In problems 1-4 Calculate $f(t)*g(t)$.

1.) $f(t)=1$ $g(t)=e^t$

$$=\int_0^t f(w)g(t-w)\,dw$$

↑
Assuming they're two piecewise functions

$$\int_0^t e^{t-w}\,dw$$

$u=t-w$

$\frac{du}{-1}=\frac{-1dw}{-1}$

$$=\int_0^t e^u\,dw$$

$$=-\int_0^t e^u\,du$$

$$=-(e^{t-w})\Big|_0^t$$

$$=-e^{t-t}-(-e^{t-0})$$

$$=\boxed{-1+e^t}$$ ← Answer

2.) $f(t)=t$, $g(t)=\sin(t)$

$$\int_0^t \underbrace{w}_{u}\,\underbrace{\sin(t-w)\,dw}_{dv}$$

$du=dw$ $v=-\left(\frac{\cos(t-w)}{-1}\right)$

$v=\cos(t-w)$

$$w(\cos(t-w))\Big|_0^t-\int_0^t \cos(t-w)\,dw$$

$-\sin(t-w)\Big|_0^t$

Answer

$$t(\cos(0))-0)-[0--\sin(t)]=\boxed{t-\sin t}$$

In problems 1-4 Calculate $f(t)*g(t)$.

3.) $f(t)=t^2,\ g(t)=e^t$

$\int_0^t w^2 e^{t-w}\,dw$

$e^t\int \underbrace{w^2}_{u}\underbrace{e^{-w}dw}_{dv}$ $v=-e^{-w}$

$du=2w\,dw$

$-w^2e^{-w}\Big|_0^t-\int_0^t -e^{-w}2w\,dw$

$\underline{-t^2e^{-t}}$

$+2\int_0^t \underbrace{e^{-w}}_{dv}\underbrace{w}_{u}dw$ $v=-e^{-w}$

$du=dw$

$-we^{-w}\Big|_0^t-\int_0^t -e^{-w}dw$

$-we^{-w}\Big|_0^t+\int_0^t e^{-w}dw$

$\uparrow$

$\underline{-2te^{-t}}$ $2(-e^{-w})\Big|_0^t$

$\downarrow$

$\underline{-2e^{-t}+2}$

Answer: $e^t\left[-t^2e^{-t}-2te^{-t}-2e^{-t}+2\right]$

4.) $f(t)=t,\ g(t)=t$

$\int_0^t w(t-w)\,dw$

$\int_0^t wt-w^2\,dw$

$t\int_0^t w\,dw-\int_0^t w^2\,dw$

$t\left(\frac{w^2}{2}\right)\Big|_0^t-\frac{w^3}{3}\Big|_0^t$

$t\left(\frac{t^2}{2}\right)-\frac{t^3}{3}$ ← Answer

5a) Show that if $f(t)=e^{at}$ & $g(t)=e^{bt}$, where a, b are constants, then $(f\cdot g)(t)=\frac{1}{a-b}(e^{at}-e^{bt})$.

$$\int_0^t e^{aw}e^{b(t-w)}\,dw$$

$$\int_0^t e^{aw}e^{bt-bw}\,dw$$

$$e^{bt}\int_0^t e^{aw}e^{-bw}\,dw$$

$$e^{bt}\int_0^t e^{aw-bw}\,dw$$

$$e^{bt}\left[\frac{1}{a-b}\left(e^{at-bt}-e^{0}\right)\right]$$

$$=\frac{1}{a-b}\left(e^{at}-e^{bt}\right)$$

5b) By direct calculation of laplace transforms show that $\mathcal{L}((f\cdot g)(t))=(\mathcal{L}(f(t))\mathcal{L}(g(t)))$ for $f(t)=e^{at}$ & $g(t)=e^{bt}$ (where $\mathcal{L}$ is the laplace transform).

Exp 1:

$$\mathcal{L}(e^{at})\mathcal{L}(e^{bt})=\boxed{\left(\frac{1}{s-a}\right)\left(\frac{1}{s-b}\right)}$$

Exp 2:

$$\mathcal{L}((f\cdot g)(t))=\frac{1}{a-b}\mathcal{L}\left[e^{at}-e^{bt}\right]$$

$$\frac{1}{a-b}(e^{at}-e^{bt})$$

$$\boxed{\frac{1}{a-b}\left(\frac{1}{s-a}-\frac{1}{s-b}\right)}$$

Let's prove their Equal by simplifying both Expressions

Exp 2: $$\left(\frac{1}{a-b}\right)\frac{\cancel{s}-b-\cancel{s}+a}{(s-a)(s-b)}=\frac{\frac{-b+a}{a-b}}{(s-a)(s-b)}=\frac{1}{(s-a)(s-b)}$$

Exp 1: $$\left(\frac{1}{s-a}\right)\left(\frac{1}{s-b}\right)=\frac{1}{(s-a)(s-b)}$$

Therefore, both Exp 1 & 2 are =.

In problems 6-8 use the convolution theorem to find the inverse Laplace transform of the given function.

6.) $F(s) = \frac{3}{s^2-1}$

$= \frac{3}{(s+1)(s-1)}$

$3\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] * \mathcal{L}^{-1}\left[\frac{1}{s-1}\right]$

$\underset{f(t)}{3e^{-t}} * \underset{g(t)}{e^{t}}$

$3\int_0^t e^{-w} e^{t-w}\,dw$

$3\int_0^t e^{t-2w}\,dw$

$-\frac{3}{2}\left(e^{t-2w}\right)\Big|_0^t$

$= \underline{-\frac{3}{2}e^{-t} + \frac{3}{2}e^{t}}$

7.) $F(s) = \frac{1}{(s-1)^2}$

$\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] * \mathcal{L}^{-1}\left[\frac{1}{s-1}\right]$

$\underset{f(t)}{e^{1t}} * \underset{g(t)}{e^{1t}}$

$\int_0^t e^{w} e^{t-w}\,dw$

$e^t\int_0^t 1\,dw$

$e^t[w]\Big|_0^t$

$= \underline{e^t[t]}$

8.) $F(s) = \frac{4}{s(s^2+4)}$

$\mathcal{L}^{-1}\left[\frac{2}{s}\right] * \mathcal{L}^{-1}\left[\frac{2}{(s^2+4)}\right]$

$\underset{f(t)}{2} * \underset{g(t)}{\sin 2t}$

$\int_0^t 2\sin 2(t-w)\,dw$

$2\int_0^t \sin 2t-2w\,dw$

$\cos 2t-2w\Big|_0^t$

$\underline{\underline{(1-\cos 2t)}}$

In problems 9-11 find the Laplace transform of the given function.

11.) $f(t) = \dfrac{e^t - 1}{t}$

use $\mathcal{L}\left[\dfrac{f(t)}{t}\right] = \int_s^\infty F(u)\,du$

$\mathcal{L}[e^t] - \mathcal{L}[1]$

$\left[\dfrac{1}{s-1} - \dfrac{1}{s}\right] = \dfrac{s}{(s-1)s} - \dfrac{(s-1)}{s(s-1)} = \dfrac{\cancel{s} - \cancel{s} + 1}{s(s-1)} = \dfrac{1}{s(s-1)}$

$\int_s^\infty \dfrac{1}{u(u-1)}\,du$ $= \dfrac{A}{u} + \dfrac{B}{(u-1)}$

$\int_s^\infty \dfrac{-1}{u} + \dfrac{1}{(u-1)}\,du$ $\quad 1 = A(u-1) + B(u)$

$1 = Au - A + Bu$

$-\ln(u) + \ln(u-1)\Big|_s^\infty$ $\quad 1 = -A \quad \boxed{B = 1}$

$0 - (-\ln(s) + \ln(s-1))$ $\quad 0 = A + B \quad \boxed{A = -1}$

$-A = B$

Answer: $\underline{\underline{\ln(s) - \ln(s-1)}}$

In problems 9-11 find the Laplace transform of the given function.

9.) $f(t) = t\cos(3t)$

$$-\frac{d}{ds}\left(\frac{s}{s^2+9}\right)$$

$$= -\frac{(s^2+9)*1 - s(2s)}{(s^2+9)^2}$$

$$= -\frac{-(s^2+9)}{(s^2+9)^2}$$

$$= \boxed{\frac{s^2-9}{(s^2+9)^2}}$$

10.) $f(t) = t^2\sin(3t)$

$$(-1)^2 \cdot F^{(2)}(s)$$

$$F(s) = \frac{3}{(s^2+9)}$$

$$F^{(1)}(s) = \frac{(0)(s^2+9) - (3)(2s)}{(s^2+9)^2}$$

$$F^{(1)}(s) = \frac{-6s}{(s^2+9)^2}$$

$$F^{(2)}(s) = \frac{(-6((s^2+9)^2) - (-6s)(2(s^2+9)\cdot 2s)}{(s^2+9)^4}$$

$$= \boxed{\frac{-6(s^2+9)^2 + 12s(s^2+9)\cdot 2s}{(s^2+9)^4}}$$

In problems 12-14 find the inverse Laplace transform function.

$$\mathcal{L}^{-1}[F(s)] = -\frac{1}{t}\mathcal{L}^{-1}[F'(s)]$$

12.) $F(s) = \ln\left(\frac{s+3}{s-3}\right)$

$$\frac{d}{ds}(\ln(s+3)) - \frac{d}{ds}(\ln(s-3))$$

$$-\frac{1}{t}\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) + \frac{1}{t}\mathcal{L}^{-1}\left(\frac{1}{s-3}\right)$$

$$x(t) = -\frac{1}{t}\left[e^{-3t} - e^{3t}\right]$$

13.) $F(s) = \ln\left(\frac{s^2+4}{s+4}\right)$

$$\frac{d}{ds}(\ln(s^2+4) - \frac{d}{ds}(\ln(s+4))$$

$$-\frac{1}{t}\mathcal{L}^{-1}\left(\frac{2s}{s^2+4}\right) + \frac{1}{t}\mathcal{L}^{-1}\left(\frac{1}{s+4}\right)$$

$$x(t) = -\frac{2}{t}\left[\cos 2t + e^{-4t}\right]$$

$2\mathcal{L}$

15.) Use the Laplace transform to transform $ty'' + (t-2)y' + y = 0$ to find a solution where $y(0)=0$ but $y(t) \not\equiv 0$.

$Y = f(t)$

{We want to find our non-trivial solution}

$$\mathcal{L}[ty'' + (t-2)y' + y] = \mathcal{L}[0]$$

$$\mathcal{L}[ty''] + \mathcal{L}[ty'] - 2\mathcal{L}[y'] + \mathcal{L}[y] = 0$$

$$\mathcal{L}[y''] = s^2 f(s) - f(0) - f'(0) = \underline{s^2 f(s) - f'(0)}$$

Apply $\rightarrow$

$$\mathcal{L}[ty''] = -\underline{(2sf(s) + s^2 f'(s))} \leftarrow \text{product rule}$$

$$\mathcal{L}[t^n f(t)] = -1^n F^{(n)}(s)$$

$$\mathcal{L}[y'] = sf(s) - f(0)$$

$$\mathcal{L}[ty'] = -\underline{(f(s) + sf'(s))}$$

$$[-2sf(s) - s^2 f'(s)] + [-f(s) - sf'(s)] - 2[sf(s) - 0] + f(s) = 0$$

$$f(s)[-2s - 1 - 2s + 1] + f'(s)[-s^2 - s] = 0$$

$$f(s)[-4s] + f'(s)[-s^2 - s] = 0$$

$$\left(\frac{1}{s^2-s}\right)\frac{f(s)[-4s]}{f(s)} = \frac{f'(s)[s^2-s]}{f(s)}\left(\frac{1}{s^2-s}\right)$$

$$-\frac{4s}{s^2-s} = \frac{f'(s)}{f(s)}$$

$$\frac{-4s}{s(s-1)} = \frac{-4}{s-1}$$

$$\int \frac{f'(s)}{f(s)}\,ds = -4\int \frac{1}{s-1}\,ds$$

$$\ln|f(s)| = -4\ln|s-1| + c$$

$$f(s) = e^{-4\ln|s-1| + c}$$

$$e^{-4\ln|s-1| + c} = |s-1|^{-4} + c$$

using $\underline{\frac{n!}{(s-a)^{n+1}} = e^{at}t^n}$ $\quad \mathcal{L}^{-1}\left[\frac{c'}{(s-1)^4}\right]$

$$\boxed{X(t) = c'' t^3 e^t} \leftarrow \underline{\text{Answer}}$$

In questions 12-14 find the inverse laplace transform of the given functions.

14.) $F(s) = \dfrac{2s}{(s^2+1)^2}$

$$\mathcal{L}^{-1}\left(\frac{2}{s^2+1}\right) * \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right)$$

$$2\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) * \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right)$$

$$2\sin(t) * \cos(t)$$

use trig formula

$$2\int_0^t \sin(w)\cos(t-w)\,dw$$

$$2\int_0^t \sin(w)[\cos t\cos w + \sin t\sin w]\,dw$$

$$2\int_0^t \sin(w)\cos(t)\cos(w)\,dw + 2\int_0^t \sin^2 w\,\sin(t)\,dw$$

$$2\cos(t)\int_0^t \sin(w)\cos(w)\,dw + 2\sin(t)\int_0^t \sin^2 w\,dw$$

use another trig identity (half Angle identity)

$$+2\sin(t)\int_0^t \frac{1-\cos 2w}{2}\,dw$$

$$\underbrace{2\cos(t)\int_0^t \sin(w)\cos(w)\,dw}_{1} + \underbrace{2\sin(t)\int_0^t \frac{1}{2}\,dw}_{2} - \underbrace{2\sin(t)\int_0^t \cos 2w\,dw}_{3}$$

$$2\cos(t)\int_0^t \sin(w)\cos(w)\,dw$$

$u = \sin w$

$\dfrac{du}{\cos w} = dw$

$$2\cos(t)\int_0^t \frac{u\cancel{\cos(w)}\,du}{\cancel{\cos(w)}}$$

$$2\cos(t)\int_0^t u\,du$$

$$\frac{u^2}{2}$$

$$2\cos(t)\left(\frac{\sin^2 w}{2}\right)\Big|_0^t = \underline{\cos(t)\sin^2(t)}$$

$$2\sin(t)\int_0^t \frac{1}{2}\,dw$$

$$\sin(t)[w]\Big|_0^t = \underline{\sin(t)(t)}$$

$$-2\sin(t)\int_0^t \cos 2w\,dw$$

$$-2\sin(t)\left[\frac{\sin 2w}{2}\right]\Big|_0^t$$

$$-\sin(t)[\sin 2w]\Big|_0^t$$

$$= \underline{-\sin(t)\sin(2t)}$$

Answer:

$$X(t) = \cos(t)\sin^2(t) + \sin(t)(t) - \sin(t)\sin(2t)$$