In Problems 1 to 3 find the Laplace transform of the given function. 1.) $x(t)=t^2e^{3t}$

$$\int_0^\infty e^{-st}e^{3t}t^2\,dt$$

$$\int_0^\infty \underbrace{e^{-st+(3t)}}_{dv}\ \underbrace{t^2}_{u}\,dt \qquad v=-\frac{1}{(s-3)}e^{-st+(3t)} \qquad du=2t\,dt$$

$$\underbrace{t^2\left(-\frac{1}{(s-3)}e^{-st+(3t)}\right)\Big|_0^\infty}_{0}-\int_0^\infty -\frac{1}{(s-3)}e^{-st+(3t)}\,2t\,dt$$

$$\frac{2}{s-3}\int_0^\infty \underbrace{e^{-st+(3t)}}_{dv}\ \underbrace{t}_{u}\,dt \qquad v=-\frac{1}{(s-3)}e^{-st+(3t)} \qquad du=dt$$

$$\frac{2}{s-3}\left(\underbrace{t\left(-\frac{1}{(s-3)}e^{-st+(3t)}\right)\Big|_0^\infty}_{0}+\int_0^\infty \frac{1}{(s-3)}e^{-st+(3t)}\,dt\right)$$

$$\left(\frac{2}{s-3}\right)\left(\frac{1}{s-3}\right)\int_0^\infty e^{-st+(3t)}\,dt$$

$$\left(\frac{2}{s-3}\right)\left(\frac{1}{s-3}\right)\left(-\frac{1}{(s-3)}\right)\left(e^{-st+(3t)}\right)\Big|_0^\infty \quad \to (-1)$$

$$=\frac{2}{(s-3)(s-3)(s-3)}$$

$$\boxed{\mathcal{L}[t^2e^{3t}]=\frac{2}{(s-3)^3}}$$ Answer

In Problems 1 & 3 find the Laplace transform of the given function. 1.) $f(t)=e^{\frac{t}{2}}\cos(5t)$

$$\int_0^\infty e^{-st}e^{\frac{t}{2}}\cos(5t)\,dt$$

$$\int_0^\infty \underbrace{e^{-st+(\frac{t}{2})}}_{D}\underbrace{\cos(5t)}_{I}\,dt$$

	D	I
+	$e^{-st+(\frac{t}{2})}$	$\cos(5t)$
−	$(-s+\frac{1}{2})e^{-st+(\frac{t}{2})}$	$\frac{\sin(5t)}{5}$
+	$(-s+\frac{1}{2})^2e^{-st+(\frac{t}{2})}$	$-\frac{\cos(5t)}{25}$

$$\frac{25}{25+(-s+\frac{1}{2})^2}\left(e^{-st+(\frac{t}{2})}\left(\frac{\sin(5t)}{5}\right)+(-s+\tfrac{1}{2})e^{-st+(\frac{t}{2})}\frac{\cos(5t)}{25}\right)\Bigg|_0^\infty$$

$$\times\left(-\frac{(-s+\frac{1}{2})}{25}\right)$$

$$\boxed{-\frac{(-s+\frac{1}{2})}{25+(-s+\frac{1}{2})^2}} = \mathcal{L}\left[e^{\frac{t}{2}}\cos(5t)\right]$$

Answer

$$\int_0^\infty e^{-st+(\frac{t}{2})}\cos(5t)\,dt = e^{-st+(\frac{t}{2})}\left(\frac{\sin(5t)}{5}\right)+(-s+\tfrac{1}{2})e^{-st+(\frac{t}{2})}\frac{\cos(5t)}{25}-\int(-s+\tfrac{1}{2})^2e^{-st+(\frac{t}{2})}\frac{\cos 5t}{25}\,dt$$

$$1+\frac{(-s+\frac{1}{2})^2}{25}\int = \frac{25+(-s+\frac{1}{2})^2}{25}$$

Reciprocal to bring to other side

$$\frac{25}{25+(-s+\frac{1}{2})^2}$$

$$-\frac{(-s+\frac{1}{2})^2}{25}\int$$

$$+\left(\frac{-s+\frac{1}{2}}{25}\right)\int$$

3.) $X(t) = e^{5t}\sin(2t)$

$$\int_0^\infty e^{-st}\left(e^{5t}\sin(2t)\right)dt$$

$$\int_0^\infty \underset{D}{\underline{e^{-st+5t}}}\ \underset{I}{\underline{\sin(2t)}}\,dt$$

	D	I
+	e^{-st+5t}	$\sin(2t)$
−	$(-s+5)e^{-st+5t}$	$-\frac{\cos(2t)}{2}$
+	$(-s+5)^2e^{-st+5t}$	$-\frac{\sin(2t)}{4}$

$$\int_0^\infty e^{-st+(5t)}\sin(2t)\,dt =$$

$$-e^{-st+(5t)}\left(\frac{\cos(2t)}{2}\right) + (-s+5)e^{-st+(5t)}\left(\frac{\sin(2t)}{4}\right) - \int_0^\infty (-s+5)^2 e^{-st+(5t)}\left(\frac{\sin(2t)}{4}\right)dt$$

$$-\frac{(-s+5)^2}{4}\int_0^\infty e^{-st+(5t)}\sin(2t)\,dt$$

Adding to other Side of Eq : $1 + \frac{(-s+5)^2}{4} = \frac{4+(-s+5)^2}{4}$

Multiply reciprocal to other side & Evaluate:

$$\frac{4}{4+(-s+5)^2}\left(-e^{-st+(5t)}\left(\frac{\cos(2t)}{2}\right) + (-s+5)e^{-st+(5t)}\left(\frac{\sin(2t)}{4}\right)\right)\Bigg|_0^\infty$$

$$\frac{4}{4+(-s+5)^2}\left(-\left(-1\left(\frac{1}{2}\right)\right)\right) = \boxed{\frac{2}{4+(-s+5)^2}}$$ Answer

In Problems 1-3 find the inverse Laplace Transform of the given function. Use Partial fractions when appropriate.

4.) $F(s) = \dfrac{1}{s^2+6s+9}$

Use Partial fractions

$$\frac{1}{(s+3)^2} = \frac{A}{(s+3)} + \frac{B}{(s+3)^2} = \frac{0}{s+3} + \frac{1}{(s+3)^2}$$

Funny (sorta redundant)

$1 = A(s+3) + B$

$s=-3$

$\boxed{1=B}$

$s=1$

$1 = 4A + 1$

$-1 \quad\quad -1$

$\frac{0}{4} = \frac{4A}{4}$

$\boxed{0=A}$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+6s+9}\right] = \mathcal{L}^{-1}\left[0 + \frac{1}{(s+3)^2}\right]$$

$= 0 + e^{-3}t^{1}$

$= e^{-3}t^{1}$ ← Answer

In problems 1-3 find the inverse laplace Transform of the given function. Use partial fractions when appropriate.

5.) $F(s) = \dfrac{s+3}{s^2+4s+8}$

$$\downarrow$$

$$\left(\frac{s+3}{(s^2+4s+4)+4}\right)$$

$$\mathcal{L}^{-1}\left[\frac{s+3}{(s+2)^2+4}\right] = e^{-3t}\cos 2t$$

6.) $F(s) = \dfrac{6s^2-s-6}{s^3-s^2-6s}$

$$\frac{6s^2-s-6}{s(s^2-s-6)} = \frac{A}{s} + \frac{B}{(s-3)} + \frac{C}{(s+2)}$$

$$\downarrow$$

$$(s-3)(s+2)$$

$$6s^2-s-6 = A(s-3)(s+2) + B(s)(s+2) + C(s)(s-3)$$

$s=3$ | $s=-2$ | $s=0$

$\dfrac{45}{15} = \dfrac{\cancel{15}B}{\cancel{15}}$ | $\dfrac{20}{10} = \dfrac{\cancel{10}C}{\cancel{10}}$ | $\dfrac{-6}{-6} = \dfrac{\cancel{-6}A}{\cancel{-6}}$

$$\mathcal{L}^{-1}\left[1\left(\frac{1}{s}\right) + 3\left(\frac{1}{s-3}\right) + 2\left(\frac{1}{s+2}\right)\right]$$

$$= (1) + 3(e^{3t}) + 2(e^{-2t})$$

Answer

7.) $F(s) = \dfrac{4}{s^3-4s}$

$$\frac{4}{s(s^2-4)} = \frac{As+B}{s^2-2^2} + \frac{C}{s}$$

$$4 = (As+B)(s) + C(s^2-4)$$

$$4 = As^2 + Bs + Cs^2 - 4C$$

$$4 = s^2(A+C) + Bs - 4C$$

$0 = A+C$ $\quad A = 1$

$4 = -4C$ $\quad C = -1$

$0 = B$

$$\mathcal{L}^{-1}\left[\left(\frac{s}{s^2-4}\right) + (-1)\left(\frac{1}{s}\right)\right]$$

$$= \cosh(2t) + (-1)$$

Answer

In Problems 1-3 find the inverse Laplace Transform of the given function. Use Partial fractions when appropriate.

8.) $F(s)=\frac{1}{s^4-2s^2+1}$

$U=s^2, \quad \frac{1}{U^2-2U+1}$

$=\frac{1}{(U-1)(U-1)}$

$=\frac{1}{(s^2-1)(s^2-1)}$

$=\frac{1}{(s+1)(s-1)(s+1)(s-1)}$

$=\frac{1}{(s+1)^2(s-1)^2}$

$=\frac{A}{(s+1)}+\frac{B}{(s+1)^2}+\frac{C}{(s-1)}+\frac{D}{(s-1)^2}$

$1=A(s+1)(s-1)^2+B(s-1)^2+C(s-1)(s+1)^2+D(s+1)^2$

s^2+2s+1

$(s-1)(s-1)$

s^2-2s+1

$s=1,$ $\frac{1}{4}=\frac{4D}{4}$ $\boxed{D=\frac{1}{4}}$

$s=-1,$ $\boxed{B=\frac{1}{4}}$

$1=As^3-As-As^2+A+s^2B-2Bs+B+s^3C-Cs$
$+s^2C-C+s^2D+2Ds+D$

$\mathcal{L}^{-1}\left[\frac{1}{4}\left(\frac{1}{s+1}\right)+\frac{1}{4}\left(\frac{1}{(s+1)^2}\right)-\frac{1}{4}\left(\frac{1}{s-1}\right)+\frac{1}{4}\left(\frac{1}{(s-1)^2}\right)\right]$

$=\frac{1}{4}\left(e^{-1t}\right)+\frac{1}{4}\left(e^{-1}t^{1}\right)-\frac{1}{4}\left(e^{1t}\right)+\frac{1}{4}\left(e^{1}t^{1}\right)$

Answer

$-\frac{3}{4}=-A$

$0=A+C$ $\quad$ $\boxed{A=\frac{1}{4}}$

$0=-A+B+C+D$ $\quad$ $-A=C$

$0=-A+C+\frac{2}{4}$ $\quad$ $-\frac{2}{4}=-A+C$

$-\frac{2}{4}=2C$

$\boxed{-\frac{1}{4}=C}$

In problems 9-11 solve the initial value problems with Laplace transforms.

9.) $x'' + 2x' + 6x = 6, \quad x(0) = x'(0) = 0$

$x = f(t)$

$\mathcal{L}[f(t)] = f(s)$

$\mathcal{L}[f'(t)] = sf(s) - f(0)$

$\mathcal{L}[f''(t)] = s^2 f(s) - sf(0) - f'(0)$

$$[s^2 f(s) - sf(0) - f'(0)] + 2[sf(s) - f(0)] + 6[f(s)] = \mathcal{L}[6]$$

$$s^2 f(s) - s(0) - (0) + 2[sf(s) - 0] + 6[f(s)] = \frac{6}{s}$$

$$s^2 f(s) + 2sf(s) + 6f(s) = \frac{6}{s}$$

$$\frac{(s^2+2s+6)f(s)}{s^2+2s+6} = \frac{\frac{6}{s}}{s^2+2s+6}$$

$$= \frac{6}{s(s^2+2s+6)} = \frac{A}{s} + \frac{bs+C}{(s^2+2s+6)},$$

$$6 = A(s^2+2s+6) + bs + C(s)$$

$$6 = As^2 + 2As + 6A + Bs^2 + Cs$$

$s = 0$

$\boxed{A = 1}$

$0 = A + B$

$\boxed{B = -1}$

$0 = 2A + C$

$\boxed{C = -2}$

$$\mathcal{L}^{-1}\left[\frac{1}{s} + \frac{-s-2}{s^2+2s+6}\right]$$

$$\mathcal{L}^{-1}\left[\frac{1}{s} + \frac{-s-2}{(s+1)^2+5}\right]$$

$$\mathcal{L}^{-1}\left[\frac{1}{s} + \frac{-1((s+1)+1)}{(s+1)^2+5}\right]$$

$$\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s+1}{(s+1)^2+5} - \frac{1}{(s+1)^2+5}\right]$$

$$1 - e^{-1t}\cos\sqrt{5}t - e^{-1t}\sin\sqrt{5}t$$

Answer

In problems 9-11 solve the initial value problems with Laplace transforms.

10.) $y''+4y'+8y=e^{-t}$, $y(0)=y'(0)=0$

$y=f(t)$

$\mathcal{L}[f(t)]=f(s)$

$\mathcal{L}[f'(t)]=sf(s)-f(0)$

$\mathcal{L}[f''(t)]=s^2f(s)-sf(0)-f'(0)$

$[s^2f(s)-sf(0)-f'(0)]+4[sf(s)-f(0)]+8[f(s)]=\mathcal{L}[e^{-t}]$

$s^2f(s)-0-0+4sf(s)-0+8f(s)=\frac{1}{s+1}$

$s^2f(s)+4sf(s)+8f(s)=\frac{1}{s+1}$

$f(s)[s^2+4s+8]=\frac{1}{s+1}$

$f(s)=\frac{1}{(s+1)(s^2+4s+8)}=\frac{A}{(s+1)}+\frac{Bs+C}{(s^2+4s+8)}$

$0=A+B$ $\quad 0=4A+B+C$ $\quad 1=A(s^2+4s+8)+Bs+C(s+1)$

$s=-1$

$\frac{1}{3}=\frac{3A}{3}$

$\boxed{A=\frac{1}{3}}$

$1=As^2+4As+8A+Bs^2+Bs+Cs+C$

$\boxed{-\frac{1}{3}=B}$

$\frac{4}{3}-\frac{1}{3}+C$

$0=\frac{3}{3}+C$

$\boxed{-1=C}$

$\mathcal{L}^{-1}\left[\frac{1}{3}\left(\frac{1}{s+1}\right)+\left(\frac{-\frac{1}{3}s-1}{s^2+4s+4+4}\right)\right]$

$\frac{-\frac{1}{3}(s+3)}{(s+2)^2+4}$ $=-\frac{1}{3}(s+2+1) \leftarrow -\frac{1}{3}(s+2)-\frac{1}{3}$

$\downarrow$

$\mathcal{L}^{-1}\left[\frac{1}{3}\left(\frac{1}{s+1}\right)+\frac{-\frac{1}{3}(s+2)-\frac{1}{3}}{(s+2)^2+4}\right]$

$\mathcal{L}^{-1}\left[\frac{1}{3}\left(\frac{1}{s+1}\right)-\frac{1}{3}\left(\frac{(s+2)+1}{(s+2)^2+4}\right)\right]$

$\mathcal{L}^{-1}\left[\frac{1}{3}\left(\frac{1}{s+1}\right)-\frac{1}{3}\left(\frac{s+2}{(s+2)^2+4}\right)-\frac{1}{3}\left(\frac{1}{(s+2)^2+4}\right)\right]$

$\frac{1}{3}(e^{-t})-\frac{1}{3}(e^{-2t}\cos 2t)-\frac{1}{3}(e^{-2t}\sin 2t)$

Answer

In problems 9-11 solve the initial value problems with Laplace transforms.

11.) $y''+2y'+2y=2\cos(t)+\sin(t)$; $y(0)=y'(0)=0$

$Y=f(t)$

$\mathcal{L}[f(t)]=f(s)$

$\mathcal{L}[f'(t)]=sf(s)-f(0)$

$\mathcal{L}[f''(t)]=s^2f(s)-sf(0)-f'(0)$

$[s^2f(s)-sf(0)-f'(0)]+2[sf(s)-f(0)]+2[f(s)]=\mathcal{L}[2\cos(t)+\sin(t)]$

$s^2f(s)-0-0+2sf(s)+2f(s)=2\left(\frac{s}{s^2+1}\right)+\left(\frac{1}{s^2+1}\right)$

$f(s)(s^2+2s+2)=\frac{2s}{s^2+1}+\frac{1}{s^2+1}$

$f(s)=\frac{2s}{(s^2+1)(s^2+2s+2)}+\frac{1}{(s^2+1)(s^2+2s+2)}$

$\frac{2s}{(s^2+1)(s^2+2s+2)}=\frac{As+B}{(s^2+1)}+\frac{Cs+D}{(s^2+2s+2)}$

$2s=(As+B)(s^2+2s+2)+(Cs+D)(s^2+1)$

$2s=As^3+2As^2+2As+Bs^2+2Bs+2B+Cs^3+Cs+Ds^2+D$

C: $0=2B+D$

s: $2=2A+2B+C$

s^3: $0=A+C$

s^2: $0=2A+B+D$

$2=2A+2B-A$

$0=2A+B-2B$

$2=A+2B$

$0=2A-B$

$2-2B=A$

$0=2(2-2B)-B$

$0=4-4B-B$

$\frac{-4}{-5}=\frac{-5B}{-5}$

$\frac{4}{5}=2A$

$\boxed{\frac{2}{5}=A}$

$\boxed{B=\frac{4}{5}}$

$\boxed{C=-\frac{2}{5}}$

$\boxed{-\frac{8}{5}=D}$

$\mathcal{L}^{-1}\left[\frac{\frac{2}{5}s+\frac{4}{5}}{(s^2+1)}+\frac{-\frac{2}{5}s-\frac{8}{5}}{(s+1)^2+1}\right]$ continued $\longrightarrow$

$\mathcal{L}^{-1}\left[\frac{2}{5}\left(\frac{s}{s^2+1}\right)+\frac{4}{5}\left(\frac{1}{s^2+1}\right)+\frac{-\frac{2}{5}(s+4)}{(s+1)^2+1}\right]$ $\frac{-\frac{2}{5}(s+1)-\frac{6}{5}}{(s+1)^2+1}$

$\mathcal{L}^{-1}\left[\frac{2}{5}\left(\frac{s}{s^2+1}\right)+\frac{4}{5}\left(\frac{1}{s^2+1}\right)-\frac{2}{5}\left(\frac{s+1}{(s+1)^2+1}\right)-\frac{6}{5}\left(\frac{1}{(s+1)^2+1}\right)\right]$

$=\frac{2}{5}(\cos t)+\frac{4}{5}(\sin t)-\frac{2}{5}(e^{-t}\cos t)-\frac{6}{5}(e^{-t}\sin t)$

(Half the Answer)

In problems 9-11 Solve the initial value problems with Laplace transforms.

11.) $y''+2y'+2y=2\cos(t)+\sin(t)$, $y(0)=y'(0)=0$

$y=f(t)$

$\mathcal{L}[f(t)]=F(s)$

$\mathcal{L}[f'(t)]=sF(s)-f(0)$

$\mathcal{L}[f''(t)]=s^2F(s)-sf(0)-f'(0)$

$$[s^2F(s)-sf(0)-f'(0)]+2[sF(s)-f(0)]+2[F(s)]=\mathcal{L}[2\cos(t)+\sin(t)]$$

$$s^2F(s)-0-0+2sF(s)+2F(s)=2\left(\frac{s}{s^2+1}\right)+\left(\frac{1}{s^2+1}\right)$$

$$F(s)(s^2+2s+2)=\frac{2s}{s^2+1}+\frac{1}{s^2+1}$$

$$F(s)=\frac{2s}{(s^2+1)(s^2+2s+2)}+\frac{1}{(s^2+1)(s^2+2s+2)}$$

$$1=\frac{As+B}{(s^2+1)}+\frac{Cs+D}{(s^2+2s+2)}$$

$$1=As^3+2As^2+2As+Bs^2+2Bs+2B+Cs^3+Cs+Ds^2+D$$

s^2: $0=2A+B+D$

c: $1=2B+D$

s: $0=2A+2B+C$

s^3: $0=A+C$

$\boxed{C=\frac{2}{5}}$

$\boxed{A=-\frac{2}{5}}$

$\boxed{\frac{2}{5}=D}$

$0=2A+B+D$

$0=2A+B+(1-2B)$

$0=2A+2B-A$

$-1=2A-B$

$0=A+2B$

$-2B=A$

$-1=2(-2B)-B$

$\boxed{B=\frac{1}{5}}$

Now we're Answering the rest

$$\mathcal{L}^{-1}\left[\frac{-\frac{2}{5}s+\frac{1}{5}}{(s^2+1)}+\frac{\frac{2}{5}s+\frac{2}{5}}{(s+1)^2+1}\right]$$

$$\mathcal{L}^{-1}\left[-\frac{2}{5}\left(\frac{s}{s^2+1}\right)+\frac{1}{5}\left(\frac{1}{(s^2+1)}\right)+\frac{\frac{2}{5}(s+1)+\frac{1}{5}}{(s+1)^2+1}\right]$$ ← all about factoring and and extracting 1

$$\mathcal{L}^{-1}\left[-\frac{2}{5}\left(\frac{s}{s^2+1}\right)+\frac{1}{5}\left(\frac{1}{s^2+1}\right)+\frac{2}{5}\left(\frac{s+1}{(s+1)^2+1}\right)+\frac{1}{5}\left(\frac{1}{(s+1)^2+1}\right)\right]$$

Second part of the Answer

$$-\frac{2}{5}(\cos t)+\frac{1}{5}(\sin t)+\frac{2}{5}(e^{-t}\cos t)+\frac{1}{5}(e^{-t}\sin t)$$

Full Answer

Full Answer for Question (11)

$$\left[\frac{2}{5}(\cos t)+\frac{4}{5}(\sin t)-\frac{2}{5}(e^{-t}\cos t)-\frac{6}{5}(e^{-t}\sin t)-\frac{2}{5}(\cos t)+\frac{1}{5}(\sin t)+\frac{2}{5}(e^{-t}\cos t)+\frac{1}{5}(e^{-t}\sin t)\right]$$

Simplified Version

$$\sin t-(e^{-t}\sin t)$$

Full Answer for Question (11)

$$\left[\frac{2}{5}(\cos t) + \frac{4}{5}(\sin t) - \frac{2}{5}(e^{-t}\cos t) - \frac{6}{5}(e^{-t}\sin t) - \frac{2}{5}(\cos t) + \frac{1}{5}(\sin t) + \frac{2}{5}(e^{-t}\cos t) + \frac{1}{5}(e^{-t}\sin t)\right]$$

Simplified Version

$$\sin t - (e^{-t}\sin t)$$