Answer the questions in the spaces provided on the exam. If you run out of room for an answer, continue on the back of the page.
Note that you must justify all answers!

Name: ______________________________

U#: ______________________________

Question	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. You have 75 minutes to complete the exam.
2. I recommend you use pencil on the exam, but blue or black ink are also acceptable.
3. You may NOT use calculators, cell phones or other electronic devices while taking the exam.
4. The exam is closed books and closed notes. If you turn in a completed feedback form at the start of class, you may use 1 page of prepared notes (single-sided).

1. [25 points] Consider the *shortest rook-path* problem. In this problem, we have some set of integer coordinates in 2D plane (at least 2), and we want to find the sequence of *legal moves* that minimizes the distance travelled from the first point (a) to the last (b), where a *legal moves* must be horizontal or vertical (of any length). Prove that the following algorithm does not find the shortest route between a and b. You may draw your counterexample or list the set of coordinates. You will receive partial credit if you can show an example with at least 4 points and describe the solution that the algorithm finds, even if it is not a counterexample.

Input: points: set of at least two 2D coordinates
Input: n: number of coordinates in points
Output: path: sequence of coordinates from points that describe the shortest legal rook-path from a to b

```
Algorithm: ShortPath
if n < 2 then
    error "Input must have at least 2 points!"
for i = 1 to n-1 do
    if i > 1 and points[i] has not been linked yet then
        continue;
    if points[n] is horizontal or vertical from points[i] then
        Add a link from points[n] to points[i];
        soln = (n);
        p = n;
        while p ≠ 1 do
            p = point linked to by points[p];
            Add p to the beginning of soln;
        end
        return soln;
    else
        for j = i+1 to n-1 do
            if points[j] is unlinked and either horizontal or vertical from points[i] then
                Add a link from points[j] to points[i];
        end
    end
end
error "There is no rook-path from a to b!"
```

2. [25 points] Use induction to prove the correctness of the following recursive algorithm to multiply two natural numbers, for all integer constants $c \geq 2$

Algorithm 1: `Multiply`(y, z)

```
/* Return the product yz */
if z == 0 then
    return 0
else
    return (Multiply(cy, ⌊z/c⌋) + y · (z mod c))
end
```

3. [25 points] Analyze the following algorithm to determine asymptotic upper bounds for its worst-case time complexity (i.e., $O(f(n))$). Make your bounds as tight as possible. You may assume that the `floor` function takes $O(1)$ time for any input.
Justify your answer. For full credit you must show all work. Explain (1) the derivation for each line of the algorithm below, (2) how you combine each derivation to represent the asymptotic worst-case time complexity of the algorithm, and (3) how you simplify the resulting expression.

```
void function(int x, int y)
   for (int i = x; i > 0; i = i - 2) {

      output "foobar"

   }

   int j = y

   while (j > 1) {

      output "foobar"

      j = floor(j / 2)

   }

   for (int k = y; k > 0; k--) {

      int m = 1

      while (m < x) {

         output "foobar"

         m = m * 2
      }
   }
}
```

Complexity

4. [25 points] Prove that $f_k(n) = O(\lg n)$ for all $k \geq 1$, where $f_k(n) = \lg(n^k)$.