Elementary Power Series Solutions- HW Problems

In problems 1-3 find a power series solution to the differential equation. Determine the radius of convergence of the power series solution.

1. $y' = 3y$
2. $(x-1)y' + 2y = 0$
3. $y'' + 4y = 0.$

In problems 4 and 5 find a recurrence relation for c_n in terms of c_0 and c_1. Use the initial conditions to find c_0 and c_1 and hence find the power series solution to the initial value problem.

4. $y'' - 9y = 0, \qquad y(0) = 4, \quad y'(0) = 0$
5. $y'' + 9y = 0, \qquad y(0) = 0, \quad y'(0) = 6.$

In problems 1-3 find a power series solution to the differential equation. Determine the radius of convergence of the power series solution.

1.) $y' = 3y$

$y' - 3y = 0$

$y = \sum_{n=0}^{\infty} C_n x^n$ & $y' = \sum_{n=1}^{\infty} n C_n x^{n-1}$

$\sum_{n=1}^{\infty} n C_n x^{n-1} - \sum_{n=0}^{\infty} C_n x^n = 0$

$\sum_{n=0}^{\infty} (n+1) C_{n+1} x^n - \sum_{n=0}^{\infty} C_n x^n = 0$

$\sum_{n=0}^{\infty} [(n+1) C_{n+1} - C_n] x^n = 0$

$(n+1) C_{n+1} - C_n = 0$

$C_{n+1} = \frac{C_n}{(n+1)}$ $n \geq 0$

$n=0$ $C_1 = \boxed{C_0}$

$n=1$ $C_2 = \frac{C_1}{2} = \boxed{C_0 \frac{1}{2}}$

$n=2$ $C_3 = \frac{C_2}{3} = C_0 \left(\frac{\frac{1}{2}}{3}\right) = \boxed{C_0 \frac{1}{6}}$

$n=3$ $C_4 = \frac{C_3}{4} = C_0 \left(\frac{\frac{1}{6}}{4}\right) = \boxed{C_0 \frac{1}{24}}$

$C_n = \frac{1}{(1+n)!} C_0$

$Y(x) = \sum_{n=0}^{\infty} \frac{1}{(1+n)!} C_0 x^n$

$Y(x) = C_0 \sum_{n=0}^{\infty} \frac{1}{(1+n)!} x^n$

$\rho = \lim_{n \to \infty} \left| \frac{C_n}{C_{n+1}} \right| = \lim_{n \to \infty} \left| \frac{\frac{1}{(1+n)!} \cdot C_0}{\frac{1}{(2+n)!} C_0} \right|$

$= \lim_{n \to \infty} \left| \frac{(2+n)!}{(1+n)!} \right|$

$= 1$

The radius of convergence is 1

(Power Series general solution)

$Y(x) = C_0 \sum_{n=0}^{\infty} \frac{1}{(1+n)!} (1)^n$

2.)

$$(x-1)y' + 2y = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^n \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$(x-1)\left(\sum_{n=1}^{\infty} n c_n x^{n-1}\right) + 2\left(\sum_{n=0}^{\infty} c_n x^n\right) = 0$$

$$\sum_{n=1}^{\infty} n c_n x^n - \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} 2 c_n x^n = 0$$

replace $n \to n+1$, subtract 1

$$\sum_{n=0}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + \sum_{n=0}^{\infty} 2 c_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[n c_n - (n+1) c_{n+1} + 2 c_n \right] x^n = 0$$

$$\frac{-(n+1)c_{n+1}}{-(n+1)} = \frac{-2c_n - n c_n}{-(n+1)}$$

$$c_{n+1} = \frac{-c_n(2+n)}{-(n+1)}$$

$$= \frac{c_n(2+n)}{n+1} \qquad n \geq 0$$

$n=0$, $c_1 = \boxed{2c_0}$

$n=1$, $c_2 = \frac{c_1(3)}{2} = \boxed{3c_0}$

$n=2$, $c_3 = \frac{c_2(4)}{3} = \boxed{4c_0}$

$n=3$, $c_4 = \frac{c_3(5)}{4} = \boxed{5c_0}$

$$c_n = (2+n)c_0$$

$$y(x) = \sum_{n=0}^{\infty} (2+n) c_0 x^n$$

$$y(x) = c_0 \sum_{n=0}^{\infty} (2+n) x^n$$

$$p = \lim_{n \to \infty} \left| \frac{2+n}{3+n} \right| = 1$$

The radius of convergence is 1

power series general solution

$$y(x) = c_0 \sum_{n=0}^{\infty} (2+n)(x)^n$$

3) $y'' + 4y = 0$

$$y=\sum_{n=0}^{\infty} c_n x^n, \quad y'=\sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y''=\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 4\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=1}^{\infty} n(n-1)c_n x^{n-1} + 4\sum_{n=0}^{\infty} c_n x^n$$

$\uparrow$

$k = n+2$

$$\lim_{n\to\infty}\left|\frac{(-1)^n\left(\frac{1}{(2n!)}-\frac{1}{(2n+1)!}\right)}{(-1)^{n+1}\left(\frac{1}{(2(n+1)!)}-\frac{1}{(2n+3)!}\right)}\right|$$

$$\sum_{n=0}^{\infty} (n+2)(n+2-1)c_{n+2}x^n + 4\sum_{n=0}^{\infty} c_n x^n$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n + 4\sum_{n=0}^{\infty} c_n x^n$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)c_{n+2} + 4c_n\right] x^n$$

$$\lim_{n\to\infty} n^{!} = \infty$$

$\uparrow$

$$c_{n+2} = \frac{-4c_n}{(n+2)(n+1)}$$

$n=0,\quad c_2 = \frac{-4c_0}{2} = -4\left(\frac{c_0}{2!}\right) \quad n \geq 0$

$n=1,\quad c_3 = \frac{-4c_1}{6} = -4\left(\frac{c_1}{3!}\right)$

$n=2,\quad c_4 = \frac{-4c_2}{12} = -4\left(\frac{c_0}{4!}\right)$

$$y(x) = -4c_0\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{(2n)!} - 4\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

$\boxed{y(x) = -4\cos x - 4\sin x}$ General Solution

{ problems 4 & 5 find a recurrence relation C_n in terms of C_0 & C_1. Use the initial conditions to find C_0 & C_1 & hence find the power series solution to the initial value problem. }

4.) $y'' - 9y = 0$, $y = \sum_{n=0}^{\infty} C_n x^n$ $y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$

$y(0) = 4$, $y'(0) = 0$

$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - 9\left(\sum_{n=0}^{\infty} C_n x^n\right)$

$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - \sum_{n=0}^{\infty} 9 C_n x^n$

$(k = n-2, n = k+2)$

$\sum_{n=0}^{\infty} n+2(n+1) C_{n+2} x^n - \sum_{n=0}^{\infty} 9 C_n x^n$

$\sum_{n=0}^{\infty} [n+2(n+1) C_{n+2} - 9C_n] x^n$

$C_{n+2} = \left(\frac{9C_n}{(n+2)(n+1)}\right)$ $n \geq 0$

Solution

$y(x) = 4\left(1 + \sum_{n=0}^{\infty} \frac{3^{n+1}}{(n+2)!} x^n\right)$

$y = C_0 + C_1 x + C_2 x^2$

$\boxed{y(0) = C_0 = 4}$

$y' = C_1 + 2C_2 x$

$\boxed{y'(0) = C_1 = 0}$

odd (C_1)

$n = 1$, $C_3 = \frac{9C_1}{(3)(2)} = \frac{9C_1}{6}$

$n = 3$, $C_5 = \frac{9C_3}{(5)(4)} = \frac{9C_3}{20} = \frac{81C_1}{120}$

$n = 5$, $C_7 = \frac{9C_5}{(7)(6)} = \frac{9C_5}{42} = \frac{729C_1}{5040}$

Even (C_0)

$n = 0$, $C_2 = \frac{9C_0}{(2)(1)} = \frac{9C_0}{2}$

$n = 2$, $C_4 = \frac{9C_2}{(4)(3)} = \frac{9C_2}{12} = \frac{81C_0}{24}$

$n = 4$, $C_6 = \frac{9C_4}{(6)(5)} = \frac{9C_4}{30} = \frac{729}{720}C_0$

$y(x) = C_0\left(1 + \sum_{n=1}^{\infty} \frac{3^{n+1}}{(n+2)!} x^n\right) + C_1\left(1 + \sum_{n=1}^{\infty} \frac{3^{n+1}}{(n+2)!} x^n\right)$

5.) $y'' + 9y = 0$, $y(0)=0$, $y'(0)=6$ (practically the same as the last question)

$$\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} + \sum_{n=0}^{\infty} 9C_n x^n$$

$(k = n-2,\ n = k+2)$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)C_{n+2} + 9C_n\right] x^n$$

$$C_{n+2} = \left(-\frac{9C_n}{(n+2)(n+1)}\right)$$

odd (C_1)	Even (C_2)
$n=1,\ C_3 = -\frac{9C_1}{(3)(2)} = \left(-\frac{9C_1}{6}\right)$	$n=0,\ C_2 = \left(-\frac{9C_0}{2}\right)$
$n=3,\ C_5 = -\frac{9C_3}{(5)(4)} = \left(\frac{81C_1}{120}\right)$	$n=2,\ C_4 = \frac{9C_2}{12} = \left(\frac{81}{24}C_0\right)$
$n=5,\ C_7 = -\frac{9C_5}{(7)(6)} = \left(-\frac{729C_1}{5040}\right)$	$n=4,\ C_6 = \frac{9C_4}{30} = \left(\frac{729}{720}C_0\right)$

$$Y(x) = C_0\left(\sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1}}{(n+2)!} x^n\right) + C_1\left(\sum_{n=0}^{\infty} \frac{(-1)\,3^{n+2}}{(n+2)!}\right)$$

$y = C_0 + C_1 x + C_2 x^2,\ y(0) = C_0 = 0$

$y = C_1 + C_2 x + C_3 x^2,\ y'(0) = C_1 = 6$

Solution

$$y(x) = 6\left(\sum_{n=0}^{\infty} \frac{(-1)\,3^{n+2}}{(n+2)!}\right)$$

Series Solutions near Ordinary Points- HW Problems

In problems 1-3 find a series solutions around 0 (ie in powers of x) for the differential equation using a recurrence relation. Find the guaranteed radius of convergence of the solution.

1. $y'' - xy' - y = 0$
2. $(x^2 + 1)y'' - 4xy' + 6y = 0$
3. $(x^2 - 3)y'' + 2xy' = 0$

In problems 4-6 solve the initial value problems with a power series.

4. $(1 + x^2)y'' + 2xy' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0$
5. $y'' + xy' - 2y = 0, \quad y(0) = 0, \quad y'(0) = 3$
6. $y'' + 2xy' + 2y = 0, \quad y(0) = 6, \quad y'(0) = 2$

7. Solve the initial value problem by first making a substitution $t = x - a$ and then finding a series solution of the form $\sum_{n=0}^{\infty} c_n t^n$. Then transform the solution back to a power series in $x - a$.

$y'' + (x - 1)y' + y = 0, \quad y(1) = 3, \quad y'(1) = -1.$

In problems 1-3 find a Series Solutions around 0 (i.e. in powers of x) for the differential equation using a recurrence relation. Find the guaranteed radius of convergence of the solution.

1.) $y'' - xy' - y = 0$, $y = \sum_{n=0}^{\infty} C_n x^n$, $y' = \sum_{n=1}^{\infty} n C_n x^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$

$$\left(\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}\right) - x\left(\sum_{n=1}^{\infty} n C_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} C_n x^n\right) = 0$$

$(k = n-2,\ n = k+2)$

$$\boxed{\sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} x^n} - \sum_{n=1}^{\infty} n C_n x^n - \left(\sum_{n=0}^{\infty} C_n x^n\right) = 0$$

$n = 0$

$$\underbrace{(0+2)(0+1)C_{0+2}x^0}_{2C_2} + \sum_{n=1}^{\infty} (n+2)(n+1) C_{n+2} x^n - \sum_{n=1}^{\infty} n C_n x^n$$

$$+ \underbrace{C_0 x^0}_{C_0} + \sum_{n=1}^{\infty} C_n x^n = 0$$

$$2C_2 + C_0 + \sum_{n=1}^{\infty} \left((n+2)(n+1) C_{n+2} - n C_n + C_n\right) x^n = 0$$

① $2C_2 + C_0 = 0 \rightarrow C_2 = -\frac{1}{2} C_0$

② $(n+2)(n+1) C_{n+2} - (n-1) C_n = 0$

$(n+2)(n+1)C_{n+2} - (n-1)C_n = 0 \Rightarrow C_{n+2} = \frac{(n-1)}{(n+2)(n+1)} C_n$

$y'' - xy' - y = 0$

$C_2 = -\frac{1}{2}C_0, \; C_{n+2} = \frac{(n-1)}{(n+2)(n+1)} C_n$

(radius of convergence)

$C_0 \leftarrow$ arbitrary

$C_1 \leftarrow$ arbitrary

$P(x) = -1x$

$Q(x) = -1$

n	$C_{n+2} = \frac{(n-1)}{(n+2)(n+1)} C_n$
0	$C_2 = \frac{-1}{(2)(1)} C_0 = -\frac{1}{2} C_0$
1	$C_3 = 0$
2	$C_4 = \frac{1}{(4)(3)} C_2 = -\frac{1}{24} C_0$
3	$C_5 = \frac{2}{(4)(5)} C_3 = 0$
4	$C_6 = \frac{3}{(6)(5)} C_4 = -\frac{3}{720} C_0$
5	$C_7 = \frac{4}{(7)(6)} C_5 = 0$

6×1

6×4

6×120

[The radius of convergence is 0]

$y = \sum_{n=0}^{\infty} a_n x^n$

Not necessary

$= C_0 + C_1 x - \frac{1}{2} C_0 x^2 + 0 - \frac{1}{24} C_0 x^4 + 0 - \frac{1}{144} C_0 x^6 + 0$

$= C_0 \left(1 + -\frac{1}{2}x^2 - \frac{1}{24}x^4 - \frac{3}{720}x^6 - \dots\right)$

$+ C_1 (1 + 0 + 0 + 0 + \dots)$

The interval of convergence is (0) zero, since our limit converges at 0

(The Solution)

$\lim = 0$

$y(x) = C_0\left(1 + \sum_{n=1}^{\infty} \frac{2n-1}{(2n)!} x^n\right) + C_1(x)$

$\in \mathbb{R}$

$\frac{2n}{(2n+2)(2n-1)} = \left| \frac{2n!}{2n-1} \cdot \frac{2n!}{(2n+1)!} \right| = \lim_{n \to \infty} \left| \frac{\frac{2n+1-1}{(2(n+1))!}}{\frac{2n-1}{(2n)!}} \right| =$

2.) $(x^2+1)y'' - 4xy' + 6y = 0$

$$y'' = \sum_{n=2}^{\infty} n(n-1)C_n x^{n-2}, \quad y' = \sum_{n=1}^{\infty} nC_n x^{n-1},$$

$$y = \sum_{n=0}^{\infty} C_n x^n$$

$$(x^2+1)\left(\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2}\right) - 4x\left(\sum_{n=1}^{\infty}(n)C_n x^{n-1}\right) + 6\left(\sum_{n=0}^{\infty} C_n x^n\right)$$

$$\sum_{n=2}^{\infty} n(n-1)(C_n x^n) + \left(\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2}\right) - \sum_{n=1}^{\infty} 4nC_n x^n + \sum_{n=0}^{\infty} 6C_n x^n$$

$(k = n-2, \; n = k+2)$

$$\sum_{n=0}^{\infty} n(n-1)C_n x^n + \sum_{n=0}^{\infty}(n+2)(n+1)C_{n+2}x^n - \sum_{n=0}^{\infty} 4(n)C_n x^n + \sum_{n=0}^{\infty} 6C_n x^n$$

$$\sum_{n=0}^{\infty}\left[n(n-1)C_n + (n+2)(n+1)C_{n+2} - 4(n)C_n + 6C_n\right]x^n = 0$$

$$C_n(n(n-1) - 4n + 6) + (n+2)(n+1)C_{n+2} = 0$$

$$\frac{(n+2)(n+1)C_{n+2}}{(n+2)(n+1)} = \frac{-C_n(n^2-5n+6)}{(n+2)(n+1)}$$

$$C_{n+2} = \frac{-C_n(n-2)(n-3)}{(n+2)(n+1)}$$

n	
0	$C_2 = -\frac{6}{2}C_0$
1	$C_3 = \frac{-2}{6}C_1 =$
2	$C_4 = 0$
3	$C_5 = 0$
4	$C_6 = -\frac{2}{30}C_4 = 0$
5	$C_7 = \frac{6}{42}C_5 = 0$
6	$C_8 = \frac{12}{56}C_6 = 0$

Solution

$$y(x) = C_0\left(\frac{6}{2}\right) + C_1\left(-\frac{2}{6}x\right)$$

Continued ↓

radius of Convergence

$$P(x)=\frac{-4x}{(x^2+1)}, \quad Q(x)=\frac{6}{(x^2+1)}$$

powers of x

$$\begin{pmatrix}\text{Singularity}\\ \text{points}\end{pmatrix}=\frac{-0\pm\sqrt{0^2-4(1)(1)}}{2(1)}=\frac{\pm\sqrt{-4}}{2}=\boxed{\pm i}$$

The radius of Convergence is at least i since solutions are around powers of zero

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3.) $(x^2-3)y''+2xy'=0$

$$y'=\sum_{n=1}^{\infty} nC_nx^{n-1}$$

$$y''=\sum_{n=2}^{\infty} n(n-1)C_nx^{n-2}$$

$$(x^2-3)\left(\sum_{n=2}^{\infty} n(n-1)C_nx^{n-2}\right)+2x\left(\sum_{n=1}^{\infty} nC_nx^{n-1}\right)=0$$

$$\left(\sum_{n=2}^{\infty} n(n-1)C_nx^n\right)-\left(\sum_{n=2}^{\infty} 3n(n-1)C_nx^{n-2}\right)+\sum_{n=1}^{\infty} 2nC_nx^n=0$$

$(k=n-2,\ n=k+2)$

$$\left(\sum_{n=0}^{\infty} n(n-1)C_nx^n\right)-\left(\sum_{n=0}^{\infty} 3(n+2)(n+1)C_{n+2}x^n\right)+\sum_{n=0}^{\infty} 2nC_nx^n=0$$

$$\sum_{n=0}^{\infty}\left[n(n-1)C_n-3(n+2)(n+1)C_{n+2}+2nC_n\right]x^n=0$$

$\downarrow$

continued

$$C_n(n(n-1)+2n)-3(n+2)(n+1)C_{n+2}=0$$

$$C_n(n^2-\cancel{1n+2n})-3(n+2)(n+1)C_{n+2}=0$$

$$C_n(n^2)-3(n+2)(n+1)C_{n+2}=0$$

$$-C_n(n^2) \qquad -C_n(n^2)$$

$$\frac{-3(n+2)(n+1)C_{n+2}}{-3(n+2)(n+1)}=\frac{-C_n(n^2)}{-3(n+2)(n+1)}$$

$$\boxed{C_{n+2}=\frac{C_n(n^2)}{3(n+2)(n+1)}}$$

radius of Convergence

$$P(x)=\frac{x}{x^2-3} \quad Q(x)=0$$

$$\frac{-0\pm\sqrt{0^2-4(1)(-3)}}{2(1)}=\frac{\pm\sqrt{12}}{2}$$

$$=\pm\frac{2\sqrt{3}}{2}$$

$$=\pm\sqrt{3}$$

The radius of covergence is less than $\sqrt{3}$

n	
0	$C_2=0$
1	$C_3=\frac{1}{18}C_1$
2	$C_4=0$
3	$C_5=\frac{9}{60}C_3=\frac{9}{1080}C_1$
4	$C_6=0$
5	$C_7=\frac{25}{126}C_5=\frac{25}{136080}C_1$

$$y(x)=C_0(0)+C_1\left(\sum_{n=1}^{\infty}\frac{n^2}{3(n+2)(n+1)}x^n\right)$$

Solution

4) $(1+x^2)y''+2xy'-2y=0,\ y(0)=1,\ y'(0)=0$

$Y=C_0+C_1X+C_0X^2$, $\underline{y(0)=1=C_0}$

$Y'=C_1+C_0X+C_1X^2$, $\underline{y'(0)=0=C_1}$

$$Y=\sum_{n=0}^{\infty} C_nX^n \quad y'=\sum_{n=1}^{\infty} nC_nX^{n-1} \quad y''=\sum_{n=2}^{\infty} n(n-1)C_nX^{n-2}$$

$$(1+x^2)\left(\sum_{n=2}^{\infty} n(n-1)C_nX^{n-2}\right)+2x\left(\sum_{n=1}^{\infty} nC_nX^{n-1}\right)-2\left(\sum_{n=0}^{\infty} C_nX^n\right)=0$$

$$\left(\sum_{n=2}^{\infty} n(n-1)C_nX^{n-2}\right)+\left(\sum_{n=2}^{\infty} n(n-1)C_nX^n\right)+\left(\sum_{n=1}^{\infty} 2nC_nX^n\right)-\left(\sum_{n=0}^{\infty} 2C_nX^n\right)=0$$

$(k=n-2,\ n=k+2)$

$$\left(\sum_{n=0}^{\infty} (n+1)(n+2)C_{n+2}X^n\right)+\left(\sum_{n=0}^{\infty} n(n-1)C_nX^n\right)+\left(\sum_{n=0}^{\infty} 2nC_nX^n\right)-\left(\sum_{n=0}^{\infty} 2C_nX^n\right)=0$$

$$\sum_{n=0}^{\infty}\left[(C_n(n(n-1)+2n-2)+2n-2)+(n+1)(n+2)C_{n+2}\right]X^n$$

$(n(n-1)+2n-2)+2n-2$

$$\sum_{n=0}^{\infty}\left[(C_n(n(n-1)+2n-2)+2n-2)+(n+1)(n+2)C_{n+2}\right]X^n$$

$n\geq 2$

$$C_{n+2}=\frac{-C_n(n^2+n-2)}{(n+1)(n+2)}=\frac{-C_n\cancel{(n+2)}(n-1)}{(n+1)\cancel{(n+2)}}$$

$$=\frac{-C_n(n-1)}{(n+1)}$$

n	
0	$C_2=C_0$
1	$C_3=-0$
2	$C_4=-\frac{1}{3}C_2=-\frac{1}{3}C_0$
3	$C_5=0$
4	$C_6=-\frac{3}{5}C_4=\frac{1}{5}C_0$
5	$C_7=-0$

Solution

$$Y(X)=1\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{1+2n}X^n\right)$$

5) $y'' + xy' - 2y = 0, \quad y(0)=0, \quad y'(0)=3$

$y = c_0 + c_1 x + c_0 x^2$
$y' = c_1 + c_0 x + c_1 x^2$

$y(0) = c_0 = 0$ $\qquad y = \sum_{n=0}^{\infty} c_n x^n, \; y' = \sum_{n=1}^{\infty} n c_n x^{n-1}, \; y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

$y'(0) = c_1 = 3$

$$\left(\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}\right) + x\left(\sum_{n=1}^{\infty} n c_n x^{n-1}\right) - 2\left(\sum_{n=0}^{\infty} c_n x^n\right)$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} 2c_n x^n$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n + \sum_{n=0}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} 2c_n x^n$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)c_{n+2} + nc_n - 2c_n\right] x^n$$

$$c_{n+2} = \frac{-nc_n + 2c_n}{(n+2)(n+1)} = \frac{c_n(-n+2)}{(n+2)(n+1)} = \frac{-c_n(n-2)}{(n+2)(n+1)}$$

n	
0	$c_2 = c_0$
1	$c_3 = \frac{1}{6}c_1$
2	$c_4 = 0$
3	$c_5 = -\frac{1}{20}c_3 = \frac{-1}{120}c_1$
4	$c_6 = 0$
5	$c_7 = \frac{-3}{42}c_5 = \frac{3}{5040}$

$$\cancel{y(x) = c_0(c_0) + \left(c_1 \tfrac{1}{6}x + \sum_{n=3}^{\infty} \frac{(-1)^n (n-2)}{(n+2)!}\right)}$$

Solution: $\quad y(x) = \left[\frac{3}{6}x + \sum_{n=3}^{\infty} \frac{(-1)^n (n-2)}{(n+2)!}\right]$

6.) $y'' + 2xy' + 2y = 0, \quad y(0) = 6, \quad y'(0) = 2$

$$y = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$\left(\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}\right) + 2x\left(\sum_{n=1}^{\infty} n c_n x^{n-1}\right) + 2\left(\sum_{n=0}^{\infty} c_n x^n\right)$$

$$\left(\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}\right) + \sum_{n=1}^{\infty} 2n c_n x^n + \sum_{n=0}^{\infty} 2 c_n x^n$$

$$\left(\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n\right) + \sum_{n=0}^{\infty} 2n c_n x^n + \sum_{n=0}^{\infty} 2 c_n x^n$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) c_{n+2} + 2n c_n + 2 c_n\right] x^n$$

$y = c_0 + c_1 x + c_0 x^2$

$y' = c_1 + c_0 x + c_1 x^2$

$$c_{n+2} = \frac{-c_n(2n+2)}{(n+2)(n+1)}$$

$$= \frac{-2c_n(n+1)}{(n+2)(n+1)} \qquad n + (n+4) \quad (n+6)$$

$$= \frac{-2c_n}{(n+2)} \qquad n+2)!\,(-n)$$

$y(0) = c_0 = 6$

$y'(0) = c_1 = 2$

n	
0	$c_2 = -c_0$
1	$c_3 = \frac{-2}{3} c_1$
2	$c_4 = \frac{-2}{4} c_2 = \frac{2}{4} c_0$
3	$c_5 = \frac{-2}{5} c_3 = \frac{4}{15} c_1$
4	$c_6 = \frac{-2}{6} c_4 = -\frac{4}{24} c_0$
5	$c_7 = \frac{-2}{7} c_5 = \frac{-8}{105} c_1$

-1

$\frac{1}{2} c_0$

$\frac{1}{6} c_0$

$$y(x) = c_1\left(\frac{(-1)^n 2n-2}{n+2} x^n\right) + c_0\left(-1 + \sum_{n=2}^{\infty} \frac{-(1)^n}{n+2)!\,(2-n)} x^n\right)$$

61 $c_8 = \frac{-2}{8} c_6 \cdots \frac{16}{1} \cdots = \frac{1}{12} c_0$

7.) Solve the initial value problem by first making a substitution $t = x - a$ & then finding a series solution of the form $\sum_{n=0}^{\infty} C_n t^n$. Then transform the solution back to a power series in $x - a$.

$$y'' + (x-1)y' + y = 0, \quad y(1) = 3, \quad y'(1) = -1.$$

Let $t = x - a$, $x = t + a$

$$\frac{dt}{dx} = 1 \qquad y' = \frac{dy}{dx} = \frac{dy}{dt}\underbrace{\frac{dt}{dx}}_{=1} = \frac{dy}{dt}$$

$$y'' + ((t+a)-1)y' + y = 0 \qquad \left\{ \begin{aligned} y &= \sum_{n=0}^{\infty} C_n t^n \\ y' &= \sum_{n=1}^{\infty} n C_n t^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) C_n t^{n-2} \end{aligned} \right\}$$

$$\sum_{n=2}^{\infty} n(n-1)C_n t^{n-2} + (t+a-1)\sum_{n=1}^{\infty} nC_n t^{n-1} + \sum_{n=0}^{\infty} C_n t^n$$

$$\sum_{n=2}^{\infty} n(n-1)C_n t^{n-2} + \sum_{n=1}^{\infty} nC_n t^{n} + \sum_{n=1}^{\infty} anC_n t^{n-1} - \sum_{n=1}^{\infty} nC_n t^{n-1} + \sum_{n=0}^{\infty} C_n t^n$$

$(a=1)$

$$\sum_{n=0}^{\infty} (n+2)(n+1)C_{n+2} t^n + \sum_{n=0}^{\infty} nC_n t^n + \cancel{\sum_{n=0}^{\infty} (1)(n+1)C_{n+1} t^n - \sum_{n=0}^{\infty} (n+1)C_{n+1} t^n} + \sum_{n=0}^{\infty} C_n t^n$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)C_{n+2} + nC_n + C_n \right] t^n$$

$$C_{n+2} = \frac{-C_n\cancel{(n+1)}}{(n+2)\cancel{(n+1)}}$$

$$C_{n+2} = \frac{-C_n}{(n+2)}$$

$$C_{n+2} = \frac{-C_n}{(n+2)}$$

$n+4$

n	
0	$C_2 = \boxed{\frac{-1}{2} C_0}$
1	$C_3 = \boxed{\frac{-1}{3} C_1}$
2	$C_4 = \frac{-1}{4} C_2 = \boxed{\frac{1}{8} C_0}$
3	$C_5 = \frac{-1}{5} C_3 = \boxed{\frac{1}{15} C_1}$
4	$C_6 = \frac{-1}{6} C_4 = \boxed{\frac{-1}{48} C_0}$
5	$C_7 = \frac{-1}{7} C_5 = \boxed{\frac{-1}{105} C_1}$
6	$C_8 = \frac{-1}{8} C_6 = \boxed{\frac{1}{384} C_0}$

$C_0 = 3$

$C_1 = -1$

$$Y = \sum_{n=0}^{\infty} C_n (t-1)^n$$

$$Y = -\frac{3}{2}(3) - \frac{1}{3}(-1)(t-1) + \frac{1}{8}(3)(t-1)^2 + \ldots$$

$$t = x+1$$

$$y = -\frac{9}{2} + \frac{1}{3}(x+1) + \frac{3}{8}(x+1)^2 + \ldots$$

(Solution)

Series Solutions Near Regular Singular Points: The Frobenius Method-HW Problems

In problems 1-3 determine if $x = 0$ is an ordinary point, a regular singular point, or an irregular singular point.

1. $x^2y'' + x(x^2+1)y' + 2y = 0$
2. $(1+x^2)y'' + xy' + x^2y = 0$
3. $x^3(1+x^2)y'' + 2(1+x^2)y' - 6x(1-x)y = 0.$

4. Make a substitution to turn the singularity at $x = 1$ into a singularity at $t = 0$. Then determine if the singuarity is regular.

$$(x^2-1)^2y'' - x(x+1)y' + 3y = 0.$$

For problems 5-8 find two linearly independent solutions for $x > 0$.

5. $4xy'' + 2y' + 4y = 0$
6. $2x^2y'' + xy' - (1+2x^2)y = 0$
7. $3x^2y'' + x(1+x)y' - y = 0$
8. $xy'' + (3-x)y' - y = 0.$

In problems 1-3 determine if $X=0$ is an ordinary point, a regular singular point, or an irregular singular point.

1) $x^2y'' + x(x^2+1)y' + 2y = 0$

$$y'' + \frac{(x)(x^2+1)y'}{x^2} + \frac{2y}{x^2} = 0$$

$$p(x) = (x)\frac{(x^2+1)}{x} \qquad q(x) = (x^2)\frac{2}{x^2}$$, both defined

$$= \underline{(x^2+1)} \qquad = \underline{2}$$

{ $x=0$ is a regular point because both $p(x)$ & $q(x)$ is analytic at $x=0$. }

2.) $(1+x^2)y'' + xy' + x^2y = 0$

$$y'' + \frac{xy'}{(1+x^2)} + \frac{x^2y}{(1+x^2)} = 0$$

$$p(x) = x\left(\frac{x}{1+x^2}\right) \qquad q(x) = x^2\left(\frac{x^2}{1+x^2}\right)$$, both defined

{ $X=0$ is a regular point because once again $p(x)$ & $q(x)$ is analytic at $x=0$. }

3.) $x^3(1+x^2)y'' + 2(1+x^2)y' - 6x(1-x)y = 0$

$$y'' + \frac{2\cancel{(1+x^2)}y'}{x^3\cancel{(1+x^2)}} - \frac{6x(1-x)y}{x^3(1+x^2)} = 0$$

undefined at $x=0$

$$p(x) = x\left(\frac{2}{x^3}\right) = \boxed{\frac{2}{x^2}} \quad q(x) = \cancel{x^2}\left(\frac{6(1-x)}{\cancel{x^2}(1+x^2)}\right) = \boxed{\frac{6(1-x)}{(1+x^2)}}$$

$p(x)$ & $q(x)$ are not both analytical at zero, therefore $x=0$ is an irregular singular point.

4.) Make a substitution to turn the singularity at $x=1$ into a singularity at $t=0$. Then determine if the singularity is regular

$(x^2-1)^2 y'' - x(x+1)y' + 3y = 0$

$t = 0 = x-1, \quad x = t+1$

$((t+1)^2-1)^2 y'' - (t+1)((t+1)+1)y' + 3y = 0$

$(t+1)(t+1)$

$(t^2+t+t+1-1)^2 y'' - (t+1)(t+2)y' + 3y = 0$

$$\frac{(t^2+2t)^2 y''}{(t^2+2t)^2} - \frac{(t+1)(t+2)y'}{(t^2+2t)^2} + \frac{3y=0}{(t^2+2t)}$$

The singularity at $t=0$ is irregular because $p(t)$ & $q(t)$ are not analytic at $t=0$

$$p(t) = (t)\frac{-(t+1)(t+2)}{(t^2+2t)^2} \quad q(t) = \cancel{(t^2)}\frac{1}{(\cancel{t^2}+2t)} = \frac{1}{2(t)}$$

For problems 5–8 find two linearly independent solutions for $X > 0$.

5.) $4xy'' + 2y' + 4y = 0$

$$y'' + \frac{2y'}{4x} + \frac{4y}{4x} = 0$$

$$p(x) = x\left(\frac{1}{2x}\right),\ q(x) = x^2\left(\frac{1}{x}\right)$$

$$p(x) = \frac{1}{2},\ q(x) = x$$

$$r(r-1) + p_0 r + q_0 = r(r-1) + \frac{1}{2}r + 0$$

$$= r^2 - r + \frac{1}{2}r$$

$$= r^2 - \frac{1}{2}r$$

$$= r\left(r - \frac{1}{2}\right)$$

$$r = 0 \quad r = \frac{1}{2}$$

$$y = \sum_{n=0}^{\infty} C_n x^{n+r},\ y' = \sum_{n=0}^{\infty} (n+r) C_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) C_n x^{n+r-2}$$

$$4x\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) C_n x^{n+r-2}\right) + 2\left(\sum_{n=0}^{\infty} (n+r) C_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} C_n x^{n+r}\right) = 0$$

$$\sum_{n=0}^{\infty} 4(n+r)(n+r-1) C_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r) C_n x^{n+r-1} + \sum_{n=0}^{\infty} 4 C_n x^{n+r} = 0$$

$$\sum_{n=1}^{\infty} 4(n+r)(n+r+1) C_{n-1} x^{n+r} + \sum_{n=0}^{\infty} 2(n+r) C_{n-1} x^{n+r} + \sum_{n=2}^{\infty} 4 C_n x^{n+r} = 0$$

$+r(\ \ +4(n+r)(n+r+1)+2(n+r+1)C_{n-1}+4C_n$

$$C_n = \frac{-C_{n-1}(4(n+r+1)(n+r)+2(n+r))}{4}$$

Case 1: $r=\frac{1}{2}$

$n \geq 1$

$$a_0 = \frac{-C_{n+1}(4(n+\frac{1}{2}+1)(n+\frac{1}{2})+2(n+\frac{1}{2}))}{4}$$

$n=1,\ a_1 = \left(-\frac{18}{4}a_0\right)$

$n=2,\ a_2 = -\frac{40}{4}a_1 = \left(\frac{45}{1}a_0\right)$

$$Y_1(x) = \left(1+\sum_{n=1}^{\infty}\frac{(18)(40)\cdots}{4}x^n\right)$$

$n=3,\ a_3 = -\frac{70}{4}a_2 = \left(\frac{3150}{4}a_0\right)$

$n=4,\ a_4 = -\frac{108}{4}a_3 = \left(\frac{340200}{1}a_0\right)$

Case 2: $r=0$

$$b_0 = \frac{-C_{n+1}(4(n+1)(n)+2(n))}{4}$$

$n=1,\ b_1 = \left(-\frac{10}{4}b_0\right)$

$n=2,\ b_2 = -\frac{28}{4}b_1 = \left(\frac{280}{1}b_0\right)$

$$Y_2(x) = \left(1+\sum_{n=1}^{\infty}\frac{(10)(28)\cdots}{4}x^n\right)$$

$n=3,\ b_3 = -\frac{54}{4}b_2 = \left(\frac{15120}{4}b_0\right)$

$n=4,\ b_4 = -\frac{88}{4}b_3 = \left(\frac{1330560}{1}b_0\right)$

General Solution

$$Y(x) = a_0 x^0\left(1+\sum_{n=1}^{\infty}\frac{(10)(28)\cdots}{4}x^n\right) + b_0 x^{\frac{1}{2}}\left(1+\sum_{n=1}^{\infty}\frac{(18)(40)\cdots}{4}x^n\right)$$

6.) $2x^2y''+xy'-(1+2x^2)y=0$

$$y''+\frac{xy'}{2x^2}-\frac{(1+2x^2)y}{2x^2}=0$$

$$y''+\frac{y'}{2x}-\left(\frac{1+2x^2}{2x^2}\right)y=0$$

$$p(x)=x\left(\frac{1}{2x}\right),\quad q(x)=x^2\cdot-\left(\frac{1+2x^2}{2x^2}\right)$$

$$=\frac{1}{2}\qquad\qquad =-\left(\frac{1+2x^2}{2}\right)=\left(\frac{1}{2}+x^2\right)$$

$$p_0=\frac{1}{2}\qquad\qquad q_0=-\frac{1}{2}$$

$$r(r-1)+\frac{1}{2}r-\frac{1}{2}=0$$

$$r^2-r+\frac{1}{2}r-\frac{1}{2}=0$$

$$r^2-\frac{1}{2}r-\frac{1}{2}=0\qquad y=\sum_{n=0}^{\infty}c_nx^{n+r}$$

$$\left(r+\frac{1}{2}\right)(r-1)=0\qquad y'=\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}$$

$$r=-\frac{1}{2},1$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2}$$

$$0=2x^2\left(\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2}\right)+x\left(\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}\right)-(1+2x^2)\left(\sum_{n=0}^{\infty}c_nx^{n+r}\right)$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)c_nx^{n+r}+\sum_{n=0}^{\infty}(n+r)c_nx^{n+r}-\sum_{n=0}^{\infty}c_nx^{n+r}-\sum_{n=0}^{\infty}c_nx^{n+r+2}=0$$

$$\sum_{n=0}^{\infty}c_{n-2}x^{n+r}$$

Continued

$$[2(n+r)(n+r-1)C_n + (n+r)C_n - C_n - \cancel{C_{n-2}}] = 0$$

$+C_{n-2}$ $+C_{n-2}$

$$\frac{C_n[2(n+r)(n+r-1)+(n+r)-1]}{[2(n+r)(n+r-1)+(n+r)-1]} = \frac{C_{n-2}}{2(n+r)(n+r-1)+(n+r)-1}$$

Case: $r_1 = -\frac{1}{2}$

$$C_n = \frac{C_{n-2}}{2(n-\frac{1}{2})(n-\frac{1}{2}-1)+(n-\frac{1}{2})-1} = \frac{C_{n-2}}{(2n-1)(n-\frac{3}{2})+(n-\frac{3}{2})}$$

$n=2$, $C_2 = \frac{C_0}{(3)(\frac{1}{2})+(\frac{1}{2})} = \frac{1}{2}C_0$

$n=4$, $C_4 = \frac{C_2}{(7)(\frac{5}{2})+(\frac{5}{2})} = \frac{1}{20}C_2 = \frac{1}{40}C_0$

$n=6$, $C_6 = \frac{C_4}{(11)(\frac{9}{2})+(\frac{9}{2})} = \frac{1}{54}C_4 = \frac{1}{2160}C_0$

$n=8$, $C_8 = \frac{C_6}{(15)(\frac{13}{2})+(\frac{13}{2})} = \frac{1}{104}C_6 = \frac{1}{224640}C_0$

General Solution

$$y(x) = C_0 x^{-\frac{1}{2}}\left(1+\sum_{n=1}^{\infty}\frac{x^{2n}}{((n+1)+5)(\frac{(n-1)+4}{2})+(\frac{(n-1)+4}{2})}\right) + B_0 x\left(1+\sum_{n=1}^{\infty}\frac{x^{2n}}{2(n+2n)+2^n}\right)$$

$$y_1(x) = C_0 x^{-(\frac{1}{2})}\left(1+\sum_{n=1}^{\infty}\frac{1}{((n+1)+5)(\frac{(n-1)+4}{2})+(\frac{(n-1)+4}{2})}\right)$$

$$B_n = \frac{B_{n-2}}{2(n+1)(n)+(n)}$$

Case: $r_2 = 1$, $n=2$, $B_2 = \frac{B_0}{2(2+1)(2)+(2)} = \frac{B_0}{14}$

$n=4$, $B_4 = \frac{B_2}{2(4+1)(4)+(4)} = \left(\frac{B_0}{616}\right)$

$n=6$, $B_6 = \frac{B_4}{2(6+1)(6)+(6)} = \left(\frac{B_0}{55440}\right)$

$$y_2(x) = B_0 x\left(1+\sum_{n=1}^{\infty}\frac{1}{2(n+2(n))+2^n}\right)$$

7.) $3x^2y'' + x(1+x)y' - y = 0$

$$y'' + \frac{x(1+x)y'}{3x^2} - \frac{y}{3x^2} = 0$$

$$y'' + \frac{(1+x)y'}{3x} - \frac{y}{3x^2} = 0$$

$$p(x) = x\left(\frac{(1+x)}{3x}\right), \quad q(x) = x^2\left(-\frac{1}{3x^2}\right)$$

$$= \frac{(1+x)}{3} \qquad = -\frac{1}{3}$$

$$= r(r-1) + \tfrac{1}{3}r - \tfrac{1}{3} = r^2 - r + \tfrac{1}{3}r - \tfrac{1}{3} = r^2 - \tfrac{2}{3}r - \tfrac{1}{3}$$

$$= (r + \tfrac{1}{3})(r-1)$$

$$y = \sum_{n=0}^{\infty} C_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) C_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) C_n x^{n+r-2} \qquad r_2 = -\frac{1}{3} \quad r_1 = 1$$

$$3x^2\left(\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n x^{n+r-2}\right) + x(1+x)\left(\sum_{n=0}^{\infty}(n+r)C_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} C_n x^{n+r}\right) = 0$$

$$\sum_{n=0}^{\infty} 3(n+r)(n+r-1)C_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)C_n x^{n+r} + \sum_{n=1}^{\infty}(n+r)C_n x^{n+r+1} - \sum_{n=0}^{\infty} C_n x^{n+r} = 0$$

$$\uparrow$$

$$\sum_{n=0}^{\infty}(n+r)C_{n-1}x^{n+r}$$

Case 1: $r=1$ $\quad C_n = \dfrac{-(n+r)C_{n-1}}{3(n+r)(n+r-1) + (n+r) - 1}$

$$a_n = \frac{-(n+1)C_{n-1}}{3(n+1)(n) + (n)}$$

$n=1$, $\quad a_1 = \dfrac{-2}{3(2)(1)+(1)} a_0 = \dfrac{-2}{7} a_0$

$n=2$ $\quad a_2 = \dfrac{-3}{3(3)(2)+(2)} a_1 = \dfrac{-3}{20} a_0$

$n=3$ $\quad a_3 = \dfrac{-4}{3(4)(3)+(3)} a_2 = \dfrac{-4}{39} a_0$

$$y_1 = \left(1 + \sum_{n=1}^{\infty} \frac{(n+1)(-1)^n}{3(n+1)(n)+(n)} x^n\right)$$

Case 2: $r = -\frac{1}{3}$

$$B_n = \frac{-(n-\frac{1}{3})B_{n-1}}{3(n-\frac{1}{3})(n-\frac{1}{3}-1)+(n-\frac{1}{3})-1} = \frac{-(n-\frac{1}{3})B_{n-1}}{3(n-\frac{1}{3})(n-\frac{4}{3})+(n-\frac{4}{3})}$$

$n=1$, $B_1 = \frac{-(\frac{2}{3})}{3(\frac{2}{3})(-\frac{1}{3})+(-\frac{1}{3})} B_0 = \frac{2}{-1} B_0$

$n=2$, $B_2 = \frac{-(\frac{5}{3})}{3(\frac{5}{3})(\frac{2}{3})+(\frac{2}{3})} B_1 = -\frac{5}{4} B_1 = \frac{10}{4} = \frac{5}{2} B_0$

$n=3$, $B_3 = \frac{-(\frac{8}{3})}{3(\frac{8}{3})(\frac{5}{3})+(\frac{5}{3})} B_2 = -\frac{8}{15} B_2 = -\frac{4}{3} B_0$

$$2 + \sum_{n=1}^{\infty} \frac{2(\frac{1}{6})^n(-1)^n}{n}$$

$$y_2 = 2 + \sum_{n=1}^{\infty} \frac{2(\frac{1}{6})^n(-1)^n}{n} x^n$$

$$y(x) = \left(2 + \sum_{n=1}^{\infty} \frac{2(\frac{1}{6})^n(-1)^n}{n} x^n\right) + \left(1 + \sum_{n=1}^{\infty} \frac{(n+1)(-1)^n}{3(2+n)(n)+(n)} x^n\right)$$

8.) $xy'' + (3-x)y' - y = 0$

$$y'' + \frac{(3-x)}{x}y' - \frac{y}{x} = 0$$

$$p(x) = x\left(\frac{(3-x)}{x}\right) \qquad q(x) = x^2\left(\frac{1}{x}\right)$$

$$= (3-x) \qquad\qquad = x$$

$$r(r-1)+3r+1 = r^2-r+3r+1 = r^2+2r+1$$

$$(r+1)(r+1)$$

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}, \qquad r_1 = 1 \quad r_2 = -1$$

$$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}$$

$$x\left(\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}\right) + (3-x)\left(\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty}c_n x^{n+r}\right) = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}3(n+r)c_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} - \sum_{n=0}^{\infty}c_n x^{n+r} = 0$$

$$\uparrow \qquad\qquad\qquad \uparrow$$

$$\sum_{n=0}^{\infty}(n+r+1)(n+r)c_{n+1}x^{n+r} + \sum_{n=1}^{\infty}3(n+r+1)c_{n-1}x^{n+r}$$

$$c_n = \frac{(n+r+1)(n+r)+3(n+r+1)c_{n-1}}{-(n+r)-1}$$

Case 1: $r_1 = 1$

$$c_n = \frac{(n+2)(n+1)+(3n+6)c_{n-1}}{-n-2}$$

$n=1,\ c_1 = \frac{18}{-3}c_0$

$n=2,\ c_2 = \frac{28}{-4}c_1 = \frac{504}{12}$

$n=3,\ c_3 = \frac{40}{-5}c_2 = \frac{-20160}{60}$

$n=4,\ c_4 = \frac{54}{-6}c_0 = \frac{1058640}{360}$

$$y_1 = \left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n(n+2)^2+(3n+6)c_n}{n+2}\right)$$

Case 2: $r=-1$ $\quad C_n = \dfrac{(n)(n-1)+3(n)\,C_{n-1}}{-n}$

$y_2 =$ $n=1,\ C_1 = \dfrac{(1)(0)+3}{-1} C_0 = -3C_0$

$n=2,\ C_2 = \dfrac{(2)(1)+6}{-2} C_1 = \dfrac{8}{-2} C_1 = \dfrac{21}{-2} C_0$

$n=3,\ C_3 = \dfrac{(3)(2)+9}{-3} C_2 = \dfrac{15}{-3} C_2 = -\dfrac{315}{6} C_0$

$n=4,\ C_4 = \dfrac{(4)(3)+12}{-4} C_3 = \dfrac{24}{-4} C_3 = \dfrac{7560}{24} C_0$

$$y_2 = \left(1 + \sum_{n=1}^{\infty} \frac{(1)(0)\cdots(n)(-1)^2}{n} x^n\right)$$

$$y(x) = \left(1 + \sum_{n=1}^{\infty} \frac{(1)(0)\cdots(n)(-1)^2}{n} x^n\right) + \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^2 (n+2)^2 + (3n+6)}{n+2} x^n\right)$$