(14) Show that the series $1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}++-\cdots$ divergent.

$$X = 1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}\cdots$$

$$= 1+\left(\frac{1}{2}-\frac{1}{3}\right)+\frac{1}{4}+\left(\frac{1}{5}-\frac{1}{6}\right)+\frac{1}{7}$$

$$> 1+\frac{1}{4}+\frac{1}{7}+\frac{1}{10}$$

$$> 1+\frac{1}{6}+\frac{1}{9}+\frac{1}{12} = 1+\frac{1}{3}\sum_{n=2}^{\infty}\frac{1}{n}$$

the given serios is bigger than $\frac{1}{n}$. $\frac{1}{n}$ is divergent, so, $X = 1+\frac{1}{2}-\frac{1}{3}\cdots$ is also divergent.

(2) If S_n is the nth partial sum of the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} z_n$, & if s denotes the sum of this series, show that $|s - S_n| \leq z_{n+1}$.

Let S_n be the partial sum of the alternating series $a_n = \sum_{n=1}^{\infty} (-1)^{n+1} z_n$ which means that:

$$S_n = z_1 - z_2 + z_3 - \dots (-1)^{n+1} z_n.$$

s denote the sum of given series, which means; $s = z_1 - z_2 + z_3 - \dots (-1)^{n+1} z_n + \dots \infty$

we have:

$$|s - S_n| = |(z_1 - z_2 + z_3 \dots (-1)^{n+1} z_n)(z_1 - z_2 + z_3 - \dots (-1)^{n+1} z_n + \dots \infty)|$$

$$= |z_{n+1} - z_{n+2} + z_{n+3} - \dots \infty|$$

$$= |z_{n+1} - (z_{n+2} - z_{n+3} + \dots \infty)|$$

z_n is a decreasing sequence

it follows that: $|z_{n+1} - (z_{n+2} - z_{n+3} + \dots \infty)| \leq z_{n+1}$

Thus, $|s - S_n| \leq z_{n+1}$

13) If (a_n) is a bounded decreasing sequence & (b_n) is a bounded increasing sequence and if $X_n = a_n + b_n$ for $n \in \mathbb{N}$, show that $\sum_{n=1}^{\infty} |X_n - X_{n+1}|$ is convergent.

$$|X_n - X_{n+1}| = |a_n + b_n - (a_{n+1} + b_{n+1})|$$
$$= |a_n + b_n - a_{n+1} - b_{n+1}|$$
$$= |a_n - a_{n+1} + b_n - b_{n+1}|$$

In general it's true, $|x+y| \leq |x| + |y|$

$$|X_n - X_{n+1}| = |a_n - a_{n+1} + b_n - b_{n+1}|$$
$$\leq |a_n - a_{n+1}| + |b_n - b_{n+1}|$$

since these are both convergent it follows that their sum is convergent

So $\sum_{n=1}^{\infty} |X_n - X_{n+1}|$ is convergent

2) If Σa_n is an absolutely convergent series, then the series $\Sigma a_n \sin(nx)$ is absolutely & uniformly convergent.

- use weirstrass M Test & comparison test

observe that $a_n \sin nx \leq a_n$ for every $n \in \mathbb{N}, x \in \mathbb{R}$
as the series Σa_n converges absolutely, with weirstrass we know $\Sigma a_n \sin(nx)$ converges uniformly & the comparison test follows that $\Sigma a_n \sin x$ converges absolutely.

13) Give an example of a function that is equal to its Taylor series expansion about $x=0$ for $x \geqslant 0$, but is not equal to this expansion for $x > 0$.

let $f(x) = \begin{cases} e^{(-1/x^2)}, & \text{if } x<0 \\ 0, & \text{if } x \geqslant 0 \end{cases}$

For $n=1$, $(e^{-1/x^2}) = e^{\frac{-1}{x^2}}(-x^{-2})$

$= x^3 e^{-1/x^2}$ when $x \neq 0$

$f'(0) = \lim_{x\to 0} \frac{e^{-1/x^2} - f(0)}{x-0}$

$= \lim x\to 0 \left(\frac{e^{-1/x^2}}{x}\right)$

$= \lim_{x\to 0} \left(\frac{\frac{1}{x}}{e^{1/x^2}}\right)$

for $t = \frac{1}{x^2}$ when $x \to 0$ that $t \to \infty$

we have: $f'(0) = \lim_{x\to 0} \frac{\frac{1}{x}}{\frac{1}{x^2}} = \lim_{t\to\infty} \frac{t^{\frac{1}{2}}}{e^t}$

$f'(0) = \lim_{t\to\infty} \frac{t^{\frac{1}{2}}}{e^t} = 0$

So it's true $n=1$

now for induction

with $n = k+1$

$\lim_{x\to\infty} \frac{a x^{-b} e^{-(x^{-2})}}{x} = \lim_{x\to\infty} \frac{a x^{-b-1}}{e^{x^{-2}}}$

For $t = \frac{1}{x^2}$ we see $x \to 0$ that $t \to \infty$

$\lim_{x\to\infty} \frac{a x^{-b} e^{-(x^{-2})}}{x} = \lim_{x\to\infty} \frac{e t^{\frac{b-1}{2}}}{e^t}$

now l'Hospital rule

$\lim_{x\to\infty} \frac{a x^{-b} e^{-(x^{-2})}}{x} = \lim_{x\to\infty} \frac{a t^{\frac{b-1}{2}}}{e^t} = 0$

$f^{(k+1)}(0) = 0$

therefore the claim is true for all $n = k+1$

(2) (a) If P is a tagged partition of $[a,b]$, show that each tag can belong to at most two subintervals in P.

(b) Are there tagged partitions in which every tag belongs to exactly two subintervals?

(a) Let tagged partition P defined as $P = \{(I_i, t_i)\}_{i=1}^{n}$ be tagged partition of $[a,b]$

observe that $t_i \in I_i$ for $I_i = [x_{i-1}, x_i]$. If $t_i \in I_j$ for some $j \neq i$ we have:

$$I_i \cap I_j \neq \emptyset$$

Thus, $j = i+1$ or $j = i-1$

As $x_i > x_{i-1}$ we have:

$$I_{i-1} \cap I_i \cap I_{i+1} = \emptyset$$

(meaning t_i can be at most two subintervals in P.)

(b) Let a tagged partition P be defined as $P = \{\{[x_0, x_1], x_1\}, \{[x_1, x_2], x_1\}\}$

we have $t_1 = x_1$ belong to two subintervals

we have $t_2 = x_2$ belong to two subintervals, so they're a tagged partition in which every tag belongs to two subintervals.

(22) Let $\Psi(x) = x^2|\cos(\pi/x)|$ for $x \in (0,1]$ & $\Psi(0)=0$. Then Ψ is continuous on $[0,1]$ & $\Psi'(x)$ exists for $x \notin E_1 = \{a_k\}$. Let $\psi(x) = \Psi'(x)$ for $x \notin E_1$, & $\psi(x) = 0$ for $x \in E_1$. Show that ψ is bounded on $[0,1]$ & that $\psi \in \mathcal{R}[0,1]$. Show that $\int_a^b \psi = \Psi(b) - \Psi(a)$ for $a, b \in [0,1]$. Also show that $|\psi| \in \mathcal{R}[0,1]$.

Let $F(x)$, $f(x)$, a_k, b_k be defined as:

$$\Psi(x) = \begin{cases} x^2|\cos(\frac{\pi}{x})|, & 0 < x \leq 1 \\ 0 & , x = 0 \end{cases}$$

$$a_k = \frac{2}{2k+1}, \quad b_k = \frac{1}{k} \quad k \in \mathbb{Z}$$

$$E = \{a_k\}_{k=1}^{\infty}$$

$$\psi(x) = \begin{cases} F'(x) & , x \notin E, x \neq 0 \\ 0 & , x \in E, x = 0 \end{cases}$$

For $x \notin E \cup \{0\}$

$$|\Psi'(0)| = |D(x^2|\cos(\tfrac{\pi}{x})|)|$$
$$= |D(x^2\cos(\tfrac{\pi}{x}))|$$
$$= |\pi\sin(\tfrac{\pi}{x}) + 2x\cos(\tfrac{\pi}{x})|$$

$[0,1]$

$$|\Psi'(x)| \leq |\pi| + 2|\cos\tfrac{\pi}{x}| \leq 2 + \pi$$

ψ is bounded.

15) If $f, g \in L[a,b]$, [redacted]
show that $|\|f\| - \|g\|| \leq \|f+g\|$.

As $|f(x)| - |g(x)| \leq |f(x) - g(x)|$

$$\int_a^b |f| - \int_a^b |g| = \int_a^b (|f| - |g|) \leq \int_a^b |f-g|$$

$$\underline{\|f\| - \|g\| \leq \|f-g\| \quad (1)}$$

$$\|g\| - \|f\| \leq \|g-f\|$$

$$\|g-f\| = \|f-g\|$$

replacing g with $-g$

$$\|f\| - \|-g\| \leq \|f+g\|$$

$$\Downarrow$$

$$\underline{\|f\| - \|g\| \leq \|f+g\| \quad (2)}$$

$$\boxed{\text{So, } \|f\| - \|g\| \leq \|f \pm g\|}$$

18) Let $g_n(x) = -1$ for $x \in [-1, -1/n]$, let $g_n(x) = nx$ for $x \in [-1/n, 1/n]$ & let $g_n(x) = 1$ for $x \in (1/n, 1]$. So that show that $\|g_m - g_n\| \to 0$ as $m, n \to \infty$, so that the Completeness Theorem 10.2.12 implies there exists $g \in L[-1,1]$ such that (g_n) converges to g in $L[-1,1]$. Find such a function g.

under the completeness theorem we have for any $m, n > \frac{8}{\varepsilon}$

$$\|g_m - g_n\| \leq \|g_m - g\| + \|g - g_n\|$$

$$= \|g_m - g\| + \|g_n - g\|$$

$$\leq \int_{-1/m}^{1/m} 2\,dx + \int_{-1/n}^{1/n} 2\,dx$$

$$\leq \frac{4}{m} + \frac{4}{n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\lim_{n\to\infty} \|g_n - g\| \leq \lim_{n\to\infty} \int_{-1/n}^{1/n} 2\,dx = \lim_{n\to\infty} \frac{4}{n} = 0$$

$$g_n \to g$$

$$g(x) = \begin{cases} -1 & \text{if } -1 \leq x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } 0 < x \leq 1 \end{cases}$$

2) let $f \in R^*[a,\gamma]$ for all $\gamma > a$. Show that $f \in R^*[a,\infty)$ if & only if for every $\varepsilon > 0$ there exists $K(\varepsilon) \geq a$ such that if $q > p \geq K(\varepsilon)$, then $|\int_p^q f| < \varepsilon$.

let $f \in R^*[a,\gamma]$ for all $\gamma \geq a$

$\Rightarrow$

given $f \in R^*[a,\gamma]$ implies that for every $\varepsilon > 0$ there exists $k(\varepsilon) \geq a$ such that if P is a subpartition of $[a,\infty]$

$$|S(f,P) - A| \leq \varepsilon$$

which implies

$$f \in R^*[a,\infty] \ \& \ \int_a^\infty f = A$$

for every $\varepsilon > 0$ there exists $K \geq a$ such that if $q \leq p \leq K(\varepsilon)$ then $\int_p^q f < \varepsilon$

$\Leftarrow$

lets suppose that if $\int_p^q f < \varepsilon$ for $q \leq p \leq k(\varepsilon)$ then.

$$\left|\int_a^q f - \int_a^p\right| \leq \varepsilon$$

which means $\lim_{\gamma \to \infty} \int_a^\gamma f$ exists. Which means $f \in R^*[a,\infty]$

15) Establish the convergence of fresnel's integral $\int_0^\infty \sin(x^2)\,dx$.
(use substitution theorem)

$\lim_{x\to\infty}\int_a^x f$, $\lim_{x\to\infty}\int_a^x g$, $\lim_{x\to\infty}\int_a^x |f|$, $\lim_{x\to\infty}\int_a^x |g|$

also:

$$\lim_{x\to\infty}\int_a^x (f+g) = \lim_{x\to\infty}\int_a^x f + \lim_{x\to\infty}\int_a^x g$$

$f+g \in \mathbb{R}^*[a,\infty)$

For any $\varepsilon>0$, $K>0$ for any $q>p>K$

$$\int_p^q |f| \leq \frac{\varepsilon}{2} \text{ \& } \int_p^q |g| \leq \frac{\varepsilon}{2}$$

$$\int_p^q |f+g| \leq \int_p^q (|f|+|g|) \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

Cauchy Criterion we have that $\lim_{x\to\infty}\int_a^x |f+g|$ exists so it follows $|f+g| \in \mathbb{R}^*[a,\infty)$
which implies that $f+g \in L[a,\infty)$

2) Consider the following sequences of functions with the indicated domains. Does the sequence converge? If so, to what? Is the convergence uniform? Is it bounded? If not bounded, is it dominated? Is it monotone? Evaluate the limit of the sequence of integrals.

(a) ~~$\frac{kx}{1+kx}$ [0,1]~~,

let $f_k(x) = \dfrac{kx}{1+k\sqrt{x}}$, $x \in [0,1]$

observe that

$$\lim_{n\to\infty} \frac{nx}{1+n\sqrt{x}} = \lim_{n\to\infty} \frac{x}{\frac{1}{n}+\sqrt{x}}$$

$$= \frac{x}{\lim_{n\to\infty} \frac{1}{n} + \sqrt{x}}$$

$$= \frac{x}{0+\sqrt{x}} = \sqrt{x}$$

~~$\frac{x^k}{1+x^k}$ [0,2]~~,

let $f(x) = \sqrt{x}$, $x \in [0,1]$ then $\lim_{n\to\infty} f_n = f$ f_k is bounded for $k \in \mathbb{R}$ so it follows that f_k is dominated. Also, it is easy to show that f_n is increasing as for $m > n$ we have: $f_m(x) - f_n(x) \geq 0$

16) If $E \in M[a,b]$, we define the (Lebesgue) measure of E to be the number $m(E) = \int_a^b 1_E$.
In this exercise, we develop a number of properties of the measure function $m: M[a,b] \to \mathbb{R}$.
(a) Show that $m(\emptyset) = 0$ & $0 \leq m(E) \leq b-a$
(b) Show that $m([c,d]) = m([c,d)) = m((c,d]) = m((c,d)) = d-c$
(c) Show that $m(E') = (b-a) - m(E)$.

(a)
$$\int_a^b 0\,dx \leq \int_a^b 1_E \leq \int_a^b 1\,dx$$
$$\Downarrow$$
$$0 \leq m(E) \leq b-a$$

As $1_\emptyset(a) = 0$ for any $x \in [a,b]$ it follow that $\int_a^b 1_\emptyset = 0$
$$m(\emptyset) = 0$$

(b) - - - - - - - - - - -

$$\int_a^b 1_{[c,d]} = \int_a^b 1\,dx = d-c$$

are all the same = to $d-c$

(c) - - - - - - - - - - - - -

$$\int_a^b 1_E + \int_a^b 1_{E'} = \int_a^b 1\,dx$$
$$m(E) + m(E') = b-a$$

2) Show that the intervals (a,∞) & $(-\infty,a)$ are open sets, & that the intervals $[b,\infty)$ & $(-\infty,b]$ are closed sets.

Let $x \in (a,\infty)$ & $\varepsilon_x < x-a$. Then if $|u-x| < \varepsilon_x$ we have:

$$u > -\varepsilon_x + x > a - x + x = a$$

which means $u \in (a,\infty)$ so it follows (a,∞) is an open set.

Let $x \in (-\infty,a)$ & $\varepsilon_x < a-x$. Then if $|u-x| < \varepsilon_x$ we have:

$$u < \varepsilon_x + x < a - x + x = a$$

which means $u \in (-\infty,a)$ is an open set

observe that $[b,\infty) = \mathbb{R} \setminus (-\infty,b)$. It's a closed set because its complement is open.

via definition

observe that $(-\infty,b] = \mathbb{R} \setminus (b,\infty)$. It's a closed set because its complement is open.

via definition

21) If in the notation used in the proof of Theorem 11.1.9. we have $a_x < y < x$, show that $y \in G$.

From the proof of Theorem 11.1.9. we know that G is an open set, $A_x \subseteq G$ & $a_x = \inf A_x$

Now, as $a_x < y < x$ & $a_x = \inf A_x$ we have that $y \in A_x$ & since $A_x \subseteq G$ it follows that $y \in G$.

2.) Exhibit an open cover of $\mathbb{N}$ that has no finite subcover.

An open cover of A is a collection of open subsets of A whose union is all of A & that a finite subcover is a subcover of A that has finitely many sets

let's observe open cover G_n where $G_n = (n - \frac{1}{2}, n + \frac{1}{2})$, $n \in \mathbb{N}$

each n open covering G_n is an open interval

each $n \in \mathbb{N}$ is contained in G_n & therefore G_n is a cover of $\mathbb{N}$ but any finite subcollection of $\{G_n\}$ will cover finitely many elements of $\mathbb{N}$

we can conclude $G_n = (n - \frac{1}{2}, n + \frac{1}{2})$, $n \in \mathbb{N}$ is an open cover of $\mathbb{N}$ with finite subc

12.) Let $K \neq \emptyset$ be compact in $\mathbb{R}$ & let $c \in \mathbb{R}$. Prove that there exists a point b in K such that $|c-b| = \sup\{|c-x| : x \in K\}$.

~~let K be a set...~~

let $\varepsilon > 0$ & $\delta = \varepsilon$ then for $x, y \in K$ if $|x-y| < \delta$:

$f(x) = |c-x| \leq |c-y| + |y-x|$

$= f(y) + |y-x|$

implies $f(x) - f(y) \leq |y-x|$

$f(y) = |c-y| \leq |c-x| + |x-y| = f(x) + |x-y|$

$f(y) - f(x) \leq |x-y| \longrightarrow |f(x) - f(y)| \leq |x-y| = \delta$

we have shown f is continuous on compact set K

Because of the Maximum Minimum theorem it follows that: there exists a point $b \in K$ such that $|c-b| = \sup\{|c-x| ; x \in K\}$

(2) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1/(1+x^2)$ for $x \in \mathbb{R}$
(a) Find an open interval (a,b) whose direct image under f is not open.
(b) Show that the direct image of the closed interval $[0,\infty)$ is not closed.

(a) Let $I = (-1,1)$ then

$$f(I) = (f(1), f(0)] = (\tfrac{1}{2}, 1]$$

set $(\frac{1}{2}, 1]$ is not open therefore, I is an open interval whose direct image is under f is not open.

(b) $$f([0,\infty]) = \left(\lim_{x\to\infty} \frac{1}{1+x^2}, f(0)\right] = (0,1]$$

set $(0,1]$ is not closed, hence, proven.

(8) Give examples of a function $f: \mathbb{R} \to \mathbb{R}$ such that the set $\{x \in \mathbb{R} : f(x) = 1\}$ is neither open nor closed in $\mathbb{R}$.

Let's observe the function f

$$f(x) = \begin{cases} -1, & x \in \mathbb{Q} \\ 1, & x \notin \mathbb{Q} \end{cases}$$

is neither open nor closed

2) Show that the functions d_∞ & d_1 defined in 11.4.2 (d, c) are metrics on $C[0,1]$.

let $f, g, h \in C[0,1]$ we have:

$$d_\infty(f,g) = \sup\{|f(x) - g(x)|\} \geq 0$$

step 1) $f, g, h \in C[0,1]$

$$d_\infty(f,g) = \sup\{|f(x) - g(x)|\} = \sup\{|g(x) - f(x)|\} = d_\infty(g,f)$$

step 2)

$$d_\infty(f,g) = \sup\{|f(x) - g(x)|\}$$
$$= \sup\{|f(x) - h(x) + h(x) - g(x)|\}$$
$$\leq \sup\{|f(x) - h(x)| + |h(x) - g(x)|\}$$
$$= d_\infty(f,h) + d_\infty(h,g)$$

10) Prove Theorem 11.4.11

let's observe sets F_1 & F_2 where:

$$F_1 = \{n ; n \in \mathbb{N}\}$$
$$F_2 = \left\{\frac{n+1}{n} ; n \in \mathbb{N}, n \geq 2\right\}$$

now let's see the difference between x_1 & x_2 for $x_1 \in F_1$ & $x_2 \in F_2$ =

$$x_1 - x_2 = n - \frac{n+1}{n} = \frac{1}{n}$$

therefore, $\inf\{|x_1 - x_2| ; x_i \in F_i\} = 0$

this means that sets $F_1 = \{n ; n \in \mathbb{N}\}$ & $F_2 = \left\{\frac{n+1}{n} ; n \in \mathbb{N}, n \geq 2\right\}$ are an example of disjoint closed sets such that $\inf\{|x_1 - x_2| ; x_i \in F_i\} = 0$

2. Show that $f(x) = x^{1/3}$, $x \in \mathbb{R}$, is not differentiable at $x = 0$.

by def, derivative of f at c given $f'(c) = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^{\frac{1}{3}} - 0}{x - 0}$$

$$= \underbrace{\lim_{x \to 0} \frac{1}{x^{2/3}}}_{\text{not a finite quantity}}$$

16) given that the restriction of the tangent function tan to $I = (-\pi/2, \pi/2)$ is strictly increasing & that $\tan(I) = \mathbb{R}$, let $\operatorname{Arctan}: \mathbb{R} \to \mathbb{R}$ be the function inverse to the restriction of tan to I. Show that Arctan is differentiable on $\mathbb{R}$ & that $D \operatorname{Arctan}(y) = (1+y^2)^{-1}$ for $y \in \mathbb{R}$.

$$D\arctan y = \frac{1}{D\tan(\arctan y)} = \frac{1}{\sec^2(\arctan y)}$$

$\arctan y = x$

$\Downarrow$

$\tan x = y$

$\Downarrow$

$\sec^2 x = 1 + \tan^2 x = 1 + y^2$

$\sec x = \sqrt{1+y^2}$

this means $D\arctan(y) = \dfrac{1}{\sec^2 x} = \dfrac{1}{1+y^2}$

$$D\arctan(y) = \frac{1}{1+y^2}$$

2) Find the points of relative extrema, the intervals on which the following functions are increasing, & those on which they are decreasing

(a) $f(x) = x^2 - 3x + 8$ (b) $g(x) = 3x - 4x^2$ (c) $h(x) = x^2 - 3x - 4$ (d) $k(x) = x^4 + 2x^2 -$

$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$

$f'(x) = 0$ for $x^2 = 1 \Rightarrow x = \pm 1$

$0 < |x| \leq 1$ then

$\frac{1}{|x|} \geq 1 \Rightarrow \frac{1}{x^2} - 1 \geq 0$

$\Rightarrow f'(x) \leq 0$

Thus f is decreasing $[-1, 0) \cup (0, 1]$

$x \geq 1 \Rightarrow 1 - \frac{1}{x^2} \geq 0 \Rightarrow f'(x) \geq 0$

increasing $[1, \infty)$

$x \leq -1 \Rightarrow 1 - \frac{1}{x^2} \geq 0 \Rightarrow f'(x) \geq 0$

increasing on $(-\infty, -1]$

$k'(x) = \frac{1}{2\sqrt{x}} - \frac{2}{2\sqrt{2+x}}$

$= \frac{1}{2}\left[\frac{\sqrt{2+x} - 2\sqrt{x}}{\sqrt{x}\sqrt{2+x}}\right]$

$= \frac{1}{\sqrt{x(2+x)}} \cdot \frac{2+x-4x}{\sqrt{2+x} + 2\sqrt{x}}$

$= \frac{2-3x}{\sqrt{x(2+x)}\left(\sqrt{2+x} + 2\sqrt{x}\right)}$

$k'(x) = 0 \Rightarrow 2 - 3x = 0$

$\Rightarrow x = \frac{2}{3}$

$(0, \frac{2}{3}]$ increasing

$\frac{2}{3} \leq x \Rightarrow 2 - 3x \leq 0$

$\Rightarrow h'(x) \leq 0$

decreasing $[\frac{2}{3}, \infty)$, $x = \frac{2}{3}$

(19) A differentiable function $f: I \to \mathbb{R}$ is said to be <u>uniformly differentiable</u> on $I := [a,b]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x-y| < \delta$ & $x, y \in I$, then

$$\left| \frac{f(x)-f(y)}{x-y} - f'(x) \right| < \varepsilon$$

Show that if f is uniformly differentiable on I, then f' is continuous on I.

let $x, y \in I$ such that $0 < |x-y| < \delta$ we have

$$f'(x) - f'(y) = f'(x) - \frac{f(x)-f(y)}{x-y} + \frac{f(x)-f(y)}{x-y} - f'(y) \Rightarrow$$

$$|f'(x)-f'(y)| \leq \left| f'(x) - \frac{f(x)-f(y)}{x-y} \right| + \left| \frac{f(x)-f(y)}{x-y} - f'(y) \right|$$

$$< \varepsilon + \varepsilon$$

$$= 2\varepsilon$$

f' is uniformly continuous on I, so continuous.

2) In addition to the suppositions of the preceding exercise, let $g(x) > 0$ for $x \in [a,b]$, $x \neq c$. If $A > 0$ & $B = 0$, prove that we must have $\lim_{x \to c} f(x)/g(x) = \infty$. If $A < 0$ & $B = 0$, prove that we must have $\lim_{x \to c} f(x)/g(x) = -\infty$.

given $M > 0$ we want to show
there exists δ such that $|x - c| < \delta$
where $x \neq c$.

$$\frac{f(x)}{g(x)} > M$$

$$f(x) > M \cdot g(x)$$

let $A > 0$, $B = 0$, $\delta > 0$ such $|x - c| < \delta$, $x \neq c$

$$f(x) \in \left(\frac{A}{2}, \frac{3A}{2}\right)$$

$$|f(x) - A| < \frac{A}{2}$$

$$g(x) < \frac{A}{2M}$$

$$\frac{1}{g(x)} > \frac{2M}{A}$$

$$\frac{\left(\frac{A}{2}\right)}{g(x)} > M$$

$$\frac{f(x)}{g(x)} > M$$

if $A > 0$ & $B = 0$ then $\lim_{x \to c} \frac{f(x)}{g(x)} = \infty$

$$\lim_{x \to c} \frac{-f(x)}{g(x)} = \infty$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = -\infty$$

if $A < 0$ & $B = 0$ then $\lim_{x \to c} \frac{f(x)}{g(x)} = -\infty$

13) Try to use L'Hospital's Rule to find the limit of $\frac{\tan x}{\sec x}$ as $x \to (\pi/2)^-$. Then evaluate directly by changing to sines & cosines.

$\lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\sec x} = \frac{\infty}{\infty}$ form, apply L'Hospital's rule:

$$\lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\sec x} = \lim_{x \to \frac{\pi}{2}} \frac{\sec^2 x}{\sec x \tan x} = \lim_{x \to \frac{\pi}{2}} \frac{\sec x}{\tan x} = \frac{\infty}{\infty} \text{ form, L'H} =$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\sec x}$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\sec x} = \lim_{x \to \frac{\pi}{2}} \frac{\left(\frac{\sin x}{\cos x}\right)}{\left(\frac{1}{\cos x}\right)} = \lim_{x \to \frac{\pi}{2}} \frac{\sin x \cdot \cancel{\cos x}}{\cancel{\cos x} \cdot 1} = \lim_{x \to \frac{\pi}{2}} \sin x = 1$$

2) Let $g(x) = |x^3|$ for $x \in \mathbb{R}$. Find $g'(x)$ & $g''(x)$ for $x \in \mathbb{R}$ & $g'''(x)$ for $x \neq 0$. Show that $g'''(0)$ does not exist.

$$g(x) = \begin{cases} x^3 & \text{if } x \geq 0 \\ -x^3 & \text{if } x < 0 \end{cases}$$

$$g'(x) = \begin{cases} 3x^2 & \text{if } x \geq 0 \\ -3x^2 & \text{if } x < 0 \end{cases}$$

$$g''(x) = \begin{cases} 6x & \text{if } x \geq 0 \\ -6x & \text{if } x < 0 \end{cases}$$

$$g'''(x) = \begin{cases} 6 & \text{if } x > 0 \\ ? & \text{if } x = 0 \\ -6 & \text{if } x < 0 \end{cases}$$

$$\lim_{x \to 0^-} \frac{g''(x) - g''(0)}{x - 0} \neq \lim_{x \to 0^+} \frac{g''(x) - g''(0)}{x - 0} \Rightarrow \lim_{x \to 0} \frac{g''(x) - g''(0)}{x - 0}$$

Does Not Exist

$$\left.\begin{aligned} \lim_{x \to 0^-} \frac{g''(x) - g''(0)}{x - 0} &= \frac{-6x - 0}{x - 0} = \frac{-6x}{x} = -6 \\ \lim_{x \to 0^+} \frac{g''(x) - g''(0)}{x - 0} &= \frac{6x - 0}{x - 0} = \frac{6x}{x} = 6 \end{aligned}\right\}$$

$$\lim_{x \to 0^-} \frac{g''(x) - g''(0)}{x - 0} \neq \lim_{x \to 0^+} \frac{g''(x) - g''(0)}{x - 0}$$

Side = Newton's method formula $\left[X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)} \right]$

9) If $g(x) = \sin x$, show that the remainder term in taylor's Theorem converges to zero as $n \to \infty$, for each fixed X_0 & X.
[Hint: see theorem 3.2.1(b)]

Taylor's theorem: $P_n(x) + R_n(x)$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

$$|R_n(x)| \leq \frac{|(x-x_0)^{n+1}|}{(n+1)!}$$

let $a_n = \frac{|(x-x_0)^{n+1}|}{(n+1)!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{|x-x_0|^{n+2}}{(n+2)!} \cdot \frac{(n+1)!}{|x-x_0|^{n+1}}$$

$$= \lim_{n\to\infty} \frac{|x-x_0|}{n+2} = 0$$

we know $\lim_{n\to\infty} a_n = 0$

by the sandwich Theorem

$$\lim_{n\to\infty} R_n(x) = 0$$

2) If $f(x)=x^2$ for $x \in [0,4]$, calculate the following Riemann sums where $\dot{P}_i$ has the same partition points as in Exercise 1, & the tags are selected as indicated.

(a) $\dot{P}_1 = (0,1,2,4)$

$[x_0=0,\ x_1=1,\ x_2=2,\ x_3=4]$

$S(f,\dot{P}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$

$= f(0)(x_1-x_0)+f(1)(x_2-x_1)+f(2)(x_3-x_2)$

$= 0^2(1-0)+1^2(2-1)+2^2(4-2)$

$= 0\cdot1+1\cdot1+4\cdot2 = 9$

(b) $\dot{P}_2 = (0,2,3,4)$

$[x_0=0,\ x_1=1,\ x_2=2,\ x_3=4]$

$S(f,\dot{P}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$

$= f(1)(x_1-x_0)+f(2)(x_2-x_1)+f(4)(x_3-x_2)$

$= 1^2(1-0)+2^2(2-1)+4^2(4-2)$

$= 1\cdot1+4\cdot1+16\cdot2 = 37$

(c) $\dot{P}_3 = (0,1,1.5,2,3,4,4)$

$[x_0=0,\ x_1=2,\ x_2=3,\ x_3=4]$

$S(f,\dot{P}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$

$= f(0)(x_1-x_0)+f(2)(x_2-x_1)+f(3)(x_3-x_2)$

$= 0^2(2-0)+2^2(3-2)+3^2(4-3)$

$= 0\cdot2+4\cdot1+9\cdot1 = 13$

(d) $\dot{P}_4 = (0,.5,2.5,3.5,4)$

$S(f,\dot{P}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$

$[x_0=0,\ x_1=2,\ x_3=3,\ x_4=4]$

$= 0^2(2-0)+3^2(3-2)+4^2(4-3)$

$= 4\cdot2+9\cdot1+16\cdot1 = 33$

13) Suppose that $c \leq d$ are points in $[a,b]$. If $\varphi:[a,b]\to\mathbb{R}$ satisfies $\varphi(x)=\alpha>0$ for $x\in[c,d]$ & $\varphi(x)=0$ elsewhere in $[a,b]$ prove that $\varphi\in\mathcal{R}[a,b]$ & that $\int_a^b \varphi = \alpha(d-c)$.

Let $\varepsilon>0$ & set $\delta=\frac{\varepsilon}{4\alpha}>0$. Let $\dot{P}=\{[x_{i-1},x_i],t_i\}$ be a tagged partition such that $\|\dot{P}\|<\delta$

$$S(\varphi:\dot{P})-\alpha(d-c)=\sum_{i=1}^{n} f(t_i)(x_i-x_{i-1})-\alpha(d-c)$$

$$\leq \alpha((d+\delta-(c-\delta))-\alpha(d-c)$$

$$=2\alpha\delta$$

$$S(\varphi;\dot{P})-\alpha(d-c)=\sum_{i=1}^{n} f(t_i)(x_i-x_{i-1})-\alpha(d-c)$$

$$=-2\alpha\delta$$

$$|S(\varphi;\dot{P})-\alpha(d-c)|<2\alpha\delta=\frac{\varepsilon}{2}<\varepsilon$$

Thus, $\varphi\in\mathcal{R}[a,b]$ & $\int_a^b \varphi=\alpha(d-c)$.

2) Consider the function h defined by $h(x) = x+1$ for $x \in [0,1]$ rational, and $h(x) = 0$ for $x \in [0,1]$ irrational. Show that h is not Reimann integrable.

If $\dot{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ is any tagged partition of $[0,1]$ with rational tags

$$S(f,\dot{P}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \geq \sum_{i=1}^{n} (x_i - x_{i-1}) = 1$$

because $f(t_i) = t_i + 1 \geq 1$

if $\dot{Q} = \{[y_{i-1}, y_i], s_i\}_{i=1}^{n}$ is any tagged partition of $[0,1]$ with irrational tags then

$$S(f,\dot{Q}) = \sum_{i=1}^{n} f(s_i)(y_i - y_{i-1}) = 0$$

$$|S(f,\dot{P}) - S(f,\dot{Q})| \geq 1$$

so f is not integrable via the Cauchy Criterion

19) Suppose that $a>0$ & that $f \in R[-a,a]$

(a) If f is even (that is, if $f(-x)=f(x)$ for all $x \in [0,a]$), show that $\int_{-a}^{a} f = 2\int_0^a f$.

$$\int_{-a}^{a} f = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$

$$\int_{-a}^{0} f = -\int_a^0 f(-y)\,dy + \int_0^a f(x)\,dx$$

$$= -\int_a^0 f(y)\,dy + \int_0^a f(x)\,dx$$

$$= \int_0^a f(y)\,dy + \int_0^a f(x)\,dx$$

$$= 2\int_0^a f(x)\,dx$$

$$= 2\int_0^a f$$

(b) If f is odd (that is, if $f(-x) = -f(x)$ for all $x \in [0,a]$), show that [redacted] $\int_{-a}^{a} f = 0$

$$\int_{-a}^{a} f = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$

$$\int_{-a}^{a} f = -\int_a^0 f(-y)\,dy + \int_0^a f(x)\,dx$$

$$= -\int_a^0 (-f(y))\,dy + \int_0^a f(x)\,dx$$

$$= \int_a^0 f(y)\,dy + \int_0^a f(x)\,dx$$

$$= -\int_0^a f(y)\,dy + \int_0^a f(x)\,dx$$

$$= 0$$

2) If $n \in \mathbb{N}$ & $H_n(x) = x^{n+1}/(n+1)$ for $x \in [a,b]$, show that the Fundamental theorem implies that $\int_a^b x^n dx = (b^{n+1} - a^{n+1})/(n+1)$. What is the set E here?

let $n \in \mathbb{N}$ & $H_n(x) = \dfrac{x^{n+1}}{n+1}$, $x \in [a,b]$. H_n is continuous on $[a,b]$

$$H_n'(x) = (\cancel{n+1})\frac{x^n}{\cancel{n+1}} = x^n$$

$x \mapsto x^n$ is a riemann integral on $[a,b]$

(we can apply the fundamental theorem of calc)

$$\int_a^b x^n dx = H_n(b) - H_n(a) = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$$

$E = \emptyset$ because H_n doesn't have discontinuity points in $[a,b]$ & $H_n'(x) = x^n$ for all $x \in [a,b]$

20) (a) If Z_1 & Z_2 are null sets, show that $Z_1 \cup Z_2$ is a null set.

$Z \subseteq \mathbb{R}$ is said to be a null set. For any $\varepsilon > 0$

$$Z \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k) \text{ \& } \sum_{k=1}^{\infty} (b_k - a_k) < \varepsilon$$

$A = \{(a_k, b_k)\}_{k=1}^{\infty}$

$B = \{(c_k, d_k)\}_{k=1}^{\infty}$

$$Z_1 \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k), Z_2 \subseteq \bigcup_{k=1}^{\infty} (c_k, d_k) \text{ \& } \sum_{k=1}^{\infty} (b_k - a_k) < \frac{\varepsilon}{2}, \sum_{k=1}^{\infty} (d_k - c_k) < \frac{\varepsilon}{2}$$

$$Z_1 \cup Z_2 \subseteq \left(\bigcup_{k=1}^{\infty} (a_k, b_k)\right) \cup \left(\bigcup_{k=1}^{\infty} (c_k, d_k)\right) = C$$

$$\sum_{k=1}^{\infty} (b_k - a_k) + \sum_{k=1}^{\infty} (d_k - c_k) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus $Z_1 \cup Z_2$ is a null set

b) More generally if Z_n is a null set for each $n \in \mathbb{N}$, show that $\bigcup_{n=1}^{\infty} Z_n$ is a null set.

$$\{(a_k^n, b_k^n)\} \; k \in \mathbb{N}$$

$$Z_n \subseteq \bigcup_{k \in \mathbb{N}} (a_k^n, b_k^n) \text{ \& } \sum_{k \in \mathbb{N}} (b_k^n - a_k^n) < \frac{\varepsilon}{2^n}$$

now it's a countable $\mathbb{R}$ collection

$$\bigcup_{n \in \mathbb{N}} Z_n \subseteq \bigcup_{k, n \in \mathbb{N}} (a_k^n, b_k^n)$$

$$\sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} (a_k^n, b_k^n) < \sum_{n \in \mathbb{N}} \frac{\varepsilon}{2^n} = \varepsilon \sum_{n \in \mathbb{N}} \frac{1}{2^n} = \varepsilon$$

Therefore $\bigcup_{n \in \mathbb{N}} Z_n$ is a null set.

2.) Prove if $f(x)=c$ for $x \in [a,b]$, then its Darboux integral is equal to $c(b-a)$.

Let
$f(x)=c$

for all $x \in [a,b]$

Let $m_k = \inf[f(x) : x \in [x_{k-1}, x_k]]$

$M_k = \sup[f(x) : x \in [x_{k-1}, x_k]]$

$I=[a,b]$, $P=(x_0, x_1, \dots, x_n)$

$L(f:P) = \sum_{k=1}^{n} m_k (x_k - x_{k-1})$

$U(f:P) = \sum_{k=1}^{n} M_k (x_k - x_{k-1})$

$L(f) = \inf(L(f:P) : P \in ([a,b]))$

$U(f) = \sup(L(f:P) : P \in ([a,b]))$

$L(f) = U(f)$

since $f(x)=c$ for all $x \in [a,b]$

$m_k = M_k = c$

$L(f:P) = \sum_{i=1}^{n} c(x_i - x_{i-1})$

$= c(x_n - x_0)$

$= c(b-a)$

so the following is true

$L(f) = \inf(L(f:P) : P \in ([a,b]))$

$= c(b-a)$

13) Let P_ε be a partition whose existence is asserted in the Integrability Criterion. Show that if P is any refinement of P_ε, then $U(f:P) - L(f:P) < \varepsilon$

If a Darboux function

There is a partition P_ε of I

$$U(f;P_\varepsilon) - L(f;P_\varepsilon) < \varepsilon$$

assum P is any refinement for P_ε

then $U(f;P) - L(f;P) < \varepsilon$

$$L(f;P_\varepsilon) \leq U(f;P_\varepsilon)$$

$$L(f;P) \leq U(f;P).$$

$$L(f;P_\varepsilon) \leq L(f;P)$$

$$\Rightarrow -L(f;P) \leq -L(f;P_\varepsilon) \quad \text{– theorem}$$

$$U(f;P) \leq U(f;P_\varepsilon)$$

$$U(f;P) - L(f;P) \leq U(f;P_\varepsilon) < \varepsilon \quad \text{– theorem}$$

$$U(f;P) - L(f;P) < \varepsilon.$$

(2) Use the simpson approximation with $n=4$ to eval $\ln 2 = \int_1^2 (1/x)dx$.
Show that $.6927 \le \ln 2 \le .6933$ & $.000016 < \frac{1}{2^5}\cdot\frac{1}{1920} \le S_4 - \ln 2 \frac{1}{1920} < .000521$

$f(x) = \frac{1}{x}$, $a=1$, $b=2$, $n=4$

$f'(x) = -\frac{1}{x^2}$

$f''(x) = \frac{2}{x^3}$

$f'''(x) = -6/x^4$

$f^4(x) = 24/x^5$

$x \geq 1 \Rightarrow \frac{1}{x} \le 1 \Rightarrow \frac{1}{x^5} \le 1 \Rightarrow f^4(x) \le 24$

$x \le 2 \Rightarrow \frac{1}{x} \geq \frac{1}{2} \Rightarrow \frac{1}{x^5} \geq \frac{1}{2^5} \Rightarrow f^4(x) \geq \frac{24}{2^5}$

f continuous on $[1,2]$

$S_4(f) - \int_1^2 f = \frac{h_4^4}{180} f^4(c) = \frac{1}{180 \cdot 4^4} f^4(c)$

$S_4(f) = \frac{1}{3} h_4 (f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2))$

$= \underline{.6933}$

$S_4(f) - \int_1^2 f = \frac{1}{180 \cdot 4^4} f^4(c)$

$\le \frac{1}{180 \cdot 4^4} \cdot 24$

$= \frac{1}{1920} = 5.2083 \times 10^{-4}$

$S_4(f) - \int_1^2 f = \frac{1}{180 \cdot 4^4} f^4(c)$

$\geq \frac{1}{180 \cdot 4^4} \cdot \frac{24}{2^5}$

$= 1.6276 \times 10^{-5}$

therefore $.0000163 \le S_4(f) - \int_1^2 f \le .000521$

$\int_1^2 \frac{1}{x} \le S_4 \frac{1}{x} - .0000163 = .6933 - .0001$

$\approx .6933$

$\int_1^2 \frac{1}{x} \geq S_4 \frac{1}{x} - .000521$

$= .6933 - .000521$

$\approx .6927$

thus $.6927 \le \int_1^2 \frac{1}{x} \le .6933$

10) $\int_0^{\pi/2} \frac{dx}{1+\sin x}$. (Approximate the indicated integrals, giving estimate the estimate error.)

f for $x \in [0, \pi/2]$

$$f'(x) = \frac{d}{dx}\left[\frac{1}{1+\sin x}\right]$$

$$= -\frac{1}{(1+\sin x)^2}\cos x$$

$$f''(x) = \frac{d}{dx}\left[-\frac{\cos x}{(1+\sin x)^2}\right]$$

$$= \frac{\sin x\,(1+\sin x)^2 + 2\cos^2 x\,(1+\sin x)}{(1+\sin x)^4}$$

$$= \frac{\sin x(1+\sin x) + 2\cos^2 x}{(1+\sin x)^3}$$

$$= \frac{\sin x + \sin^2 x + 2\cos^2 x}{(1+\sin x)^3} = \frac{\sin x + 1 + \cos^2 x}{(1+\sin x)^3}$$

$x \in [0, \pi/2]$

$$|f''(x)| \le \frac{3}{1} = 3$$

$$|f''(x)| \le \max\{|f''(0)|, |f''(\pi/2)|\} = \max\{2,1\} = 2$$

$$\left|T_n(f) - \int_0^{\pi/2} f\right| \le \frac{(\pi/2)\,2\,h_n^2}{12} = \frac{\pi^3}{48n^2}$$

$$\frac{\pi^3}{48n^2} \le 10^{-2} \longleftrightarrow n \geq 8.03$$

$n = 9$

$$T_9(f) = h_9\left(\tfrac{1}{2}f(0) + f(h_9) + f(2h_9) + \dots + f(8h_9) + \tfrac{1}{2}f(\pi/2)\right) = 1.0025$$

where $h_9 = \frac{\pi/2}{9}$

$$\int_0^{\pi/2} \frac{dx}{1+\sin x} \approx 1$$

If $f^{(?)}$ is a polynomial with degree 3 show that the Simpson approximations are exact.

let $n \in \mathbb{N}$

$$S_{2n}(f) = \frac{2}{3} M_n(f) + \frac{1}{3} T_n(f)$$

$$\left| S_{2n} - \int_a^b f \right| = \left| \frac{2}{3} M_n(f) + \frac{1}{3} T_n(f) - \left(\frac{2}{3} + \frac{1}{3}\right) \int_a^b f \right|$$

$$\leq \frac{2}{3} \left| M_n - \int_a^b f \right| + \frac{1}{3} \left| T_n - \int_a^b f \right|$$

$$\leq \left(\frac{2}{3} \cdot \frac{1}{24} + \frac{1}{3} \frac{1}{12} \right) * \left(\frac{(b-a)^3}{n^2} \right)$$

$$= \frac{(b-a)^3}{18 n^2} B_2$$

used $\left(|f''(x)| \leq B_2 , b_2 \in [a, b] \right)$ — might be dat error

$$\left| S_{2n} - \int_a^b f \right| \leq \frac{(b-a)^3}{18 n^2} B_2$$

2) Show that $\lim(nx/(1+n^2x^2))=0$ for all $x \in \mathbb{R}$.

$$\lim_{x\to\infty} \frac{nx}{1+n^2x^2} = \lim_{x\to\infty} \frac{nx}{x^2\left(\frac{1}{x^2}+n^2\right)}$$

$$= \lim_{x\to\infty} \frac{n}{x\left(\frac{1}{x^2}+n^2\right)}$$

$$\lim_{x\to\infty} \frac{n}{x\left(\frac{1}{x^2}+n^2\right)} = \frac{n}{\infty\left(\frac{1}{\infty}+n^2\right)} = \frac{n}{\infty} = 0$$

17) Show that the convergence of the sequence in Exercise 7 is uniform on the interval $[a, \infty)$, but is not uniform on the interval $[0, \infty)$.

$$f(x) = \begin{cases} 1, & x = 0 \\ 0, & x > 0 \end{cases}$$

let $a > 0$. For $x \in [a, \infty)$ & $n \in \mathbb{N}$

$$|f_n(x)| = |e^{-nx}| \leq e^{-na}$$

$$\|f_n\|_{[a,\infty)} \leq e^{-na} \xrightarrow{n \to \infty} 0$$

show that the convergence isn't uniform on $[0, \infty)$, $f_n(1/n) = e^{-1}$!

$$\|f_n\|_{[0,\infty)} \geq |f_n(1/n)| = e^{-1}$$

2) Prove that the sequence in Ex is an example of a sequence of continuous functions that converges nonuniformly to a continuous limit.

$$f_n(x) = \begin{cases} n^2 x, & 0 \le x \le 1/n \\ -n^2(x - 2/n), & 1/n \le x \le 2/n \\ 0, & 2/n \le x \le 1 \end{cases}$$

$$\lim_{x \to 1/n^-} f_n(x) = \lim_{x \to 1/n^-} n^2 x = n$$

&

$$\lim_{x \to 1/n^+} f_n(x) = \lim_{x \to 1/n^+} -n^2(x - 2/n) = -n^2(-1/n) = n$$

$$\lim_{n \to \infty} f_n(x) = 0, \ \forall x \in [0,1]$$

$$\|f_n\|_{[0,1]} = \sup_{x \in [0,2]} |f_n(x)| = n$$

that is $\|f_n\|_{[0,1]} \not\to 0$.

5) let $f_n(x) = x^n$ for $x \in [0,1]$, $n \in \mathbb{N}$. Show that (f_n) is a decreasing sequence of continuous function that converges to a continuous limit function, but the convergence is not uniform on $[0,\infty)$.

For $x \in [0,1]$

$$x^{n+1} \leq x^n \Rightarrow f_n(x) - f_{n+1}(x) = x^n - x^{n+1} \geq 0,$$

(f_n) is of decreasing functions on $[0,1]$

$x \in [0,1)$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0$$

$f_n(1) = 1$ for all $n \in \mathbb{N}$ so

$$\lim_{n\to\infty} f_n(1) = 1$$

thus, it converges pointwise with

$$f(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases}$$

which is not a continuous function.

the convergence isn't uniform since for all n

$$\|f_n - f\|_{[0,1]}$$

$$= \sup\{|f_n(x) - f(x)| : x \in [0,1]\}$$

$$= \sup\{|x^n| : x \in [0,1)\}$$

$$= 1$$

(a) Calculate e correct to 5 decimal places.

$$|E_m(x) - E_n(x)| = \left| e^x - \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}\right) \right.$$

$$\leq \frac{x^{n+1}}{(n+1)!}\left[1 + \frac{x}{n} + \frac{x^2}{n^2} + \dots + \left(\frac{x}{n}\right)^{m-n-1}\right]$$

$$\leq \frac{2x^{n+1}}{(n+1)!}$$

lets let $m \to \infty$

$$\left| e^x - \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}\right) \right| \leq \frac{2x^{n+1}}{(n+1)!}$$

for $n = 9$ & $x = 1$

$$\frac{2x^{n+1}}{(n+1)!} = 5.5115 \times 10^{-7} < 10^{-5}$$

$$E_9(1) =$$

$$e \approx 1 + \frac{1}{1!} + \frac{1^2}{2!} + \dots + \frac{1^9}{9!} \approx 2.71828$$

15) If $a>0$, $a\neq 1$, show that the function $x \mapsto \log_a x$ is differentiable on $(0,\infty)$ & that $D\log_a x = 1/(x\ln(a))$ for $x \in (0,\infty)$.

Let $a>0$ & $a \neq 1$. For $x \in (0,\infty)$ let $f(x) = \log_a x$. Then

$$f(x) = \frac{\ln x}{\ln a}, x \in (0,\infty)$$

$$f'(x) = \frac{d}{dx}\left[\frac{\ln x}{\ln a}\right] = \frac{1}{\ln a}(\ln)'(x) = \frac{1}{x\ln a}$$

$$D\log_a x = \frac{1}{x\ln a}, x \in (0,\infty)$$

2) Show that $|\sin x| \leq 1$ & $|\cos x| \leq 1$ for all $x \in \mathbb{R}$.

use corollary

$$(\cos x)^2 + (\sin x)^2 = 1 \text{ for al } x \in \mathbb{R}$$

$$(|\cos x|)^2 + (|\sin x|)^2 = 1 \text{ for all } x \in \mathbb{R}$$

Suppose there exists $c \in \mathbb{R}$ such that $|\cos c| > 1$

$$1 = \cos^2 c + \sin^2 c > 1 + 0 = 1$$

which is a contradiction, therefore

$$|\cos x| \leq 1, \forall x \in \mathbb{R}$$

$$|\sin x| \leq 1, \forall x \in \mathbb{R}$$

10) Show that $c(x) \geq 1$ for all $x \in \mathbb{R}$, that both c & s are strictly increasing on $(0, \infty)$ & that $\lim_{x\to\infty} c(x) = \lim_{x\to\infty} s(x) = \infty$.

$$c(x) = \frac{e^x + e^{-x}}{2} > \frac{e^x}{2} > 0$$

therefore, $c(x) > \frac{e^x}{2} > 0$, for all $x \in \mathbb{R}$

Since $s'(x) = c(x)$, it follows s is strictly increasing on $\mathbb{R}$.
Since $s(0) = 0$, $c'(x) = s(x)$ & s is strictly increasing, means c is strictly increasing on $(0, \infty)$. Thus $c(x) > c(0) = 1$ for all $x \in (0, \infty)$
Also since c is even we conclude $c(x) \geq 1$ for all $x \in \mathbb{R}$

So,
$$\lim_{n\to\infty} c(x) = \lim_{n\to\infty} \frac{e^x + e^{-x}}{2} = \infty$$

$$\lim_{n\to\infty} s(x) = \lim_{n\to\infty} \frac{e^x - e^{-x}}{2} = \infty$$

2) Show that if a series is conditionally convergent, then the series obtained from its positive terms is divergent, & the series obtained from its negative terms is divergent.

$$a_n^+ = \frac{a_n + |a_n|}{2}$$ (a series obtained by positive)

$$a_n^- = \frac{a_n - |a_n|}{2}$$ (a series obtained by negative)

Case 1:

$|a_n| = 2a_n^+ - a_n$

$\sum |a_n| = 2\sum a_n^+ - \sum a_n$

the sum of 2 convergent series is convergent
S being conditionally convergent

Case 2: $|a_n| = a_n - 2a_n^-$

$\sum |a_n| = \sum a_n - 2\sum a_n^-$

This is a contradiction (being convergent) since S is conditionally convergent.

13) (a) Does this series $\sum_{n=1}^{\infty}\left(\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}}\right)$ converge?

$$\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}}\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(n+1)-n}{\sqrt{n}\cdot(\sqrt{n+1}+\sqrt{n})}$$

$$= \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}\cdot(\sqrt{n+1}+\sqrt{n})} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)}+n} \approx \frac{1}{2n}, \quad \lim_{n\to\infty}\frac{\frac{1}{\sqrt{n(n+1)}+n}}{\frac{1}{2n}} = \lim_{n\to\infty}\frac{2n}{\sqrt{n(n+1)}+n}$$

$$= \lim_{n\to\infty}\frac{2n}{\sqrt{n(n+1)}+n}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{2}{\sqrt{1(1+\frac{1}{n})}+1} = \frac{2}{\sqrt{1(1+0)}+1} = 1$$

due to $\frac{1}{n}$ diverges this does

(b) Does the series $\sum_{n=1}^{\infty}\left(\frac{\sqrt{n+1}-\sqrt{n}}{n}\right)$ converge?

$$\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{n} = \sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{n}\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \sum_{n=1}^{\infty}\frac{(n+1)-n}{n\cdot(\sqrt{n+1}+\sqrt{n})} < \frac{1}{n\cdot\sqrt{n}} = \frac{1}{n^{3/2}}$$

$\sum \frac{1}{n^{3/2}}$ converges via comparison test

2) Establish the convergence or divergence of the series whose nth t[erm]
is:

(a) $(n(n+1))^{-1/2}$

$$= \frac{1}{\sqrt{n^2+n}} > \frac{1}{\sqrt{2n^2}} = \frac{1}{n\sqrt{2}}$$

diverges

comparison test

(b) $(n^2(n+1))^{-1/2}$

$$= \frac{1}{\sqrt{n^3+n^2}} < \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$$

converges

comparison test

(c) $n!/n^n$

Corollary

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{(n+1)!\, n^n}{(n+1)^{n+1}\, n!}$$

$$= \frac{n^n}{(n+1)^n}$$

$$\lim_{n\to\infty} = \frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

converges

(d) $(-1)^n\, n/(n+1)$.

term test

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n n}{n+1} \neq 0$$

$\sum \frac{(-1)^n n}{n+1}$ diverges