

MIS Compensation: Optimizing Sampling Techniques in Multiple Importance Sampling

Seminar-Ausarbeitung von

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1. Abstract

Here is some sample text.

- (Write abstract (at the end)) ToDo
- (Write introduction. Structure as in the paper (overview, problem, others, solution proposal)) ToDo
- (Citation before the dot. In more than one sentence, name in the first sentence and reference in the last sentence.) ToDo
- (are all the citations correct?) ToDo
- (Does the date of access need to be in the references?) ToDo

2. Introduction

Since Veach and Guibas introduced Multiple Importance Sampling (MIS) [VG95] it had a hugh impact in a variety of directions in computer graphics. Not only did it help to construct more advanced rendering algorithms like Vertex Connection and Merging [GKDS12], it also helped in calculating direct illumination from area light sources and environment maps [PJH18, Chapter 14.3] and subsurface scattering [KKCF13]. Besides being introduced in computer graphics MIS has also been used in other fields [HO14].

For the proof of MIS being almost optimal the assumption was made, that the distribution of samples among the techniques and the sampling densities are given and fixed. The only things left for fine tuning was the weighting function with the balance heuristic being a popular and widely adopted choice because of its simplicity and proven tight bounds when only using non-negative weights [Vea97, Theorem 9.2]. Recently researchers found that using not only non-negative weights can result in a truly optimal weighting function [IVG⁺19].

No previous work examined the effect of designing a sampling density itself for application in the MIS framework. Karlík et al. proposed a method to do exactly that and show that it can improve the MIS framework and reduce variance even further. They assumed multiple sampling techniques and pick one probability density function (pdf) of a technique and modify it in a way that it compensates for the averaging introduced by the balance heuristic. Their optimization can be applied beforehand and doesn't need adaptive updates as the work of Cappé et al. [CDG⁺08]. Others created product sampling methods [HEV⁺16] that try to match the integrand closer, but MIS compensation as, Karlík et al. call their method, on the other hand can even cause a pdf to be further away from the integrand but still reduce the variance [KŠV⁺19].

Karlík et al. show the effects of their MIS compensation with direct illumination using a HDR environment map and a scene with surfaces that have multiple different bidirectional reflectance distribution functions (BRDF). The standard approach with MIS would take samples proportional to the BRDF and the environment map, but they assume this approach to be too defensive and modify the pdf for the environment map in a way that the variance is reduced and therefore render times can be improved. Their modification is done by applying a formula to the tabulated pdf in a preprocessing step after which any renderer able to use HDR maps for MIS sampling can use the updated environment map for sampling without any further adjustment.

They also used their approach in path guiding [VHH⁺19] which learns a guiding density to improve the algorithm of [MGN17] by adjusting that density [KŠV⁺19].

3. Related Work

The MIS framework was introduced in [VG95] and combines multiple different sampling techniques with a weighting function to sample a more complex function better. In their work they proposed a few different weighting functions the most famous being the balance heuristic, but also the power and cutoff heuristic which perform better in certain scenarios where one technique matches the integrand well. Despite the balance heuristic being provably good Kondapaneni et al. [IVG⁺19] developed a weighting function that is truly optimal and uses negative weights to archive even better results.

A lot of work has been put into optimizing the weighting functions, but Karlík et al. followed a different approach and worked on a technique to adjust one sampling density to reduce the variance.

Another direction of research in the field of MIS is to figure out how to distribute the samples over the different sampling techniques. Pajot et al. [PBPP11] proposed a method where they calculate a so called representativity based on common rendering information such as the BRDF, which is a measure of how well a technique matches the integrand. In a one-sample estimator this representativity can also be used to derive a probability for each sampling strategy.

Lu et al. [LPG13] created a method where no prior knowledge about the scene is needed to generate a sample distribution over two sampling strategies. They start by sampling with a small number of samples evenly distributed over the strategies and calculate a value to partition the total number of samples over the sampling techniques.

He and Owen [HO14] presented another method to get a partitioning of samples for more than two strategies by proving that the variance is jointly convex in the distribution of samples. They also proposed an improved method for convex optimization to find such a distribution.

Sbert et al. [SHSK16] worked on a modification of the balance heuristic that doesn't take the distribution of samples into account for calculating the weights. Their method works in two phases. The first phase uses 20% of the total samples equally distributed over all sampling methods to calculate the variance σ_i^2 . After that the following stage is subdivided into eight substages each using 10% of the total number of samples and iteratively updating σ_i^2 and calculating the distribution α over the techniques for the next substage.

What all the above techniques have in common is that they all rely on initial samples to fine tune their succeeding execution. The proposed method of Karlík et al. on the other hand doesn't need any prior samples to be taken and therefore their performance impact should be much smaller [KŠV⁺19].

One widely used application of MIS is image based lighting with BSDF and HDR map sampling [PJH18]. When using a HDR environment map for direct illumination the pdf for that map is usually proportional to the brightness of the HDR map, Karlík et al. [KŠV⁺19] created a method that alters that sampling density to further decrease the variance. Agarwal et al. came up with the idea of stratifying the environment map and taking into account the visibility in the scene to create a method that can require up to two orders of magnitude less samples for the same quality [ARBJ03].

Another approach from Clarberg and Akenine-Möller uses product importance sampling and creating the BSDF on the fly during rendering. Their precomputation step uses quadtree-based multiplication while during rendering they build the approximation of the BSDF and evaluate the product in only one single tree traversal which makes it perform faster than other methods [CAM08].

The method of Karlík et al. does not change the sampling procedure itself and therefore can work without modifications of the environment map sampler [KŠV⁺19].

Herholz et al. worked on a method to create a pdf the resembles the whole product of the integral (BSDF and reflected light) we want to solve by training a Gaussian mixture model (GMM) for the individual factors with initial samples, which they then combine to draw samples during the integration [HEV⁺16].

Müller et al. also worked on path guiding, where they used an algorithm that iteratively learns the radiance distribution in the scene to get better samples from their distribution [MGN17].

The work of Karlík et al. builds on the algorithm proposed by Müller et al. to further improve the guiding density [KŠV⁺19].

4. Multiple Importance Sampling

In this chapter we will introduce all the theoretical background necessary to follow the upcoming chapters. We will start with the basic probability essentials followed by a section about drawing samples after an arbitrary function. Then we introduce the concept of Monte Carlo integration and have a short look at variance in that section too. After that importance sampling is explained and lastly multiple importance sampling will be introduced with the balance heuristic being the final section.

4.1 Probability Basics

For our use case we need to define random variables, probability distribution functions (pdf) and cumulative distribution functions (cdf). A random variable maps an event to a real number $X : \Omega \rightarrow \mathbb{R}$. In a discrete scenario our random variable corresponds to one exact event e.g. a dice throw showing three pips on the top. The probability for this can be expressed as $P(X = 3) = \frac{1}{6}$ or as $X(3) = \frac{1}{6}$ since the random variable X can take six different values which are all equally likely. To express the probability that an event is within a range of values we can use the cdf which is defined as

$$F(x) = P(X \leq x), x \in \mathbb{R}.$$

For the probability of a random variable in an interval $[a, b)$ we can write

$$P(a \leq X < b) = F(b) - F(a) \text{ [Sch18].}$$

When we are talking about continuous events e.g. turning a wheel of fortune, every exact angle has a probability of nearing zero. But we can still express our probability for an angle interval with the cdf from above. In the continuous case we have a pdf that is used to calculate the cdf with

$$F(x) = \int_{-\infty}^x p(\tilde{x})d\tilde{x}.$$

The pdf has to be non-negative ($\forall x : p(x) \geq 0$) and integrate to one ($\int_{-\infty}^{\infty} p(x)dx = 1$) to be valid and “[...] describes the relative probability of a random variable taking on a particular value” [PJH18, Chapter 13.1].

4.2 Generating Samples after a specific Function

Knowing how to sample after a specific distribution is essential for the next section 4.3, since this is our adjustment wheel to manipulate the quality of our rendering (for a given amount of time). Most of the time we don't need uniformly distributed samples, rather our samples should follow a specific characteristic of a material or a scene. For this we need to know how to generate samples that correspond to that characteristic.

Given a function $f(x)$ from which we would like to draw samples from the first step is to make sure it fulfills the constraints for a pdf in an interval $[a, b]$ we want to use. First we check that $\forall x \in [a, b] : f(x) \geq 0$ and then calculate the integral with

$$F = \int_a^b f(x)dx. \quad (4.1)$$

To Do F can be used to normalize our goal function which yields us $\tilde{f}(x) = \frac{f(x)}{F}$ as a valid pdf. The next step is to calculate our cdf as $F(x) = \int_a^x \tilde{f}(t)dt$ which we will then invert to get $F^{-1}(x)$. Now we only need to draw an equally distributed random number (which most programming languages have a library for) in the interval $[0, 1]$ which we call ξ and evaluate $F^{-1}(\xi) = X$ to get X which is distributed proportional to $f(.)$. This method is called "inverse cdf" [Sch18]. (**Should I also cover 2D sampling?**)

To Do

(**Should I cover more sampling methods (rejection sampling e.g.)?**)

4.3 Monte Carlo Integration

Monte Carlo integration is a technique to approximate the integral of an arbitrary function $f(x)$ by taking N random samples X_i from a pdf $p(X_i)$. As we know the expectation of a random variable is calculated by

$$E(X) = \int_{-\infty}^{\infty} xp(x)dx$$

or more generally

$$E(g(x)) = \int_{-\infty}^{\infty} g(x)p(x)dx.$$

For the next step we need the law of large numbers which states that

$$P\left[\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N X_i = E(X)\right] = 1$$

with X_i drawn from $p(.)$ which means that the average of our samples will converge to the expected value [Vea97, Chapter 2.4.1]. From here we can formulate

$$\int_a^b g(x)p(x)dx \stackrel{\text{definition}}{=} E(g(x)) \stackrel{\text{law of large numbers}}{\approx} \frac{1}{N} \sum_{i=1}^N g(x_i)$$

and with $g(x) = \frac{f(x)}{p(x)}$

$$\begin{aligned} \int_a^b \frac{f(x)}{p(x)} p(x)dx &\approx \frac{1}{N} \sum_{i=1}^N \frac{f(x_i)}{p(x_i)} \\ \Leftrightarrow \int_a^b f(x)dx &\approx \frac{1}{N} \sum_{i=1}^N \frac{f(x_i)}{p(x_i)} \end{aligned} \quad (4.2)$$

which is the Monte Carlo estimator [Sch18].

4.3.1 Variance

Generally the variance is described by

$$V(X) = \frac{1}{N} \sum_{i=1}^N (x_i - E(X))^2$$

for discrete random variables [Sch18] and

$$V(X) = \int_a^b (x - I)^2 p(x) dx$$

with I being the mean of the random variable X for continuous variables [wyz19]. When we look at the variance of our Monte Carlo integral we get

$$\begin{aligned} V(g(x)) &= \int_a^b (g(x) - F)^2 p(x) dx \\ &= \int_a^b \left(\frac{f(x)}{p(x)} - F \right)^2 p(x) dx \end{aligned} \tag{4.3}$$

with F being the integral from equation 4.1.

4.4 Importance Sampling

With the variance from equation 4.3 let's see how we can improve it. We can easily see that reducing the term $\frac{f(x)}{p(x)} - F$ will finally reduce the variance the most since it will be exponentiated. Since F and $f(\cdot)$ are fixed (that is the integral we want to calculate) only $p(\cdot)$ is left for modification. When we choose

$$p(x) = \frac{f(x)}{\int_a^b f(x) dx}$$

we get

$$\begin{aligned} \frac{f(x)}{p(x)} - F &= \frac{f(x)}{\int_a^b f(x) dx} \int_a^b f(x) dx - F \\ &\stackrel{4.1}{=} \int_a^b f(x) dx - \int_a^b f(x) dx = 0 \end{aligned}$$

which is ideal as we get a variance of 0 and are done. The problem is that we don't know the integral at the start since that is what we want to calculate in the first place. When we look at our chosen pdf we see that $p(\cdot)$ is proportional to $f(\cdot)$ by a factor $\frac{1}{\int_a^b f(x) dx} = \frac{1}{F} := c$. So we should at least try to chose $p(\cdot)$ somewhat proportional to $f(\cdot)$. The difficulty is that $f(\cdot)$ can be arbitrary complex so we might only be able to approximate some part of it with one specific pdf.

4.5 Multiple Importance Sampling

The rendering equation

$$L_o(x, \omega) = L_e(x, \omega) + \int_{\Omega^+} f_s(x, \omega, \omega_o) L_r(x, \omega_o) |n \cdot \omega_o|^+ d\omega_o \tag{4.4}$$

consists of the emitted light at point x $L_e(\cdot)$, the bidirectional scattering distribution function (BSDF) $f_s(\cdot)$, the reflected light at point x in direction ω expressed as $L_r(\cdot)$ and the cosine to account for looking at an angle onto the surface $|n \cdot \omega_o|^+$ where the $^+$ means

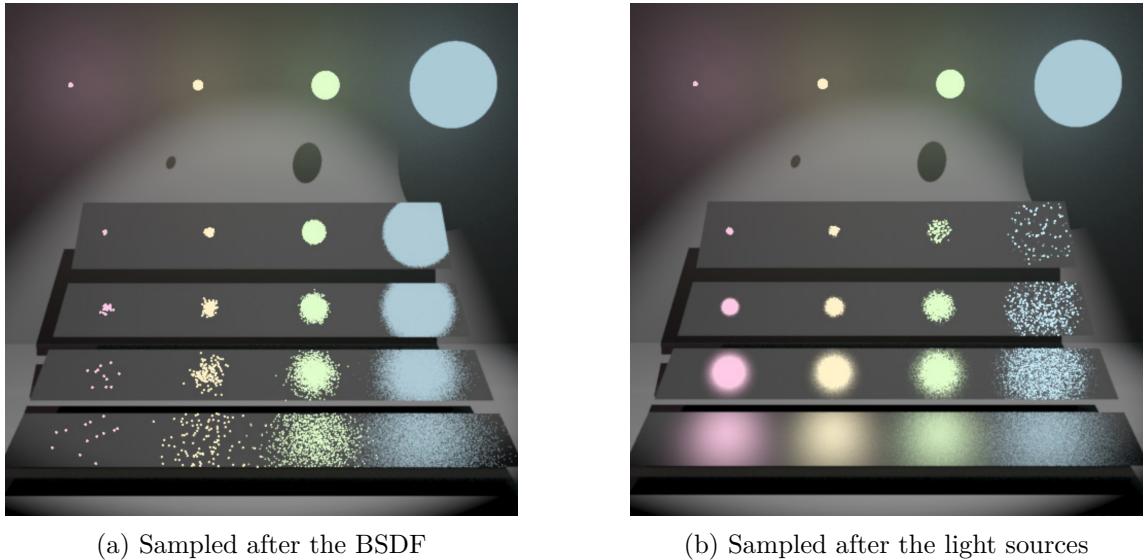


Figure 4.1: A scene with four light sources getting larger in diameter from left to right and four horizontal objects with BSDFs from diffuse at the bottom to more and more specular at the top.

- (a) This scene is sampled using a pdf that is proportional to the BSDF of the surface.
- (b) Here the scene was sampled with a pdf that generated samples on the light source. These graphics were taken from [Vea97, Figure 9.2].

we are only taking values > 0 so only directions pointing into the upper hemisphere. With this we can see that a single pdf that only is proportional to e.g., the BSDF leaves out the $L_r(\cdot)$ term so it can't be optimal. Having multiple pdfs where each samples a different part of the rendering equation well a combination of them should create a good result. E. Veach created a good example of this problem in his thesis [Vea97] which can be seen in figure 4.1. He also introduced Multiple Importance Sampling to use a combination of pdfs to get better results than would be possible with only one pdf.

When sampling only the BSDF we see that in figure 4.1a the bottom left reflection is very noisy, because sampling a diffuse BSDF can give a wide range of directions and since the light source on the left is very small only very few directions will point directly on that light source. The more specular surfaces in that figure are well sampled which makes sense, because there are less possible directions to sample so the reflection will be well sampled.

On the other hand figure 4.1b shows the same scene but sampled after the light sources by sampling a random point on the light source and then evaluating the render equation to get the contribution. Here the opposite happened in the two corners we looked at before. The bottom left reflection is very well sampled since the direction to the light source is still valid (has some contribution in the BSDF) and therefore we can see the whole reflection. But now the top right reflection is noisy because the direction to a random point on the light source has no contribution in the BSDF.

If we could combine both approaches the result should cover all cases and show all reflections well. To do this Veach introduced Multiple Importance Sampling [Vea97, Chapter 9]. The updated estimator now looks like this:

$$F = \sum_{i=1}^k \frac{1}{n_i} \sum_{j=1}^{n_i} w_i(X_{i,j}) \frac{f(X_{i,j})}{p_i(X_{i,j})}. \quad (4.5)$$

Compared to the initial estimator from equation 4.2 we now have additional weight

functions $w_i(\cdot)$ and split up our samples over the k different sampling technique with $n = \sum_{i=1}^k n_i$. To keep the estimator unbiased the weighting function need to fulfill two constraints:

- $\sum_{i=1}^n w_i(x) = 1$ when $f(x) \neq 0$
- $w_i(x) = 0$ when $p_i(x) = 0$.

The first constraint makes sure that every point is neither reduced nor increased in its contribution which is important to keep the estimator unbiased. The second constraint states that a pdf that could not create this sample also must have no weight so it will be ignored. A concrete weighting function will be introduced in the next subsection.

4.5.1 Balance Heuristic

A very popular weighting function also introduced by Veach is the balance heuristic [Vea97, Chapter 9.2.2]. The formula to calculate it is as follows:

$$w_i(x) = \frac{n_i p_i(x)}{\sum_{j=1}^k n_j p_j(x)}. \quad (4.6)$$

So the weight for a sample consists of the amount of samples n_i drawn with that technique, the probability $p_i(x)$ for that sample divided by the sum of all techniques. **(show how it falls down to a simple standard monte carlo estimator when inserting into the formula (when more text is needed?))** ToDo

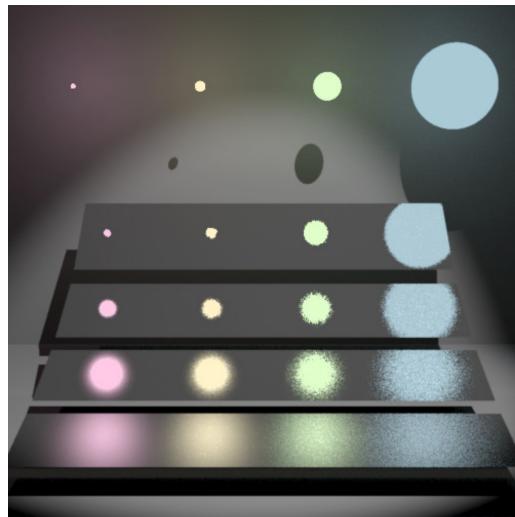


Figure 4.2: This image shows the same scene as in figure 4.1 but rendered using the balance heuristic. Taken from [Vea97, Figure 9.4].

Figure 4.2 show that with the balance heuristic the noisy areas we observed while only using a single pdf in figure 4.1 disappeared and all combinations of surface roughness and light source size are well represented. When we insert the weighting function from

equation 4.6 into equation 4.5 we get the following:

$$\begin{aligned}
F &= \sum_{i=1}^k \frac{1}{n_i} \sum_{j=1}^{n_i} \frac{n_i p_i(X_{i,j})}{\sum_{m=1}^k n_m p_m(X_{i,j})} \frac{f(X_{i,j})}{p_i(X_{i,j})} \\
&= \sum_{i=1}^k \sum_{j=1}^{n_i} \frac{f(X_{i,j})}{\sum_{m=1}^k n_m p_m(X_{i,j})} \\
&= \frac{1}{N} \sum_{i=1}^N \frac{f(X_i)}{\sum_{m=1}^k c_m p_m(X_i)} \\
&= \frac{1}{N} \sum_{i=1}^N \frac{f(X_i)}{p_{eff}(X_i)}.
\end{aligned}$$

The $1/N$ term comes from $n_m = c_m * N$. When we compare the last line with the initial estimator from equation 4.2 we see that they are identical with $p(x) = p_{eff}(x) = \sum_{m=1}^k c_m p_m(x)$. With that we can conclude that using MIS with the balance heuristic corresponds to a normal MIS estimator drawing samples from a combined sampling distribution [Vea97, Chapter 9.2.2.1].

5. Multiple Importance Sampling Compensation

Short intro with reference that all this is from the paper

mis balance heuristic too defensive -> oversampling of low valued and undersampling of high valued areas name comes from modifying one pdf directly to compensate the averaging effect in the balance heuristic

Set T of samplers, one is chosen (p_t) called free to be modified to create a pdf that minimized the combined variance with the balance heuristic

optimal solution has zero variance so p_{eff} should exactly match the integrand $p_{eff} = f/F$ for the balance heuristic $p_{eff} = q + ct * p_t$ with q all techniques except the free one + the free one that gives definition of p_t p_t is not necessarily a valid pdf has to be positive and integrate to one clamping and normalizing

sampling parts that other samplers cover well is wasted effort so the free pdf fills the missing part to create the perfect (at least better) pdf in regard to matching the integrand needs to know F beforehand, but can be approximated and will be shown still unbiased as for $f > 0$ p_t is not 0 when q is zero. can only be zero when q is non zero but then we still have a valid pdf as only p_{eff} has to be > 0 which is the case when q is not zero. The pdf we created so far doesn't guarantee to reduce variance because of the max operator and the re-normalization

in the next section we will derive a truly optimal pdf that provably does reduce variance in the one-sample estimator We will also show that our compensated free pdf is often close to being an optimal estimator.

5.1 Optimality

write variance as second moment - F^2 with second moment being formula and $p_{opt\dots}$ To avoid max and normalization we set two constraints... Formula from appendix to derive λ . Since we need the max there is no analytical solution to the optimal pdf so it needs to be calculated iteratively which makes it impractical If the mis-compensated solution (from above in square braces in the paper) is close-enough to the optimal solution, which is often the case that can be used instead

5.1.1 Bounds for the MIS-compensated solution

To show the similarity of the optimal solution and the MIS-compensated one we derive following bounds ... by ... (appendix B) With eq. 9 we can easily see that the optimal and the compensated solution are equal and the lower bound applies. (just input into 9, fulfill conditions) with this we can assume that the optimal solution is often not far from the compensated one. In case they are not the same we want to see how badly using the compensated pdf over the optimal one can increase variance the highest increase would be $1 / ct$ over the optimal solution. to get a concrete number for the introduced variance multiple scenes were tested with the worst result being 1.6 times

They showed that the compensated pdf is often similar to the optimal one. In case they are not the same it is expensive to calculate the optimal pdf since it involves iteratively evaluating the integral and checking if the conditions hold true the mis-compensated solution can be used because it is easy to calculate with an approximation of F and is still a good overall solution.

- ToDo** (is it okay to refer to the original paper for the derivation of the optimal solution (appendix prooofs))
- ToDo** (show how F will be approximated by λ . maybe no approach to approximation is shown)
- ToDo** (Explain their work. Formulas should be clearly explained.)

6. Application: Image Based Lighting

In this chapter we start by describing image based lighting shortly. Following that the approach of Karlík et al. is explained to use in image based lighting. (Explain the ToDo contents of this chapter (at the end))

6.1 Image Based Lighting

Creating realistic lighting for a scene can be very complex and time consuming. To avoid modelling a lighting environment a commonly used method is to create an image of a real world environment and using that for the direct illumination of a scene. This approach is called image based lighting and was introduced back in 1998 by Debevec in [Deb98].

To get the lighting of a scene in the real world you could place a mirror sphere in the middle of your scene and the a photo of that sphere ideally from far away so the camera itself doesn't take up a large part of the scene. The name of this method is sphere mapping. During rendering you place your scene in the center of that textured sphere and if a shadow ray doesn't hit any object of the scene you read out the texel on the sphere that corresponds to the direction of the ray.

A different method is to take six pictures of the scene from the top, bottom, left, right, front and back and then use the same evaluation method as with the sphere with the direction of the shadow ray direction to get the texel from the environment map. For more information on how to use environment maps/image based lighting consider [env19].

(Information about sampling the environment map)

ToDo

6.2 MIS Compensation in Image Based Lighting

(Explain how their approach improves ibl variance)

ToDo

(If space left show results)

ToDo

7. Application: Path Guiding

In this chapter we will give a short overview of what path guiding is. After that we will look at how Kerlík et al. [KŠV⁺19] applied MIS compensation on path guiding to further improve render quality.

7.1 Path Guiding

The idea behind importance sampling is to let the sampling density be proportional to the integrand to get a small as possible variance in the result or image in our use case. Normally we can use the BRDF and the direct illumination to create two sampling distributions, but one important part is the indirect illumination which is not known beforehand and therefore can't be used easily. That's exactly what path guiding is for. It learns the incident radiance and builds a distribution based on this. This distribution is then used to create a new image from scratch to create a better distribution until the final iterations yields the image after combining all iterations weighted by their inverse variance [VHH⁺19]. During rendering the distribution can be normally used in MIS.

7.2 MIS Compensation in Path Guiding

ToDo

(Explain how their approach improves path guiding)

8. Summary

(Write a short summary of what they achieved)

ToDo

8.1 Future Work

(Mention what could be done next based on this work)

ToDo

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Erklärung

Ich versichere, dass ich die Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe, die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht und die Satzung des KIT zur Sicherung guter wissenschaftlicher Praxis in der jeweils gültigen Fassung beachtet habe. Die Arbeit wurde in gleicher oder ähnlicher Form noch keiner anderen Prüfungsbehörde vorgelegt und von dieser als Teil einer Prüfungsleistung angenommen.

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