Second Order Linear Differential Equations- HW Problems

1. Verify that $y_1 = e^x$ and $y_2 = xe^x$ are solutions of $y'' - 2y' + y = 0$. Show that y_1 and y_2 are linearly independent and find a particular solution to the differential equation, y, such that $y(0) = 2$ and $y'(0) = 5$.

In problem 2 and 3 determine if the functions are linearly independent.

2. $f(x) = e^x \cos(x)$, $\quad g(x) = e^x \sin(x)$
3. $f(x) = 1 + \cos(2x)$, $\quad g(x) = \cos^2(x)$.

In problems 4-7 find the general solution to the given differential equation.

4. $y'' - 4y = 0$
5. $y'' + y' - 2y = 0$
6. $y'' - 8y' + 16y = 0$
7. $4y'' - 12y' + 9y = 0$.

In problems 8 and 9 find the particular solutions to the initial value problems.

8. $y'' - y' - 6y = 0;$ $\quad y(0) = 6,\ \ y'(0) = 8$
9. $y'' - 6y' + 9y = 0;$ $\quad y(0) = 4,\ \ y'(0) = 9$.

Christian 9 Barden

1.) Verify $y_1 = e^x$ & $y_2 = xe^x$ are solutions of $y'' - 2y' + y = 0$. Show y_1 & y_2 are linearly independent and find the particular solution to the differential equation, y, such that $y(0) = 2$ & $y'(0) = 5$.

a) $y_1 = e^x$ $\quad$ $y_2 = xe^x$

$y_1' = e^x$ $\quad$ $y_2' = xe^x + e^x$

$y_1'' = e^x$ $\quad$ $y_2'' = (xe^x + e^x) + e^x = xe^x + 2e^x$

$e^x = xe^x + 2e^x$

{dif Eq = 0, gives you zero when substituted, so they are solution}

$y_1'' - 2y_1' + y_1 = e^x - 2(e^x) + e^x$

$= 2e^x - 2e^x = \boxed{0}$

$y_2'' - 2y_2' + y_2 = (xe^x + 2e^x) - 2(xe^x + e^x) + (xe^x)$

$= (2xe^x + 2e^x) - (2xe^x + 2e^x) = \boxed{0}$

{W(y1, y2) ≠ 0, Therefore its independent}

b) $W(y_1, y_2) = \begin{bmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{bmatrix} = \cancel{x(e^x)^2} + (e^x)^2 - \cancel{x(e^x)^2}$

$W(y_1, y_2) = (e^x)^2$

e) $y = c_1 y_1 + c_2 y_2 = c_1 e^x + c_2(xe^x)$

$2 = y(0) = c_1 e^0 + c_2(0)(e^0)$

$\boxed{2 = c_1}$

$5 = y'(0) = c_1(e^0) + c_2((0)e^{(0)} + e^0)$

note: $c_2(0+1)$

$5 = c_1 + c_2(1)$

$5 = c_1 + c_2$

$5 = 2 + c_2$

$-2 \quad -2$

$\boxed{3 = c_2}$

{Determine if the functions are linearly independent}

2)

$f(x) = e^x \cos x$
$g(x) = e^x \sin x$

$$W(f,g) = \begin{bmatrix} e^x \cos x & e^x \sin x \\ -e^x \sin x + e^x \cos x & (e^x \cos x + e^x \sin x) \end{bmatrix}$$

$$(e^x \cos x)(e^x \cos x + e^x \sin x) - ((-e^x \sin x + e^x \cos x)(e^x \sin x))$$

$$(e^x \cos x)^2 + (e^x)^2 \cos x \sin x + (e^x \sin x)^2 - (e^x)^2 \cos x \sin x$$

(Therefore the functions are linearly independent)

$W(f,g) \neq 0$ $\quad W(f,g) = (e^x \cos x)^2 + (e^x \sin x)^2$

---

3) $f(x) = 1 + \cos(2x)$

$g(x) = \cos^2(x)$

$$W(f,g) = \begin{bmatrix} 1 + \cos(2x) & \cos^2(x) \\ -2\sin(2x) & -2\cos x \sin x \end{bmatrix}$$

$\frac{d}{dx}[\cos x]^2$

$2(\cos x) \cdot -\sin x$

$$= (1 + \cos(2x))(-2\cos x \sin x) - (-2\sin(2x)(\cos^2 x))$$

$W(f,g) = -2\cos x \sin x - 2\cos x \sin x \cos 2x + 2\sin(2x)(\cos^2 x)$

$W(f,g) \neq 0$ {doesn't simplify to zero so, it's linearly independent}

4) Find the General Solution

$y'' - 4y = 0, \; y = e^{rx}$

$P(x)y'' + Q(x)y' + R(x)y = G(x)$

$y'' + 0y' + 4y = 0$

$ar^2 + br + c = 0$

$ay'' + by' + cy = 0$

$\Downarrow$

$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$

$\Downarrow$

$e^{rx}[ar^2 + br + c] = 0$

$\Downarrow$

$ar^2 + br + c = 0$

$y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2e^{rx}$

$1r^2 + 0r - 4 = 0$

$r^2 - 4 = 0$

$+4 \quad +4$

$\sqrt{r^2} = \sqrt{4}$

$\boxed{r = \pm 2}$

$\boxed{\begin{matrix} r_1 = 2 \\ r_2 = -2 \end{matrix}}$

discriminant is greater than zero, 2 real solutions occur

$y = C_1e^{r_1x} + C_2e^{r_2x}$

$\boxed{y = C_1e^{2x} + C_2e^{-2x}}$

General solution

Find the general solution

(5.) $y'' + y' - 2y = 0,\ y = e^{rx}$

$r_1 = \alpha + \beta i$

$r_2 = \alpha - \beta i$

$ar^2 + br + c = 0$

$1r^2 + 1r - 2 = 0$

$(r+2)(r-1)$

$\begin{Bmatrix} r_1 = -2 \\ r_2 = 1 \end{Bmatrix}$

$\boxed{y = C_1 e^{-2x} + C_2 e^{1x}}$ ← General solution

(6.) $y'' - 8y' + 16y = 0,\ y = e^{rx}$

$ar^2 + br + c = 0$

$1r^2 - 8r + 16 = 0$

$(r-4)(r-4) = 0$

$\begin{Bmatrix} r_1 = 4 \\ r_2 = 4 \end{Bmatrix}$

$\boxed{y = C_1 e^{4x} + C_2 e^{4x}}$ ← General solution

(7.) $4y'' - 12y' + 9y = 0,\ y = e^{rx}$

$4r^2 - 12r + 9 = 0$

General solution

↓

$\boxed{y = C_1 e^{\left(\frac{3\sqrt{2}+3}{2}\right)} + C_2 e^{\left(\frac{-3\sqrt{2}+3}{2}\right)}} \Leftarrow \frac{-(-12) \pm \sqrt{(-12)^2 + 4(9)(9)}}{2(4)} = \left(\frac{12}{8} \pm \frac{3\sqrt{2}}{2}\right)$

$r_1 = \left(\frac{3\sqrt{2}+3}{2}\right)$

$r_2 = \left(-\frac{3\sqrt{2}+3}{2}\right)$

(Find the particular solution)

8) $y''-y'-6y=0;\ y(0)=6,\ y'(0)=8\ ,\ y=e^{rx}$

$(r^2e^{rx}-re^{rx}-6e^{rx})=0$

$e^{rx}(r^2-r-6)=0$

Particular Solution

$(r-3)(r+2)=0$

$\boxed{y=4e^{3x}+2e^{-2x}}$

$\{r=3,\ r=-2\}$

$y=C_1e^{3x}+C_2e^{-2x}$

$y'=(3C_1e^{3x})+(-2C_2e^{-2(x)})$

$2(6=C_1+C_2)\quad 12=2C_1+2C_2$

$8=3C_1-2C_2\quad 8=3C_1-2C_2$

$20=5C_1$

$\frac{20}{5}=\frac{5C_1}{5}$

$\boxed{4=C_1}$

$8=12-2C_2$

$-12\quad -12$

$\frac{-4}{-2}=\frac{-2C_2}{-2}$

$\boxed{2=C_2}$

9) $y''-6y'+9y=0;\ y(0)=4,\ y'(0)=9\ ,\ y=e^{rx}$

$(r^2e^{rx}-6re^{rx}+9e^{rx})=0$

$e^{rx}(r^2-6r+9)=0$

$(x-3)(x-3)=0$

$x=3$

$y=C_1e^{3x}+C_2e^{3x}$

$y'=(3C_1e^{3x})+(3C_2e^{3x})$

$3(4=C_1+C_2)=12=3C_1+3C_2$

$9=3C_1+3C_2$

There's no solution to the system

$12\neq 9$

(So no constants can be derived for particular solution)

Particular Solution

DNE

Higher Order Linear Differential Equations- HW Problems

1. Show that $f(x) = x$, $g(x) = x^2 - 2x$, and $h(x) = 8x - 2x^2$ are linearly dependent by finding real numbers c_1, c_2, and c_3 such that $c_1 f(x) + c_2 g(x) + c_3 h(x) = 0$ for all x.

2. Use the Wronskian to show that the functions in problem number 1 are linearly dependent on $\mathbb{R}$.

3. Use the Wronskian to show that e^{2x}, e^{4x}, and e^{6x} are linearly independent on $\mathbb{R}$.

4. $y^{(3)} - 2y'' - 5y' + 6y = 0$ has $y_1 = e^x$, $y_2 = e^{-2x}$, and $y_3 = e^{3x}$ as linearly independent solutions. Find the particular solution where $y(0) = 0$, $y'(0) = -4$, and $y''(0) = 14$.

In problems 5 and 6 you are given the complementary solution y_c and a particular solution y_p of a differential equation. Find the solution to the given initial value problem.

5. $y'' + 9y = 18x$, $y_c = c_1 \cos(3x) + c_2 \sin(3x)$, $y_p = 2x$,
   $y(0) = 3$, $y'(0) = 7$.

6. $y'' + 4y' + 13y = 13x + 17$,
   $y_c = c_1 e^{-2x} \cos(3x) + c_2 e^{-2x} \sin(3x)$; $y_p = x + 1$,
   $y(0) = 5$, $y'(0) = -1$.

1) Show $f(x)=x$, $g(x)=x^2-2x$, & $h(x)=8x-2x^2$ are linearly dependent by finding real numbers c_1, c_2 & c_3, such that $c_1f(x)+c_2g(x)+c_3h(x)=0$ for all x.

We need to show we can find c_1, c_2, c_3, not all 0, such that:

$$c_1(x)+c_2(x^2-2x)+c_3(8x-2x^2)=0$$

$$\underline{c_1(x)}+c_2x^2\underline{-2c_2x}+\underline{8c_3x}-2c_3x^2=0$$

$$x(c_1-2c_2+8c_3)+x^2(c_2-2c_3)=0$$

$$c_1-2c_2+8c_3=0 \quad \& \quad c_2-2c_3=0$$

let

$$c_1-2(2)+8=0 \qquad \boxed{c_2=2}$$

$$c_1=-4 \qquad \boxed{c_3=1}$$

$$\boxed{c_1=-4}$$

$$-4(x)+2(x^2-2x)+1(8x-2x^2)=0$$

Thus, $f(x)$, $g(x)$, and $h(x)$ are linearly dependent on $\mathbb{R}$.

Question 2 H.W. 3

$f(x) = x$

$g(x) = x^2 - 2x$

$h(x) = 8x - 2x^2$

$$W(f,g,h) = \begin{bmatrix} x & x^2-2x & 8x-2x^2 \\ 1 & 2x-2 & 8-4x \\ 0 & 2 & -4 \end{bmatrix}$$

$$W(f,g,h) = \begin{bmatrix} x & x^2-2x & 8x-2x^2 & x & x^2-2x \\ 1 & 2x-2 & 8-4x & 1 & 2x-2 \\ 0 & 2 & -4 & 0 & 2 \end{bmatrix}$$

$0 \quad 2x(8-4x) \quad -4(x^2-2x) \quad \Big| \quad -4x(2x-2) \quad 0 \quad 2(8x-2x^2)$

$= (-8x^2+8x)+(16x-4x^2)-((16x-8x^2)-4x^2+8x$

$= \cancel{-8x^2}+\cancel{8x}+\cancel{16x}-\cancel{4x^2}-\cancel{16x}+\cancel{8x^2}+\cancel{4x^2}-\cancel{8x}$

$= 0$

$W(f,g,h) = 0$, linearly dependendent

3.) Use the Wronskian to show that the functions e^{2x}, e^{4x}, and e^{6x} are linearly independent on $\mathbb{R}$.

$f(x) = e^{2x}$
$g(x) = e^{4x}$
$h(x) = e^{6x}$

$$W(f,g,h) = \begin{bmatrix} e^{2x} & e^{4x} & e^{6x} \\ 2e^{2x} & 4e^{4x} & 6e^{6x} \\ 4e^{2x} & 16e^{4x} & 36e^{6x} \end{bmatrix}$$

$$W(f,g,h) = \begin{bmatrix} e^{2x} & e^{4x} & e^{6x} & e^{2x} & e^{4x} \\ 2e^{2x} & 4e^{4x} & 6e^{6x} & 2e^{2x} & 4e^{4x} \\ 4e^{2x} & 16e^{4x} & 36e^{6x} & 4e^{2x} & 16e^{4x} \end{bmatrix}$$

$(16e^{2x}e^{4x}e^{6x})$ $(72e^{6x}e^{2x}e^{4x})$ $(144e^{6x}e^{4x}e^{2x})$ $(24e^{2x}e^{6x}e^{4x})$ $(32e^{4x}e^{2x}e^{6x})$

$(96e^{4x}e^{6x}e^{2x})$

$$(32e^{4x}e^{2x}e^{6x}) + (24e^{2x}e^{6x}e^{4x}) + (144e^{6x}e^{4x}e^{2x}) - (e^{6x}e^{4x}e^{2x})$$

Let's factor for readability $\Downarrow$ $(16+96+72)$

$$(e^{4x}e^{2x}e^{6x})(32+24+144) - (e^{6x}e^{4x}e^{2x})(16+96+72)$$

$$(e^{4x}e^{2x}e^{6x})(200) - (e^{6x}e^{4x}e^{2x})(184)$$

$$W(f,g,h) = 16(e^{4x}e^{2x}e^{6x})$$

$W(f,g,h) \neq 0$ therefore your functions are Independent

4) $y^{(3)} - 2y'' - 5y' + 6y = 0$ has $y_1 = e^x$, $y_2 = e^{-2x}$ & $y_3 = e^{3x}$ as linearly independent solutions. Find the particular solution when $y(0) = 0$, $y'(0) = -4$, & $y''(0) = 14$.

particular solution $\quad y = e^{rx}$, $y' = re^{rx}$, $y'' = r^2 e^{rx}$, $y''' = r^3 e^{rx}$

$$\boxed{y = \frac{8}{11}e^{3x} + \frac{30}{11}e^{-2x} - \frac{38}{11}e^{1x}}$$

$$e^{rx}(r^3 - 2r^2 - 5r + 6) = 0$$

reduce polynomial with synthetic division

1	1	-2	-5	6
	↓	+1	-1	-6
	1	-1	-6	0

$\pm 1 \pm 2 \pm 3 \pm 6$

$r = 1$

$(1r^2 - 1r - 6)(r - 1)$

$(r-3)(r+2)(r-1)$

$$\begin{Bmatrix} r = 3 \\ r = -2 \\ r = 1 \end{Bmatrix}$$

$14 = 8c_1 + 3\left(\frac{30}{11}\right)$

$\left(\frac{64}{11} = 8c_1\right) = \boxed{\frac{8}{11} = c_1}$

$16 = -8c_1 + 8c_2$

$14 = 8c_1 + 3c_2$

$\frac{30}{11} = \frac{11c_2}{11}$

$\boxed{\frac{30}{11} = c_2}$

$\boxed{-\frac{38}{11} = c_3}$

$0 = \frac{38}{11} + c_3$

$0 = \frac{8}{11} + \frac{30}{11} + c_3$

$$\begin{Bmatrix} 0 = c_1 + c_2 + c_3 \\ -4 = 3c_1 - 2c_2 + c_3 \\ 14 = 9c_1 + 4c_2 + c_3 \end{Bmatrix}$$

$$\boxed{\begin{aligned} y &= c_1 e^{3x} + c_2 e^{-2x} + c_3 e^{1x} \\ y' &= 3c_1 e^{3x} - 2c_2 e^{-2x} + c_3 e^{x} \\ y'' &= 9c_1 e^{3x} + 4c_2 e^{-2x} + c_3 e^{x} \end{aligned}}$$

$$\begin{aligned} -(0 &= c_1 + c_2 + c_3) \\ -4 &= 3c_1 - 2c_2 + c_3 \\ \hline -4 &= 2c_1 - 2c_2 \end{aligned}$$

$$\begin{aligned} -(0 &= c_1 + c_2 + c_3) \\ 14 &= 9c_1 + 4c_2 + c_3 \\ \hline 14 &= 8c_1 + 3c_2 \end{aligned}$$

$-4(-4 = 2c_1 - 2c_2)$

$14 = 8c_1 + 3c_2$

In problem 5 & 6 you are given a complementary solution y_c & a particular solution y_p of a differential eq. find the solution to the given initial value problem.

5.)

Conditions

$y'' + 9y = 18x,$

$y_c = C_1 \cos(3x) + C_2(\sin(3x))$

$y_p = 2x,\ y(0) = 3,\ y'(0) = 7$

$y_c + y_p$ = General Solution to the differential Equation

$y = C_1 \cos(3x) + C_2 \sin(3x) + 2x$

$y' = -3C_1 \sin(3x) + 3C_2 \cos(3x) + 2$

$\boxed{\frac{2}{6} = C_1}$

$-3\,(3 = C_1 + C_2)$ $\Rightarrow$ $-9 = -3C_1 - 3C_2$

$7 = -3C_1 + 3C_2$ $\quad$ $7 = -3C_1 + 3C_2$

$\frac{-2}{-6} = \frac{-6C_1}{-6}$ $\quad \frac{2}{6}$

$\frac{8}{3} = \frac{3C_2}{3} \Rightarrow \boxed{\frac{8}{3} = C_2}$

$7 = -1 + 3C_2 \Leftarrow (7 = -3(\frac{2}{6}) + 3C_2)$

$+1 \quad +1$

Particular Solution: $y = \frac{2}{6}\cos(3x) + \frac{8}{3}\sin(3x) + 2x$

6.)

Conditions

$y'' + 4y' + 13y = 13x + 17$

$y_c = C_1 e^{-2x}\cos(3x) + C_2 e^{-2x}\sin(3x)$

$y(0) = 5,\ y'(0) = -1,\ y_p = x + 1$

$y = C_1 e^{-2x}\cos(3x) + C_2 e^{-2x}\sin(3x) + (x+1)$

$y' = -3C_1 e^{-2x}\sin(3x) - 2C_1 e^{-2x}\cos(3x)$

$+ 3C_2 e^{-2x}\cos(3x) - 2C_2 e^{-2x}\sin(3x) + 1$

$5 = C_1 + 1$ $\quad \boxed{4 = C_1}$

$-1 = -2C_1 + 3C_2$

$-1 = -2(4) + 3C_2$

$-1 = -8 + 3C_2$

$+8 \quad +8$

$\boxed{\frac{7}{3} = C_2}$ $\quad \frac{7}{3} = \frac{3C_2}{3}$

$$\boxed{y = 4e^{-2x}\cos(3x) + \frac{7}{3}e^{-2x}\sin(3x) + (x+1)}$$

particular Solution

Homogeneous Equations with Constant Coefficients- HW Problems

In problems 1-5 find the general solution to the differential equation.

1. $y'' + 8y' + 16y = 0$
2. $y'' + 4y' + 20y = 0$
3. $y^{(3)} + 3y'' = 0$
4. $y^{(4)} + 6y'' + 9y = 0$
5. $y^{(4)} + 2y^{(3)} + y'' = 0.$

In problems 6-8 solve the initial value problem.

6. $y'' + 4y' + 13y = 0, \quad y(0) = 2, \quad y'(0) = 5$
7. $2y^{(3)} - y'' - y' = 0, \quad y(0) = 9, \quad y'(0) = 1, \quad y''(0) = 4.$
8. $y^{(3)} + y'' = 0, \quad y(0) = 6, \quad y'(0) = 1, \quad y''(0) = 1$

9. $y = e^{2x}$ is one solution to $y^{(3)} + 4y'' + y' - 26y = 0$. Find the general solution.

{ In problems 1-5 find the general solution to the differential equation. }

1.) $y'' + 8y' + 16y = 0$, $y = e^{rx}$, $y' = re^{rx}$, $y'' = r^2e^{rx}$

$e^{rx}(r^2 + 8r + 16) = 0$

$(r+4)(r+4) = 0$, thus $-4 = r$

$\boxed{y = C_1e^{-4x} + C_2e^{-4x}}$ General solution

2.) $y'' + 4y' + 20y = 0$, $y = e^{rx}$, $y' = re^{rx}$, $y'' = r^2e^{rx}$

$e^{rx}(r^2 + 4r + 20) = 0$

Extra:

Euler Formula: $e^{ix} = (\cos x + i\sin x)$

$= e^{(a+bi)x}$

$= e^{ax+ibx}$

$= e^{ax}e^{ibx}$

$= e^{ax}(\cos bx + i\sin bx)$

$\frac{-4 \pm \sqrt{(4)^2 - 4(1)(20)}}{2(1)} = \frac{-2 \pm \sqrt{-64}}{2} = -2 \pm \frac{8}{2}i = -2 \pm 4i$

$\boxed{y = e^{-2x}[C_1\cos(4x) + C_2\sin(4x)]}$

General Solution

$r = -2 + 4i$

$r = -2 - 4i$

$r_1 = \alpha + \beta i$

Sub in values from your pos conjugate

3.) $y^{(3)} + 3y'' = 0$, $y'' = r^2e^{rx}$ $y''' = r^3e^{rx}$

$e^{rx}(r^3 + 3r^2) = 0$

$\sqrt{r^2} = \sqrt{0}$

General Solution $r^2(r^2 + 3) = 0$

$y = C_1e^0 + e^{(0)}[C_2\cos(\sqrt{3}x) + C_3\sin(\sqrt{3}x)]$

$\sqrt{r^2} = \sqrt{-3}$

$r = \pm\sqrt{3}i$

$\left\{ \begin{array}{l} r = 0 \\ r = +\sqrt{3}i \\ r = -\sqrt{3}i \end{array} \right\}$

4) $y^4 + 6y'' + 9y = 0$, $y = e^{rx}$, $y'' = r^2 e^{rx}$, $y^4 = r^4 e^{rx}$

$e^{rx}(r^4 + 6r^2 + 9) = 0$

$r^4 + 6r^2 = -9$

$\sqrt{r^2} = \sqrt{0}$

$r^2(r^2 + 6) = -9$

$\underline{r = 0}$

$r^2 + 6 = -9 \Rightarrow \sqrt{r^2} = \pm\sqrt{-15}$ $r = \pm\sqrt{15}\, i$
$\quad -6 \quad -6$

General Solution

$y = e^x(C_1 \cos(\sqrt{15}x) + C_2 \sin(\sqrt{15}x)) + xe^x(C_3 \cos(\sqrt{15}x) + C_4 \sin(\sqrt{15}x))$

5) $y^4 + 2y^3 + y'' = 0$, $y'' = r^2 e^{rx}$, $y^3 = r^3 e^{rx}$, $y^4 = r^4 e^{rx}$

$e^{rx}(r^4 + 2r^3 + r^2) = 0$

$r^2(r^2 + 2r + 1) = 0$

$\sqrt{r^2} = \sqrt{0}$

$(r+1)(r+1) = 0$

$r = -1$

$\underline{r = 0}$

General Solution

$y = e^x(C_1 \cos(-x) + C_2 \sin(-x)) + xe^x(C_3 \cos(-x) + C_4 \sin(-x))$

$\frac{9}{5} = C_2$

$\frac{27}{5} = 3C_2$

{6-8 Solve the initial value problems}

6.) $y'' + 4y' + 13y = 0,\ y(0) = 2,\ y'(0) = 5$ $\quad 5 = \frac{-2}{5} + 3C_2$

$y = e^{rx},\ y' = re^{rx},\ y'' = r^2 e^{rx}$

$e^{rx}(r^2 + 4r + 13) = 0$

$= \frac{-(4) \pm \sqrt{(4)^2 - 4(1)(13)}}{2(1)}$

$= -2 \pm \frac{\sqrt{36}}{2} i$

$= -2 \pm 3i$

$-3(2 = C_1 + C_2) \Rightarrow -6 = -3C_1 - 3C_2$

$5 = -2C_1 + 3C_2 \quad 5 = -2C_1 + 3C_2$

$\frac{-1}{-5} = \frac{-5C_1}{-5}$

$\boxed{\frac{1}{5} = C_1}$

$y = e^{-2x}[C_1 \cos(3x) + C_2 \sin(3x)]$

$y' = e^{-2x}[-3C_1 \sin(3x) + 3C_2 \cos(3x)] - 2e^{-2x}[C_1 \cos(3x) + C_2 \sin(3x)]$

Particular Solution

$y = e^{-2x}[\frac{1}{5}\cos(3x) + \frac{9}{5}\sin(3x)]$

7.) $2y^{(3)} - y'' - y' = 0,\ y(0) = 9,\ y'(0) = 1,\ y''(0) = 4$

Particular sol: $\frac{-3\sqrt{5}+15}{2} e^0 + \frac{3\sqrt{5}+7}{2} e^{\left(\frac{\sqrt{5}+1}{2}\right)x} - 2e^{\left(\frac{\sqrt{5}-1}{2}\right)x}$

$y' = re^{rx},\ y'' = r^2 e^{rx},\ y^3 = r^3 e^{rx}$

$9 = C_1 + C_2 + C_3 \quad e^{rx}(r^3 - r^2 - r) = 0$

$1 = \left(\frac{\sqrt{5}+1}{2}\right)C_2 + \left(\frac{\sqrt{5}-1}{2}\right)C_3 \quad r(r^2 - r - 1) = 0$

$4 = \left(\frac{\sqrt{5}+1}{2}\right)^2 C_2 + \left(\frac{\sqrt{5}-1}{2}\right)^2 C_3 \quad r = 0$

$-\left(\frac{\sqrt{5}+1}{2}\right) = \left(\frac{\sqrt{5}-1}{2}\right)\left(\frac{\sqrt{5}+1}{2}\right)C_3$

$\boxed{-2 = C_3} \quad 4 = \left(\frac{\sqrt{5}-1}{2}\right)^2 C_3$

$= \frac{-(-1) \pm \sqrt{(1)^2 - 4(1)(-1)}}{2(1)}$

$r = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$

$y = C_1 e^0 + C_2 e^{\left(\frac{\sqrt{5}+1}{2}\right)x} + C_3 e^{\left(\frac{\sqrt{5}-1}{2}\right)x}$

$y' = \left(\frac{\sqrt{5}+1}{2}\right)C_2 e^{\left(\frac{\sqrt{5}+1}{2}\right)x} + \left(\frac{\sqrt{5}-1}{2}\right)C_3 e^{\left(\frac{\sqrt{5}-1}{2}\right)x}$

$y'' = \left(\frac{\sqrt{5}+1}{2}\right)^2 C_2 e^{\left(\frac{\sqrt{5}+1}{2}\right)x} + \left(\frac{\sqrt{5}-1}{2}\right)^2 C_3 e^{\left(\frac{\sqrt{5}-1}{2}\right)x}$

$\boxed{C_2 = \frac{3\sqrt{5}+7}{2}} \quad \boxed{C_1 = \frac{-3\sqrt{5}+15}{2}}$

8.) $y^{(3)} + y'' = 0,\ y(0) = 6,\ y'(0) = 1,\ y''(0) = 1$

(Particular Solution)

$y = 7 + 7e^{-1x} - 8e^{-1x}$

$y^3 = r^3 e^{rx},\ y'' = r^2 e^{rx}$

$e^{rx}(r^3 + r^2) = 0$

$r^2(r+1)$

$r = 0 \quad r = -1$

$y = C_1 e^0 + C_2 e^{-1x} + C_3 e^{-1x}$

$y' = -C_2 e^{-1x} - C_2 e^{-1x}$

$y'' = C_2 e^{-1x} + C_2 e^{-1x}$

$6 = C_1 + C_2 + C_3$

$+\ 1 = -C_2 - C_3$

$\boxed{7 = C_1}$

$6 = C_1 - 1 \Leftarrow 6 = C_1 + 7 - 8 \quad \boxed{C_3 = -8} \Leftarrow 1 + C_3 = -7 \Leftarrow 1 = -7 - C_3$

$+1 \quad +1 \qquad -1 \quad -1 \qquad +C_3 \quad +C_3$

$\boxed{7 = C_2}$

9.) $y = e^{2x}$ is one solution to $y^{(3)} + 4y'' + y' - 26y = 0$
Find the general solution.

$$y = e^{rx},\ y' = re^{rx},\ y'' = r^2e^{rx},\ y^3 = r^3e^{rx}$$

$$e^{rx}(r^3 + 4r^2 + r - 26) = 0$$

$\pm 1 \pm 2 \pm 13$

$$\begin{array}{c|cccc} 2 & 1 & 4 & 1 & -26 \\ & \downarrow & 2 & 12 & 26 \\ \hline & 1 & 6 & 13 & 0 \end{array}$$

$$(r^2 + 6r + 13) = 0$$

$$= \frac{-6 \pm \sqrt{(6)^2 - 4(1)(13)}}{2(1)} = -3 \pm \frac{4}{2}i$$

$r = 2 \quad r = -3 \pm 2i$

$$y = C_1 e^{2x} + e^{-3(x)}\left[C_2 \cos(-3x) + C_3 \sin(-3x)\right]$$

General Solution