibility
Modularity

FUTURE WORK & IMPLEMENTATION

- PRH implementation will use order operator structures to:
  - Determine, if given two simplices, which was born first in full-space (or subspace) filtration; for reordering rows of the boundary matrix.
  - Determine, if given a chain, if or when it was born as a relative cycle/boundary; for reordering columns of T and S to get A and B.
- Many next steps to explore!

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THANK YOU!