0.1 Clover Action

The process of discretizing QCD onto a grid introduces errors, relative to the continuum action, which are polynomial in the lattice spacing, a. The process of systematically removing these effects, order by order in a, is called *improvement*.

For orientation it is useful to consider a finite difference derivative of some smooth function f. The forward difference derivative is defined as $f'(x) = \frac{1}{h}\left[f(x+h) - f(x)\right]$ and has an $\mathcal{O}(h)$ error relative to the continuous derivative. Another, slightly better, definition of a discretized derivative is the central difference derivative, $f'(x) = \frac{1}{2h}\left[f(x+h) - f(x-h)\right]$. Here the error is $\mathcal{O}(h^2)$. This captures the essence of improvement. Simply put, we want to remove discretization error up to some order in the lattice spacing.

On the lattice we will mainly be interested in improving the Euclidean action,

$$\bar{\Psi}\left(m_0 + \not{D}\right)\Psi. \tag{1}$$

From an *effective field theory* approach, improvement amounts to adding *irrelevant*[1] operators to the action multiplied by powers of the spacing and coefficients chosen to cancel the discretization artifacts. Since these operators are multiplied by powers of a they disappear as the continuum limit is taken ($a \to 0$).

The lattice action we use [?], can be obtained by using the field transformation

$$\begin{aligned}
\Psi &= \left(1 + \frac{1}{2}\Omega_m a_t m + \frac{1}{2}\Omega_t a_t \gamma_4 \overrightarrow{D}_4 + \frac{1}{2}\Omega_s a_s \gamma_j \overrightarrow{D}_j\right)\psi \\
\bar{\Psi} &= \bar{\psi}\left(1 + \frac{1}{2}\bar{\Omega}_m a_t m + \frac{1}{2}\bar{\Omega}_t a_t \gamma_4 \overleftarrow{D}_4 + \frac{1}{2}\bar{\Omega}_s a_s \gamma_j \overleftarrow{D}_j\right),
\end{aligned} \tag{2}$$

where [?] describes a method for non-perturbative tuning of the improvement parameters, $\Omega_{m,t,s}, \bar{\Omega}_{m,t,s}$. Here $a_{t,s}$ are the lattice spacings in the temporal and spatial directions which owing to our anisotropic formulation are not the same.

[1] In the context of QCD in four dimensions irrelevant operators are those which have mass dimension greater than four. $\bar{\psi}\sigma_{\mu\nu}F^{\mu\nu}\psi$ is an example of a dimension five operator while $(\bar{\psi}\psi)^2$ is a local dimension six operator. Dimension five operators can be used to eliminate $\mathcal{O}(a)$ effects, dimension six $\mathcal{O}(a^2)$. We will only concern ourselves with $\mathcal{O}(a)$ improvement.