Homework 4 sample solution

Due 09/10/15

September 3, 2015

1. Use the *formal definition* of Big-Oh to prove that if $f(n) = O(g(n))$, then $f(n) + g(n) = O(g(n))$.

   Answer:

   Proof. Since $f(n) = O(g(n))$, there exist positive constants c and n_0 such that $f(n) \leq cg(n)$ for all $n \geq n_0$. As such, $f(n) + g(n) \leq (c+1)g(n)$ for all $n \geq n_0$. Thus, there exist constants $c' = c + 1$ and n_0 such that $f(n) + g(n) \leq c'g(n)$ for all $n \geq n_0$, so $f(n) + g(n) = O(g(n))$. $\square$

2. Prove that if $f(n)$ is a polynomial of the form $\displaystyle\sum_{i=1}^{d} a_i n^{x_i}$, for some coefficients $a_1, a_2, \ldots, a_d$ and exponents $x_1, x_2, \ldots, x_d$, then $f(n) = \Theta(n^{\max\{x_1, x_2, \ldots, x_d\}})$. *Hint*: you may use any property of Big-Oh notation listed in the slides. You may wish to use induction for this problem.

   Answer:

   Proof. We prove the claim by induction on d.

   (*Base case*) When $d = 1$, $f(n) = a_1 n^{x_1}$. There exist constants $c_1 = c_2 = a_1$ and $n_0 = 1$ such that $c_1 n^{x_1} \leq f(n) \leq c_2 n^{x_1}$, so $f(n) = \Theta(n^{x_1})$.

   (*Inductive step*) Suppose that $f(n) = \Theta(n^{\max\{x_1, x_2, \ldots, x_d\}})$ for all polynomials $f(n)$ with d terms, and suppose that $f(n)$ is a polynomial with $k+1$ terms: $f(n) = \sum_{i=1}^{k+1} a_i n^{x_i}$. Note that $f(n) = a_{k+1} n^{x_{k+1}} + g(n)$, where $g(n) = \sum_{i=1}^{k} a_i n^{x_i}$. Since $g(n)$ has d terms, $g(n) = \Theta(n^{x'})$, where $x' = \max\{x_1, x_2, \ldots, x_k\}$, by the Inductive Hypothesis. There are two possibilities here: either $x' \geq x_{k+1}$ or $x' < x_{k+1}$. We consider these cases individually.

   (*Case 1*: $x' \geq x_{k+1}$) Since $x' \geq x_{k+1}$, $n^{x'} \leq n^{x_{k+1}}$ for all $n \geq 1$, so $n^{x_{k+1}} = O(n^{x'})$. Since $a_{k+1} n^{x_{k+1}} = \Theta(n^{x_{k+1}})$ and $n^{x_{k+1}} = O(n^{x'})$, $a_{k+1} n^{x_{k+1}} + g(n) = \Theta(g(n)) = \Theta(n^{x'})$. Also, $\max\{x_1, x_2, \ldots, x_{k+1}\} = x'$ since $x' = \max\{x_1, \ldots, x_k\}$ and $x' \geq x_{k+1}$.

   (*Case 2*: $x' < x_{k+1}$) Since $x' < x_{k+1}$, $n^{x_{k+1}} \geq n^{x'}$ for all $n \geq 1$, so $n^{x'} = O(n^{x_{k+1}})$. Since $g(n) = \Theta(n^{x'})$ and $g(n) = O(n^{x_{k+1}})$, $a_{k+1} n^{x_{k+1}} + g(n) =$

$\Theta(n^{x_{k+1}})$. Also, $\max\{x_1, \ldots, x_{k+1}\} = x_{k+1}$ since $x' = \max\{x_1, \ldots, x_k\}$ and $x_{k+1} > x'$.

Thus, in either case, $f(n) = \Theta(n^{\max\{x_1, x_2, \ldots, x_{k+1}\}})$. Therefore, the claim holds for all polynomials with 1 or more terms, by induction. $\square$

3. (*Bonus*) Prove that $2^n = \Omega(n^k)$ for all integers $k \geq 1$.

Answer:

Proof. First, we prove that $\lim_{n\to\infty} \frac{2^n}{n^k} = \infty$ for all $k \geq 1$ by induction.

(*Base case*) Consider $\lim_{n\to\infty} \frac{2^n}{n^1}$. Since both 2^n and n^1 approach ∞ as n approaches ∞, we can apply L'Hôpital's Rule:

$$\begin{aligned}
\lim_{n\to\infty} \frac{2^n}{n} &= \lim_{n\to\infty} \frac{\frac{d}{dn}2^n}{\frac{d}{dn}n} \\
&= \lim_{n\to\infty} \frac{2^n \ln 2}{1} \\
&= \infty
\end{aligned}$$

(*Inductive step*) Suppose that $\lim_{n\to\infty} \frac{2^n}{n^k} = \infty$, for some $k \geq 1$, and consider $\lim_{n\to\infty} \frac{2^n}{n^{k+1}}$. Since both 2^n and n^{k+1} approach ∞ as n approaches ∞, we can apply L'Hôpital's Rule:

$$\begin{aligned}
\lim_{n\to\infty} \frac{2^n}{n^{k+1}} &= \lim_{n\to\infty} \frac{\frac{d}{dn}2^n}{\frac{d}{dn}n^{k+1}} \\
&= \lim_{n\to\infty} \frac{2^n \ln 2}{(k+1)n^k} \\
&= \frac{\ln 2}{k+1} \lim_{n\to\infty} \frac{2^n}{n^k} \\
&= \infty
\end{aligned}$$

(Note that we applied the inductive hypothesis between the last two lines.) Therefore, $\lim_{n\to\infty} \frac{2^n}{n^k} = \infty$ for all $k \geq 1$, by induction. As a result, $2^n = \Omega(n^k)$, for all $k \geq 1$. $\square$