Introduction to Probability

J. Alexander Branham

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- There are other ways of thinking about probability, but we'll stick with this one

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  - The probability of disjoint (mutually exclusive) sets is equal to their sums

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  - π represents the probability of success

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- PMF:

$$\binom{n}{k} p^k (1-p)^{n-k}$$

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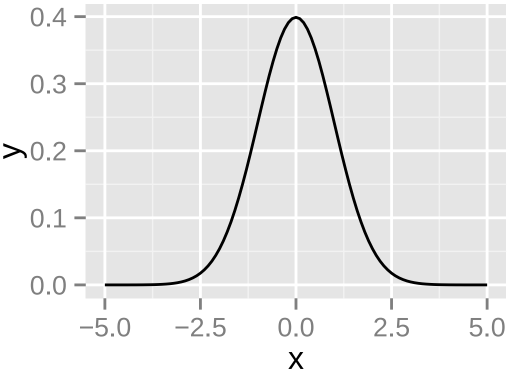
    - Kinda...

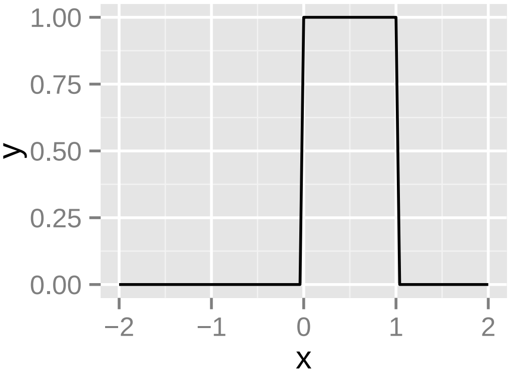
Continuous distributions - Normal

```
ggplot(data.frame(x = c(-5, 5)), aes(x)) +
  stat_function(fun = dnorm)
```



Continuous distributions - Uniform

```
ggplot(data.frame(x = c(-2, 2)), aes(x)) +
  stat_function(fun = dunif)
```



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- For uniform from previous slide, $Pr(0 < y < .5) = 0.5$

CDF

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- $F_X(x) = Pr(X \leq x)$

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- $$\sum_{y_i < y} Pr(Y = y_i)$$

$Pr(y \leq 5)$

$$\binom{10}{1}.5^1(1-.5)^{10-1}+$$
$$\binom{10}{2}.5^2(1-.5)^{10-2}+$$
$$\binom{10}{3}.5^3(1-.5)^{10-3}+$$
$$\binom{10}{4}.5^4(1-.5)^{10-4}+$$
$$\binom{10}{5}.5^5(1-.5)^{10-5}$$

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- $F_Y(y) = \int_{-\infty}^{y} f(y)dy$