Vibrating Springs- HW Problems

1. A mass of 1000g is attached to a spring which is stretched 25cm by a force of 9N. At time $t = 0$ the mass is moved stretching the spring 2m to the right, and setting it in motion with an intial velocity of 6m/sec to the left. Find the position of the mass at time $t, x(t)$. Write $x(t)$ in the form $x(t) = Ccos(\omega t - \alpha)$. Find the amplitude and period of the motion.

In problems 2-5 a mass is attached to a spring, with spring constant k, and a dashpot, with damping constant c. The initial velocity of the mass is v_0 and the initial position is x_0. Find the position function of the mass, $x(t)$, and identify if the motion is overdamped, critically damped, or underdamped. If the motion is underdamped, write the position in the form $x(t) = Ce^{-pt}(\cos(\omega t - \alpha))$. Also find the undamped position function (ie where $c = 0$), $u(t) = Ccos(\omega t - \alpha)$.

2. $m = 2, \quad k = 50, \quad c = 12, \quad x_0 = -1, \quad v_0 = -1$

3. $m = 2, \quad k = 6, \quad c = 8, \quad x_0 = 3, \quad v_0 = 2$

4. $m = 1, \quad k = 9, \quad c = 6, \quad x_0 = -2, \quad v_0 = -6$

5. $m = 1, \quad k = 50, \quad c = 10, \quad x_0 = 0, \quad v_0 = 10.$

1.) A mass of 1000g is attached to a spring which is stretched 25 cm by force of 9N. At time $t=0$ the mass is moved stretching the spring 2m to the right, and setting it in motion with an initial velocity of 6m/sec to the left. Find the position of the mass at time t, $X(t)$. Write $X(t)$ in the form $x(t) = C\cos(\omega t - \alpha)$. Find the amplitude & period of the motion.

$F = 9N \quad X = .25m$

$m = 1kg \quad A = 2m$

1000g = 1kg

① $K = \frac{F}{X} = \frac{9N}{.25m} = 36 N/m$

② $W_0 = \sqrt{\frac{K}{m}} = \sqrt{\frac{36N}{1kg}} = 6\sqrt{\frac{N}{K}}$

③ $\frac{V(0)}{-Aw_0} = \frac{-Aw_0 \sin(\theta)}{-Aw_0}$

$\Downarrow$

$\frac{-(-6m/s)}{(2m * 6\sqrt{\frac{N}{K}})} = \boxed{.5}$

④ $\alpha = \tan^{-1}(.5) = \boxed{\frac{\pi}{6}}$

⑤ $X(t) = 2m\cos(6t + \frac{\pi}{6})$

$\boxed{X(t) = 2\cos(6t + \frac{\pi}{6})}$

$\boxed{\text{period} = \frac{2\pi}{6} = \frac{\pi}{3}}$

$\boxed{\text{Amplitude} = |2| = 2}$

In problems 2-5 a mass is attached to a spring with spring constant k, & a dashpot, with damping constant c. The initial velocity of the mass is v_0 and the initial position is x_0. Find the position function of the mass, $x(t)$, & identify if the motion is overdamped, critically damped, or underdamped. If the motion is underdamped, write the position in the form $x(t)=Ce^{-pt}(C\cos(\omega t-\alpha))$. Also find the undamped position function (ie where $c=0$), $u(t)=C\cos(\omega t-\alpha)$.

answer 1

2.) $m=2,\quad k=50,\quad c=12,\quad x_0=-1,\quad v_0=-1$

$c^2-4km=12^2-4(50)(2)=-256$

$\boxed{x(t)=\sqrt{\frac{65}{16}}e^{-6t}\cos(25t-1.0516)}$

motion is underdamped

$w_0=\sqrt{\frac{K}{M}}=\frac{50}{2}=25$

$mx''+cx'+kx=0$

$2x''+12x'+50x=0$

$\alpha=\tan^{-1}\left(\frac{B}{A}\right)=1.0516$

$(2r^2+12r+50)=0$

$r=\frac{-(12)\pm\sqrt{(12)^2-4(2)(50)}}{2(2)}$

$-1=x'(0)=-6A+4B$
$-1=x(0)=A$

$r=-6\pm\frac{16}{4}i$

$C=\sqrt{A^2+B^2}$ $\quad -1=-6(-1)+4B$

$r=-6\pm4i$

$C=\sqrt{\frac{65}{16}}$ $\quad \frac{-7}{4}=\frac{4B}{4}$

$\boxed{B=\frac{-7}{4}}$

$\boxed{A=-1}$

$x(t)=e^{-6t}(A\cos4t+B\sin4t)$

$x'(t)=-6(e^{-6t})(A\cos4t+B\sin4t)+e^{-6t}(-4A\sin(4t)+4B\cos(4t))$

(undamped)
$c = 0$

$x(t) = mx'' + kx = 0$

$= 2x'' + 50x = 0$

$2r^2 + 50 = 0$

$\frac{2r^2}{2} = \frac{-50}{2}$

$\sqrt{r^2} = \sqrt{-25}$

2nd Answer

$r = \pm 5i$

$\boxed{u(t) = \sqrt{\frac{26}{25}} \cos(25t - 1.3734)}$

$\boxed{u(t) = A\sin 5t + B\cos 5t}$

$\boxed{u'(t) = 5A\cos 5t - 5B\sin 5t}$

$-1 = u(0) = B$

$-1 = u'(0) = 5A$

$\boxed{-1 = B}$

$\boxed{\frac{-1}{5} = A}$

$\sqrt{\frac{k}{m}} = 25$

$c = \sqrt{A^2 + B^2} = \sqrt{\frac{26}{25}}$

$\alpha = \tan^{-1}\left(\frac{B}{A}\right) = 1.3734$

3) $m=2,\ k=6,\ c=8,\ x_0=3,\ V_0=2$

$$c^2-4km = 8^2-4(6)(2)=16$$

(It's overdampened motion)

$$mx''+cx'+kx=0$$

$$2x''+8x'+6x=0$$

$$(2r^2+8r+6=0)$$

$$r=\frac{-(8)\pm\sqrt{4(2)(6)}}{2(2)}$$

$$-2\pm 4\sqrt{3}$$

$$x(t)=Ae^{(-2-4\sqrt{3}t)}+Be^{(-2+4\sqrt{3}t)}$$

$$x'(t)=(-2-4\sqrt{3})Ae^{(-2-4\sqrt{3}t)}+(-2+4\sqrt{3})Be^{(-2+4\sqrt{3}t)}$$

1st answer

$$\boxed{X(t)=\left(\frac{7\sqrt{}+13}{11}\right)e^{(-2-4\sqrt{3}t)}+\left(\frac{-7\sqrt{3}+20}{11}\right)e^{(-2+4\sqrt{3}t)}}$$

$$3=X_0=A+b$$

$$2=V_0=(-2-4\sqrt{3})A+(-2+4\sqrt{3})B$$

$$(3-B)=A$$

$$\boxed{A=\left(\frac{+7\sqrt{3}+13}{11}\right) \quad B=\left(\frac{-7\sqrt{3}+20}{11}\right)}$$

$$2=(-2-4\sqrt{3})(3-B)+(-2+4\sqrt{3})B$$

$$2=(12\sqrt{3}-6)-(-2-4\sqrt{3})B+(-2+4\sqrt{3})B$$

$$2=(12\sqrt{3}-6)-8\sqrt{3}-4B$$

$$\frac{-12\sqrt{3}+8}{-8\sqrt{3}-4}=\frac{-8\sqrt{3}-4B}{-8\sqrt{3}-4}$$

$\begin{pmatrix} c = 0 \\ \text{undampened} \end{pmatrix}$

$x(t) = mx'' + kx = 0$

$= 2x'' + 6x = 0$

$\frac{2r^2}{2} + 6 = 0 \quad -6 \quad \frac{-6}{6}$

$\sqrt{r^2} = \sqrt{-3}$

$r = \pm\sqrt{3}\, i$

$u(t) = A(\cos\sqrt{3}t) + B(\sin\sqrt{3}t)$

$u'(t) = -\sqrt{3}A(\sin\sqrt{3}t) + \sqrt{3}B(\cos\sqrt{3}t)$

$3 = x_0 = A$

$\frac{2}{\sqrt{3}} = v_0 = \frac{\sqrt{3}B}{\sqrt{3}}$

$C = \sqrt{A^2+B^2} = \sqrt{\frac{31}{3}}$

$\boxed{u(t) = \sqrt{\frac{31}{3}}\cos(\sqrt{3}t - .3674)}$

↑ 2nd answer

$\boxed{B = \frac{2}{\sqrt{3}}}$

$\boxed{A = 3}$

$\boxed{C = \sqrt{\frac{31}{3}}}$

$\boxed{\alpha = \tan^{-1}\left(\frac{B}{A}\right) = .36742}$

$\boxed{w = \sqrt{\frac{k}{m}} = \sqrt{3}}$

4)

$m=1,\ k=9,\ c=6,\ x_0=-2,\ v_0=-6$

$$c^2-4km=6^2-4(9)(1)=0$$

critically dampened

$$mx''+cx'+kx=0$$

$$x''+6x'+9x=0$$

$$r^2+6r+9=0$$

$$(r+3)(r+3)=0$$

$$r=-3$$

$$x(t)=c_1e^{-3t}+c_2e^{-3t}t=e^{-3t}(c_1+c_2t)$$

$$x'(t)=e^{-3t}(c_2)-3e^{-3t}(c_1+c_2t)$$

$$x_0=-2=c_1+c_2$$
$$v_0=-6=c_2-3(c_1)$$

$$\boxed{c_2=-2-c_1}$$

$$-6=(-2-c_1)-3c_1$$

$$-6=-2-2c_1$$
$$+2\quad +2$$

$$\frac{-4}{-2}=\frac{-2c_1}{-2}$$

$$\boxed{2=c_1}$$

$$-6=c_2-3(-2)$$

$$\boxed{c_2=-12}$$

$$\boxed{x(t)=e^{3t}(2-12t)}$$

1st answer

$\begin{pmatrix} c=0 \\ \text{undamped} \end{pmatrix}$

$x(t) = mx'' + kx = 0$

$= 1x'' + 9x = 0$

$r^2 = -9$

$\sqrt{r^2} = \pm\sqrt{-9}$

$r = \pm 3i$

$u(t) = A\cos(3t) + B\sin(3t)$

$u'(t) = -3A\sin(3t) + 3B\cos(3t)$

$x_0 = -2 = A$

$v_0 = -6 = 3B$

$A = -2$	$w = \sqrt{\frac{k}{m}} = 3$
$B = -2$	$c = \sqrt{A^2+B^2} = \sqrt{8}$
$\alpha = \tan^{-1}\left(\frac{B}{A}\right) = .7853$	
$u(t) = \sqrt{8}\cos(3t - .7853)$	

5) $m=1$, $k=50$, $c=10$, $X_0=0$, $V_0=10$

$$c^2-4km = 10^2-4(50)(1) = -100$$

Motion is underdamped

$$mx''+cx'+kx=0$$

$$x''+10x'+50x=0$$

$$r^2+10r+50=0$$

$$r=\frac{-(10)\pm\sqrt{(10)^2-4(1)(50)}}{2(1)}$$

$$r=(-5\pm 5i)$$

$$v(t)=e^{-5t}(A\cos(5t)+B\sin(5t))$$

$$v'(t)=-5e^{-5t}(A\cos(5t)+B\sin(5t))$$
$$+e^{-5t}(-5A\sin(5t)+5B\cos(5t))$$

$$X_0=0=A$$
$$V_0=10=-5A+5B$$

$$\boxed{x(t)=2e^{-5t}(\cos(\sqrt{50}t))}$$

← 1st Answer

$A=0$	$w=\sqrt{\frac{K}{M}}=\sqrt{50}$
$B=2$	$C=\sqrt{A^2+B^2}=2$
$\alpha=\tan^{-1}(\frac{B}{A})=0$	

$(0 = c)$

(undamped)

$$x(t) = mx'' + kx = 0$$

$$x'' + 50x = 0$$

$$r^2 + 50 = 0$$
$$-50 \quad -50$$

$$\sqrt{r^2} = \sqrt{50}$$

$$r = \pm\sqrt{50}\, i$$

$$u(t) = \underline{A\cos\sqrt{50}t + b\sin\sqrt{50}t}$$

$$u'(t) = \underline{-\sqrt{50}A\sin\sqrt{50}t + \sqrt{50}\,b\cos\sqrt{50}t}$$

$$X_0 = 0 = A$$

$$V_0 = 10 = \sqrt{50}\,B$$

$A = 0$	$W = \sqrt{\frac{k}{m}} = \sqrt{50}$
$B = \frac{10}{\sqrt{50}}$	$C = \sqrt{A^2 + B^2} = \sqrt{2}$
$\alpha = \tan^{-1}\left(\frac{B}{A}\right) = 0$	

$$\boxed{u(t) = \sqrt{2}\left(\cos\left(\sqrt{50}t\right)\right)}$$

2nd answer

Nonhomogeneous Equations: The Methods of Undetermined Coefficients and Variation of Parameters- HW Problems

In problems 1-5 find a particular solution y_p of the given equation.

1. $y'' - y' - 2y = 4x^2$
2. $y'' + 4y' + 13y = 20e^{-x}$
3. $y' - 3y = -4xe^x$
4. $y'' - y' - 2y = -10\cos(x)$
5. $y^{(3)} + y'' = 12x + 4.$

In problems 6-8 solve the initial value problem.

6. $y'' + y' - 2y = -10\sin(x);\quad y(0) = 4,\quad y'(0) = 3.$
7. $y'' - 2y' + 2y = 2x + 1;\quad y(0) = 3,\quad y'(0) = 1.$
8. $y^{(4)} - y^{(3)} = 24x - 6;$
   $y(0) = 4,\ y'(0) = 6,\ y''(0) = 6,\ y^{(3)} = -10.$

In problems 9-11 use the method of variation of parameters to find a particular solution to the differential equation.

9. $y'' - \frac{3}{x}y' + \frac{3}{x^2}y = \frac{1}{x};$ where $y_1 = x$ and $y_2 = x^3$ are solutions to $y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 0.$
10. $y'' - 2y' + y = \frac{e^x}{x}$
11. $y'' + 4y = \sin(2x).$

(In problems 1-5 find a particular solution y_p of the given equation)

1) $y'' - y' - 2y = 4x^2$

$f(x) = 4x^2$

$y_p = Ax^2$

$y'_p = 2AX$

$y''_p = 2A$

$(2A) - (2AX) - 2(AX^2) = 4x^2$

$(2A) - 2A(x) - 2A(x^2) = 4x^2$

$$\frac{2A(1 - \cancel{x} - x^2)}{(1 - \cancel{x} - x^2)} = \frac{4x^2}{(1 - x - x^2)}$$

$$\left(\frac{1}{2}\right) 2A = \frac{4x^2}{(1 - x - x^2)} \left(\frac{1}{2}\right)$$

$$A = \left(\frac{2x^2}{1 - x - x^2}\right)$$

$$\boxed{y_p = \left(\frac{2x^4}{1 - x - x^2}\right)}$$

2.) $y'' + 4y' + 13y = 20e^{-x}$

$$F(x) = 20e^{-x}$$

$$y_p = Ae^{-x}$$

$$y_p' = -Ae^{-x}$$

$$y_p'' = Ae^{-x}$$

$$(Ae^{-x}) + 4(-Ae^{-x}) + 13(Ae^{-x}) = 20e^{-x}$$

$$A(e^{-x} + 4e^{-x} + 13e^{-x}) = 20e^{-x}$$

$$\frac{A(10e^{-x})}{10e^{-x}} = \frac{20e^{-x}}{10e^{-x}}$$

$$A = 2$$

$$\boxed{y_p = 2e^{-x}}$$

3) $y' - 3y = -4xe^{x}$

$$F(x) = -4xe^{x}$$

$$y_p = Axe^{x}$$

$$y_p' = Ae^{x} + Axe^{x}$$

$$(Ae^{x} + Axe^{x}) - 3(Axe^{x}) = -4xe^{x}$$

$$Ae^{x} - 2(Axe^{x}) = -4xe^{x}$$

$$\frac{(Ae^{x})(1-\cancel{2x})}{(1-\cancel{2x})} = \frac{-4xe^{x}}{(1-2x)}$$

$$\left(\frac{1}{e^x}\right)(Ae^{x}) = \frac{4x\cancel{e^{x}}}{(1-2x)}\cancel{\left(\frac{1}{e^{x}}\right)}$$

$$\boxed{y_p = \left(\frac{4x}{1-2x}\right)xe^{x}}$$

$$A = \frac{4x}{1-2x}$$

4) $y'' - y' - 2y = -10\cos x$

$$\left| \begin{array}{l} y_p = A\cos x + B\sin x \\ y'_p = -A\sin x + B\cos x \\ y''_p = -A\cos x - B\sin x \end{array} \right|$$

$$(-A\cos x - B\sin x) - (-A\sin x + B\cos x) - 2(A\cos x + B\sin x) = -10\cos x$$

$$(-A + B - 2A)\cos x + (-B - A - 2B)\sin x = -10\cos x + 0\sin x$$

$$\begin{array}{l} -A + B - 2A = -10 \\ -B - A - 2B = 0 \end{array} \Rightarrow \left\{ \begin{array}{l} B - 3A = -10 \\ -3B - A = 0 \end{array} \right\}$$

$$B = -10 + 3A$$

$$-3(-10 + 3A) - A = 0$$

$$30 - 9A - A = 0$$

$$30 - 10A = 0$$

$$\frac{30}{10} = \frac{10A}{10}$$

$$\boxed{A = 3}$$

$$-3B - 3 = 0$$

$$+3 \quad +3$$

$$\frac{-3B}{-3} = \frac{3}{-3}$$

$$\boxed{B = -1}$$

$$\boxed{y_p = 3\cos x - 1\sin x}$$

5) $y''' + y'' = 12x + 4$

$y_p = Ax + B$

$y'_p = A$

$y''_p = 0$

$y'''_p = 0$

$0 = 12x + 4$

$\frac{4}{12} = x$

$\boxed{y_p = A\left(\frac{-4}{12}\right) + B}$

In problems 6-8 solve the initial value problem

6) $y'' + y' - 2y = -10\sin(x)$; $y(0) = 4$, $y'(0) = 3$

$y_p = A\sin(x) + B\cos(x)$

$y'_p = A\cos(x) - B\sin(x)$

$y''_p = -A\sin(x) - B\cos(x)$

$(-A\sin(x) - B\cos x) + (A\cos(x) - B\sin(x)) - 2(A\sin x + B\cos x) = -10\sin x$

$(-A - B - 2A)\sin x + (-B + A + B)\cos x = -10\sin x + 0\cos x$

$\left\{ \begin{array}{l} -B - 3A = -10 \\ A = 0 \end{array} \right\}$ $A = 0$ $B = 10$

$y_p = 10\cos(x)$

6.)Continued

$$y'' + y' - 2y = 0$$

$$r^2 + r - 2 = 0 = (r+2)(r-1)$$

$$\frac{-(1) \pm \sqrt{(1)^2 - 4(1)(-2)}}{2(1)}$$

$$r = (-2, 1)$$

$$y_c = C_1 e^{-2x} + C_2 e^{1x}$$

$$y(x) = C_1 e^{-2x} + C_2 e^{1x} + 10\cos x$$

$$4 = y(0) = C_1 + C_2 + 10 \Rightarrow -6 = C_1 + C_2$$

$$3 = y'(0) = -2C_1 + C_2 - 10 \Rightarrow 13 = -2C_1 + C_2$$

$$\boxed{y(x) = \left(\frac{17}{3}\right) e^{-2x} + \left(\frac{1}{3}\right) e^{1x} + 10\cos x}$$

$$C_1 = -6 - C_2$$

$$13 = -2(-6 - C_2) + C_2$$

$$13 = 12 + 2C_2 + C_2$$

$$13 = 12 + 3C_2$$
$$-12 \quad -12$$

$$-6 = C_1 + \frac{1}{3} \qquad \frac{1}{3} = \frac{\cancel{3}C_2}{\cancel{3}}$$
$$-\frac{1}{3} \quad -\frac{1}{3}$$

$$\boxed{C_1 = \frac{17}{3}} \qquad \boxed{\frac{1}{3} = C_2}$$

7) $y'' - 2y' + 2y = 2x + 1$; $y(0) = 3$, $y'(0) = 1$

$y_p = Ax + B$

$y'_p = A$

$y'' = 0$

$-2(A) + 2(Ax + B) = 2x + 1$

$(-2A + 2B) + (2Ax) = 2x + 1$

$\boxed{y_p = x + 2}$

$\frac{2A}{2} = \frac{2}{2}$

$\boxed{A = 1}$

$-2 + 2B = 1$
$+3 \qquad +3$

$-2B = 4$
$\frac{}{2} \quad \frac{}{2}$

$\boxed{B = 2}$

$y'' - 2y' - 2y = 0$

$r^2 - 2r - 2 = 0$

$\frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = (1 \pm 1i)$

$y_c = e^{1x}(\cos x + \sin x)$

$\boxed{y = x + 2 + e^{1x}(\cos x + \sin x)}$

8) $y^{(4)} - y^{(3)} = 24x - 6$

$r^4 - r^3 = 0$

$r^3(r-1) = 0$

$r^3 = 0 \quad 1 = r$

$$\boxed{y_c = C_1 + C_2x + C_3x^2 + C_4e^{1x}}$$

$y_p = Ax + b$

$y_p' = A$

$y'' = 0$

$y''' = 0$

$y^4 = 0$

$$\boxed{y = -2 + 5x - 5x^2 + 6e^x}$$

$\boxed{y_p = 0}$ $\quad \boxed{C_1 = -2}$

$4 = C_1 + C_2x + C_3x^2 + C_4e^x$ $\quad \boxed{C_2 = 5}$

$\boxed{6 = C_4}$ $\quad \boxed{C_3 = -5}$

$6 = C_2 + 2C_3x + 6$

$-6 \qquad -6$

$0 = C_2 + 2C_3x \qquad \frac{-10}{2} = \frac{2C_3}{2}$

$C_2 = -2C_3x$

in problems 9-11 use the method of variation of parameters to find a particular solution to the differential equation.

9.) $y'' - \frac{3}{x}y' + \frac{3}{x^2}y = \frac{1}{x}$; where $y_1 = x$ \$ $y_2 = x^3$ are solutions to $y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 0$

$$y_c = C_1 x + C_2 x^3$$

$$\begin{vmatrix} y_1 = x & y_2 = x^3 \\ y_1' = 1 & y_2' = 3x^2 \end{vmatrix}$$

$$W(y_1, y_2) = \det \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}$$

$$= 3x^3 - x^3 = \underline{2x^3}$$

$$y_p = -x \int \frac{\cancel{x}^3 \left(\frac{1}{x}\right)}{2\cancel{x}^3} dx + x^3 \int \frac{x\left(\frac{1}{x}\right)}{2x^3} dx$$

$$y_p = \frac{-x}{2} \int \frac{1}{x} dx + \frac{x^3}{2} \int \frac{1}{x^3} dx$$

$$\boxed{y_p = -\frac{x}{2}(\ln|x|) + \left(\frac{x^3}{2}\right)\frac{x^{-2}}{-2}}$$

10.) $y'' - 2y' + y = \frac{e^x}{x}$

$= r^2 - 2r + 1 = 0$

$= (r-1)(r-1)$

$r = 1$

$y_c = C_1 e^x + C_2 x e^x$

$y_1 = e^x \quad y_2 = xe^x$

$y_1' = e^x \quad y_2' = e^x + xe^x$

$W(y_1, y_2) = \det \begin{vmatrix} e^x & xe^x \\ e^x & (e^x + xe^x) \end{vmatrix}$

$(e^x)^2 + \cancel{x(e^x)^2} - \cancel{x(e^x)^2}$

$= (e^x)^2$

$y_p = -e^x \int \frac{(xe^x)\left(\frac{e^x}{x}\right)}{(e^x)^2} dx + xe^x \int \frac{(e^x)\left(\frac{e^x}{x}\right)}{(e^x)^2} dx$

$-e^x \int 1 \, dx + xe^x \int \frac{\left(\frac{e^x}{x}\right)}{e^x} dx$

$-e^x \int 1 \, dx + xe^x \int \frac{1}{x} dx$

$$\boxed{y_p = -e^x(x) + xe^x(\ln(x))}$$

11) $y'' + 4y = \sin(2x)$

$y'' + 4y = 0$

$r^2 + 4 = 0$

$\quad -4 \quad -4$

$\sqrt{r^2} = \sqrt{-4}$

$r = \pm 2i$

$y_c = C_1 \sin(2x) + C_2 \cos(2x)$

$y_1 = \sin 2x \qquad y_2 = \cos 2x$

$y'_1 = 2\cos 2x \qquad y'_2 = -2\sin 2x$

$$W(y_1, y_2) = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix}$$

$= -2\sin^2(2x) - (2\cos^2(2x))$

$\sin^2 2x + \cos^2 2x = 1 \qquad = -2(\sin^2(2x) + \cos^2(2x))$

$$y_p = -\frac{\sin 2x}{-2} \int \frac{\cos 2x (\sin 2x)}{(\sin^2 2x + \cos^2 2x)} dx + \frac{\cos 2x}{-2} \int \frac{\sin 2x (\sin 2x)}{(\sin^2 2x + \cos^2 2x)} d$$

$$y_p = \frac{\sin 2x}{2} \int \cos 2x (\sin 2x) dx + \frac{\cos 2x}{-2} \int \sin^2(2x) dx$$

11.) Continued

$u = \sin 2x$

$du = 2\cos 2x$

$$y_p = \frac{\sin 2x}{2}\int \frac{\cancel{\cos 2x}\, u}{2\cancel{\cos 2x}}\,du + \frac{\cos 2x}{-2}\int \frac{1-\cos(4x)}{2}\,dx$$

$$= \frac{\sin 2x}{4}\int u\,du + \frac{\cos 2x}{-4}\int 1-\cos(4x)\,dx$$

$$= \frac{\sin 2x}{4}\int \frac{u^2}{2}\,du + \frac{\cos 2x}{-4}\int 1\,dx - \left(\frac{\cos 2x}{-4}\right)\int \cos(4x)\,dx$$

$$y_p = \left(\frac{\sin 2x}{4}\right)\left(\frac{(\sin 2x)^2}{2}\right) + \left(\frac{\cos 2x}{-4}\right)(x) - \frac{\cos 2x}{-16}\left(\sin(4x)\right)$$

Electrical Circuits- HW Problems

In problems 1 and 2 assume we have an RL circuit (ie there is no capacitor) where at time $t = 0$, $I(0) = 0$. Assume that $L = 5H$ and $R = 20ohms$.

1. If $E(t) = 60$ find $I(t)$. Find $\lim_{t \to \infty} I(t)$.
2. If $E(t) = 60\sin(30t)$ find $I(t)$.

3. Assume we have an RC circuit (ie, $L = 0$) where the resistance is $R = 10ohms$ and the capacitance is $C = 0.01Farads$. Suppose the charge at time $t = 0$ is $Q(0) = 10Coulombs$ and $E(t) = 50sin(20t)$. Find $Q(t)$ and the current $I(t)$.

In problems 4 and 5 we have an RLC circuit where $R, L, C, E(t), I(0)$, and $Q(0)$ are given. Find $I(t)$.

4. $L = 2H$, $R = 12ohms$, $C = 0.02F$, $E(t) = 100V$, $I(0) = 0$, $Q(0) = 4$
5. $L = 2H$, $R = 22ohms$, $C = \frac{1}{60}F$, $E(t) = 10e^{-t}$, $I(0) = 0$, $Q(0) = 0$.

In problems 1 & 2 assume we have an RL circuit (i.e. there is no capacitor) where at time $t=0$, $I(0)=0$. Assume that $L=5H$ and $R=20$ ohms.

1.) If $E(t)=60$, find $I(t)$. & find $\lim_{t\to\infty} I(t)$.

$LI' + RI = E(t)$

$5I' + 20I = 60$

$\frac{5(I'+4I)}{5} = \frac{60}{5}$

$I' + 4I = 12$

$\cancel{\frac{dt}{I}} \frac{dI}{\cancel{dt}} = -4I \left(\frac{dt}{\cancel{I}}\right)$

$\int \frac{1}{I} dI = \int -4\,dt$

$\ln|I| = -4t + C_1$

$e^{(\ln|I|)} = e^{(-4t+C_1)}$

$\boxed{I_c(t) = C_1 e^{(-4t)}}$

$E(t) = 60$

$I_p = A \quad I' + 4I = 12$

$I_p' = 0 \quad 0 + 4(A) = 12$

$\frac{4(A)}{4} = \frac{12}{4}$

$\boxed{I_p(t) = 3} \quad \boxed{A = 3}$

$I(t) = C_1 e^{(-4t)} + 3$

$I(0) = 0 = C_1 e^{(-4t)} + 3$

$\boxed{C_1 = -3}$

$\boxed{I(t) = -3e^{(-4t)} + 3}$

$\lim_{x\to\infty} -3e^{(-4t)} + 3 = \underline{-9}$

2.) If $E(t) = 60\sin(30t)$ find $I(t)$

(Complementary solution is the same)

$I_c(t) = C_1 e^{(-4t)}$

$E(t) = 60\sin(30t)$

$I_p = A\sin(30t) + B\cos(30t)$

$I_p' = 30A\cos(30t) - 30B\sin(30t)$

$(30A\cos(30t) - 30B\sin(30t)) + 4(A\sin(30t) + B\cos(30t)) = 60\sin(30t)$

$(30A + 4B)\cos(30t) + (-30B + 4A)\sin(30t) = 60\sin(30t)$

2.) continued

$$\begin{cases} 30A + 4B = 0 \\ -30B + 4A = 60 \end{cases}$$

Solve A + B

$B = -\frac{30A}{4}$

$-30\left(\frac{-30A}{4}\right) + 4A = 60$

$\boxed{B = \left(-\frac{450}{229}\right)}$

$\left(\frac{900}{4}\right)A + 4A = 60$

$\boxed{A = \frac{60}{229}}$

$I_p(t) = \frac{60}{229}\sin(30t) - \frac{450}{229}\cos(30t)$

$I_c(t) = C_1 e^{(-4t)}$

$I(t) = C_1 e^{(-4t)} + \frac{60}{229}\sin(30t) - \frac{450}{229}\cos(30t)$

$I(0) = 0 = C_1 - \frac{450}{229} = C_1 = \frac{450}{229}$

Answer

$$\boxed{I(t) = \left(\frac{450}{229}\right)e^{(-4t)} + \frac{60}{229}\sin(30t) - \frac{450}{229}\cos(30t)}$$

3.) Assume we have an RC circuit (i.e., $L=0$) where the resistance is $R=10$ Ohms & the capacitance is $C=.01$ Farads. Suppose the charge at time $t=0$ is $Q(0)=10$ Coulombs & $E(t)=50\sin(20t)$. Find $Q(t)$ & the current $I(t)$.

$R\frac{dQ}{dt}+\frac{1}{C}Q=E(t)$

$R=10,\ C=.01$

$E(t)=50\sin(20t)$

$10\frac{dQ}{dt}+\frac{1}{.01}Q=50\sin(20t)$

$$\frac{10\left(\frac{dQ}{dt}+10Q\right)}{10}=\frac{50\sin(20t)}{10}$$

$Q'+10Q=5\sin(20t)$

$Q'+10Q=0$

$-10Q \quad -10Q$

$Q'=-10Q$

$\left(\frac{dt}{Q}\right)\frac{dQ}{dt}=-10Q\left(\frac{dt}{Q}\right)$

$\frac{dQ}{Q}=-10dt$

$\int\frac{1}{Q}dQ=-10\int 1\,dt$

$\ln|Q|=((-10t)+C_1)$

$e^{(\ln|Q|)}=e^{((-10t)+C_1)}$

$\boxed{Q_c(t)=C_1e^{(-10t)}}$

$E(t)=50\sin(20t)$,

$Q_p=A\sin(20t)+B\cos(20t)$

$Q_p'=20A\cos(20t)-20B\sin(20t)$

(remember)

$Q'+10Q=5\sin(20t)$

$(20A\cos(20t)-20B\sin(20t))+10(A\sin(20t)+B\cos(20t))$
$=5\sin(20t)$

$(20A+B)\cos(20t)+(-20B+10A)\sin(20t)=5\sin(20t)$

$\begin{Bmatrix}-20B+10A=5\\20A+B=0\end{Bmatrix}$

Solve for A

$B=-20A$

$-20(-20A)+10A=5$

$400A+10A=5$

$\frac{410A}{410}=\frac{5}{410}$

$\boxed{A=\frac{1}{82}}$

Solve for B

$20\left(\frac{1}{82}\right)+B=0$

$\boxed{B=-\frac{10}{41}}$

$\boxed{Q_p(t)=\frac{1}{82}\sin(20t)-\frac{10}{41}\cos(20t)}$

In problems 4 & 5 we have an RLC circuit where $R, L, C, E(t), I(0)$, & $Q(0)$ are given. Find $I(t)$.

4.) $L = 2H$, $R = 12$ ohms, $C = .02F$
$E(t) = 100V$, $I(0) = 0$, $Q(0) = 4$

$LI'' + RI' + \frac{1}{C} I = E'(t)$

$2I'' + 12I' + \frac{1}{.02} I = 0$

$2I'' + 12I' + 50I = 0$

$2(I'' + 6I' + 25I) = 0$

$I'' + 6I' + 25I = 0$

$r^2 + 6r + 25 = 0$

$$r = \frac{-(6) \pm \sqrt{(6)^2 - 4(1)(25)}}{2(1)} = -3 \pm 4i$$

$I(t) = e^{-3t}(C_1 \cos(4t) + C_2 \sin(4t))$

$I'(t) = -3e^{-3t}(C_1 \cos(4t) + C_2 \sin(4t)) + e^{-3t}(-4C_1 \sin 4t + 4C_2 \cos(4t))$

$I(0) = 0 = C_1 \Rightarrow \boxed{C_1 = 0}$

$I'(0)$

$LI'(0) + RI(0) + \frac{1}{C} Q(0) = E(0)$

$2(I'(0)) + 0 + \frac{4}{50} = 100$

$\left(\frac{1}{2}\right) 2(I'(0)) = \frac{2498}{25}\left(\frac{1}{2}\right)$

$\boxed{I'(0) = \frac{1249}{25}}$

$\frac{1249}{25} = I'(0) = -3C_1 + 4C_2$

$\left(\frac{1}{4}\right)\frac{1249}{25} = 4C_2\left(\frac{1}{4}\right)$

$C_2 = \frac{1249}{100}$

$$\boxed{I(t) = e^{-3t}\left(\frac{1249}{100}\right)\sin(4t)}$$

In problems 4 & 5 we have an RLC circuit where $R, L, C, E(t), I(0)$, & $Q(0)$ are given. Find $I(t)$.

5.) $L = 2H$, $R = 22$ ohms, $C = \frac{1}{60}$ F
$E(t) = 10e^{-t}$ V, $I(0) = 0$, $Q(0) = 0$

$LI'' + RI' + \frac{1}{C}I = E'(t)$

$2I'' + 22I' + 60I = -10e^{-t}$

$\frac{2(I'' + 11I' + 30I)}{2} = \frac{-10e^{-t}}{2}$

$I'' + 11I' + 30I = -5e^{-t}$

$r^2 + 11r + 30 = 0$

$(r+5)(r+6) = 0$

$r = -5 \quad r = -6$

$I(t) = C_1 e^{-5t} + C_2 e^{-6t}$

$I'(t) = -5C_1 e^{-5t} - 6C_2 e^{-6t}$

$I(0) = 0 = C_1 + C_2 \qquad C_1 = C_2$

$\frac{2I'(0)}{2} = \frac{10e^{-t}}{2}$

$I'(0) = 5 = -5C_1 - 6C_2$

$5 = -5C_2 - 6C_2$

$\boxed{\frac{5}{-11} = \frac{-11C_2}{-11}}$

$$\boxed{I(t) = \left(\frac{5}{-11}\right)e^{-5t} + \left(\frac{5}{-11}\right)e^{-6t}}$$