Some Substitution Methods and Exact Equations- HW Problems

In problems 1-3 show the equation is exact and find the general solution to the differential equation.

1. $(2x^2 - 3y^2)dx + (y^2 - 6xy)dy = 0$
2. $(\cos(x) + \ln(y))dx + \left(\frac{x}{y} + e^y\right)dy = 0$
3. $\left(x^3 + \frac{y}{x}\right) + (y^2 + \ln(x))\frac{dy}{dx} = 0.$

In problems 4-6 find the general solution to the differential equation.

4. $xy' = y + 2\sqrt{xy}$
5. $(xy)y' - x^2 - y^2 = 0$
6. $x^2y' = xy + y^2$

In problems 7 and 8 find the general solution by reducing the order of the differential equation.

7. $y'' = 2(y')^2$
8. $x^2y'' + 4xy' = 1$

1)

Solve: $(2x^2-3y^2)dx + (y^2-6xy)dy = 0$

$$\underbrace{(2x^2-3y^2)}_{M(x,y)}dx + \underbrace{(y^2-6xy)}_{N(x,y)}dy = 0$$

Identify M & N (check if exact)

① partially differentiate your M in terms of y and N in terms of x, should be the same

Exact!!

$$\boxed{\frac{\partial M}{\partial y} = -6y \qquad \frac{\partial N}{\partial x} = -6y}$$

② Follow stipulation that there's a function with $f_x(x,y) = M(x,y)$, $f_y(x,y) = N(x,y)$

$$f_x(x,y) = 2x^2 - 3y^2$$

$$f_y(x,y) = y^2 - 6xy$$

$$\int f_x(x,y)\,dx = \int 2x^2 - 3y^2\,dx$$

③ then integrate the funtion with partial differential on both sides with respect to x.

$$f(x,y) = 2\frac{x^3}{3} - 3y^2(x) + h(y)$$

④ now you have a function containing $(h(y))$ & what you want to do is solve for that $h(y)$, you do this by partially diffrentiating the function you just integrated by y

$$f_y(x,y) = -6xy + \boxed{h'(y)}$$

⑤ then identify that your partial for y up top is equal to the one we discovered

$$f_y(x,y) = \boxed{y^2} - 6xy$$

$$h'(y) = y^2$$

⑥ then you integrate the $h(y)$ and the term it's = to, and sub answer to $f(x,y)$ & set = C

$$\int h'(y)\,dy = \int y^2\,dy$$

$$h(y) = \frac{y^3}{3}$$

$$\boxed{f(x,y) = 2\frac{x^3}{3} - 3y^2(x) + \left(\frac{y^3}{3}\right) = C}$$

General Solutution

2.) Show the equation is exact & find solution of differential Eq.

$$\underbrace{(\cos(x) + \ln(y))}_{M(x,y)}dx + \underbrace{\left(\frac{x}{y} + e^{y}\right)}_{N(x,y)}dy = 0$$

Exact!!

$$\left[\frac{\partial M}{\partial y} = \frac{1}{y} \qquad \frac{\partial N}{\partial x} = \frac{1}{y}\right]$$

$f_x(x,y) = \cos(x) + \ln(y)$

$f_y(x,y) = \frac{x}{y} + e^{y}$

$f(x,y) = \int \cos(x) + \ln(y)\,dx$

$f(x,y) = \sin(x) + h(y)$

$f_x(x,y) = \cos(x) + h'(y)$

$h'(y) = \ln(y)$

$\frac{d}{dy}(h'(y)) = \frac{d}{dy}(\ln(y))$

$h(y) = \frac{1}{y}$

general solution $\boxed{y = \cos(x) + \frac{1}{y}}$

3.) Show it's exact and find general solution

$$\underbrace{\left(x^3+\frac{y}{x}\right)}_{M(x,y)}+\underbrace{\left(y^2+\ln(x)\right)}_{N(x,y)}\frac{dy}{dx}=0$$

exact!!

$$\boxed{\frac{\partial M}{\partial y}=\frac{1}{x} \quad \frac{\partial N}{\partial x}=\frac{1}{x}}$$

$f_x(x,y)=x^3+\frac{y}{x}$

$f_y(x,y)=y^2+\ln(x)$

$f(x,y)=\int x^3+\frac{y}{x}\,dx$

$f(x,y)=\frac{x^4}{4}+y\ln x+h(y)$

$f_y(x,y)=\frac{\partial}{\partial y}\left(\frac{x^3}{3}+y\ln x+h(y)\right)$

$f_y(x,y)=\ln(x)+h'(y)$

$f_y(x,y)=y^2+\ln(x)$

$h'(y)=y^2$

$\int h'(y)\,dy=\int y^2\,dy$

$\boxed{h(y)=\frac{y^3}{3}}$

<u>general solution</u>

$$\boxed{y=\frac{x^4}{4}+y\ln x+\left(\frac{y^3}{3}\right)}$$

Find the general Solution to the differential equation.

7.)

$$xy' = y + 2\sqrt{xy}$$

$$\frac{x\frac{dy}{dx}}{x} = \frac{y + 2\sqrt{xy}}{x}$$

$$\frac{dy}{dx} = \left(\frac{y}{x}\right) + \frac{2\sqrt{xy}}{x}$$

redefine $\sqrt{x^2}$

$$\frac{dy}{dx} = \left(\frac{y}{x}\right) + \left(2\frac{\sqrt{xy}}{\sqrt{x^2}}\right)$$

$$= 2\sqrt{\frac{xy}{x^2}}$$

$$\frac{dy}{dx} = \left(\frac{y}{x}\right) + 2\sqrt{\frac{y}{x}}$$

$$\boxed{v = \frac{y}{x} \qquad \frac{dy}{dx} = \frac{dv}{dx}x + v \qquad y = vx}$$

$$\frac{dv}{dx}x + v = v + 2\sqrt{v}$$

$$-v \qquad -v$$

$$\left(\frac{dx}{x2\sqrt{v}}\right)\frac{dv}{dx}x = 2\sqrt{v}\left(\frac{dx}{2\sqrt{v}x}\right)$$

$$\int \frac{1}{2\sqrt{v}}\,dv = \int \frac{1}{x}\,dx$$

$$\frac{1}{2}\int v^{-\frac{1}{2}}\,dv = \ln|x|$$

$$v^{\frac{1}{2}} = \ln|x| + c$$

$$\left(\frac{y}{x}\right)^{\frac{1}{2}} = \ln|x| + c$$

$$\sqrt{\left(\frac{y}{x}\right)^{\frac{1}{2}}} = \pm\sqrt{\ln|x| + c}$$

$$(x)\ \frac{y}{x} = (x) \pm \sqrt{\ln|x| + c}$$

$$\boxed{y = x \pm \sqrt{\ln|x| + c}}$$

general solution

5) Solve and find general solution:

$(xy)y' - x^2 - y^2 = 0$

$+x^2 + y^2$

$$\frac{(xy)y'}{(xy)} = \frac{x^2+y^2}{(xy)}$$

$$y' = \frac{x^2+y^2}{(xy)}$$

$$\frac{dy}{dx} = \frac{x^2}{xy} + \frac{y^2}{xy}$$

$$\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}$$

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^{-1} + \frac{y}{x}$$

$v = \frac{y}{x}$ $\quad \frac{dy}{dx} = \frac{dv}{dx}x + v$

$y = vx$

$$\frac{dv}{dx}x + v = (v)^{-1} + v$$

$-v \qquad -v$

$$\left(\frac{\cancel{(dx)}(v)}{\cancel{x}}\right)\frac{dv}{\cancel{dx}}\cancel{x} = \frac{1}{\cancel{v}}\left(\frac{\cancel{(v)}(dx)}{x}\right)$$

$$\int v\,dv = \int \frac{1}{x}\,dx$$

$$\frac{v^2}{2} = \ln|x| + c$$

$$\frac{\left(\frac{y}{x}\right)^2}{2} = \ln|x| + c$$

$$\sqrt{\left(\frac{y}{x}\right)^2} = \pm\sqrt{2\ln|x| + 2c}$$

$$\frac{y}{x} = \pm\sqrt{2\ln|x| + 2c}$$

$$\boxed{y = x\pm\sqrt{2\ln|x| + 2c}}$$

6) Solve & find General Solution: $x^2 y' = xy + y^2$

$$\frac{\cancel{x^2} y'}{\cancel{x^2}} = \frac{xy + y^2}{x^2}$$

$$y' = \frac{xy}{x^2} + \frac{y^2}{x^2}$$

$$y' = \frac{y}{x} + \left(\frac{y}{x}\right)^2$$

$$\boxed{\begin{array}{ll} v = \frac{y}{x} & \frac{dy}{dx} = \frac{dv}{dx}x + v \\ y = vx & \end{array}}$$

$$\frac{dv}{dx}x + v = v + (v)^2$$

$$-v \qquad -v$$

$$\left(\frac{\cancel{dx}}{\cancel{x}\, v^2}\right)\frac{dv}{\cancel{dx}}\cancel{x} = \cancel{(v)^2}\left(\frac{dx}{x\,\cancel{v^2}}\right)$$

$$\int \frac{1}{v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\int v^{-2}\,dv = \ln|x| + c$$

$$-v^{-1} = \ln|x| + c$$

$$-\left(\frac{y}{x}\right)^{-1} = \ln|x| + c$$

$$-\ln|x| - c = \left(\frac{y}{x}\right)^{-1}$$

general solution

$$\boxed{y = \frac{x}{-\ln|x| - c}} \Leftarrow \frac{(y) - \ln|x| - c}{-\ln|x| - c} = \frac{\frac{x}{\cancel{y}}\cancel{(y)}}{-\ln|x| - c}$$

7.)

① Reducible Second Order differential Eq (Missing X)

$$y'' = 2(y')^2$$

② substitute differentials for p & $\frac{dp}{dx}$

$$\boxed{\begin{aligned} p &= y' = \frac{dy}{dx} \\ y'' &= \frac{dp}{dx} = p' \end{aligned}}$$

③ Now Dif Eq. in terms of p

$$\cancel{\left(\frac{dx}{2p^2}\right)} \frac{dp}{dx} = 2(p)^2 \left(\frac{dx}{2p^2}\right)$$

④ Now separate into 2 integrals & integrate in terms of p the other x

$$\Rightarrow \int \frac{1}{2p^2}\,dp = \int 1\,dx$$

$$\frac{1}{2}\int \frac{1}{p^2}\,dp = \int 1\,dx$$

⑤ = power rule

$$(2)\ \frac{1}{2}\left(\frac{p^{-1}}{-1}\right) = x + c^1\ (2)$$

⑥ = algabraic isolation

$$(-1)\left(\frac{p^{-1}}{-1}\right) = 2x + 2c^1\ (-1)$$

$$(p)\ \frac{1}{p} = -2x - 2c^1\ (p)$$

$$\frac{1}{(-2x-2c^1)} = \frac{(-2x-2c^1)(p)}{\cancel{(-2x-2c^1)}}$$

⑦ equation equal to P.

⑧ placed back $\frac{dy}{dx}$ to separate into integrals

$$p = \frac{1}{(-2x-2c^1)}$$

$$(dx)\ \frac{dy}{dx} = \frac{1}{(-2x-2c^1)}\ (dx)$$

$$\int 1\,dy = \int \frac{1}{-2x-2c^1}\,dx$$

⑨ U-sub =

$$y = -\frac{1}{2}\int \frac{1}{u}\,dx \qquad u = -2x - 2c^1, \quad \frac{du}{-2} = dx$$

$$\boxed{y = -\frac{1}{2}\ln|-2x - 2c^1| + c^2}$$

general solution

★ If there are not 2 constants you're wrong

8)

① Second Order Differential Eq (Missing Y)

$$x^2 y'' + 4x y' = 1$$

② Substitute differentials for p & $\frac{dp}{dx}$

$$\boxed{\boxed{p} = y' = \frac{dy}{dx} \qquad p' = y'' = \boxed{\frac{dp}{dx}}}$$

$-\frac{1}{x}$

③ Now Dif Eq in terms of x p

$$x^2 \frac{dp}{dx} + 4xp = 1$$

$$\left(\frac{1}{\cancel{x^2}p}\right) \cancel{x^2}\frac{dp}{dx} = 1 - 4xp \left(\frac{1}{(x^2)}\right)$$

$$\cancel{(dx)}\ \frac{1}{p}\frac{dp}{\cancel{dx}} = \frac{1-4x}{x^2}(dx)$$

$$\int \frac{1}{p}\,dp = \int \frac{1-4x}{x^2}\,dx$$

$$\int \frac{1}{p}\,dp = \int \frac{1}{x^2}\,dx - 4\int \frac{x}{x^2}\,dx$$

$$\frac{x^{-1}}{-1} - 4\int \frac{1}{x}\,dx$$

$$e^{(\ln|p|)} = e^{\left(\frac{x^{-1}}{-1} - 4\ln|x| + c'\right)}$$

$$p = e^{(-x^{-1} - 4\ln|x| + c')}$$

$$K = (e^{(c')})$$

$$p = Ke^{(-x^{-1} - 4\ln|x|)}$$

$$\cancel{(dx)}\ \frac{dy}{\cancel{dx}} = Ke^{(-x^{-1} - 4\ln|x|)}\,(dx)$$

$$\int 1\,dy = K\int e^{(-x^{-1} - 4\ln|x|)}\,dx$$

$$y = \frac{Ke^{(-x^{-1} - 4\ln|x|)}}{(x^{-2} - \frac{4}{x})} + c^{2}$$

Population Models- HW Problems

In problems 1 and 2 solve the logistic differential equation (without using the general formula).

1. $\frac{dx}{dt} = x(5 - x); \quad x(0) = 3$
2. $\frac{dx}{dt} = 20x - x^2; \quad x(0) = 2$

3. Suppose a population of mice, $P(t)$, has a birth rate of zero and a death rate proportional to $\frac{1}{\sqrt[3]{P}}$ (in deaths per month). If at time $t = 0$ the population of mice is 1000 and the population is 216 after 3 months, when will the population equal 0?

4. A population of squirrels, $P(t)$, sastisfies the logistic equation $\frac{dP}{dt} = aP - bP^2$. Suppose that at time $t = 0$ the population is 120, there are 4 births per month and 3 deaths per month. How many months does it take for the population to reach 95% of the limiting population?

Solve logistic differential equation (without using the general formula).

1.)

(Part I)

Solve $\frac{dx}{dt} = x(5-x)$; $x(0)=3$

$$\left(\frac{dt}{x(5-x)}\right)\frac{dx}{dt} = x(5-x)\left(\frac{dt}{x(5-x)}\right)$$

$$\int \frac{1}{x(5-x)}\,dx = \int 1\,dt$$

$$= t + c'$$

$$x(5-x)\cdot\left(\frac{1}{x(5-x)}\right) = \left(\frac{A}{x} + \frac{B}{5-x}\right)\cdot x(5-x)$$

$$1 = A(5-x) + Bx$$

Put $x=0$ | Put $x=5$

$\frac{1}{5} = \frac{5A}{5}$ | $\frac{1}{5} = \frac{5B}{5}$

$A = \left(\frac{1}{5}\right)$ | $B = \left(\frac{1}{5}\right)$

$$\int \frac{\left(\frac{1}{5}\right)}{x} + \frac{\left(\frac{1}{5}\right)}{5-x}\,dx$$

$$\frac{1}{5}\int \frac{1}{x}\,dx + \frac{1}{5}\int \frac{1}{5-x}\,dx$$

$$\frac{1}{5}\ln|x| + \frac{1}{5}\int \frac{1}{u(-1)}\,du \qquad u = 5-x\,dx$$

$$\frac{1}{5}\ln|x| - \frac{1}{5}\ln|5-x| + C \qquad \frac{du}{-1} = dx$$

(Part II)

$$\frac{1}{5}(\ln|x| - \ln|5-x|) = t + c$$

$$(5)\ \frac{1}{5}\ln\left|\frac{x}{5-x}\right| = t + c\ (5)$$

$$e^{\left(\ln\left|\frac{x}{5-x}\right|\right)} = e^{(5t+5c)}$$

$$\frac{x}{5-x} = e^{(5t+5c)}$$

$$= Ae^{(5t)}$$

Conditions

$\frac{x}{5-x}$

$t>0$

$x(0)=3$ (Part III)

$M>P>0$

$$\left|\frac{3}{5-3}\right| = Ae^{0} = A$$

$$\boxed{A = \frac{3}{2}}$$

$$\left|\frac{x}{5-x}\right| = \frac{3}{2}e^{(5t)}$$

$$(5-x)\left(\frac{x}{\cancel{5-x}}\right) = \frac{3}{2}e^{(5t)}(5-x)$$

$$x = \frac{3}{2}e^{(5t)}(5-x)$$

$$x = \left(\frac{3}{2}e^{(5t)}(5-x)\right)$$

$$x = \left(\frac{15}{2}e^{(5t)} - \frac{3x}{2}e^{(5t)}\right)$$

2) Solve the logistic differential equation (without using the general formula).

(Part I) $\frac{dx}{dt} = 20x - x^2$; $x(0) = 2$

① (Separated dx & dt, then factored out x from $\frac{1}{x(20-x)}$)

$$\left(\frac{\cancel{dt}}{20x-x^2}\right)\frac{dx}{\cancel{dt}} = 20x - x^2 \left(\frac{dt}{20x-x^2}\right)$$

$$\int \frac{1}{20x-x^2}\,dx = \int 1\,dt$$

$$\int \frac{1}{x(20-x)}\,dx = t + c$$

Conditions $\left\{ \frac{x}{20-x} > 0,\ t > 0 \right\}$

$$(x(20-x))\frac{1}{x(20-x)} = \frac{A}{x} + \frac{B}{(20-x)}\,(x(20-x))$$

② (partial fraction time!!) (Multiplied each side by LCD) (Sub values for x to solve for A & b)

$$1 = A(20-x) + Bx$$

put $x = 0$

$$\frac{1}{20} = \frac{20A}{20}$$

$$\boxed{A = \frac{1}{20}}$$

put $x = 20$

$$\frac{1}{20} = \frac{20B}{20}$$

$$\boxed{B = \frac{1}{20}}$$

(Part II)

$$\left(\frac{1}{20}\right)\int \frac{1}{x}\,dx + \left(\frac{1}{20}\right)\int \frac{1}{20-x}\,dx$$

③ Sub in our values to the partial fraction,
- integrate both sides w/ u-sub
- multiply by 20 both sides
- get rid of ln by making each side base e

$$\frac{1}{20}(\ln|x|) - \left(\frac{1}{20}\right)\int \frac{1}{u}\,du \qquad u = 20-x,\quad \frac{du}{-1} = dx$$

$$\frac{1}{20}(\ln|x| - \ln|20-x|) = t + c$$

$$(20)\frac{1}{20}\ln\left|\frac{x}{20-x}\right| = t + c\ (20)$$

$$\ln\left|\frac{x}{20-x}\right| = 20t + 20c$$

$$\left|\frac{x}{20-x}\right| = e^{(20t+20c)}$$

$$\left|\frac{x}{20-x}\right| = Ae^{(20t)}$$

(Part III)

4) Solve for Constant A

$x(0) = 2$

$$\left|\frac{2}{20-2}\right| = Ae^0 = A$$

$$A = \frac{2}{18} = \boxed{\frac{1}{9}}$$

pop capacity $M > P > 0$, $t > 0$

$$\left|\frac{x}{20-x}\right| = \frac{1}{9}e^{(20t)}\ (20-x)$$

$$x = \frac{1}{9}e^{(20t)}(20-x)$$

$$x = \left(\frac{20}{9}e^{(20t)} - \frac{1}{9}xe^{(20t)}\right)$$

⑤ interpret conditions, solve for x & we're done.

3. Suppose a population of mice, $P(t)$, has a birth rate of zero and a death rate proportional to $\frac{1}{\sqrt[3]{P}}$ (in deaths per month). If at time $t=0$ the population of mice is 1000 and the population is 216 after 3 months, when will the population equal 0?

(Part I)

(first I found the birth & death rate) ① Beta $\beta = 0$ ← Birth rate; delta $\delta = b\left(\frac{1}{\sqrt[3]{p}}\right)$ ← Death Rate

(then, sub β & δ into the formula $\frac{dp}{dt}$) ② $\frac{dp}{dt} = (\beta - \delta)p = \left(0 - \frac{b}{\sqrt[3]{p}}\right)p$

(Simplify my base & exponents) ③ $\frac{dp}{dt} = (-b)\frac{p}{\sqrt[3]{p}} = (-b)\,p^{1-\frac{1}{3}} = (-b)\,p^{\frac{2}{3}}$

Then, I start to separate variables ④ $\left(\frac{1}{\sqrt[3]{p^2}}\right)\frac{dp}{dt} = (-b)\sqrt[3]{p^2}\left(\frac{1}{\sqrt[3]{p^2}}\right)$

$(dt)\,\frac{1}{\sqrt[3]{p^2}}\frac{dp}{dt} = -b\,(dt)$

I'm left with an equation that can be integrated ⑤ $\int \frac{1}{\sqrt[3]{p^2}}\,dp = \int -b\,dt$

$\int p^{-\frac{2}{3}}\,dp = (-b)t + C$

Then, I divide both integrated sides by 3 & cube both sides ⑥ $\frac{3(\sqrt[3]{p})}{3} = \frac{(-b)t + C}{3}$

$(\sqrt[3]{p})^3 = \left(\frac{(-b)t}{3} + \frac{C}{3}\right)^3$

⑦ $p = \left(\frac{(-b)t}{3} + \frac{C}{3}\right)^3$

$k = -b$

Following that I substitute $-b$ for k, which is the rate for my function p: $p = \left(\frac{1}{3}kt + \frac{C}{3}\right)^3$

⑧ (Part II) (use value of population at time 0)

Find constant C

$1000 = p(0) = \left(\frac{C}{3}\right)^3$

$\sqrt[3]{1000} = \frac{C}{3}$

$(3)\,10 = \frac{C}{3}(3)$

$\boxed{C = 30}$

⑨ (use population after 3 months)

Solve for k

$216 = \left(\frac{1}{3}k(3) + 10\right)^3$

$\sqrt[3]{216} = \sqrt[3]{(k+10)^3}$

$6 = k + 10$
$-10 \qquad -10$

$\boxed{-4 = k}$

(Part III) 10

(Finally I bring everything together) I plack in all the constants for the general solution found previously & set it $=0$ to zero. Applied Algebra & Done!

$p(t) = \left(\frac{1}{3}(-4)t + 10\right)^3$

$\sqrt[3]{0} = \sqrt[3]{\left(\frac{1}{3}(-4)t + 10\right)^3}$

$0 = \left(-\frac{4}{3}t\right) + 10$
$-10 \qquad -10$

$\left(-\frac{3}{4}\right) - 10 = \left(-\frac{4}{3}t\right)\left(-\frac{3}{4}\right)$

$7.5 = t$

(So in 7.5 months the population of mice is = to zero.)

4. A population of squirrels, $P(t)$, sastisfies the logistic equation $\frac{dP}{dt} = aP - bP^2$. Suppose that at time $t = 0$ the population is 120, there are 4 births per month and 3 deaths per month. How many months does it take for the population to reach 95% of the limiting population?

$P_0 = 120$

initial birth rate $= 4 = aP_0 = 120a \rightarrow a = \left(\frac{4}{120}\right) = \left(\frac{1}{30}\right)$

initial Death rate $= 3 = bP_0^2 = 120^2 b \rightarrow b = \left(\frac{3}{14400}\right) = \frac{1}{4800}$

$$\frac{dP}{dt} = aP - bP^2 = bP\left(\frac{a}{b} - P\right);$$

Let $K = b = \frac{1}{4800}$, $M = \frac{a}{b} = \frac{\left(\frac{1}{30}\right)}{\left(\frac{1}{4800}\right)} = 160$

$$P(t) = \left(\frac{MP_0}{P_0 + (M - P_0)e^{-KMt}}\right)$$

$$.95(M) = \frac{MP_0}{P_0 + (M-P_0)e^{-KMt}}$$

$$.95 = \frac{P_0}{P_0 + (M-P_0)e^{-KMt}} = \frac{120}{120 + ((160) - 120)e^{-\left(\frac{1}{4800}\right)(160)t}}$$

$$.95 = \frac{120}{120 - 40e^{-\frac{4}{120}t}}$$

$$(.95)\left(120 - 40e^{-\frac{4}{120}t}\right) = 120$$

$$114 - 40e^{-\frac{4}{120}t} = 120$$

$$-114 \qquad -114$$

$$-\left(\frac{1}{40}\right) - 40e^{-\frac{4}{120}t} = 6 - \left(\frac{1}{40}\right)$$

$$t = \ln\left(-\frac{3}{20}\right) \times -\frac{120}{4} \Leftarrow \left(-\frac{120}{4}\right) \times -\frac{4}{120}t = \ln\left|\frac{3}{20}\right| \Leftarrow \times\left(-\frac{120}{4}\right) \ln\left(e^{-\frac{4}{120}t}\right) = \ln\left(-\frac{3}{20}\right)$$

$\boxed{t \approx 56.914}$

Velocity and Acceleration Models- HW Problems

1. A car's acceleration is proportional to the difference between 120mph and the car's velocity. It takes the car 10 seconds to accelerate from 0mph to 60mph. How long will it take the car to accelerate to 100mph?

2. A car is travelling at 50ft/sec and the engine shuts off. After 10 seconds the car is travelling at 25ft/sec. Suppose that the resistance is proportional to the velocity so that $\frac{dv}{dt} = -\rho v, \quad \rho > 0$. Find the velocity of the car, $v(t)$, and the distance travelled, $x(t)$, t seconds after the engine shuts off.

3. Solve problem number 2 where $\frac{dv}{dt} = -\rho v^2, \quad \rho > 0$.

(1)

Goal: use velocity function to find acceleration function and solve for time when $a(t) = 100$

part III

part I

$$\left(\frac{dt}{(120-v)}\right)\frac{dv}{dt} = k(120-v)\left(\frac{dt}{(120-v)}\right)$$

part II

$$\boxed{\frac{\ln|2|}{10} = K}$$

$\ln|2| = 10K$

$2 = e^{K \cdot 10}$

$\frac{1}{.5} = e^{K \cdot 10}$

$1 = 60\left(\left(\frac{1}{120}\right)e^{K10}\right)$

$\left(\frac{-\left(\frac{1}{120}\right)e^{Kt}}{1}\right)\frac{1}{\left(\frac{1}{120}\right)e^{Kt}} = -60\left(\frac{-\left(\frac{1}{120}\right)e^{K10}}{1}\right)$

$V(10) = -\frac{1}{\left(\frac{1}{120}\right)e^{Kt}} + 120 = 60$

$$\boxed{A = \frac{1}{120}}$$

$\frac{1}{120} = \frac{120A}{120}$

$\int \frac{1}{(120-v)}dv = \int k\,dt$

$-\int \frac{1}{u}du$

$e^{(-\ln|120-v|)} = e^{(Kt+C)}$

$(120-V)\frac{1}{120-V} = Ae^{Kt}(120-V)$

$\frac{1}{Ae^{Kt}} = \frac{Ae^{Kt}(120-V)}{Ae^{Kt}}$

$\frac{1}{Ae^{Kt}} = 120 - V$
$-120 \qquad -120$

$(-1)\frac{1}{Ae^{Kt}} - 120 = -V(-1)$

$-\frac{1}{Ae^{Kt}} + 120 = V(t)$

$\left(-\frac{1}{A} + 120\right) = V(0)$
$-120 \qquad -120$

$(A)\frac{1}{A} = 120(A) \qquad = -\frac{1}{A} = -120$

$V(t) = -\frac{1}{\left(\frac{1}{120}\right)e^{\left(\frac{\ln|2|}{10}\right)t}} + 120$

$\frac{dv}{dt} = \frac{d}{dt}\left(-\frac{1}{\left(\frac{1}{120}\right)e^{\left(\frac{\ln|2|}{10}\right)t}} + 120\right)$

$= -\frac{d}{dt}\left(\frac{1}{\left(\frac{1}{120}\right)e^{\left(\frac{\ln|2|}{10}\right)t}}\right)$

$= -120\frac{d}{dt}\left(\frac{1}{e^{\left(\frac{\ln|2|}{10}\right)t}}\right)$

$a(t) = 120\left(\frac{1}{\left(e^{\left(\frac{\ln|2|}{10}\right)t}\right)^2}\right)$ *

$e^{\frac{\ln|2|}{10}t}$ *

$\frac{\ln|2|}{10}$

$$\ln\left(e^{\frac{\ln(2)}{10}t}\right) = \ln\left(\frac{120\left(\frac{\ln(2)}{10}\right)}{100}\right)$$

$$\frac{\frac{\ln(2)}{10}t}{\frac{\ln(2)}{10}} = \boxed{\frac{\ln\left(\frac{120\left(\frac{\ln(2)}{10}\right)}{100}\right)}{\frac{\ln(2)}{10}}} \qquad \boxed{t>0}$$

$t \approx 35.8$ seconds

$$\frac{100 = \cancel{120}\left(\frac{1}{\left(e^{\frac{\ln(2)}{10}t}\right)^2}\right) * e^{\frac{\ln(2)}{10}t} * \cancel{\frac{\ln(2)}{10}}}{120\left(\frac{\ln(2)}{10}\right)}$$

$$\left(\frac{100}{120\left(\frac{\ln(2)}{10}\right)} = \frac{e^{\frac{\ln(2)}{10}t}}{\left(e^{\frac{\ln(2)}{10}t}\right)^2}\right) \qquad \left(e^{\frac{\ln(2)}{10}t}\right)\frac{100}{120\left(\frac{\ln(2)}{10}\right)} = \frac{1}{\cancel{e^{\frac{\ln(2)}{10}t}}}\left(e^{\frac{\ln(2)}{10}t}\right)$$

$$e^{\frac{\ln(2)}{10}t} = \frac{120\left(\frac{\ln(2)}{10}\right)}{100}$$

(2)

$$\frac{dv}{dt} = -\rho v$$

$$\left(\frac{dt}{v}\right)\frac{dv}{dt} = -\rho v\left(\frac{dt}{v}\right)$$

$$\int \frac{1}{v}\,dv = \int -\rho\,dt$$

$$\ln|v| = -\rho t + C$$

$$e^{(\ln|v|)} = e^{(-\rho t + C)}$$

$$V = Ae^{(-kt)}$$

Remember: $V(0) = 50$ & $V(10) = 25$

$$50 = Ae^{(-\rho \cdot 0)}$$

$$\boxed{50 = A}$$

$$25 = 50e^{(-\rho \cdot 10)}$$

$$\frac{25}{50} = e^{(-\rho \cdot 10)}$$

$$\frac{\ln\left|\frac{25}{50}\right|}{-10} = \frac{-10\rho}{-10}$$

$$\boxed{\frac{\ln\left(\frac{50}{25}\right)}{10} = \rho}$$

$$V(t) = 50e^{\left(-\left(\frac{\ln\left(\frac{50}{25}\right)}{10}\right)t\right)}$$

Velocity is 25 ft/sec 10 seconds after braking

$V(10) = \underline{25 \text{ ft/sec}}$

$X(10) = \underline{360.67 \text{ ft}}$

(total distance traveled is 360.67 ft traveled after braking)

$$\frac{dx}{dt} = v$$

$$(dt)\frac{dx}{dt} = v\,(dt)$$

$$\int_0^x 1\,dx = \int_0^t 50e^{-\rho t}\,dt$$

$u = -\rho t$

$$x = 50\int_0^t \frac{e^u}{-\rho}\,du \qquad \frac{du}{-\rho} = dt$$

$$x(t) = \frac{50}{-\rho}\left(e^{-\rho t}\right)\Big|_0^t$$

$$\Leftarrow$$

$$x(t) = \frac{50}{-\rho}\left(e^{-\rho t} - 1\right)$$

$$\Uparrow$$

$$x(t) = \left(\frac{50}{-\rho}\right)e^{-\rho t} + \left(\frac{50}{\rho}\right)$$

$$\Uparrow$$

$$x(t) = \frac{50}{-\left(\frac{\ln\left(\frac{50}{25}\right)}{10}\right)}\left(e^{-\left(\frac{\ln\left(\frac{50}{25}\right)}{10}\right)t} - 1\right)$$

(3)

$$\left(\frac{dt}{v^2}\right)\frac{dv}{dt} = -p v^2 \left(\frac{dt}{v^2}\right)$$

$$\int \frac{1}{v^2}\,dv = \int -p\,dt$$

$$-v^{-1} = -pt + c$$

$$(-1)\; -\frac{1}{v} = -pt + c\;(-1)$$

$$(v)\;\frac{1}{v} = pt - c\;(v)$$

$$\frac{1}{pt - c} = \frac{pt - c\,(v)}{pt - c}$$

$$v(t) = \frac{1}{pt - c}$$

$v(0) = 50$, $v(10) = 25$

$$(-c)\,50 = \frac{1}{\cancel{-c}}\,(-c)$$

$$\frac{-c\;50}{50} = \frac{1}{50}$$

$$(-1)\; -c = \frac{1}{50}\,(-1)$$

$$\boxed{c = \frac{1}{50}}$$

$$\left(10p - \frac{1}{50}\right)25 = \frac{1}{\cancel{(10)p - \frac{1}{50}}}\;\cancel{\left(10p - \frac{1}{50}\right)}$$

$$\frac{\left(10p - \frac{1}{50}\right)25}{25} = \frac{1}{25}$$

$$10p - \frac{1}{50} = \frac{1}{25}$$

$$+\frac{1}{50} \qquad +\frac{1}{50}$$

$$10p = \frac{3}{50}$$

$$\boxed{p = \frac{3}{500}}$$

$$v(t) = \frac{1}{\left(\frac{3}{500}\right)t - \frac{1}{50}}$$

$$\int \frac{dx}{dt}\,dt = \int \frac{1}{\left(\frac{3}{500}\right)t - \frac{1}{50}}\,dt$$

$$\int \frac{50}{\left(\left(\frac{3}{10}\right)t\right) - 1}\,dt$$

$$50\int \frac{1}{\left(\frac{3}{10}t\right) - 1}\,dt$$

$u = \frac{3}{10}t - 1$

$\frac{du}{\frac{3}{10}} = dt$

$$\frac{500}{3}\ln\left|\frac{3}{10}t - 1\right| + C$$

$$x(t) = \frac{500}{3}\ln\left|\frac{3}{10}t - 1\right| + 50$$

$v(10) = 25$ ft/sec

$x(10) = 165$ ft

Answers