Introduction to Matrix Algebra

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Scalars

- Let's start with something familiar, with a new word
- One number (12, for example) is referred to as a *scalar*
- This can be thought of as a 1*x*1 matrix
  - more on that in a bit...

$$\begin{bmatrix} 12 \end{bmatrix} = c$$

Vectors

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$$\begin{bmatrix} 12 \\ 14 \\ 15 \end{bmatrix}$$

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Since this is a column of numbers, we cleverly refer to it as a *column vector*

Row vectors

If we take b and arrange it so that it it a row of numbers instead of a column, we refer to it as a *row vector*:

$$\begin{bmatrix} 12 & 14 & 15 \end{bmatrix} = d$$

Matrix

We can put multiple vectors together to get a *matrix*:

$$\begin{bmatrix} 12 & 14 & 15 \\ 115 & 22 & 127 \\ 193 & 29 & 219 \end{bmatrix} = A$$

Matrices, cntd

- We refer to the *dimensions* of matrices by *row* x *column*
- So A is a 3x3 matrix.
- Note that matrices are usually designated by capital letters (and sometimes bolded as well)

Dimensions

ROW x COLUMN

Matrix Indices

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- Matrix A is an mxn matrix where $m = n = 3$.
- More generally, matrix B is an mxn matrix where the elements look like this:

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1m} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2m} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{nm} \end{bmatrix}$$

Addition and subtraction are EASY

- Requirement: Must have *exactly* the same dimensions
- To do the operation, just add or subtract each element with the corresponding element from the other matrix:

$$A \pm B$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \pm \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & a_{13} \pm b_{13} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & a_{23} \pm b_{23} \\ a_{31} \pm b_{31} & a_{32} \pm b_{32} & a_{33} \pm b_{33} \end{bmatrix}$$

Scalar Multiplication

Easy - just multiply each element of the matrix by the scalar

$$cA = \begin{bmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \\ ca_{31} & ca_{32} & ca_{33} \end{bmatrix}$$

Matrix Multiplication

- Requirement: the two matrices must be *conformable*
- This means that the number of columns in the first matrix equals the number of rows in the second
- The resulting matrix will have the number of rows in the first, and the number of columns in the second!

Question time

Which can we multiply? What will the resulting dimensions be?

$$b = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix} M = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 4 \\ 2 & 3 & 2 \end{bmatrix} L = \begin{bmatrix} 6 & 5 & -1 \\ 1 & 4 & 3 \end{bmatrix}$$

Question time

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ONLY LM and NOT ML

The dimensions will be $2x3$

How to actually do this?

Multiply each row by each column
(Board examples)

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```
L <- matrix(c(6,5,-1, 1,4,3),
            nrow=2, byrow=TRUE)
M <- matrix(c(1,0,2, 1,2,4, 2,3,2),
            nrow=3, byrow=TRUE)
L%*%M

##      [,1] [,2] [,3]
## [1,]    9    7   30
## [2,]   11   17   24
```

Matrix Division

Matrix Division

HAHAHA . . . NOPE

Properties of matrix operations

- Addition and subtraction
  - Associative $(A \pm B) \pm C = A \pm (B \pm C)$
  - Communicative $A \pm B = B \pm A$
- Multiplication
  - $AB \neq BA$
  - $A(BC) = (AB)C$
  - $A(B + C) = AB + AC$
  - $(A + B)C = AC + BC$

- Switch the rows and columns
- so a *nxm* matrix becomes *mxn*
- typically denoted L' or L^T

```
L

##      [,1] [,2] [,3]
## [1,]    6    5   -1
## [2,]    1    4    3

t(L)

##      [,1] [,2]
## [1,]    6    1
## [2,]    5    4
## [3,]   -1    3
```

Properties of transposition

- Matrix is *always* conformable for multiplication with its transpose in both directions
- $(A \pm B)' = A' \pm B'$
- $A'' = A$
- $(AB)' = B'A'$
- $(cA)' = cA'$ where c is a scalar

Special types of matrices

Some matricies get more love than others

Square matrix

Any *nxn* matrix (same number rows and columns)

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 4 \\ 2 & 3 & 2 \end{bmatrix}$$

Symmetric matrix

A square matrix that is the same as its transpose

$$\begin{bmatrix} 2 & 5 & 7 \\ 5 & 9 & 6 \\ 7 & 6 & 7 \end{bmatrix}$$

Diagonal matrix

A symmetric matrix with zeros everywhere but the main diagonal

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

Scalar matrix

A diagonal matrix with the same number all along the diagonal

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Identity matrix

A scalar matrix where the diagonal elements are 1.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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A scalar matrix where the diagonal elements are 1.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- This is a super important type of matrix.
- It gets its own notation: I_n where n is the number of rows and columns
- Note that $I_n A = A$ and also $A I_n = A$