

Series Solutions Near Regular Singular Points:  $r_1 - r_2$  is an Integer-  
HW Problems

In problems 1-4 find all possible Frobenius solutions. If a Frobenius solution does not exist, show why it doesn't.

1.  $x^2y'' + xy' + (x^2 - 4)y = 0$
2.  $x^2y'' + (6x + x^2)y' + xy = 0$
3.  $x^2y'' - xy' + y = 0$
4.  $xy'' + (4 - x)y' - y = 0.$

In problems 1-4 find all possible Frobenius solutions.  
 If a Frobenius solution does not exist, show why it  
 doesn't.

$$1) x^2 y'' + xy' + (x^2 - 4)y = 0$$

$$\therefore y'' + \frac{xy'}{x^2} + \frac{(x^2 - 4)}{x^2} y = 0$$

$$f(x) = x \left(\frac{1}{x}\right)$$

$$g(x) = x^2 \left( x^2 \left( \frac{x^2 - 4}{x^2} \right) \right)$$

$$f(x) = 1$$

$$g(x) = -4$$

$$r(r-1) + n = 4$$

$$r^2 - r + r - 4$$

$$(r^2 - 4)$$

$$r_1 = 2 \quad r_2 = -2$$

$$r_1 - r_2 = 4$$

$$x^2 \sum_{n=0}^{\infty} (n+r-1)(n+r) (n x^{n+r-2}) - x \sum_{n=0}^{\infty} (n+r)(n x^{n+r-1} + (x^2 - 4) \sum_{n=0}^{\infty} (n x^{n+r}) = 0$$

$$\sum_{n=0}^{\infty} (n+r-1)(n+r) (n x^{n+r-2}) - \sum_{n=0}^{\infty} (n+r)(n x^{n+r-1}) + \sum_{n=0}^{\infty} (n x^{n+r+1})$$

$$- \sum_{n=0}^{\infty} 4(n x^{n+r}) = 0$$

Corvinus

$$\sum_{n=0}^{\infty} [(n+r-1)(n+r) - (n+r) - 4] \binom{n}{r} x^{n+r} + \sum_{n=0}^{\infty} \binom{n}{r-2} x^{n+r+2} = 0$$

$\uparrow$   
 $\boxed{[(n+r)^2 - 2(n+r) - 4]}$

$\uparrow$   
 $\sum_{n=0}^{\infty} \binom{n-2}{r} x^{n+r}$

$$n=0, c_0=0 \Rightarrow n=1, c_1=0$$

$$n \geq d$$

$$[(n+m)^2 - 2(n+m) - 4] \{ n + (n-1) \} = 5$$

Start with Smaller root  $r = 2$

$$C_n = -\frac{1}{(n+2)^2 - 2(n+2) + 4} = -\frac{1}{n^2 + 4n + 4 - 2n - 4} =$$

$$C_1 = 0, \text{ all odds are zero} = \frac{1}{n^2 + 1 + n + 4}$$

$$n = \delta \quad L = -\frac{1}{\delta} L_0$$

$$n=4 \quad l_4 = -\frac{1}{28} \quad l_2 = \frac{1}{336} \quad l_0$$

$$w=6 \quad b_1 = -\frac{1}{5.2} \quad b_4 = -\frac{1}{17.47} \quad b_6$$

$$z_n = \frac{(-1)^n a}{2^{2n-1}}$$

$$\text{but: } l_2 = -\frac{1}{2}l_0 \neq 0$$

therefore  $\rightarrow$  doesn't satisfy forbenius theorem  $\rightarrow$  doesn't exist

## Solution

$$Y_1(x) = C_0 x^2 \left( 1 + \sum_{n=1}^{\infty} \left( \frac{(-1)^n x^{2n}}{P_n^{2n-1}} \right) \right)$$

$$2) x^2 y'' + (6x + x^2) y' + xy = 0$$

$$y'' + \frac{(6x+x^2)}{x^2} y' + \frac{xy}{x^2} = 0$$

$$P(x) = x \left( \frac{6x+x^2}{x^2} \right) \quad q(x) = x^2 \left( \frac{x}{x^2} \right)$$

$$P(x) = \frac{6x+x^2}{x} \quad q(x) = x$$

Find their r-values at 0

$$P(0) = \frac{0}{0} \quad q(0) = 0$$

The point 0 is not analytic

So, we can't proceed to set up our indicial equation, or find its R-values.

Therefore, there is no Frobenius Solution.

$$3) \begin{cases} x^2 y'' - x y' + y = 0 \\ y'' - \frac{x}{x^2} y' + \frac{1}{x^2} y = 0 \end{cases} \quad \left. \begin{array}{l} y_1 + y_1' x + y_1'' x^2 (\ln x) + y_1''' x^3 + \sum_{n=2}^{\infty} (n+2)(n+1) b_n x^{n+2} - y_1 - y_1' x \ln x \\ + \sum_{n=1}^{\infty} n(n+2) b_n x^n + y_1 (\ln x) + \sum_{n=0}^{\infty} b_n x^{n+2} \end{array} \right\}$$

$$\begin{aligned} f(x) &= x \left( \frac{y_1}{x^2} \right) \quad g(x) = x^2 \left( \frac{1}{x^2} \right) \\ f(0) &= -1 \quad g(0) = 1 \\ f(r-1) - r + 1 &= 0 \\ f^2 - f - r + 1 &= 0 \\ r - 2r + 1 &= 0 \\ (r-1)(r-0) &= 0 \\ r_1 = r_2 &= 1 \end{aligned}$$

Just a Sol

$$\ln x \left[ \frac{y'' x^2 - y'_1 x + y_1}{x^2} \right] + 2 y'_1 x + \sum_{n=2}^{\infty} (n+2)(n+1) b_n x^{n+2} + \sum_{n=1}^{\infty} n(n+2) b_n x^n + \sum_{n=0}^{\infty} b_n x^{n+2} = 0$$

$$2 \left( \sum_{n=1}^{\infty} \frac{(-1)^n d_n x^{2n}}{2^{2n} (n!)^2} \right) + 3 b_0 x^2 + 6 b_1 x^3 + \sum_{n=2}^{\infty} b_n x^{n+2} = 0$$

$$2 \left( \sum_{n=1}^{\infty} \frac{(-1)^n d_n x^{2n}}{2^{2n} (n!)^2} \right) + 3 b_0 x^2 + 6 b_1 x^3 + \sum_{n=2}^{\infty} n^2 (n+6n-2) x^n = 0$$

$$n \geq 2, \quad b_n = -\frac{6n-2}{n^2} \frac{1}{6n^2} \left[ -d_{2n-2} + \frac{2(-1)^{n+1}}{2^{2n} (n!)^2} x \right]$$

$$n=2: b_2 = \left[ -\frac{1}{6} + \frac{-6}{64} \right] = -\frac{7}{32}$$

$$n=3: b_6 = \left[ \frac{7}{32} + \frac{13}{2304} \right] = \frac{11}{64}$$

There is only 1 frobenius  
Solution since  $r_1 = r_2$   
 $y = y_1(\ln x) + \sum_{n=0}^{\infty} b_n x^{n+2}$

$$y' = y_1 \left( \frac{1}{x} \right) + y_1' \left( \ln x \right) + \sum_{n=1}^{\infty} n(n+2) b_n x^{n+1}$$

$$y'' = y_1 \left( \frac{1}{x^2} \right) + y_1' \left( \frac{1}{x} \right) + y_1'' \left( \ln x \right) + y_1''' \left( \frac{1}{x} \right) + \sum_{n=2}^{\infty} (n+2)(n+1) b_n x^n$$

Answer:  $(J_0(x)) \ln x + \frac{x^2}{6} - \frac{7}{24} x^4 + \frac{19}{64} x^6 - \dots = 0$

$$x^2 \left( y_1 \left( \frac{1}{x^2} \right) + y_1' \left( \frac{1}{x} \right) + y_1'' \left( \ln x \right) + y_1''' \left( \frac{1}{x} \right) + \sum_{n=2}^{\infty} (n+2)(n+1) b_n x^n \right) - x \left( y_1 \left( \frac{1}{x} \right) + y_1' \left( \ln x \right) + \sum_{n=1}^{\infty} n(n+2) b_n x^{n+1} \right) + \left( y_1 \left( \ln x \right) + \sum_{n=0}^{\infty} b_n x^{n+2} \right)$$

$$4) XY'' + 4(4-x)Y' - Y = 0 \quad | \quad X\left(\sum_{n=0}^{\infty} (n+r)(n+r-1)x^{n+r-2}\right) + 4(4-x)\sum_{n=0}^{\infty} n+r(nx^{n+r-1})$$

$$Y'' + \frac{4(4-x)}{X}Y' - \frac{Y}{X} = 0 \quad | \quad -\left(\sum_{n=0}^{\infty} (n+r)x^{n+r}\right) = 0 \quad \text{(16-4x)}$$

$$P(x) = X\left(\frac{4(4-x)}{x}\right) \quad Q(x) = x\left(\frac{1}{x}\right)$$

$$P(x) = 4(4-x) \quad Q(x) = x$$

$$P(0) = 16$$

$$Q(0) = 0$$

$$\frac{\sum_{n=0}^{\infty} (n+r)(n+r-1)x_n x^{n+r-1} + 16 \sum_{n=0}^{\infty} n+r(nx^{n+r-1})}{-4 \sum_{n=0}^{\infty} n+r(nx^{n+r}) - \sum_{n=0}^{\infty} (nx^{n+r})}$$

$$r(r-1) + 16r = 0$$

$$r^2 - r + 16r = 0 \quad | \quad \sum_{n=0}^{\infty} \left[ n \left[ (n-r)(n+r) + (n+r) \right] x^{n+r-1} - \sum_{n=1}^{\infty} [4(n+r-1)-1] \binom{n}{n-1} x^{n+r-1} \right]$$

$$r^2 + 15r = 0$$

$$r(r+15) = 0$$

$$r_1 = 0 \quad r_2 = -15$$

$$r_1 - r_2 = 15, \quad r_1 - r_2 = N$$

$$[Y = \sum_{n=0}^{\infty} (n+r)x^{n+r}]$$

$$[Y' = \sum_{n=0}^{\infty} n+r(nx^{n+r-1})]$$

$$[Y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)(nx^{n+r-2})]$$

$$\frac{C_n[(n+15)(n+15-1)+(n-15)]}{[(n+15)(n+15-1)+(n-15)]} = \frac{C_{n-1}[4n+13]}{[(n+15)(n+14)+(n-15)]}$$

$$n=1 : C_1 = C_0 \frac{17}{226} \quad n \geq 1$$

$$n=2 : C_2 = C_1 \frac{3}{37} = C_0 \frac{51}{8562}$$

$$\text{Answer: } n=3 : C_3 = C_2 \frac{25}{241} = C_0 \frac{1375}{2517228}$$

$$Y(x) = C_0 \left( \frac{17}{226} \right) + C_1 \left( \sum_{n=2}^{\infty} \frac{25(5n^2+5n+1)}{2517228} x^n \right)$$

## The Gamma Function and Bessel Functions- HW Problems

1. Use the fact that  $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$  to find  $\Gamma\left(\frac{5}{2}\right)$ .
2. Show that  $\Gamma(x + 5) = (x + 4)(x + 3)(x + 2)(x + 1)x\Gamma(x)$ .
3. Find  $J_3(x)$  in terms of  $J_0(x)$  and  $J_1(x)$ .
4. Show that  $y = xJ_1(x)$  is a solution to the differential equation  

$$xy'' - y' - x^2J'_0(x) = 0.$$

Hint:  $J_1(x)$  satisfies Bessel's equation:

$$x^2J_1''(x) + xJ_1'(x) + (x^2 - 1)J_1(x) = 0,$$

and  $J'_0(x) = -J_1(x)$ .

5. Write  $\int x^4J_0(x)dx$  in terms of Bessel functions and  $\int J_0(x)dx$ .

1.) Use the fact  $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$  to find  $\Gamma\left(\frac{5}{2}\right)$

Definition

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, x > 0$$

remember:

$$\Gamma(x+1) = x\Gamma(x)$$

$$\Gamma\left(\frac{5}{2}\right) = \left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right) = \left(\frac{3}{2}\right)\underline{\Gamma\left(\frac{3}{2}\right)}$$

$$\Gamma\left(\frac{3}{2}\right) = \left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right) = \left(\frac{1}{2}\right)\underline{\Gamma\left(\frac{1}{2}\right)}$$

$$\left\{ \text{Thus, } \Gamma\left(\frac{5}{2}\right) = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi} = \left(\frac{3}{4}\right)\sqrt{\pi} \right.$$

2.) show that  $\Gamma(x+5) = (x+4)(x+3)(x+2)(x+1)x\Gamma(x)$

$$\Gamma(x+5) = ((x+5)-1)\underline{\Gamma((x+5)-1)}$$

$$\Gamma(x+4) = ((x+4)-1)\underline{\Gamma((x+4)-1)}$$

$$\Gamma(x+3) = ((x+3)-1)\underline{\Gamma((x+3)-1)}$$

$$\Gamma(x+2) = ((x+2)-1)\underline{\Gamma((x+2)-1)}$$

$$\Gamma(x+1) = x\underline{\Gamma(x)}$$

$$\text{Therefore, } \Gamma(x+5) = (x+4)(x+3)(x+2)(x+1)x\underline{\Gamma(x)}$$

3.) Find  $J_3(x)$  in terms of  $J_0(x) \& J_1(x)$

use bessel function relationship

$$0 = J_{p-1}(x) - \left(\frac{2}{x}\right) J_p(x) + J_{p+1}(x)$$

$$\underline{J_{p+1}(x) = \left(\frac{2}{x}\right) J_p(x) - J_{p-1}(x)}$$

$$n=1$$

$$\underline{J_2(x) = \left(\frac{2}{x}\right) J_1(x) - J_0(x)}$$

$$n=2$$

$$\underline{J_3(x) = \left(\frac{4}{x}\right) J_2(x) - J_1(x)}$$

$$\underline{J_3(x) = \left(\frac{4}{x}\right) \left( \left(\frac{2}{x}\right) J_1(x) - J_0(x) \right) - J_1(x)}$$

4.) Show that  $y = x J_1(x)$  is a solution to the differential equation  $x^2 y'' - y' - x^2 J_0'(x) = 0$

Hint  
 $J_1(x)$  satisfies  $x^2 J_1''(x) + x J_1'(x) + (x^2 - 1) J_1(x) = 0$ , and  $J_0'(x) = -J_1(x)$ .  
 Bessel's equation  
 (Then sub into eq)

$$(First \text{ find } y' \text{ & } y'') \quad | \quad x(J_1(x) + J_1'(x) + xJ_1''(x)) - (J_1(x) + xJ_1'(x)) - x^2(J_1'(x)) = 0$$

$$y' = J_1(x) + xJ_1'(x) \quad | \quad xJ_1(x) + xJ_1'(x) + x^2J_1''(x) - J_1(x) - x^2(-J_1(x)) = 0$$

$$y'' = J_1'(x) + J_1'(x) + xJ_1''(x) \quad | \quad x^2J_1''(x) + xJ_1'(x) + (x^2 - 1)J_1(x) = 0$$

It takes this form, therefore a solution

Answer

Simplifies to

Bessel Eq

5) Write  $\int x^4 J_0(x) dx$  in terms of Bessel functions

$$\int \frac{x^4}{u} \frac{J_0(x)}{dv} dx = x J_1(x) - \int \frac{J_1(x)}{dv} \frac{4x^3}{u} dx$$

$$du = 4x^3 dx$$

$$v = J_1(x)$$

$$v = -J_0(x)$$

$$du = 3x^2 dx$$

$$x J_1(x) - 4 \left( -x^3 J_0(x) - \int -J_0(x) 3x^2 dx \right)$$

$$x J_1(x) - 4 \left( -x^3 J_0(x) + 3 \int \frac{J_0(x)}{dv} \frac{x^2}{u} dx \right)$$

$$v = J_1(x)$$

$$x J_1(x) - 4 \left( -x^3 J_0(x) + 3 \left( x^2 J_1(x) - \int J_1(x) 2x dx \right) \right)$$

$$x J_1(x) - 4 \left( -x^3 J_0(x) + 3 \left( x^2 J_1(x) - 2 \int \frac{J_1(x)}{dv} \frac{x}{u} dx \right) \right)$$

$$v = -J_0(x)$$

$$du = dx$$

$$\int x^4 J_0(x) dx = x J_1(x) - 4 \left( -x^3 J_0(x) + 3 \left( x^2 J_1(x) - 2(-x J_0(x) + \int J_0(x) dx) \right) \right)$$

## Laplace Transforms- HW Problems

In problems 1-5 use the definition of a Laplace transform to find the Laplace transform of the given function.

1.  $f(t) = t^2 + t$
2.  $f(t) = \cos^2(t)$
3.  $f(t) = e^{(2t-1)}$
4.  $f(t) = 2t \quad 0 \leq t \leq 1$   
 $= 0 \quad 1 < t$
5.  $f(t) = 1 \quad 0 \leq t \leq 2$   
 $= 0 \quad 2 < t.$

In problems 6-10 use the Laplace transforms of functions developed in class and the linearity properties of the Laplace transform to find the Laplace transforms of the following functions.

6.  $f(t) = 4 - 3t^2$
7.  $f(t) = 3 \cos(2t) + 2e^{3t}$
8.  $f(t) = te^{2t}$  (Hint: Use the definition and integrate by parts first)
9.  $f(t) = 4 + \sinh(2t)$
10.  $f(t) = t^{\frac{5}{2}} + \sin(3t)$

In problems 11-17 use the Laplace transforms developed in class to find the inverse Laplace transforms of the following functions.

$$11. \quad F(s) = \frac{3}{s}$$

$$12. \quad F(s) = \frac{1}{s^5}$$

$$13. \quad F(s) = \frac{s}{s^2+9}$$

$$14. \quad F(s) = \frac{3}{s-4}$$

$$15. \quad F(s) = \frac{4-2s}{s^2+16}$$

$$16. \quad F(s) = \frac{s-5}{9-s^2}$$

$$17. \quad F(s) = -3s^{-1}e^{-4s}$$

In problems 1-5 use the definition of a Laplace transform to find the Laplace transform of a given function.

$$1.) f(t) = t^2 + t$$

$$\mathcal{L}(f) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) dt$$

$$\mathcal{L}[f(t)] = \mathcal{L}[t^2] + \mathcal{L}[t]$$

$$\mathcal{L}[t^2] = \int_0^\infty \frac{e^{-st}}{dv} \frac{t^2}{u} dt$$

$$v = -\frac{1}{s} \cdot e^{-st}$$

$$du = dt$$

$$(t^2)(-\frac{1}{s} \cdot e^{-st}) \Big|_0^\infty - \int_0^\infty -\frac{1}{s} \cdot e^{-st} (2t) dt$$

$$\left[ \omega^2 \left( -\frac{1}{s} e^{-\infty} \right) - 0 \cdot \left( -\frac{1}{s} e^0 \right) \right] + \frac{2}{s} \int_0^\infty \frac{e^{-st}}{dv} \frac{t}{u} dt$$

$$dv \quad u \quad du = dx$$

$$\mathcal{L}[t] = \int_0^\infty \frac{e^{-st}}{dv} \frac{t}{u} dt$$

$$v = -\frac{1}{s} e^{-st}$$

$$du = dt$$

$$= +\frac{1}{s} \int_0^\infty e^{-st} dt$$

$$(t)(-\frac{1}{s} e^{-st}) \Big|_0^\infty - \int_0^\infty -\frac{1}{s} \cdot e^{-st} dt$$

$$\frac{2}{s} \left( \frac{1}{s} \left( -\frac{1}{s} e^{-\infty} + \frac{1}{s} e^0 \right) \right)$$

$$\frac{2}{s} \cdot \frac{1}{s} \cdot \frac{1}{s} = \frac{2}{s^3}$$

$$= \frac{1}{s} \int_0^\infty e^{-st} dt$$

$$= \mathcal{L}[t^2] = \frac{2}{s^3}$$

Answer  $\mathcal{L}[t^2 + t] = \frac{1}{s^3} + \frac{1}{s^2}$   
 $s > 0$

$$\frac{1}{s} (-\frac{1}{s} e^{-st}) \Big|_0^\infty = \frac{1}{s} (0 + \frac{1}{s} e^0) \Rightarrow \mathcal{L}[t] = \frac{1}{s^2}$$

$$2.) f(t) = \cos^2(t)$$

$$\mathcal{L}[\cos^2(t)] = \int_0^\infty e^{-st} (\cos^2(t)) dt$$

$$\mathcal{L}\left[\frac{1 + \cos(2t)}{2}\right] = \int_0^\infty e^{-st} \left(\frac{1 + \cos(2t)}{2}\right) dt$$

↑  
Break it up =  $\int_0^\infty e^{-st} \frac{1}{2} dt + \int_0^\infty e^{-st} \frac{\cos(2t)}{2} dt$

$$\frac{1}{2} \int_0^\infty e^{-st} dt = \frac{1}{2s} (e^{-st}) \Big|_0^\infty$$

$$-\frac{1}{2s} [ -e^{-st} + e^0 ] dt$$

remember

$$\frac{1}{2} \lim_{N \rightarrow \infty} \left( \int_0^\infty e^{-st} \cdot ((\cos(2t)) dt \right) \quad |$$

$$\mathcal{L}\left[\frac{1}{2}\right] = \frac{1}{2s}$$

D	I
$+ e^{-st}$	$(\cos(2t))$
$- s e^{-st}$	$\frac{1}{2} \sin(2t)$
$+ s^2 e^{-st}$	$-\frac{1}{4} \cos(2t)$

Make  
denom  
common  $\frac{4}{4}$

$$1 \int e^{-st} \cos(2t) dt = \frac{e^{-st}}{2} \sin(2t) - \frac{s e^{-st}}{4} \cos(2t)$$

$$+ \frac{s^2}{4} \int e^{-st} \cos(2t) dt$$

$$- \frac{s^2 - s}{4} e^{-st} \cos(2t)$$

$$\left[ \frac{4+s^2}{4} \int e^{-st} \sin(2t) dt = \frac{e^{-st}}{2} \sin(2t) - \frac{s e^{-st}}{4} \cos(2t) \right] + \frac{s^2 - s}{4} e^{-st} \cos(2t)$$

$$\int e^{-st} \sin(2t) dt = \frac{4}{4+s^2} \left[ \frac{e^{-st}}{2} \sin(2t) - \frac{s e^{-st}}{4} \cos(2t) \right]$$

↓  
Continued

$$= \lim_{N \rightarrow \infty} \left( \frac{4}{4+s^2} e^{-st} \sin(2t) - \frac{s}{4+s^2} e^{-st} \cos 2t \right)^N$$

$$\lim_{N \rightarrow \infty} \left( \frac{4}{4+s^2} e^{-sN} \overset{0}{\underset{0}{\sin}}(2N) - \frac{s}{4+s^2} e^{-sN} \overset{0}{\underset{0}{\cos}}(2N) \right)$$

$$- \left( \frac{4}{4+s^2} e^0 (0) - \frac{s}{4+s^2} e^{-s(0)} (1) \right)$$

$$= 0 - \left( 0 - \frac{s}{4+s^2} \right)$$

$$= \frac{1}{2} \left( \frac{s}{4+s^2} \right)$$



remember

$$\mathcal{L}[\cos^2(t)] = \frac{1}{2s} + \frac{1}{2} \left( \frac{s}{4+s^2} \right) \quad s > 0$$



ANSWER

$$3.) f(t) = e^{(2t-1)}$$

$$\mathcal{L}[e^{(2t-1)}] = \int_0^\infty e^{-st} e^{(2t-1)} dt$$

$$\int_0^\infty e^{(-st + (2t-1))} dt$$

$$\int_0^\infty e^{(-s+2t-1)} dt$$

$$= \frac{1}{-s+2} \left( e^{(-s+2t-1)} \right) \Big|_0^\infty$$

$$= -\frac{1}{-(s-2)} \left( \lim_{t \rightarrow \infty} \left( e^{(-s(\infty) + 2(\infty) - 1)} - e^{(-1)} \right) \right)$$

$$= e^{(-1)} \left( \frac{1}{s-2} \right)$$

$s \geq 2$

$$4.) f(t) = 2t \quad 0 \leq t \leq 1 \quad \mathcal{L}[2t] = \int_0^1 e^{-st} \frac{2t}{u} du$$

$= 0$

$1 < t$

$$v = \underbrace{\frac{1}{s} e^{-st}}_{u}, du = 2dt$$

$$2t + \left( -\frac{1}{s} e^{-st} \right) - \int_0^1 -\frac{1}{s} e^{-st} 2dt$$

$$2t + \left( -\frac{1}{s} e^{-st} \right) + \frac{2}{s} \int_0^1 e^{-st} dt$$

$$2\left(-\frac{1}{s} e^{-s}\right) + \frac{2}{s} \left( -\frac{1}{s} e^{-s} \right) \Big|_0^1$$

$$+ \frac{2}{s} \left( -\frac{1}{s} e^{-s} + \frac{1}{s} e^0 \right)$$

$$s > 0 \quad \boxed{\mathcal{L}[2t] = -\frac{2}{s^2} e^{-s} + \frac{1}{s^2} + 2 \left( -\frac{1}{s} e^{-s} \right)}$$

$$5.) f(t) = \begin{cases} 1 & 0 \leq t \leq 2 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L}[1] = \int_0^{\infty} e^{-st} 1 dt = \int_0^{\infty} e^{-st} dt$$

$$\left( -\frac{1}{s} e^{-st} \right) \Big|_0^{\infty}$$

$$= -\frac{1}{s} e^{-s \cdot 0} + \frac{1}{s} e^{-s \cdot \infty}$$

$$\mathcal{L}[1] = -\frac{1}{s} e^{-2s} + \frac{1}{s}$$

$$s > 0$$

In problems 6-10 use the Laplace transforms of functions developed in class & the linearity properties of the Laplace transform to find the Laplace transforms of the following functions.

6.)  $f(t) = 4 - 3t^2$

$$L[4] - 3L[t^2]$$

| first find  $L[4]$ ,

Then find  $L[t^2]$

$$\begin{aligned} &\cdot \int_0^\infty e^{-st} t^2 dt \\ &\frac{d}{dt} \left[ \frac{e^{-st}}{s} t^2 \right] \end{aligned}$$

$$\begin{aligned} v &= -\frac{1}{s} e^{-st} \\ du &= 2t dt \end{aligned}$$

$$\int_0^\infty e^{-st} 4 dt$$

$$\begin{aligned} &+ \int_0^\infty -\frac{1}{s} e^{-st} 2t dt \\ &= 4 \left( -\frac{1}{s} e^{-st} \right) \Big|_0^\infty \end{aligned}$$

$$+ 2 \left( -\frac{1}{s} e^{-st} \right) - \int_0^\infty -\frac{1}{s} e^{-st} 2t dt$$

$$4 \left( 0 + \frac{1}{s} e^{-s \cdot 0} \right) = \boxed{\frac{4}{s}}$$

$$\begin{aligned} &\uparrow + \frac{2}{s} \int_0^\infty e^{-st} \frac{1}{s} dt \\ &\downarrow \frac{d}{dt} \left[ \frac{e^{-st}}{s} \right] \end{aligned}$$

remember

$$+ \left( -\frac{1}{s} e^{-st} \right) - \int_0^\infty -\frac{1}{s} e^{-st} dt$$

$$+ \left( -\frac{1}{s} e^{-st} \right) + \frac{1}{s} \int_0^\infty e^{-st} dt$$

$$\frac{2}{s} \cdot \frac{1}{s} \left( -\frac{1}{s} e^{-st} \right) \Big|_0^\infty$$

Answer

$$L[4 - 3t^2] = \frac{4}{s} - 3 \left( \frac{2}{s^3} \right)$$

$s > 0$

$$\frac{2}{s} \cdot \frac{1}{s} \left( 0 + \frac{1}{s} \right) = \boxed{\frac{2}{s^3}}$$

$$7) f(t) = 3\cos(2t) + 2e^{3t} \quad | \quad d[e^{st}] = \int_0^{\infty} e^{-st} \cdot e^{st} dt$$

$$3d[\cos(2t)] + 2d[e^{3t}] \quad | \quad \int_0^{\infty} (-st + (3+)) dt +$$

$$\left( -\frac{1}{s+3} \right) e^{-st + (3+)} \Big|_0^{\infty}$$

$$d[\cos(2t)] \quad | \quad -(s-3)e^{-st + (3+)} \Big|_0^{\infty}$$

$$\int_0^{\infty} e^{-st} \cos(2t) dt \quad | \quad -\left(\frac{1}{s-3}\right)(e^{-\infty} - e^0)$$

$$+\boxed{\begin{array}{c} D \\ e^{-st} \end{array}} \quad \boxed{\begin{array}{c} D \\ -s e^{-st} \end{array}} \quad \boxed{\begin{array}{c} I \\ \sin(2t) \\ 2 \end{array}}$$

$$-\boxed{\begin{array}{c} D \\ -s^2 e^{-st} \end{array}} \quad \boxed{\begin{array}{c} I \\ -\frac{\cos(2t)}{4} \end{array}}$$

$$\boxed{\text{Answer} \rightarrow d[3\cos(2t) + 2e^{3t}] = 3\left(\frac{4}{s^2+4}\right) + 2\left(\frac{1}{s-3}\right)}$$

$$S > 0$$

$$(e^{-st})\left(\frac{\sin(2t)}{2}\right) - (s e^{-st})\left(\frac{\cos(2t)}{4}\right) \rightarrow \left(s^2 \cdot e^{-st}\right)\left(\frac{\cos(2t)}{4}\right) dt$$

$$\int_0^{\infty} e^{-st} \cos(2t) dt = (e^{-st})\left(\frac{\sin(2t)}{2}\right) - (s e^{-st})\left(\cos(2t)\right) + \frac{s^2}{4} \int_0^{\infty} e^{-st} \cos(2t) dt$$

$$+\frac{9}{4} \int_0^{\infty} e^{-st} \cos(2t) dt \quad | \quad + \frac{s^2}{4} \int_0^{\infty} e^{-st} / \cos(2t) dt$$

$$\boxed{\frac{5}{4} + \frac{4}{4}} \quad \boxed{\frac{-5+4}{4} \int_0^{\infty} e^{-st} \cos(2t) dt = (e^{-st})\left(\frac{\sin(2t)}{2}\right) - (s e^{-st})\left(\cos(2t)\right)}$$

$$= \frac{4}{s^2+4} \left( (e^{-st})\left(\frac{\sin(2t)}{2}\right) - (s e^{-st})\left(\cos(2t)\right) \right) \Big|_0^{\infty}$$

$$\frac{4}{s^2+4} \left( (e^{-\infty})\left(\frac{\sin(\infty)}{2}\right) - \left(\frac{1}{4} e^{-\infty}\right)\left(\cos(2t)\right) - \left(e^0 \left(\frac{\sin(0)}{2}\right) - \left(\frac{1}{4}\right)(1)\right) \right)$$

$$\left(\frac{1}{4}\right) \left(\frac{4}{s^2+4}\right) = \boxed{\frac{1}{s^2+4} = d[\cos(2t)]}$$

$$8.) f(t) = t e^{2t}$$

$$\mathcal{L}[t e^{2t}] = \int_0^\infty e^{-st} \cdot t e^{2t} dt$$

$$\int_0^\infty \frac{t}{u} \frac{e^{(-st)+(2t)}}{du} dt$$

$$du = dt$$

$$v = \left(\frac{1}{-s+2}\right) \left(e^{(-st)+(2t)}\right)$$

$$(+) \left( \frac{e^{(-st)+(2t)}}{-s+2} \right) \Big|_0^\infty - \int_0^\infty \left( \frac{1}{-s+2} \right) \left( e^{(-st)+2t} \right) dt +$$

$$(+) \left( \frac{e^{(-st)+(2t)}}{-s+2} \right) \Big|_0^\infty + \int_0^\infty \left( \frac{1}{s-2} \right) \left( e^{(-st)+2t} \right) dt +$$

$$(+) \left( \frac{e^{(-st)+(2t)}}{-s+2} \right) \Big|_0^\infty + \left. -\left( \frac{1}{(s-2)^2} \right) \left( e^{(-st)+2t} \right) \right|_0^\infty$$

$$(-( \infty)(0)) - (0) + -\left( \frac{1}{(s-2)^2} \right) (0 - 1)$$

$$= \boxed{\frac{1}{(s-2)^2}} \quad s > 2$$

$$9.) f(t) = 4 + \sinh(2t)$$

$$\mathcal{L}[4] + \mathcal{L}[\sinh(2t)] = \mathcal{L}[4] + \left( 2 \left[ \frac{e^{kt} - e^{-kt}}{s} \right] \right)$$

$$\mathcal{L}[\sinh(2t)] = \int_0^\infty e^{st} \sinh(2t) dt \quad \frac{1}{2} \left( \mathcal{L}[e^{kt}] - \mathcal{L}[e^{-kt}] \right)$$

$$\mathcal{L}[e^{kt}] = \int_0^\infty e^{-st} e^{kt} dt \quad \mathcal{L}[e^{-kt}] = \int_0^\infty e^{-st} e^{-kt} dt$$

$$\int_0^\infty e^{-s+t+kt} dt \quad \int_0^\infty e^{-s-t-kt} dt$$

$$\left( \frac{1}{s+k} \right) e^{-st+kt} \Big|_0^\infty - \left( \frac{1}{s-k} \right) e^{-st-kt} \Big|_0^\infty$$

$$\mathcal{L}[4] + \left( \frac{1}{s-k} - \frac{1}{s+k} \right) \frac{1}{2}$$

$$4 \int e^{-st} dt \\ \frac{4}{s}$$

$$\boxed{\frac{4}{s} + \frac{1}{2} \left( \frac{1}{s-2} - \frac{1}{s+2} \right)} = \mathcal{L}[4 + \sinh(2t)]$$

$$|s > 0 \\ \text{or}$$

$$\frac{1}{2} \left( \frac{s+2}{(s-2)(s+2)} - \frac{(s-2)}{(s-2)(s+2)} \right)$$

Another Form

$$\frac{1}{2} \left( \frac{4}{s^2 - 4} \right)$$

$$\frac{1}{s} + \left( \frac{2}{s^2 - 4} \right) = \mathcal{L}[4 + \sinh(2t)]$$

$$\frac{2}{s^2 - 4}$$

$$10) f(t) = t^{\frac{5}{2}} + \sin(3t)$$

~~$$\int_0^\infty e^{-st} t^{\frac{5}{2}} dt$$~~

~~$$\int_0^\infty e^{-st} t^{\frac{5}{2}} dt$$~~

~~$$\int_0^\infty e^{-st} t^{\frac{5}{2}} dt = \int_0^\infty u^{\frac{5}{2}} du$$~~

$$\int_0^\infty e^{-st} t^{\frac{5}{2}} dt = \int_0^\infty t^{\frac{5}{2}} dt e^{-st}$$

$$2\int_0^\infty t^{\frac{5}{2}} dt = \frac{\Gamma(\frac{7}{2})}{s^{\frac{7}{2}-1}}$$

$$\frac{\Gamma(\frac{5}{2})}{s^{\frac{5}{2}-1}} = \frac{\Gamma(\frac{7}{2})}{s^{\frac{7}{2}}}$$

Next page

$$\text{remember } (s-i3)(s+i3) = s^2 + 9$$

$$= \frac{1}{2i} \left( \frac{1}{s-i3} - \frac{1}{s+i3} \right)$$

$$= \frac{\left( \frac{1}{s-i3} - \frac{1}{s+i3} \right)}{2i(-i)}$$

$$= -i \frac{\left( \frac{1}{s-i3} - \frac{1}{s+i3} \right)}{2}$$

$$= \frac{(i3)}{s^2 + 9} = \frac{3}{s^2 + 9}$$

$$L[\sin(3t)] = \frac{3}{s^2 + 9}$$

$$\frac{(s-i3)(s+i3)}{(s-i3)(s+i3)} \cdot \frac{2i((s-i3) - 2i(s+i3))}{2i(-i)} = \frac{2i(-i)}{(s-i3)(s+i3)}$$

→ continued

$T(1+x) = xT(x)$ , we want  $T(\frac{\pi}{2})$ , we know  $T(\frac{1}{2}) = \sqrt{91}$

$$T\left(\frac{3}{2}\right) = T\left(1 + \frac{1}{2}\right) = \frac{1}{2}T\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{91} \approx \frac{\sqrt{91}}{2}$$

$$T\left(\frac{5}{2}\right) = T\left(1 + \frac{3}{2}\right) = \frac{3}{2}T\left(\frac{3}{2}\right) = \frac{3}{2}\left(\frac{\sqrt{91}}{2}\right) = \frac{3\sqrt{91}}{4}$$

$$T\left(\frac{7}{2}\right) = T\left(1 + \frac{5}{2}\right) = \frac{5}{2}T\left(\frac{5}{2}\right) = \frac{5}{2}\left(\frac{3\sqrt{91}}{4}\right) \approx \frac{15\sqrt{91}}{8}$$

$$\boxed{L[s^2 + s \sin(3s)] = \frac{\frac{15\sqrt{91}}{8}}{s^2 + 9} + \frac{3}{s^2 + 9}} \quad s > 0$$

↑  
Answer

In problems 11-17 use the Laplace transforms developed in class to find inverse Laplace transforms of the following functions.

$$11.) F(s) = \frac{3}{s} \quad L^{-1}\left[\frac{3}{s}\right] = 3, L[a] = \frac{a}{s}$$

$$12.) F(s) = \frac{1}{ss} \quad L^{-1}\left[\frac{1}{ss}\right] = \frac{1}{24}(t^4), \text{ remember } \frac{n!}{s^{n+1}} = L[t^n]$$

$$13.) F(s) = \frac{s}{s^2+9} \quad L^{-1}\left[\frac{s}{s^2+9}\right] = \underline{\cos(3t)}, \quad L[\cos kt] = \frac{s}{s^2+k^2}$$

$$14.) F(s) = \frac{3}{s-4} \quad L^{-1}\left[\frac{3}{s-4}\right] = \underline{3(e^{4t})}, \quad L[e^{at}] = \frac{1}{s-a}$$

$$15.) F(s) = \frac{4-2s}{s^2+16} \quad L^{-1}\left[\frac{4-2s}{s^2+16}\right] = \underline{(\sin(4t)-2\cos(4t))}$$

$$F(s) = \frac{4}{s^2+16} - \frac{2s}{s^2+16}, \quad L^{-1}\left[\frac{4}{s^2+16}\right] - 2L^{-1}\left[\frac{s}{s^2+16}\right]$$

$$\sin(4t) - 2\cos(4t)$$

$$16.) F(s) = \frac{s-5}{9-s^2}, \quad L^{-1}\left[\frac{s}{9-s^2}\right] - L^{-1}\left[\frac{5}{9-s^2}\right]$$

$$\underline{(\cosh(t) - \frac{5}{3}\sinh(t))}$$

$$17.) F(s) = -3s^1 e^{4s}, \quad L^{-1}\left[-3 \frac{e^{-4s}}{s}\right] = -3 L^{-1}\left[\frac{e^{-4s}}{s}\right]$$

$$U(+-\alpha)f(+-\alpha) \rightarrow$$

sorta  
confusing

$$\overbrace{-3 U(+-\alpha) L^{-1}\left[\frac{1}{s}\right]}$$

$$= \underline{-3 U(+-\alpha)(1)}$$