1)

[redacted]

Ch. 5: Eigen values & eigen vectors

§ 5.1 : Determinants

∟ used to be very imp't

Kramer Rule was an alternative to Gaussian

computer made it not relevant except one bit is needed

The definition is recursive like matrix multiplication

Eq 1) $A = [a]_{1\times1}$ $\det(A) = a$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\det(A) = a_{11}a_{22} - a_{12}a_{21}$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\det(A) = 1(4) - 3(2) = -2$

b)

$$-1[0-(-12)] - 5[0 - -6] + 2[6-(1)]$$

$$-1(12) - 5(6) + 2(5)$$

$$-12 - 30 + 10$$

$$-42 + 10 = \boxed{-32}$$

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3) $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & 4 & 1 \end{bmatrix}$

We make 3 submatrices called "Minors"

$A_{11} = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$ $A_{12} = \begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix}$ $A_{13} = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}$

$\det(A) = 1\det(A_{11}) - 2\det(A_{12}) + 1\det(A_{13})$

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & 4 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & 4 & 1 \end{bmatrix}$ $[1 \; 2 \; 1$

$=$

Eg $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 5 & 2 \\ 3 & 3 & 1 & -6 \\ 2 & 1 & 2 & 0 \end{bmatrix}$

$$\det(A) = 1 \det \underbrace{\begin{bmatrix} -1 & 5 & 2 \\ 3 & 1 & -6 \\ 1 & 2 & 0 \end{bmatrix}}_{\text{break it down further}} - 2 \det \begin{bmatrix} 0 & 5 & 2 \\ 3 & 1 & -6 \\ 2 & 2 & 0 \end{bmatrix} + 0 - 1 \det \begin{bmatrix} 0 & -1 & 5 \\ 3 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

Properties of determinants

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

1) If A is upper triangular or lower triangular $\det(A) =$

B) $\det($

$$-1[21-30]-3[-14-25]$$

$$= 2[21-0]-3[7-0]+5[6-15]$$

$$= -24$$

$$\text{Proj}_S(\bar{v}) = \frac{\bar{v} \circ \bar{b}_1}{b_1 \circ \bar{b}_1}\bar{b}_1 + \frac{\bar{v} \circ \bar{b}_2}{b_2 \circ b_2}\bar{b}_2$$

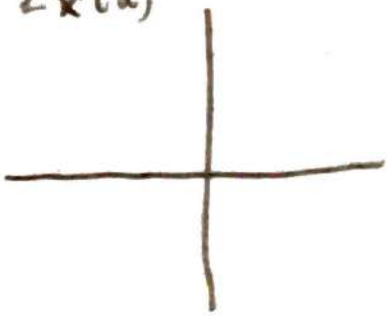
$$\boxed{\frac{6}{14}} = \frac{-1(3) + 3(-1) + 6(2)}{3^2 + -1^2 + 2^2}\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} +$$

$$\boxed{\frac{-16}{6}} = \frac{-1(1) + 3(-1) + 6(-2)}{(1)^2 + (-1)^2 + (-2)^2}\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 9/7 \\ -3/7 \\ 6/7 \end{bmatrix} + \begin{bmatrix} -8/3 \\ 8/3 \\ \frac{16}{3} \end{bmatrix}$$

$$\text{Proj}_S(\bar{v}) = \begin{bmatrix} -29/21 \\ \frac{47}{21} \\ \frac{130}{21} \end{bmatrix}$$

Ex (2)



$V = \mathbb{R}^2$

$\bar{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Can you find c_1 c_2 such that $c_1\bar{v}_1 + c_2\bar{v}_2 = 0$?

Ans NO

$$\boxed{c_1\bar{v}_1 + c_2\bar{v}_2 = 0}$$

Eg (3) $V = \mathbb{R}^3$

independent set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Ind. set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Dependent Set

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ; \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \right\}$

Show that $\bar{w}$ is a linear combin of v_1, v_2

$\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Strategy to determine if $\{\bar{v}\,\bar{v}_2 \ldots \bar{v}_k\}$ is dependent or independent

$c_1\bar{v}_1 + c_2\bar{v}_2 + \ldots c_k\bar{v}_k = 0$

$[\bar{v}_1\ \bar{v}_2 \ldots \bar{v}_k] \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = \bar{0}$

$$\underset{c_1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \underset{c_2}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \underset{c_3}{1}\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = 0$$

Solve this matrix eq using G.E.
If some or all $c_i \neq 0$ then dep
If all c_i are 0 then ind

look at Eq again

$V = \mathbb{R}^3$ $\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\bar{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\bar{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \bar{0}$$

$c_1 = 0$
$c_2 = 0$
$c_3 = 0$

Solve:

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \bar{0}$$

$$\begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{array}\right]$$

$$\begin{array}{l} \\ \\ R_3 - 2R_2 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 is free

$-3x_2 - 3x_3 = 0$

$x_2 = -x_3$

$x_1 + 4x_2 + 2x_3 = 0$

$x_1 - 4x_3 + 2x_3 = 0$

$x_1 = 2x_3$

$x_3 = 1$ $x_2 = -1$ $x_1 = 2$

$$2\bar{v}_1 - \bar{v}_2 + \bar{v}_3$$

c_1 c_2 c_3

$4\bar{v} - 2\bar{v}_2 + 2\bar{v}_3 = 0$ It's dependent

Infinite possibilities for c_1, c_2, c_3

Ex 6) $V = \mathbb{R}^3$

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \bar{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

$$\begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 4 \\ 3 & 3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 2 \end{bmatrix}$$

$$\xrightarrow{} \begin{array}{l} \\ \\ R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right]$$

$$\bar{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\{\bar{v}_1\ \bar{v}_2\ \bar{v}_3\}$ is n…

Now we can write less

s pir… matri…

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 4 \\ 3 & 3 & 5 \end{bmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 2 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{bmatrix}$$

10/19/22

2.3)

Recall Matrix Equation $A\bar{x} = \bar{b}$

Three different ways of looking at the same thing

$$\begin{cases} 2x_1 + 2x_2 - x_3 = 5 \\ x_1 + x_2 + 4x_3 = 0 \\ 3x_1 + 2x_2 + 5x_3 = 2 \end{cases}$$

$$\begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 4 \\ 3 & 2 & 5 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{=\bar{x}} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} = \bar{b}$$

$$x_1 \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}$$

To find $\vec{x}$ we need $\bar{b}$ to be a linear combination of $\bar{a}_1, \bar{a}_2, \bar{a}_3$

(Today we will do another perspective sh

2.3. B-coordinate vector

Let V be a vector space

Let $B = \{\bar{v}_1, \bar{v}_2, \bar{v}_3, \ldots \bar{v}_n\}$ be a basis of V

$\Leftrightarrow$ independent
spans V

Prop 1 from 2.2

Prop 1: Every vector v can be uniquely written as a linear combination of basis vectors

Let $\bar{v} \in \mathbb{R}^n$ prop 1 $\Rightarrow \bar{V} = c\bar{v}_1 + c_2\bar{v}_2$

here we have a lis of $c_1 \dots c_n$ associated with $\bar{V}$

Think of $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ as a vector in V

We have a new vector associated with $\bar{v}$

This is called the B-coordinate

vector of v | we write $[\bar{v}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Ex $V = \mathbb{R}^3$ $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right.$

$\bar{V} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ we can write $\bar{V} = 2\bar{v}_1 + 3\bar{v}_2 + 5\bar{v}_3$

so have $[\bar{v}]_B = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$

Ex 1) $V = \mathbb{R}^2$ $B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$ — If you did gaussian

$\bar{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ with a little bit of work we can

gauss

$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\bar{v} = -1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$$[\bar{v}]_B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

R^2 $\boxed{8}$ $\boxed{4}$ basc

Ex $V =$ Polynomial of degree at most 3

$\bar{v} = f(x) = \underline{5}x^3 + \underline{4}x^2 + \underline{2}x + \underline{5}$

what is the basis of V

Ans : $B = \{x^3, x^2, x, 1\}$

Every poly in V can be written as a linear combination of $x^3, x^2, x, 1$

$$[\bar{v}]_B = \begin{bmatrix} 5 \\ 4 \\ 2 \\ 5 \end{bmatrix} \qquad [\bar{v}]_B = \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}$$

B-coord vector

$f(x) = 8x^3 + 2x + 1$

$B = \{x^3, x^2, x, 1\}$

$\boxed{\text{Representing coefficients of your vector}}$

$$[\bar{v}]_B = \begin{bmatrix} 8 \\ 2 \\ 0 \\ 1 \end{bmatrix} \qquad [\bar{v}]_B = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 8 \end{bmatrix}$$

↳ n of the c_i in

$\bar{v} = c_1\bar{v}_1 + \dots c_k\bar{v}_k$

Ex let V Polynomials of degree at most 3

Determine if $\{2x^2+1, 5x^2+x+4, 2x+3, x^2+2x+1\}$ is dependent or independent

Solve:

det $B = \{x^2, x, 1\}$

Proposition 4

$\bar{v} = 2x^2+1 \quad [\bar{v}]_B = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \mathbb{R}^3$

$\bar{v} = 5x^2+x+4 \quad [\bar{v}]_B = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$

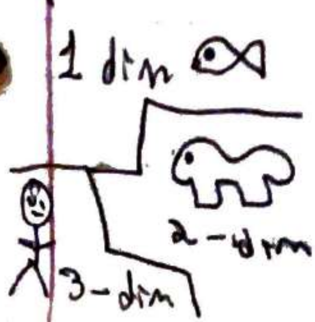
$V = 2x+3 \quad [\bar{v}]_B = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$

$\bar{v} = x^2+2x+1 \quad [v]_B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

These vectors are dependent because it is greater than $\mathbb{R}^3$

dimension of vector space is size of basis

<u>Note</u> <u>Proposition 3</u>: Every basis has the same number of vector



The dimension of V is the no of vectors in a basis

$\dim(\mathbb{R}^3) = 3$

linear Algebra is the study of n-dimensional space $\mathbb{R}^n$

Change of basis

Since a vector space has many bases sometime we need to change from one basis B to answer basis B'. How do we do this?

Gaussian Elimination

Ex $V = \mathbb{R}^2$ has many bases

$$B = \left\{ \begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix} \right\} \qquad B' = \left\{ \begin{bmatrix}1\\2\end{bmatrix} \begin{bmatrix}-1\\2\end{bmatrix} \right\}$$

$$\begin{bmatrix}1 & \\ -4 & 5\end{bmatrix} = \begin{bmatrix}6 & \frac{22}{7} \\ -5 & -\frac{19}{7}\end{bmatrix} \underbrace{\begin{bmatrix}-9 & -5 \\ 1 & 1\end{bmatrix}}_{B'}$$

Ex Find the "Change of basis matrix" from

vectors

$$B = \left\{ \begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}3\\-5\end{bmatrix} \right\} \text{ to } B' = \left\{ \begin{bmatrix}-9\\1\end{bmatrix} \begin{bmatrix}-5\\1\end{bmatrix} \right\}$$

Solve:

$$\left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ 0 & -5 & 1 & 1 \end{array}\right] \xrightarrow{R_2 + 4R_1} \left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ 0 & 7 & -33 & -19 \end{array}\right]$$

$B \qquad B'$

Turn B into identity

$$\xrightarrow{\frac{R_2}{7}} \left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ 0 & 1 & -5 & -\frac{19}{7} \end{array}\right]$$

$$\xrightarrow{R_1 - 3R_2} \left[\begin{array}{cc|cc} 1 & 0 & 6 & \frac{22}{7} \\ 0 & 1 & -5 & -\frac{19}{7} \end{array}\right]$$

The change of bases matrix is $\begin{bmatrix}6 & \frac{22}{7} \\ -5 & -\frac{19}{7}\end{bmatrix}$

Solution $\quad$ single coeff $\quad$ vectors

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Ex Find the change of basis matrix from B' to B

<u>solve</u>

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 3 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix} \quad v_3 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} \quad \overline{W} = \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 5 & 0 \\ -2 & 5 & 8 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$$

$$\begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & - & \overline{w} \end{array} \\ \begin{array}{c} R_2-3R1 \\ R_3+2R1 \end{array} & \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 3 & 5 & 0 & 1 \\ -2 & 5 & 8 & 6 \end{array}\right] \end{array}$$

The set of vectors is a basis for $\mathbb{R}^3$ due to proposition 4 (vectors match dimension)

$$R_3-\frac{7}{2}R2 \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 2 & -6 & 1 \\ 0 & 7 & 12 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 2 & -6 & 1 \\ 0 & 0 & -9 & \frac{5}{2} \end{array}\right]$$

$$x_1 + x_2 + 2x_3 = 1$$

$$2x_2 - 6x_3 = 1$$

$$-9x_3 = \frac{5}{2}$$

$$2.1\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + -.833\begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix} - \frac{2.5}{9}\begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$$

All $c_i \neq 0$

Therefore, it is linear dependent

$c_1 = \boxed{v_1 = 2.1}$

$c_2 = \boxed{v_2 = -.833}$

$c_3 = \boxed{v_3 = -\frac{2.5}{9}}$

part I — part 2

multiple fill blanks — Quiz problems

$\mathbb{R}^3$

$$b = \left\{ \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\bar{v} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{array}\right]$$

2.A: Column Space, Row Space, Null Space

Let A be an $m \times n$ matrix

$$A = \begin{array}{c} \begin{array}{ccc} \bar{a}_1 & \bar{a}_2 & \bar{a}_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{mn} \end{bmatrix} \end{array}$$

Column vectors $\bar{a}_1, \bar{a}_2 \ldots \bar{a}_n$ are in $\mathbb{R}^m$

we can also of the rows as vectors in $\mathbb{R}^n$

Define 1) The set of linear combinations of the columns of A from a subspace of $\mathbb{R}^n$ called the column space

2) The set of linear combinations of the rows of a form a subspace of $\mathbb{R}^n$ called the row space Row(A)

Note: the rows of A are columns of A^T

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basis → linear independent and spans V

Find a basis for col(A) and a basis for row(A)

given $A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{bmatrix}$ Do Guassian

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{bmatrix} \xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & -6 & -6 \end{bmatrix} \xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

indep

indep

not independent

dependent

where you

pivot is your base

Basis for Row(A) = $\left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \\ -3 \end{bmatrix} \right\}$

Null space of A

Recall H.W. 2.1 #6

let A be an $m \times n$

matrix show that $S = \{ x \in \mathbb{R}^n / A\bar{x} = \bar{0} \}$ is a

Subspace of $\mathbb{R}^n$

proof ① For every $\bar{x}, \bar{y} \in S$ $\bar{x} + \bar{y} \in S$

② $c\bar{x} \in S$

Null space

$\bar{x} \in S \Rightarrow A\bar{x} = \bar{0}$

$\bar{y} \in S \Rightarrow A\bar{y} = \bar{0}$

$A(\bar{x} + \bar{y}) = A\bar{x} + A\bar{y} = \bar{0} + \bar{0} = \bar{0}$

$\bar{x} + \bar{y} \in S$

$A(c\bar{x}) = c(A\bar{x}) = c(0) = \bar{0}$

1)

(a) A Vector space is a non-empty set V of objects called vectors and two operatotions (vector addition) and (scalar multiplication) that satisfy the following properties for add $u, v,$ $w,$ in V and scalars c, d.

(b) [redacted] Let V be a vector space. A subspace of V is a subset of V that is also a vector space.

(c) Adding 2 or more vectors which are multiplied by scalar values

(d) The linear space formed by all the vectors that can be written as linear combinations of vectors belonging to a given set

2)

7.1)

$$V = (x, y)$$

8:

$$(c+d)\cdot\overset{v}{(x,y)} = c\cdot\overset{v}{(x,y)} + d\cdot\overset{v}{(x,y)}$$

9: $c\cdot(d\cdot\overset{v}{(x,y)}) = (c\cdot d)\cdot\overset{v}{(x,y)}$

3) a) $S = \{f(x) = ax^n\}$ is a subspace

$g(x) = bx^n$ $\quad f(x) + g(x) = ax^n + bx^n = (a+b)x^n \in S$

$$cf(x) = c(ax^n) = (ca)x^n \in S$$

b) $S = \{ f(x) = x^n + a \}$

This is not a subspace

$$g(x) = x^n + b$$

$$f(x) + g(x) = x^n + a + x^n + b = 2x^n + (a+b) \notin S$$

c) Are subspaces given that in example

a) ax^n is a subspace given

addition and [redacted] scalar multiplication

producing a subspace from set S.

D)

[redacted]

Not all polynomials to degree n is a subspace given that $x^n + a$ is not a subspace however ax^n is a subspace

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4) (Vector addition: adding 2 or more vectors for a vector sum)

(Vector multiplication: Producing a scalar quantity by multiplying 2 or more vectors)

1) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$

2) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

3) $\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right) + w = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + w \right)$

) $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + 0 = v$, element denoted by zero such that this exist

5) $-\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, such that $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \left(-\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = 0$

6) $c \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

) $c \cdot \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right) = c \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + c \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_2 \end{bmatrix}$

8) $(c \cdot d) \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = c \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + d \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

) $c \cdot \left(d \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right) = (c \cdot d) \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

10) $1 \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \neq$ multiplicable inverse

↑ except 10

5)

$$S = \mathbb{R}^3$$

$\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$ because

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$$

1. $f(x) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$, $g(x) = \begin{bmatrix} c \\ d \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ d \\ 0 \end{bmatrix} = \begin{bmatrix} x+c \\ y+d \\ 0+0 \end{bmatrix} \in S$

2. $c \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} c \cdot x \\ c \cdot y \\ c \cdot 0 \end{bmatrix} \in S$

---

6)

$A_{m \times n}$ $\qquad S = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$

$$(c \cdot d) \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} \times \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} cdm_1 \\ cdm_2 \\ cdm_3 \end{bmatrix} \times \begin{bmatrix} c \cdot dn_1 \\ c \cdot dn_2 \\ c \cdot dn_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

7)

$V = ax^n$

Show $S = \{ f \in V \mid \int_0^1 f(x)dx = 0 \}$ is a subspace of V

$f(x) = 2x^2$

$\int_0^1 c \cdot f(x)dx \quad \int_0^1 5 \cdot f(x)\,dx$

$5 \int_0^1 \frac{2x^3}{3}$

$10 \int_0^1 \frac{x^3}{3}\,dx$ is a subspa

8) which are subsets of $\mathbb{R}^2$?

a) $S = \left\{ \begin{bmatrix} x_1 \\ y \end{bmatrix} \right\}$

is a subspace

$\begin{bmatrix} x_1 \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} x_1 + c \\ y + d \end{bmatrix} \in S$

$c \begin{bmatrix} x_1 \\ y \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy \end{bmatrix} \in S$

b) $S = \left\{ \begin{bmatrix} x \\ x \end{bmatrix} \right\}$

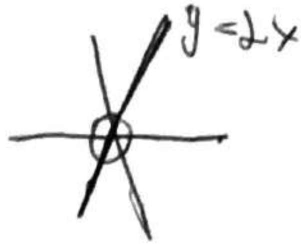
$\begin{bmatrix} u \\ u \end{bmatrix} + \begin{bmatrix} v \\ v \end{bmatrix} = \begin{bmatrix} u+v \\ u+v \end{bmatrix} \in S$

is a subspace

d) $S = \left\{ \begin{bmatrix} x \\ mx + b \end{bmatrix} \right\}$ Not a subspace

$\begin{bmatrix} u \\ u+3 \end{bmatrix} + \begin{bmatrix} v \\ v+3 \end{bmatrix} = \begin{bmatrix} u+v \\ u+v+6 \end{bmatrix} \notin S$ Not a subspace

c) $S = \left\{ \begin{bmatrix} x \\ kx \end{bmatrix} \right\}$



S is a subspace

$\begin{bmatrix} x \\ 2x \end{bmatrix}$

e) $S = \left\{ \begin{bmatrix} x \\ x^2 \end{bmatrix} \right\}$

let $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \in S$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \notin S$ since $5 \neq 3^2$

Not a Subspace

9)

(a) $S = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \right\}$ $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

is not a subspace

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \in S$

(b) $S = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \right\}$ $v = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}, d = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$

not a subspace $v + d = \begin{bmatrix} 5 & 5 \\ -5 & 5 \end{bmatrix}$

c) isn't a subspace $\mathbb{R}$

$$S = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right\}$$

d) is a subspace $\mathbb{R}$

$$S = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \right\}$$

e) isn't a subspace $\mathbb{R}$

$$S = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \right\}$$

1) Define:

Rank(A) - The rank denotes the "information content" of the matrix. A matrix may be very large, but if its rank is small then it can be replaced by a small matrix that carries the same information.

Col(A) - Set of all linear combinations of columns of A is a subspace $\mathbb{R}^m$. We call it the column space of A and we denote it as Col(A)

Row(A) - The set of all linear combinations of rows of (A) is a subspace of $\mathbb{R}^n$. We call this subspace the row space of A and we denote it as Row(A)

Null(A) - If $A(cu) = cA(u) = c(0) = 0$

This means cv is a solution to $Av = 0$ and therefore $cv \in S$.
We call this subspace the null space of A

2)

1 pivot

a) $\begin{bmatrix} 2 & 5 \\ -3 & -4 \end{bmatrix} \xrightarrow{R_2 + R_1\left(\frac{3}{2}\right)}$ [redacted] $\begin{bmatrix} 2 & 5 \\ 0 & \frac{7}{2} \end{bmatrix}$

★ basis of $Col(A) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ [redacted]

★ basis of $Row(A) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

1 pivot

b) $\begin{bmatrix} 6 & 9 \\ 4 & 5 \end{bmatrix} \xrightarrow{R_2 - R_1\left(\frac{4}{6}\right)}$ [redacted] $\begin{bmatrix} 6 & 9 \\ 0 & -1 \end{bmatrix}$

★ basis of $Col(A) = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

★ basis of $Row(A) = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$

c) $\begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}$

2 pivots

★ basis of $Col(A) = \begin{bmatrix} 3 \\ -3 \\ 6 \end{bmatrix} \begin{bmatrix} -7 \\ 5 \\ 4 \end{bmatrix}$

★ basis of $Row(A) = \begin{bmatrix} 3 \\ -7 \\ -2 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}$

basis of $Null(A) = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$

$\begin{array}{l} R_2 + R_1 \\ \\ R_3 - 2R_1 \end{array} \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{bmatrix}$

$R_3 + R_2(5) \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$R_3 - R_2$

$X_1 = \frac{1}{2} X_3$

$X_2 = -\frac{X_3}{2}$

$X_3 =$ free

d) $\begin{bmatrix} 4 & 3 & -5 \\ -4 & -5 & 7 \\ 8 & 6 & -8 \end{bmatrix}$

$R_2 + R_1$ $\quad R_3 - 2R_1$

$\begin{bmatrix} 4 & 3 & -5 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 4 & 3 & -5 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

1 pivot

Basis of Null(A) $= \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}$

Basis of Col(A) $= \begin{bmatrix} 4 \\ -4 \\ 8 \end{bmatrix}$

Basis of Row(A) $= \begin{bmatrix} 4 \\ 3 \\ -5 \end{bmatrix}$

$\boxed{x_1 = -\frac{1}{2}x_3}$

$x_2 = \frac{-2x_3}{-2}$ $\quad \boxed{x_2 = x_3}$

$\boxed{x_3 = \text{free}}$

2 pivots

e) $\begin{bmatrix} 2 & -1 & 2 \\ -6 & 0 & -2 \\ 8 & -1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

$R_2 + R_1(3)$ $\quad R_3 + R_2$

$R_3 - R_1(4)$

Basis of Col(A) $= \begin{bmatrix} 2 \\ -6 \\ 8 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$

Basis of Null(A) $= 0$ $\quad$ Basis of Row(A) $= \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \\ 4 \end{bmatrix}$

f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

No pivots required therefore

Col(A) $= 0$ $\quad$ Basis of Null(A) $= 0$

Row(A) $= 0$

g)

$$A = \begin{bmatrix} 0 & 0 & -3 \\ 0 & 5 & 4 \\ 2 & -8 & 1 \end{bmatrix}$$

No Pivot Required

Basis of Row$(A) = \emptyset$

Basis of Col$(A) = \emptyset$

Basis of Null$(A) = \emptyset$

h)

$$A = \begin{bmatrix} 0 & 0 & -3 \\ 0 & 5 & 4 \\ 2 & -8 & 1 \end{bmatrix}$$

No Pivot Required

Basis of Row$(A) = \emptyset$

Basis of Col$(A) = \emptyset$

Basis of Null$(A) = \emptyset$

i)

1 pivot

$$A = \begin{bmatrix} -1 & -3 & 3 & -2 \\ -3 & 7 & -1 & 2 \\ 0 & 1 & -4 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & -3 & 3 & -2 \\ 0 & 16 & -10 & 8 \\ 0 & 1 & -4 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -3 & 3 & -2 \\ 0 & 16 & -10 & 8 \\ 0 & 0 & -3 & \frac{1}{3} \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -3 & 3 & -2 \\ 0 & 16 & -10 & 8 \\ 0 & 0 & 0 & - \end{bmatrix}$$

$R_2 - 3R_1$

$R_3 + R_1 \frac{1}{3}$

$R_3 + R_2\left(-\frac{3}{10}\right)$

Basis of Row$(A) = \begin{bmatrix} -2 \\ 3 \\ -1 \\ 3 \end{bmatrix}$

Basis of Col$(A) = \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix}$

Basis of Null$(A) = \emptyset$

1) <u>chemistry</u>

chemical equation describes the quantities of substances consumed and produced by chemical reactions. It is based on the principal that atoms are neither created or destroyed.

Propane and Oxygen combine to form carbon dioxide and water.

$$x_1 \underset{\text{propane}}{C_3H_8} + x_2 \underset{\text{oxygen}}{O_2} \longrightarrow x_3 \underset{\text{carbon dioxide}}{CO_2} + x_4 \underset{\text{water}}{H_2O}$$

key idea is that you can represent a molecule as a vector.

C - carbon

H - hydrogen

O - oxygen

Think of this as a vector $\begin{bmatrix} C \\ H \\ O \end{bmatrix}$

we can write C_3H_8 in vector form as

$$\begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} \begin{array}{l} \longrightarrow 3 \text{ carbons} \\ \longrightarrow 8 \text{ hydrogens} \\ \longrightarrow 0 \text{ oxygen} \end{array}$$

Carbon
Hydrogen
oxygen

We can write H_2O as $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ $\begin{array}{l} - C \text{ atoms} \\ - H \text{ atoms} \\ - O \text{ atoms} \end{array}$

<u>CHO order</u>

$$\underset{C_3H_8}{x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}} + \underset{O_2}{x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}} = \underset{CO_2}{x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}} + \underset{H_2O}{x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}$$

$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} - x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} 3 & 0 & -1 & 0 \\ 8 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{0}$$

$$\bar{x} = \begin{bmatrix} \frac{1}{4}x_4 \\ \frac{5}{4}x_4 \\ \frac{3}{4}x_4 \\ x_4 \end{bmatrix}$$

<u>Ans:</u> Take $x_4 = 4$ $\bar{x} = \begin{bmatrix} 1 \\ 5 \\ 3 \\ 4 \end{bmatrix}$

$$\boxed{C_3H_8 + 5O_2 \rightarrow 3CO_2 + 4_2O}$$

Boron sul

Solve

2) Nutritious Food Combinations

(Problems of formulating specialized diets for humans and livestock is important)

Ex: The Cambridge diet is a powder that consist of protein, carbs, fats and other nutrients. It recommends 33 grams of protein, 45 grams of carbs, and 3 grams of fat per 100 grams of powder. Among the food items that goes into the powder are, fat-free milk, soy flour, whey.

	x_1	x_2	x_3
protein	36	51	13
carb	52	34	74
fat	0	7	1.1

$$\begin{bmatrix} \text{protein} \\ \text{carb} \\ \text{fat} \end{bmatrix} \qquad \text{milk} = \begin{bmatrix} 36 \\ 52 \\ 0 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 36 \\ 52 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 51 \\ 34 \\ 7 \end{bmatrix} + x_3 \begin{bmatrix} 13 \\ 74 \\ 1.1 \end{bmatrix} = \begin{bmatrix} 33 \\ 45 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.277 \\ 0.392 \\ 0.233 \end{bmatrix}$$

per 100 grams $\boxed{\begin{array}{l} 27.7\% \text{ milk} \\ 39.2\% \text{ is soy} \\ 23.3\% \text{ is whe} \end{array}}$

Christian Mardeb

Matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 5 & 0 \\ 2 & 5 & 8 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 2 \\ -2 & 5 & 0 \\ 2 & 5 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & 4 \\ 0 & 5 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 + 2R_1$

$R_3 - 2R_1$

$R_3 - R_2$

Basis of $Col(A) = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}$

Basis of $Row(A) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix}$

Basis of $Null(A) = \begin{bmatrix} -2 \\ -4/5 \\ 1 \end{bmatrix}$

$\boxed{x_1 = -2x_3}$

$x_1 + 0x_2 + 2x_3 = 0$

$0x_1 + 5x_2 + 4x_3 = \ldots$

$\boxed{x_3 = fr\ldots}$

$\begin{bmatrix} -2x_3 \\ -\frac{4x_3}{5} \\ x_3 \end{bmatrix}$

$\boxed{x_2 = -\frac{4x_3}{5}}$

$\frac{5x_2}{5} = -4\ldots$

§ 3.1 continued

Recall Def'n of Linear Transformation

Let V and W be vector spaces.
A Linear transformation is a function
$T: V \to W$ that satisfies two prop

1) $T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v})$

7) Show that $\boxed{\begin{array}{l} T: V \to W \\ T(\bar{x}) = \bar{x} \end{array}}$

identity function

Prove: 1) $T(\bar{a} + \bar{b}) = T(\bar{a}) + T(\bar{b})$

2) $T(c\bar{a}) = cT(\bar{a})$

$= T(\bar{a} + \bar{b}) = \bar{a} + \bar{b}$

$= T(\bar{a}) + T(\bar{b})$

$T(c\bar{a}) = c\bar{a} = cT(\bar{a})$

8) Show that $T: V \to W$

BD ∇

0
4
0
3
-1

Proof: $T(\bar{a}+\bar{b}) = \bar{0}_W = \bar{0}_W + \bar{0}_W$

$= T(\bar{a}) + T(\bar{b})$

$T(c\bar{a}) = \bar{0}_W = c\bar{0}_W = cT(\bar{a})$

T is l.t.

3.1.6 A^n

Alternate defenetion of linear transformation

$\mathbb{R}^3$

$$\begin{bmatrix} 1 & 2 & 3 \\ 8 & 1 & 9 \\ 0 & 4 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 7 \\ 1 & 4 & 6 \\ 2 & 3 & 4 \end{bmatrix}$$

a) $S = \left\{ \begin{bmatrix} a & b & c \\ d & a & 6 \\ f & g & a \end{bmatrix} \right\}$

(additive)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 6 \\ 7 & 8 & 1 \end{bmatrix} + \begin{bmatrix} 18 & 11 & 12 \\ 13 & 18 & 15 \\ 16 & 17 & 18 \end{bmatrix} = \begin{bmatrix} 19 & 13 & 15 \\ 17 & 19 & 21 \\ 23 & 25 & 19 \end{bmatrix} \checkmark$$

(multiplicative)

$$3 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 6 \\ 7 & 8 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 3 & 18 \\ 21 & 24 & 3 \end{bmatrix} \checkmark$$

This (S) is a subspace in $\mathbb{R}^3$ with same diagonal entries after closure

b) $S = \left\{ \begin{bmatrix} 2 & 2 & 2 \\ a & b & c \\ d & e & f \end{bmatrix} \right\}$

different form in h_1

$$\begin{bmatrix} 2 & 2 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 2 \\ 8 & 9 & 10 \\ 11 & 12 & 13 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 4 \\ 12 & 13 & 15 \\ 17 & 19 & 21 \end{bmatrix} \notin S$$

not a subspace

$$2 \begin{bmatrix} 2 & 2 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 4 \\ 6 & 8 & 10 \\ 12 & 14 & 16 \end{bmatrix} \notin S$$

Both does not have number 2 for diagonal entr

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & 5 \\ 10 & 5 & -1 \\ 3 & 2 & 1 \end{bmatrix}$$

long Way

first Column

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 10 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + 10\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 6 \\ 37 \end{bmatrix}$$

2nd Column

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} = -1\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 7 \\ 15 \end{bmatrix}$$

3rd Column

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + -1\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 \\ -11 \\ 8 \end{bmatrix}$$

$\boxed{\text{Answer}}$

$$\begin{bmatrix} 5 & 1 & 6 \\ 6 & 7 & -11 \\ 37 & 15 & 8 \end{bmatrix}$$

short cut

R1

$1(2) + 0(10) + 1(3)$

$1(-1) + 0(5) + 1(2)$

$1(5) + 0(-1) + 1(1)$

R2

$-2(2) + 1(10) + 0(3)$

$-2(-1) + 1(5) + 0(2)$

$-2(5) + 1(-1) + 0(1)$

R3

$2(2) + 3(10) + 1(3)$

$2(-1) + 3(5) + 1(2)$

$2(5) + 3(-1) + 1(1)$

$\boxed{\text{Solution}}$

$$\begin{bmatrix} 5 & 1 & 6 \\ 6 & 7 & -11 \\ 37 & 15 & 8 \end{bmatrix}$$

Use LU to solve $ax=b$

$$A = \begin{bmatrix} 4 & 3 & 5 \\ -4 & -5 & 2 \\ 8 & 6 & 2 \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} 1 \\ 1 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$R_2 + R_1$

$R_3 - 2R_1$

$$\begin{bmatrix} 4 & 3 & 5 \\ 0 & -2 & 7 \\ 0 & 0 & -8 \end{bmatrix}$$

U

$$Y = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

L

$$(L)^{-1}(\bar{B}) = Y$$

$$\bar{X} = \begin{bmatrix} 6.0625 \\ -5.25 \\ -1.25 \end{bmatrix} \qquad \bar{X} = \bar{Y}(U)^{-1}$$

$$\begin{bmatrix} 1 & 0 & 6 \\ -2 & 5 & 0 \\ 2 & 5 & 12 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 6 \\ 0 & 0 & 12 \\ 0 & 5 & 0 \end{bmatrix}$$

$R_2 + 2R_1$

$R_3 - 2R_1$

$\text{Rank}(A) = 3$

Row space $= \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$

Col Space $= \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}$

$\cancel{12 x_3} \quad \cancel{5 = x_2}$

There is No Null

No free variable

Christian T Burden

Linear Transformations

Ch 3: Linear Transformations

§ 3.1 Definition and Examples

— a function from $\mathbb{R}^n$ — $\mathbb{R}^m$ that satisfies certain properties

elevator pitch

Precalculus Functions

Vertical line

check for a 1 to 1 function

onto function

B is the range

function

$f: A \to B$

pre-image $\xrightarrow{f}$ image

domain A, b codomain

A function maps every element in set A to exactly one element in B

one to one function

A, B

Every element in A is mapped to exactly one element in B and vice versa

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = 2x + b$

restriction maybe

Range

y-axis
the set of elements that have pre-image

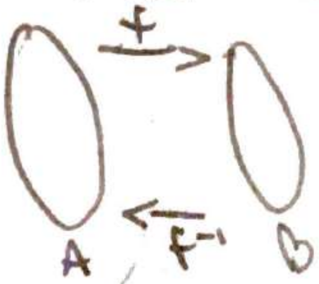
only maps to 0, ∞

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

Range $[0, \infty)$

Inverse function



f^{-1} is the function from B to A

Function Algebra

$(f \cdot g)(x) = f(x)g(x)$

$(f \circ g)(x) = f(g(x))$

$\frac{f}{g}(x) = \frac{f(x)}{g(x)}$

$(f \pm g)(x) = f(x) \pm g(x)$

Defn: let V and W be vector spaces

A linear transformation is a function

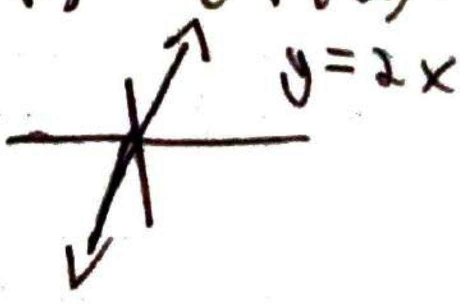
$T : V \longrightarrow W$ that satisfies

a) $T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v})$

b) $T(c\bar{v}) = cT(\bar{u})$

Ex. 1 $\mathbb{R} \longrightarrow \mathbb{R}$

$T(x) = 2x$



Proof: $T(a+b) = 2(a+b)$

$= 2a + 2b$

$T(a) + T(b)$

$T(ca) = 2(ca) = c2a = cT(a)$

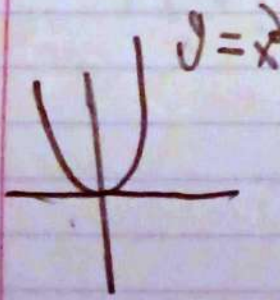
Therefor T is a linear trasformation

Ex 2. $T: \mathbb{R} \to \mathbb{R}$

$T(x) = x^2$

(curves are not linear transformations)

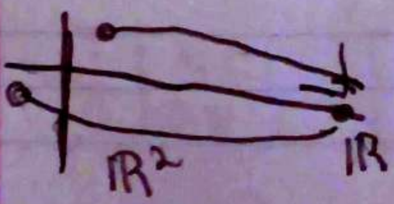
let's try to prove this is a lin trans



$T(a+b) = (a+b)^2 \neq a^2 + b^2$

T is not a linear transformation

Ex 3) $T: \mathbb{R}^2 \to \mathbb{R}$



$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 7x_1 - 5x_2$

what sort of function is this

$14 - 15 = -1$

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 7x_1 - 5x_2$$

show $T(\bar{a}+\bar{b}) = T(\bar{a}) + T(\bar{b})$

$T(c\bar{a}) = cT(\bar{a})$

proof: $T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} a_1+b_1 \\ a_2+b_2 \end{bmatrix}\right)$

$= 7(a_1+b_1) - 5(a_2+b_2)$

$= 7a_1 + 7b_1 - 5a_2 - 5b_2$

$= 7a_1 - 5a_2 + 7b_1 - 5b_2$

$T\left(c\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} ca_1 \\ ca_2 \end{bmatrix}\right) = 7\boxed{ca_1} - 5\boxed{ca_2}$

$= c(7a_1 - 5a_2)$

$= cT\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

Ex 4) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$

$\begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_1+b_1 & a_2+b_2 \end{bmatrix}$

$$T\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}\right) = T\begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1+b_1+a_2+b_2 \\ a_2+b_2 \\ a_3+b_3 \end{bmatrix}$$

$$T\left(c\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} ca_1 \\ ca_2 \\ ca_3 \end{bmatrix}\right) = \begin{bmatrix} ca_1+ca_2 \\ ca_2 \\ ca_3 \end{bmatrix}$$

$$c\begin{bmatrix} a_1+a_2 \\ a_2 \\ a_3 \end{bmatrix}$$

linear transformation

just fail 1 property

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 + 2x_3 \\ x_3 \end{bmatrix}$$

$$T\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}$$

This is a linear transformation

$$\begin{bmatrix} (a_1+b_1) + 2(a_2+b_2) \\ (a_2+b_2) + 2(a_3+b_3) \\ a_3+b_3 \end{bmatrix}$$

$$\begin{bmatrix} a+b_1 \\ a_2+b_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} 2(a_2+b_2) \\ 2(a_3+b_3) \\ b_3 \end{bmatrix}$$

$$T\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + T\begin{bmatrix} 2a_2 \\ 2a_3 \\ a_3 \end{bmatrix}$$

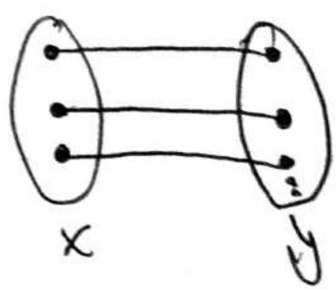
3.2: Isomorphism between vector spaces

1 to 1 linear transformation

Define: A let $T: V \to W$ is one-to-one if every vector

(every input has exactly one unique output & vice versa

& Every domain element has to be used



Define: let $T: V \to W$ be a L.t.

let T be a linear transformation from V to W

The kernal of T is the set of all vectors $\bar{v} \in V$ that are mapped to $\bar{0}_W$

$$Ker(T) = \{\bar{v} \in V \mid T(v) = \bar{0}_W\}$$

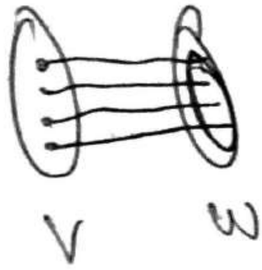
$$Ker(T) = \bar{0}_V$$

96
57 50%

35% final

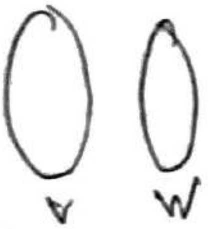
Minimum = 70%

Define: $\text{Range}(T) = \{ T(\vec{v}) \mid \vec{v} \in V \}$



are the elements of W that have pre-images of V

Ex $f : \mathbb{R} \to \mathbb{R}$



$f(x) = x^2$

$\text{Range}(f) = [0, \infty)$

Def: T is an isomorphis (bijection) if T is one-to-one

The way you prove a function is 1 to 1 if

$a = b \longleftrightarrow f(a) = f(b)$

(onto is when positive numbers are used)

Prop: Let $T: V \to W$ be a L.T

a) $Ker(T)$ is a subspace of V

b) $Range(T)$ is a subspace of W

Proof a)

poly of degree n = n solutions

Fundamental Theorem of Linear Algebra

Let V be a vector space of finite dimension n. Then V is isomorphism to $\mathbb{R}^n$.

This is why it makes sense to focus on $\mathbb{R}^n$

Proof: Let v be a vector space of dim n. Let $B = \{\bar{b}_1, \bar{b}_2 \ldots \bar{b}_n\}$ be a basis to V. Let $\bar{v} \in V$

any vector can be written as a linear combination of its vector

$$[V]_b = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \to \text{from } 2.3$$

$$T(\vec{v}) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

we will prove ① T is a l.t

② T is one-to-one

③ T is onto

T

Kernal must map to zero vector

to be considered a 1 to 1

Christian 9 Barden

(Use a counter example if it fails)

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}x_1+x_2\\x_2\\x_3\end{bmatrix}$$

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}x_1^2\\x_2\end{bmatrix}$$ square/root not a linear trasfofmation

$\boxed{\text{Wednesday (computer graphics)} \quad R^2 \to R^2}$

$$\begin{bmatrix}x_1+5\\x_2\end{bmatrix}$$ — constant

Kert(T) = 0 = One to One

T is on if range(T) = W

whole range is used onto

Poly at most 3 dim 4

$S = \{2x+1, 5x^2+x+4, 2x+3\}$

$T(\vec{x}) = A(\vec{x})$

Prop 2: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be any L.t.
then there exists a uniq

$\mathbb{R}^3 \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$

$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

5 4
2 -1
-1
6

$(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$

Only find inverse of a square Matri

Def: T is an isomorphism (byjection) if T is one to one

Def: Let $T: V \rightarrow W$ be a L.t.

The kernal of T is the set of all vectors $\bar{v} \in V$ that are mapped to $\bar{0}_w$

Def: A 1 to 1 linear transformation occurs when every input has a unique output and vice versa

Every domain element must be used

Def: A function is onto when its codomain is restricted to only positive number like in example x^2.

(2) assuming $V = \mathbb{R}^4$ & $\begin{bmatrix} x^3 \\ x^2 \\ x \\ 1 \end{bmatrix}$

(2a) Set of Vectors

$\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 3 \end{bmatrix}$ Dependent free-variable at x_3

After Gaussian

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

[redacted]

(These vectors are not bases of $\mathbb{R}^4$ only 3 pivots)

(2b) Set of Vectors

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -1 \\ 0 \end{bmatrix}$ — [redacted]

After Gassian

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & \text{[illegible]} \\ 0 & 0 & \text{[illegible]} \end{bmatrix}$ → independent

[redacted]

(These vectors are not bases of $\mathbb{R}^4$ only 3 pivot points)

2c) Set of Vectors

$$4\times 3 \quad \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 3 \\ 1 \end{bmatrix}$$

[redacted]

After Gaussian

$$\begin{bmatrix} \underline{1} & 0 & 1 \\ 0 & \underline{1} & 3 \\ \boxed{0 \quad 0 \quad 0} \\ \boxed{0 \quad 0 \quad 0} \end{bmatrix}$$

x – free

1 – free

Therefore a dependent matrix

(There are 2 pivot points these vectors are not bases, and dousnt span $\mathbb{R}^4$)

2d) Set of Vectors

$$\begin{bmatrix} 0 \\ 0 \\ 7 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -16 \\ 16 \\ 1 \end{bmatrix}$$

4x4

[redacted]

After gaussian

$$\begin{bmatrix} \underline{1} & 0 & 0 & 1 \\ 0 & \underline{1} & 0 & 1 \\ 0 & 0 & \underline{1} & 8 \\ 0 & 0 & 0 & \text{[redacted]} \end{bmatrix} \leftarrow \text{independent system}$$

$\uparrow$

$\underline{-7}$

no free variable

(There are 4 pivot points these vector's are bases of $\mathbb{R}^4$)

2 e) Set of Vectors

$$\begin{bmatrix} 1 \\ 0 \\ -12 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

[redacted]

After Gaussian

$$\begin{bmatrix} \underline{1} & 0 & 0 & 0 \\ 0 & \underline{1} & 0 & 0 \\ 0 & 0 & \underline{1} & 0 \\ 0 & 0 & 0 & \underline{1} \end{bmatrix} \leftarrow \text{independent}$$

4 pivots

There are 4 pivot points After Guassian indicating that the set of Vectors Span space $\mathbb{R}^4$ therefore they are bases

2 f) Set of Vectors

$$\begin{bmatrix} -1 \\ 9 \\ -18 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

[redacted]

After Gaussian

$$\begin{bmatrix} \underline{1} & 0 & 0 & 0 \\ 0 & \underline{1} & 0 & 0 \\ 0 & 0 & \underline{1} & 0 \\ 0 & 0 & 0 & \underline{1} \end{bmatrix} \leftarrow \text{independent}$$

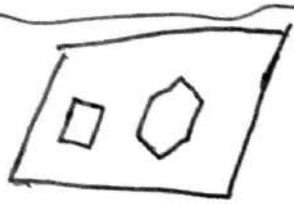
4 pivots

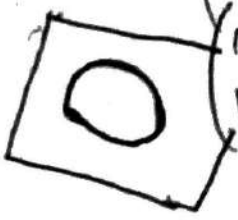
There are 4 pivot points after gaussian therefore they are bases of $\mathbb{R}^4$.

Chapter 4 : ORTHOGNALITY

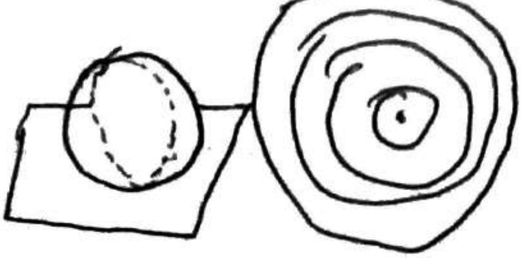
perpendicular

Flatland



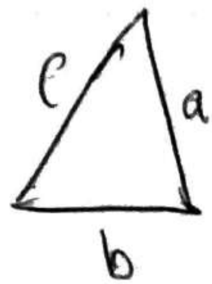


(more sides
higher class)



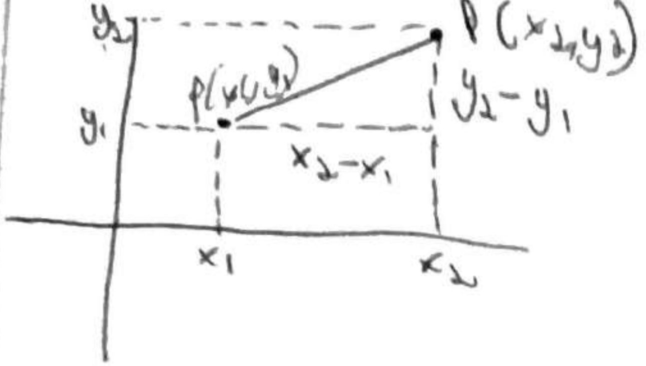
(Pythagorean theorem in
4 dimensions)

Pre-Calc Higher Dimensions



$a^2+b^2=c^2$

$D=\sqrt{(x_2-x_1)^2+(y_2-}$



$P(x_1\ y_1 z_1)$ $\qquad$ $Q(x_2\ y_2\ z_2)$

$$\text{distance}(PQ) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

§ 4.1: Inner Product

dot product

<u>Define</u>: let $\bar{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\bar{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ be two vectors in $\mathbb{R}^n$

The inner product of $\bar{u}$ and $\bar{v}$ is defined

as $\bar{u} \cdot \bar{v} = \bar{u}^T \bar{v} = [u_1\ u_2 \cdots u_n]_{1\times n} \begin{bmatrix} v_1 \\ \vdots \\ \vdots \\ v_n \end{bmatrix}_{n\times 1}$

dot product NO times

$= u_1 v_1 + u_2 v_2 + \cdots u_n v_n$

Ex $\bar{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \end{bmatrix}$ $\quad \bar{v} = \begin{bmatrix} 2 \\ -4 \\ 1 \\ 5 \end{bmatrix}$

$\bar{u} \cdot \bar{v} = 2 - 8 - 1 + 25 = 18$

Must have same
row for space

Defin ② length of a vector $\bar{v}$ is defined as

$$\|v\| = \sqrt{\vec{v} \circ \vec{v}} = \sqrt{v_1^2 + v_2^2 + \dots v_n^2}$$

$$\bar{v} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

$$\|v\| = \sqrt{3^2 + 8^2}$$

$$= \sqrt{9 + 64} = \sqrt{73}$$

Define ③ Distance between vectors $\bar{u}$ and $\bar{v}$ is

$$\|\bar{u} - \bar{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \dots (u_n - v_n)^2}$$

Prop 1 let $\bar{u}$

- inner product
- distance
- properties

$$\bar{u} \circ \bar{v} = \sqrt{u_1 v_1 + u_n v_n} = v.$$

Transformation $T \circ S$

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$S\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_1 \end{bmatrix}$$

S to T in-person tutoring

$$S\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad S\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$S\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$T \circ S(\bar{x}) = T(S(\bar{x}$$

$$= T\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0(1)+1(1) & 0(1)+1(1) \\ 1(1)+0(1) & 1(1)+0(1) \end{bmatrix}$$

$$T \circ S = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Nov 23rd (Meet Up)

(Week 11)

Ch. 5: Orthogonality

Ch. 4: Orthogonality

§4.1 Inner product. (continued)

Defin ① $\bar{u} \circ \bar{v} = \bar{u}^T \bar{v} = u_1 v_1 + u_2 v_2$

② $\|v\| = \sqrt{v \circ v} = \sqrt{v_1^2 + v_2^2}$

③ $\|\bar{u} - \bar{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 \cdots (u_n - v_n)^2}$

④ unit vector: $\dfrac{\bar{v}}{\|\bar{v}\|}$

⑤ $\bar{u}$ and $\bar{v}$ are orthagonal if $\bar{u} \circ \bar{v} = 0$

orthagonal means perpendicular

where we stopped

Resuming with pg 100 in typed Notes

Goal: Pyth Theorem for $\mathbb{R}^n$

Pythagorean theorem is special case of law of Cosines

2 Coordinates

Prop (2) let $\bar{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\bar{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

are vectors

Then $\bar{u} \circ \bar{v} = \|\bar{u}\| \, \|\bar{v}\| \cos\theta$ in $\mathbb{R}^2$

Highlight that

$\theta = 90^\circ$ $\quad \bar{u} \circ \bar{v} = \|u\|\,\|v\|(0) = 0$

Application:

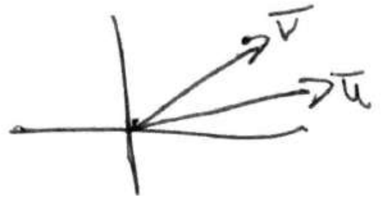
Suppose you have 2 documents

Doc 1 Doc 2

How similar are these 2 documents?

Ans 1 See how many words the have in common

Ans 2 Turn documents into vectors $\bar{u}$ and $\bar{v}$



$$\|\bar{u}-\bar{v}\| = \sqrt{(u_1-v_1)^2+\cdots(u_n-v_n)}$$

Ans 3 Focus on Angle

Prop. 3 Pythagorean Theorem for $\mathbb{R}^n$

let $\bar{u}$ and $\bar{v}$ be two orthogonal vector in $\mathbb{R}^n$

$$\|\bar{u}+\bar{v}\|^2 = \|u\|^2 + \|v\|^2$$

Defn: let S be a subspace of a vector space $\mathbb{R}^n$ we say $\bar{v} \in \mathbb{R}^n$ is orthogonal S

S

orthogonal to all vectors in the subspace

$\bar{v} \cdot \bar{w} = 0$ for all $\bar{w} \in S$

② The set of all vectors orthogonal to S is called the orthogonal complement of S.

(1)

Dist

length

Dot

$\bar{U} \circ \bar{V} = 0$

orthogonal

are vector in $\mathbb{R}^2$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ coz 2 entries

Orthogonal Means linear independence

Theorem 8 (Orthogonal Decomposition Theorem)

let S be a subspace of $\mathbb{R}^n$ and let $B=\{b_1\ b_2 --- b_k\}$ be an orthogonal basis for S. let $v \in \mathbb{R}^n$ $k<n$ Then $v = proj_S(\bar{v}) + \bar{z}$

where $proj_S(\bar{v}) = \frac{\bar{v}\circ\bar{b}_1}{\bar{b}_1\circ b_1} b_1 + \cdots \frac{\bar{v}\circ\bar{b}_k}{\bar{b}_k\circ\bar{b}_k}$

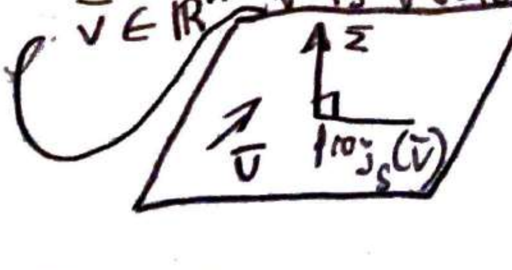
Projection of $\bar{v}$ on subspace S

and $\bar{z}$ is orthogonal to S, / leng of z less than

Moreover for all $\bar{u} \in S$, $\|\bar{z}\| \le \|\bar{v} - \bar{u}\|$

and $\|z\| = \|\bar{v} - \bar{u}\| \iff \bar{u} = proj_S(v)$

$\bar{v} \in \mathbb{R}^n$ v is vector in $\mathbb{R}^3$



S is a subspace of $\mathbb{R}^n$

$\bar{z}$ orthogonal to S

S is a plane in $\mathbb{R}^3$

length of z less than dis from

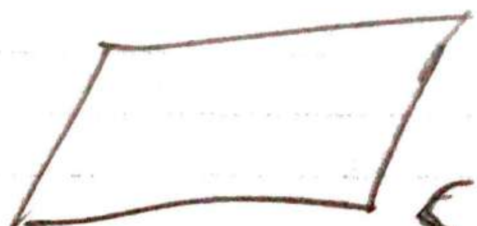
High Dimensional Data

FB 500 datapoints on each of U.S.

each of us is a

$\mathbb{R}^{500}$

(address, - - - - - - - -)



S

Ex. Let S be a Subspace of $\mathbb{R}^3$

Masters in Data Science

Math 3111: Graph Theory and Applications

Ex. page rank Calculator (inktomi)

linear Algebra Graph theory

Ex. let S be a subspace of $\mathbb{R}^3$

$$B = \left\{ \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$ be an

orthogonal bas

§ HW 4.2 2) Find the orthogonal projection $\hat{V}$ or $Proj_S(\bar{v})$ of the following vectors $\bar{v}$ on to The subspace S and the corresponding $\bar{z}$

a) $\bar{v} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis of S

b) $\bar{v} = \begin{bmatrix} -1 \\ 7 \\ 6 \end{bmatrix}$ $B = \left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$

4.3 Gram-Schmidt Process

This section we learn an algorithm for converting any basis into an orthogonal basis This is called the Gram-Schmidt Process

Theorem. Let $b = \{b_1\ b_2 \ldots b_n\}$ be a basis for $\mathbb{R}^n$ We can obtain an orthogonal basis $\{\bar{v}_1, \bar{v}_2 \ldots \bar{v}_n\}$ as follows:

$$\bar{v} = \bar{b}_1 \qquad \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$\bar{v}_2 = \bar{b}_2 - \frac{b_2 \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1} \bar{v}_1 \qquad \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} -$$

$$v_3 = \bar{b}_3 - \left[\frac{b_3 \cdot v_1}{v_1 \cdot v_1} \bar{v}_1 + \frac{b_3 \cdot v_2}{v_2 \cdot \bar{v}_2} \bar{v}_2 \right.$$

$\vdots$

$$v_n = b_n -$$

11.9 Representations of functions as Power Series

recall: $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \dots$

$= \frac{1}{1-r}$ if $|r|<1$

e: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ when $|x|<1$

ex: Write $\frac{1}{1+x}$ as a power series, and find the interval of convergence

$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$

$= \sum_{n=0}^{\infty} (-1)^n x^n$

$|-x|<1$

$|x|<1$

$-1 < x < 1$

• x: $\frac{2}{3-x} = \frac{2}{3}\left(\frac{1}{1-\frac{x}{3}}\right)$ common ratio

$$\frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{2x^n}{3^{n+1}}$$

converges when $\left|\frac{x}{3}\right| < 1$

Calculus of Power Series

$-1 < \frac{x}{3} < 1$

If $f(x) = \sum_{n=0}^{\infty} c_n x^n$

Then (i) $f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n x^n\right)$

$$= \sum_{n=0}^{\infty} \frac{d}{dx}(c_n x^n)$$

$\frac{d}{dx}(c_0 + c_1 x + c_2 x^2 + c_3 x^3$

$0 + c_1 + 2c_2 x + 3c_3 x^2$

is sum of

$\frac{d}{dx}$ of sum derivativ

(iii) $\int f(x)\,dx = \int \sum_{n=0}^{\infty} c_n x^n dx$

$$= \sum_{n=0}^{\infty} c_n x^n dx$$

ex: write $\frac{1}{1-x} = 1 + x + x^2 + \ldots$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) - \frac{d}{dx}(1-x)^{-1}$$

$$= (1-x)^{-2}$$

$$= \frac{1}{(1-x)^2}$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(1 + x + x^2 + x^3 + \ldots\right.$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}(1-x)^{-1} \quad \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3$$

$$\boxed{= \sum_{n=0}^{\infty} (n+1) x^n} \text{ - power}$$

$$f'(x) = \frac{1}{1+(x^2)}$$

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(x^2)} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$f(x) = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx = \sum_{n=0}^{\infty} (-1)^n x^{2n} dx$$

$$\sum_{n=0}^{\infty} (-1)^n \int x^{2n} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

$$f(0) = \tan^{-1} 0 = 0$$

$$f(0) = \sum_{n=0}^{\infty} \frac{0}{1} + C = 0 + C \Rightarrow C = 0$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

§ 11.1 Taylor and Maclarin Series:

If a funct $f(x)$ has a power series representation, then

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$= c_0 + c_1(x) + c_2(x^2) + c_3(x^3) + \ldots$$

Notice: $f(0) = c_0$ — first term

$f'(0) = c_1$

$f''(0) = 2c_2$

$f'''(0) = 6c_3$

$f'(x) = c_1 + 2c_2x + 3c_3x$

$f''(x) = 2c_2 + 6c_3x + 12c_4$

$f'''(x) = 6c_3 + 24c_4x$

In general $n!\,c_n = f^{(n)}(0)$

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad c_n = \frac{f^{(n)}(0)}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \leftarrow \text{Maclarin Series}$$

1)

	Goods	Services
Goods	.2	.7
Services	.8	.3

$$p_g = .2p_g + .7p_s$$

$$p_s = .8p_g + .3p_s$$

$$0 = -.8p_g + .7p_s$$

$$0 = .8p_g - .7p_s$$

5)

$$B_2S_3 + H_2O \longrightarrow H_3BO_3 + H_2S$$

$$\begin{array}{l} \text{Boron} \\ \text{Sulfur} \\ \text{hydrogen} \\ \text{oxygen} \end{array} \begin{bmatrix}2\\3\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\2\\1\end{bmatrix} \overset{=}{\longrightarrow} \begin{bmatrix}1\\0\\3\\3\end{bmatrix} + \begin{bmatrix}0\\1\\2\\0\end{bmatrix}$$

Balancing Equation

$$\begin{bmatrix}2\\3\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\2\\1\end{bmatrix} - \begin{bmatrix}1\\0\\3\\3\end{bmatrix} - \begin{bmatrix}0\\1\\2\\0\end{bmatrix} = \vec{0}$$

$$\begin{bmatrix}2 & 0 & -1 & 0\\3 & 0 & 0 & 1\\0 & 2 & -3 & 2\\0 & 1 & 3 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \vec{0}$$

$$\overline{x} = \vec{0}$$

$$NaHCO_3 + H_3C_6H_5O_7 \rightarrow Na_3C_6H_5O_7 + H_2O + CO_2$$

$$\begin{array}{l} \text{Sodium} \\ \text{Hydrogen} \\ \text{Carbon} \\ \text{oxygen} \end{array} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 8 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 6 \\ 7 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \quad O_2 +$$

1

move to one side and set = to $\bar{0}$ +

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 1 & 8 & -5 & -2 & 0 \\ 1 & 6 & -6 & 0 & -1 \\ 3 & 7 & -7 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \bar{0}$$

$$\bar{X} = \begin{bmatrix} -11x_5 \\ 5/3\, x_5 \\ 11/3\, x_5 \\ -3x_5 \\ x_5 \end{bmatrix}$$

$$PbN_6 + CrMn_2O_8 \longrightarrow Pb_3O_4 + Cr_2O_3 + MnO_2 + NO$$

$$\begin{array}{c} Pb \\ N \\ Mn \\ Cr \\ O \end{array} \begin{bmatrix} 1 \\ 6 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

move it to one side and set equal to $\bar{0}$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 8 & -4 & -3 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \bar{0}$$

$$\bar{X} = \begin{bmatrix} 1/6\, x_6 \\ 22/45\, x_6 \\ 1/18\, x_6 \\ 11/45\, x_6 \\ 44/45\, x_6 \\ x_6 \end{bmatrix}$$