

Balanced Trees

Balanced trees have height $O(\log n)$.

★ Height-balanced trees

At each node, **height** of left and right subtrees are “close”.
e.g. AVL trees, B-trees, red-black trees, splay trees

★ Weight-balanced trees

At each node, **number of nodes** in left and right subtrees are “close”.
e.g. BB[α] trees

★ Search, Predecessor, Successor, Minimum, Maximum
 $O(\log n)$ time.



★ Insert and Delete may have to **rebalance** the tree.

Splay Trees

Self-adjusting binary search tree (Sleator & Tarjan 1985)



Recently accessed elements
are quick to access again.

Amortized running time: $O(\log n)$

Averaged running time per operation
over a **worst-case sequence** of
operations.

Advantages

- ✱ **Simple implementation** — easier to implement than other self-balancing BSTs such as red-black trees and AVL trees.
- ✱ **Comparable performance** — average-case performance is as efficient as other self-balancing BSTs.
- ✱ **Small memory footprint** — no need to store bookkeeping data.
- ✱ **Working well with nodes containing identical keys.**

Splaying

Restructuring heuristics:

- ★ Performs rotations bottom-up along the access path and moves the accessed item **all the way to the root**.
- ★ Does the rotations **in pairs**, in an order depending on the structure of the access path.

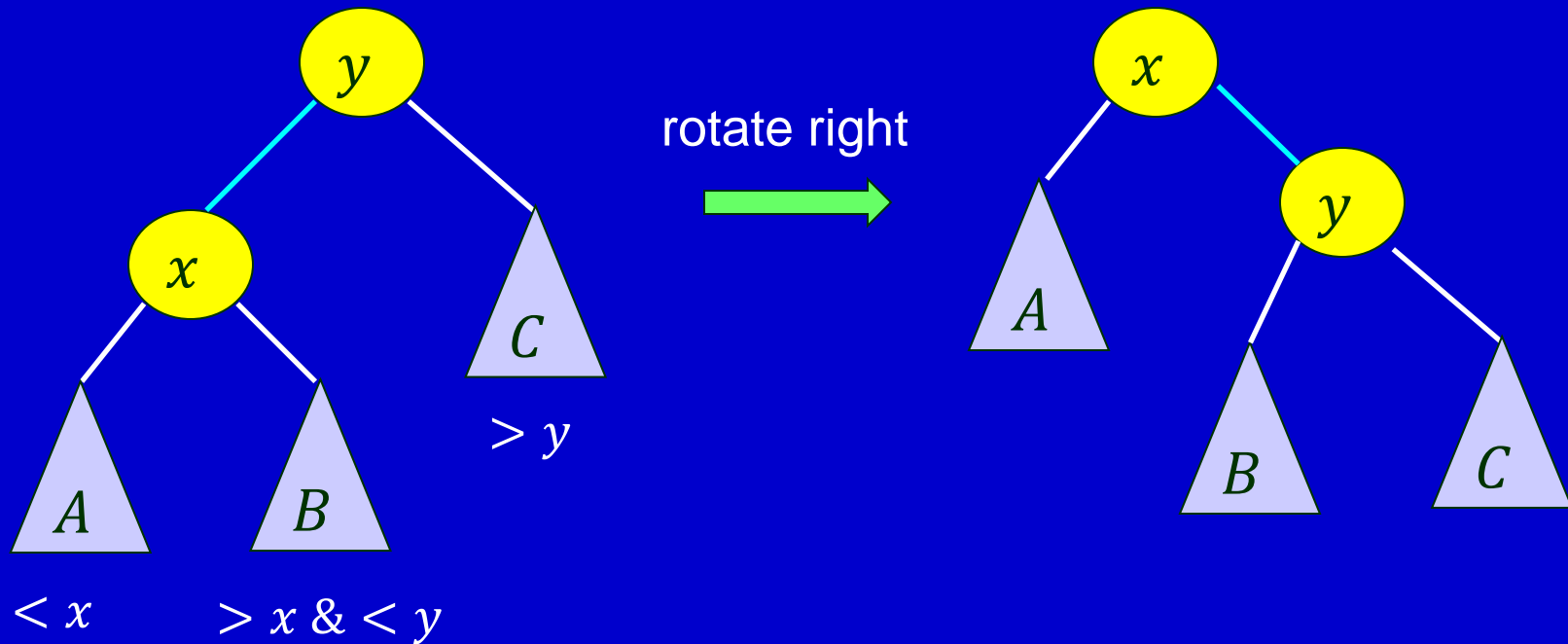
How to splay a tree at a node x ?

Repeat a sequence of splaying steps until x is the root of tree.

zig, zig-zig, or zig-zag.

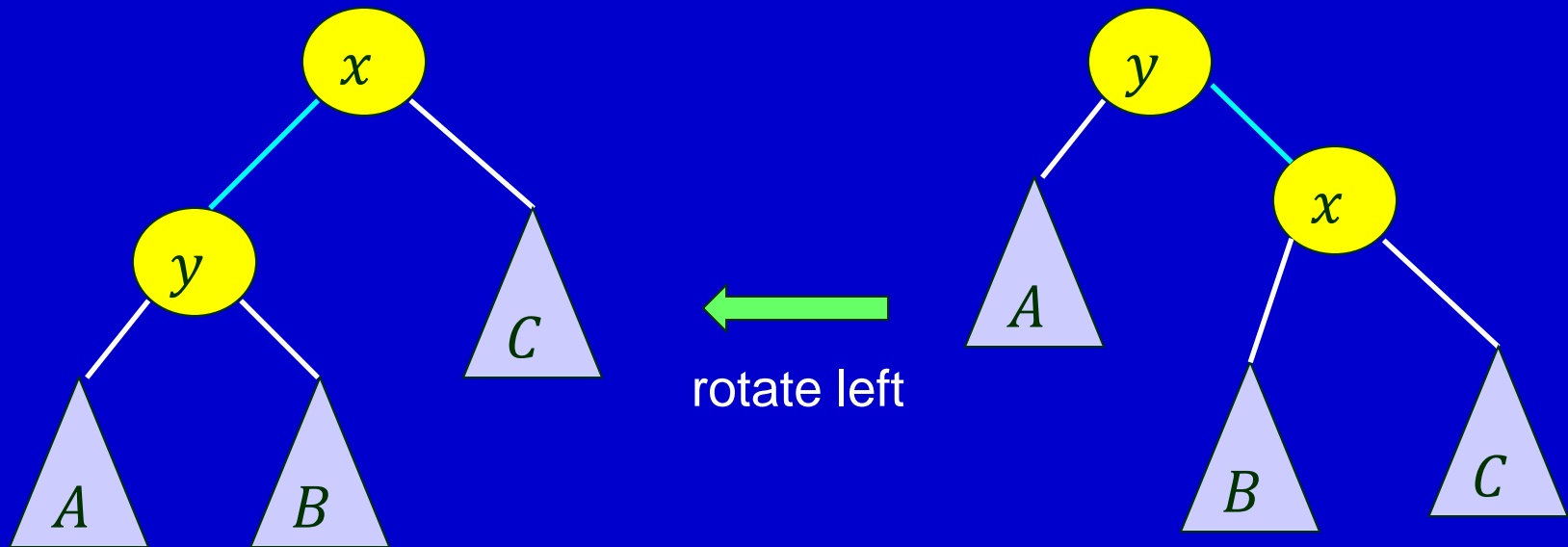
Case 1: zig

Parent y of x is the tree root.
Rotate the edge $\langle x, y \rangle$.



Terminal case.

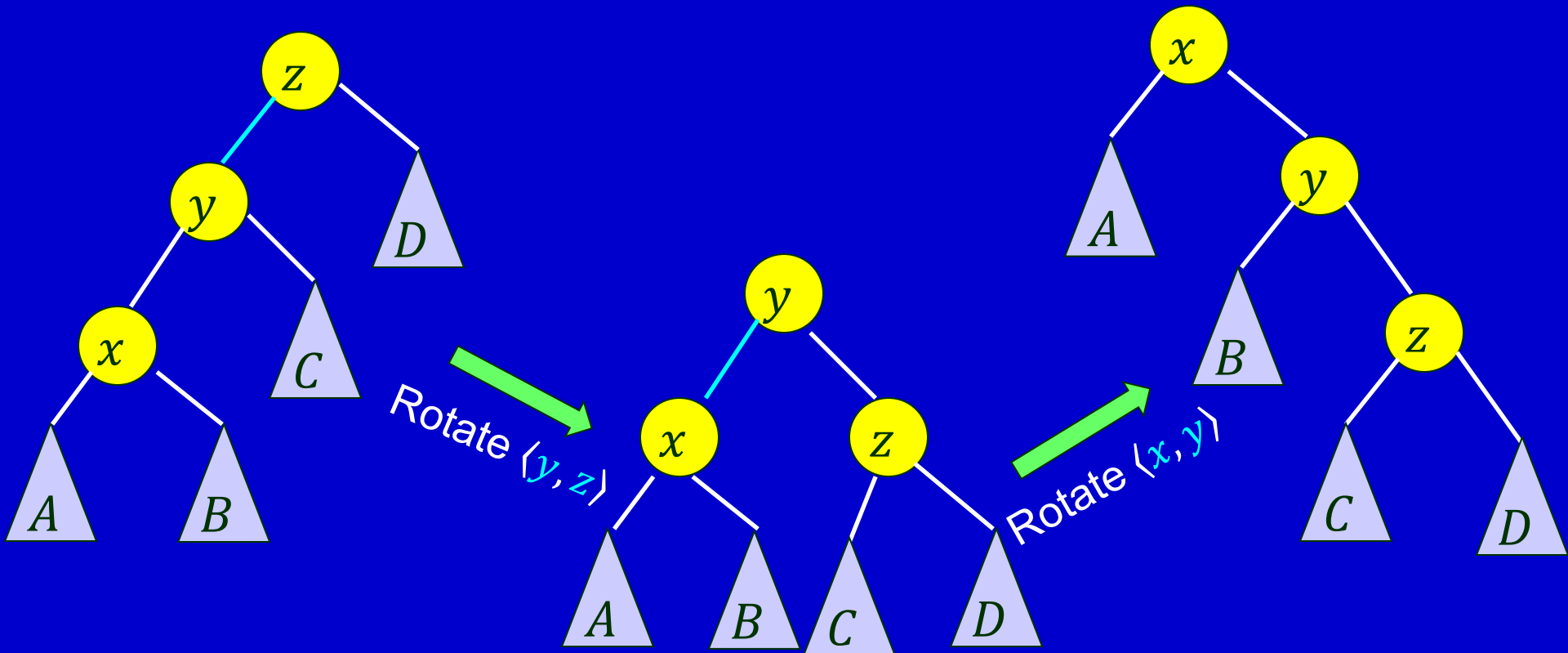
Case 1: zig (cont'd)



Case 2: zig-zig

- a) Parent y of x is not the root (x has grandparent z).
- b) x and y are both left or both right children.

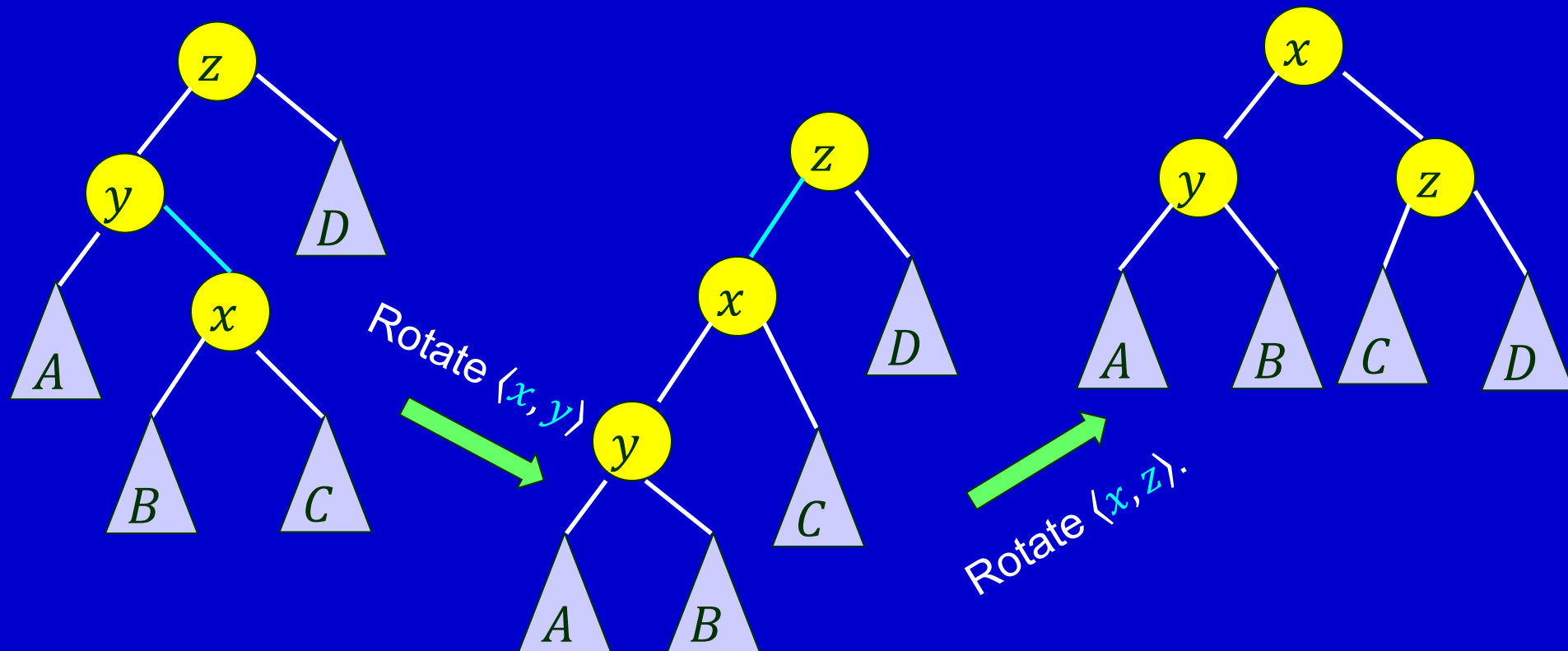
Symmetric rotations when x and y are both right children.



Case 3: zig-zag

- a) Parent y of x is not the root (x has grandparent z).
- b) x is a left child and y is a right child, or vice versa.

Symmetric rotations when x is a left child and y is a right child.



Time for Splaying a Node

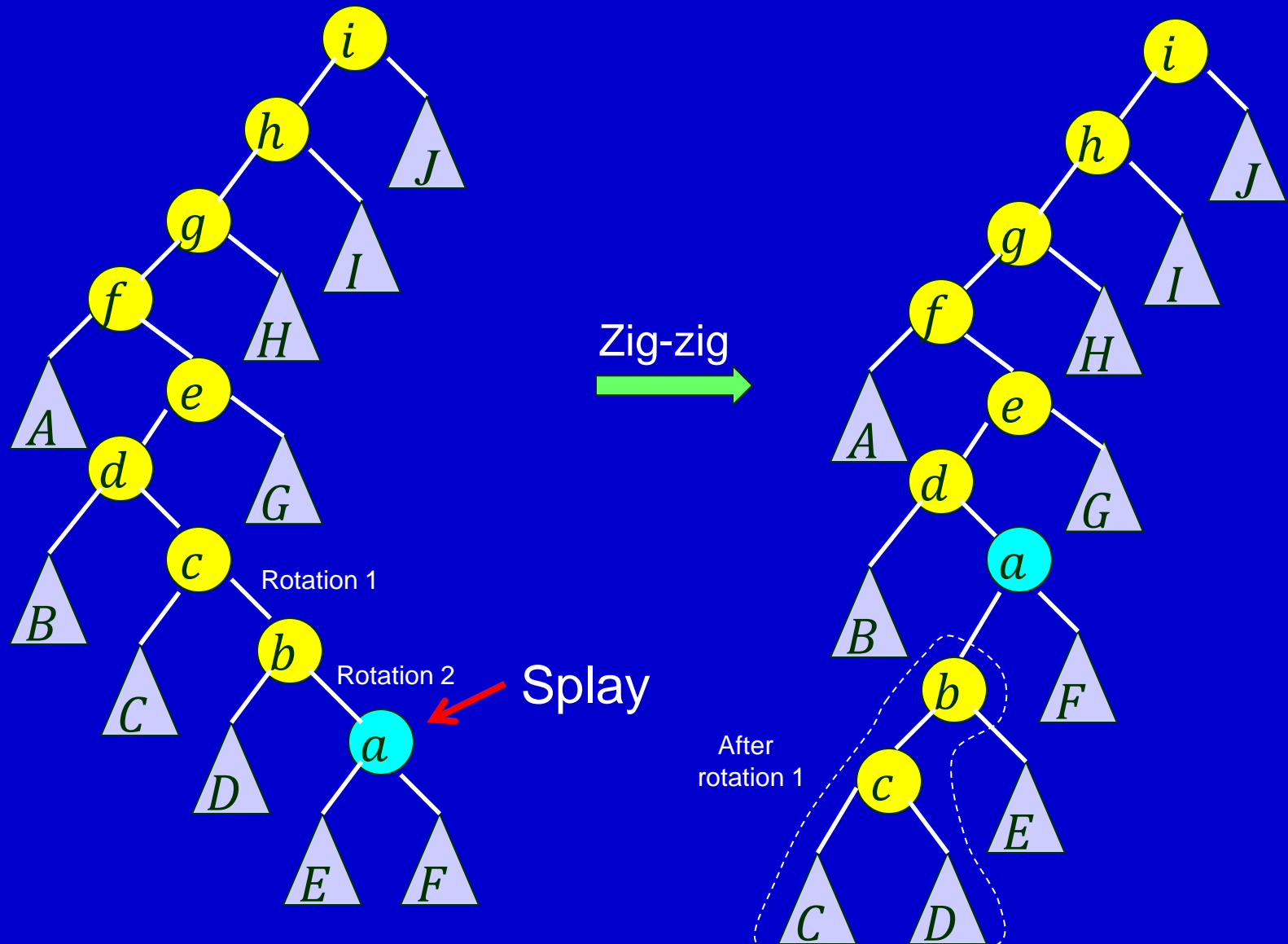
$\Theta(d)$: where d is the depth of the node x .

Proportional to the time to access the item stored in x .

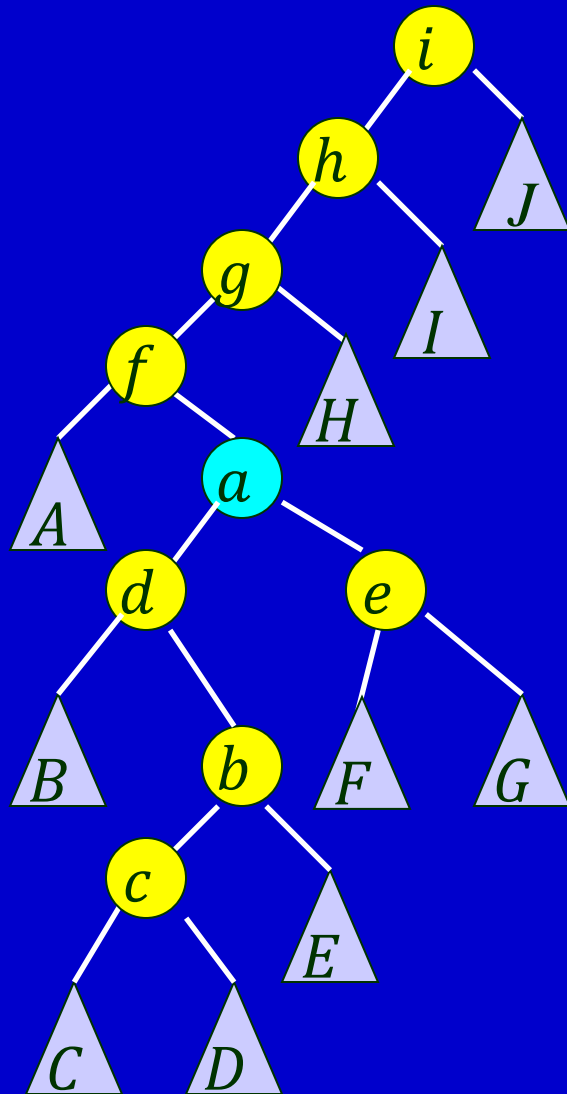
Effects of splaying:

- ✱ Moves x to the root.
- ✱ Roughly *halves* the depth of every node along the access path (shown in the next two examples).

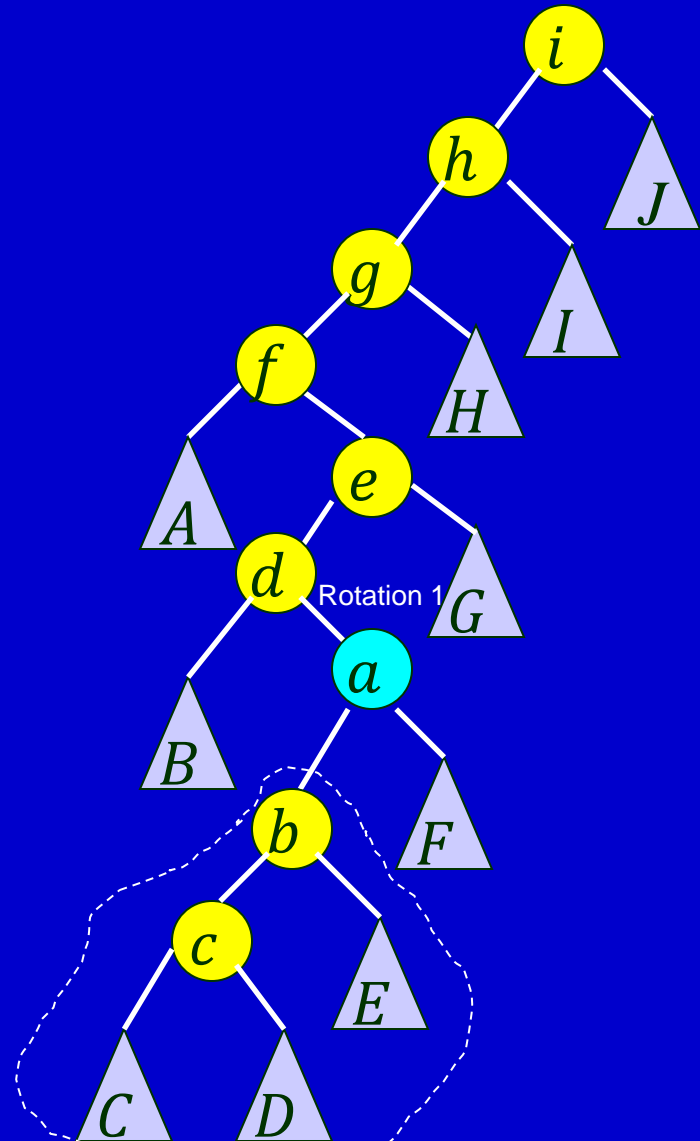
Example 1 – Splay Step 1



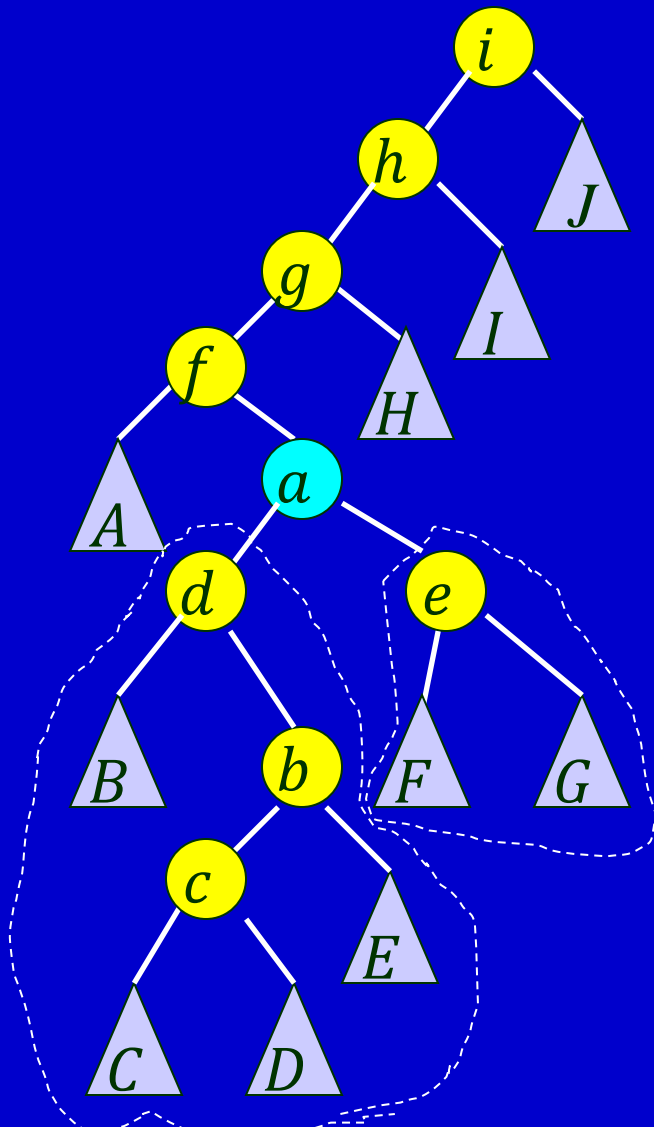
Splay Step 2



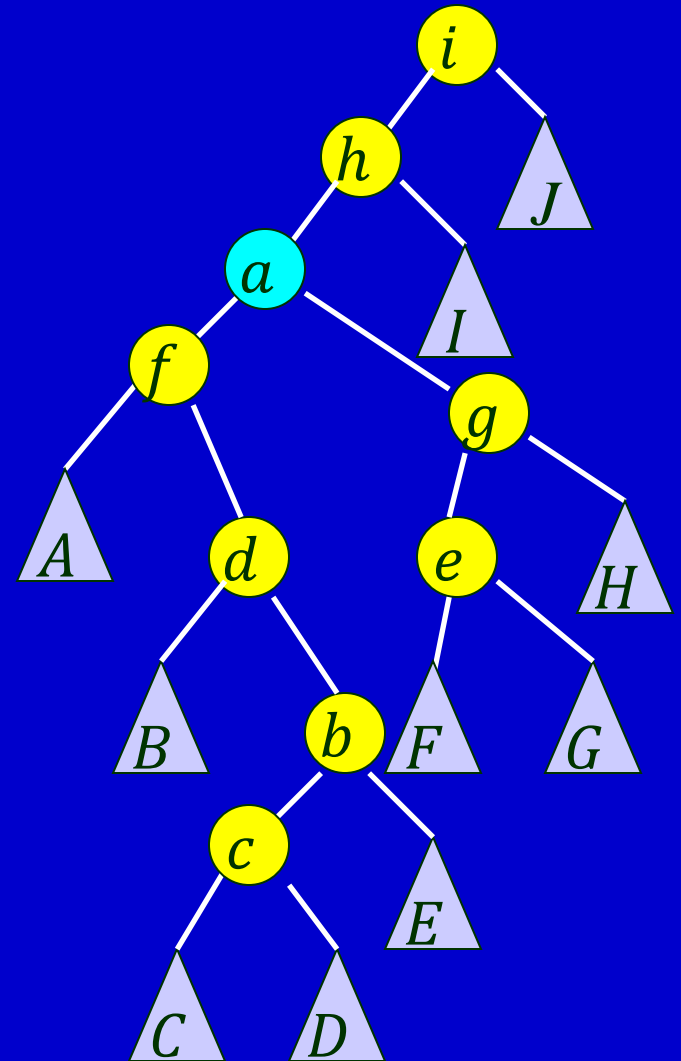
Zig-zag



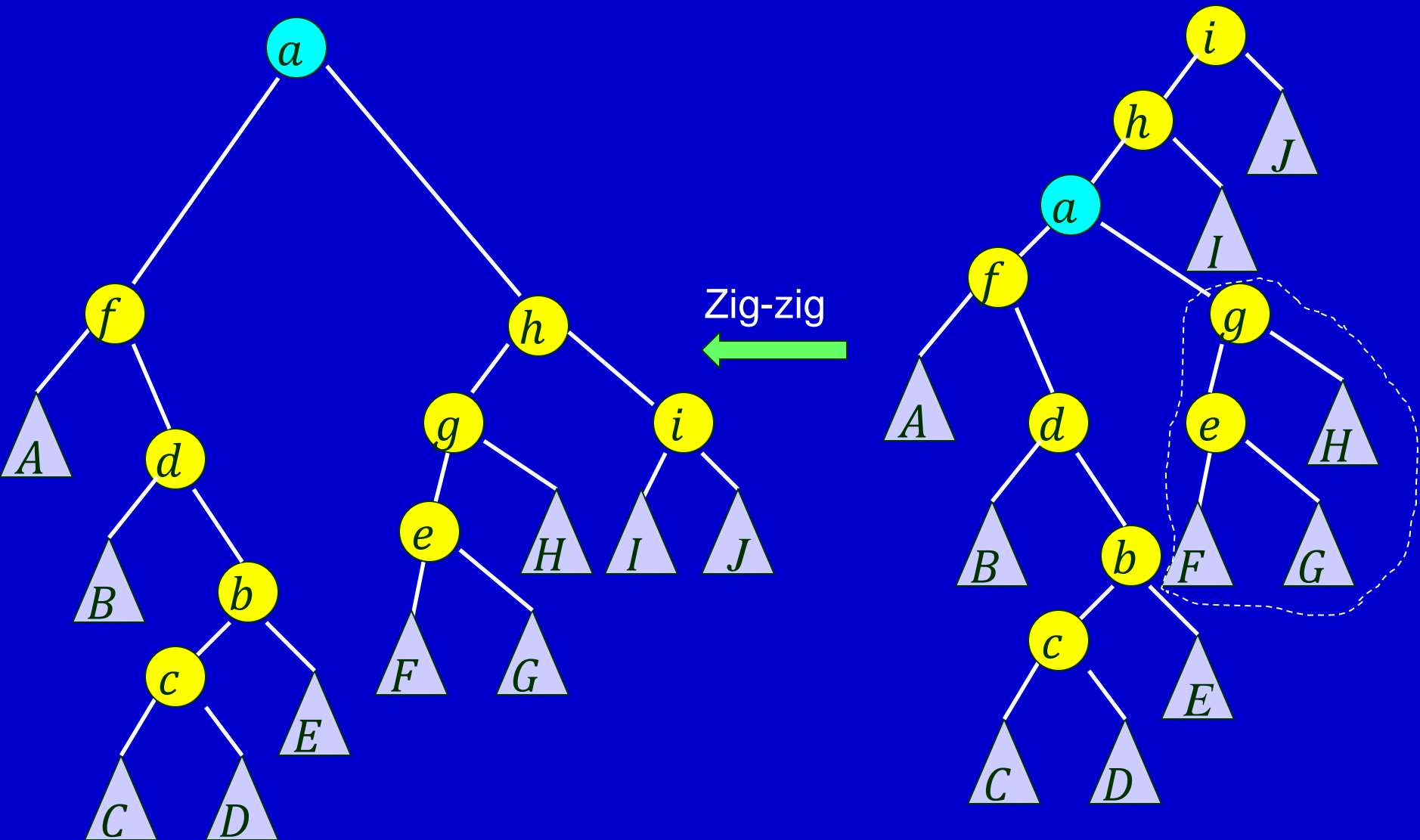
Splay Step 3



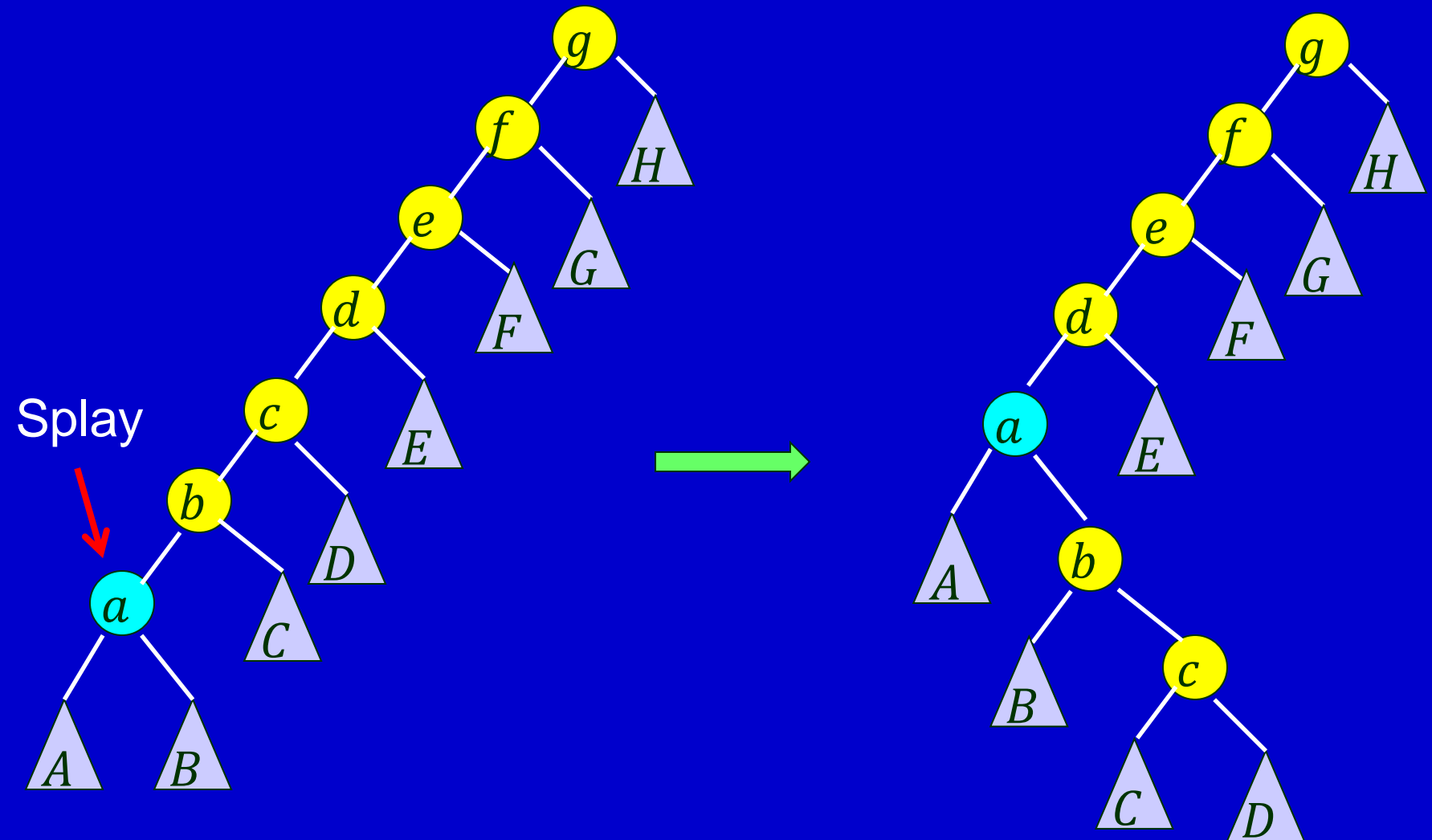
Zig-zag
→



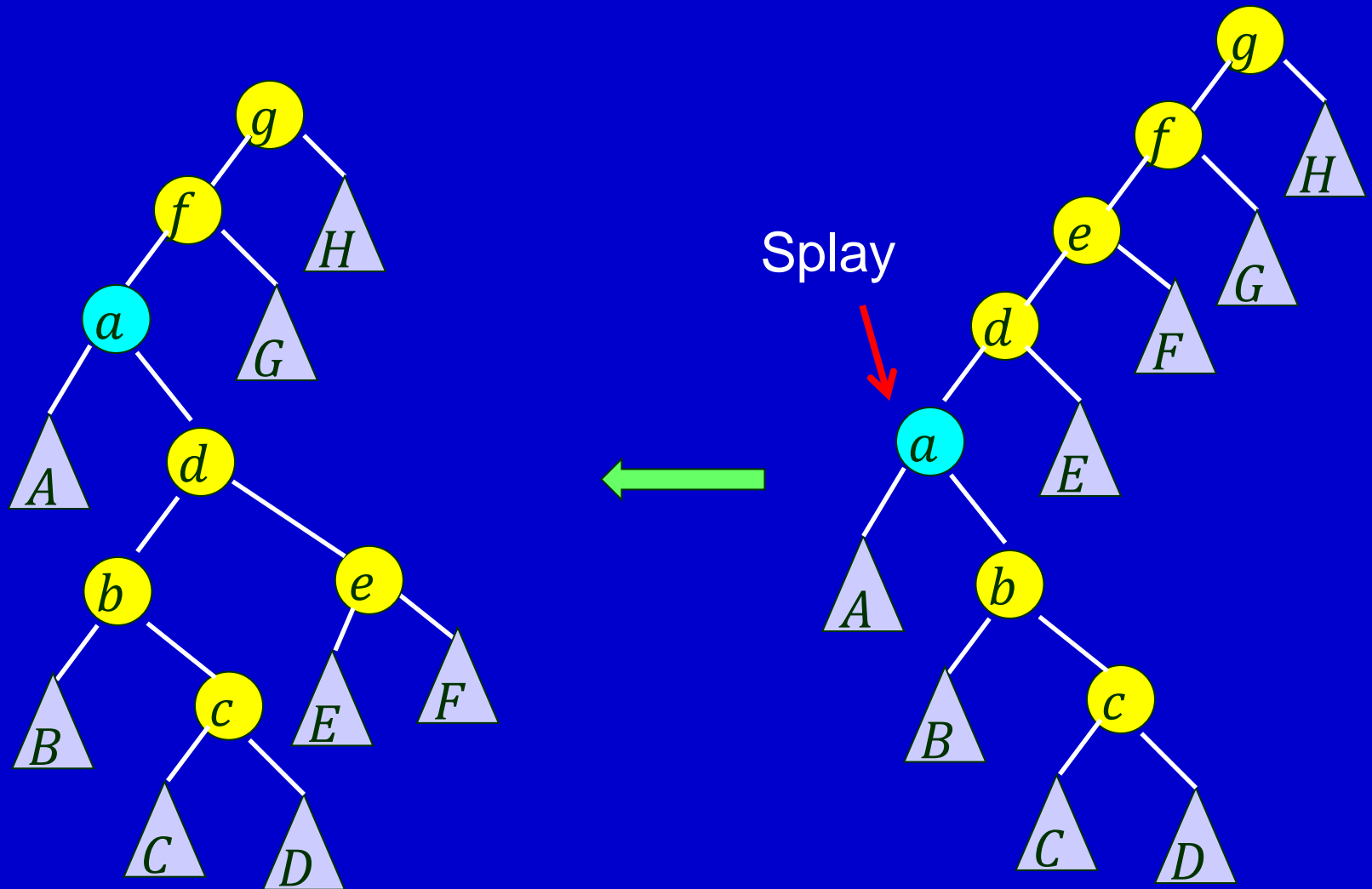
Splay Step 4



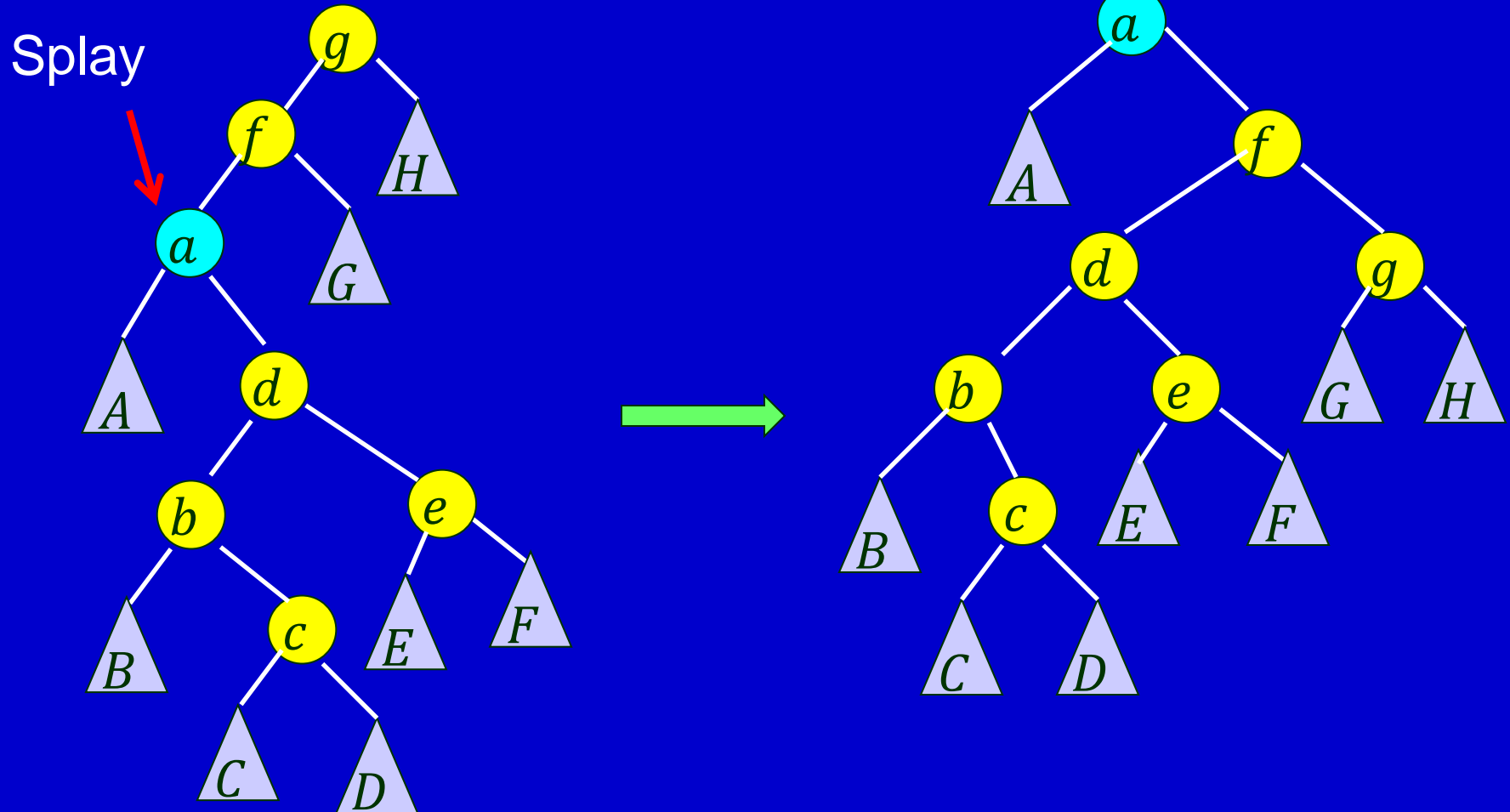
Example 2 – All zig-zig Steps



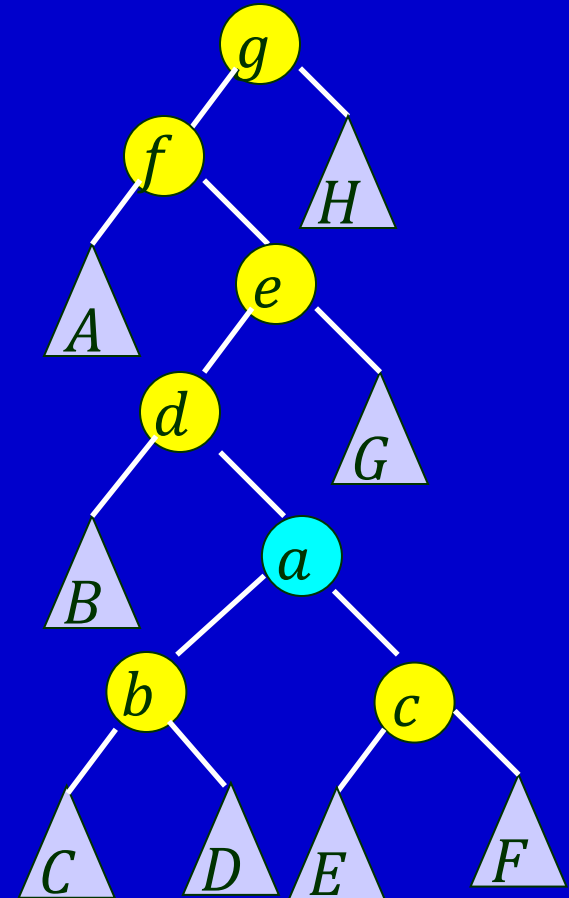
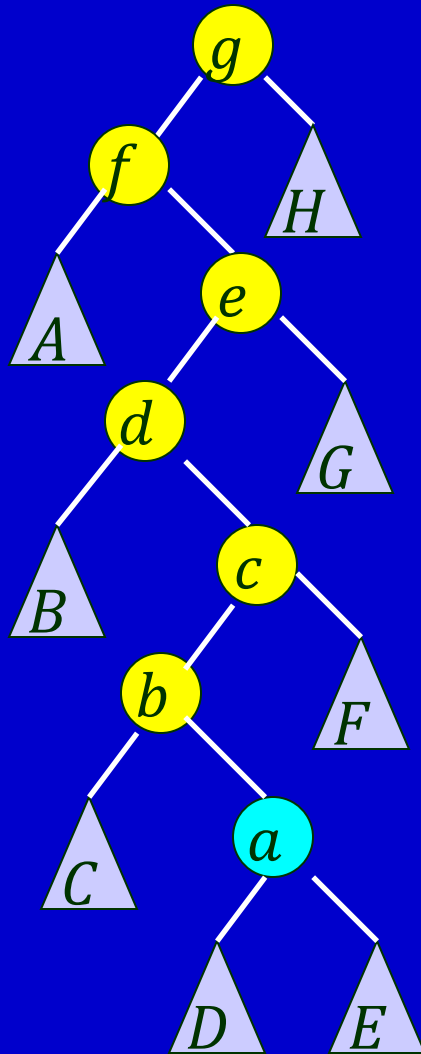
Example 2 (cont'd)



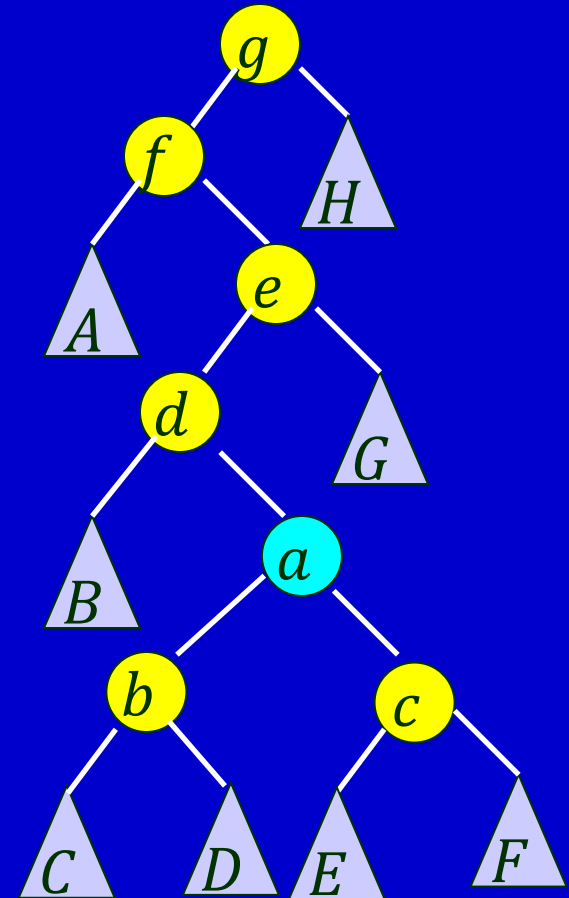
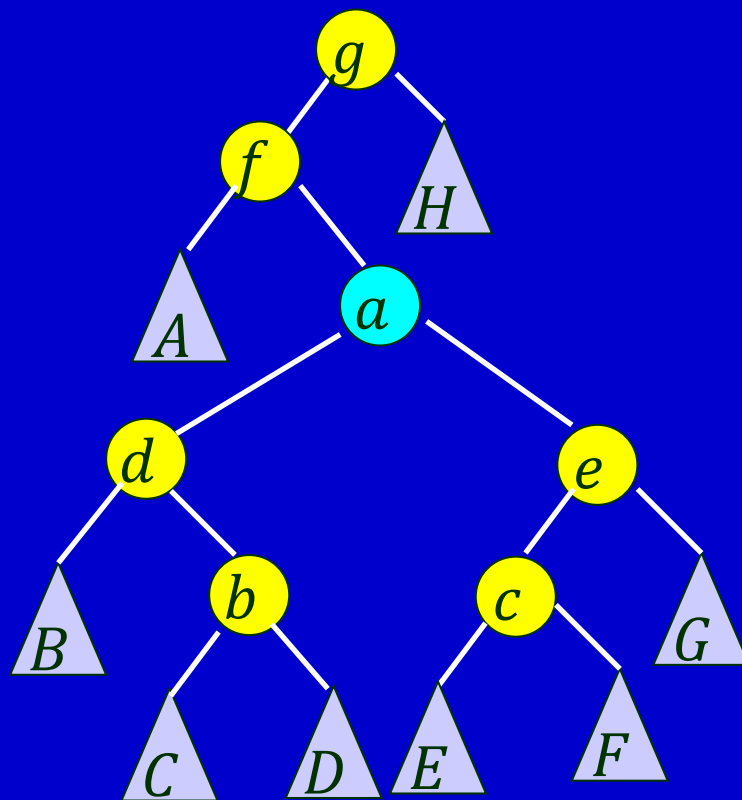
Example 2 (cont'd)



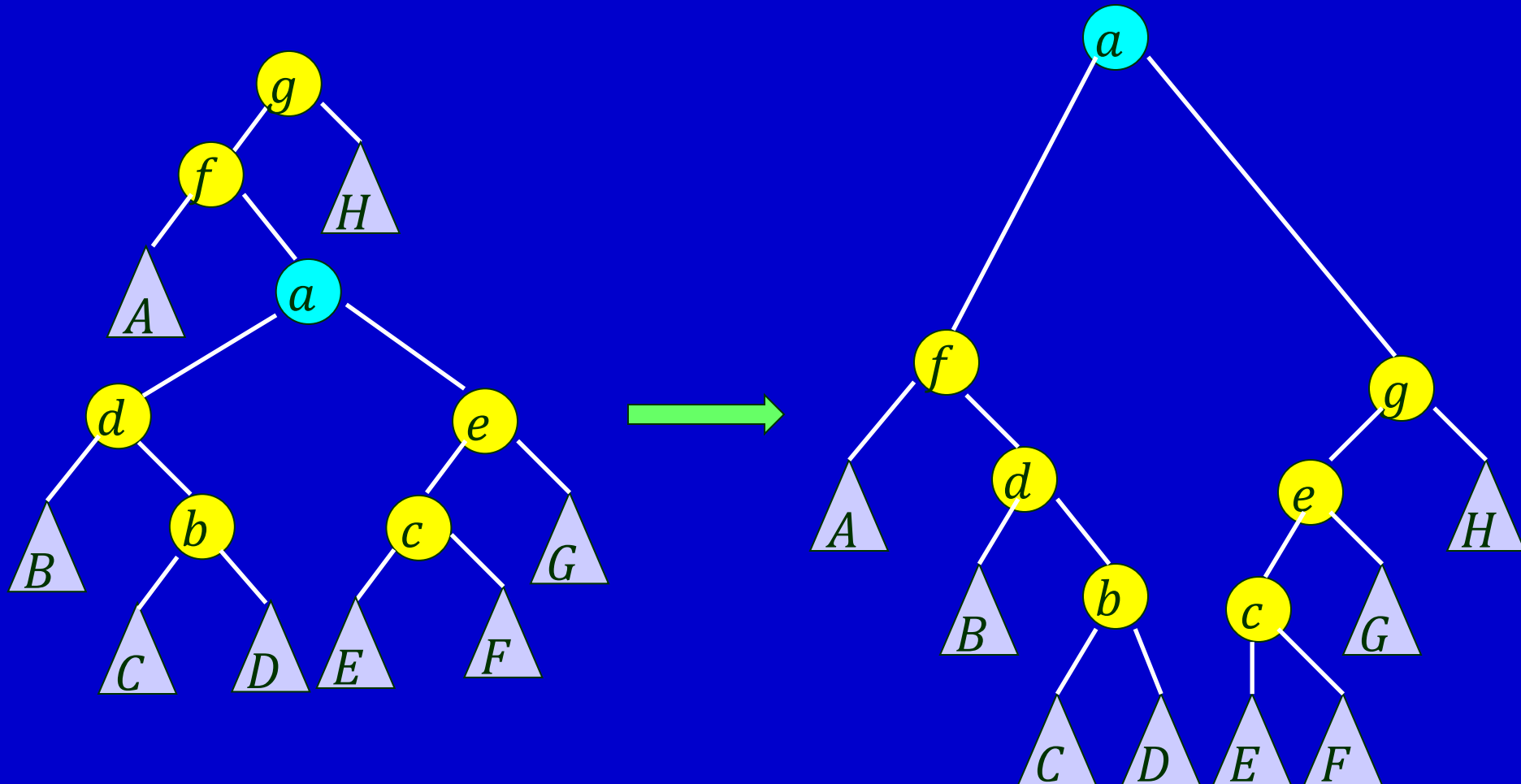
Example 3 – All zig-zag Steps



Example 3 (cont'd)



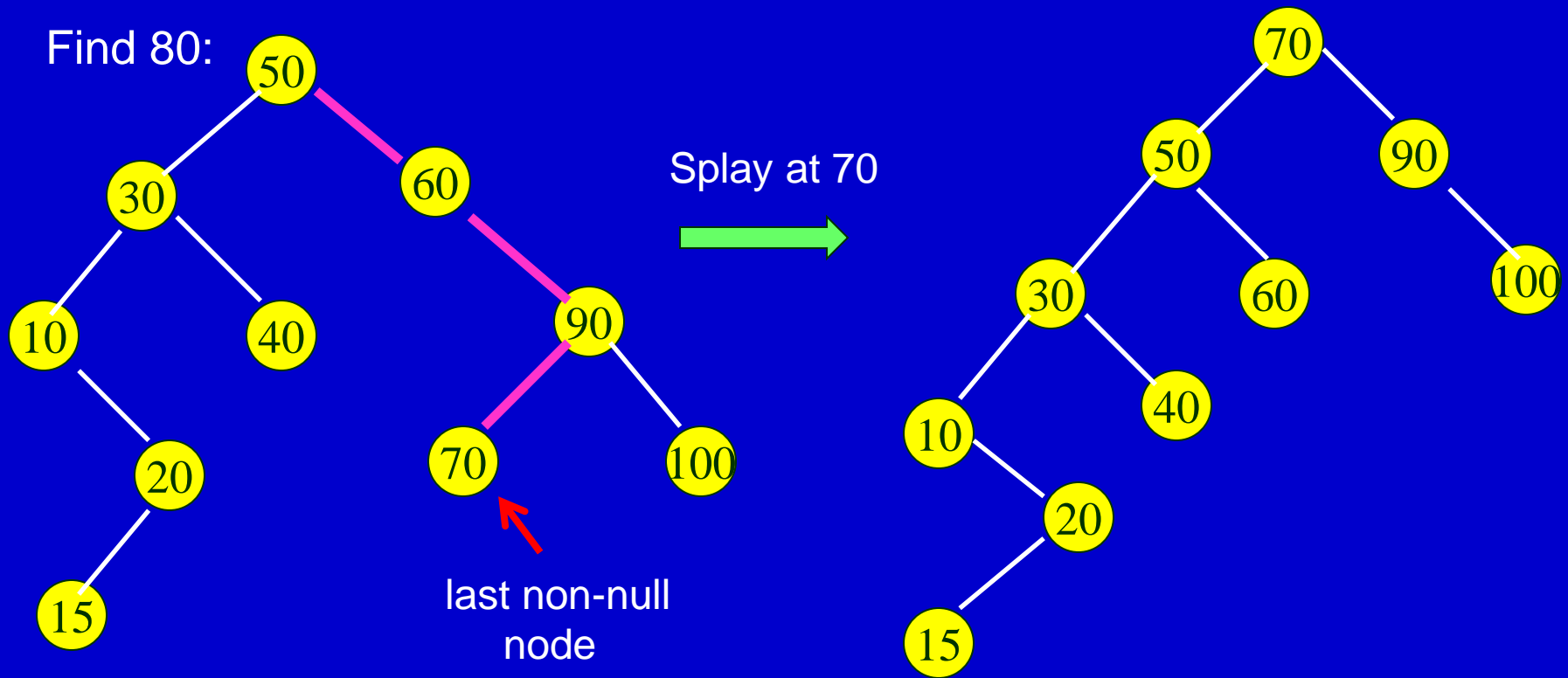
Example 3 (cont'd)



Data Access

All BST update operations can be implemented by splaying.

Access: if the item is stored in node x , splay at x ;
otherwise, splay at the last *non-null* node (i.e., the leaf node) on the search path.

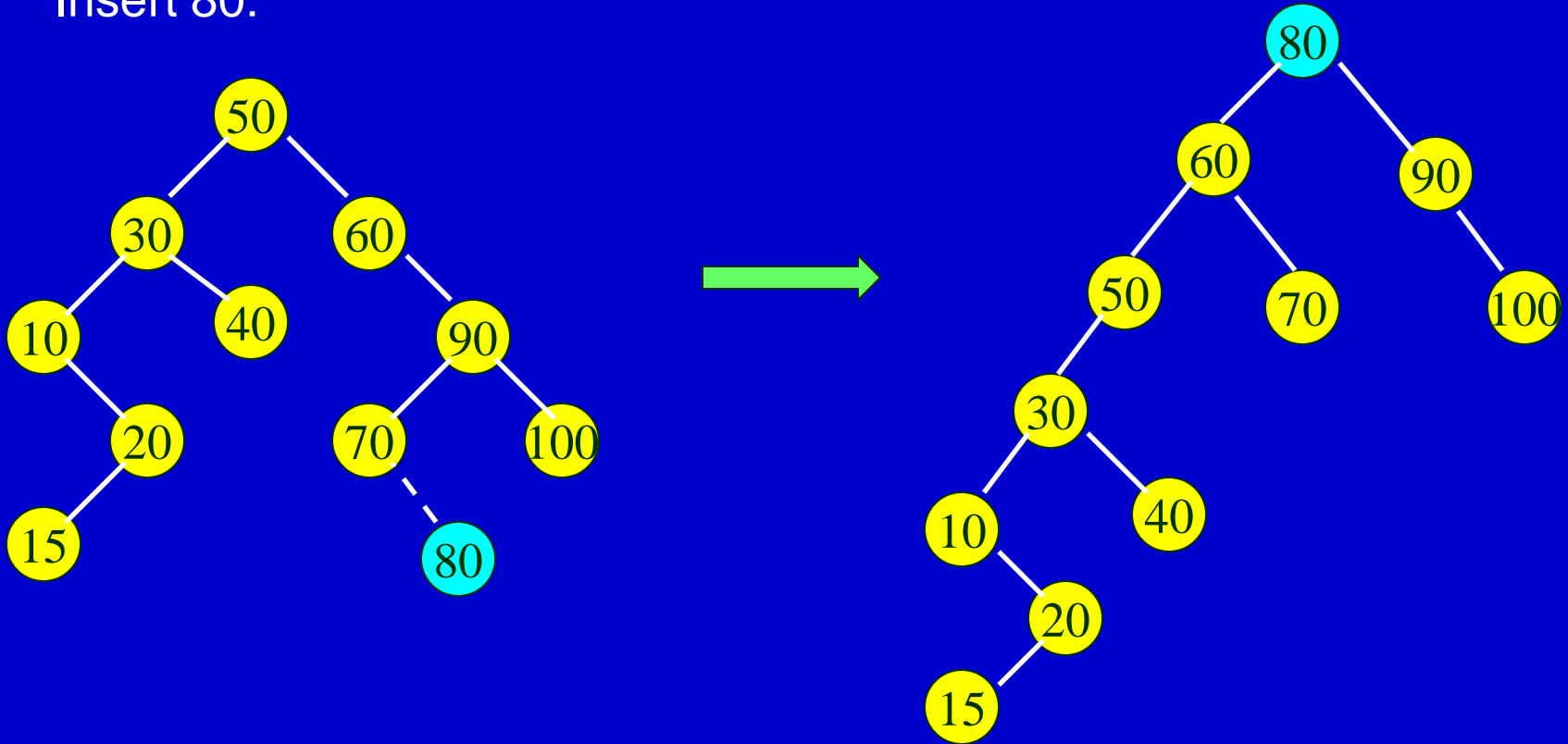


Insertion

1. Search for the item.
2. Create a new node containing the item if not in the tree.
3. Splay the tree at the new node.

BST insertion + Splaying.

Insert 80:



Joining Two BSTs

Two BSTs S and T such that all items in S are less than those in T . Combine S and T into a single tree.

1. Access the largest item in S . It is stored in node x .
After the access:

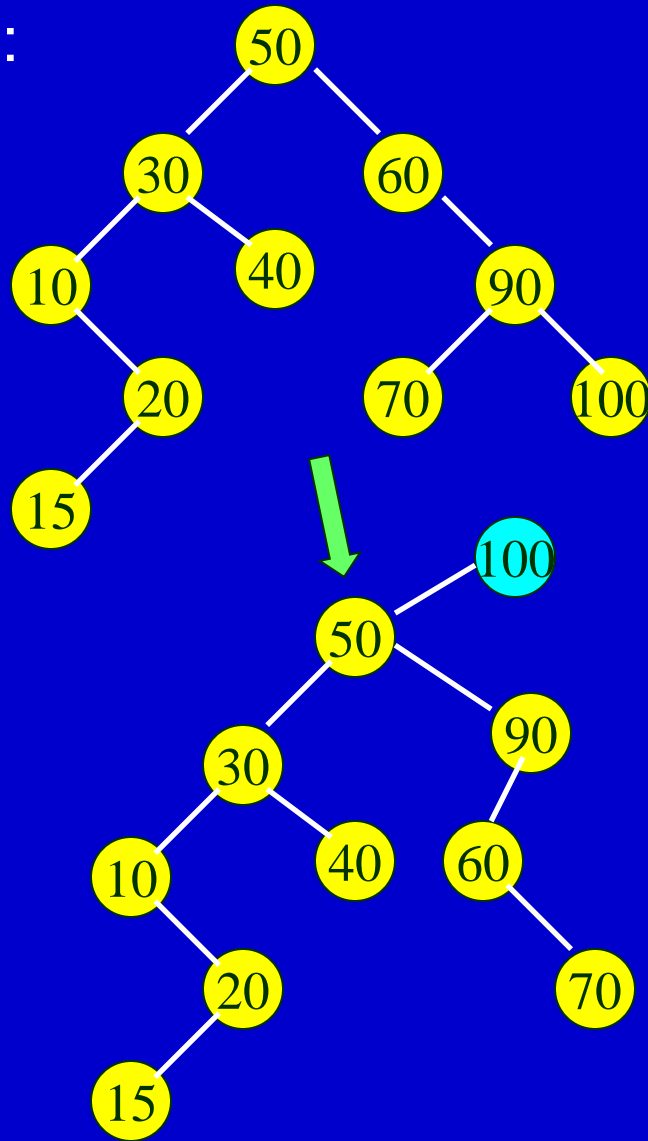
- ✱ x is at the root.

- ✱ x has no right child.

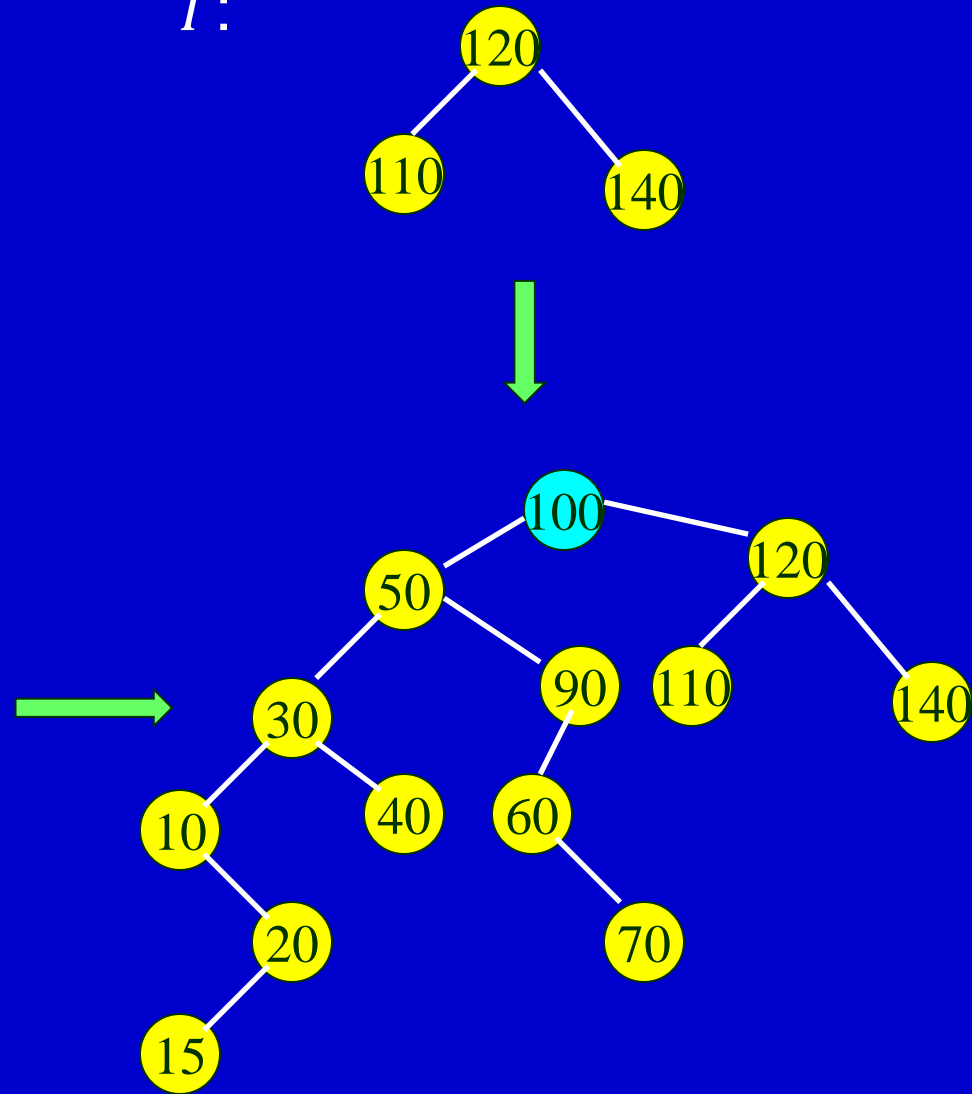
2. Making T the right subtree of this root.

Example 4 (of Join)

S:



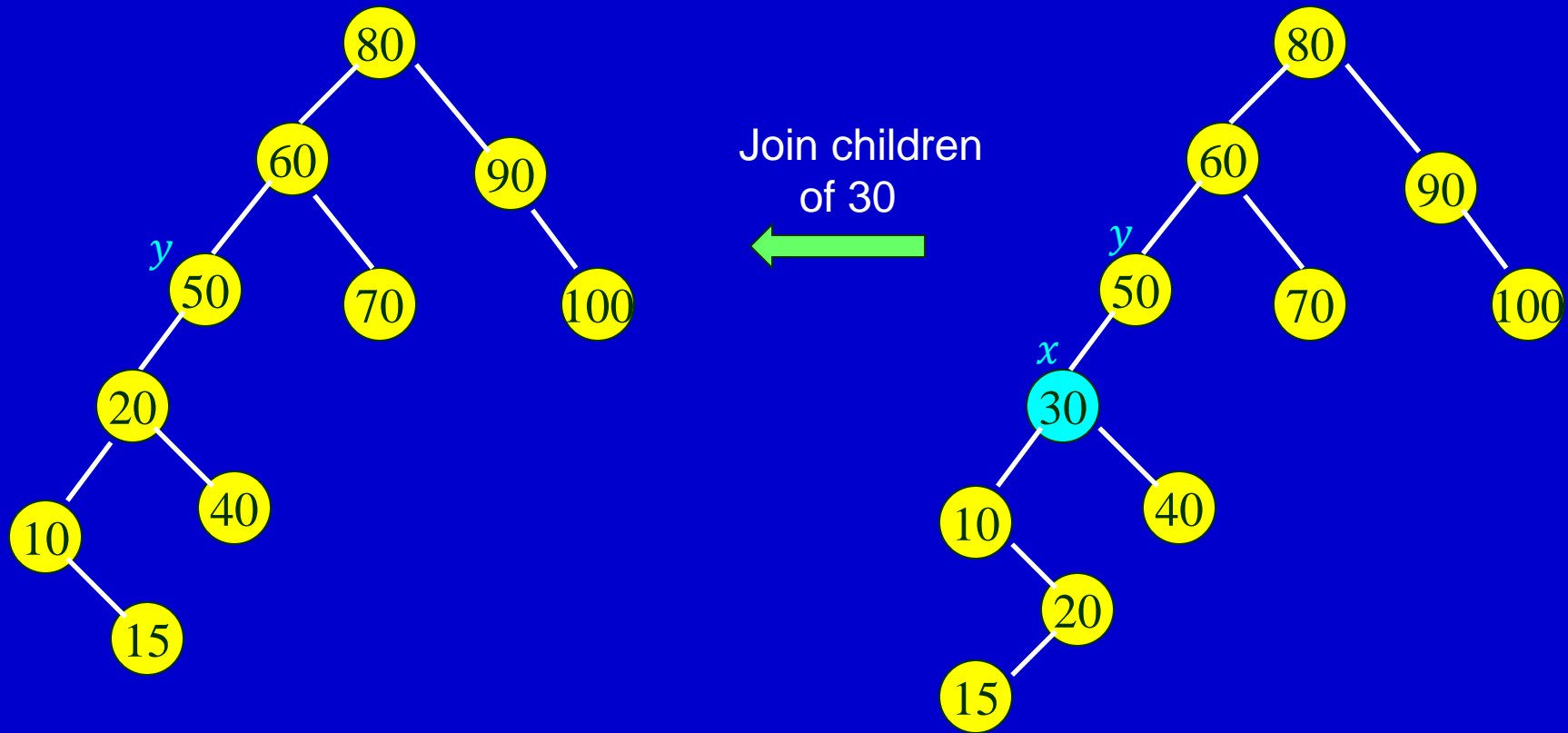
T:



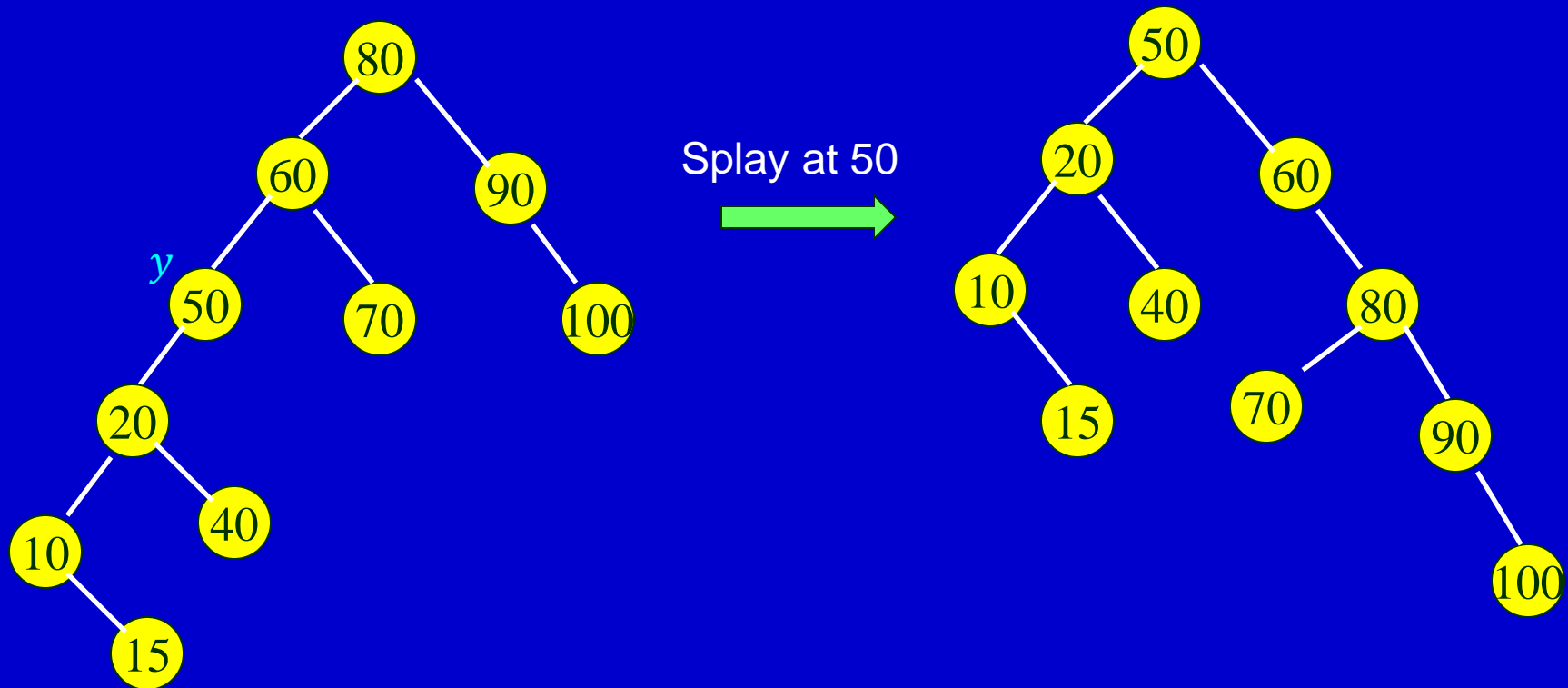
Deletion

1. Search for the item (stored at x with parent y).
2. Replace x by the join of the left and right subtrees of x .
3. Splay at y (or the last node on the search path if the item is not found).

Example Delete 30:



Example 5 (cont'd)



For more on splay trees, we refer to the following paper (where all examples but the joint one are from).

D. D. Sleator and R. E. Tarjan. Self-adjusting binary search trees. *Journal of the ACM*, 32(3):652-686, 1985.