An Application of the Theory of Probabilities to the Study of a priori *Pathometry.*—Part II.

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CONTENTS.

Errata in Part I. ('Proceedings,' A, vol. 92.)

Page.

211, line 23, *write* $\mathrm{Z/P}$ *for* ZP.

213, equation 17, *omit* factor $(1-x)$.

equation 18, *omit* factor $(\mathrm{L}+1-2x)$.

line 23, *omit* "since $\frac{1}{2}(\mathrm{L}+1)$ is greater than L."

215, line 15, *write* $d\mathrm{Z}/dt$ *for* dz/dt.

line 21, *write* $-(\mathrm{V}-\mathrm{N}-r)/(\mathrm{K}-\mathrm{V})$ *for* $-(\mathrm{V}-\mathrm{N}+r)$.

217, equation 29, *write* $x = \mathrm{L}-(\mathrm{L}'-\mathrm{L})/(y-1)$.

218, line 22, *write* $\mathrm{V}-v$ *for* $v-\mathrm{V}$.

line 2 from bottom, *write* $\displaystyle\int\frac{2\mathrm{K}\beta\epsilon^{-2\mathrm{K}\beta t}}{y_0-\epsilon^{-2\mathrm{K}\beta t}}\,dt$.

219, equation 36, *write* $\epsilon^{-\mathrm{KL}t}$ *for* $\epsilon^{-2\mathrm{KL}t}$.

equation 38, *write* P_0 *for* P (twice).

equation 39, *write* ϵ^{-vt} *for* $\epsilon^{-\mathrm{KL}}$ and *for* $\epsilon^{-\mathrm{KL}'}$.

Page.

221, line 8, last term, *write* L^2 for L.

lines 4, 3 from bottom, *write* "seen that all the tangentials of x are zero."

222, lines 6, 5 from bottom *write* "$dx_0/dt = K(LP_0-1)/P_0^2$, or nearly KL/P_0."

223, line 10, *write* KL/P_0 *for* K/P_0.

225, equation 67, *write* $(1-L)$ *for* $(-L)$.

229, equation 80, *write* x/x_0 *for* L/x_0.

230, 3, *write* Part VI *for* Part IV.

Preface to Part II.

Part I of this paper was published in the 'Proceedings,' A, vol. 92 (1916), having been read on November 11, 1915. In June last, the Royal Society was kind enough to give a Government Grant for providing me with assistance in order to complete the paper, and for carrying on further studies upon the subject; and Miss Hilda P. Hudson, M.A., Sc.D., was appointed for the work from May 1, 1916. The continuation of the paper has accordingly been written in conjunction with her; and I should like to take the opportunity to express my obligations to her for her valuable assistance, especially in regard to Part III—which is to appear shortly.

I must apologise for the rather numerous small errors in Part I—due to the fact that the proofs were received by me when I was abroad on active service.

The entire paper here presented is still limited to the theoretical side of the subject, as defined in the Preface to Part I. Records of epidemics are now being examined in order to find how far the theoretical results which we have reached may be applied to them; but these studies must be reserved entirely for future discussion.—R. R.

VIII.

(i) *Hypothetical Epidemics: Deduction of Constants.*—In any case of independent happening we can use equations 8 and 12 with V, v, N, r, all zero. Then

$$\begin{aligned} dx/dt &= h(1-x), \qquad \text{and, if } x_0 = 0, \\ ht &= -\log_e(1-x), \\ &= x+\tfrac{1}{2}x^2+\tfrac{1}{3}x^3+\ldots \text{ if } x \text{ is small}, \\ &= 2{\cdot}30 \log_{10}\frac{1}{1-x} \text{ for all values of } x. \end{aligned}$$

If the value of x is known for any value of t, say a year, this equation enables us to find h, and thence x for any other period, hour, day, week, etc.

Thus suppose that child-birth, death, or migration occurs to 20 *per mille* of a population of all ages during one year, to what proportion do they occur in one day? We have

$$365h = 0{\cdot}02 + 0{\cdot}0002 + 0{\cdot}0000027 + \ldots,$$
$$h = 0{\cdot}000055.$$

To find the proportion for one hour, we divide by 24; and to find it for 7 or 30 days, we multiply by these numbers respectively.

Or suppose that 85 per cent. of plague cases, taken at any stage of the illness, die within 30 days, and we require the daily death-rate of such mixed cases. Then we have

$$30h = -2{\cdot}30 \log_{10}(1 - 0{\cdot}85),$$
$$h = 0{\cdot}063,$$

giving a constant death-rate of 6·3 per cent. *per diem*. If the cases are all taken from the commencement of the illness, the curves are quite different, and show the highest death-rate some days after the onset—but this does not concern us at present.

If the annual birth-rate is 30 *per mille*, and the annual death-rate is 20 *per mille*, then the corresponding daily rates are

$$n = 0{\cdot}000083 \quad \text{and} \quad m = 0{\cdot}000055.$$

Hence if the immigration and emigration rates counterbalance each other we may take for the daily variation element $v = n - m = 0{\cdot}000028$.

In this example the increase of the population at the end of the year would amount to 1 per cent.—that is, P would vary from P_0 at the beginning of the year to $1{\cdot}01\,P_0$ at the end of it. But if we use the value of v just given for equation 13, namely, $P = P_0 e^{vt}$ (putting $t = 365$ and $vt = 0{\cdot}0103$), we obtain $P = 1{\cdot}0103\,P_0$ at the end of the year—which is a little greater than the figure just given. The reason for this is, of course, that in equation 13 we assume that the propagation is continuous—that is, that during the year the progeny of the original population will themselves have progeny. But this is not assumed in the value of P first obtained.

We may also calculate v from equation 13 if we know the values of P_0 and P at the beginning and end of t. Thus the population of England and Wales was 10,164,256 in 1811, and 36,070,492 in 1911. Then since $vt = \log_\epsilon P - \log_\epsilon P_0$ and $t = 365 \times 100$, the daily variation-element is $v = 0{\cdot}0000347$, the yearly variation-element is 0·012666, and the ten-yearly variation-element 0·12666. If a population doubles itself in a period t, we have $vt = \log_\epsilon 2 = 0{\cdot}693$, so that at the above rate the English population ought to double itself in 19,975 days, or nearly 55 years—which is about the case.

If the daily death-rate of plague is taken at the figure just calculated, of 0·063, and the plague birth-rate is a normal one of 30 *per mille* a year (so that $\mathrm{N} = n$), then $\mathrm{V} = \mathrm{N}-\mathrm{M} = -0{\cdot}063$; so that, if $v = 0{\cdot}000028$, $v-\mathrm{V} = 0{\cdot}063\ldots$ daily. Such figures give us some concrete ideas of what the constants are likely to be in cases of human epidemics. [Compare Sections III and V (iii).]

The reversion-element r is particularly elusive since we seldom have known figures regarding it to deal with. It is best to proceed as we did in connection with equation 20, and to assume that 90 per cent. of the affected individuals revert in the time t, where $t = 1, 2, 3, \ldots, 10$, or more years.

Then for a daily element $365r = \frac{1}{t}\, 2{\cdot}30$. Hence if 90 per cent. of the cases revert in one year, the daily reversion-element is $r = 0{\cdot}00631$. Thus the reversion-element is likely to be large compared with v or N and correspondingly important, and will be by no means negligible even when the considered disease is one in which the acquired immunity is supposed to be very lasting [compare Section III (iii)]. For a case reverts not only when it becomes again capable of showing a recognisable infection, but when it is again able to harbour the infective agents sufficiently to afford them a *nidus* whence they may infect others; and it is quite possible and indeed probable that this may often occur much earlier than we imagine. Thus persons vaccinated against smallpox may acquire the disease in a very mild and modified form after only a few years, and may then infect non-protected persons in full force; and there is no proof that a mild first attack of measles or other diseases may not give quite as short-lived an immunity—that is, that those who have had the disease in infancy may not comparatively early acquire it again in an indistinguishable form and then spread it. In such cases, r may be considerably larger than we might otherwise expect.

Regarding immigration and emigration, it will suffice to note that in 1911 there were 350,429 immigrants into the United Kingdom and 454,527 emigrants from it.

Further remarks on the constants will be found in Section X.

(ii) We may now proceed to consider numerical examples of proportional happenings. For this purpose it is convenient to adopt the suggestion made towards the end of Section VII (vi), and to take c as a function of an independent parameter γ which is such that when γ is positive KL is positive and x always increases with the time, and when γ is negative KL is negative and x always diminishes with the time (Section IX). We shall also write D for $v-\mathrm{V}$ and R for $\mathrm{N}+r$. The principal constants of x and f may then be written

$$c = \mathrm{D}+(\gamma+1)\,\mathrm{R}, \qquad \mathrm{K} = (\gamma+1)\,\mathrm{R}, \qquad \mathrm{KL} = \gamma\mathrm{R},$$

$$\mathrm{L} = \frac{\gamma}{\gamma+1}, \qquad l = c\frac{\gamma}{(\gamma+1)^2} \qquad \lambda = \frac{(\gamma+1)^2}{4\gamma}.$$

The reader is reminded that KL is the coefficient of l in x, the proportion of affected individuals (equation 50), and that the x-curve reaches its centre of symmetry when $x = \frac{1}{2}\mathrm{L}$, and its maximum value L when t is very large [Section VII (iii)]. The period at which x reaches its centre of symmetry and dx/dt reaches its maximum is $t = \tau = \frac{1}{\mathrm{KL}}\log_\epsilon\frac{\mathrm{L}-x_0}{x_0}$. If $x = 1/\mathrm{P}_0$ and LP_0 is large compared with unity, then, as suggested in connection with equation 58, we may now write

$$\gamma\mathrm{R}\tau = \log_\epsilon \mathrm{P}_0+\log_\epsilon\gamma-\log_\epsilon(\gamma+1),$$

and at twice this period the maximum number of affected individuals less one is reached.

Similarly, when t is very large, the ultimate value of the proportion of new cases is l. But if $\mathrm{L} > \frac{1}{2}$, f rises to a previous maximum value $\frac{1}{4}c$ when $x = \frac{1}{2}$, and thereafter falls to its ultimate value l [Section VII (v)]. The ratio between the maximum and the ultimate value is λ (equation 67). The period at which the maximum value is reached is

$$t = \tau' = \frac{1}{\mathrm{KL}}\log_\epsilon\frac{1}{x_0}\frac{\mathrm{L}-x_0}{2\mathrm{L}-1}$$

(equation 66); and when $x_0 = 1/\mathrm{P}_0$ and LP_0 is large, this may be written

$$\gamma\mathrm{R}\tau' = \log_\epsilon \mathrm{P}_0+\log_\epsilon\gamma-\log_\epsilon(\gamma-1).$$

But this maximum of f (previous to its ultimate value) occurs only if $\gamma > 1$.

It is evident that when c is expressed as a function of γ in this manner, K, KL, L, τ, τ', and λ are independent of v and V, and that l contains them only as parameters of c.

On the other hand, τ and τ' are functions of the original population P_0 and increase as it increases—because, obviously, x and f require a longer period to reach their corresponding values if the population is large. It is convenient therefore to adopt a definite figure for P_0 in the following Table; and we put $\mathrm{P}_0 = 10{,}000$, which is, say, the population of an ordinary town. We may also assume that unity is negligible compared with LP_0 unless $\gamma < 0{\cdot}1$

If P_0 is not 10,000, but some multiple or fraction of it, say 10,000 p, all

Table of Hypothetical Epidemics.

Ex. No.	γ.	L.	$\frac{l}{c}$.	λ.	Reversion rate 90 per cent. per annum. R = 0 ·0064.				Reversion rate negligible. R = 0 ·00083.			
					c.	τ.	τ'.	lP_0.	c.	τ.	τ'.	lP_0.
						years.	days.			years.	years.	
1	0 ·001	$\frac{1}{1001}$	0 ·000998	—	0 ·0064	940	—	0 ·064	0 ·00008	72,100	—	0 ·00083
2	0 ·01	$\frac{1}{101}$	0 ·00980	—	0 ·0065	196	—	0 ·64	0 ·00008	15,100	—	0 ·0082
3	0 ·1	$\frac{1}{11}$	0 ·0826	—	0 ·0070	29	—	5 ·8	0 ·00009	2,200	—	0 ·075
4	1	$\frac{1}{2}$	$\frac{1}{4}$	1	0 ·0128	3 ·7	—	32	0 ·00017	280	—	0 ·41
5	2	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{9}{8}$	0 ·0192	—	774	43	0 ·00025	—	163	0 ·55
6	3	$\frac{3}{4}$	$\frac{3}{16}$	$\frac{4}{3}$	0 ·0256	—	501	48	0 ·00033	—	105	0 ·62
7	4	$\frac{4}{5}$	$\frac{4}{25}$	$\frac{25}{16}$	0 ·0320	—	371	51	0 ·00042	—	78	0 ·66
8	5	$\frac{5}{6}$	$\frac{5}{36}$	$\frac{9}{5}$	0 ·0384	—	295	53	0 ·00050	—	62	0 ·69
9	6	$\frac{6}{7}$	$\frac{6}{49}$	$\frac{49}{24}$	0 ·0448	—	245	55	0 ·00058	—	51	0 ·71
10	7	$\frac{7}{8}$	$\frac{7}{64}$	$\frac{16}{7}$	0 ·0512	—	209	56	0 ·00067	—	44	0 ·73
11	8	$\frac{8}{9}$	$\frac{8}{81}$	$\frac{81}{32}$	0 ·0576	—	183	57	0 ·00075	—	38	0 ·74
12	9	$\frac{9}{10}$	$\frac{9}{100}$	$\frac{25}{9}$	0 ·0640	—	162	58	0 ·00083	—	34	0 ·75
13	10	$\frac{10}{11}$	$\frac{10}{121}$	3 ·03	0 ·0704	—	146	58	0 ·00092	—	31	0 ·76
14	100	$\frac{100}{101}$	$\frac{100}{101^2}$	5 ·5	0 ·646	—	14 ·4	63	0 ·00843	—	3	0 ·82
15	1,000	$\frac{1000}{1001}$	$\frac{1000}{1001^2}$	251	6 ·41	—	1 ·44	64	0 ·0835	—	days. 110	0 ·83
16	10,000	$\frac{10000}{10001}$	$\frac{10000}{10001^2}$	2500	64 ·1	—	0 ·14	64	0 ·835	—	11	0 ·83

we have to do to find τ and τ' is to add to or subtract from the figures given in the corresponding columns of the Table the quantity $\log_\epsilon p/\gamma R$. Thus, if $P_0 = 1000$, we subtract $2{\cdot}30/\gamma R$. As γ increases above 100, τ and τ' approximate more and more closely to the value $\log_\epsilon P_0/\gamma R$, τ being always less than this and τ' greater than it.

The actual curves depend largely on the values of R. If we suppose the case of a human disease, of which the affected birth-rate N equals the natural birth-rate n, which is taken at 0·000083 daily, corresponding, according to the previous sub-section, with an annual birth-rate of 30 *per mille*—then R equals the former figure if r is so small as to be negligible. But if r is a daily rate corresponding to an annual reversion of 90 per cent., it will amount to as much as 0·0063, so that $R = 0{\cdot}0064$. We will suppose that these two values of R are a minimum and a maximum.

Now if we suppose also that $c = 1$, that is, that each affected individual daily infects or reinfects one other individual, then we have $\gamma = (1-D)/R-1$. That is, if $D = 0$, $\gamma = 12{,}000$ if R is a minimum, and $\gamma = 155{\cdot}5$ (or roughly 150) if R is a maximum. Thus, if the infection rate is near unity, the value of γ will reach from three to five figures, and the epidemic will resemble the last examples given in the Table.

In Example 1, if the population remains the same during the whole of this long period (which is, of course, never likely to occur), only about ten living persons out of the 10,000 will be found unaffected at the end of it; and the ultimate and maximum daily number of new cases lP_0 reaches only 0·00083, that is, only one new case in 3·3 years.

If, however, R has its maximum value, there is one new case in about sixteen days.

In Example 4, half the population is ultimately affected, but still f has no maximum before its ultimate value. When, however, in Example 5 γ increases above unity, this maximum begins to appear, and the ratio λ between the maximum and ultimate values of f constantly increases with γ. At the same time, τ and τ' diminish and approach each other.

If c is nearly unity, $\tau' = 2{\cdot}30 \log_{10} P_0 = 9{\cdot}2$ days, and f reaches its maximum in a short time, whatever the value of R. More generally, if $D = 0$, $c = (\gamma+1)R$, and if γ is large (say over 50), $\gamma+1 = \gamma$ roughly, and $\tau = 1/c \,.\, \log_\epsilon P_0$. In other words, if γ is considerable, not only does $L = 1$ nearly, but the maximum $(\frac{1}{4}c)$ of f and the mode, or time (τ') when that maximum is attained, depend roughly on c only, the population being fixed.

If $L > \frac{1}{2}$ the f-curve slopes downward from its summit more slowly than it rose towards that summit. Put $x = \frac{1}{2} + \xi$, then $f = c\,(\frac{1}{4} - \xi^2)$, and has the same value for equal and opposite values of ξ.

But $\left|\frac{df}{dt}\right| = 2c \left|\xi\right| \frac{d\xi}{dt}; \quad \frac{d\xi}{dt} = \frac{dx}{dt} = \frac{\mathrm{K}}{c} f - \mathrm{K}(1-\mathrm{L})(1+\xi),$

and is less when ξ is positive than when ξ has the equal and opposite value. Hence, at two points of the f-curve whose ordinates are equal, the descending side is less steep than the ascending side.

Since $f = \frac{c}{\mathrm{K}} \frac{dx}{dt} + c(1-\mathrm{L})x$, and the first term on the right gives a perfectly symmetrical bell-shaped curve, we may call $c(1-\mathrm{L})x$ the *excess* of the f-curve. When $\mathrm{D} = 0$ and γ is considerable, the excess varies roughly from $\frac{1}{2}\mathrm{R}$ at the centre of symmetry to R, when t is very large—that is, roughly, from $\frac{1}{2}l$ to l. Both of these become small when r, the reversion-rate, is small.

The important result, therefore, follows that, if also c is near unity, and therefore γ is large—that is, if the infection-rate is high while the reversion-rate is low—then the f-curve becomes a nearly symmetrical bell-shaped curve. This, according to Dr. Brownlee (Section I), is just the kind of curve presented by epidemics of certain zymotic diseases. The *prima facie* inference is therefore that such epidemics may be wholly or chiefly mere cases of proportional happening, as defined in Section VII (i). It is even the case that, in such diseases, the fall of the curve is often more slow than its rise, as happens with the f-curve.

When γ is large and $\mathrm{L} = 1$ nearly, the two changes of curvature of the curve occur, according to equation 65, when $6x = 3 \mp \sqrt{3}$, that is, when (equation 53)

$$\begin{aligned} \gamma \mathrm{R} t &= \log_\epsilon \mathrm{P}_0 + \log_\epsilon (1/x - 1), \\ &= \log_\epsilon \mathrm{P}_0 - 1{\cdot}32 \quad \text{and} \quad \log_\epsilon \mathrm{P}_0 + 1{\cdot}03, \end{aligned}$$

the latter being nearer to the summit than the former.

(iii) Several methods may be employed for calculating the curves of x, f, Z and F in detail. The simplest is to divide the period during which x varies from x_0 to $\frac{1}{2}\mathrm{L}$ into a number (say 10) of equal parts and then to calculate the ordinate of the curve at each section. In equation 58 we used τ to express this interval when $x_0 = 1/\mathrm{P}_0$, but it may be employed more generally to express the abscissa of the centre of symmetry, when x_0 has any value, so that $\mathrm{KL}\tau = \log_\epsilon(\mathrm{L}/x_0 - 1)$. We now put $t = \tau\mathrm{T}/10$, give to T the successive values 1, 2, 3, ..., 10, and calculate the corresponding values of x, namely $x_1, x_2, x_3, \ldots$, and so on. Thus we have from equations 50

$$x_{\mathrm{T}} = \mathrm{L}\{1 + \sqrt[10]{(\mathrm{L}/x_0 - 1)^{10-\mathrm{T}}}\}^{-1}.$$

The root can be easily evaluated by means of logarithms, and the value of

x then obtained from a table of reciprocals (*e.g.*, as in Barlow's Tables). Owing to the symmetry of the x-curve, its values when T lies between 10 and 20 can be easily found—for example $x_{13} = L - x_7$. When we have ascertained the ordinates of x, we can quickly calculate those of f from equation 61, and, unless the case-mortality is high, we can generally assume for short and sharp epidemics that the original population remains constant and that $Z = xP_0$ and $F = fP_0$. It will be observed that, by this method, when τ has been first calculated we need subsequently deal only with the values of L and of $L/x_0 - 1$. Thus suppose that $L = 1$ nearly, that $P = 10{,}000$, and that $x_0 = 1/P_0$, then we have approximately,

$T = 0$	1	2	3	4	5	6	7	8	9	10
$Z = 1$	2·5	6·3	15·8	39·7	99	245	593	1368	2847	5000
$F/c = 1$	2·5	6·3	15·8	39·6	98	239	558	1181	2037	2500

$T =$	11	12	13	14	15	16	17	18	19	20
$Z =$	7153	8632	9407	9755	9901	9960	9984	9994	9997	9999
$F/c =$	2037	1181	558	239	98	39·6	15·8	6·3	2·5	1

This is, of course, an ultimate case where f is quite symmetrical—so that, for example $f_7 = f_{13} = cx_7x_{13}$.

Another useful method for calculating x and f is to transfer the origin to the centre of symmetry (equation 56). But if the ordinates are required for successive natural units of time, such as days or weeks, in order to compare them with statistics of epidemics in which such units have been used, then we must of course give to t the successive values 1, 2, 3, ..., and calculate the ordinates directly from equations 50. Here the symmetry of x will not be so useful, because the centre of symmetry is not likely to coincide with any one of the periods of time for which the ordinates have been found. It will, therefore, be generally easier to plot the curve by the method first indicated, to measure off the natural units of time along the time-axis, and then to estimate the corresponding ordinates geometrically.

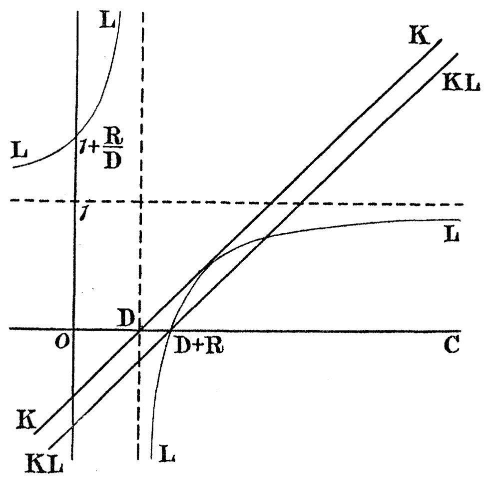
IX.

(i) *Hypometric Happening:* KL *Negative.*—We have hitherto always supposed that the happening-element is constant during the whole of the considered period, but it is now necessary to examine the changes which will be produced in the functions by certain changes in this element. We here study only the cases of dependent happenings already discussed, and these only for hypometric happening. It is no longer very useful to take c as a function of γ.

In Section VII (vi), while considering the constants K, L, c, we saw from equations 50 that if KL is positive, x always increases and is therefore always greater than x_0, its value when $t = 0$. This implies that $c > \mathrm{D}+\mathrm{R}$. When this inequality becomes an equality, KL vanishes and $dx_0/dt = 0$, that is, x remains near its original value, when t is not very large, so that the happening may be said to be *isometric* [see (v) below]. When, however, $c < \mathrm{D}+\mathrm{R}$, KL becomes negative; and it is seen from equations 50 that, as $\epsilon^{-\mathrm{KL}t}$ then always increases with the time, x always diminishes, and must therefore be always less than x_0.

We call such cases *hypometric happening*. That is, we now suppose that, after having been greater than $\mathrm{D}+\mathrm{R}$, c becomes less than it, or even zero. What, then, will be the nature of the curves?

The solutions of equations 44, 45, 47, 51, 52 remain valid for all real values of c, and we have therefore only to interpret results already known.



It is first necessary to examine K, KL, and L more particularly as functions of c, but we shall do so only in the case when D is positive (injurious happenings) or zero. Then D, R are positive constants. We have

$$\mathrm{K} = c-\mathrm{D}, \qquad \mathrm{KL} = c-\mathrm{D}-\mathrm{R},$$

represented by parallel straight lines meeting the axis of c at an angle of 45° on the positive side of the origin,

$$\mathrm{L} = 1-\mathrm{R}/(c-\mathrm{D}), \quad \text{or} \quad (\mathrm{L}-1)(c-\mathrm{D}) = -\mathrm{R},$$

and the graph of L is a rectangular hyperbola, with asymptotes $\mathrm{L} = 1$, $c = \mathrm{D}$, dotted lines in the figure, parallel to the axes of co-ordinates.

The last portion of this L curve, for $c > \mathrm{D}+\mathrm{R}$, gives cases of hypermetric happening already studied; but we have to consider the range $0 \leqq c < \mathrm{D}+\mathrm{R}$; since L becomes infinite when $c = \mathrm{D}$, we have to consider the two portions:

$0 \leqq c < \mathrm{D}$, for which L is positive and greater than 1; and $\mathrm{D} < c < \mathrm{D}+\mathrm{R}$, for which L is negative.

(ii) First let $c = 0$; that is, suppose that the happening suddenly ceases altogether, while all the other elements remain the same. It is advisable to use marked letters for the quantities after this event. We take it then that when the original happening c ceases, $t' = 0$, and that the values which x, Z, P, had reached up to this time are now expressed by x_0', Z_0', P_0', while x', Z', P' denote these functions as t' increases. Similarly, K' and L' are now the values of K and L without the element c. Then, since $\mathrm{K}' = -\mathrm{D}$, equations 51 and 52 become

$$\mathrm{P}' = \mathrm{P}_0' \epsilon^{vt'} \left\{ 1 - \frac{x_0'}{\mathrm{L}'} (1 - \epsilon^{-(\mathrm{D}+\mathrm{R})t'}) \right\}, \tag{81}$$

$$\mathrm{Z}' = \mathrm{Z}_0' \epsilon^{(\mathrm{V}-\mathrm{R})t'}. \tag{82}$$

As $\mathrm{V}-\mathrm{R} = -\mathrm{M}+\mathrm{I}-\mathrm{E}-r$, Z' decreases from its original value Z_0' or Z, and ultimately approaches zero—unless the immigration is greater than the sum of the other elements (as may happen in a war area)—and its diminution is more rapid when M, E, r are large. With regard to equation 81, we observe that $\mathrm{P}_0'\epsilon^{vt'}$ is the natural increase of the population, living when the happening ceased, after that event; and the equation shows that P' equals this quantity less a function of t' which is zero when $t' = 0$, which increases with t', and which finally slowly approaches the limit $x_0'/\mathrm{L}' \,.\, \mathrm{P}_0'\epsilon^{vt'}$ when t' is large. That is to say, the population which remained when the happening ceased continues to suffer from its effects for some time afterwards, and then finally reaches the value

$$\mathrm{P}' = \mathrm{P}_0' \epsilon^{vt'} (1 - x_0'/\mathrm{L}').$$

Here x_0'/L' is always less than unity, for x_0' is the value of x at the moment when the happening ceased, and this could not have been greater than L, which is less than unity [Section VII (vi)], while, as shown in the previous sub-section, L' is greater than unity when $c = 0$. Hence P' is always less than $\mathrm{P}_0'\epsilon^{vt'}$, and the difference gives the loss of population due to the cases which continue after the happening ceased.

The fraction L'/x_0' occurs in the denominator of x' in equation 50. As it is greater than unity, the coefficient of $\epsilon^{-\mathrm{K}'\mathrm{L}'t'}$ is positive, so that x' always diminishes as t' increases—as could have been already inferred from equations 81 and 82.

(iii) The next case is when c' is some quantity between zero and D (which is generally very small). For the study of this case, the first of equations 50 may be written

$$x' = x_0' \left\{ \frac{x_0'}{\mathrm{L}'} + \left(1 - \frac{x_0'}{\mathrm{L}'}\right) \epsilon^{-\mathrm{K}'\mathrm{L}'t'} \right\}^{-1}, \tag{83}$$

which diminishes indefinitely as t' increases. If $c' = \mathrm{D}$ and L' is consequently infinite, this becomes $x_0\epsilon^{\mathrm{K'L'}t'}$, which vanishes.

The values of P' and Z' are easily obtained from equations 51 and 52, if we remember that the indices D/K' and c'/K' are now negative. When $c' = \mathrm{D}$, the expressions for P' and Z' become indeterminate, but can be easily evaluated, and we have

$$\mathrm{P}' = \mathrm{P}_0\,\epsilon^{vt'-q}, \quad \text{where} \quad q = x_0'\,\mathrm{D}(1-\epsilon^{-\mathrm{R}t'})/\mathrm{R}.$$

(iv) When c' lies between D and D+R, K' is positive and L' is negative, and varies from $-\infty$ to zero. Then it will be seen from equation 83 that x' is still positive, and, as in the previous cases, diminishes indefinitely as t' increases. For it may be written

$$x' = x_0'\left\{\epsilon^{-\mathrm{K'L'}t'} - \frac{x_0'}{\mathrm{L}'})\epsilon^{-\mathrm{K'L'T'}} - 1)\right\}^{-1} \tag{84}$$

where $\mathrm{K'L'}$ is negative. For P' we have

$$\mathrm{P}' = \mathrm{P}_0'\,\epsilon^{vt'}\left\{1 - \frac{x_0'}{\mathrm{L}'}(1-\epsilon^{\mathrm{K'L'}t'})\right\}^{-\mathrm{D}/\mathrm{K}'}, \tag{85}$$

which is also always positive since L' is negative, and increases without limit when t' is large. P' is less than $\mathrm{P}_0'\epsilon^{vt'}$, because the effects of the original happening continue to be felt by the population after the happening itself has been reduced from c to c'.

(v) When $c' = \mathrm{D}+\mathrm{R}$ exactly, L' and $\mathrm{KL}' = 0$, and, when t' is small, $x = x_0'$ nearly, and the happening may be said to be *isometric*. The values of x' and P' become indeterminate, but can be ascertained by finding the limits when $\mathrm{L}' = 0$ of the expressions in equations 84 and 85, so that

$$x' = \frac{x_0'}{1+x_0'\mathrm{K}'t'},$$

$$\mathrm{P}' = \mathrm{P}_0'\,\epsilon^{vt'}(1+x_0'\mathrm{K}'t')^{-\mathrm{D}/\mathrm{K}'}.$$

Or the same equations can be obtained by integrating equation 47 after putting $\mathrm{L} = 0$ in it, and then integrating both sides of equation 44. In this case also, then, x' tends to decrease as t' increases, and finally approaches zero, when t' is very large.

(vi) If $\mathrm{K'L'}$ is positive, however small it may be, x' always increases, as already seen in Section VII (iii).

It must be remembered that, though x', the proportion of affected individuals, diminishes, this does not mean that new cases cease to occur. On the contrary, unless c is absolute zero, f' is the same function of x' as before. And rx' expresses, as before, the reversions of the old cases.

When $\mathrm{D} = 0$, the phase described in (iii) above is suppressed, and we begin, when $c' = 0$, at the point when $K' = 0$, $K'L' = -R$, and $L' = -\infty$, and the functions will be easily understood.

X.

(i) *Parameter Analysis.*—We have now considered x, f, and P as functions of the time, but it remains to examine how they vary if we give different values to the parameters h, r, v, V, n, m, i, e, N, M, I, E. The symbol l has been used to denote the limit of f when t is very large. This is an important quantity in disease happenings because it gives the number of new cases which continue to occur when x has reached its limit, that is, after epidemic manifestations have ceased. It is, by hypothesis, these final new cases which keep the infection alive permanently in the population; we may call l the *endemic ratio*. First consider l as a function of r

$$\frac{\partial l}{\partial r} = \frac{c}{K}(2L-1) = \frac{c}{K^2}(c-D-2R).$$

Thus l is positive when $r = 0$ (if $KL > 0$) and increases with r until this reaches a value which makes $L = \frac{1}{2}$ and $l = \frac{1}{4}c$ [Section VII (vi)]. After this l diminishes as r increases. But if $c-D > 2R$ then r never reaches this value and l always increases. Thus l considered as a function of r has forms similar to those of f considered as a function of t.

The behaviour of l as a function of c is of the same nature. A similar procedure applies to f and P.

The effect upon the total population P due to variations in the parameters is important. The fundamental equation is 44:

$$dP/dt = vP - DxP.$$

As P and x are always positive, the sign of dP/dt depends on v and D. If $D = 0$, the happening is equivariant and the natural change of population due to v is not affected. If D is negative, the happening is beneficial and the natural change of population is augmented by it. If D is positive, the happening is injurious and the natural change is reduced. If $v-Dx$ is negative, especially when $x = L$, the total population will diminish indefinitely.

It is found that $\partial P/\partial r$ is always positive; that is, in infectious diseases, a quicker loss of affectedness, including loss of immunity, is beneficial. This may seem surprising until we reflect that, as $v > V$, the longer the affectedness continues in the individual the greater will be the loss of life due to it, since we generally suppose in such cases that $M > m$ and that these elements act during, not one, but every time-unit lived by the affected part of the population. Of course, the greater r is the more quickly on the average will affectedness, including immunity, be lost.

Next, $\partial P/\partial c$ is zero or negative, and the higher the infection rate the more injurious is the happening.

$\partial P/\partial V$ is zero when $t = 0$. It generally increases at first with t and afterwards decreases; it is negative when $x = L$ provided $1 - c(1-L)/K$ is negative. In other words, in these cases an increase of V is beneficial at the beginning of the outbreak, but injurious later on. Now $V = N - M + I - E$ and diminishes if the case-mortality M increases; in fact $\partial P/\partial M = -\partial P/\partial V$. Thus an increase in the case-mortality is injurious at first, but may be ultimately *beneficial*, because an affected individual who dies ceases to be infective.

An Application of the Theory of Probabilities to the Study of a priori *Pathometry.*—Part III.

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CONTENTS.

XI.

(i) *Variable Happening.*—In Section IX, we commenced by touching upon the necessity of studying the effect of changes in the happening-element, but there dealt only with the case of hypometric happening. We now proceed to examine other changes in this element.