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1. Solve the initial-value problem

$$\begin{aligned} u_t + 2tx^2 u_x &= 0, \\ u(0,x) &= f(x), \end{aligned}$$

for an arbitrary function f.

2. Solve the the initial-value problem

$$\begin{aligned} u_t + u_x + u &= e^{t+2x} \\ u(0,x) &= e^{-x^2}. \end{aligned}$$

Be careful: The solution is *not* constant along characteristics, but instead satisfies an ODE.

3. Solve the wave equation

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \\ u(0,x) &= e^x, \\ u_t(0,x) &= \sin x. \end{aligned}$$

4. Use the conservation of energy identity from class to show that the only solution to the wave equation

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \\ u(0,x) &= 0, \text{ Displacement} \\ u_t(0,x) &= 0, \text{ Velocity} \end{aligned}$$

show

is $u = 0$. Use this to conclude that there is at most one solution to the wave equation

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \\ u(0,x) &= u_0(x), \\ u_t(0,x) &= u_1(x), \end{aligned}$$

The energy stays the same and is determined by init cond

for arbitrary u_0, u_1.

Hint: You may need to use the following fact, which you can use without proof (but you should be able to explain intuitively why it is true): If f is a function with the property that

$$\int_{-\infty}^{\infty} (f(x))^2 \, dx = 0,$$

then $f(x) \equiv 0$.

There is only one solution because variability in conditions makes each created wave unique show

For the second part of the problem, if you have two such solutions u, v, what equation does their difference $w = u - v$ satisfy? What are the initial conditions?

Then swing into this & show

1)

$$\begin{cases} U_t + 2t^2 U_x = 0 \\ U(0,x) = f(x) \end{cases}$$

$$\cancel{(dx * 2t)} \frac{1}{\cancel{2t}x^2} = \frac{dt}{\cancel{dx}} \cancel{(dx} * 2t)$$

$$\int \frac{1}{x^2} dx = \int 2t\,dt$$

$$\int \frac{1}{x^2} dx = 2\int t\,dt$$

$$f(x) + \frac{x^{-1}}{-1} = \cancel{2}\left[\frac{t^2}{\cancel{2}}\right]$$

$$f(x) - \frac{1}{x} = t^2$$

$$f(x) = t^2 + \frac{1}{x}$$

Thus, $\boxed{u(x,t) = t^2 + \frac{1}{x}}$ ← Solution

Checking

$$2t + \cancel{t^2} + 2t + \cancel{x^2}(-\cancel{x^{-2}})$$

$$2t - (2t) = 0$$

2)

$$u_t + u_x = e^{t+2x} - u$$

$$\begin{cases} \frac{d}{dt} c(t) = 1 \\ c(0) = z \end{cases} \Rightarrow c(t) = t + z$$

$$U(t) = u(t, c(t))$$

$$\frac{d}{dt}(U(t)) = u_t(t, c(t)) + c'(t) u_x(t, c(t))$$

$$= u_t(t, c(t)) + (1) u_x(t, c(t))$$

$$= (1) e^{t+2(c(t))} - u(t)$$

$$\frac{d}{dt}(u(t)) = (1) e^{t+2(t+z)} - u(t)$$

$$\frac{d}{dt}(u(t)) = (1) e^{t+2t+2z} - u(t)$$

$$\frac{d}{dt}(u(t)) = (1) e^{3t+2z} - u(t)$$

$$\frac{dU}{dt} = e^{3t+2z} - u(t)$$

$$\frac{dU}{dt} + u(t) = e^{3t+2z}$$

$$\frac{du}{dt} + u(t) = e^{3t+2z}$$

$$I = e^{\int 1\,dt} = e^{t} \quad \begin{cases} p(t) = 1 \\ g(t) = e^{(3t+2z)} \end{cases}$$

$$\frac{d}{dt}\left[e^{t} * u\right] = e^{3t+2z} * e^{t}$$

$$e^{t} * u = e^{2z} \int e^{4t}\,dt$$

$u(0) = u_0(z)$

$$u(t, c(t)) = \left[e^{2z}\left(\tfrac{1}{4}e^{4t}\right) + u(0)\right]\left(\frac{1}{e^{t}}\right)$$

$z = x - t$

Solution

$$u(t, c(t)) = \left[e^{2(x-t)}\left(\tfrac{1}{4}e^{4t}\right) + u_0(x-t)\right]\left(\frac{1}{e^{t}}\right)$$

3)

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(0,x) = e^x \\ u_t(0,x) = \sin x \end{cases}$$ given

$$t^2 - x^2 = (t+x)(t-x)$$

$$0 = U_{tt} - U_{xx} = \left(\frac{d}{dt} + \frac{d}{dx}\right)\underbrace{(U_t - U_x)}_{V}$$

let $V = U_t - U_x$

Then $V_t + V_x = 0 \to$ Transport PDE

Sol: $v(t,x) = f(t-x)$

thus; $u_t - u_x = f(t-x)$

$$\left.\begin{array}{l} \frac{d}{dt}(c(t)) = -1 \\ c(0) = z \end{array}\right\} \Rightarrow c(t) = -t + z$$

$$U(t) = u(t, z-t)$$

$$\frac{dU}{dt} = f(t-(z-t))$$

$$= f(2t-z)$$

Thus,

$$u(t) = e^{z} + \int_0^t f(2s-z)\,ds$$

$$u(t,x) = e^{(x+t)} + \int_0^t f(2s-(x+t))\,ds$$

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$\varsigma = \underline{2s-t-x}$

$s=0,\ \varsigma = -(t+x)$

$s=t,\ \varsigma = t-x$

$\frac{d\varsigma}{ds} = 2$, so

$$\int_0^t f(2s-t-x)\,ds = \frac{1}{2}\int_{-(t+x)}^{t-x} f(\varsigma)\,d\varsigma$$

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$$\boxed{u(x,t) = e^{x+t} + \frac{1}{2}\int_{x-t}^{x+t} \sin(\varsigma) - e^{\varsigma}\,d\varsigma}$$

unknown

Remember Condition: $u_t(0,x) = \sin x$

$$u_t(x,t) = e^{(x+t)}(1) + \frac{1}{2}\left(f(x+t) - f(x-t)(-1)\right)$$

$$= e^{(x+t)} + \frac{1}{2}\left(f(x+t) - f(x-t)\right)$$

$t=0$ $\quad \sin x = e^{(x+0)} + \frac{1}{2}\left(f(x) + f(x)\right)$

$$\sin x = e^{(x)} + \frac{1}{2}\left(2f(x)\right)$$

$$f(x) = \sin x - e^{(x)}$$

Therefore,

$$u(x,t) = e^{x+t} + \frac{1}{2}\int_{x-t}^{x+t} \sin(s) - e^s \, ds$$

$$= e^{x+t} + \frac{1}{2}\int_{x-t}^{x+t} \sin(s)\,ds - \frac{1}{2}\left(e^{x+t} - e^{x-t}\right)$$

$$-\frac{1}{2}\left(e^{x+t}\right) + \frac{1}{2}\left(e^{x-t}\right)$$

$$u(x,t) = \frac{1}{2}\left(e^{x+t} + e^{x-t}\right) + \frac{1}{2}\int_{x-t}^{x+t} \sin(s)\,ds$$

D'Alembert's Formula

$$u(x,t) = \frac{1}{2}\left(e^{x+t} + e^{x-t}\right) + \frac{1}{2}\int_{x-t}^{x+t} \sin(s)\,ds$$

$$+ \frac{1}{2}\left[-\left(\cos(x+t) - \cos(x-t)\right)\right]$$

Solution for the Wave Equation. $u(x,t) = \frac{1}{2}\left(e^{x+t} + e^{x-t}\right) - \frac{1}{2}\left[\cos(x+t) - \cos(x-t)\right]$

4)

Given the conservation of energy Identity ($\mathcal{E}(t)=\mathcal{E}(0)$), the only solution to the wave equation

$$\begin{cases} U_{tt}-U_{xx}=0 \\ U(0,X)=0 \leftarrow \text{Displacement} \\ U_t(0,X)=0 \leftarrow \text{Velocity} \end{cases}$$

is $u(t,X)=0$. This is due to the initial conditions (displacement $[U(0,X)=0]$/Velocity$[U_t(0,X)=0]$) being both equal to 0. These initial conditions imply the wave won't propagate in Space-time. Since there is no propagation, the Kenetic & potential energy should result in the wave having a conserved energy of 0 for $\mathcal{E}(0)$ & $\mathcal{E}(t)$.

<u>This is Shown here mathematically:</u>

$$U(0,X)=0,\ \mathcal{E}(0)=\int_{-\infty}^{\infty} \underbrace{\overset{\text{kenetic}}{\left|\frac{du}{dt}\right|^2}}_{0} + \underbrace{\overset{\text{potential}}{\left|\frac{du}{dx}\right|^2}}_{0} dx = X\Big|_{-\infty}^{\infty} = \underline{\underline{0}} \quad \text{Therefore,}$$

$$\underline{\underline{\mathcal{E}(t)=\mathcal{E}(0)}}$$

$$u(t,X)=0,\ \mathcal{E}(t)=\int_{-\infty}^{\infty} \underbrace{\left|\frac{du}{dt}\right|^2}_{0} + \underbrace{\left|\frac{du}{dx}\right|^2}_{0} dx = X\Big|_{-\infty}^{\infty} = \underline{\underline{0}}$$

Also, Given the conserved energy being 0 there is no propogation

We know that the specific constant for $U(t,X)=0$ because of the initial displacement $U(0,X)=0$ & since there is no propogation. That means the wave's initial posistion stays for all time.

In addition, this translates over to another similar wave function denoted as

$$\begin{cases} U_{tt} - U_{xx} = 0 \\ U(0,x) = U_0(x) \\ U_t(0,x) = U_1(x) \end{cases}$$

the only difference between this wave function & the last one are the initial conditions. This means the conserved energy might differ depending on whether "$U_0(x) = U_1(x) = 0$", or it strays from the previous initial conditions. But, Just like the last wave function the solution remains to be singular because the initial conditions make every wave unique in its behavior (oscillation). Therefore, there's a most 1 solution for every wave, which is predetermined by its initial conditions.

Also, You can look at it from a conservation of energy standpoint

if I found 2 Solutions (2 different functions) $u = xt$

$u = x$

& I decided to find their conserved energy. They would produce different constant values, which based on the conservation of energy's Means that I'm Wrong.

However, if I have 1 sol like previously shown indeed satisfy $\mathcal{E}(t) = \mathcal{E}(0)$

For the second part, if I have u, v the equation that their difference ($W = u - v$) satisfy is

$$W_{tt} - W_{xx} = 0$$

this is shown through

$$\frac{d}{dt^2}(W) - \frac{d}{dx^2}(W)$$

$$W_{tt} - W_{xx}$$

then substituting ($u - v = W$)

$$\underbrace{(U-V)}_{}{}_{tt} - \underbrace{(U-V)}_{}{}_{xx}$$

$$U_{tt} - V_{tt} - U_{xx} + V_{xx}$$

$$= U_{tt} - U_{xx} - V_{tt} + V_{xx}$$

Then let's assume

$$= 0$$

thus $W_{tt} - W_{xx} = 0$

you can have the <u>initial conditions</u> as any integer or function but for simplicity

$$W(0, X) = 0$$

$$W_t(0, X) = 0$$