Christian G. Barden

1. Let p be any real number. Solve the problem

$$y' + \frac{y}{t^p} = 0, \qquad y(1) = 1.$$

Note: You answer will look different when $p = 1$ and otherwise.

2. Find the general solution of the equation

$$y'' + y = 0.$$

Note: There should be two arbitrary constants.

3. Find a condition on λ that ensures that every solution of the equation

$$y'' + \lambda y = 0,$$

is periodic with period 2π, meaning $y(t) = y(t + 2\pi)$ for all t.

4. Find a function $f = f(t, y)$ so that the solution to the first-order ODE

$$y' = f(t, y), \qquad y(0) = y_0$$

"breaks down in finite time" for some y_0, meaning there is a time $T > 0$ so that $\lim_{t \to T} y(t) = \infty$. Explain your answer. Why does this not contradict the existence and uniqueness theorem?

Hint: Try $f(t, y) = y^2$ and use "separation of variables".

5. Find a function $f = f(t, y)$ so that the initial-value problem

$$y' = f(t, y), \qquad y(0) = 0$$

has not just one, but *two* solutions; that is, this ODE exhibits non-uniqueness. Why does your example not contradict the existence and uniqueness theorem?

Hint: Try $f(t, y) = t\sqrt{y}$. Then $y(t) \equiv 0$ is a solution. Can you find another?

$$y' + \frac{y}{t^p} = 0$$

1) $$\frac{dy}{dt} + \frac{y}{t^p} = 0$$

$$\frac{dy}{dt} + \frac{y}{t^p} = 0$$

$$\left(\frac{dt}{-y}\right)\frac{dy}{dt} = -\frac{y}{t^p}\left(\frac{dt}{-y}\right)$$

$$-\int \frac{1}{y}\,dy = \int \frac{1}{t^p}\,dt$$

$$(-1)(-\ln|y|) = \left(\frac{t^{-p+1}}{-p+1} + c\right)(-1)$$

$$e^{(\ln|y|)} = e^{\left(-\left(\frac{t^{-p+1}}{-p+1}\right) - c\right)}$$

General Solution: $y = Ae^{\left(-\left(\frac{t^{-p+1}}{-p+1}\right)\right)}$

$$\frac{y(1) = A\cancel{e^{\left(-\left(\frac{1^{-p+1}}{-p+1}\right)\right)}}}{\cancel{e^{\left(-\left(\frac{1^{-p+1}}{-p+1}\right)\right)}}} = \frac{1}{e^{\left(-\left(\frac{1^{-p+1}}{-p+1}\right)\right)}}$$

particular solution: $$y(t) = \frac{e^{\left(-\left(\frac{t^{-p+1}}{-p+1}\right)\right)}}{e^{\left(-\left(\frac{1^{-p+1}}{-p+1}\right)\right)}}$$

2) $y'' - y = 0$

$y = e^{rt}$

$e^{rt}(r^2 - 1) = 0$ Solve the characteristic Eq

$r^2 - 1 = 0$

$(r+1)(r-1) = 0$

$r = -1, 1$

gen sol: $y = C_1 e^{-t} + C_2 e^{t}$

3.) {Condition on λ so every solution of $y'' + \lambda y = 0$ is periodic with period 2π, meaning $y(t) = y(t + 2\pi)$ for all t}

Character Equation

$r^2 + \lambda = 0$

$\frac{-(0) \pm \sqrt{(0)^2 - 4(1)(\lambda)}}{2}$

$\pm \sqrt{\lambda} i = \pm \frac{\sqrt{-4\lambda}}{2}$

The question is asking for a periodic solution with period 2π. Thus implies the answer must repeat every 2π. after solving the characteristic Equation the gen sol is $y = e^{0}(C_1 \sin(\sqrt{\lambda} t) + C_2 \cos(\sqrt{\lambda} t))$

In order to maintain a period of 2π, λ must equal 1. This is because the parent function $\cos x + \sin x$ inherently has the period 2π. If λ was anything else it would shift the period of the parent function out of 2π.

So the condition is $\lambda = 1$ which means $\sqrt{\lambda} = 1$

4.) A function $f = (t, y)$, that is a solution to the first order ODE

$\{ y' = f(t, y), \; y(0) = y_0 \}$ is the function y^2.

This is because, indeed, at some finite time when I separate the variables for $y' = y^2$, then apply its initial condition. I get a particular solution which if taken the limit of its variable, t, to some finite number $\frac{1}{y_0}$, I get a discontinuity to infinity (∞).

This can be shown mathematically as:

Solving as an IVP $\Rightarrow$ $y' = y^2$

$$\left(\frac{dt}{y^2}\right)\frac{dy}{dt} = \cancel{y^2}\left(\frac{dt}{\cancel{y^2}}\right)$$

$$\int \frac{1}{y^2}\,dy = \int dt$$

$$\int y^{-2}\,dy = \int dt$$

$$\cancel{(-1)}\frac{y^{-1}}{\cancel{-1}} = (t + c)(-1)$$

$$\left(\frac{\cancel{y}}{(-t-c)}\right)\frac{1}{\cancel{y}} = \cancel{-t-c}\left(\frac{y}{\cancel{(-t-c)}}\right)$$

$y(0) = y_0$, $y = \frac{1}{(-t-c)}$ = Gen sol

$$y_0 = y(0) = \frac{1}{-c}$$

$$(-c)\,y_0 = \frac{1}{\cancel{-c}}\cancel{(-c)}$$

$$c = -\frac{1}{y_0}$$

particular solution:

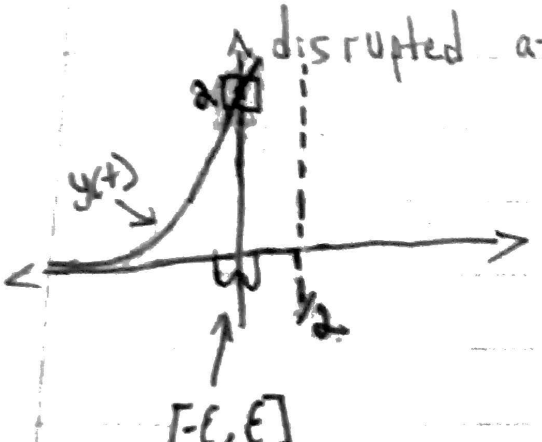
$$y(t) = \frac{1}{-t + \frac{1}{y_0}}$$

So if you

$$\lim_{t \to \frac{1}{y_0}} \frac{1}{-t + \frac{1}{y_0}} = \infty$$

The fact that the particular solution of $y' = y^2$ acted in a way that diverges to infinity (∞) when approaching $\frac{1}{y_0}$, it does **NOT** mean it contradicts the existence & uniqueness theorem.

The theorem states that the function must be continuous within an interval $[-\varepsilon, \varepsilon]$ "near" a point of solution which in our case is $(0, y_0)$, and given that the interval $[-\varepsilon, \varepsilon]$ has the capability being very small the existence & uniqueness theorem is not disrupted at all by $y(t) = \dfrac{1}{-t + \frac{1}{y_0}}$



Ex:
lets say
$y_0 = 2$

So point $(0, 2)$ is a solution that's inside a boxed interval of the $y(t)$ function that's continuous!!!

The discontinuity did not disrupt the interval, so the theorem holds!!!

5.) The function $f(t,y)=t\sqrt{y}$ is a function that satisfies the IVP $\{y'=f(t,y),\ y(0)=0\}$ & it contains two solutions that exhibit non-uniqueness. When you find the particular solution for $y'=t\sqrt{y}$ you see it produces 2 solutions & more due to the function being a higher order polynomial to the 4th degree. So, your evaluation of the function at $t=0$, $y(t)$ maps to 0. Thus the conditions stated are met. It doesn't contradict the existence & uniqueness theorem because it's non-uniqueness is not a critera for contradiction it's just a statement of the function's behavior, for which the theorem identifies (non-uniqueness).
This is shown Mathematically as:

$$y'=t\sqrt{y}$$

$$\cancel{\left(\frac{dt}{\sqrt{y}}\right)}\frac{dy}{\cancel{dt}}=t\sqrt{y}\cancel{\left(\frac{dt}{\sqrt{y}}\right)}$$

$$\int\frac{1}{\sqrt{y}}\,dy=\int t\,dt$$

$$\int y^{-\frac{1}{2}}\,dy=\frac{t^2}{2}+c$$

$$\cancel{\left(\frac{1}{2}\right)}\frac{y^{\frac{1}{2}}}{\cancel{\frac{1}{2}}}=\left(\frac{t^2}{2}+c\right)\left(\frac{1}{2}\right)$$

$$(\sqrt{y})^2=\left(\frac{t^2}{4}+\frac{c}{2}\right)^2$$

$$\boxed{y=\frac{t^4}{16}+\left(\frac{c}{2}\right)^2}\text{ : Gen solution}$$

$$y(0)=\left(\frac{0^4}{4}+\frac{c}{2}\right)^2=0$$

$$y(0)=\left(\frac{c}{2}\right)^2=0$$

$$\sqrt{\left(\frac{c}{2}\right)^2}=\sqrt{0}$$

$$(2)\frac{c}{2}=0(2)$$

$$\boxed{c=0}$$

particular sol: $y=\frac{t^4}{16}$

Another function that fits the conditions of the IVP is $t^{-\frac{1}{3}}\sqrt[3]{y}$. It basically produces similar behavior of $t\sqrt{y}$ in that 2 solutions can be found within the ODE $y'=t^{-\frac{1}{3}}\sqrt[3]{y}$. In addition, both of it's evaluations at $t=0$ maps to $y(t)=0$.

Shown here mathematically:

$$y'=t^{-\frac{1}{3}}\sqrt[3]{y}$$

$$\left(\frac{\cancel{dt}}{\sqrt[3]{y}}\right)\frac{dy}{\cancel{dt}}=t^{-\frac{1}{3}}\cancel{\sqrt[3]{y}}\left(\frac{dt}{\cancel{\sqrt[3]{y}}}\right)$$

$$\int y^{-\frac{1}{3}}dy=\int t^{-\frac{1}{3}}dt$$

$$\left(\frac{2}{3}\right)\frac{y^{\frac{2}{3}}}{\frac{2}{3}}=\left(\frac{t^{\frac{2}{3}}}{\frac{2}{3}}+c\right)\left(\frac{2}{3}\right)$$

$$\sqrt[3]{y^{\frac{2}{3}}}=\sqrt[3]{t^{\frac{2}{3}}+\frac{2(c)}{3}}$$

$$y^2=t^2+\sqrt[3]{\frac{2(c)}{3}}$$

gen sol: $y(t)=\pm\sqrt{t^2+\sqrt[3]{\frac{2(c)}{3}}}$

$$0=y(0)=\pm\sqrt{\sqrt[3]{\frac{2(c)}{3}}}$$

$$0=y(0)=\pm\sqrt[6]{\frac{2(c)}{3}}$$

2-Cases

$(0)^6=-\left(\sqrt[6]{\frac{2(c)}{3}}\right)^6$	$(0)^6=\left(\sqrt[6]{\frac{2(c)}{3}}\right)^6$
$0=-\frac{2(c)}{3}$	$0=\frac{2(c)}{3}$
$c=0$	$c=0$

particular sol: $y(t)=\pm t$

$y(t)=\pm\sqrt{t^2}$