Solving Initial Value Problems with Laplace Transforms- HW Problems

In problems 1-5 solve the initial value problem using Laplace transforms.

1. $y' - 4y = 0;$ $\quad y(0) = 3$
2. $y'' + 9y = 0;$ $\quad y(0) = 3, \quad y'(0) = 6$
3. $x'' + 2x' - 3x = 0;$ $\quad x(0) = 5, \quad x'(0) = -3$
4. $x'' + 9x = \sin(2t)\,;$ $\quad x(0) = x'(0) = 0$
5. $x'' + 6x' + 8x = 8;$ $\quad x(0) = x'(0) = 0.$

In problems 1-5 solve the initial value problem using Laplace transforms.

$f(t) = y$

1.) $y' - 4y = 0; \quad y(0) = 3$

$\mathcal{L}[f(t)] = f(s)$

$\mathcal{L}[y'] - \mathcal{L}[4y] = \mathcal{L}[0]$

$\mathcal{L}[f'(t)] = sf(s) - f(0)$

$\mathcal{L}[f''(t)] = s^2 f(s) - sf(0) - f'(0)$

$\mathcal{L}[y'] - 4\mathcal{L}[y] = 0$

$sf(s) - f(0) - 4(f(s)) = 0$

$sf(s) \cancel{-3} - 4f(s) = 0$

$\cancel{+3} \qquad +3$

$sf(s) - 4f(s) = 3$

$\frac{f(s)(s\cancel{-4})}{(s\cancel{-4})} = \boxed{\left(\frac{3}{s-4}\right)}$

$x(t) = \mathcal{L}^{-1}\left[\frac{3}{s-4}\right] = 3\left[\mathcal{L}^{-1}\left[\frac{1}{s-4}\right]\right] = 3\left[e^{4t}\right]$

2.) $y'' + 9y = 0; \quad y(0) = 3, \quad y'(0) = 6 \qquad f(t) = y$

$(s^2 f(s) - s f(0) - f'(0)) + 9(f(s)) = \mathcal{L}[0]$

$s^2 f(s) - s(3) - 6 + 9f(s) = 0$

$+6 \qquad\qquad +6$

$s^2 f(s) - 3s + 9f(s) = 6$

$+3s \qquad\qquad +3s$

$s^2 f(s) + 9f(s) = 6 + 3s$

$$\frac{f(s)\cancel{(s^2+9)}}{\cancel{(s^2+9)}} = \boxed{\frac{6+3s}{(s^2+9)}}$$

$$\mathcal{L}^{-1}\left[\frac{6+3s}{s^2+9}\right] = 6\sin(3t) + 3\cos(3t)$$

$f(t) = x$

3.) $x'' + 2x' - 3x = 0; \quad x(0) = 5, \; x'(0) = -3$

$s^2 f(s) - sf(0) - f'(0) + 2(sf(s) - f(0)) - 3(f(s)) = \mathcal{L}[0]$ | $\mathcal{L}[f(t)] = f(s)$

$s^2 f(s) - s(5) - (-3) + 2sf(s) - 2(5) - 3f(s) = 0$ | $\mathcal{L}[f'(t)] = sf(s) - f(0)$

$+5s \quad -3 \quad +10$ | $\mathcal{L}[f''(t)] = s^2 f(s) - sf(0) - f'(0)$

$s^2 f(s) + 2sf(s) - 3f(s) = 7 + 5s$

$$\frac{f(s)(s^2 + 2s - 3)}{(s^2 + 2s - 3)} = \boxed{\frac{7 + 5s}{(s^2 + 2s - 3)}}$$

$$(s+3)(s-1)\left(\frac{7+5s}{(s+3)(s-1)}\right) = \left(\frac{A}{(s+3)} + \frac{B}{(s-1)}\right)(s+3)(s-1)$$

$$7 + 5s = A(s-1) + B(s+3)$$

put $s = 1$ put $s = -3$

$\frac{12}{4} = \frac{4B}{4}$ $7 - 15 = -4A$

$\boxed{B = \frac{12}{4}}$ $\frac{-8}{-4} = \frac{-4A}{-4}$

$\boxed{\frac{8}{4} = A}$

$$\frac{8}{4}\mathcal{L}^{-1}\left[\frac{1}{(s+3)}\right] + \frac{12}{4}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right]$$

$$x(t) = \boxed{\frac{8}{4}\left[e^{-3t}\right] + \frac{12}{4}\left[e^{1t}\right]}$$

↑

Answer

4.) $x'' + 9x = \sin(2t);\ x(0) = x'(0) = 0$

$\mathcal{L}[x''] + 9\mathcal{L}[x] = \mathcal{L}[\sin(2t)]$

$(s^2 f(s) - s f(0) - f'(0)) + 9(f(s)) = \frac{2}{s^2+4}$

$s^2 f(s) - s(0) - (0) + 9(f(s)) = \frac{2}{s^2+4}$

$s^2 f(s) + 9 f(s) = \frac{2}{s^2+4}$

$\frac{f(s)\cancel{(s^2+9)}}{\cancel{(s^2+9)}} = \frac{\left(\frac{2}{s^2+4}\right)}{s^2+9} = \boxed{\frac{2}{(s^2+4)(s^2+9)}}$

Partial fractions

$s^4 + 4s^2 + 9s^2 + 36$

$s^4 + 13s^2 + 36$

$\mathcal{L}\left[\mathcal{L}^{-1}\left[\frac{1}{(s^2+2^2)(s^2+3^2)}\right]\right]$

$\downarrow$

$\frac{1}{(s^2+2^2)(s^2+3^2)} \Rightarrow \frac{As+B}{(s^2+2^2)} + \frac{(s+D)}{(s^2+3^2)}$

$\mathcal{L}\left[\mathcal{L}^{-1}\left[\frac{(-\frac{3}{5})s + (\frac{11}{3})}{s^2+4} + \frac{(\frac{8}{5})s + (-\frac{8}{3})}{s^2+9}\right]\right]$

$\mathcal{L}\left(\frac{11}{9}(\sin(2t)) - \frac{3}{5}\cos(2t) + (-\frac{8}{9}\sin(3t)) + \frac{8}{5}(\cos(3t))\right)$

$\uparrow$

Answer

$1 = As + B(s^2+3^2) + (s+D)(s^2+2^2)$

$1 = As(s^2+9) + B(s^2+9) + Cs(s^2+4) + D(s^2+4)$

$1 = As^3 + 9As + s^2B + 9B + Cs^3 + 4Cs + Ds^2 + 4D$

$1 = s^3(A+C) + s^2(B+D) + s(9A+4C) + (9B+4D)$

sys1

$1 = A + C \quad 1 = 9A + 4C$

$1 = B + D \quad 1 = 9B + 4D$

sys2

Cleaner version

$x(t) = \frac{11}{6}(\sin(2t)) - \frac{3}{5}\cos(2t) + (-\frac{8}{9}\sin 3t) + (\frac{8}{5}\cos 3t)$

$1 = 9A + 4C$

$-9A \quad -9A$

$\frac{1-9A}{4} = \frac{4C}{4}$

$1 - D = B$

$\boxed{\frac{11}{3} = B}$

$1 = B - \frac{8}{3}$

$1 = 9(1-D) + 4D$

$1 = -\frac{3}{5} + C$

$4(1) = \left(A + \left(\frac{1-9A}{4}\right)\right)$

$\boxed{D = -\frac{8}{3}}$

$\frac{-8}{3} = \frac{3D}{3}$

$\boxed{\frac{8}{5} = C}$

$\boxed{-\frac{3}{5} = A}$

$4 = 4A + (1-9$

$3 = -5A$

$4 = 1 - 5A$

$-1 \quad -1$

5.) $x'' + 6x' + 8x = 8; \quad x(0) = x'(0) = 0$

$\mathcal{L}[x''] + 6\mathcal{L}[x'] + 8\mathcal{L}[x] = \mathcal{L}[8]$

$\mathcal{L}[f(t)] = f(s)$

$\mathcal{L}[f'(t)] = sf(s) - f(0)$

$\mathcal{L}[f''(t)] = s^2 f(s) - sf(0) - f'(0)$

$s^2 f(s) - sf(0) - f'(0) + 6(sf(s) - f(0)) + 8(f(s)) = \frac{8}{s}$

$s^2 f(s) - \cancel{s(0)} - \cancel{0} + 6sf(s) - \cancel{6(0)} + sf(s) = \frac{8}{s}$

$\frac{f(s)\cancel{(s^2+6s+s)}}{\cancel{(s^2+6s+s)}} = \frac{\left(\frac{8}{s}\right)}{(s^2+6s+s)} = \frac{8}{s(s^2+6s+s)} = \boxed{\frac{8}{s^3+6s^2+s^2}}$

partial fractions

$8\mathcal{L}^{-1}\left[\frac{1}{s^3+6s^2+s^2}\right]$

$\frac{1}{s^2(s+6+1)} = \frac{1}{s^2(s+7)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s+7)}$

$8\mathcal{L}^{-1}\left[-\frac{1}{49}\left[\frac{1}{s}\right] + \frac{1}{7}\left[\frac{1}{s^2}\right] + \frac{1}{49}\left[\frac{1}{s+7}\right]\right]$

$1 = A(s)(s+7) + B(s+7) + C(s^2)$

$x(t) = 8\left[-\frac{1}{49}[1] + \frac{1}{7}[t^1] + \frac{1}{49}[e^{-7t}]\right]$

put $s=0$

$\frac{1}{7} = \frac{7B}{7}, \boxed{B = \frac{1}{7}}$

put $s=-7$

$\frac{1}{49} = \frac{49C}{49}, \boxed{C = \frac{1}{49}}$

put $s=1$

$1 = A(1)(1+7) + \frac{1}{7}(8) + \frac{1}{49}(1)$

$1 = 8A + \frac{8}{7} + \frac{1}{49}$

$-\frac{8}{7} - \frac{1}{49} \qquad -\frac{8}{7} - \frac{1}{49}$

$\frac{\left(-\frac{8}{49}\right)}{8} = \frac{8A}{8}$

$= A$

Translation of Laplace Transforms and Partial Fractions- HW Problems

In problems 1-3 find the Laplace transform of the given function.

1. $x(t) = t^2 e^{3t}$
2. $f(t) = e^{\left(\frac{t}{2}\right)} \cos(5t)$
3. $x(t) = e^{5t} \sin(2t)$

In problems 4-8 find the inverse Laplace Transform of the given function. Use partial fractions when appropriate.

4. $F(s) = \frac{1}{s^2+6s+9}$
5. $F(s) = \frac{s+3}{s^2+4s+8}$
6. $F(s) = \frac{6s^2-s-6}{s^3-s^2-6s}$
7. $F(s) = \frac{4}{s^3-4s}$
8. $F(s) = \frac{1}{s^4-2s^2+1}$

In problems 9-11 solve the initial value problems with Laplace transforms.

9. $x'' + 2x' + 6x = 6;$ $\quad x(0) = x'(0) = 0$
10. $y'' + 4y' + 8y = e^{-t};$ $\quad y(0) = y'(0) = 0$
11. $y'' + 2y' + 2y = 2\cos(t) + \sin(t)\,;$ $\quad y(0) = y'(0) = 0$

In Problems 1 to 3 find the Laplace transform of the given function. 1.) $x(t)=t^2e^{3t}$

$$\int_0^\infty e^{-st}e^{3t}t^2\,dt$$

$$\left(\frac{2}{s-3}\right)\left(\frac{1}{s-3}\right)\left(-\frac{1}{(s-3)}\right)\left(e^{-st+(3t)}\right)\Big|_0^\infty$$

$\to (-1)$

$$\int_0^\infty \underbrace{e^{-st+(3t)}}_{dv}\underbrace{t^2}_{u}\,dt \quad v=-\frac{1}{(s-3)}e^{-st+(3t)} \quad du=2t\,dt$$

$$=\frac{2}{(s-3)(s-3)(s-3)}$$

$$t^2\left(-\frac{1}{(s-3)}e^{-st+(3t)}\right)\Big|_0^\infty-\int_0^\infty -\frac{1}{(s-3)}e^{-st+(3t)}\,2t\,dt$$

$\uparrow 0$

$$X(s)=\boxed{\mathcal{L}[t^2e^{3t}]=\frac{2}{(s-3)^3}}$$ Answer

$$\frac{2}{s-3}\int_0^\infty \underbrace{e^{-st+(3t)}}_{dv}\underbrace{t}_{u}\,dt \quad v=-\frac{1}{(s-3)}e^{-st+(3t)} \quad du=dt$$

$$\frac{2}{s-3}\left(t\left(-\frac{1}{(s-3)}e^{-st+(3t)}\right)\Big|_0^\infty+\int_0^\infty\frac{1}{(s-3)}e^{-st+(3t)}\,dt\right)$$

$\uparrow 0$

$$\left(\frac{2}{s-3}\right)\left(\frac{1}{s-3}\right)\int_0^\infty e^{-st+(3t)}\,dt$$

In Problems 1 & 3 find the Laplace transform of the given function. 1.) $f(t) = e^{\frac{t}{2}} \cos(5t)$

$$\int_0^\infty e^{-st} e^{\frac{t}{2}} \cos(5t)\,dt$$

$$\int_0^\infty \underbrace{e^{-st+(\frac{t}{2})}}_{D} \underbrace{\cos(5t)}_{I}\,dt$$

	D	I
+	$e^{-st+(\frac{t}{2})}$	$\cos(5t)$
−	$(-s+\frac{1}{2}) e^{-st+(\frac{t}{2})}$	$\frac{\sin(5t)}{5}$
+	$(-s+\frac{1}{2})^2 e^{-st+(\frac{t}{2})}$	$-\frac{\cos(5t)}{25}$

$$\frac{25}{25+(-s+\frac{1}{2})^2}\left(e^{-st+(\frac{t}{2})}\left(\frac{\sin(5t)}{5}\right)+(-s+\tfrac{1}{2})e^{-st+(\frac{t}{2})}\frac{\cos(5t)}{25}\right)\Bigg|_0^\infty$$

$$\left(-\frac{(-s+\frac{1}{2})}{25}\right) \times$$

$$X(s) = \boxed{-\frac{(-s+\frac{1}{2})}{25+(-s+\frac{1}{2})^2}} = \mathcal{L}\left[e^{\frac{t}{2}}\cos(5t)\right]$$

← Answer

$$\int_0^\infty e^{-st+(\frac{t}{2})}\cos(5t)\,dt = e^{-st+(\frac{t}{2})}\left(\frac{\sin(5t)}{5}\right)+(-s+\tfrac{1}{2})e^{-st+(\frac{t}{2})}\frac{\cos(5t)}{25} - \int (-s+\tfrac{1}{2})^2 e^{-st+(\frac{t}{2})}\frac{\cos 5t}{25}\,dt$$

$$1+\frac{(-s+\frac{1}{2})^2}{25}\int = \frac{25+(-s+\frac{1}{2})^2}{25}$$ reciprocal

$$\downarrow$$

$$\frac{25}{25+(-s+\frac{1}{2})^2}$$

to bring to other side

$$-\frac{(-s+\frac{1}{2})^2}{25}\int$$

$$+\left(\frac{-s+\frac{1}{2}}{5\cdot 5}\right)\int$$

3.) $X(t) = e^{5t}\sin(2t)$

$$\int_0^\infty e^{-st}\left(e^{5t}\sin(2t)\right)dt$$

$$\int_0^\infty \underset{D}{\underline{e^{-st+5t}}}\ \underset{I}{\underline{\sin(2t)}}\,dt$$

	D	I
+	e^{-st+5t}	$\sin(2t)$
−	$(-s+5)e^{-st+5t}$	$-\frac{\cos(2t)}{2}$
+	$(-s+5)^2e^{-st+5t}$	$-\frac{\sin(2t)}{4}$

$$\int_0^\infty e^{-st+(5t)}\sin(2t)\,dt =$$

$$-e^{-st+(5t)}\left(\frac{\cos(2t)}{2}\right) + (-s+5)e^{-st+(5t)}\left(\frac{\sin(2t)}{4}\right) - \int_0^\infty (-s+5)^2 e^{-st+(5t)}\left(\frac{\sin(2t)}{4}\right)dt$$

$$-\frac{(-s+5)^2}{4}\int_0^\infty e^{-st+(5t)}\sin(2t)\,dt$$

Adding to other Side of Eq: $1 + \frac{(-s+5)^2}{4} = \frac{4+(-s+5)^2}{4}$

Multiply reciprocal to other side: $\frac{4}{4+(-s+5)^2}\left(-e^{-st+(5t)}\left(\frac{\cos(2t)}{2}\right) + (-s+5)e^{-st+(5t)}\left(\frac{\sin(2t)}{4}\right)\right)\Big|_0^\infty$

& Evaluate $X(s) = \frac{4}{4+(-s+5)^2}\left(-\left(-1\left(\frac{1}{2}\right)\right)\right) = \boxed{\frac{2}{4+(-s+5)^2}}$ Answer

In Problems 1-3 find the inverse Laplace Transform of the given function. Use partial fractions when appropriate.

4.) $F(s) = \dfrac{1}{s^2+6s+9}$

Use partial fractions

$$\frac{1}{(s+3)^2} = \frac{A}{(s+3)} + \frac{B}{(s+3)^2} = \frac{0}{s+3} + \frac{1}{(s+3)^2}$$

funny (sorta redundant)

$$1 = A(s+3) + B$$

$s=-3$

$\boxed{1=B}$

$s=1$

$$1 = 4A + 1$$
$$-1 \qquad -1$$

$$\frac{0}{4} = \frac{4A}{4}$$

$\boxed{0=A}$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+6s+9}\right] = \mathcal{L}^{-1}\left[0 + \frac{1}{(s+3)^2}\right]$$

$$= 0 + e^{-3}t^{1}$$

$x(t) = e^{-3}t^{1}$ ← Answer

In problems 1-3 find the inverse laplace Transform of the given function. Use partial fractions when appropriate.

5.) $F(s) = \dfrac{s+3}{s^2+4s+8}$

$\downarrow$

$\left(\dfrac{s+3}{(s^2+4s+4)+4}\right)$

$X(t) = \mathcal{L}^{-1}\left[\dfrac{s+3}{(s+2)^2+4}\right] = e^{-3t}\cos 2t$

6.) $F(s) = \dfrac{6s^2-s-6}{s^3-s^2-6s}$

$\dfrac{6s^2-s-6}{s(s^2-s-6)} = \dfrac{A}{s} + \dfrac{B}{(s-3)} + \dfrac{C}{(s+2)}$

$\downarrow$

$(s-3)(s+2)$

$6s^2-s-6 = A(s-3)(s+2) + B(s)(s+2) + C(s)(s-3)$

$s=3$ $\quad$ $s=-2$ $\quad$ $s=0$

$\dfrac{45}{15} = \dfrac{\cancel{15}B}{\cancel{15}}$ $\quad$ $\dfrac{20}{10} = \dfrac{\cancel{10}C}{\cancel{10}}$ $\quad$ $\dfrac{-6}{-6} = \dfrac{\cancel{-6}A}{\cancel{-6}}$

$\mathcal{L}^{-1}\left[1\left(\frac{1}{s}\right) + 3\left(\frac{1}{s-3}\right) + 2\left(\frac{1}{s+2}\right)\right]$

$X(t) = \underline{\underline{(1) + 3(e^{3t}) + 2(e^{-2t})}}$

$\uparrow$

Answer

7.) $F(s) = \dfrac{4}{s^3-4s}$

$\dfrac{4}{s(s^2-4)} = \dfrac{As+B}{s^2-2^2} + \dfrac{C}{s}$

$4 = (As+B)(s) + C(s^2-4)$

$4 = As^2 + Bs + Cs^2 - 4C$

$4 = s^2(A+C) + Bs - 4C$

$0 = A+C$ $\quad$ $A=1$

$4 = -4C$ $\quad$ $C=-1$

$0 = B$

$\mathcal{L}^{-1}\left[\left(\dfrac{s}{s^2-4}\right) + (-1)\left(\dfrac{1}{s}\right)\right]$

$X(t) = \underline{\cosh(2t) + (-1)}$

$\uparrow$

Answer

In Problems 1-3 find the inverse laplace Transform of the given function. Use Partial fractions when appropriate.

8.) $F(s) = \frac{1}{s^4 - 2s^2 + 1}$

$$\mathcal{L}^{-1}\left[\frac{1}{4}\left(\frac{1}{s+1}\right) + \frac{1}{4}\left(\frac{1}{(s+1)^2}\right) - \frac{1}{4}\left(\frac{1}{s-1}\right) + \frac{1}{4}\left(\frac{1}{(s-1)^2}\right)\right]$$

$$x(t) = \frac{1}{4}\left(e^{-1t}\right) + \frac{1}{4}\left(e^{-1}t^{1}\right) - \frac{1}{4}\left(e^{1t}\right) + \frac{1}{4}\left(e^{1}t^{1}\right)$$

Answer

$-\frac{3}{4} = -A$

$u = s^2$, $\frac{1}{u^2 - 2u + 1}$

$= \frac{1}{(u-1)(u-1)}$

$= \frac{1}{(s^2-1)(s^2-1)}$

$= \frac{1}{(s+1)(s-1)(s+1)(s-1)}$

$= \frac{1}{(s+1)^2(s-1)^2}$

$= \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s-1)} + \frac{D}{(s-1)^2}$

$1 = A(s+1)(s-1)^2 + B(s-1)^2 + C(s-1)(s+1)^2 + D(s+1)^2$

s^2+2s+1

$(s-1)(s-1)$

s^2-2s+1

$s=1,$

$\frac{1}{4} = \frac{4D}{4}$ $\boxed{D = \frac{1}{4}}$

$s=-1,$

$\boxed{B = \frac{1}{4}}$

$1 = As^3 - As - As^2 + A + s^2B - 2Bs + B + s^3C - Cs + s^2C - C + s^2D + 2Ds + D$

$0 = A + C$

$0 = -A + B + C + D$

$0 = -A + C + \frac{2}{4}$

$\boxed{A = \frac{1}{4}}$

$-A = C$

$-\frac{2}{4} = -A + C$

$-\frac{2}{4} = 2C$

$\boxed{-\frac{1}{4} = C}$

In problems 9-11 solve the initial value problems with Laplace transforms.

9.) $x'' + 2x' + 6x = 6$, $x(0) = x'(0) = 0$

$x = f(t)$

$\mathcal{L}[f(t)] = f(s)$

$\mathcal{L}[f'(t)] = sf(s) - f(0)$

$\mathcal{L}[f''(t)] = s^2 f(s) - sf(0) - f'(0)$

$[s^2 f(s) - sf(0) - f'(0)] + 2[sf(s) - f(0)] + 6[f(s)] = \mathcal{L}[6]$

$s^2 f(s) - s(0) - (0) + 2[sf(s) - 0] + 6[f(s)] = \frac{6}{s}$

$s^2 f(s) + 2sf(s) + 6f(s) = \frac{6}{s}$

$\frac{(s^2+2s+6)f(s)}{s^2+2s+6} = \frac{\frac{6}{s}}{s^2+2s+6}$

$= \frac{6}{s(s^2+2s+6)} = \frac{A}{s} + \frac{Bs+C}{(s^2+2s+6)}$,

$6 = A(s^2+2s+6) + Bs + C(s)$

$6 = As^2 + 2As + 6A + Bs^2 + Cs$

$s = 0$

$\boxed{A = 1}$

$0 = A + B$

$\boxed{B = -1}$

$0 = 2A + C$

$\boxed{C = -2}$

$\mathcal{L}^{-1}\left[\frac{1}{s} + \frac{-s-2}{s^2+2s+6}\right]$

$\mathcal{L}^{-1}\left[\frac{1}{s} + \frac{-s-2}{(s+1)^2+5}\right]$

$\mathcal{L}^{-1}\left[\frac{1}{s} + \frac{-1((s+1)+1)}{(s+1)^2+5}\right]$

$\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s+1}{(s+1)^2+5} - \frac{1}{(s+1)^2+5}\right]$

$x(t) = 1 - e^{-1t}\cos\sqrt{5}t - e^{-1t}\sin\sqrt{5}t$

Answer

In problems 9-11 solve the initial value problems with Laplace transforms.

10.) $y''+4y'+8y=e^{-t}$, $y(0)=y'(0)=0$

$y=f(t)$

$\mathcal{L}[f(t)]=f(s)$

$\mathcal{L}[f'(t)]=sf(s)-f(0)$

$\mathcal{L}[f''(t)]=s^2f(s)-sf(0)-f'(0)$

$[s^2f(s)-sf(0)-f'(0)]+4[sf(s)-f(0)]+8[f(s)]=\mathcal{L}[e^{-t}]$

$s^2f(s)-0-0+4sf(s)-0+8f(s)=\frac{1}{s+1}$

$s^2f(s)+4sf(s)+8f(s)=\frac{1}{s+1}$

$f(s)[s^2+4s+8]=\frac{1}{s+1}$

$f(s)=\frac{1}{(s+1)(s^2+4s+8)}=\frac{A}{(s+1)}+\frac{Bs+C}{(s^2+4s+8)}$

$1=A(s^2+4s+8)+Bs+C(s+1)$

$s=-1$

$1=3A$

$\frac{1}{3}=\frac{3A}{3}$

$\boxed{A=\frac{1}{3}}$

$1=As^2+4As+8A+Bs^2+Bs+Cs+C$

$0=A+B$ $\quad$ $0=4A+B+C$

$\boxed{-\frac{1}{3}=B}$

$\frac{4}{3}-\frac{1}{3}+C$

$0=\frac{3}{3}+C$

$\boxed{-1=C}$

$\mathcal{L}^{-1}\left[\frac{1}{3}\left(\frac{1}{s+1}\right)+\left(\frac{-\frac{1}{3}s-1}{s^2+4s+4+4}\right)\right]$

$\frac{-\frac{1}{3}(s+3)}{(s+2)^2+4}=-\frac{1}{3}(s+2+1)$

$-\frac{1}{3}(s+2)-\frac{1}{3}$

$\mathcal{L}^{-1}\left[\frac{1}{3}\left(\frac{1}{s+1}\right)+\frac{-\frac{1}{3}(s+2)+\frac{1}{3}}{(s+2)^2+4}\right]$

$\mathcal{L}^{-1}\left[\frac{1}{3}\left(\frac{1}{s+1}\right)-\frac{1}{3}\left(\frac{(s+2)+1}{(s+2)^2+4}\right)\right]$

$\mathcal{L}^{-1}\left[\frac{1}{3}\left(\frac{1}{s+1}\right)\ \frac{1}{3}\left(\frac{s+2}{(s+2)^2+4}\right)-\frac{1}{6}\left(\frac{2}{(s+2)^2+4}\right)\right]$

$x(t)=\frac{1}{3}(e^{-t})-\frac{1}{3}(e^{-2t}\cos 2t)+\frac{1}{6}(e^{-2t}\sin 2t)$

Answer

In problems 9-11 solve the initial value problems with Laplace transforms.

11.) $y''+2y'+2y=2(\cos(t)+\sin(t)),\ y(0)=y'(0)=0$

$Y=f(t)$

$\mathcal{L}[f(t)]=f(s)$

$\mathcal{L}[f'(t)]=sf(s)-f(0)$

$\mathcal{L}[f''(t)]=s^2f(s)-sf(0)-f'(0)$

$[s^2f(s)-sf(0)-f'(0)]+2[sf(s)-f(0)]+2[f(s)]=\mathcal{L}[2(\cos(t)+\sin(t))]$

$s^2f(s)-0-0+2sf(s)+2f(s)=2\left(\frac{s}{s^2+1}\right)+\left(\frac{1}{s^2+1}\right)$

$f(s)(s^2+2s+2)=\frac{2s}{s^2+1}+\frac{1}{s^2+1}$

$f(s)=\frac{2s}{(s^2+1)(s^2+2s+2)}+\frac{1}{(s^2+1)(s^2+2s+2)}$

$\frac{2s}{(s^2+1)(s^2+2s+2)}=\frac{As+B}{(s^2+1)}+\frac{Cs+D}{(s^2+2s+2)}$

$2s=As+B(s^2+2s+2)+(Cs+D)(s^2+1)$

$2s=As^3+2As^2+2As+Bs^2+2Bs+2B+Cs^3+Cs+Ds^2+D$

$c:\ 0=2B+D$

$s:\ 2=2A+2B+C$

$s^3:\ 0=A+C$

$s^2:\ 0=2A+B+D$

$2=2A+2B-A$

$0=2A+B-2B$

$2=A+2B$

$0=2A-B$

$2-2B=A$

$0=2(2-2B)-B$

$0=4-4B-B$

$\frac{-4}{-5}=\frac{-5B}{-5}$

$\boxed{B=\frac{4}{5}}$

$\boxed{C=-\frac{2}{5}}$

$\boxed{-\frac{8}{5}=D}$

$\boxed{\frac{2}{5}=A}$

$\frac{4}{5}=2A$

$\mathcal{L}^{-1}\left[\frac{\frac{2}{5}s+\frac{4}{5}}{(s^2+1)}+\frac{-\frac{2}{5}s-\frac{8}{5}}{(s+1)^2+1}\right]$ continued $\longrightarrow$

$\mathcal{L}^{-1}\left[\frac{2}{5}\left(\frac{s}{s^2+1}\right)+\frac{4}{5}\left(\frac{1}{s^2+1}\right)+\frac{-\frac{2}{5}(s+4)}{(s+1)^2+1}\right]$ $\frac{-\frac{2}{5}(s+1)-\frac{6}{5}}{(s+1)^2+1}$

$\mathcal{L}^{-1}\left[\frac{2}{5}\left(\frac{s}{s^2+1}\right)+\frac{4}{5}\left(\frac{1}{s^2+1}\right)-\frac{2}{5}\left(\frac{s+1}{(s+1)^2+1}\right)-\frac{6}{5}\left(\frac{1}{(s+1)^2+1}\right)\right]$

$=\frac{2}{5}(\cos t)+\frac{4}{5}(\sin t)-\frac{2}{5}(e^{-t}\cos t)-\frac{6}{5}(e^{-t}\sin t)$

(Half the Answer)

In problems 9-11 solve the initial value problems with Laplace transforms.

11.) $y''+2y'+2y=2\cos(t)+\sin(t)$, $y(0)=y'(0)=0$

$y=f(t)$

$\mathcal{L}[f(t)]=f(s)$

$\mathcal{L}[f'(t)]=sf(s)-f(0)$

$\mathcal{L}[f''(t)]=s^2f(s)-sf(0)-f'(0)$

$[s^2f(s)-sf(0)-f'(0)]+2[sf(s)-f(0)]+2[f(s)]=\mathcal{L}[2\cos(t)+\sin(t)]$

$s^2f(s)-0-0+2sf(s)+2f(s)=2\left(\frac{s}{s^2+1}\right)+\left(\frac{1}{s^2+1}\right)$

$f(s)(s^2+2s+2)=\frac{2s}{s^2+1}+\frac{1}{s^2+1}$

$f(s)=\frac{2s}{(s^2+1)(s^2+2s+2)}+\frac{1}{(s^2+1)(s^2+2s+2)}$

$1=\frac{As+B}{(s^2+1)}+\frac{Cs+D}{(s^2+2s+2)}$

$1=As^3+2As^2+2As+Bs^2+2Bs+2B+Cs^3+Cs+Ds^2+D$

$s^2: 0=2A+B+D$

$C: 1=2B+D$

$s: 0=2A+2B+C$

$s^3: 0=A+C$

$\boxed{C=\frac{2}{5}}$

$\boxed{A=-\frac{2}{5}}$

$\boxed{\frac{2}{5}=D}$

$0=2A+B+D$

$0=2A+B+(1-2B)$

$0=2A+2B-A$

$-1=2A-B$

$0=A+2B$

$-2B=A$

$-1=2(-2B)-B$

$\boxed{B=\frac{1}{5}}$

Now we're Answering the rest

$\mathcal{L}^{-1}\left[\frac{-\frac{2}{5}s+\frac{1}{5}}{(s^2+1)}+\frac{\frac{2}{5}s+\frac{2}{5}}{(s+1)^2+1}\right]$

$\mathcal{L}^{-1}\left[-\frac{2}{5}\left(\frac{s}{s^2+1}\right)+\frac{1}{5}\left(\frac{1}{(s^2+1)}\right)+\frac{\frac{2}{5}(s+1)+\frac{1}{5}}{(s+1)^2+1}\right]$ ← all about factoring and and extracting 1

$\mathcal{L}^{-1}\left[-\frac{2}{5}\left(\frac{s}{s^2+1}\right)+\frac{1}{5}\left(\frac{1}{s^2+1}\right)+\frac{2}{5}\left(\frac{s+1}{(s+1)^2+1}\right)+\frac{1}{5}\left(\frac{1}{(s+1)^2+1}\right)\right]$

Second part of the Answer: $-\frac{2}{5}(\cos t)+\frac{1}{5}(\sin t)+\frac{2}{5}(e^{-t}\cos t)+\frac{1}{5}(e^{-t}\sin t)$

Full Answer

Full Answer for Question (11)

$$X(t)=\left[\frac{2}{5}(\cos t)+\frac{4}{5}(\sin t)-\frac{2}{5}(e^{-t}\cos t)-\frac{6}{5}(e^{-t}\sin t)-\frac{2}{5}(\cos t)+\frac{1}{5}(\sin t)+\frac{2}{5}(e^{-t}\cos t)+\frac{1}{5}(e^{-t}\sin t)\right]$$

Simplified Version

$$X(t)=\sin t-(e^{-t}\sin t)$$

The Convolution Theorem/Derivatives & Integrals of Transforms-
HW Problems

In problems 1-4 calculate $f(t) * g(t)$.

1. $f(t) = 1, \quad g(t) = e^t$
2. $f(t) = t, \quad g(t) = \sin(t)$
3. $f(t) = t^2, \quad g(t) = e^t$
4. $f(t) = t, \quad g(t) = t$

5a. Show that if $f(t) = e^{at}$ and $g(t) = e^{bt}$, where a, b are constants, then

$$(f * g)(t) = \frac{1}{a-b}(e^{at} - e^{bt}).$$

b. By direct calculation of Laplace transforms, show that

$$\mathcal{L}\big((f * g)(t)\big) = (\mathcal{L}(f(t))(\mathcal{L}(g(t))$$

for $f(t) = e^{at}$ and $g(t) = e^{bt}$ (where $\mathcal{L}$ is the Laplace transform).

In problems 6-8 use the convolution theorem to find the inverse Laplace transform of the given function.

6. $F(s) = \frac{3}{s^2-1}$
7. $F(s) = \frac{1}{(s-1)^2}$
8. $F(s) = \frac{4}{s(s^2+4)}$

In problems 9-11 find the Laplace transform of the given function.

9. $f(t) = tcos(3t)$

10. $f(t) = t^2 \sin(3t)$

11. $f(t) = \frac{e^t - 1}{t}$

In problems 12-14 find the inverse Laplace transform of the given function.

12. $F(s) = \ln(\frac{s+3}{s-3})$

13. $F(s) = \ln(\frac{s^2+4}{s+4})$

14. $F(s) = \frac{2s}{(s^2+1)^2}$

15. Use the Laplace transform to transform $ty'' + (t-2)y' + y = 0$ to find a solution where $y(0) = 0$ but $y(t) \not\equiv 0$.

In problems 1-4 Calculate $f(t) * g(t)$.

1.) $f(t)=1 \quad g(t)=e^t$

$$=\int_0^t f(w)g(t-w)\,dw$$

Assuming they're two piecewise functions

$$\int_0^t e^{t-w}\,dw$$

$u=t-w$

$\frac{du}{-1}=\frac{-1dw}{-1}$

$$=\int_0^t e^u\,dw$$

$$=-\int_0^t e^u\,du$$

$$=-(e^{t-w})\Big|_0^t$$

$$=-e^{t-t}-(-e^{t-0})$$

$$=\boxed{-1+e^t} \leftarrow \text{Answer}$$

2.) $f(t)=t,\ g(t)=\sin(t)$

$$\int_0^t \underbrace{w}_{u}\ \underbrace{\sin(t-w)\,dw}_{dv}$$

$du=dw \quad v=-\left(\frac{\cos(t-w)}{-1}\right)$

$v=\cos(t-w)$

$$w(\cos(t-w))\Big|_0^t-\int_0^t \cos(t-w)\,dw$$

$$-\sin(t-w)\Big|_0^t$$

Answer

$$t(\cos(0))-0)-[0--\sin(t)] = \boxed{t-\sin t}$$

In problems 1-4 Calculate $f(t)*g(t)$.

3.) $f(t)=t^2$, $g(t)=e^t$

$$\int_0^t w^2 e^{t-w}\,dw$$

$$e^t\int \underbrace{w^2}_{u}\,\underbrace{e^{-w}dw}_{dv} \qquad v=-e^{-w}$$

$$du=2w\,dw$$

$$-w^2e^{-w}\Big|_0^t-\int_0^t -e^{-w}\,2w\,dw$$

$$\underline{-t^2e^{-t}} \qquad +2\int_0^t \underbrace{e^{-w}}_{dv}\,\underbrace{w}_{u}\,dw \qquad v=-e^{-w}$$

$$du=dw$$

$$-we^{-w}\Big|_0^t-\int_0^t -e^{-w}\,dw$$

$$-we^{-w}\Big|_0^t+\int_0^t e^{-w}\,dw$$

$$\uparrow$$

$$\underline{-2te^{-t}} \qquad 2(-e^{-w})\Big|_0^t$$

$$\downarrow$$

$$\underline{-2e^{-t}+2}$$

Answer: $e^t\left[-t^2e^{-t}-2te^{-t}-2e^{-t}+2\right]$

4.) $f(t)=t$, $g(t)=t$

$$\int_0^t w(t-w)\,dw$$

$$\int_0^t wt-w^2\,dw$$

$$t\int_0^t w\,dw-\int_0^t w^2\,dw$$

$$t\left(\frac{w^2}{2}\right)\Big|_0^t-\frac{w^3}{3}\Big|_0^t$$

$$t\left(\frac{t^2}{2}\right)-\frac{t^3}{3} \leftarrow \text{Answer}$$

5a) Show that if $f(t)=e^{at}$ & $g(t)=e^{bt}$, where a,b are constants, then $(f\cdot g)(t)=\frac{1}{a-b}(e^{at}-e^{bt})$.

$$\int_0^t e^{aw}e^{b(t-w)}dw$$

$$\int_0^t e^{aw}e^{bt-bw}dw$$

$$e^{bt}\int_0^t e^{aw}e^{-bw}dw$$

$$e^{bt}\int_0^t e^{aw-bw}dw$$

$$e^{bt}\left[\frac{1}{a-b}\left(e^{at-bt}-e^{0}\right)\right]$$

$$=\frac{1}{a-b}\left(e^{at}-e^{bt}\right)$$

5b) By direct calculation of laplace transforms show that $\mathcal{L}((f\cdot g)(t))=(\mathcal{L}(f(t))\mathcal{L}(g(t)))$ for $f(t)=e^{at}$ & $g(t)=e^{bt}$ (where $\mathcal{L}$ is the laplace transform).

Exp 1:

$$\mathcal{L}(e^{at})\mathcal{L}(e^{bt})=\boxed{\left(\frac{1}{s-a}\right)\left(\frac{1}{s-b}\right)}$$

Exp 2:

$$\mathcal{L}((f\cdot g)(t))=\frac{1}{a-b}\mathcal{L}\left[e^{at}-e^{bt}\right]$$

$$\frac{1}{a-b}(e^{at}-e^{bt}) \qquad \boxed{\frac{1}{a-b}\left(\frac{1}{s-a}-\frac{1}{s-b}\right)}$$

Let's prove their Equal by simplifying both Expressions

$$\text{Exp 2: }\left(\frac{1}{a-b}\right)\frac{\cancel{s}-b-\cancel{s}+a}{(s-a)(s-b)}=\frac{\frac{-b+a}{a-b}}{(s-a)(s-b)}=\frac{1}{(s-a)(s-b)}$$

$$\text{Exp 1: }\left(\frac{1}{s-a}\right)\left(\frac{1}{s-b}\right)=\frac{1}{(s-a)(s-b)}$$

Therefore, both Exp 1 & 2 are =.

In problems 6-8 use the convolution theorem to find the inverse Laplace transform of the given function.

6.) $F(s)=\frac{3}{s^2-1}$

$$=\frac{3}{(s+1)(s-1)}$$

$$3\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] * \mathcal{L}^{-1}\left[\frac{1}{s-1}\right]$$

$$\underset{\substack{\uparrow \\ f(t)}}{3e^{-t}} * \underset{\substack{\uparrow \\ g(t)}}{e^{t}}$$

$$3\int_0^t e^{-w} e^{t-w}\,dw$$

$$3\int_0^t e^{t-2w}\,dw$$

$$-\frac{3}{2}\left(e^{t-2w}\right)\Big|_0^t$$

$$\underline{= -\frac{3}{2}e^{-t}+\frac{3}{2}e^{t}}$$

7.) $F(s)=\frac{1}{(s-1)^2}$

$$\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] * \mathcal{L}^{-1}\left[\frac{1}{s-1}\right]$$

$$\underset{\substack{\uparrow \\ f(t)}}{e^{1t}} * \underset{\substack{\uparrow \\ g(t)}}{e^{1t}}$$

$$\int_0^t e^{w} e^{t-w}\,dw$$

$$e^{t}\int_0^t 1\,dw$$

$$e^{t}[w]\Big|_0^t$$

$$\underline{=e^{t}[t]}$$

8.) $F(s)=\frac{4}{s(s^2+4)}$

$$\mathcal{L}^{-1}\left[\frac{2}{s}\right] * \mathcal{L}^{-1}\left[\frac{2}{(s^2+4)}\right]$$

$$\underset{\substack{\uparrow \\ f(t)}}{2} * \underset{\substack{\uparrow \\ g(t)}}{\sin 2t}$$

$$\int_0^t 2\sin 2(t-w)\,dw$$

$$2\int_0^t \sin 2t-2w\,dw$$

$$\cos 2t-2w\Big|_0^t$$

$$\underline{\underline{(1-\cos 2t)}}$$

In problems 9-11 find the Laplace transform of the given function.

9.) $f(t) = t\cos(3t)$

$$-\frac{d}{ds}\left(\frac{s}{s^2+9}\right)$$

$$= -\frac{(s^2+9)*1 - s(2s)}{(s^2+9)^2}$$

$$= -\frac{-(s^2+9)}{(s^2+9)^2}$$

$$= \boxed{\frac{s^2-9}{(s^2+9)^2}}$$

10.) $f(t) = t^2\sin(3t)$

$$(-1)^2 \cdot F^{(2)}(s)$$

$$F(s) = \frac{3}{(s^2+9)}$$

$$F^{(1)}(s) = \frac{(0)(s^2+9) - (3)(2s)}{(s^2+9)^2}$$

$$F^{(1)}(s) = \frac{-6s}{(s^2+9)^2}$$

$$F^{(2)}(s) = \frac{(-6(s^2+9)^2) - (-6s)(2(s^2+9)\cdot 2s)}{(s^2+9)^4}$$

$$= \boxed{\frac{-6(s^2+9)^2 + 12s(s^2+9)\cdot 2s}{(s^2+9)^4}}$$

In problems 9-11 find the Laplace transform of the given function.

11.) $f(t) = \dfrac{e^t - 1}{t}$

use $\mathcal{L}\left[\dfrac{f(t)}{t}\right] = \int_s^\infty F(u)\,du$

$\mathcal{L}[e^t] - \mathcal{L}[1]$

$\left[\dfrac{1}{s-1} - \dfrac{1}{s}\right] = \dfrac{s}{(s-1)s} - \dfrac{(s-1)}{s(s-1)} = \dfrac{\cancel{s} - \cancel{s} + 1}{s(s-1)} = \dfrac{1}{s(s-1)}$

$\int_s^\infty \dfrac{1}{u(u-1)}\,du \quad = \dfrac{A}{u} + \dfrac{B}{(u-1)}$

$\int_s^\infty \dfrac{-1}{u} + \dfrac{1}{(u-1)}\,du$

$1 = A(u-1) + B(u)$

$1 = Au - A + Bu$

$-\ln(u) + \ln(u-1)\Big|_s^\infty$

$1 = -A \quad \boxed{B = 1}$

$0 - (-\ln(s) + \ln(s-1))$

$0 = A + B \quad \boxed{A = -1}$

$-A = B$

Answer: $\ln(s) - \ln(s-1)$

In problems 12-14 find the inverse Laplace transform function.

$\mathcal{L}^{-1}[F(s)] = -\frac{1}{t}\mathcal{L}^{-1}[F'(s)]$

12.) $F(s) = \ln\left(\frac{s+3}{s-3}\right)$

$\frac{d}{ds}(\ln(s+3)) - \frac{d}{ds}(\ln(s-3))$

$-\frac{1}{t}\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) + \frac{1}{t}\mathcal{L}^{-1}\left(\frac{1}{s-3}\right)$

$x(t) = -\frac{1}{t}\left[e^{-3t} - e^{3t}\right]$

13.) $F(s) = \ln\left(\frac{s^2+4}{s+4}\right)$

$\frac{d}{ds}(\ln(s^2+4) - \frac{d}{ds}(\ln(s+4))$

$-\frac{1}{t}\mathcal{L}^{-1}\left(\frac{2s}{s^2+4}\right) + \frac{1}{t}\mathcal{L}^{-1}\left(\frac{1}{s+4}\right)$

$x(t) = -\frac{2}{t}\left[\cos 2t + e^{-4t}\right]$

$\mathcal{L}\mathcal{L}$

In questions 12-14 find the inverse laplace transform of the given functions.

14.) $F(s)=\frac{2s}{(s^2+1)^2}$

$\mathcal{L}^{-1}\left(\frac{2}{s^2+1}\right) * \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right)$

$2\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) * \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right)$

$2\sin(t) * \cos(t)$

use trig formula

$2\int_0^t \sin(w)\cos(t-w)\,dw$

$2\int_0^t \sin(w)[\cos t\cos w + \sin t \sin w]\,dw$

$2\int_0^t \sin(w)\cos(t)\cos(w)\,dw + 2\int_0^t \sin^2 w \sin(t)\,dw$

$2\cos(t)\int_0^t \sin(w)\cos(w)\,dw + 2\sin(t)\int_0^t \sin^2 w\,dw$

use another trig identity (half Angle identity)

$+2\sin(t)\int_0^t \frac{1-\cos 2w}{2}\,dw$

$\underbrace{2\cos(t)\int_0^t \sin(w)\cos(w)\,dw}_{1} + \underbrace{2\sin(t)\int_0^t \frac{1}{2}\,dw}_{2} - \underbrace{2\sin(t)\int_0^t \cos 2w\,dw}_{3}$

$2\cos(t)\int_0^t \sin(w)\cos(w)\,dw$

$u=\sin w$

$\frac{du}{\cos w}=dw$

$2\cos(t)\int_0^t \frac{u\cancel{\cos(w)}\,du}{\cancel{\cos(w)}}$

$2\cos(t)\int_0^t u\,du$

$\frac{u^2}{2}$

$\cancel{2}\cos(t)\left(\frac{\sin^2 w}{\cancel{2}}\right)\Big|_0^t = \underline{\cos(t)\sin^2(t)}$

$2\sin(t)\int_0^t \frac{1}{2}\,dw$

$\sin(t)[w]\Big|_0^t = \underline{\sin(t)(t)}$

$-2\sin(t)\int_0^t \cos 2w\,dw$

$-\cancel{2}\sin(t)\left[\frac{\sin 2w}{\cancel{2}}\right]\Big|_0^t$

$-\sin(t)[\sin 2w]\Big|_0^t$

$= \underline{-\sin(t)\sin(2t)}$

Answer:

$\boxed{X(t)=\cos(t)\sin^2(t)+\sin(t)(t)-\sin(t)\sin(2t)}$

15.) use the Laplace transform to transform $ty''+(t-2)y'+y=0$ to find a solution where $y(0)=0$ but $y(t)\not\equiv 0$.

$\underline{Y=f(t)}$

{We want to find our non-trivial solution}

$$\mathcal{L}[ty''+(t-2)y'+y]=\mathcal{L}[0]$$

$$\mathcal{L}[ty'']+\mathcal{L}[ty']-2\mathcal{L}[y']+\mathcal{L}[y]=0$$

$$\mathcal{L}[y'']=s^2f(s)-f(0)-f'(0)=\underline{s^2f(s)-f'(0)}$$

$$\mathcal{L}[ty'']=-\underline{(2sf(s)+s^2f'(s))}$$ ← product rule

Apply →

$$\mathcal{L}[t^nf(t)]=-1^nF^{(n)}(s)$$

$$\mathcal{L}[y']=sf(s)-f(0)$$

$$\mathcal{L}[ty']=-\underline{(f(s)+sf'(s))}$$

$$[-2sf(s)-s^2f'(s)]+[-f(s)-sf'(s)]-2[sf(s)-0]+f(s)=0$$

$$f(s)[-2s-1-2s+1]+f'(s)[-s^2-s]=0$$

$$f(s)[-4s]+f'(s)[-s^2-s]=0$$

$$\left(\frac{1}{s^2-s}\right)\frac{f(s)[-4s]}{f(s)}=\frac{f'(s)[s^2-s]}{f(s)}\left(\frac{1}{s^2-s}\right)$$

$$-\frac{4s}{s^2-s}=\frac{f'(s)}{f(s)}$$

$$\frac{-4s}{s(s-1)}=\frac{-4}{s-1}$$

$$\int\frac{f'(s)}{f(s)}ds=-4\int\frac{1}{s-1}ds$$

$$\ln|f(s)|=-4\ln|s-1|+c$$

$$f(s)=e^{-4\ln|s-1|+c}$$

$$e^{-4\ln|s-1|+c}=|s-1|^{-4}+c$$

using $\frac{n!}{(s-a)^{n+1}}=e^{at}t^n$ $\quad\mathcal{L}^{-1}\left[\frac{c'}{(s-1)^4}\right]$

$$\boxed{X(t)=c''t^3e^t}$$ ← Answer

Piecewise Continuous Functions- HW Problems

In problems 1-4 find the inverse Laplace transform of the given function.

1. $F(s) = s^{-4}e^{-s}$
2. $F(s) = \frac{e^{-3s}}{s-3}$
3. $F(s) = \frac{e^{-2s}}{s^2+4}$
4. $F(s) = \frac{se^{-s}}{s^2+9}$

In problems 5-7 find the Laplace transform of the given function.

5. $f(t) = 3 \qquad 0 \le t \le 4$

   $= 0 \qquad t > 4$

6. $f(t) = t \qquad 0 \le t < 2$

   $= 0 \qquad t \ge 2$

7. $f(t) = \sin(2t) \qquad 0 \le t \le 2\pi$

   $= 0 \qquad t > 2\pi.$

In problems 8-10 solve the initial value problem.

8. $x'' + 9x = f(t)$, where

$$f(t) = 1 \qquad 0 \le t < 2$$
$$= 0, \qquad t \ge 2$$

and $x(0) = x'(0) = 0$.

Must be a typo since there are only 8-9 questions

9. $x'' + 4x = f(t)$, where

$$f(t) = \sin(t) \qquad 0 \le t \le 2\pi$$
$$= 0 \qquad t > 2\pi$$

and $x(0) = x'(0) = 0$.

.

In Problems 1-4 find the Inverse Laplace transform of the given function.

Rule: $\mathcal{L}^{-1}(e^{-as}(G(s))) = u(t-a)g(t-a)$

1.) $F(s) = s^{-4}e^{-s}$

$\mathcal{L}^{-1}(F(s)) = u(t-4)\frac{1}{6}((t-4)^3)$

$\frac{1}{6}\mathcal{L}^{-1}\left(\frac{6}{s^4}\right) = \frac{1}{6}(t^3)$

$X(t) = u(t-1)\frac{(t-1)^3}{6}$

↑ answer

2.) $F(s) = \frac{e^{-3s}}{s-3}$

$\mathcal{L}^{-1}(F(s)) = u(t-3)e^{3(t-3)}$

$X(t) = u(t-3)e^{3t-9}$

↑ answer

$g(s) = \frac{1}{s-3}$

$\mathcal{L}^{-1}(g(s))\ \ e^{3t}$

3.) $F(s) = \frac{e^{-2s}}{s^2+4}$

$\mathcal{L}^{-1}(F(s)) = u(t-2)\frac{1}{2}\sin 2(t-2)$

$g(s) = \frac{1}{s^2+4}$

$\frac{1}{2}\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right) = \frac{1}{2}(\sin 2t)$

$X(t) = u(t-2)\frac{\sin(2t-4)}{2}$

↑ answer

4.) $F(s) = \frac{se^{-s}}{s^2+9}$

$\mathcal{L}^{-1}(F(s)) = u(t-1)\cos 3(t-1)$

$g(s) = \frac{s}{s^2+9} = \cos 3t$

Answer: $u(t-1)\cos(3t-3)$

In Problems 5-7 find the Laplace transform of the given function.

5.) $f(t) = \begin{cases} 3 & 0 \le t \le 4 \\ 0 & t > 4 \end{cases}$

$f(t) = 3[u(t) - u(t-4)] + 0[\cancel{u(t-4)}]$

$f(t) = 3u(t) - 3u(t-4)$

$\mathcal{L}[f(t)] = \frac{3}{s} - 3e^{-4s}\mathcal{L}[1]$

$= \frac{3}{s} - e^{-4s}\left[\frac{1}{s}\right]$

6.) $f(t) = \begin{cases} t & 0 \le t \le 2\pi \\ 0 & t > 2\pi \end{cases}$

$f(t) = t[u(t) - u(t-2\pi)] + 0[\cancel{u(t-2\pi)}]$

$f(t) = tu(t) - tu(t-2\pi)$

$\mathcal{L}[f(t)] = e^{-s}\mathcal{L}[t] - e^{-s}\mathcal{L}[t-2\pi]$

$= e^{-s} \cdot \frac{1}{s^2} - e^{-s}\left[\frac{1}{s^2} - \frac{2\pi}{s}\right]$

7.) $f(t) = \begin{cases} \sin(2t) & 0 \le t \le 2\pi \\ 0 & t > 2\pi \end{cases} = f(t) = \sin(2t)[u(t) - u(t-2\pi)] + 0[\cancel{u(t-2\pi)}]$

$f(t) = \sin(2t)u(t) - \sin(2t)u(t-2\pi)$

$\mathcal{L}[f(t)] = e^{0} \cdot \frac{2}{s^2+2} - e^{-2\pi s} \cdot \frac{2}{s^2+2}$

In problems 8-10 Solve the initial Value Problems

↑ must be a typo because there is only 8-9 problems

8.) $\left\{ x'' + 9x = f(t), \text{ where } \begin{array}{ll} f(t) = 1 & 0 \le t < 2 \\ = 0 & t \ge 2 \end{array} \text{ and } x(0) = x'(0) = 0 \right\}$

$x = f(t)$

$\mathcal{L}[f(t)] = \underline{f(s)}$

$\mathcal{L}[f''(t)] = \underline{s^2 f(s) - s f(0) - f'(0)}$

$$[s^2 f(s) - s f(0) - f'(0)] + 9[f(s)] = \mathcal{L}[f(t)]$$

$$\mathcal{L}[f(t)] = \mathcal{L}[(1)u(t) - u(t-2) + 0(\cancel{u}(t-2))]$$

$$\mathcal{L}[u(t) - u(t-2)]$$

$$= \underline{\frac{1}{s} - e^{-2s}\frac{1}{s}}$$

$$s^2 f(s) + 9 f(s) = \frac{1}{s} - \frac{e^{-2s}}{s}$$

$$s^2 + 9(f(s)) = \frac{1}{s} - \frac{e^{-2s}}{s}$$

$$\mathcal{L}^{-1}\left[f(s) = \frac{1}{s(s^2+9)} - \frac{e^{-2s}}{s(s^2+9)}\right]$$

↓ continued

9.) $\left\{ x'' + 4x = f(t), \text{ where } \begin{array}{ll} f(t) = \sin t & 0 \le t \le 2\pi \\ = 0 & t > 2\pi \end{array} \text{ and } x(0) = x'(0) = 0 \right\}$

$$[s^2 f(s) - s f(0) - f'(0)] + 4[f(s)] = \mathcal{L}[f(t)]$$

$$\mathcal{L}[f(t)] = \mathcal{L}[\sin t(u(t) - u(t-2\pi)) + 0(u(\cancel{t}-2\pi))]$$

$$= e^{0}\cdot\frac{1}{s^2+1} - e^{2\pi s}\cdot\frac{1}{s^2+1} = \underline{\frac{1}{s^2+1} - \frac{e^{2\pi s}}{s^2+1}}$$

$$s^2 f(s) + 4 f(s) = \frac{1}{s^2+1} - \frac{e^{2\pi s}}{s^2+1}$$

$$(s^2+4) f(s) = \frac{1}{s^2+1} - \frac{e^{2\pi s}}{s^2+1}$$

$$\mathcal{L}^{-1}\left[f(s) = \frac{1}{s^2+1(s^2+4)} - \frac{e^{2\pi s}}{s^2+1(s^2+4)}\right]$$

↓ continued

In problems 8-10 Solve the initial Value Problems

must be a typo because there is only 8-9 problems

8.) Continued

$$\mathcal{L}^{-1}\left[f(s)=\frac{1}{s(s^2+9)}-\frac{e^{-2s}}{s(s^2+9)}\right]$$

$$\frac{1}{s(s^2+9)}=\frac{A}{s}+\frac{Bs+D}{s^2+9}$$

$$1=A(s^2+9)+Bs+D(s)$$

$$1=s^2A+9A+Bs^2+Ds$$

$$\frac{1}{9}=\frac{9A}{9} \quad \boxed{A=\frac{1}{9}} \quad 0=A+B$$

$$\boxed{B=-\frac{1}{9}} \quad \boxed{0=D}$$

$$f(t)=\frac{1}{9}\mathcal{L}^{-1}\left[\frac{1}{s}\right]-\frac{1}{9}\mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right]-\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s(s^2+9)}\right]$$

$$\boxed{\begin{aligned}f(t)=&\ \frac{1}{9}[u(t)]-\frac{1}{9}[\cos 3t]\\ &-u(t-2)\left[\frac{1}{9}[u(t-2)]-\frac{1}{9}[\cos 3(t-2)]\right]\end{aligned}}$$

Answer

9.) Continued

$$\mathcal{L}^{-1}\left[f(s)=\frac{1}{s^2+1(s^2+4)}-\frac{e^{2\pi s}}{s^2+1(s^2+4)}\right]$$

$$\frac{1}{s^2+1(s^2+4)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+4}$$

$$1=As+B(s^2+4)+Cs+D(s^2+1)$$

$$1=As^3+4As+Bs^2+4B+Cs^3+Cs+Ds^2+D$$

$$C:\ 1=4B+D$$

$$s^3:\ 0=A+C$$

$$s^2:\ 0=B+D$$

$$s:\ 0=4A+C$$

$$\frac{1}{3}=\frac{3B}{3} \quad \boxed{B=\frac{1}{3}} \quad \boxed{D=-\frac{1}{3}}$$

$$\boxed{\begin{aligned}A&=0\\ C&=0\end{aligned}}$$

$$f(t)=\frac{\frac{1}{3}}{s^2+1}+\frac{-\frac{1}{3}}{s^2+4}-\mathcal{L}^{-1}\left[\frac{e^{2\pi s}}{s^2+1(s^2+4)}\right]$$

$$\boxed{\begin{aligned}f(t)=&\ \frac{1}{3}[\sin t]-\frac{1}{6}[\sin 2t]\\ &-u(t+2\pi)\left[\frac{1}{3}[\sin t+2\pi]-\frac{1}{6}[\sin 2(t+2\pi)]\right]\end{aligned}}$$

Answer