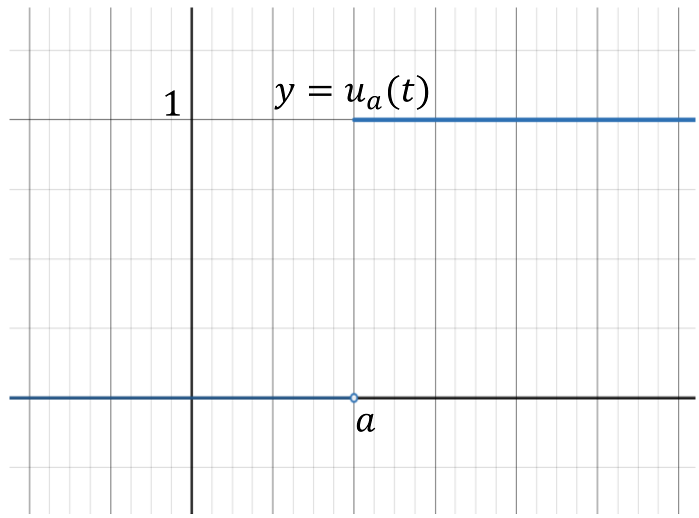
Piecewise Continuous Functions

Recall if:

$$u(t) = 0 \text{ if } t < 0$$
$$= 1 \text{ if } t \geq 0$$
$$u_a(t) = u(t-a) = 0 \text{ if } t < a$$
$$= 1 \text{ if } t \geq a$$



Then:

$$\mathcal{L}\big(u(t)\big) = \frac{1}{s}$$
$$\mathcal{L}\big(u(t-a)\big) = \frac{e^{-as}}{s}.$$

Theorem: Translation on the t-axis.

If $\mathcal{L}(f(t))$ exists for $s > c$, then $\mathcal{L}\big(u(t-a)f(t-a)\big) = e^{-as}F(s)$ and $\mathcal{L}^{-1}\big(e^{-as}F(s)\big) = u(t-a)f(t-a)$.

Notice that:

$$u(t-a)f(t-a) = 0 \text{ if } t < a$$
$$= f(t-a) \text{ if } t \geq a.$$

Proof:

$$e^{-as}F(s) = e^{-as}\int_{w=0}^{w=\infty} e^{-sw}f(w)\,dw$$

$$= \int_{w=0}^{w=\infty} e^{-s(w+a)}f(w)\,dw$$

Let $t = w + a$

$$dt = dw$$

$$e^{-as}F(s) = \int_{t=a}^{t=\infty} e^{-st}f(t-a)\,dt$$

$$= \int_{t=0}^{t=\infty} e^{-st}u(t-a)f(t-a)\,dt$$

$$= \mathcal{L}\big(u(t-a)f(t-a)\big).$$

Ex. Let $F(s) = \frac{e^{-s}}{s+2}$. Find $\mathcal{L}^{-1}(F(s))$.

$$\mathcal{L}^{-1}\left(e^{-as}\big(G(s)\big)\right) = u(t-a)g(t-a)$$

where $\mathcal{L}\big(g(t)\big) = G(s)$.

So for $\mathcal{L}^{-1}\left(\frac{e^{-s}}{s+2}\right)$, $a = 1$ and $G(s) = \frac{1}{s+2}$.

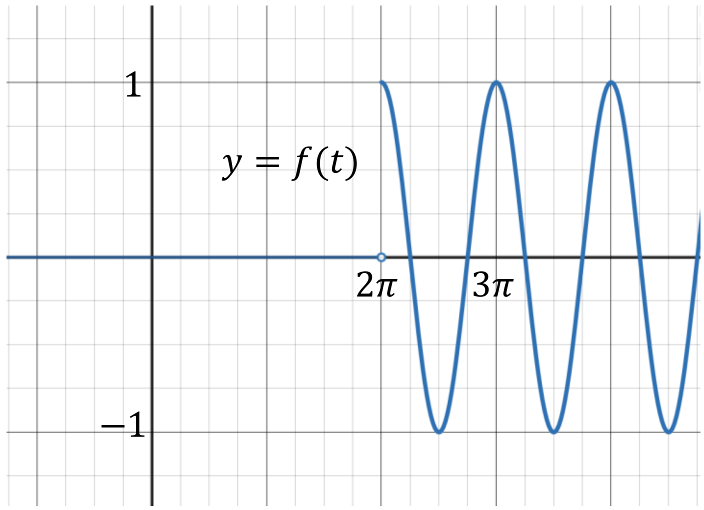
From a Laplace transform table, if $g(t) = e^{-2t}$, then $G(s) = \frac{1}{s+2}$.

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{s+2}\right) = u(t-1)g(t-1) = u(t-1)e^{-2(t-1)}.$$

Ex. Find $\mathcal{L}(f(t))$ when

$$f(t) = 0 \qquad \text{if } t < 2\pi$$

$$= \cos 2t \quad \text{if } t \geq 2\pi.$$



$$f(t) = u(t - 2\pi)\cos 2t$$

$$= u(t - 2\pi)\cos(2t - 2\pi)$$

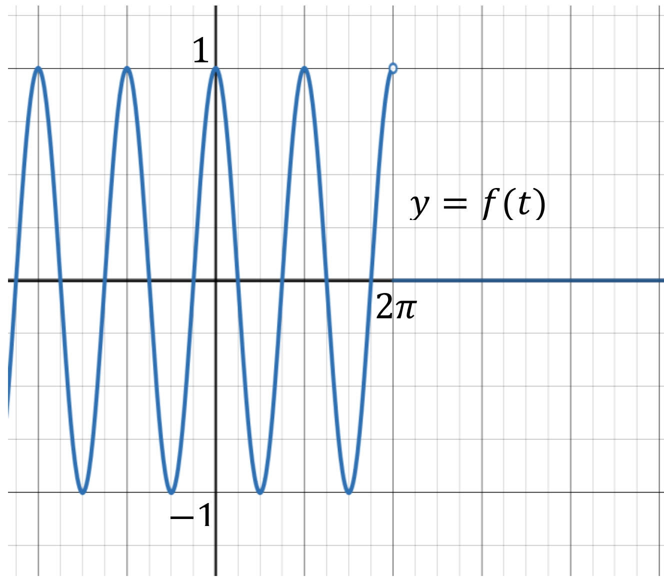
$$\mathcal{L}(u(t - 2\pi)\cos(2t - 2\pi)) = e^{-2\pi s}\mathcal{L}(\cos 2t)$$

$$= e^{-2\pi s}\left(\frac{s}{s^2+4}\right).$$

Ex. Find $\mathcal{L}(f(t))$ when

$$f(t) = \cos 2t \qquad \text{if } t < 2\pi$$

$$= 0 \qquad \text{if } t \geq 2\pi.$$



$$u(t - 2\pi) = 0 \text{ if } t < 2\pi$$

$$= 1 \text{ if } t \geq 2\pi.$$

So

$$1 - u(t - 2\pi) = 1 \text{ if } t < 2\pi$$

$$= 0 \text{ if } t \geq 2\pi.$$

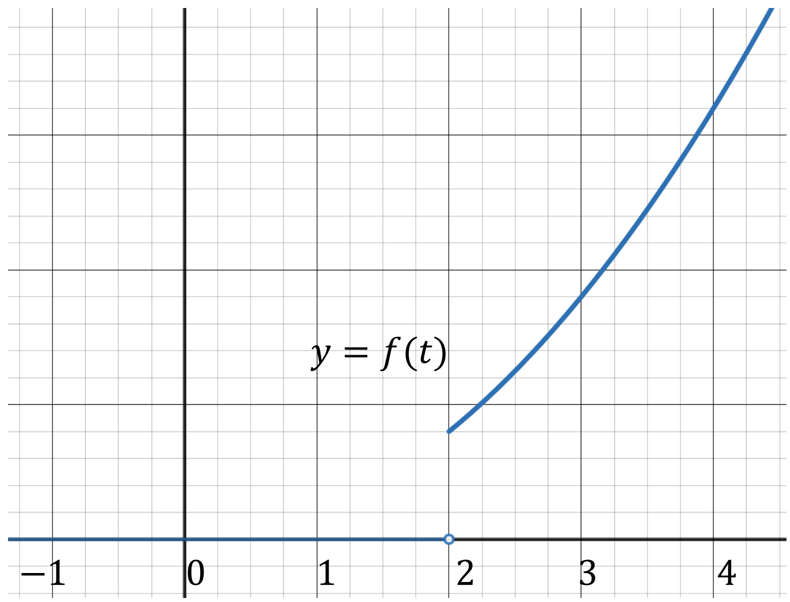
Thus,

$$f(t) = \big(1 - u(t - 2\pi)\big)\cos 2t = \cos 2t - u(t - 2\pi)\cos[2(t - 2\pi)].$$

$$\mathcal{L}(\cos 2t - u(t-2\pi)\cos[2(t-2\pi)])$$
$$= \mathcal{L}(\cos 2t) - \mathcal{L}(u(t-2\pi)\cos[2(t-2\pi)])$$
$$\mathcal{L}(f(t)) = \frac{s}{s^2+4} - \frac{e^{-2\pi s}s}{s^2+4} = \frac{s(1-e^{-2\pi s})}{s^2+4}.$$

Ex. Find $\mathcal{L}(g(t))$ when

$$g(t) = 0 \quad \text{if } t < 2$$
$$= t^2 \quad \text{if } t \geq 2.$$



We need to write $g(t) = u(t-2)f(t-2)$.

So after translating $f(t)$ two units to the right we get t^2.

$f(t) = (t+2)^2$ works (i.e. $f(t-2) = t^2$).

$$\mathcal{L}(g(t)) = \mathcal{L}\big(u(t-2)f(t-2)\big) = e^{-2s}\mathcal{L}(t^2+2t+4)$$
$$= e^{-2s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{4}{s}\right).$$

Ex. A mass weighing 32 lbs. is attached to the end of a spring that is stretched one foot by a force of 4 lbs (i.e. $k = 4lb/ft$). Initially, The mass is at rest. At time $t = 0$ seconds, an external force $f(t) = 4\cos 2t$ is applied to the mass. At time $t = 2\pi$ seconds the force is turned off and the mass continues its motion. Find $x(t)$, the position of the mass at time t.

So we need to solve: $x'' + 4x = f(t), \quad x(0) = x'(0) = 0$

where $f(t) = 4\cos 2t$ if $0 \le t < 2\pi$

$= 0$ if $t \ge 2\pi$.

Taking the Laplace transform of this equation we get:

$$\left(s^2X(s) - sx(0) - x'(0)\right) + 4X(s) = \frac{4s(1-e^{-2\pi s})}{s^2+4}$$

where the RHS comes from a previous example.

$$(s^2 + 4)X(s) = \frac{4s(1-e^{-2\pi s})}{s^2+4}$$

$$X(s) = \frac{4s(1-e^{-2\pi s})}{(s^2+4)^2} = \frac{4s}{(s^2+4)^2} - \left(e^{-2\pi s}\right)\frac{4s}{(s^2+4)^2}.$$

Recall that:

$$g(t) = \mathcal{L}^{-1}\big(G(s)\big) = t\mathcal{L}^{-1}(\int_{w=s}^{\infty} G(w)\, dw)$$

$$\mathcal{L}^{-1}\left(\frac{4s}{\left(s^2+4\right)^2}\right) = t\mathcal{L}^{-1}\left(\int_{w=s}^{\infty} \frac{4w}{\left(w^2+4\right)^2}\ dw\right)$$

$$= t\mathcal{L}^{-1}\left(-\frac{2}{w^2+4}\Big|_{w=s}^{w=\infty}\right)$$

$$= t\mathcal{L}^{-1}\left(\left(\frac{2}{s^2+4}\right)\right) = t\sin 2t$$

$$\mathcal{L}^{-1}\big(e^{-as}F(s)\big) = u(t-a)f(t-a)$$

$$\mathcal{L}^{-1}\left(e^{-2\pi s}\frac{s}{\left(s^2+4\right)^2}\right) = u(t-2\pi)(t-2\pi)(\sin 2(t-2\pi))$$

Since: $$X(s) = \frac{4s}{\left(s^2+4\right)^2} - \left(e^{-2\pi s}\right)\frac{4s}{\left(s^2+4\right)^2}\,.$$

we have:

$$x(t) = \mathcal{L}^{-1}\left(\frac{4s}{\left(s^2+4\right)^2}\right) - \mathcal{L}^{-1}\left(e^{-2\pi s}\frac{s}{\left(s^2+4\right)^2}\right)$$

$$x(t) = t(\sin 2t) - (u(t-2\pi))(t-2\pi)(\sin 2t)$$

$$x(t) = \big[t - \big(u(t-2\pi)\big)(t-2\pi)\big]\sin 2t.$$

In other words:

$x(t) = t\sin 2t$ if $t < 2\pi$ since $\big(u(t-2\pi)\big)(t-2\pi) = 0$

$= 2\pi\sin 2t$ if $t \geq 2\pi$ since $\big(u(t-2\pi)\big)(t-2\pi) = t-2\pi.$