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(2) If $f$ & $g$ are increasing functions on an interval $I \subseteq \mathbb{R}$, show that $f+g$ is an increasing function on $I$. If $f$ is also strictly increasing on $I$, then $f+g$ is strictly increasing on $I$.

Let $f, g : I \to \mathbb{R}$ be increasing functions.
Let us consider $x, y \in I$ with $x \leq y$.
So $f(x) \leq f(y)$ & $g(x) \leq g(y)$. Thus
$f(x) + g(x) \leq f(y) + g(y)$ is
$(f+g)(x) \leq (f+g)(y)$. This proves
$f+g$ is increasing.

Now strictly increasing, (take into account above notation)
$x < y$ we have $f(x) < f(y)$ & $g(x) \leq g(y)$.
So $f(x) + g(x) < f(y) + g(y)$.
This proves that $f+g$ is a strictly increasing function.

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(14) Let $x \in \mathbb{R}$, $x > 0$, & if $r, s \in \mathbb{Q}$ show that
$$x^r x^s = x^{r+s} = x^s x^r \ \& \ (x^r)^s = x^{rs} = (x^s)^r.$$

We can write $\left(x^{\frac{1}{n}}\right)^m$ (theorem)

$$\left(x^{\frac{1}{n}}\right)^m = x^{\frac{m}{n}}$$

As $\frac{m}{n} = \frac{p}{q}$

$$x^{\frac{m}{n}} = x^{\frac{p}{q}} \quad \text{(Def)}$$

$$x^{\frac{p}{q}} = \left(x^{\frac{1}{q}}\right)^p.$$

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,b] & $f: I \to \mathbb{R}$. We say that $f$ is "locally increasing" at $c \in I$ if there exists $\delta(c) > 0$ such that $f$ is increasing on $I \cap [c-\delta(c), c+\delta(c)]$.
Prove that if $f$ is locally increasing at every point of $I$, then $f$ is increasing on $I$.

$$I \subseteq \bigcup_{i=1}^{n} (c_i - \delta(c_i), c_i + \delta(c_i)) \subseteq \bigcup_{i=1}^{n} [c_i - \delta(c_i), c_i + \delta(c_i)]$$

1.

Let $x, y \in [c_i - \delta(c_i), c_i + \delta(c_i)]$

As $f$ is increasing on $[c_i - \delta(c_i), c_i + \delta(c_i)]$

$$x < y \Rightarrow f(x) \leq f(y)$$

2.

$x \in [\ldots, \ldots]$ & $y \in [\ldots, \ldots]$

$$f(x) \leq f(c_i + \delta(c_i)) \leq f(c_j - \delta(c_j)) \leq f(y)$$

$$\Downarrow$$

$$f(x) \leq f(y)$$

So $x < y \Rightarrow f(x) \leq f(y)$ & it follows that $f$ is increasing on $I$.

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(13) Let $A \subseteq \mathbb{R}$ & Suppose that $f: A \to \mathbb{R}$ has the property: for each $\varepsilon > 0$ there exists a function $g_\varepsilon : A \to \mathbb{R}$ such that $g_\varepsilon$ is uniformly continuous on $A$ & $|f(x) - g_\varepsilon(x)| < \varepsilon$ for all $x \in A$. Prove that $f$ is uniformly continuous on $A$.

Assume $\varepsilon > 0$ so there exists uniformly continuous function.

$$|f(x) - g_{\frac{\varepsilon}{3}}(x)| < \frac{\varepsilon}{3}$$

$\delta > 0$

$$|g_{\frac{\varepsilon}{3}}(x) - g_{\frac{\varepsilon}{3}}(x)| < \frac{\varepsilon}{3} \text{ whenever } |x-y| < \delta$$

$$\begin{aligned}
|f(x) - f(y)| &= |f(x) - g_{\frac{\varepsilon}{3}}(x) + g_{\frac{\varepsilon}{3}}(x) - g_{\frac{\varepsilon}{3}}(y) + g_{\frac{\varepsilon}{3}}(y) - f(y)| \\
&\le |\ \text{"} - \text{"}\ | + |\ \text{"} - \text{"}\ | + |\ \text{"} - \text{"}\ | \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

, $|f(x) - f(y)| < \varepsilon$ whenever $|x-y| < \delta$

So, $|f(x) - g_{\frac{\varepsilon}{3}}(x)| < \frac{\varepsilon}{3}$

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(13) Let $A \subseteq \mathbb{R}$ & Suppose that $f: A \to \mathbb{R}$ has the property: for each $\varepsilon > 0$ there exists a function $g_\varepsilon : A \to \mathbb{R}$ such that $g_\varepsilon$ is uniformly continuous on $A$ & $|f(x) - g_\varepsilon(x)| < \varepsilon$ for all $x \in A$. Prove that $f$ is uniformly continuous on $A$.

Assume $\varepsilon > 0$ so there exists uniformly continuous function.

$$|f(x) - g_{\frac{\varepsilon}{3}}(x)| < \frac{\varepsilon}{3}$$

$\delta > 0$

$$|g_{\frac{\varepsilon}{3}}(x) - g_{\frac{\varepsilon}{3}}(x)| < \frac{\varepsilon}{3} \text{ whenever } |x-y| < \delta$$

$$|f(x) - f(y)| = |f(x) - g_{\frac{\varepsilon}{3}}(x) + g_{\frac{\varepsilon}{3}}(x) - g_{\frac{\varepsilon}{3}}(y) + g_{\frac{\varepsilon}{3}}(y) - f(y)|$$

$$\leq |\quad{}'' \quad - \quad{}'' \quad| + |\quad{}'' \quad - \quad{}'' \quad| + |\quad{}'' \quad - \quad{}'' \quad|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

, $|f(x) - f(y)| < \varepsilon$ whenever $|x-y| < \delta$

So, $|f(x) - g_{\frac{\varepsilon}{3}}(x)| < \frac{\varepsilon}{3}$

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(2) Suppose that $\delta_1$ is the gauge defined by $\delta_1(0) = \frac{1}{4}$, $\delta(t) = \frac{3}{4}t$ for $t \in (0,1]$. Are the partitions given in Exercise 1 $\delta_1$-fine? Note that $\delta(t) \leq \delta_1(t)$ for all $t \in [0,1]$.

$$0 \in \left[0,\tfrac{1}{4}\right] \subseteq [0-\delta(0), 0+\delta(0)] \subseteq [0-\delta_1(0), 0+\delta_1(0)]$$

$$\tfrac{1}{2} \in \left[\tfrac{1}{4},\tfrac{1}{2}\right] \subseteq \left[\tfrac{1}{2}-\delta(\tfrac{1}{2}), \tfrac{1}{2}+\delta(\tfrac{1}{2})\right] \subseteq \left[\tfrac{1}{2}-\delta_1(\tfrac{1}{2}), \tfrac{1}{2}+\delta_1(\tfrac{1}{2})\right]$$

then

$$\left[\tfrac{3}{5}-\delta_1(\tfrac{3}{5}), \tfrac{3}{5}+\delta_1(\tfrac{3}{5})\right] = \left[\tfrac{3}{5}-\tfrac{9}{20}, \tfrac{3}{5}+\tfrac{9}{20}\right] = \left[\tfrac{3}{20}, \tfrac{21}{20}\right]$$

Now:

$$\tfrac{3}{5} \in \left[\tfrac{1}{2}, 1\right] \subseteq \left[\tfrac{3}{20}, \tfrac{21}{20}\right] = \left[\tfrac{3}{5}-\delta_1(\tfrac{3}{5}), \tfrac{3}{5}+\delta_1(\tfrac{3}{5})\right]$$

thus, $P_2$ is $\delta_1$-fine.

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(2) Show that function $f(x)=1/x$ is uniformly continuous on $A=[1,\infty)$, but that is not uniformly continuous on $B=(0,\infty)$.

$x, y \in A$

$$\left|\frac{1}{x^2}-\frac{1}{y^2}\right| = \frac{|x^2-y^2|}{xy}$$

$$= \frac{|x-y|\cdot|x+y|}{xy}$$

$$= \left(\frac{1}{x}+\frac{1}{y}\right)\cdot|x-y|$$

$x, y \in [1,\infty]$ follows $\frac{1}{x}\leq 1$, $\frac{1}{y}\leq 1$

$$\left(\frac{1}{x}+\frac{1}{y}\right)\leq(1+1) \Rightarrow \left(\frac{1}{x}+\frac{1}{y}\right)\cdot|x-y| \leq (1+1)\cdot|x-y|$$

$$\left|\frac{1}{x^2}-\frac{1}{y^2}\right| = \left(\frac{1}{x}+\frac{1}{y}\right)\cdot|x-y| \leq (1+1)\cdot|x-y|$$

$$\Downarrow$$

$$\left|\frac{1}{x^2}-\frac{1}{y^2}\right| \leq 2\cdot|x-y| \quad \forall x, y \in A$$

So $f$ is uniformly continuous on $A$

$x_n, y_n \in B$

Such that $x_n = \frac{1}{\sqrt{n}}$, $y_n = \frac{1}{\sqrt{n+1}}$

$$\left|\frac{1}{x^2}-\frac{1}{y^2}\right| = \left|(\sqrt{n})^2-(\sqrt{n+1})^2\right|$$

$$= |n+1-n|$$

$$= 1$$

so $f$ is not uniformly continuous on $B$

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(17) If $f : [0,1] \to \mathbb{R}$ is continuous & has only rational values, must $f$ be constant? Prove your assertion.

Lets prove $f$ is constant & a continuous function that takes rational values

Suppose $f(x)$ is not a constant

Then there exists $a, b \in \mathbb{Q}$ Such that $f(a) \neq f(b)$.

Via density theorem there's an irrational # between rational values $c \in [f(a), f(b)]$, such that $c \notin \mathbb{Q}$

we know $f$ is continuous & by Bolzano's intermediate value theorem, $f$ must take value $c$ which is a contradiction. Thus $f$ only takes rational vals.

So, $f(a) = f(b)$ $\forall a, b \in [0,1]$

Thus $f$ is constant.

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(17) If $f : [0,1] \to \mathbb{R}$ is continuous & has only rational values, must $f$ be constant? Prove you assertion.

Lets prove $f$ is constant & a continuous function that takes rational values

Suppose $f(x)$ is not a constant

Then there exists $a, b \in Q$
Such that $f(a) \neq f(b)$.

Via density theorem there's an irrational # between rational values
$c \in [f(a), f(b)]$, such that $c \notin Q$

we know $f$ is continuous & by Bolzano's intermediate value theorem, $f$ must take value $c$ which is a contradiction.
Thus $f$ only takes rational vals.

So, $f(a) = f(b)\ \forall a, b \in [0,1]$
Thus $f$ is constant.

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(13) Suppose that $f$ is a continuous additive function on $\mathbb{R}$. If $c = f(1)$, show that we have $f(x) = cx$ for all $x \in \mathbb{R}$. [Hint: First show that if $r$ is a rational number, then $f(r) = cr$.]

$$f(0) = f(0+0) = f(0) + f(0)$$

$$\Rightarrow f(0) = 0$$

$n=1$ $\quad f(1) = c = c \cdot 1$

$n \in \mathbb{N}$ $\quad f(n) = cn \quad (*)$

by additivity $(*)$ & $f(1) = c$

$$f(n+1) = f(n) + f(1)$$
$$= cn + c$$
$$= c(n+1)$$

$$\boxed{f(n) = cn, \forall n \in \mathbb{N}}$$

For $m, n \in \mathbb{N}$, $n \neq 0$ $f$ additive, $f(1) = c$

$$f\left(\frac{m}{n}\right) = f\Big(\underbrace{\frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}}_{m \text{ times}}\Big)$$
$$= m \cdot f\left(\frac{1}{n}\right)$$
$$= \frac{m}{n} \cdot n\left(f\left(\frac{1}{n}\right)\right)$$
$$= \frac{m}{n} f\left(\frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}\right)$$
$$= \frac{m}{n} f\left(\frac{n}{n}\right)$$
$$= \frac{m}{n} f(1)$$
$$= c\frac{m}{n}$$

$f(0) = f(r + (-r)) = f(r) + f(-r)$

$\Rightarrow f(-r) = -f(r), \forall r \in \mathbb{Q}, r > 0$ $\quad \boxed{f(r) = cr, \forall r \in \mathbb{Q}}$

Since $\mathbb{Q}$ is dense on $\mathbb{R}$

use $x_n \in \mathbb{Q}, \forall n \Rightarrow f(x_n) = cx_n$

$$f(x) = f\left(\lim_{x_n \to x} x_n\right) = \lim_{x_n \to x} f(x_n) = \lim_{x_n \to x} f(x_n)$$
$$= \lim_{x_n \to x} cx_n$$
$$= c \lim_{x_n \to x} x_n$$
$$= cx, \forall x \in \mathbb{R}$$

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(2) Let $I=[a,b]$ & let $f:I\to\mathbb{R}$ & $g:I\to\mathbb{R}$ be continuous functions on $I$. Show that the set $E=\{x\in I: f(x)=g(x)\}$ has the property that if $(x_n)\subseteq E$ & $x_n\to x_0$, then $x_0\in E$.

Let $(x_n)\subseteq E$ & $x_n\to x_0$.

Since $(x_n)\subseteq E$, we have $f(x_n)=g(x_n)$ for all $n$. By Sequential Criterion for continuity, taking limit $n\to\infty$ we get

$$\lim_n g(x_n)=\lim_n f(x_n) \Leftrightarrow g(\lim_{n\to\infty} x_n)=f(\lim_{n\to\infty} x_n)$$

which implies

$$g(x_0)=f(x_0)$$

Therefore, $x_0\in E$

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(2) let $I = [a, b]$ & let $f: I \to \mathbb{R}$ & $g: I \to \mathbb{R}$ be continuous functions on $I$. Show that the set $E = \{x \in I : f(x) = g(x)\}$ has the property that if $(x_n) \subseteq E$ & $x_n \to x_0$, then $x_0 \in E$.

let $(x_n) \subseteq E$ & $x_n \to x_0$.

Since $(x_n) \subseteq E$, we have $f(x_n) = g(x_n)$ for all $n$. By Sequential Criterion for continuity, taking limit $n \to \infty$ we get

$$\lim_n g(x_n) = \lim_n f(x_n) \Leftrightarrow g(\lim_{n\to\infty} x_n) = f(\lim_{n\to\infty} x_n)$$

which implies

$$g(x_0) = f(x_0)$$

Therefore, $x_0 \in E$

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For $m, n \in \mathbb{N}$, $n \neq 0$ f additive, $f(1) = c$

$$f\left(\frac{m}{n}\right) = f\left(\underbrace{\frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}}_{m \text{ times}}\right)$$

$$= m \cdot f\left(\frac{1}{n}\right)$$

$$= \frac{m}{n} \cdot n\left(f\left(\frac{1}{n}\right)\right)$$

$$= \frac{m}{n} f\left(\frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}\right)$$

$$= \frac{m}{n} f\left(\frac{n}{n}\right)$$

$$= \frac{m}{n} f(1)$$

$$= c\frac{m}{n}$$

$f(0) = f(r + (-r)) = f(r) + f(-r)$

$\Rightarrow f(-r) = -f(r), \forall r \in \mathbb{Q}, r > 0$ $\boxed{f(r) = cr, \forall r \in \mathbb{Q}}$

(13) Suppose that $f$ is a continuous additive function on $\mathbb{R}$. If $c = f(1)$, show that we have $f(x) = cx$ for all $x \in \mathbb{R}$. [Hint: First show that $r$ is a rational ⇒ then $f(r) = cr$.]

$$f(0) = f(0+0) = f(0) + f(0)$$

$$\Rightarrow f(0) = 0$$

$n=1$ $\quad f(1) = c = c \cdot 1$

$n \in \mathbb{N}$ $\quad f(n) = cn \; (*)$

by additivity $(*)$ & $f(1) = c$

$$f(n+1) = f(n) + f(1)$$

$$= cn + c$$

$$= c(n+1)$$

$$\boxed{f(n) = cn, \forall n \in \mathbb{N}}$$

since $\mathbb{Q}$ is dense on $\mathbb{R}$

$$f(x) = f\left(\lim_{x_n \to x} x_n\right) = \lim_{x_n \to x} f(x_n) = \lim_{x_n \to x} f(x_n)$$

use $x_n \in \mathbb{Q}, \forall n \Rightarrow f(x_n) = cx_n$

$$= \lim_{x_n \to x} cx_n$$

$$= c \lim_{x_n \to x} x_n$$

$$= cx, \forall x \in \mathbb{R}$$

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(2) Show that if $f: A \to \mathbb{R}$ is continuous on $A \subseteq \mathbb{R}$ & if $n \in \mathbb{N}$, then the function $f^n$ defined by $f^n(x) = (f(x))^n$, for $x \in A$, is continuous at $A$.

$f^1 = f$

Suppose $f^n$ continuous at $A$

$f^{n+1} = f^n f$

$f^{n+1}$ being the product of two continuous functions on $A$ is continuous.

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(13) Define $g:\mathbb{R}\to\mathbb{R}$ by $g(x)=2x$ for $x$ rational & $g(x)=x+3$ for $x$ irrational. Find all points at which $g$ is continuous.

$$g(c)=\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} 2x_n = 2c$$

$$g(c)=\lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} y_n + 3 = c+3$$

$$2c=c+3$$
$$c=3$$

$$|g(x)-g(3)| = |g(x)-6|$$
$$\leq \sup\{|2x-6|, |(x+3)-x|\}$$
$$= \sup\{2|x-3|, |x-3|\}$$
$$= 2|x-3|$$

$\varepsilon>0$ we can choose $\delta=\frac{\varepsilon}{2}$

$$|x-3|<\delta \Rightarrow |g(x)-g(3)|<\varepsilon$$

($g$ is only continuous at 3.)

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(13) Let $f$ & $g$ be defined on $(a, \infty)$ & suppose $\lim_{x\to\infty} f = L$ & $\lim_{x\to\infty} g = \infty$. Prove that $\lim_{x\to\infty} f\circ g = L$.

(Step 1)

Since $\lim_{x\to\infty} f(x) = L$, for any $\varepsilon > 0$ $\exists$ $K(\varepsilon) > 0$ such that whenever $x > K(\varepsilon)$, then $|f(x) - L| < \varepsilon$

(Step 2)

but as $\lim_{x\to\infty} g(x) = \infty$, there is $K'(K(\varepsilon)) > 0$ such that if $x > K'(K(\varepsilon))$, then $g(x) > K(\varepsilon)$.

(Step 3)

Combining these two observations we get that $\delta > 0$, $\exists$ $K(\delta) > 0$ such that if $x > K(\delta)$, then $|f\circ g(x) - L| < \delta$, thus $\lim_{x\to\infty} f\circ g(x) = L$.

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(13) Let $f$ & $g$ be defined on $(a, \infty)$ & suppose $\lim_{x\to\infty} f = L$ & $\lim_{x\to\infty} g = \infty$. Prove that $\lim_{x\to\infty} f\circ g = L$.

(Step 1)

Since $\lim_{x\to\infty} f(x) = L$, for any $\varepsilon > 0$ $\exists\, K(\varepsilon) > 0$ such that whenever $x > K(\varepsilon)$, then $|f(x) - L| < \varepsilon$

(Step 2)

But as $\lim_{x\to\infty} g(x) = \infty$, there is $K'(K(\varepsilon)) > 0$ such that if $x > K'(K(\varepsilon))$, then $g(x) > K(\varepsilon)$

(Step 3)

Combining these two observations we get that $\delta > 0$, $\exists\, K(\delta) > 0$ such that if $x > K(\delta)$, then $|f\circ g(x) - L| < \delta$, thus $\lim_{x\to\infty} f\circ g(x) = L$.

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(2) Establish the discontinuity criterion 5.1.4.

Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ & $c \in A$

Let $f$ be discontin at $c$.

Suppose every sequence $(x_n)$ in $A$ converging to $c$, the sequence $(f(x_n))$ converges to $f(c)$.
then through sequential criterion for continuity $f$ is continuous at $c$, which is not possible.
There is a sequence $x_n$ in $A$ such that $(x_n)$ converges to $c$, but $f(x_n)$ does not converg to $c$.

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(2) give an example of a function that has a right-hand limit but not a left hand limit at a point.

right hand L :

$$f(x) = \begin{cases} \sin(\frac{1}{x}), & x<0 \\ 0, & x \geq 0 \end{cases}$$

clearly $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 0 = 0$

so $x=0$ exists

left hand L :

$x_n, y_n < 0$ & $x_n, y_n \to 0$

$$\sin\left(\frac{1}{x_n}\right) = \sin(-2n\pi) = 0 \neq -1 = \sin\left(2n\pi + \frac{\pi}{2}\right) = \sin\left(\frac{1}{y_n}\right)$$

So it does not exist.

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(13) Functions $f$ & $g$ are defined on $\mathbb{R}$ by $f(x)=x+1$ & $g(x)=2$ if $x\neq 1$ & $g(1)=0$.

(a) Find $\lim_{x\to 1} g(f(x))$ & compare with $g(\lim_{x\to 1} f(x))$.

(b) Find $\lim_{x\to 1} f(g(x))$ & compare with $f(\lim_{x\to 1} g(x))$.

(a)

$$g(x)=\begin{cases} 2 & \text{if } x\neq 1 \\ 0 & \text{if } x=1 \end{cases}$$

$$g\circ f = g(f(x))=\begin{cases} 2 & \text{if } \neq 0 \\ 0 & \text{if } =0 \end{cases}$$

$$\lim_{x\to 1} g(f(x)) = 2$$

$$g(\lim_{x\to 1} f(x)) = g(2)=2$$

(b) $\lim_{x\to 1} f(g(x))=3$

$$f(\lim_{x\to 1} g(x))=3$$

$$f\circ g = f(g(x))=\begin{cases} 3 & \text{if } x\neq 1 \\ 1 & \text{if } x=1 \end{cases}$$

3.5- Cauchy Criterion 3 13
3.6- Properly divergent sequences 2
3.7- Intro to Infinite series 2 13 $\frac{10}{16}$
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(b) $\lim_{x\to 2} \frac{x^2-4}{x-2}$, $x>0$

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}}$$

$$\lim_{x\to 2}(x+2)$$

$$= 4$$

(c) $\lim_{x\to 0}\frac{(x+1)^2-1}{x}$, $x>0$

$$\lim_{x\to 0}\frac{(x+1)^2-1}{x} = \lim_{x\to 0}\frac{(x^2+2x+1)-1}{x}$$

$$= \lim_{x\to 0}\frac{x^2+2x}{x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to 0}(x+2) = 2$$

(2) Determine the following limits & State which theorems are use in each case.

(a) $\lim_{x\to 2}\sqrt{\frac{2x+1}{x+3}}$, $x>0$

$$\lim_{x\to 2}(2x+1)(x+3) = \frac{\lim_{x\to 2}(2x+1)}{\lim_{x\to 2}(x+3)}$$

$$= \frac{5}{5}$$

$$= 1$$

(d) $\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$, $x>0$

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1}\frac{\sqrt{x}-1}{(\sqrt{x}+1)(\sqrt{x}-1)}$$

$$= \lim_{x\to 1}\frac{1}{\sqrt{x}+1}$$

$$= \frac{\lim_{x\to 1} 1}{\lim_{x\to 1}(\sqrt{x}+1)}$$

$$= \frac{1}{\lim_{x\to 1}\sqrt{x} + \lim_{x\to 1} 1}$$

$$= \frac{1}{\sqrt{\lim_{x\to 1} x} + 1}$$

$$= \frac{1}{\sqrt{1}+1}$$

$$= \frac{1}{2}$$

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(16) use the Cauchy Condensation Test to discuss the $p$-series $\sum_{n=1}^{\infty}(1/n^p)$ for $p>0$.

Cauchy Condensation test states

$$\sum_{n=1}^{\infty} a_n,$$

where $a_n$ is decreas, pos, sequence, converges if and only if:

$$\sum_{n=1}^{\infty} 2^n a_{2^n},$$

The $n^{th}$ term of our series is $a_n=\frac{1}{n^p}$. Since $p>0$ we obtain that $a_n$ is strictly, pos, & decreasing

$$\sum_{n=1}^{\infty} 2^n a_{2^n} = \sum_{n=1}^{\infty} 2^n \frac{1}{(2^n)^p} = \sum_{n=1}^{\infty} 2^{n(1-p)}$$

this is a geometric series

only Converges for

$$|r|<1$$

$$2^{n(1-p)}<1$$

$$n(1-p)<0$$

$$1-p<0$$

$$p>1.$$

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(2) Determine a condition $|x-4|$ that will assure:

(a) $|\sqrt{x}-2|<\frac{1}{2}$ (b) $|\sqrt{x}-2|<10^{-2}$

Given $x\geq 0$ we have that $x-4=(\sqrt{x}-2)(\sqrt{x}+2)$ & $0\leq\sqrt{x}$ which gives us that $|\sqrt{x}-2|=\frac{|x-4|}{\sqrt{x}+2}$ & $\frac{1}{\sqrt{x}+2}\leq\frac{1}{2}$ & so

$$|\sqrt{x}-2|\leq\frac{|x-4|}{2}$$

(a) if $|x-4|<1$ then $|\sqrt{x}-2|<\frac{1}{2}$ &

(b) if $|x-4|<2\cdot 10^{-2}$ then $|\sqrt{x}-2|<10^{-2}$.

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(15) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by setting $f(x) = x$ if $x$ is rational, & $f(x) = 0$ if $x$ is irrational.

(a) Show that $f$ has a limit at $x = 0$

(b) Use a sequential argument to show that if $c \neq 0$, then $f$ does not have a limit at $c$.

(a)

For any $\varepsilon > 0$ choose $\delta = \varepsilon$, then, for any $x \in \mathbb{R}$ with $|x| < \delta$

$$|f(x) - 0| = |f(x)| = \begin{Bmatrix} |x|, & x \in Q \\ 0, & x \notin Q \end{Bmatrix}$$

$$\Rightarrow |f(x)| < \varepsilon$$

therefore $x = 0$ has a limit which gives 0

(b) For $0 \neq c \in \mathbb{R}$ let $(x_n)$ & $(y_n)$ be sequences both converging to $c$ such that $(x_n) \in \mathbb{R} \setminus Q$ & $(y_n) \in Q$ for all $n \in \mathbb{N}$. This is possible because $\mathbb{R} \setminus Q$ & $Q$ are dense sets.

So

$$f(x_n) = 0 \ \forall n$$
$$f(y_n) = y_n \ \forall n$$

So $\lim\limits_{n \to \infty} f(x_n) = 0$ but $\lim\limits_{n \to \infty} f(y_n) = y_n$

& thus $f$ does not have a limit at $x = c \neq 0$

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$$\sum_{n=1}^{\infty} 2^n a_{2^n},$$

The $n^{th}$ term of our series is $a_n = \frac{1}{n^p}$. Since $p>0$ we obtain that $a_n$ is strictly, pos, & decreasing

$$\sum_{n=1}^{\infty} 2^n a_{2^n} = \sum_{n=1}^{\infty} 2^n \frac{1}{(2^n)^p} = \sum_{n=1}^{\infty} 2^{n(1-p)}$$

this is a geometric series

only Converges for

$$|r|<1$$

$$2^{n(1-p)}<1$$

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$$1-p<0$$

$$p>1.$$

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(13) If $\sum a_n$ with $a_n > 0$ is convergent, then is $\sum \sqrt{a_n}$ always convergent? Either prove or give a counter example.

By inequality of arithmetic means & geometric means;

$$\frac{a_n + a_{n+1}}{2} > \sqrt{a_n a_{n+1}}.$$

$$\sum_{n=1}^{\infty}(a_n) > \sum_{n=1}^{\infty}\left(\frac{a_n + a_{n+1}}{2}\right) > \sum_{n=1}^{\infty}\sqrt{a_n a_{n+1}}$$

$$\sum_{n=1}^{\infty}\sqrt{a_n a_{n+1}}$$

$$\frac{a_1}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n + a_{n+1}}{2}\right)$$

$$= \frac{a_1}{2} + \frac{a_1}{2} + \frac{a_2}{2} + \frac{a_2}{2} + \frac{a_3}{2} + \dots$$

defined as $\sum_{n=1}^{\infty}(a_n)$

So given that $\sum_{n=1}^{\infty} a_n$

$$\sum_{n=1}^{\infty}(a_n) > \sum_{n=1}^{\infty}\left(\frac{a_n + a_{n+1}}{2}\right) > \sum_{n=1}^{\infty}\sqrt{a_n a_{n+1}}$$

$$\sum_{n=1}^{\infty}\sqrt{a_n a_{n+1}}$$

is always convergent

3.5- Cauchy Criterion $\underline{2}$ $\underline{13}$
3.6- Properly divergent sequences $\underline{2}$ $\underline{10}$
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(2) Show that the convergence of a series is not affected by changing a finite number of its terms (of course, the value of the sum may be changed)

Let $S = \sum_{i=1}^{\infty} x_n$ be an arbitrary series. Let $A = \{n : x_n \text{ is changed}\}$ be a finite set, & let $S' = \sum_{i=1}^{\infty} x_n'$ be a new series, created by the changes from $A$.

Since $A$ is a finite set, $n_0 = \max\{n : x_n \in A\}$ exists. So, for $n > n_0$, $x_n$ is not changed, which means that $S'' = \sum_{i=n_0+1}^{\infty} x_n$ is the $n_0$-tail of $S'$ & $S$.

By theorem 3.14 that $S$ converges $\Leftrightarrow$ $S''$ converges, & that $S'$ converges $\Leftrightarrow$ $S''$ converges. These two statements together say:

$S$ converges $\Leftrightarrow$ $S'$ converges

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(13) If $x_1 = 2$ & $x_{n+1} = 2 + \frac{1}{x_n}$ for $n \geq 1$, show that $(x_n)$ is a contractive sequence. What is its limit?

$(x_n)$ is contractive if $0 < C < 1$ exists so:

$$|x_{n+2} - x_{n+1}| \leq C \cdot |x_{n+1} - x_n|, \forall n$$

$$|x_{n+2} - x_{n+1}| = \left|\left(2 + \frac{1}{x_{n+1}}\right) - \left(2 + \frac{1}{x_n}\right)\right|$$

$$= \left|\frac{1}{x_{n+1}} - \frac{1}{x_n}\right|$$

$$= \left|\frac{x_n - x_{n+1}}{x_{n+1} \cdot x_n}\right|$$

$$\leq \left|\frac{x_n - x_{n+1}}{2 \cdot 2}\right|$$

$$= \frac{1}{4}|x_n - x_{n+1}|$$

$$C = \frac{1}{4}$$

$x_n \geq 2, \forall n$

$$x_{n+1} = 2 + \frac{1}{x_n}$$

$$\geq 2 + 0$$

$$= 2$$

So $x_n \geq 2 \ \forall n$

$$\lim(x_{n+1}) = \lim(x_n) = x$$

$$x_{n+1} = 2 + \frac{1}{x_n} \quad \Big| \lim$$

$$x = 2 + \frac{1}{x} \quad \Big| \cdot x$$

$$x^2 = 2x + 1$$

$$x^2 - 2x - 1 = 0$$

$$x = \frac{2 \pm \sqrt{4 - 4 \cdot 1 \cdot (-1)}}{2}$$

$$x_a = 1 - \sqrt{2} < 2$$

$$x_b = 1 + \sqrt{2} > 2$$

Since $x_n \geq 2$ $\lim(x_n) = 1 + \sqrt{2}$

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(2) Give examples of properly divergent sequences $(x_n)$ & $(y_n)$ with $x_n \neq 0$ for all $n \in \mathbb{N}$ such that:
(a) $(x_n/y_n)$ is convergent (b) $(x_n/y_n)$ is properly divergent.

(a) $x_n = \sqrt{n}$, $y_n = n$, $n \in \mathbb{N}$

$(x_n), (y_n)$ are properly divergent since they're unbounded

$$\frac{x_n}{y_n} = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} \xrightarrow{n \to \infty} 0$$

Therefore, $\left(\frac{x_n}{y_n}\right)$ is convergent & $\lim_{n \to \infty} \frac{x_n}{y_n} = 0$

(b) $x_n = n$, $y_n = \sqrt{n}$, $n \in \mathbb{N}$

$(x_y), (y_n)$ are properly divergent sequences since they are unbounded

$$\frac{x_n}{y_n} = \frac{n}{\sqrt{n}} = \sqrt{n}$$

therefore, it is properly divergent.

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(10) Show that if $\lim (a_n/n) = L$, where $L > 0$, then $\lim (a_n) = +\infty$.

We know $\lim_{n\to\infty} x_n y_n$ exists therefore $(x_n y_n)$ is a convergent series that's bounded.

So $\exists M > 0$ such that $|x_n y_n| < M$ $n \in \mathbb{N}$. let us $\varepsilon > 0$. Then $\frac{\varepsilon}{M} > 0$.

Archimedean property

$$0 < \frac{1}{m} < \frac{\varepsilon}{M} \Rightarrow \frac{M}{m} < \varepsilon$$

Since $(x_n)$ is properly divergent meaning $\lim_{n\to\infty} x_n = \pm\infty$ then $\exists K \in \mathbb{N}$ such that $\forall n \geq K$

$$x_n > m$$

$$\Rightarrow |x_n| > m$$

$$\Rightarrow \frac{|x_n||y_n|}{m} > |y_n|$$

$$\Rightarrow |y_n| < \frac{|x_n| \cdot |y_n|}{m} < \frac{M}{m} < \varepsilon$$

Thus, $\lim_{n\to\infty} y_n = 0$

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$X_n = \sqrt{n}$, Show that $(X_n)$ satisfies $\lim |X_{n+1} - X_n| = 0$, but that is not a Cauchy Sequence.

( $X_n$ is unbounded & is not a Cauchy sequence )

$$|X_{n+1} - X_n| = \sqrt{n+1} - \sqrt{n} = \sqrt{n+1} - \sqrt{n}\left(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)$$

$$= \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

now apply the limit

$$\lim_{n\to\infty} |X_{n+1} - X_n| = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$$

So $\sqrt{n+1} - \sqrt{n} = \dfrac{1}{\sqrt{n+1}+\sqrt{n}}$

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(2) Show directly from the definition that the following are not Cauchy Sequences.

(a) $\left(\frac{n+1}{n}\right)$ (b) $\left(1+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)$

(a) for $\varepsilon > 0$, $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \frac{\varepsilon}{2}$$

for every $m, n \in \mathbb{N}, m > n \geq n_0$

$$\left|\frac{m+1}{m} - \frac{n+1}{n}\right| = \left|\frac{1}{m} - \frac{1}{n}\right|$$

use triangle inequality

$$\left|\frac{m+1}{m} - \frac{n+1}{n}\right| \leq \frac{1}{m} + \frac{1}{n}$$

use property $\frac{1}{m} < \frac{1}{n}$

$$\left|\frac{m+1}{m} - \frac{n+1}{n}\right| \leq \frac{2}{n}$$

Using $n \geq n_0$

$$\left|\frac{m+1}{m} - \frac{n+1}{n}\right| \leq \frac{2}{n_0}$$, we now apply $\frac{1}{n_0} < \frac{\varepsilon}{2}$, $\left|\frac{m+1}{m} - \frac{n+1}{n}\right| < \varepsilon$, Thus $\left(1+\frac{1}{n}\right)$ Cauchy Sequence

(b) for $\varepsilon > 0$, $n_0 \in \mathbb{N}$ for which $\frac{1}{2^{n_0-1}} < \varepsilon$

for $m, n \in \mathbb{N}, m > n \geq n_0$

$$= \left|\left(1+\frac{1}{2!}+\cdots+\frac{1}{m!}\right) - \left(1+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)\right|$$

$$= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{m!}$$

use property $2^k < k!$ for $k \geq 4$

$$\left|\left(1+\frac{1}{2!}+\cdots+\frac{1}{m!}\right) - \left(1+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)\right| \leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots + \frac{1}{2^m}$$

$$= \frac{1}{2^n}\left(\frac{1}{2} + \frac{1}{2^2} + \cdots \frac{1}{2^{m-n}}\right)$$

The sum of $\sum_{n=0}^{\infty} \frac{1}{2^n}$

$$= \left|\left(1+\frac{1}{2!}+\cdots+\frac{1}{m!}\right) - \left(1+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)\right| \leq \frac{1}{2^n}\left(1+\frac{1}{2}+\frac{1}{4}+\cdots\right)$$

$$= \frac{1}{2^n} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{2^{n-1}} \leq \frac{1}{2^{n_0-1}} < \varepsilon$$, Thus $\left(1+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)$ is the Cauchy Sequence.

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5.4- Uniform Continuity 2 5 9 13
5.5- Continuity & Groups 2 5 9 13
5.6- monotone & inverse functions 2 5 9 13

(2) Show directly from the definition that the following are not Cauchy sequences.

(a) $\left(\frac{n+1}{n}\right)$ (b) $\left(1+\frac{1}{2!}+\dots+\frac{1}{n!}\right)$

(a) for $\varepsilon > 0$, $n_0 \in \mathbb{N}$ such that

$\frac{1}{n_0} < \frac{\varepsilon}{2}$

for every $m, n \in \mathbb{N}, m > n \geq n_0$

$\left|\frac{m+1}{m} - \frac{n+1}{n}\right| = \left|\frac{1}{m} - \frac{1}{n}\right|$

use triangle inequality

$\left|\frac{m+1}{m} - \frac{n+1}{n}\right| \leq \frac{1}{m} + \frac{1}{n}$

use property $\frac{1}{m} < \frac{1}{n}$

$\left|\frac{m+1}{m} - \frac{n+1}{n}\right| \leq \frac{2}{n}$

Using $n \geq n_0$

$\left|\frac{m+1}{m} - \frac{n+1}{n}\right| \leq \frac{2}{n_0}$, we now apply $\frac{1}{n_0} < \frac{\varepsilon}{2}$, $\left|\frac{m+1}{m} - \frac{n+1}{n}\right| < \varepsilon$, Thus $\left(1+\frac{1}{n}\right)$ Cauchy Sequence.

(b) for $\varepsilon > 0$, $n_0 \in \mathbb{N}$ for which $\frac{1}{2^{n_0-1}} < \varepsilon$

for $m, n \in \mathbb{N}, m > n \geq n_0$

$= \left|\left(1+\frac{1}{2!}+\dots+\frac{1}{m!}\right) - \left(1+\frac{1}{2!}+\dots+\frac{1}{n!}\right)\right|$

$= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots + \frac{1}{m!}$

use property $2^k < k!$ for $k \geq 4$

$\left|\left(1+\frac{1}{2!}+\dots+\frac{1}{m!}\right) - \left(1+\frac{1}{2!}+\dots+\frac{1}{n!}\right)\right| \leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \dots + \frac{1}{2^m}$

$= \frac{1}{2^n}\left(\frac{1}{2} + \frac{1}{2^2} + \dots \frac{1}{2^{m-n}}\right)$

The sum of $\sum_{n=0}^{\infty} \frac{1}{2^n}$

$= \left|\left(1+\frac{1}{2!}+\dots+\frac{1}{m!}\right) - \left(1+\frac{1}{2!}+\dots+\frac{1}{n!}\right)\right| \leq \frac{1}{2^n}\left(1+\frac{1}{2}+\frac{1}{4}+\dots\right)$

$= \frac{1}{2^n} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{2^{n-1}} \leq \frac{1}{2^{n_0-1}} < \varepsilon$, Thus $\left(1+\frac{1}{2!}+\dots+\frac{1}{n!}\right)$ is the Cauchy Sequence.

2 is of Supremum 2,6,9,

2.5) In Is 2,6,9,13

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3.3) Monotone Sequences 2,6,9,13

3.4) Subsequences & the Bolzano-Weierstrass Theorem 2,6,9,13

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(9) Suppose that $X_n \geq 0$ for all $n \in \mathbb{N}$ & that $\lim((-1)^n X_n)$ exists. Show that $(X_n)$ converges.

Let $X = (X_n)$ be such that every subsequence of $X$ has a subsequence that converges to 0. We want to prove that $\lim X = 0$.

Let's assume $\lim X \neq 0$. By def $\varepsilon_0 > 0$ & $N > 0$ & $n_k > N$

$$|X_{n_k} - 0| = |X_{n_k}| \geq \varepsilon_0$$

Lets choose $y_1 = X_{n_k}$

we know there exists $n_{k+1} > n_k$ so:

$$|X_{n_{k+1}} - 0| = |X_{n_{k+1}}| \geq \varepsilon_0$$

choose $y_2 = X_{n_{k+1}}$

(by continuing procedure we have $(y_n)$ which is a subsequence of $(X_n)$. $|y_n| > \varepsilon_0 > 0$, so $y_n$ doesnt converge to zero.)

Since this contradicts subsequent X converging to zero

$\lim X = 0$

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3.4) Subsequences & the Bolzano-Weierstrass Theorem 2,6,9,13

(13) If $X_n = (-1)^n/n$, find the subsequence of $(X_n)$ that is constructed in the second proof of the Bolzano-Weierstrass Theorem 3.4.8, when we take $I_1 = [-1,1]$.

$x_n = \frac{(-1)^n}{n}$, $I_1 = [-1,1]$, $n_1 = 1$

First step to bisect $I_1$ into intervals $I_1'$ & $I_1''$ of = length

$I_1' = [-1,0]$, $I_1'' = [0,1]$

$A_1 = \{n \in \mathbb{N} : n > n_1, x_n \in I_1'\}$

$B_1 = \{n \in \mathbb{N} : n > n_1, x_n \in I_1''\}$

$(x_1 = -1, x_2 = \frac{1}{2}, x_3 = -\frac{1}{3}, x_4 = \frac{1}{4}, x_5 = -\frac{1}{5}, \ldots)$

$A_2$ is empty, so finite, so we choos $I_3 = I_2'' = [-\frac{1}{2}, 0]$.

we choose $n_3$ as the smallest $\mathbb{N}$

$I_3 \Rightarrow n_3 = 5$

For $k \geq 4$ we choose

$$I_k = \left[-\frac{1}{2^{k-2}}, 0\right]$$

Since $I_{k-1}' = \left[-\frac{1}{2^{k-1}}, -\frac{1}{2^{k-2}}\right]$ has finitely many elements & $n_k = 2k-1$, $k \in \mathbb{N}$

So the subsequence of $(X_{2k-1}) = -\frac{1}{2k-1}$, $k \in \mathbb{N}$

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$x_n = \frac{(-1)^n}{n}$, $I_1 = [-1, 1]$, $n_1 = 1$

First step to bisect $I_1$ into intervals $I_1'$ & $I_1''$ of = length

$I_1' = [-1, 0]$, $I_1'' = [0, 1]$

$A_1 = \{n \in \mathbb{N} : n > n_1, x_n \in I_1'\}$

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$(x_1 = -1, x_2 = \frac{1}{2}, x_3 = -\frac{1}{3}, x_4 = \frac{1}{4}, x_5 = -\frac{1}{5}, \ldots)$

$A_2$ is empty, so finite, so we choos $I_3 = I_2'' = [-\frac{1}{2}, 0]$.

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$$I_k = \left[-\frac{1}{2^{k-2}}, 0\right]$$

Since $I_{k-1}' = \left[-\frac{1}{2^{k-1}}, -\frac{1}{2^{k-2}}\right]$ has finitely many elements & $n_k = 2k-1, k \in \mathbb{N}$

So the subsequence of $(X_{2k-1}) = -\frac{1}{2k-1}, k \in \mathbb{N}$

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(6) Let $x_n = n^{1/n}$ for $n \in \mathbb{N}$

(a) Show that $x_{n+1} < x_n$ if & only if $(1+1/n)^n < n$, & infer that the inequality is valid for $n \geq 3$. Conclude that $(x_n)$ is ultimately decreasing & that $x = \lim(x_n)$ exists.

(b) Use the fact that the subsequence $(x_{2n})$ also converges to $x$ to conclude that $x = 1$.

(a)

$$x_{n+1} < x_n \leftrightarrow (n+1)^{\frac{1}{n+1}} < n^{\frac{1}{n}} / (\cdot)^{n(n+1)}$$
$$\leftrightarrow (n+1)^{\frac{n(n+1)}{n+1}} < n^{\frac{n(n+1)}{n}}$$
$$\leftrightarrow (n+1)^n < n^{n+1}$$
$$\leftrightarrow (n+1)^n < n^n \cdot n / \cdot \frac{1}{n^n}$$
$$\leftrightarrow \frac{(n+1)^n}{n^n} < n$$
$$\leftrightarrow \left(\frac{n+1}{n^n}\right)^n < n$$
$$\leftrightarrow \left(1+\frac{1}{n}\right)^n < n \quad (1)$$

$e_n$ // $n \geq 3$

& $(x_n)$ is decreasing for $n \geq 3$

Since $X_n$ is bounded by zero, then it's convergent.

(b) Every subsequence of convergent sequence converges to the same limit. Thus,

$$x = \lim_n x = \lim_n x_{2n}$$
$$= \lim_n \sqrt{2n}^{\frac{1}{2n}}$$
$$= \lim_n \sqrt{2}^{\frac{1}{n}} \cdot \sqrt{n}^{\frac{1}{n}}$$
$$= \lim_n \sqrt{2}^{\frac{1}{n}} \cdot \sqrt{\lim_n n^{\frac{1}{n}}}$$
$$= \sqrt{x}$$

Therefore the relation is

$x = \sqrt{x} \leftrightarrow x^2 - x = 0 \leftrightarrow x(x-1) = 0 \leftrightarrow x = 0$ or $x = 1$

Since $X_n \geq 1$ for all $n$, it has to be $x \geq 1 \Rightarrow x = 1$

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(2) Use the method of Ex 3.4.3(b) to show that if $0<c<1$, then $\lim(c^{1/n})=1$.

For $0<c<1$, note that $c^{1/n} < c^{1/(n+1)}$. Hence the sequence $(c^{1/n})$ is an increasing sequence. Also each term is bounded by one. Hence by Monotone Converge theorem, the sequence converges.
So $\lim c^{1/2n} = \lim c^{1/n}$ & hence if $x$ is the limit then $\sqrt{x}=x$ & hence $x=x^2$. Therefore $x=0$ or $x=1$. But the sequence is increasing & each term is positive (+), therefore we have $x=1$. Done!!

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3.4) Subsequences & the Bolzano-Weierstrass Theorem 2, 6, 9, 13

(13) Use the method in Example 3.3.5 to calculate $\sqrt{2}$, correct to within 4 decimals.

$s_1 > 0$, Sequence $(s_n)$, where $s_{n+1} = \frac{1}{2}\left(s_n + \frac{2}{s_n}\right)$ converges to $\sqrt{2}$

using inequality

$$0 \leq s_n - \sqrt{2} \leq s_n - \frac{2}{s_n} \leq \frac{s_n^2 - 2}{s_n}$$

Now let's use $s_1 = 1$

$$s_2 = \frac{1}{2} \cdot \left(1 + \frac{2}{1}\right) = 1.5$$

$$\frac{s_2^2 - 2}{s_2} = \frac{1.5^2 - 2}{1.5} \approx .16667$$

$$s_3 = \frac{1}{2}\left(1.5 + \frac{2}{1.5}\right) \approx 1.41667$$

$$\frac{s_3^2 - 2}{s_3} = \frac{1.41667^2 - 2}{1.41667} \approx .004908$$

$$s_4 = \frac{1}{2}\left(1.41667 + \frac{2}{1.41422}\right) \approx 1.41422$$

$$\frac{s_4^2 - 2}{s_4} = \frac{1.41422^2 - 2}{1.41422} \approx .000013$$

Thus $\sqrt{2} \approx \underline{\underline{.000013}}$

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(9) Let $A$ be an infinite subset of $\mathbb{R}$ that is bounded above & let $u = \sup A$. Show that there exists an increasing sequence $(X_n)$ with $X_n \in A$ for all $n \in \mathbb{N}$ such that $u = \lim(X_n)$.

Let $A \subset \mathbb{R}$ be infinite & bounded above. Let $u = \sup A$

Step 1) $u \in A$

By defining $X_n = u \in A, \forall n$ we get the sequence $(X_n)$ that is clearly increasing & bouded above by $u$ so the Monotone Convergence Theorem States:

$$\lim(X_n) = u$$

---

Step 2) By definitione of the supremum, $\varepsilon > 0$, there exists $x \in A$

$u - \varepsilon < x < u$, Now let $n_1 \in \mathbb{N}$ for $\varepsilon = \frac{1}{n_1}$ there exists $X_1 \in A$ $u - \frac{1}{n_1} < x_1 < u$, by choosing $n_2 \in \mathbb{N}$ so that $X_1 < u - \frac{1}{n_2}$ & for $\varepsilon = \frac{1}{n_2}$ there exists $X_2 \in A$ so that: $X_1 < u - \frac{1}{n_2} < X_2 < u \Rightarrow x_1 < X_2$, we created a decreasing sequence $(X_n)$ bounded by $u$ so, $\lim(X_n) = u$

So consider
$u \in A$
$u \notin A$

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$u \in A$
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Step 2) By definition of the supremum, $\varepsilon > 0$, there exists $x \in A$

$u - \varepsilon < x < u$, Now let $n_1 \in \mathbb{N}$ for $\varepsilon = \frac{1}{n_1}$ there exists $x_1 \in A$ $u - \frac{1}{n_1} < x_1 < u$, by choosing $n_2 \in \mathbb{N}$ so that $x_1 < u - \frac{1}{n_2}$ & for $\varepsilon = \frac{1}{n_2}$ there exists $x_2 \in A$ so that: $x_1 < u - \frac{1}{n_2} < x_2 < u \Rightarrow x_1 < x_2$, we created a decreasing sequence $(x_n)$ bounded by $u$ so, $\lim(x_n) = u$

So consider
$u \in A$
$u \notin A$

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So consider
$u \in A$
$u \notin A$

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1) Follow this given sequence:

$z_1 > 0,\ z_{n+1} = \sqrt{a + z_n},\ n \in \mathbb{N},\ a > 0$

2) use induction to show $0 \leq z_n$.

$z_1 > 0$

Assume $z_n \geq 0$ for arbitrary $n$.

$$z_{n+1} = \sqrt{a+z_n} \geq \sqrt{a+0} = \sqrt{a} > 0$$

So by induction the sequence is bound below by 0.

(6) let $a > 0$ & let $z_1 > 0$. Define $z_{n+1} = \sqrt{a + z_n}$ for $n \in \mathbb{N}$. Show that $(z_n)$ converges & find the limit.

(3) This sequence is monotone because:

$$(z_{n+1} - z_n)(z_{n+1} + z_n) = z_{n+1}^2 - z_n^2 = (a + z_n) - (a + z_{n-1}) = z_n - z_{n-1}$$

$z_{n+1} > z_n \Rightarrow z_n > z_{n-1}$

$z_{n+1} < z_n \Rightarrow z_n < z_{n-1}$

So, in both cases its monotone.

(Let's check 2 cases)

$$z_{n+1} - z_n = \sqrt{a+z_n} - z_n = \left(\sqrt{a+z_n}\right) \cdot \frac{\sqrt{a+z_n} + z_n}{\sqrt{a+z_n} + z_n}$$

$$= \frac{(a+z_n) - z_n^2}{\left(\sqrt{a+z_n}\right) + z_n} = \frac{-z_n^2 + z_n + a}{\sqrt{a+z_n} + z_n}$$

$$= -\frac{\left(z_n - \frac{1+\sqrt{1+4a}}{2}\right)\left(z_n - \frac{1-\sqrt{1+4a}}{2}\right)}{\sqrt{a+z_n} + z_n}$$

Since $\frac{1-\sqrt{1+4a}}{2} < 0 \Rightarrow z_n - \frac{1-\sqrt{1+4a}}{2} > 0$

$\frac{1+\sqrt{1+4a}}{2} \leq z_1 \Rightarrow \frac{1+\sqrt{1+4a}}{2} \leq z_n = z_{n+1} \leq z_n$

$\frac{1+\sqrt{1+4a}}{2} \geq z_1 \Rightarrow \frac{1+\sqrt{1+4a}}{2} \geq z_n = z_{n+1} \geq z_n$

bound above decrease, below increases so its converge

Now find limit of $z_{n+1}$

$\lim(z_n) = \lim(z_{n+1}) = z$

$z = \sqrt{a+z}$

$z^2 = a + z$

$z^2 - z - a = 0$

$z = \frac{1 \pm \sqrt{1+4a}}{2}$

So $\lim(z_n) = \frac{1+\sqrt{1+4a}}{2}$

Since we showed $z_n \geq 0$, $\frac{1-\sqrt{1+4a}}{2}$ is not the limit

2.4 Applications of Supremum $\underline{2}, \underline{6}, \underline{9}, \underline{13}$ (Advanced Calculus)

2.5) Intervals $\underline{2}, \underline{6}, \underline{9}, \underline{13}$

3.1) Sequences & their limits $\underline{2}, \underline{6}, \underline{9}, \underline{13}$

3.2) Limit Theorems $\underline{2}, \underline{6}, \underline{9}, \underline{13}$

3.3) Monotone Sequences 2, 6, 9, 13

3.4) Subsequences & the Bolzano-Weierstrass Theorem 2, 6, 9, 13

(13) If $a>0$, $b>0$, show that $\lim\left(\sqrt{(n+a)(n+b)}-n\right)=\frac{(a+b)}{2}$

First Step rewrite the sequence

$$\sqrt{(n+a)(n+b)}-n = \left(\sqrt{(n+a)(n+b)}-n\right)\cdot\frac{\sqrt{(n+a)(n+b)}+n}{\sqrt{(n+a)(n+b)}+n}$$

$$=\frac{\sqrt{(n+a)(n+b)}-n^2}{\sqrt{(n+a)(n+b)}+n}$$

$$=\frac{(n+a)(n+b)-n^2}{\sqrt{(n+a)(n+b)}+n}$$

$$=\frac{n^2+n(a+b)+ab-n^2}{\sqrt{(n+a)(n+b)}+n}$$

$$=\frac{n^2+n(a+b)+ab-n^2}{\sqrt{(n+a)(n+b)}+n}$$

$$=\frac{n(a+b)+ab}{\sqrt{(n+a)(n+b)}+n}\cdot\frac{\frac{1}{n}}{\frac{1}{n}}=\frac{(a+b)+\frac{ab}{n}}{\sqrt{(1+\frac{a}{n})(1+\frac{b}{n})}+1}$$

Now find the limit

$$\Rightarrow \lim\left(\frac{ab}{n}\right)=0$$

$$\Rightarrow \lim\left(1+\frac{1}{n}\right)=1$$

$$\Rightarrow \lim\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)=1\cdot 1=1$$

$$\Rightarrow \lim\sqrt{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}=\sqrt{1}=1$$

$$\Rightarrow \lim\sqrt{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}+1=1+1=2$$

$$\Rightarrow \frac{(a+b)+\frac{ab}{n}}{\sqrt{(1+\frac{a}{n})(1+\frac{1}{n})}+1}$$

$$=\frac{(a+b)+0}{2}=\frac{a+b}{2}$$

2.4 Applications of Supremum. 2, 6, 9, 13 (Advanced Calculus)
2.5) Intervals 2, 6, 9, 13
3.1) Sequences & their Limits 2, 6, 9, 13
3.2) Limit Theorems 2, 6, 9, 13
3.3) Monotone Sequences 2, 6, 9, 13
3.4) Subsequences & the Bolzano-Weierstrass Theorem 2, 6, 9, 13

(2) Let $x_1 > 1$ & $x_{n+1} = 2 - 1/x_n$ for $n \in \mathbb{N}$. Show that $(x_n)$ is bounded & monotone. Find the limit.

$$x_2 = 2 - \frac{1}{x_1}$$

Now $x_1 > 1 \Rightarrow \frac{1}{x_1} < 1$ & thus $x_2 = 2 - \frac{1}{x_1} > 1$ & $x_2 < 2$, so $1 < x_2 < 2$.

we show by induction $1 < x_n < 2$ for all $n \geq 2$. We did the basis $n = 2$

$$1 < x_n < 2$$
$$1 > \frac{1}{x_n} > \frac{1}{2}$$
$$-1 < \frac{-1}{x_n} < \frac{-1}{2}$$
$$-1 < \frac{-1}{x_n} < \frac{-1}{2}$$
$$1 < 2 - \frac{1}{x_n} < -\frac{1}{2}$$
$$1 < x_{n+1} < 2$$

$$x_{n+1} - x_n = 2 - \frac{1}{x_n} - x_n$$
$$= 2 - 2 - \left(\frac{1}{x_n} + x_n\right)$$
$$= 2 - \left(\frac{1 + x^2}{x_n}\right)$$
$$= \frac{-(x_n^2 - 2x_n + 1)}{x_n} = -\frac{(x_n - 1)^2}{x_n} < 0$$

$$x_{n+1} = 2 - \frac{1}{x_n} \xrightarrow{n \to \infty} x = 2 - \frac{1}{x} \to x^2 = 2x - 1 \to (x-1)^2 = 0 \to \boxed{x = 1}$$

$$\boxed{\text{Thus } \lim_{n \to \infty} x_n = 1}$$

2.4) Applications of Supremum $\underline{2}, \underline{6}, \underline{9}, \underline{13}$ (Advanced Calculus)
2.5) Intervals $\underline{2}, \underline{6}, \underline{9}, \underline{13}$
3.1) Sequences & their limits $\underline{2}, \underline{6}, \underline{9}, \underline{13}$
3.2) Limit Theorems $\underline{2}, \underline{6}, \underline{9}, \underline{13}$
3.3) Monotone Sequences $\underline{2}, \underline{6}, 9, 13$
3.4) Subsequences & the Bolzano-Weierstrass Theorem 2, 6, 9, 13

(6) Let $a>0$ & let $z_1>0$. Define $z_{n+1}=\sqrt{a+z_n}$ for $n\in\mathbb{N}$. Show that $(z_n)$ converges & find the limit.

1) Follow this given sequence:

$z_1>0,\ z_{n+1}=\sqrt{a+z_n},\ n\in\mathbb{N},\ a>0$

2) use induction to show $0\leq z_n$.

$z_1>0$

Assume $z_n\geq 0$ for arbitrary $n$.

$$z_{n+1}=\sqrt{a+z_n} \geq \sqrt{a+0} = \sqrt{a} > 0$$

So by induction the sequence is bound below by 0.

(3) This sequence is monotone because:

$$(z_{n+1}-z_n)(z_{n+1}+z_n)=z_{n+1}^2-z_n^2 = (a+z_n)-(a+z_{n-1}) = z_n-z_{n-1}$$

$z_{n+1}>z_n \Rightarrow z_n>z_{n-1}$

$z_{n+1}<z_n \Rightarrow z_n<z_{n-1}$

So, in both cases its monotone.

(Let's check 2 cases)

$$z_{n+1}-z_n=\sqrt{a+z_n}-z_n = \left(\sqrt{a+z_n}\right)\cdot\frac{\sqrt{a+z_n}+z_n}{\sqrt{a+z_n}+z_n}$$

$$=\frac{(a+z_n)-z_n^2}{\left(\sqrt{a+z_n}\right)+z_n} = \frac{-z_n^2+z_n+a}{\sqrt{a+z_n}+z_n}$$

$$=-\frac{\left(z_n-\frac{1+\sqrt{1+4a}}{2}\right)\left(z_n-\frac{1-\sqrt{1+4a}}{2}\right)}{\sqrt{a+z_n}+z_n}$$

Since $\frac{1-\sqrt{1+4a}}{2}<0 \Rightarrow z_n-\frac{1-\sqrt{1+4a}}{2}>0$

$\frac{1+\sqrt{1+4a}}{2}\leq z_1 \Rightarrow \frac{1+\sqrt{1+4a}}{2}\leq z_n \Rightarrow z_{n+1}\leq z_n$

$\frac{1+\sqrt{1+4a}}{2}\geq z_1 \Rightarrow \frac{1+\sqrt{1+4a}}{2}\geq z_n \Rightarrow z_{n+1}\geq z_n$

bound above decrease, below increase so its monotone

Now find limit of $z_{n+1}$

$$\lim(z_n)=\lim(z_{n+1})=z$$

$$z=\sqrt{a+z}$$

$$z^2=a+z$$

$$z^2-z-a=0$$

$$z=\frac{1\pm\sqrt{1+4a}}{2}$$

So $\lim(z_n)=\frac{1+\sqrt{1+4a}}{2}$

Since we showed $z_n\geq 0$, $\frac{1-\sqrt{1+4a}}{2}$ is not the limit

2.4 Applications of Supremum $\underline{2},\underline{6},\underline{9},\underline{13}$ (Advanced Calculus)

2.5) Intervals $\underline{2},\underline{6},\underline{9},\underline{13}$

3.1) Sequences & their limits $\underline{2},\underline{6},\underline{9},\underline{13}$

3.2) Limit Theorems $\underline{2},\underline{6},\underline{9}$, 13

3.3) Monotone Sequences 2, 6, 9, 13

3.4) Subsequences & the Bolzano-Weierstrass Theorem 2, 6, 9, 13

(9) Let $y_n = \sqrt{n+1} - \sqrt{n}$ for $n \in \mathbb{N}$. Show that $(\sqrt{n}\, y_n)$ converges. Find the limit.

- First find the limit of $y_n$

Let's rewrite $y_n = \sqrt{n+1} - \sqrt{n}$

$$= (\sqrt{n+1} - \sqrt{n}) \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{\sqrt{n+1}^2 - \sqrt{n}^2}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$0 \le \frac{1}{\sqrt{n+1} + \sqrt{n}} \le \frac{1}{\sqrt{n}} \;\&\; \lim \frac{1}{n} = \lim \frac{1}{\sqrt{n}} = 0$$

(with squeeze theorem)

$\lim y_n = 0$

- Now we look for $\lim \sqrt{n}\, y_n$

Let's rewrite $(\sqrt{n}\, y_n)$:

$$\sqrt{n}\, y_n = \sqrt{n} \cdot \frac{1}{\sqrt{n+1} + \sqrt{n}} = \frac{\sqrt{n}}{\sqrt{1+\frac{1}{n}} + 1} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}}$$

$$= \frac{1}{\sqrt{1+\frac{1}{n}} + 1}$$

$$\lim \frac{1}{n} = 0 \Rightarrow \lim \left(\frac{1}{n} + 1\right) = 1$$

$$\Rightarrow \lim \left(\sqrt{\frac{1}{n} + 1}\right) = \sqrt{1} = 1$$

$$\Rightarrow \lim \frac{1}{\sqrt{\frac{1}{n}+1} + 1} = \frac{1}{1+1} = \boxed{\frac{1}{2}}$$

Answer

2.4 Applications of Supremum. 2, 6, 9, 13 (Advanced Calculus)
2.5) Intervals 2, 6, 9, 13
3.1) Sequences & their limits 2, 6, 9, 13
3.2) Limit Theorems 2, 6, 9, 13
3.3) Monotone Sequences 2, 6, 9, 13
3.4) Subsequences & the Bolzano-Weierstrass Theorem 2, 6, 9, 13

(6) Find the limits of the following sequences.

(a) $\lim\left((2+1/n)^2\right)$ (b) $\lim\left(\frac{(-1)^n}{n+2}\right)$

(c) $\lim\left(\frac{\sqrt{n}-1}{\sqrt{n}-1}\right)$ (d) $\lim\left(\frac{n+1}{n\sqrt{n}}\right)$

(a) $\lim\left(\left(2+\frac{1}{n}\right)^2\right)=\lim\left(\left(2+\frac{1}{n}\right)\cdot\left(2+\frac{1}{n}\right)\right)$

$=\lim\left(2+\frac{1}{n}\right)\cdot\lim\left(2+\frac{1}{n}\right)$

$=\left(\lim 2+\lim\frac{1}{n}\right)\cdot\left(\lim 2+\lim\frac{1}{n}\right)$

$=(2+0)\cdot(2+0)$

$=4$

(b) use squeeze theorem

$$\frac{-1}{n+2}\leq\frac{(-1)^n}{n+2}\leq\frac{1}{n+2}$$

since $\lim\left(\frac{-1}{n+2}\right)=\lim\left(\frac{1}{n+2}\right)=0$, the squeeze theorem states

$$\lim\left(\frac{(-1)^n}{n+2}\right)=0$$

(c)

$$\lim\left(\frac{\sqrt{n}-1}{\sqrt{n}+1}\right)=\lim\left(\frac{\sqrt{n}-1}{\sqrt{n}+1}\cdot\frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}}\right)=\frac{\lim\left(1-\frac{1}{\sqrt{n}}\right)}{\lim\left(1+\frac{1}{\sqrt{n}}\right)}$$

$$=\frac{\lim(1)-\lim\left(\frac{1}{\sqrt{n}}\right)}{\lim(1)+\lim\left(\frac{1}{\sqrt{n}}\right)}=\frac{1-\lim\left(\sqrt{\frac{1}{n}}\right)}{1+\lim\left(\sqrt{\frac{1}{n}}\right)}$$

we know $\lim\left(\frac{1}{n}\right)=0$, if $\lim\sqrt{\frac{1}{n}}=\sqrt{\lim\frac{1}{n}}=0$

So $\lim\left(\frac{\sqrt{n}-1}{\sqrt{n}+1}\right)=1$

(d) $\lim\left(\frac{n+1}{n\sqrt{n}}\right)=\lim\left(\frac{n}{n\sqrt{n}}+\frac{1}{n\sqrt{n}}\right)$

$=\lim\left(\frac{n}{n\sqrt{n}}\right)+\lim\left(\frac{1}{n\sqrt{n}}\right)$

$=\lim\left(\frac{1}{\sqrt{n}}\right)+\lim\left(\frac{1}{n}\cdot\frac{1}{\sqrt{n}}\right)$

$=\lim\sqrt{\frac{1}{n}}+\lim\left(\frac{1}{n}\right)\cdot\lim\sqrt{\frac{1}{n}}$

$=0+0\cdot 0$

$=0$

2.4 Applications of Supremum 2, 6, 9, 13 (Advanced Calculus)
2.5) Intervals 2, 6, 9, 13
3.1) Sequences & their Limits 2, 6, 9, 13
3.2) Limit Theorems 2, 6, 9, 13
3.3) Monotone Sequences 2, 6, 9, 13
3.4) Subsequences & the Bolzano-Weierstrass Theorem 2, 6, 9, 13

(6) Find the limits of the following sequences.

(a) $\lim((2+1/n)^2)$

(b) $\lim\left(\frac{(-1)^n}{n+2}\right)$

(c) $\lim\left(\frac{\sqrt{n}-1}{\sqrt{n}-1}\right)$

(d) $\lim\left(\frac{n+1}{n\sqrt{n}}\right)$

(a) $\lim\left(\left(2+\frac{1}{n}\right)^2\right)=\lim\left(\left(2+\frac{1}{n}\right)\cdot\left(2+\frac{1}{n}\right)\right)$

$=\lim\left(2+\frac{1}{n}\right)\cdot\lim\left(2+\frac{1}{n}\right)$

$=\left(\lim 2+\lim\frac{1}{n}\right)\cdot\left(\lim 2+\lim\frac{1}{n}\right)$

$=(2+0)\cdot(2+0)$

$=4$

(b) use squeeze theorem

$$\frac{-1}{n+2}\leq\frac{(-1)^n}{n+2}\leq\frac{1}{n+2}$$

since $\lim\left(\frac{-1}{n+2}\right)=\lim\left(\frac{1}{n+2}\right)=0$, the squeeze theorem states $\lim\left(\frac{(-1)^n}{n+2}\right)=0$

(c)

$\lim\left(\frac{\sqrt{n}-1}{\sqrt{n}+1}\right)=\lim\left(\frac{\sqrt{n}-1}{\sqrt{n}+1}\cdot\frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}}\right)=\frac{\lim\left(1-\frac{1}{\sqrt{n}}\right)}{\lim\left(1+\frac{1}{\sqrt{n}}\right)}$

$=\frac{\lim(1)-\lim\left(\frac{1}{\sqrt{n}}\right)}{\lim(1)+\lim\left(\frac{1}{\sqrt{n}}\right)}=\frac{1-\lim\left(\sqrt{\frac{1}{n}}\right)}{1+\lim\left(\sqrt{\frac{1}{n}}\right)}$

we know $\lim\left(\frac{1}{n}\right)=0$, if $\lim\sqrt{\frac{1}{n}}=\sqrt{\lim\frac{1}{n}}=0$

So $\lim\left(\frac{\sqrt{n}-1}{\sqrt{n}+1}\right)=1$

(d) $\lim\left(\frac{n+1}{n\sqrt{n}}\right)=\lim\left(\frac{n}{n\sqrt{n}}+\frac{1}{n\sqrt{n}}\right)$

$=\lim\left(\frac{n}{n\sqrt{n}}\right)+\lim\left(\frac{1}{n\sqrt{n}}\right)$

$=\lim\left(\frac{1}{\sqrt{n}}\right)+\lim\left(\frac{1}{n}\cdot\frac{1}{\sqrt{n}}\right)$

$=\lim\sqrt{\frac{1}{n}}+\lim\left(\frac{1}{n}\right)\cdot\lim\sqrt{\frac{1}{n}}$

$=0+0\cdot 0$

$=0$

2.4) Applications of Supremum. 2, 6, 9, 13 (Advanced Calculus)

2.5) Intervals 2, 6, 9, 13

3.1) Sequences & their limits 2, 6, 9, 13

3.2) Limit Theorems 2, 6, 9, 13

3.3) Monotone Sequences 2, 6, 9, 13

3.4) Subsequences & the Bolzano-Weierstrass Theorem 2, 6, 9, 13

(13) Show that $\lim(1/3^n) = 0$.

Since $n \leq 3^n \longleftrightarrow \frac{1}{3^n} \leq \frac{1}{n}$ we have

$$\left|\frac{1}{3^n} - 0\right| \leq \frac{1}{n}$$

given theorem 3.1.1 & $\lim_n \frac{1}{n} = 0$ we get

$$\lim_n \frac{1}{3^n} = 0$$

2.4 Applications of Supremum. 2, 6, 9, 13 (Advanced Calculus)
2.5) Intervals 2, 6, 9, 13
3.1) Sequences & their Limits 2, 6, 9, 13
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3.3) Monotone Sequences 2, 6, 9, 13
3.4) Subsequences & the Bolzano-Weierstrass Theorem 2, 6, 9, 13

(2) Give an example of 2 divergent sequences $X, Y$ such that:

(a) their sum $X+Y$ converges

(b) their product $XY$ converges

{Let $X = (0,1,0,1,\dots)$ & $Y = (1,0,1,0,1,\dots)$. Both sequences diverge because the difference of 2 consecutive terms is 1.}

(a)

$X + Y = (1,1,1,1,\dots)$, hence converges as its constant sequence.

(b)

$XY = (0,0,0,0,\dots)$, hence converges.

2.4 Applications of Supremum. 2, 6, 9, 13 (Advanced Calculus)
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(13) Show that $\lim(1/3^n) = 0$.

Since $n \leq 3^n \longleftrightarrow \frac{1}{3^n} \leq \frac{1}{n}$ we have

$$\left|\frac{1}{3^n} - 0\right| \leq \frac{1}{n}$$

given theorem 3.1.1 & $\lim_n \frac{1}{n} = 0$ we get

$$\lim_n \frac{1}{3^n} = 0$$

2.4) Applications of Supremum 2, 6, 9, 13 (Advanced Calculus)

2.5) Intervals 2, 6, 9, 13

3.1) Sequences & their limits 2, 6, 9, 13

3.2) Limit Theorems 2, 6, 9, 13

3.3) Monotone Sequences 2, 6, 9, 13

3.4) Subsequences & the Bolzano-Weierstrass Theorem 2, 6, 9, 13

(6) Show that

(a) $\lim\left(\frac{n}{n^2+1}\right)=0$

(b) $\lim\left(\frac{2n}{n+1}\right)=2$

(c) $\lim\left(\frac{3n+1}{2n+5}\right)=\frac{3}{2}$

(d) $\lim\left(\frac{n^2-1}{2n^2+3}\right)=\frac{1}{2}$

(a) observe $n<n+7$ for $n\in\mathbb{N}$,
so $\sqrt{n}<\sqrt{n+7}$ for all $n\in\mathbb{N}$
so $\frac{1}{\sqrt{n+7}}<\frac{1}{\sqrt{n}}$ for all $n\in\mathbb{N}$
$\varepsilon>0$ given Corollary 2.4.5 $k_\varepsilon\in\mathbb{N}$ $0<\frac{1}{k_\varepsilon}<\varepsilon^2$
so $\frac{1}{\sqrt{k_\varepsilon}}<\varepsilon$
thus $n\geq k_\varepsilon$ then $\sqrt{k_\varepsilon}\leq\sqrt{n}$, $\frac{1}{\sqrt{n+7}}<\frac{1}{\sqrt{n}}\leq\frac{1}{\sqrt{k_\varepsilon}}$
$\left|\frac{1}{\sqrt{n+7}}-0\right|=\left|\frac{1}{\sqrt{n+7}}\right|=\frac{1}{\sqrt{n+7}}<\varepsilon$ for all $n\geq k_\varepsilon$
thus $\lim\left(\frac{1}{\sqrt{n+7}}\right)=0$

(b) Observe that $n<n+2$ for all $n\in\mathbb{N}$ which gives us that $\frac{1}{n+2}<\frac{1}{n}$ for all $n\in\mathbb{N}$
$\varepsilon>0$ it follows Corollary 2.4.5 that there exists $K_\varepsilon\in\mathbb{N}$ such that $0<\frac{1}{K_\varepsilon}<\frac{\varepsilon}{4}$ which gives $\frac{4}{K_\varepsilon}<\varepsilon$
Thus if $n\geq K_\varepsilon$ we have that $\frac{1}{n+2}<\frac{1}{n}\leq\frac{1}{K_\varepsilon}$ which gives that $\frac{4}{n+2}<\frac{4}{n}\leq\frac{4}{K_\varepsilon}<\varepsilon$ & therefore, $\left|\frac{2n}{n+2}-2\right|=\left|\frac{2n-2n-4}{n+2}\right|$
$=\left|\frac{-4}{n+2}\right|=\frac{4}{n+2}<\varepsilon$ for all $n\geq K_\varepsilon$
thus, $\lim\left(\frac{2n}{n+2}\right)=2$

(c) observe $0<n<n+1$ for $n\in\mathbb{N}$ & then $n^2<(n+1)^2$ for all $n\in\mathbb{N}$ which gives $\frac{n}{(n+1)^2}<\frac{1}{n}$ for all $n\in\mathbb{N}$
& thus $\frac{\sqrt{n}}{n+1}<\frac{1}{\sqrt{n}}$ for all $n\in\mathbb{N}$
Given $\varepsilon>0$ for corollary 2.4.5 there exists $k_\varepsilon\in\mathbb{N}$ such that $0<\frac{1}{K_\varepsilon}<\varepsilon^2$ which gives us that $\frac{1}{\sqrt{K_\varepsilon}}<\varepsilon$
Thus $n\geq k_\varepsilon$ we have that $\sqrt{k_\varepsilon}\leq\sqrt{n}$ which gives us $\frac{\sqrt{n}}{n+1}<\frac{1}{\sqrt{n}}\leq\frac{1}{\sqrt{K_\varepsilon}}<\varepsilon$, thus $\left|\frac{\sqrt{n}}{n+1}-0\right|=\left|\frac{\sqrt{n}}{n+1}\right|=\frac{\sqrt{n}}{n+1}<\varepsilon$ $n\geq K_\varepsilon$
thus, $\lim\left(\frac{\sqrt{n}}{n+1}\right)=0$

(d) observe $n^2<n^2+1$, $n\in\mathbb{N}$ which gives $\frac{n}{n^2+1}<\frac{1}{n}$ for all $n\in\mathbb{N}$
since $\varepsilon>0$ it follows from corollary 2.4.5, there exists $k_\varepsilon\in\mathbb{N}$ such that $0<\frac{1}{K_\varepsilon}<\varepsilon$
Thus if $n\geq k_\varepsilon$, we have that $\frac{n}{n^2+1}<\frac{1}{n}\leq\frac{1}{K_\varepsilon}<\varepsilon$ & therefore,
$\left|\frac{(-1)^n n}{n^2+1}-0\right|=\left|\frac{(-1)^n n}{n^2+1}\right|=\frac{n}{n^2+1}<\varepsilon$ for all $n\geq K_\varepsilon$.
Thus, $\lim\left(\frac{(-1)^n n}{n^2+1}\right)=0$

2.4 Applications of Supremum. 2, 6, 9, 13 (Advanced Calculus)

2.5) Intervals 2, 6, 9, 13

3.1) Sequences & their limits 2, 6, 9, 13

3.2) Limit Theorems 2, 6, 9, 13

3.3) Monotone Sequences 2, 6, 9, 13

3.4) Subsequences & the Bolzano-Weierstrass Theorem 2, 6, 9, 13

(9) Show that if $x_n \geq 0$ need not imply the convergence of $(x_n)$.

let $(x_n)$ be such a sequence that $x_n > 0$ & $\lim(x_n) = 0$

let $\varepsilon > 0$. The definition of a limit of a sequence states $K(\varepsilon)$ such that $n \geq K(\varepsilon)$, this is true:

$$|x_n - 0| = |x_n| = |x_n \geq 0| = x_n < \varepsilon^2$$

This means that $n \geq K(\varepsilon)$:

$$|\sqrt{x_n} - 0| = |\sqrt{x_n}| = \sqrt{x_n} < \sqrt{\varepsilon^2} = \varepsilon$$

which again by limit of a sequence means that $\lim(\sqrt{x_n}) = 0$

2.4) Applications of Supremum 2, 6, 9, 13 (Advanced Calculus)

2.5) Intervals 2, 6, 9, 13

3.1) Sequences & their Limits 2, 6, 9, 13

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(6) Show that

(a) $\lim\left(\frac{n}{n^2+1}\right)=0$

(b) $\lim\left(\frac{2n}{n+1}\right)=2$

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(d) $\lim\left(\frac{n^2-1}{2n^2+3}\right)=\frac{1}{2}$

(a) observe $n<n+7$ for $n\in\mathbb{N}$,

so $\sqrt{n}<\sqrt{n+7}$ for all $n\in\mathbb{N}$

so $\frac{1}{\sqrt{n+7}}<\frac{1}{\sqrt{n}}$ for all $n\in\mathbb{N}$

$\varepsilon>0$ given Corollary 2.4.5 $K_\varepsilon\in\mathbb{N}$ $0<\frac{1}{K_\varepsilon}<\varepsilon^2$

so $\frac{1}{\sqrt{K_\varepsilon}}<\varepsilon$

thus $n\geq K_\varepsilon$ then $\sqrt{K_\varepsilon}\leq\sqrt{n}$, $\frac{1}{\sqrt{n+7}}<\frac{1}{\sqrt{n}}\leq\frac{1}{\sqrt{K_\varepsilon}}<\varepsilon$

$\left|\frac{1}{\sqrt{n+7}}-0\right|=\left|\frac{1}{\sqrt{n+7}}\right|=\frac{1}{\sqrt{n+7}}<\varepsilon$ for all $n\geq K_\varepsilon$

thus $\lim\left(\frac{1}{\sqrt{n+7}}\right)=0$

(b) Observe that $n<n+2$ for all $n\in\mathbb{N}$ which gives us that $\frac{1}{n+2}<\frac{1}{n}$ for all $n\in\mathbb{N}$

$\varepsilon>0$ it follows Corollary 2.4.5 that there exists $K_\varepsilon\in\mathbb{N}$ such that $0<\frac{1}{K_\varepsilon}<\frac{\varepsilon}{4}$ which gives $\frac{4}{K_\varepsilon}<\varepsilon$

Thus if $n\geq K_\varepsilon$ we have that $\frac{1}{n+2}<\frac{1}{n}\leq\frac{1}{K_\varepsilon}$ which gives that $\frac{4}{n+2}<\frac{4}{n}\leq\frac{4}{K_\varepsilon}<\varepsilon$ & therefore, $\left|\frac{2n}{n+2}-2\right|=\left|\frac{2n-2n-4}{n+2}\right|=\left|\frac{-4}{n+2}\right|=\frac{4}{n+2}<\varepsilon$ for all $n\geq K_\varepsilon$

thus, $\lim\left(\frac{2n}{n+2}\right)=2$

(c) observe $0<n<n+1$ for $n\in\mathbb{N}$ & then $n^2<(n+1)^2$ for all $n\in\mathbb{N}$ which gives $\frac{n}{(n+1)^2}<\frac{1}{n}$ for all $n\in\mathbb{N}$ & thus $\frac{\sqrt{n}}{n+1}<\frac{1}{\sqrt{n}}$ for all $n\in\mathbb{N}$

Given $\varepsilon>0$ for corollary 2.4.5 there exists $K_\varepsilon\in\mathbb{N}$ such that $0<\frac{1}{K_\varepsilon}<\varepsilon^2$ which gives us that $\frac{1}{\sqrt{K_\varepsilon}}<\varepsilon$

Thus $n\geq K_\varepsilon$ we have that $\sqrt{K_\varepsilon}\leq\sqrt{n}$ which gives us $\frac{\sqrt{n}}{n+1}<\frac{1}{\sqrt{n}}\leq\frac{1}{\sqrt{K_\varepsilon}}<\varepsilon$, thus $\left|\frac{\sqrt{n}}{n+1}-0\right|=\left|\frac{\sqrt{n}}{n+1}\right|=\frac{\sqrt{n}}{n+1}<\varepsilon$ $n\geq K_\varepsilon$

thus, $\lim\left(\frac{\sqrt{n}}{n+1}\right)=0$

(d) observe $n^2<n^2+1$, $n\in\mathbb{N}$ which gives $\frac{n}{n^2+1}<\frac{1}{n}$ for all $n\in\mathbb{N}$

Since $\varepsilon>0$ it follows from corollary 2.4.5, there exists $K_\varepsilon\in\mathbb{N}$ such that $0<\frac{1}{K_\varepsilon}<\varepsilon$

Thus if $n\geq K_\varepsilon$, we have that $\frac{n}{n^2+1}<\frac{1}{n}\leq\frac{1}{K_\varepsilon}<\varepsilon$ & therefore, $\left|\frac{(-1)^n n}{n^2+1}-0\right|=\left|\frac{(-1)^n n}{n^2+1}\right|=\frac{n}{n^2+1}<\varepsilon$ for all $n\geq K_\varepsilon$.

Thus, $\lim\left(\frac{(-1)^n n}{n^2+1}\right)=0$

2.4 Applications of Supremum 2,6,9,13 (Advanced Calculus)
2.5) Intervals 2,6,9,13
3.1) Sequences & their limits 2,6,9,13
3.2) Limit Theorems 2,6,9,13
3.3) Monotone Sequences 2,6,9,13
3.4) Subsequences & the Bolzano-Weierstrass Theorem 2,6,9,13

(6) If $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq$ is a nested sequence of i & if $I_n = [a_n, b_n]$, Show that $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$ & $b_1 \geq b_2 \geq b_n \geq \cdots$

let $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq \ldots$ be a nested sequence of intervals with $I_n = [a_n, b_n]$.

Since $I_{n+1} \subseteq I_n$, $I_n = [a_n, b_n]$ for all $n \in \mathbb{N}$, it follows from Exercise 1 that $a_n \leq a_{n+1}$ & $b_n \geq b_{n+1}$ for all $n \in \mathbb{N}$.

Thus, $a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots$ and $b_1 \geq b_2 \ldots \geq b_n \geq \ldots$

2.4 Applications of Supremum. 2,6,9,13 (Advanced Calculus)
2.5) Intervals 2,6,9,13
3.1) Sequences & their Limits 2,6,9,13
3.2) Limit Theorems 2,6,9,13
3.3) Monotone Sequences 2,6,9,13
3.4) Subsequences & the Bolzano-Weierstrass Theorem 2,6,9,13

(13) (a) Give the first four digits in the binary representation of 3/8 & 7/16.
(b) Give the complete binary representation of $\frac{1}{3}$.

(a)

(b)

[look at Numerical Analysis Notes]

2.4 Applications of Supremum $\underline{2}, \underline{6}, \underline{9}, \underline{13}$ (Advanced Calculus)
2.5) Intervals $\underline{2}, \underline{6}, \underline{9}, \underline{13}$
3.1) Sequences & their limits $\underline{2}$, 6, 9, 13
3.2) Limit Theorems 2, 6, 9, 13
3.3) Monotone Sequences 2, 6, 9, 13
3.4) Subsequences & the Bolzano-Weierstrass Theorem 2, 6, 9, 13

(2) The first few terms of a sequence $(x_n)$ are given below. Assuming that the "natural pattern" indicated by these terms persists, give a formula for the nth term $x_n$.

(a) 5, 7, 9, 11, . . . .

$$\boxed{x_n = 2n+3}$$

(b) $1/2, -1/4, 1/8, -1/16, \ldots$

$$\boxed{x_n = \frac{(-1)^{n+1}}{2^n}}$$

(c) $1/2, 2/3, 3/4, 4/5, \ldots$

$$\boxed{x_n = \frac{n}{n+1}}$$

(D) 1, 4, 9, 16, . . . .

$$\boxed{x_n = n^2}$$

2.4 Applications of Supremum $\underline{2}, \underline{6}, \underline{9}, \underline{13}$ (Advanced Calculus)
2.5) Intervals $\underline{2}, 6, 9, 13$
3.1) Sequences & their Limits 2, 6, 9, 13
3.2) Limit Theorems 2, 6, 9, 13
3.3) Monotone Sequences 2, 6, 9, 13
3.4) Subsequences & the Bolzano-Weierstrass Theorem 2, 6, 9, 13

(2) If $S \subseteq \mathbb{R}$ is nonempty, show that $S$ is bounded if only if there exists a closed bounded interval $I$ such that $S \subseteq I$.

Suppose that $S$ is bounded. Then $S$ is bounded above & below giving $a, b \in \mathbb{R}$ such that $a \leq s \leq b$ for all $s \in S$ & so $s \in [a,b]$ for all $s \in S$. Thus, $S \subseteq I$ where $I = [a,b]$.

Suppose there's an interval $I = [a,b]$ such that $S \subseteq I$. Then we have $s \in I$ for all $s \in S$ which gives $a \leq s \leq b$ for all $s \in S$ & therefore $S$ is bounded above & below.

2.4 Applications of Supremum 2, 6, 9, 13 (Advanced Calculus)
2.5) Intervals 2, 6, 9, 13
3.1) Sequences & their Limits 2, 6, 9, 13
3.2) Limit Theorems 2, 6, 9, 13
3.3) Monotone Sequences 2, 6, 9, 13
3.4) Subsequences & the Bolzano-Weierstrass Theorem 2, 6, 9, 13

(2) If $S \subseteq \mathbb{R}$ is nonempty, show that $S$ is bounded if & only if there exists a closed bounded interval $I$ such that $S \subseteq I$.

Suppose that $S$ is bounded. Then $S$ is bounded above & below giving $a, b \in \mathbb{R}$ such that $a \leq s \leq b$ for all $s \in S$ & so $s \in [a,b]$ for all $s \in S$. Thus, $S \subseteq I$ where $I = [a,b]$.

Suppose there's an interval $I = [a,b]$ such that $S \subseteq I$. Then we have $s \in I$ for all $s \in S$ which gives $a \leq s \leq b$ for all $s \in S$ & therefore $S$ is bounded above & below.

2.4 App • $\underline{2},\underline{6},9$

2.5) Intervals 2, 6, 9, 13

3.1) Sequences & their Limits 2, 6, 9, 13

3.2) Limit Theorems 2, 6, 9, 13

3.3) Monotone Sequences 2, 6, 9, 13

3.4) Subsequences & the Bolzano-Weierstrass Theorem 2, 6, 9, 13

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$2x+y$ [(12) Given any $x \in \mathbb{R}$, show that there exists $n \in \mathbb{N}$ such that $1/2^n < y$.]

If $x \in \mathbb{Z}$, set

$n = x+1$

If $x \notin \mathbb{Z}$ observe two cases:

1. $x > 0$

by corollary 2.4.6, there exists $n \in \mathbb{N}$ such that $n-1 \leq x <$

uniqueness:

· if $n, m$ are integers such that $n < m$, then $n \leq m-1$

Therefore, the sets $\{z : n-1 \leq z < n\}$ & $\{z : m-1 \leq z < m\}$ are disjoint.

Therefore, $n$ such that $n-1 \leq x < n$ is unique.

2. $x < 0$

For $-x > 0$ there exists unique $n \in \mathbb{N}$ such that

$n-1 \leq -x < n \Rightarrow -n < x \leq 1-n$ (case 1)

$-n < x \leq 1-n \Leftrightarrow -n \leq x < 1-n$

where $x \notin \mathbb{Z} \Rightarrow x \neq -n, 1-n$

$\Leftrightarrow m-1 \leq x < m$

where $m = 1-n \in \mathbb{Z} \Rightarrow -n = m-1$

2.4 Applications of Supremum. 2,6,9,13 (Advanced Calculus)
2.5) Intervals 2,6,9,13
3.1) Sequences & their limits 2,6,9,13
3.2) Limit Theorems 2,6,9,13
3.3) Monotone Sequences 2,6,9,13
3.4) Subsequences & the Bolzano-Weierstrass Theorem 2,6,9,13

(6) Let $X$ be a nonempty set & let $f: X \to \mathbb{R}$ have bounded range in $\mathbb{R}$. If $a \in \mathbb{R}$, Show that Ex. 2.4.1(a) implies

$$\sup\{a+f(x): x \in X\} = a + \sup\{f(x): x \in X\}$$

Show that we also have

$$\inf\{a+f(x): x \in X\} = a + \inf\{f(x): x \in X\}$$

First let's prove $\inf\{a+S\} = a + \inf S$, let $u = \inf S$ then for any $s \in S$

$$\inf S \leq s$$

$$a + \inf S \leq a + s$$

$$a + u \leq a + s \quad \forall s \in S$$

$$\inf\{a+u\} = a + u \leq \inf\{a+S\} \ (1)$$

Now if $v$ is the lower bound of set $a+S$ then

$$a + s \geq v \quad s \in S$$

$$s \geq v - a \quad s \in S$$

Therefore $v-a$ is lower bound of $S$ meaning $v - a \leq \inf S \Rightarrow v \leq \inf S + a$

Now we take $v = \inf\{a+S\}$. Then

$$v = \inf\{a+S\} \leq \inf S + a \ (2)$$

$$\inf\{a+S\} = a + \inf S$$

2.4 App ions of Supremum. 2,6,9,13 (Advanced Calculus)
2.5) Intervals 2,6,9,13
3.1) Sequences & their Limits 2,6,9,13
3.2) Limit Theorems 2,6,9,13
3.3) Monotone Sequences 2,6,9,13
3.4) Subsequences & the Bolzano-Weierstrass Theorem 2,6,9,13

(9) Let $X = Y = \{x \in \mathbb{R} : 0 < x < 1\}$. Define $h : X \times Y \to \mathbb{R}$ by $h(x,y) = 2x + y$.
(a) For each $x \in X$, find $f(x) = \sup\{h(x,y) : y \in Y\}$; find $\inf\{f(x) : x \in X\}$
(b) For each $y \in Y$, find $g(y) = \inf\{h(x,y) : x \in X\}$; then find $\sup\{g(y) : y \in Y\}$. Compare with the result found in part (a).

(a) Lets use a common hypothesis $X = Y = \{x \in \mathbb{R} : 0 < x < 1\}$ & define $h : X \times Y \to \mathbb{R}$ as $h(x,y) = 2x+y$ given the def of $x$ & $y$ $0 < x < 1$, $0 < y < 1$. Thus, $\sup\limits_{y \in Y}\{h(x,y)\} = \sup\limits_{y \in Y}\{2x+y\} = 2x + \sup\limits_{y \in Y}\{y\} = 2x+1$

$$\inf_{x \in X}\left\{\sup_{y \in Y}\{h(x,y)\}\right\} = 1$$

(There is ordered set literature behind this if confused)

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(b) in the second part we change the order

$\sup\limits_{y \in Y}\left\{\inf\limits_{x \in X}\{h(x,y)\}\right\}$, $\inf\limits_{x \in X}\{h(x,y)\} = \inf\limits_{x \in X}\{2x+y\} = \inf\limits_{x \in X}\{2x\} + y = y$, $\sup\limits_{y \in Y}\left\{\inf\limits_{x \in X}\{h(x,y)\}\right\} = \sup\limits_{y \in Y}\{y\} = 1$

$0 < x < 1$
$0 < y < 1$

$$\sup_{y \in Y}\left\{\inf_{x \in X}\{h(x,y)\}\right\} = 1, \quad \inf_{x \in X}\left\{\sup_{y \in Y}\{h(x,y)\}\right\} = \sup_{y \in Y}\left\{\inf_{x \in X}\{h(x,y)\}\right\} = 1$$

2.4 A $\quad$ 2, 6, 9

2.5) Intervals 2, 6, 9, 13
3.1) Sequences & their Limits 2, 6, 9, 13
3.2) Limit Theorems 2, 6, 9, 13
3.3) Monotone Sequences 2, 6, 9, 13
3.4) Subsequences & the Bolzano-Weierstrass Theorem 2, 6, 9, 13

$2x+y$ [(12) Given any $x \in \mathbb{R}$, show that there exists $n \in \mathbb{N}$ such that $1/2^n < y$.]

If $x \in \mathbb{Z}$, set

$n = x+1$

If $x \notin \mathbb{Z}$ observe two cases:

1. $x > 0$

by corollary 2.4.6, there exists $n \in \mathbb{N}$ such that $n-1 \leq x <$

uniqueness:

- if $n, m$ are integers such that $n < m$, then $n \leq m-1$

Therefore, the sets $\{z : n-1 \leq z < n\}$ & $\{z : m-1 \leq z < m\}$ are disjoint.

Therefore, $n$ such that $n-1 \leq x < n$ is unique.

2. $x < 0$

For $-x > 0$ there exists unique $n \in \mathbb{N}$ such that $n-1 \leq -x < n \Rightarrow -n < x \leq 1-n$ (case 1)

$-n < x \leq 1-n \leftrightarrow -n \leq x < 1-n$

where $x \notin \mathbb{Z} \Rightarrow x \neq -n, 1-n$

$\leftrightarrow m-1 \leq x < m$

where $m = 1-n \in \mathbb{Z} \Rightarrow -n = m-1$

1.1) 1, 6, 15, 19 - sets & functions

(Advanced Calculus)

1.2) 11, 4, 14, 20 - Mathematical induction

1.3) 1, 6, 10, 13 - Finite & Infinite sets

2.1) 1, 3, 16, 24 - The Algebra & order principle of $\mathbb{R}$

2.2) 3, 7, 13, 19 - Absolute Value & real line

2.3) 3, 6, 9, 13 - The Completeness of property $\mathbb{R}$

(13) Show that a nonempty finite set $S \subseteq \mathbb{R}$ contains its supremum.
[hint: use induction & preceding exercise.]

Let $|S| = n < \infty$

Base Case: $n=1$, Say $S = \{a_1\} \Rightarrow \sup S = a_1 \in S$

Induction: Suppose $n \geq 2$ & that

$\sup S' \in S$, for all finite sets $S'$ where $|S'| = n$

$S = \{a_1, a_2, \dots, a_n, a_{n+1}\}$

& define $S_1$ as $S_1 = \{a_1, a_2, \dots, a_n\}$

$S_1$ is a subset of $S$ & $S = S_1 \cup \{a_{n+1}\}$

$|S_1| = n$, so by induction, $\sup S_1 = s_1 = S_1 \subseteq S$.

By previous exercise, $s_1 = \sup S_1$

$\sup S = \sup\{S_1, a_{n+1}\}$ As $s_1, a_{n+1} \in S$

$= \sup\{s_1, a_{n+1}\}$ ,

$\sup S \in S$

2.4 App ions o Supremum 2,6,9,13 (Advanced Calculus)
2.5) Intervals 2,6,9,13
3.1) Sequences & their Limits 2,6,9,13
3.2) Limit Theorems 2,6,9,13
3.3) Monotone Sequences 2,6,9,13
3.4) Subsequences & the Bolzano-Weierstrass Theorem 2,6,9,13

(2) If $S = \{1/n - 1/m : n, m \in \mathbb{N}\}$, find inf $S$ & sup $S$.

(Supremum) Sup - least upper bound
(Infimum) inf - greatest lower bound

Let's see sup $S = 1$

given $\varepsilon > 0$ there exists $m_\varepsilon \in \mathbb{N}$ such $\frac{1}{m_\varepsilon} < \varepsilon$ that given $-\varepsilon < -\frac{1}{m_\varepsilon}$ & therefore
$1 - \varepsilon < 1 - \frac{1}{m_\varepsilon}$

Observe

$$s_\varepsilon = \frac{1}{1} - \frac{1}{m_\varepsilon} = 1 - \frac{1}{m_\varepsilon} \in S$$

given $\varepsilon > 0$, $s_\varepsilon \in S$ such that $1 - \varepsilon < s_\varepsilon$ & therefore $\sup S = 1$.

Since $0 < \frac{1}{m} \leq 1$ for all $m \in \mathbb{N}$ we have $-1 \leq -\frac{1}{m} < 0$ for $m \in \mathbb{N}$

Since $0 < \frac{1}{n} \leq 1$ for all $n \in \mathbb{N}$

$$-1 \leq \frac{1}{n} - 1 < \frac{1}{n} - \frac{1}{m} < \frac{1}{n} < 1$$

[Since $S$ bounded by $-1$ & $1$ then inf & sup exists.]

Let's see inf $S = 1$

given $v > -1$, we get $0 < v+1$, $n_v \in \mathbb{N}$

$\frac{1}{n_v} < v+1$

see $s_v = \frac{1}{n_v} - 1 = \frac{1}{n_v} - \frac{1}{1}$ so $s_v \in S$

So $s_v \in S$ exists, where $s_v < v$, so $v$ is not a lowerbound of $S$ & $v \neq \inf S$

Since $v > -1$, so there's no $v > -1$ is a lower bound of $S$, thus $\inf S = -1$.

2.4 Applications of Supremum 2,6,9,13 (Advanced Calculus)
2.5) Intervals 2,6,9,13
3.1) Sequences & their limits 2,6,9,13
3.2) Limit Theorems 2,6,9,13
3.3) Monotone Sequences 2,6,9,13
3.4) Subsequences & the Bolzano-Weierstrass Theorem 2,6,9,13

(2) If $S=\{1/n-1/m : n, m\in\mathbb{N}\}$, find inf $S$ & Sup $S$.

(supremum) Sup - least upper bound
inf - greatest lower bound (infimum)

Since $0<\frac{1}{m}\leq 1$ for all $m\in\mathbb{N}$ we have $-1\leq -\frac{1}{m}<0$ for $m\in\mathbb{N}$

Since $0<\frac{1}{n}\leq 1$ for all $n\in\mathbb{N}$

$-1\leq \frac{1}{n}-1<\frac{1}{n}-\frac{1}{m}<\frac{1}{n}<1$

[Since $S$ bounded by $-1$ & $1$ then inf & sup exists.]

let's see sup $S=1$

given $\varepsilon>0$ there exists $m_\varepsilon\in\mathbb{N}$ such $\frac{1}{m_\varepsilon}<\varepsilon$ that gives $-\varepsilon<-\frac{1}{m_\varepsilon}$ & therefore $1-\varepsilon<1-\frac{1}{m_\varepsilon}$

Observe

$s_\varepsilon=\frac{1}{1}-\frac{1}{m_\varepsilon}=1-\frac{1}{m_\varepsilon}\in S$

given $\varepsilon>0$, $s_\varepsilon\in S$ such that $1-\varepsilon<s_\varepsilon$ & therefore $\sup S=1$.

let's see inf $S=1$

given $v>-1$, we get $0<v+1$, $n_v\in\mathbb{N}$ $\frac{1}{n_v}<v+1$

see $s_v=\frac{1}{n_v}-1=\frac{1}{n_v}-\frac{1}{1}$ so $s_v\in S$

So $s_v\in S$ exists, where $s_v<v$, so $v$ is not a lowerbound of $S$ & $v\neq \inf S$

Since $v>-1$, so there's no $v>-1$ is a lower bound of $S$, thus $\inf S=-1$.

1.1) 1, 6, 15, 19 - Sets & functions

(Advanced Calculus)

1.2) 11, 4, 14, 20 - Mathematical induction

1.3) 1, 6, 10, 13 - Finite & Infinite sets

2.1) 1, 3, 16, 24 - The Algebra & Order principle of $\mathbb{R}$

2.2) 3, 7, 13, 19 - Absolute Value & real line

2.3) 3, 6, 9, 13 - The Completeness of property $\mathbb{R}$

(9) Let $S \subseteq \mathbb{R}$ be nonempty. Show that if $u = \sup S$, then for every number $n \in \mathbb{N}$ the number $u - 1/n$ is not an upper bound of $S$, but the number $u + 1/n$ is an upper bound of $S$. (The Converse is also true)

Let $u = \sup S$. $u$ is by definition an upperbound of $S$, $u + \frac{1}{n}$ is also an upperbound of $S$, since $u + \frac{1}{n} > u$, $\forall n \in \mathbb{N}$

for $\varepsilon = \frac{1}{n}$ by lemma 2.3.4 there exists an $s_\varepsilon \in S$ such that

$$u - \frac{1}{n} = u - \varepsilon < s_\varepsilon$$

Therefore, $u - \frac{1}{n}$ is not an upper bound of $S$.

(Advanced Calculus)

1.1) 1, 6, 15, 19 - Sets & functions
1.2) 11, 4, 14, 20 - Mathematical induction
1.3) 1, 6, 10, 13 - Finite & Infinite sets
2.1) 1, 3, 16, 24 - The Algebra & Order principle of $\mathbb{R}$
2.2) 3, 7, 13, 19 - Absolute Value & real line
2.3) 2, 6, 9, 13 - The Completeness of property $\mathbb{R}$

(6) Let $S$ be a nonempty subset of $\mathbb{R}$ that is bounded below. Prove that $\inf S = -\sup\{-s : s \in S\}$.

Consider the set $S' = \{-s : s \in S\}$.

Since $\inf S \leq s$ for all $s \in S$ we have that $-s \leq -\inf S$ for all $s \in S$ which gives us that $S'$ is bounded above by $-\inf S$ & therefore there exists $\sup S'$ which gives us that $S' \leq -\inf S$.

We will show that $\sup S' = -\inf S$.

Suppose that $\sup S' < -\inf S$. Then we have that $\inf S < -\sup S'$ which gives that $-\sup S'$ is not a lower bound of $S$ & therefore there exists $s \in S$ such that $s < -\sup S'$. $-\sup S'$ is not a lower bound of $S$ & therefore there exists $s \in S$ such that

Thus, $\sup S' < -s$ which is a contradiction since $-s \in S'$

So, $\sup S' = -\inf S$.

1. 1, 6, 15, 19 - 3 5

1.2) 11, 4, 14, 20 - Mathematical induction

1.3) 1, 6, 10, 13 - Finite & Infinite sets

2.1) 1, 3, 16, 24 - The Algebra & Order principle of $\mathbb{R}$

2.2) 5, 7, 13, 19 - Absolute Value & real line

2.3) 3, 5, 9, 13 - The Completeness of property $\mathbb{R}$

(2) Let $S = \{x \in \mathbb{R} : x > 0\}$. Does $S_2$ have lower bounds? Does $S_2$ have upper bounds? Does $\inf S_2$ exist? Does $\sup S_2$ exist? Prove your statements.

- Since $0 < x$ for all $x \in S_1$, we have that 0 is a lower bound of $S_2$ & therefore, a real number smaller than 0 is also a lower bound of $S_2$.

- Suppose now that $S_2$ has at least one upper bound. Then, there exists $u \in \mathbb{R}$ such that $x \leq u$ for all $x \in S_2$. Since $0 < x$ & $x \leq u$ for all $x \in S_2$ it follows that $0 < u$ which gives us that $0 < 1 \leq u + 1$ & therefore $u, u+1 \in S_2$. Since $u$ is an upper bound of $S_2$ & $u+1 \in S_2$ we have that $u + 1 \leq u$ which gives us that $1 \leq 0$ which is not true. (So no upper bound for $S_2$)

- We already know that 0 is a lower bound of $S_2$. Suppose there exists a lower bound $s$ of $S_2$ such that $0 < s$. Then we have in particular that $s \in S_2$.

- Observe now that $0 < \frac{s}{2} < s$. Thus, we have that $\frac{s}{2} \in S$ & $\frac{s}{2} < s$ which contradicts the fact that $s$ is a lower bound of $S_2$.

- Hence, $S_2$ has no lower bound bigger than 0 & therefore $\inf S_2 = 0$. Since the supremum of a nonempty set is always an upper bound of the set & $S_2$ has no upper bounds, it follows that $\sup S_2$ does not exist.

1.1) 1, 6, 15, 19 - sets & functions

(Advanced Calculus)

1.2) 11, 4, 14, 20 - Mathematical induction

1.3) 1, 6, 10, 13 - Finite & Infinite sets

2.1) 1, 3, 16, 24 - The Algebra & Order principle of $\mathbb{R}$

2.2) 3, 7, 13, 19 - Absolute Value & real line

2.3) 2, 5, 9, 13 - The Completeness of property $\mathbb{R}$

(13) Find all $x \in \mathbb{R}$ that satisfy both $|2x-3| < 5$ & $|x+1| > 2$ simultaneously.

We have that

$$|2x-3| < 5 \leftrightarrow -5 < 2x-3 < 5 \leftrightarrow -2 < 2x < 8 \leftrightarrow -1 < x < 4$$

the solution set of the inequality

$|2x-3| < 5$ is $(-1,4)$.

We have that $|x+1| > 2 \leftrightarrow x+1 < -2$ or $x+1 > 2 \leftrightarrow x < -3$ or $x > 1$.

Thus the solution set of the inequality $|x+1| > 2$ is $(-\infty,-3) \cup (1,\infty)$

Observe that

$$\begin{aligned}(-1,4) \cap ((-\infty,-3) \cup (1,\infty)) &= ((-1,4) \cap (-\infty,-3)) \cup ((-1,4) \cap (1,\infty)) \\ &= \emptyset \cup (1,4) \\ &= (1,4)\end{aligned}$$

So the subset of $\mathbb{R}$ that satisfy $|2x-3| < 5$ & $|x+1| > 2$ simultaneously is $(1,4)$.

(Advanced Calculus)

1.1) 1, 6, 15, 19 - sets & functions
1.2) 11, 4, 14, 20 - Mathematical induction
1.3) 1, 6, 10, 13 - Finite & Infinite sets
2.1) 1, 3, 16, 24 - The Algebra & order principle of $\mathbb{R}$
2.2) 3, 7, 13, 19 - Absolute Value & real line
2.3) 2, 5, 9, 13 - The completeness of property $\mathbb{R}$

(19) Show that if $a, b, c \in \mathbb{R}$, then the middle number is $\text{mid}\{a,b,c\} = \min\{\max\{a,b\}, \max\{b,c\}, \max\{c,a\}\}$.

From each pair in two elements in set $\{a,b,c\}$ we choose a larger one.

$$\{\max\{a,b\}, \max\{b,c\}, \max\{c,a\}\}$$

choosing the smallest is choosing the middle

$$\text{mid}\{a,b,c\} = \min\{\max\{a,b\}, \max\{b,c\}, \max\{c,a\}\}$$

for example suppose $a \le b \le c$. Then

$$\max\{a,b\} = b$$
$$\max\{b,c\} = c$$
$$\max\{c,a\} = c$$

$$\min\{\max\{a,b\}, \max\{b,c\}, \max\{c,a\}\} = \min\{b,c,c\} = b$$

1.1) 1, 6, 15, 19 - Sets & functions

(Advanced Calculus)

1.2) 11, 4, 14, 20 - Mathematical induction

1.3) 1, 6, 10, 13 - Finite & Infinite sets

2.1) 1, 3, 16, 24 - The Algebra & Order principle of $\mathbb{R}$

2.2) 3, 7, 13, 19 - Absolute Value & real line

2.3) 2, 5, 9, 13 - The Completeness of property $\mathbb{R}$

(2) Let $S=\{x\in\mathbb{R} : x>0\}$. Does $S_2$ have lower bounds? Does $S_2$ have upper bounds? Does inf $S_2$ exist? Does sup $S_2$ exist? Prove your statements.

- Since $0<x$ for all $x\in S_1$, we have that 0 is a lower bound of $S_2$ & therefore, any real number smaller than 0 is also a lower bound of $S_2$.

- Suppose now that $S_2$ has at least one upper bound. Then, there exists $u\in\mathbb{R}$ such that $x\leq u$ for all $x\in S_2$. Since $0<x$ & $x\leq u$ for all $x\in S_2$ it follows that $0<u$ which gives us that $0<1\leq u+1$ & therefore $u, u+1\in S_2$.
Since $u$ is an upper bound of $S_2$ & $u+1\in S_2$ we have that $u+1\leq u$ which gives us that $1\leq 0$ which is not true. (So no upper bound for $S_2$)

- We already know that 0 is a lower bound of $S_2$. Suppose there exists a lower bound $s$ of $S_2$ such that $0<s$. Then we have in particular that $s\in S_2$.

- Observe now that $0<\frac{s}{2}<s$. Thus, we have that $\frac{s}{2}\in S$ & $\frac{s}{2}<s$ which contradicts the fact that $s$ is a lower bound of $S_2$.

- Hence, $S_2$ has no lower bound bigger than 0 & therefore inf $S_2=0$.
Since the supremum of a nonempty set is always an upper bound of the set & $S_2$ has no upper bounds, it follows that sup $S_2$ does not exist.

(Advanced Calculus)

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2.1) 1, 3, 16, 24 - The Algebra & Order principle of $\mathbb{R}$
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(3) If $x, y, z \in \mathbb{R}$ & $x \leq z$, show that $x \leq y \leq z$ if & only if $|x-y| + |y-z| = |x-z|$. Interpret this geometrically.

$x, y, z \in \mathbb{R}$, $x \leq z$

P.T: $x < y \leq z$ if $|x-y| + |y-z| = |x-z|$

Proof: $x \leq y \leq z \Rightarrow \begin{cases} |x-y| = -(x-y) \\ |y-z| = -(y-z) \\ |x-z| = -(x-z) \end{cases}$

$\Rightarrow |x-y| + |y-z| = |x-z|$

& $x \leq z$, suppose that (i) $y < x$
(ii) $y > z$

then eq(1) becomes: $-(x-y) + (y-z) \neq z - x$

thus $x \leq y \leq z$

{geometrically this ($|x-y| + |y-z| = |x-z|$) holds only if $y \in [x, z]$.}

(Advanced Calculus)

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(7) find all $x \in \mathbb{R}$ that satisfy the equation $|x+1| + |x-2| = 7$.

Consider equation ↑

(Step 1)

If $x < -1$ then $x+1 < 0$ & $x-2 < -3 < 0$ which gives us that $|x+1| = -(x+1) = -x-1$ & $|x-2| = -(x-2) = -x+2$

Thus $|x+1| + |x-2| = -x-1-x+2 = -2x+1$

& the eq becomes $-2x+1 = 7$

$-2x+1 = 7 \leftrightarrow -2x = 6 \leftrightarrow x = -3$.

observe that $|-3+1| + |-3-2|$
$= |-2| + |-5| = 2+5 = 7$

& therefore $-3$ is a Solution to the eq.

$-3$ is the only solution to $(-\infty, -1)$.

(Step 2)

If $-1 \leq x < 2$ then $0 \leq x+1$ & $x-2 < 0$ which gives us that $|x+1| = x+1$ & $|x-2| = -(x-2) = -x+2$

Thus, $|x+1| + |x-2| = x+1-x+2 = 3$

& so no $x \in [-1, 2)$ is a Solution to the equation.

(step 3) If $2 \leq x$ then $0 < 3 \leq x+1$ & $0 \leq x-2$ which gives us that $|x+1| = x+1$ & $|x-2| = x-2$

Thus $|x+1| + |x-2| = x+1+x-2 = 2x-1$ & then the eq becomes $2x-1 = 7$.

$2x-1 = 7 \leftrightarrow 2x = 8 \leftrightarrow x = 4$

observe that $|4+1| + |4-2| = |5| + |2| = 5+2 = 7$ and so 4 is a solution to the eq.

thus 4 is the only Solution to $[2, \infty)$

Thus the solution of the set eq is $3, 4\}$

(Advanced Calculus)

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2.3) 2, 5, 9, 13 - The Completeness of property $\mathbb{R}$

(7) find all $x \in \mathbb{R}$ that satisfy the equation $|x+1|+|x-2|=7$.

Consider equation ↑ (step 1)

If $x<-1$ then $x+1<0$ & $x-2<-3<0$ which gives us that $|x+1|=-(x+1)=-x-1$ & $|x-2|=-(x-2)=-x+2$

Thus $|x+1|+|x-2|=-x-1-x+2=-2x+1$

& the eq becomes $-2x+1=7$

$-2x+1=7 \Leftrightarrow -2x=6 \Leftrightarrow x=-3$.

observe that $|-3+1|+|-3-2|$
$=|-2|+|-5|=2+5=7$

& therefore $-3$ is a Solution to the eq.

$-3$ is the only solution to $(-\infty,-1)$.

(step 2)

If $-1\leq x<2$ then $0\leq x+1$ & $x-2<0$ which gives us that $|x+1|=x+1$ & $|x-2|=-(x-2)=-x+2$

Thus, $|x+1|+|x-2|=x+1-x+2=3$

& so no $x\in[-1,2)$ is a solution to the equation.

(step 3) If $2\leq x$ then $0<3\leq x+1$ & $0\leq x-2$ which gives us that $|x+1|=x+1$ & $|x-2|=x-2$

Thus $|x+1|+|x-2|=x+1+x-2=2x-1$ & then the eq becomes $2x-1=7$.

$2x-1=7 \Leftrightarrow 2x=8 \Leftrightarrow x=4$

observe that $|4+1|+|4-2|=|5|+|2|=5+2=7$ and so 4 is a solution to the eq.

thus 4 is the only Solution to $[2,\infty)$

Thus the solution of the set eq is 3

(Advanced Calculus)

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(b) If $0<c<1$ & $m, n \in \mathbb{N}$, Show that $c^n \leq c$ for all $n \in \mathbb{N}$, & that $c^n < c$ for $n > 1$.

(b) Let $m, n \in \mathbb{N}$ arbitrary such that $m > n$ & $0<c<1$. Then $\frac{1}{c} > 1$

So we can apply part (a)

$$\left(\frac{1}{c}\right)^m \geq \left(\frac{1}{c}\right)^n$$

$$\Leftrightarrow c^{-m} \geq c^{-n} / \cdot c^{m+n}$$

$$\Leftrightarrow c^n \geq c^m$$

(24) (a) If $c > 1$, Show that $c^n \leq c$ for all $n \in \mathbb{N}$, & that $c^n < c$ for $n > 1$.

Suppose $m > n \Rightarrow m-n > 0 \Rightarrow m-n \in \mathbb{N}$

$$c^{m-n} \geq c > 1 / \cdot c^n$$

$$c^m > c^n$$

we proved one direction

now suppose for the other

$c > 1$ & $c^m > c^n$.

If $m \leq n \Rightarrow$

$1^\circ$ $m = n \Rightarrow c^m = c^n$

$2^\circ$ $m < n \Rightarrow c^m < c^n$

in both cases we contradicted $c^m > c^n$. So the assumption $m \leq n$ is wrong; it must be $m > n$.

(Advanced Calculus)

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(b) If $0<c<1$ & $m, n \in \mathbb{N}$, show that $c^n \leq c$ for all $n \in \mathbb{N}$, & that $c^n < c$ for $n>1$.

b) Let $m, n \in \mathbb{N}$ arbitrary such that $m>n$ & $0<c<1$. Then $\frac{1}{c}>1$

So we can apply part (a)

$$\left(\tfrac{1}{c}\right)^m \geq \left(\tfrac{1}{c}\right)^n$$

$$\leftrightarrow c^{-m} \geq c^{-n} / \cdot c^{m+n}$$

$$\leftrightarrow c^n \geq c^m$$

(24) (a) If $c>1$, show that $c^n \leq c$ for all $n \in \mathbb{N}$, & that $c^n < c$ for $n>1$.

Suppose $m>n \Rightarrow m-n>0 \Rightarrow m-n \in \mathbb{N}$

$$c^{m-n} \geq c > 1 / \cdot c^n$$

$$c^m > c^n$$

we proved one direction

now suppose for the other $c>1$ & $c^m > c^n$.

If $m \leq n \Rightarrow$

1° $m=n \Rightarrow c^m = c^n$

2° $m<n \Rightarrow c^m < c^n$

in both cases we contradicted $c^m > c^n$. So the assumption $m \leq n$ is wrong; it must be $m>n$.

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2.3) 2, 5, 9, 13 - The Completeness of property $\mathbb{R}$

(16) Find all real numbers $x$ that satisfy the following inequalities.

(a) $x^2 > 3x+4$

$x^2 > 3x+4 \Leftrightarrow x^2-3x-4>0 \Leftrightarrow (x+1)(x-4)$

two possibilities

- $x+1>0,\ x-4>0 \Rightarrow x>-1,\ x>4 \Rightarrow \boxed{x>4}$
- $x+1<0,\ x-4<0 \Rightarrow x<-1,\ x<4 \Rightarrow \boxed{x<-1}$

Sol: $x \in (-\infty,-1) \cup (4,\infty)$

(b) $1 < x^2 < 4$

$1<x^2<4 \Leftrightarrow 1<|x|<2$

$\Rightarrow x \in (-2,-1) \cup (1,2)$

(c) $1/x < x$

$\frac{1}{x} < x \Leftrightarrow \frac{1}{x} - x < 0$

$\frac{1}{x}(1-x^2)<0$

$\frac{1}{x}(1-x)(1+x)<0$

If $x>0$ then

$\frac{1}{x}(1-x)(1+x)<0 \Leftrightarrow (1-x)(1+x)<0 \Leftrightarrow 1-x<0 \Leftrightarrow x>1$

If $x<0$ then

$\frac{1}{x}(1-x)(1+x)<0 \Leftrightarrow (1-x)(1+x)>0 \Leftrightarrow 1+x>0 \Leftrightarrow x>-1$

$\boxed{\text{Sol: } x \in (-1,0) \cup (1,\infty)}$

(d) $1/x < x^2$

$\frac{1}{x} < x^2 \Leftrightarrow \frac{1}{x} - x^2 < 0 \Leftrightarrow \frac{1}{x}(1-x^3)<0$

if $x>0$, $\frac{1}{x}(1-x^3)<0 \Leftrightarrow 1-x^3<0 \Leftrightarrow x^3>1 \Leftrightarrow x>1$

if $x<0$, $\frac{1}{x}(1-x^3)<0 \Leftrightarrow 1-x^3>0 \Leftrightarrow x^3<1 \Leftrightarrow x<1$

$\boxed{\text{Sol: } x \in (-\infty,0) \cup (1,\infty)}$

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2.3) 2, 5, 9, 13 - The completeness of property $\mathbb{R}$

(1) If $a, b \in \mathbb{R}$, prove the following

(a) If $a+b=0$, then $b=-a$

Let $a, b \in \mathbb{R}$

If $a+b=0$ the adding $-a$ to both sides of the equation we have that $(-a)+(a+b)=(-a)+0$

Hence since

$$(-a)+(a+b)=((-a)+a)+b$$
$$=0+b$$
$$=b$$

& $(-a)+0=-a$

we have that $\boxed{b=-a.}$

(b) $-(-a)=a$

We have that $-(-a)+(-a)=0$

adding $a$ to both sides of $-(-a)+(-a)=0$, then $(-(a)+(-a))+a=0+a$

$$(-(-a)+(-a))+a=-(-a+((-a)+a)$$
$$=-(-a)+0=-(-a)$$

& $0+a=a$ we result with $\boxed{-(-a)=a}$

(c) $(-1)a=-a$

By Theorem 2.1.2 (c) we have that $a \cdot b=0$

Thus since $1+(-1)=0$ & $1 \cdot a=a$ we have that

$$0 \cdot a=0 \cdot a=(1+(-1))a=1 \cdot a+(-1) \cdot a$$
$$=a+(-1)a$$

therefore $a+(-1)a=0$

now add $-a$ on both sides

$$-a+(a+(-1)a)=-a+0$$
$$\downarrow$$
$$(-a+a)+(-1)a=(-1)a \quad , \quad \boxed{(-1)a=-a}$$

(d) $(-1)(-1)=1$

by item (c) we have that $(-1)(-1)=-(-1)$

& by item (b) we have that $-(-1)=1$

Hence, $(-1)(-1)=1.$

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2.3) 2, 5, 9, 13 - The completeness of property $\mathbb{R}$

Solve the following equations, justifying each step by referring to an appropriate property or theorem.

(a) $2x+5=8$, There are 2 binary operations addition & multiplication, $+, \cdot$.

$$2x+5=8 \;/ +(-5)$$
$$2x+5+(-5)=3+5+(-5)$$
$$2x+0=3+0$$
$$2x=3$$
$$2x=3 \;/ \cdot\frac{1}{2}$$
$$\frac{1}{2}\cdot 2\cdot x=\frac{1}{2}\cdot 3$$
$$1\cdot x=\frac{3}{2}$$
$$\boxed{x=\frac{3}{2}}$$

(b) $x^2=2x$

$$x^2=2x \;/ -2x$$
$$x^2-2x=2x+(-2x)$$
$$x^2-2x=0$$
$$x(x-2)=0$$
$$x-2=0$$
$$x=2$$
$$\boxed{x=0 \text{ or } x=2}$$

(c) $x^2-1=3$

$$x^2-1=3 \;/ +(-3)$$
$$x^2-4=0$$
$$(x-2)(x+2)=0$$
$$x-2=0 \;/ +2$$
$$x=2$$
$$x+2=0 \;/ +(-2)$$
$$x=-2$$

$\{2,-2$

(d) $(x-1)(x+2)=0$

$$x-1=0 \;/ +1$$
$$x=1$$
$$x+2=0 \;/ +(-2)$$
$$x=-2$$
$$\boxed{\{1,-2\}}$$

* when you have factors split them into conditions

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# (Advanced Calculus)

Solve the following equations, justifying each step by referring to an appropriate property or theorem.

(a) $2x+5=8$, There are 2 binary operations addition & multiplication, $+, \cdot$.

$$2x+5=8 \;/+(-5)$$
$$2x+5+(-5)=3+5+(-5)$$
$$2x+0=3+0$$
$$2x=3$$
$$2x=3 \;/\cdot\tfrac{1}{2}$$
$$\tfrac{1}{2}\cdot 2\cdot x=\tfrac{1}{2}\cdot 3$$
$$1\cdot x=\frac{3}{2}$$
$$\boxed{x=\frac{3}{2}}$$

(b) $x^2=2x$

$$x^2=2x \;/-2x$$
$$x^2-2x=2x+(-2x)$$
$$x^2-2x=0$$
$$x(x-2)=0$$
$$x-2=0$$
$$x=2$$
$$\boxed{x=0 \text{ or } x=2}$$

(c) $x^2-1=3$

$$x^2-1=3 \;/+(-3)$$
$$x^2-4=0$$
$$(x-2)(x+2)=0$$
$$x-2=0 \;/+2$$
$$x=2$$
$$x+2=0 \;/+(-2)$$
$$x=-2$$
$$\{2,-2\}$$

(d) $(x-1)(x+2)=0$

* when you have factors split them into conditions

$$x-1=0 \;/+1$$
$$x=1$$
$$x+2=0 \;/+(-2)$$
$$x=-2$$
$$\boxed{\{1,-2\}}$$

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(6) Exhibit a bijection between $\mathbb{N}$ & a proper subset of itself.

The set of all even integers $2\mathbb{N} = \{2n : n \in \mathbb{N}\}$ is a proper subset of $\mathbb{N}$.

Function $f$ defined by

$$f(n) = 2n$$

is a bijection from $\mathbb{N}$ into $2\mathbb{N}$

injectivity

suppose $f(n) = f(m) \Rightarrow 2n = 2n = 2m \Rightarrow n = m$

Surjectivity

An arbitrary even integer $x$ is equal to $2n$, for some $n \in \mathbb{N}$. Then $f(n) = 2n = x$.

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(b) Given that $h(m,3)=19$, find $m$.

- Let's see the value of $h(m,3)$ using the above counting formula

$$h(m,3)=\frac{1}{2}(m+3-2)(m+3-1)+m$$
$$=\frac{1}{2}(m+1)(m+2)+m$$
$$=\frac{1}{2}(m^2+3m+2)+m$$
$$=\frac{1}{2}m^2+\frac{5}{2}m+1$$

solve the quadratic

$$\frac{1}{2}m^2+\frac{5}{2}m+1=19$$
$$m^2+5m-36=0$$
$$m_{1,2}=\frac{-5\pm\sqrt{25+144}}{2}$$
$$\boxed{m_1=4}$$
$$m_2=-9\notin\mathbb{N}$$

(10)

(a) If $(m,n)$ is the 6th point down the 9th diagonal of the array in Figure 1.3.1, calculate its number according to the counting method given for Theorem 1.3.8.

- we can see the 6th point down the 9th diagonal is $(6,4)$ since it satisfies $k=m+n-1$ from diagonal procedure. from $n=9-6+1=4$

we count the point $(m,n)$ via first counting point in first $k-1=m+n-2$ diagonals & then adding $m$. As a result we have counting function $h:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ defined as

$$h(6,4)=\frac{1}{2}(m+n-2)(m+n-1)+m$$
$$h(m,n)=\frac{1}{2}(6+4-2)(6+4-1)+6$$
$$=\frac{1}{2}\cdot 8\cdot 9+6$$
$$=36+6$$
$$\boxed{=42}$$

(Advanced Calculus)

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(13) Prove that the collection $\mathcal{F}(\mathbb{N})$ of all finite subsets of $\mathbb{N}$ is countable.

A set is countable if it's finite.

1.- Let's define $\mathcal{F}(\mathbb{N})$ as a collection of all finite subsets of $\mathbb{N}$, we must prove that $\mathcal{F}(\mathbb{N})$ is countable.

2.- The finite subsets of $\mathbb{N}$ are not necessarily in a consecutive sequence, thus we may find any order or sequence, but they must have a maximum since they are finite sets. Thus any finite subset $A \subset \mathbb{N}$ is forced to satisfy

$$A \in A_k = P(\{1, 2, 3, \dots, k\}),$$

where $k = \max_{i \in A} i$ & $P$ denotes the power set.

So $\mathcal{F}(\mathbb{N})$ is $\subset$ to $A_k$ subsets.

3.- $\mathcal{F}(\mathbb{N}) = \bigcup_{k=0}^{\infty} A_k$

$A_0 = \emptyset$ & $A_k = P(\{1, 2, 3, \dots, k\})$

Now prove 2 directional inclusions

$$\bigcup_{k=0}^{\infty} A_k \subset \mathcal{F}(\mathbb{N})$$

$A \in \mathcal{F}(\mathbb{N})$, then $|A| = n \in \mathbb{N}$, where $|A|$ stands for # elements in set $A$, which implies $A$ is a finite set.

$M \in \mathbb{N}$ such that

$$M = \max_{i \in A} i$$

$$A \in A_M = P(\{1, 2, 3, \dots, M\}) \subset \bigcup_{k=0}^{\infty} A_k$$

$$A \in \bigcup_{k=0}^{\infty} A_k$$

$$\mathcal{F}(\mathbb{N}) \subset \bigcup_{k=0}^{\infty} A_k$$

4. Now prove inverse inclusion

$A \in \bigcup_{k=0}^{\infty} A_k$ then $A \in A_j$ for at least some $j \in \mathbb{N}$

$A \in P(\{1, 2, 3, \dots, j\})$

$A \subset \{1, 2, 3, \dots, j\} \to A \in \mathcal{F}(\mathbb{N}) \to \mathcal{F}(\mathbb{N}) = \bigcup_{k=0}^{\infty} A_k$

5. Since $A_k$ is a finite set then each set $\bigcup_{k=0}^{\infty} A_k$ is countebele $\mathcal{F}(\mathbb{N})$

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(1) Prove that the non-empty set $T_1$ is finite if & only if there is a bijection from $T_1$ onto a finite set $T_2$.

Suppose $T_1$ is a finite set. Then $T_1 = \{a_1, a_2, \dots, a_n\}$

Define $T_2 = T_1$ & function $f$ to be

$$f : T_1 \to T_2 \qquad \text{for } i = 1, 2, \dots, n$$
$$a_i \to a_i$$

$f$ is identity map so its a bijection

Conversely $T_2$ be a finite set

$f : T_1 \to T_2$ be a bijection

Claim: $T_1$ is finite.

Define $g = f^{-1}$. $g : T_2 \to T_1$ is a bijection. ($f$ is a bijection so is $f^{-1}$)

Since $g$ is onto,

$$T_1 = g(T_2) = \{g(t_2) : t_2 \in T_2\}$$

Let $T_2 = \{a_1, a_2, \dots, a_n\}$. Then $T_1$ is equal to

$$T_1 = g(T_2) = \{g(a_1), g(a_2), \dots, g(a_n)\}$$

thus $T_1$ is finite.

1.1) 1, 6, 15, 19 - Sets & functions

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(1) Prove that the non-empty set $T_1$ is finite if & only if there is a bijection from $T_1$ onto a finite set $T_2$.

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Conversely $T_2$ be a finite set

$f : T_1 \to T_2$ be a bijection

Claim: $T_1$ is finite.

Define $g = f^{-1}$. $g : T_2 \to T_1$ is a bijection. ($f$ is a bijection so is $f^{-1}$)

since $g$ is onto,

$$T_1 = g(T_2) = \{g(t_2) : t_2 \in T_2\}$$

Let $T_2 = \{a_1, a_2, \dots, a_n\}$. Then $T_1$ is equal to

$$T_1 = g(T_2) = \{g(a_1), g(a_2), \dots, g(a_n)\}$$

thus $T_1$ is finite.

1.1) 1, 6, 15, 19 - sets 1

# Advanced

1.2) 11, 4, 14, 20 - Mathematical induction

1.3) 1, 6, 10, 13 - Finite & Infinite sets

2.1) 1, 3, 16, 24 - The Algebra & order principle of $\mathbb{R}$

2.2) 3, 7, 13, 19 - Absolute Value & real line

2.3) 2, 5, 9, 13 - The completeness of property $\mathbb{R}$

(11) Conjecture a formula for the sum $1/1\cdot3 + 1/3\cdot5 + \cdots + 1/(2n-1)(2n+1)$ & prove you conjecture by induction

$1 = 1^2$

$1+3 = 4 = 2^2$

$1+3+5 = 9 = 3^2$

$1+3+5+7 = 16 = 4^2$

So we conjecture $1+3+5+7+\ldots+(2n-1) = n^2$ for all $n \in \mathbb{N}$

Base case $n=1$, $2(1)-1 = 1 = 1^2$

$S_n = 1+3+7+\ldots+(2n-1) = n^2$

Now add $n+1$ both sides

$S_{n+1} = 1+3+5+7+\ldots+(2n-1)+(2(n+1)-1) = n^2+(2(n+1)-1)$

$n^2+(2(n+1)-1) = n^2+(2n+2-1)$

$n^2+2n+1 = (n+1)^2$

$\ldots + (2n-1)+(2(n+1)-1) = (n+1)^2$ therefore its true.

(Advanced Calculus)

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(20) Let the numbers $x_n$ be defined as follows: $x_1 = 1$, $x_2 = 2$ & $x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)$ for all $n \in \mathbb{N}$. Use the principle of strong induction (1.2.5) to show that $1 \leq x_n \leq 2$ for all $n \in \mathbb{N}$.

$x_1 = 1$
$x_2 = 2$
$x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)$, $n \in \mathbb{N}, n \geq 2$

For $n = 1, 2$ the property $1 \leq x_n \leq 2$, $n \in \mathbb{N}$, $n \geq 2$. Suppose $1 \leq x_k \leq 2$, $k = 1, 2, \ldots, n-1$

(1) $x_n = \frac{1}{2}(x_{n-1} + x_{n-2}) \geq \frac{1}{2}(1+1) =$ ($1$ $x_k \geq 1$ by assumption, $k < n$)

(2) $x_n = \frac{1}{2}(x_{n-1} + x_{n-2}) \leq \frac{1}{2}(2+2) =$ ($2$ $x_k \leq 2$ by assumption, $k < n$)

So $1 \leq x_n < 2$, $\forall n \in \mathbb{N}$ by principle of Strong Induction.

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(4) prove that $1^2+3^2+\dots+(2n-1)^2 = (4n^3-n)/3$ for all $n \in \mathbb{N}$.

$n=1,\ 1^2 = \frac{4(1)^3-1}{3}$

$S_n = 1^2+3^2+\dots+(2n-1)^2 = \frac{4n^2-n}{3}$

$S_{n+1} = 1^2+3^2+\dots+(2n-1)^2+(2(n+1)-1)^2 = \frac{4n^3-n}{3}+(2(n+1)-1)^2$

$$= \frac{4n^3-n}{3}+(2n+2-1)^2$$
$$= \frac{4n^3-n}{3}+(2n+1)^2$$
$$= \frac{4n^3-n}{3}+4n^2+4n+1$$
$$= \frac{4n^3-n+12n^2+12n+3}{3}$$
$$= \frac{4n^3+12n^2+12n+4-n-1}{3}$$
$$= \frac{4(n^3+3n^2+3n+1)-(n+1)}{3}$$
$$= \frac{4(n+1)^3-(n+1)}{3}$$

1.1) 1, 6, 15, 19 - Sets & functions

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(14) Prove that $2^n < n!$ for all $n \geq 4$, $n \in \mathbb{N}$.

Base.
Case. $n=4$, $2^4 = 16 < 24 = 4!$
which makes the statement true.

Induction:

$S_{n+1} = 2^{k+1} = 2(2^k) < 2k!$

Since $4 \leq k$ $2 < 5 \leq k+1$

$2k! < (k+1)k! = (k+1)!$

$2^{k+1} = (2(2^k)) < 2k! < (k+1)!$

true for all $k+1$.

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(15) Show that if $f: A \to B$ & $G, H$ are subsets of $B$, then $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$ & $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$.

Step 1: $f: A \to B$, $G, H \subseteq B$

$y \in f^{-1}(G \cup H) \longleftrightarrow f(y) \in G \cup H$

$\longleftrightarrow f(y) \in G$ or $f(y) \in H$

$\longleftrightarrow y \in f^{-1}(G)$ or $y \in f^{-1}(H)$

$\longleftrightarrow y \in f^{-1}(G) \cup f^{-1}(H)$

Analogously

$y \in f^{-1}(G \cap H) \longleftrightarrow f(y) \in G \cap H$

$\longleftrightarrow f(y) \in G$ & $f(y) \in H$

$\longleftrightarrow y \in f^{-1}(G)$ & $y \in f^{-1}(H)$

$\longleftrightarrow y \in f^{-1}(G) \cap f^{-1}(H)$

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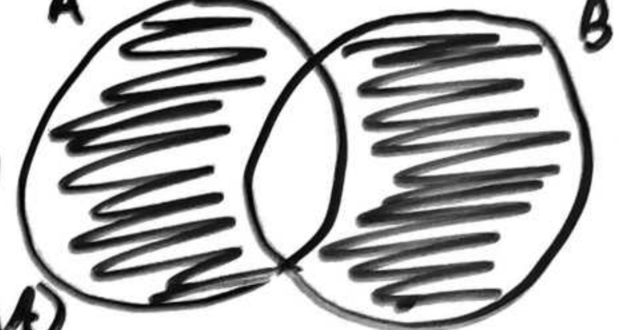
(6) The symmetric difference of two sets A & B is the set D of all elements that belong to either A or B but not both. Represent D with a Diagram.

Step 1:




By reflexivity they are equivalent.

(a) Show that $D=(A\setminus B)\cup(B\setminus A)$

Let's directly prove this

$D=\overline{A\cap B}$, But let's use equivalency to make it easier

thus $\overline{A\cap B}=D$, thus $\overline{A\cap B}=(A\setminus B)\cup(B\setminus A)$

- Lets use the Difference definition of sets

$\overline{A\cap B}=(x\in A,\ x\notin B)\cup(x\in B,\ x\notin A)$

- Let's define it in terms of it's elements

$(x\notin A \,\&\, B)=(x\in A,\ x\notin B)\cup(x\in B,\ x\notin A)$

Let's use union definition

$-(x\notin A \,\&\, B)=(x\in A,\ x\notin B)$ or $(x\in B,\ x\notin A)$

$=(x\in A)$ or $(x\in B)$

$=(x\notin A \,\&\, B)$

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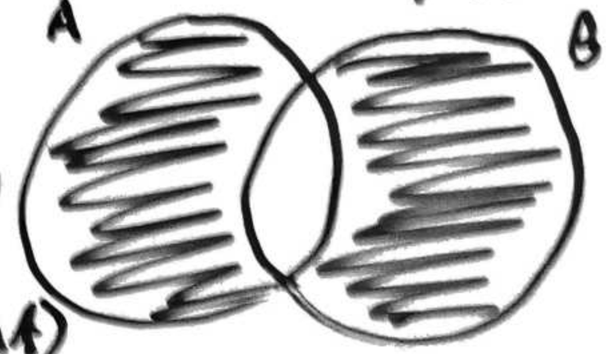


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(Advanced Calculus)

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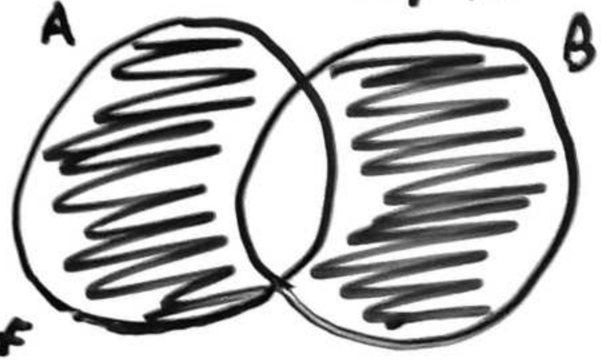


(b) Show that $D = (A \cup B) \setminus (A \cap B)$

$D = \overline{A \cap B}$,

so $\overline{A \cap B} = (A \cup B) \setminus (A \cap B)$

let use union def & intersect def

$(\overline{A \cap B}) = (x \in A \text{ or } x \in B) \setminus (x \in A \;\&\; x \in B)$

$(\overline{A \cap B}) = (x \in (x \in A \text{ or } x \in B), x \notin (x \in A \;\&\; x \in B))$

so

$(\overline{A \cap B}) = (x \in A \text{ or } x \in B)$

$= (x \in A \text{ or } x \in B) = (x \in A \text{ or } x \in B)$ by reflexivity they are equal

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1) Let $A = \{k : k \in \mathbb{N}, k \leq 20\}$, $B = \{3k-1 : k \in \mathbb{N}\}$ & $C = \{2k+1 : k \in \mathbb{N}\}$.

Step 1:

$A := \{1, 2, 3, \ldots, 20\}$

$B := \{2, 5, 8, 11, \ldots, 23, \ldots\}$

$C := \{3, 5, 7, 9, 11, \ldots 21, \ldots\}$

Step 2:

(a) $A' = A \cap B = \{x : x \in A \text{ \& } x \in B\} = \{2, 5, 8, 11, 14, 17, 20\}$

$A' \cap C = \{x : x \in A' \text{ \& } x \in C\} = \{5, 11, 17\}$

$* \; A \cap B \cap C = \{5, 11, 17\}$

(b) $A' \setminus C = \{x : x \in A' \text{ \& } x \notin C\} = \{2, 8, 14, 20\}$

$* \; (A \cap B) \setminus C = \{2, 8, 14, 20\}$

(c) $C' = A \cap C = \{x : x \in A \text{ \& } x \in C\} = \{3, 5, 7, 9, 11, 13, 15, 17, 19\}$

$C' \setminus B = \{x : x \in C' \text{ \& } x \notin B\} = \{3, 7, 9, 13, 15, 19\}$

$(A \cap C) \setminus B = \{3, 7, 9, 13, 15, 19\}$

(Advanced Calculus)

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(14) Show that if $f: A \to B$ is surjective & $H \subseteq B$, then $f(f^{-1}(H)) = H$. Give an example to show that equality need not hold if $f$ is not surjective.

Tst: $f^{-1}(f(E)) = E$

Let $x \in f^{-1}(f(E)) = \{x \in A : f(x) \in f(E)\}$. Let $f(x) = f(y)$ for some $y \in E$.

But since $f$ is injective then $x = y \in E \Rightarrow x \in E \Rightarrow f^{-1}(f(E)) \subseteq E$

Now let $x \in E$, we claim that $x \in f^{-1}(f(E))$.

$x \in E$ then $f(x) \in f(E)$, $f^{-1}(f(E))$

& $x \in f^{-1}(f(E))$ thus

$E \subseteq f^{-1}(f(E)) \Rightarrow E = f^{-1}(f(E))$

disprov $f$ is not injective Ex:

$f: A \to B$, $f(x) = 1$ $x \in \{0,1\}$

$E = \{0\}$ note $f(E) = \{f(0)\} = 1$

$f^{-1}(f(E)) = \{x \in A : f(x) = 1\} \Rightarrow f^{-1}(f(E)) = \{0,1\}$

(14) (a) Show that if $f: A \to B$ is injective & $E \subseteq A$, then $f^{-1}(f(E)) = E$. Give an example to show that equality need not hold if $f$ is not injective.

Tst: $f(f^{-1}(H)) = H$

Same flow

$f(f^{-1}(H)) = \{y : y = f(x) \text{ for some } x \in f^{-1}(H)\}$

$\{y : y = f(x),\ f(x) \in H\}$

$\{y : y \in H\} \subseteq H$

$f(f^{-1}(H)) \subseteq H$

Since $f$ is surjective

$\exists x \in A$ such $f(x) = h \in H$

$\Rightarrow H \subseteq f(f^{-1}(H)) \Rightarrow H = f(f^{-1}(H))$

$f(x) \in f(f^{-1}(H)) \Rightarrow h \in f^{-1}(f(H))$

Ex: $A = \{0,1\}$, $B = \{1,2\}$ & $H = \{2\}$

$f: A \to B$ such that $f(0) = f(1) = 1$

$f^{-1}(H) = \{x \in A : f(x) \in H\}$ thus $f^{-1}(f(H)) = \emptyset$

$f(f^{-1}(H)) \neq H$

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(b) Show that if $f: A \to B$ is surjective & $H \subseteq B$, then $f(f^{-1}(H)) = H$. Give an example to show that equality need not hold if $f$ is not surjective.

Tst: $f^{-1}(f(E)) = E$

let $x \in f^{-1}(f(E)) = \{x \in A : f(x) \in f(E)\}$. let $f(x) = f(y)$ for some $y \in E$.

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disprove $f$ is not injective Ex:

$f: A \to B$, $f(x) = 1$ $x \in \{0,1\}$

$E = \{0\}$ note $f(E) = \{f(0)\} = 1$

$f^{-1}(f(E)) = \{x \in A : f(x) = 1\} \Rightarrow f^{-1}(f(E)) = \{0,1\}$

(19) (a) Show that if $f: A \to B$ is injective & $E \subseteq A$, then $f^{-1}(f(E)) = E$. Give an example to show that equality need not hold if $f$ is not injective.

Tst: $f(f^{-1}(H)) = H$

Same flow

$f(f^{-1}(H)) = \{y : y = f(x) \text{ for some } x \in f^{-1}(H)\}$

$\{y : y = f(x),\ f(x) \in H\}$

$\{y : y \in H\} \subseteq H$

$f(f^{-1}(H)) \subseteq H$

Since $f$ is surjective $\exists x \in A$ such $f(x) = h \in H$

$\Rightarrow H \subseteq f(f^{-1}(H)) \Rightarrow H = f(f^{-1}(H))$

$f(x) \in f(f^{-1}(H)) \Rightarrow h \in f^{-1}(f(H))$

Ex: $A = \{0,1\}$, $B = \{1,2\}$ & $H = \{2\}$

$f: A \to B$ such that $f(0) = f(1) = 1$

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