

## Solving Initial Value Problems with Laplace Transforms- HW Problems

In problems 1-5 solve the initial value problem using Laplace transforms.

$$1. \quad y' - 4y = 0; \quad y(0) = 3$$

$$2. \quad y'' + 9y = 0; \quad y(0) = 3, \quad y'(0) = 6$$

$$3. \quad x'' + 2x' - 3x = 0; \quad x(0) = 5, \quad x'(0) = -3$$

$$4. \quad x'' + 9x = \sin(2t); \quad x(0) = x'(0) = 0$$

$$5. \quad x'' + 6x' + 8x = 8; \quad x(0) = x'(0) = 0.$$

In problems 1-5 solve the initial value problem  
using Laplace transforms.

$$\mathcal{L}[f(t)] = F(s)$$

$$1.) y' - 4y = 0; \quad y(0) = 3$$

$$\mathcal{L}[f(t)] = F(s)$$

$$\mathcal{L}[y'] - \mathcal{L}[4y] = \mathcal{L}[0]$$

$$\mathcal{L}[f'(t)] = sF(s) - f(0)$$

$$\mathcal{L}[y] - 4\mathcal{L}[y] = 0$$

$$\mathcal{L}[f''(t)] = s^2F(s) - sf(0) - f'(0)$$

$$sf(s) - f(0) - 4(f(s)) = 0$$

$$sf(s) - 3 - 4f(s) = 0$$

$$sf(s) - 4f(s) = 3$$

$$\frac{f(s)(s-4)}{(s-4)} = \boxed{\frac{3}{s-4}}$$

$$x(t) = \mathcal{L}^{-1}\left[\frac{3}{s-4}\right] = 3\left[\mathcal{L}^{-1}\left[\frac{1}{s-4}\right]\right] = 3\left[e^{4t}\right]$$

$$2.) y'' + 9y = 0; \quad y(0) = 3, \quad y'(0) = 6 \quad f(t) = y$$

$$(s^2 f(s) - s f(0) - f'(0)) + 9(f(s)) = L[0]$$

$$s^2 f(s) - s(3) - 6 + 9f(s) = 0$$

$+6 \qquad \qquad \qquad +6$

$$s^2 f(s) - 3s + 9f(s) = 6$$

$+3s \qquad \qquad \qquad +3s$

$$s^2 f(s) + 9f(s) = 6 + 3s$$

$$\frac{f(s)(s^2 + 9)}{(s^2 + 9)} = \boxed{\frac{6 + 3s}{(s^2 + 9)}}$$

$$L^{-1} \left[ \frac{6+3s}{s^2+9} \right] = 6\sin(3t) + 3\cos(3t)$$

$$f(t) = X$$

$$3.) x'' + 2x' - 3x = 0; \quad x(0) = 5, \quad x'(0) = -3$$

$$s^2 f(s) - s f(0) - f'(0) + 2(s f(s) - f(0)) - 3(f(s)) = 1[0] + 1[f(t)] = f(s)$$

$$\begin{array}{l} s^2 f(s) - s(5) - (-3) + 2s f(s) - 2(5) - 3f(s) = 0 \\ \quad + 5s \quad - 3 \qquad \qquad \quad + 10 \end{array} \quad \begin{array}{l} 1[2][f'(t)] = sf(s) - f(0) \\ 1[2][f''(t)] = s^2 f(s) - sf(0) - f'(0) \end{array}$$

$$s^2 f(s) + 2s f(s) - 3f(s) = 7 + 5s$$

$$\frac{f(s)(s^2 + 2s - 3)}{(s^2 + 2s - 3)} = \boxed{7 + 5s}$$

$$(s+3)(s-1) \left( \frac{7 + 5s}{(s+3)(s-1)} \right) = \left( \frac{A}{(s+3)} + \frac{B}{(s-1)} \right) (s+3)(s-1)$$

$$7 + 5s = A(s-1) + B(s+3)$$

$$\text{put } s=1 \quad \text{put } s=-3$$

$$\frac{12}{4} = \frac{4B}{4}$$

$$7 - 15 = -4A$$

$$B = \frac{12}{4}$$

$$\frac{-8}{4} = -4A$$

$$\frac{8}{4} = A$$

$$\frac{8}{4} \int^{-1} \left[ \frac{1}{(s+3)} \right] + \frac{12}{4} \int^{-1} \left[ \frac{1}{s-1} \right]$$

$$x(t) = \boxed{\frac{8}{4} [e^{-3t}] + \frac{12}{4} [e^{t}]}$$

↑  
Answer

$$4.) x'' + 9x = \sin(2t); x(0) = x'(0) = 0$$

$$2[x''] + 92[x] = L[\sin(2t)]$$

$$(s^2 f(s) - sf(0) - f'(0)) + 9(f(s)) = \frac{2}{s^2 + 4}$$

$$s^2 f(s) - s(0) - 0 + 9(f(s)) = \frac{2}{s^2 + 4}$$

$$s^2 f(s) + 9f(s) = \frac{2}{s^2 + 4}$$

$$\frac{f(s)(s^2 + 9)}{(s^2 + 4)} = \frac{\frac{2}{s^2 + 4}}{s^2 + 9} = \boxed{\frac{2}{(s^2 + 4)(s^2 + 9)}}$$

Partial fractions

$$2 \left[ \mathcal{L}^{-1} \left[ \frac{1}{(s^2 + 2^2)(s^2 + 3^2)} \right] \right]$$

$$s^4 + 4s^2 + 9s^4 + 36$$

$$s^4 + 13s^2 + 36$$

$$\frac{1}{(s^2 + 2^2)(s^2 + 3^2)} \Rightarrow \frac{As + B}{(s^2 + 2^2)} + \frac{(s+1)}{(s^2 + 3^2)}$$

$$2 \left[ \mathcal{L}^{-1} \left[ \frac{(\frac{-2}{3})s + (\frac{11}{3})}{s^2 + 4} + \frac{(\frac{8}{3})s + (-\frac{8}{3})}{s^2 + 9} \right] \right] | = As + B(s^2 + 3^2) + (s+1)(s^2 + 2^2)$$

$$2 \left[ \frac{(\frac{11}{6}\sin(2t) - \frac{3}{5}\cos(2t) + (-\frac{8}{9}\sin(3t) + \frac{8}{5}\cos(3t))}{s^2 + 4} \right] | = A(s^3 + 9As + s^2B + 9B + (s^3 + 4(s+1)s^2 + 4))$$

$$1 = A(s^3 + 9As + s^2B + 9B + (s^3 + 4(s+1)s^2 + 4))$$

$$1 = s^3(A + C) + s^2(B + D) + s(9A + 4C) + (9B + 4D)$$

ANSWER

Clearer version

$$x(t) = \frac{11}{6}(\sin(2t)) - \frac{3}{5}(\cos(2t)) + \left(-\frac{8}{9}\sin(3t) + \frac{8}{5}\cos(3t)\right)$$

$$\begin{array}{l|l} 1 = A + C & 1 = 9A + 4C \\ 1 = B + D & 1 = 9B + 4D \\ \hline & \text{sys 1} \\ & \text{sys 2} \end{array}$$

$$1 = 9A + 4C$$

$$-9A - 4C$$

$$1 - 9A = 4C$$

$$\boxed{\frac{11}{3} = B}$$

$$1 - D = B$$

$$1 = B - \frac{8}{3}$$

$$\boxed{D = -\frac{8}{3}}$$

$$\frac{-8}{3} = \frac{3D}{3}$$

$$\boxed{\frac{8}{3} = C}$$

$$\boxed{-\frac{3}{5} = A}$$

$$3 = -5A$$

$$4 = 4A + (1 - 9)$$

$$4 = 1 - 5A$$

$$5.) \quad x'' + 6x' + 8x = 8; \quad x(0) = x'(0) = 0$$

$$\mathcal{L}[x''] + 6\mathcal{L}[x'] + 8\mathcal{L}[x] = \mathcal{L}[8] \quad | \quad \mathcal{L}[f(t)] = f(s)$$

$$s^2 f(s) - sf(0) - f'(0) + 6(sf(s) - f(0)) + 8(f(s)) = \frac{8}{s}$$

$$s^2 f(s) - s(0) - 0 + 6sf(s) - 6(0) + 8f(s) = \frac{8}{s}$$

$$\frac{f(s)(s^2 + 6s + 8)}{(s^2 + 6s + 8)} = \frac{\frac{8}{s}}{s(s^2 + 6s + 8)} = \boxed{\frac{8}{s^3 + 6s^2 + 8s}}$$

Partial fractions

$$8 \mathcal{L}^{-1} \left[ \frac{1}{s^3 + 6s^2 + 8s} \right]$$

$$\left( \frac{1}{s(s+6+1)} = \frac{1}{s^2(s+7)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s+7)} \right)$$

$$8 \mathcal{L}^{-1} \left[ -\frac{1}{49} \left[ \frac{1}{s} \right] + \frac{1}{7} \left[ \frac{1}{s^2} \right] + \frac{1}{49} \left[ \frac{1}{s+7} \right] \right]_1 = A(s)(s+7) + B(s+7) + C(s^2)$$

$$x(t) = 8 \left[ -\frac{1}{49} [1] + \frac{1}{7} [t] + \frac{1}{49} [e^{-7t}] \right] \quad \begin{array}{l} \text{put } s=0 \\ \frac{1}{7} = 7B, B = \frac{1}{7} \\ \text{put } s=1 \end{array} \quad \begin{array}{l} \text{put } s=-7 \\ \frac{1}{49} = 49C, C = \frac{1}{49} \end{array}$$

$$1 = A(1)(1+7) + \frac{1}{7}(8) + \frac{1}{49}(1)$$

$$\begin{array}{l} 1 = 6A + \frac{8}{7} + \frac{1}{49} \\ -\frac{8}{7} - \frac{1}{49} \end{array} \quad \begin{array}{l} -\frac{8}{7} - \frac{1}{49} \\ (-\frac{8}{49}) = \frac{8A}{8} \end{array}$$

$$- = A$$

## Translation of Laplace Transforms and Partial Fractions- HW Problems

In problems 1-3 find the Laplace transform of the given function.

1.  $x(t) = t^2 e^{3t}$
2.  $f(t) = e^{(\frac{t}{2})} \cos(5t)$
3.  $x(t) = e^{5t} \sin(2t)$

In problems 4-8 find the inverse Laplace Transform of the given function. Use partial fractions when appropriate.

4.  $F(s) = \frac{1}{s^2 + 6s + 9}$
5.  $F(s) = \frac{s+3}{s^2 + 4s + 8}$
6.  $F(s) = \frac{6s^2 - s - 6}{s^3 - s^2 - 6s}$
7.  $F(s) = \frac{4}{s^3 - 4s}$
8.  $F(s) = \frac{1}{s^4 - 2s^2 + 1}$

In problems 9-11 solve the initial value problems with Laplace transforms.

9.  $x'' + 2x' + 6x = 6; \quad x(0) = x'(0) = 0$
10.  $y'' + 4y' + 8y = e^{-t}; \quad y(0) = y'(0) = 0$
11.  $y'' + 2y' + 2y = 2\cos(t) + \sin(t); \quad y(0) = y'(0) = 0$

In Problems 1 & 3 find the Laplace transform of the given function.

$$1.) X(t) = t^2 e^{3t}$$

$$\int_0^\infty e^{-st} e^{3t} + t^2 dt$$

$$\left( \frac{2}{s-3} \right) \left( \frac{1}{s-3} \right) \left( \frac{1}{(s-3)^2} \right) \left( e^{-st+3t} \right) \Big|_0^\infty \Rightarrow (1)$$

$$\int_0^\infty \frac{e^{-st+3t}}{u} \frac{t^2}{u} dt \quad v = -\frac{1}{(s-3)} e^{-st+3t}$$

$$du = 2t dt$$

$$= \frac{1}{(s-3)(s-3)(s-3)}$$

$$+ \left. t^2 \left( -\frac{1}{(s-3)^2} e^{-st+3t} \right) \right|_0^\infty - \int_0^\infty -\frac{1}{(s-3)} e^{-st+3t} 2t dt$$

$$\stackrel{\text{1}}{\cancel{0}} \qquad \qquad \qquad \stackrel{\text{X(s)}}{=} \boxed{d \left[ t^2 e^{3t} \right] = \frac{2}{(s-3)^3} \quad \text{Answer}}$$

$$\frac{1}{s-3} \int_0^\infty \frac{e^{-st+3t}}{u} \frac{t}{u} dt \quad v = -\frac{1}{(s-3)} e^{-st+3t}$$

$$du = dt$$

$$\frac{1}{s-3} \left( t \left( -\frac{1}{(s-3)} e^{-st+3t} \right) \Big|_0^\infty + \int_0^\infty \frac{1}{(s-3)} e^{-st+3t} dt \right)$$

$$\stackrel{\text{0}}{\cancel{1}}$$

$$\left( \frac{1}{s-3} \right) \left( \frac{1}{s-3} \right) \int_0^\infty e^{-st+3t} dt$$

In Problems 1 & 3 find the Laplace transform of the given function. 2.)  $f(t) = e^{\frac{t}{2}} \cos(5t)$

$$\int_0^\infty e^{-st} e^{\frac{t}{2}} \cos(5t) dt$$

$$\int_0^\infty \frac{e^{-st + (\frac{t}{2})}}{D} \frac{(os(5t))}{I} dt$$

$e^{-st + (\frac{t}{2})}$	$\cos(5t)$
$(-s + \frac{1}{2}) e^{-st + (\frac{t}{2})}$	$\frac{\sin(5t)}{5}$
$(-s + \frac{1}{2})^2 e^{-st + (\frac{t}{2})}$	$-\frac{\cos(5t)}{25}$

$$\frac{25}{25 + 6s + \frac{1}{4}} \left[ \left( e^{-st + (\frac{t}{2})} \left( \frac{\sin(5t)}{5} \right) + (s + \frac{1}{2}) e^{-st + (\frac{t}{2})} \frac{\cos(5t)}{25} \right) \Big|_0^\infty - \left( -\frac{s + \frac{1}{2}}{25} \right) \right]$$

$$X(s) = -\frac{(s + \frac{1}{2})}{25 + 6s + \frac{1}{4}} = L[e^{\frac{t}{2}} \cos(5t)]$$

∴ Answer

$$\int_0^\infty e^{-st + (\frac{t}{2})} \cos(5t) dt = e^{-st + (\frac{t}{2})} \left( \frac{\sin(5t)}{5} \right) + (6s + \frac{1}{2}) e^{-st + (\frac{t}{2})} \frac{\cos(5t)}{25} \Big|_0^\infty$$

$$1 + \frac{(-s + \frac{1}{2})^2}{25} \Big|_0^\infty = \frac{25 + 6s + \frac{1}{4}}{25} \xrightarrow{\text{Reciprocal to bring } -\frac{(-s + \frac{1}{2})^2}{25} \text{ to other side}} + \left( \frac{-s + \frac{1}{2}}{25} \right) \Big|_0^\infty$$

$$3.) X(t) = e^{5t} \sin(2t)$$

$$\int_0^{\infty} e^{-st} (e^{st} \sin(2t)) dt$$

$$\int_0^{\infty} \frac{e^{-st+st}}{D} \frac{\sin(2t)}{I} dt$$

	D	I
+	$e^{-st+5t}$	$\sin(2t)$
-	$(s+5)e^{-st+5t}$	$-\frac{\cos(2t)}{2}$
+	$(-s+5)e^{-st+5t}$	$-\frac{\sin(2t)}{4}$

$$\begin{aligned} & \int_0^{\infty} e^{-st+(5t)} \sin(2t) dt = \\ & -e^{-st+(5t)} \left( \frac{\cos(2t)}{2} \right) + (-s+5) e^{-st+(5t)} \left( \frac{\sin(2t)}{4} \right) - \int_0^{\infty} (-s+5)^2 e^{-st+(5t)} \left( \frac{\sin(2t)}{4} \right) dt \\ & -\frac{(-s+5)^2}{4} \int_0^{\infty} e^{-st+(5t)} \sin(2t) dt \end{aligned}$$

Adding to other side of Eq :  $1 + \frac{(-s+5)^2}{4} = \frac{4+(-s+5)^2}{4}$

Multiply reciprocal to other side

& Evaluate  $X(s) =$

$$\frac{4}{4+(-s+5)^2} \left( -e^{-st+(5t)} \left( \frac{\cos(2t)}{2} \right) + (-s+5) e^{-st+(5t)} \left( \frac{\sin(2t)}{4} \right) \right) \Big|_0^{\infty}$$

$$\frac{4}{4+(-s+5)^2} \left( -\left( -1 \left( \frac{1}{2} \right) \right) \right) = \boxed{\frac{2}{4+(-s+5)^2}}$$

Answer

In problems 1-3 find the inverse Laplace Transform of the given function.  
use partial fractions when appropriate.

4.)  $Q(s) = \frac{1}{s^2 + 6s + 9}$

use partial fractions

$$\frac{1}{(s+3)^2} = \frac{A}{(s+3)} + \frac{B}{(s+3)^2} = \frac{0}{s+3} + \frac{1}{(s+3)^2}$$

$$1 = A(s+3) + B$$

$$s = -3$$

$$s = 1$$

$$\boxed{1=6}$$

$$\begin{matrix} 1 = 4A + 1 \\ -1 \end{matrix}$$

$$\frac{0}{4} = \frac{4A}{4}$$

$$\boxed{0=4}$$

funny (sorta redundant)

$$\begin{aligned} L^{-1}\left[\frac{1}{s^2 + 6s + 9}\right] &= L^{-1}\left[0 + \frac{1}{(s+3)^2}\right] \\ &= 0 + t^3 f^1 \end{aligned}$$

$$X(t) = t^3 f^1 \leftarrow \underline{\text{Answer}}$$

In problems 1-3 find the inverse Laplace Transform of the given function.  
Use partial fractions when appropriate.

$$5.) \mathcal{F}(s) = \frac{s+3}{s^2+4s+8}$$

$$\downarrow$$

$$\left( \frac{s+3}{(s^2+4s+4)+4} \right)$$

$$\mathcal{L}^{-1} \left[ \left( \frac{s}{s^2-4} \right) + (-1) \left( \frac{1}{s} \right) \right]$$

$$x(t) = \mathcal{L}^{-1} \left[ \frac{s+3}{(s+2)^2+4} \right] = e^{-3t} (\cos 2t + ) \quad | \quad x(t) = \underbrace{(\cosh 2t) + (-1)}$$

$$6.) \mathcal{F}(s) = \frac{6s^2-s-6}{s^3-s^2-6s}$$

Answer

$$\frac{6s^2-s-6}{s(s^2-s-6)} = \frac{A}{s} + \frac{B}{(s-3)} + \frac{C}{(s+2)}$$

$$(s-3)(s+2)$$

$$6s^2-s-6 = A(s-3)(s+2) + B(s)(s+2) + C(s)(s-3)$$

$$s=3$$

$$s=-2$$

$$s=0$$

$$\frac{45}{15} = 15b$$

$$\frac{20}{10} = 10C$$

$$\frac{-6}{-6} = -1A$$

$$7.) \mathcal{F}(s) = \frac{4}{s^3-4s}$$

$$\frac{4}{s(s^2-4)} = \frac{As+b}{s^2-4} + \frac{C}{s}$$

$$4 = (As+b)s + C(s^2-4)$$

$$4 = As^2 + bs + C s^2 - 4C$$

$$4 = s^2(A+C) + bs - 4C$$

$$0 = A+C \quad A=1$$

$$4 = -4C \quad C=-1$$

$$0 = B$$

$$\mathcal{L}^{-1} \left[ 1 \left( \frac{1}{s} \right) + 3 \left( \frac{1}{s-3} \right) + 2 \left( \frac{1}{s+2} \right) \right]$$

$$x(t) = \underline{\underline{(1) + 3(e^{3t}) + 2(e^{-2t})}}$$

Answer

In problems 1-3 find the inverse Laplace Transform of the given function.  
Use partial fractions when appropriate.

$$8.) F(s) = \frac{1}{s^4 - 2s^2 + 1}$$

$$U = s^2, \frac{1}{U^2 - 2U + 1}$$

$$= \frac{1}{(U-1)(U-1)}$$

$$= \frac{1}{(s^2-1)(s^2-1)}$$

$$= \frac{1}{(s+1)(s-1)(s+1)(s-1)}$$

$$= \frac{1}{(s+1)^2(s-1)^2}$$

$$= \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s-1)} + \frac{D}{(s-1)^2}$$

$$1 = A(s+1)(s-1)^2 + B(s-1)^2 + C(s-1)(s+1)^2 + D(s+1)^2$$

$$\begin{matrix} s^2 + 2s + 1 \\ (s-1)(s-1) \end{matrix}$$

$$s^2 - 2s + 1$$

$$s = 1,$$

$$\frac{1}{4} = \frac{4}{4}$$

$$D = \frac{1}{4}$$

$$\begin{aligned} s = -1, \quad & 1 = 1s^3 - As - As^2 + A + s^2B - 2Bs + B + s^3C - Cs \\ & + s^2C - C + s^2D + 2Ds + D \end{aligned}$$

$$\mathcal{L}^{-1} \left[ \frac{1}{4} \left( \frac{1}{s+1} \right) + \frac{1}{4} \left( \frac{1}{(s+1)^2} \right) - \frac{1}{4} \left( \frac{1}{s-1} \right) + \frac{1}{4} \left( \frac{1}{(s-1)^2} \right) \right]$$

$$x(t) = \underbrace{\frac{1}{4}(e^{-t})}_{\text{---}} + \underbrace{\frac{1}{4}(e^{-t}t)}_{\text{---}} - \underbrace{\frac{1}{4}(e^{t})}_{\text{---}} + \underbrace{\frac{1}{4}(e^{t}t)}_{\text{---}} - \frac{3}{4} = -A$$

Answer

$$\therefore 0 = A + C$$

$$\therefore 0 = -A + B + C + D$$

$$A = \frac{1}{4}$$

$$-A = C$$

$$\therefore 0 = -A + (\frac{1}{4}) \quad -\frac{1}{4} = -A + C$$

$$-\frac{1}{4} = 2C$$

$$-\frac{1}{4} = C$$

In problems 9-11 solve the initial value problems with Laplace transforms.

$$9.) x'' + 2x' + 6x = 6, \quad x(0) = x'(0) = 0$$

$$x = f(t)$$

$$\mathcal{L}[f(t)] = f(s)$$

$$\mathcal{L}[f'(t)] = s f(s) - f(0)$$

$$\mathcal{L}[f''(t)] = s^2 f(s) - s f(0) - f'(0)$$

$$[s^2 f(s) - s f(0) - f'(0)] + 2[s f(s) - f(0)] + 6[f(s)] = \mathcal{L}[6]$$

$$s^2 f(s) - s f(0) - f'(0) + 2[s f(s) - 0] + 6[f(s)] = \frac{6}{s}$$

$$s^2 f(s) + 2s f(s) + 6 f(s) = \frac{6}{s}$$

$$\frac{(s^2 + 2s + 6)f(s)}{s^2 + 2s + 6} = \frac{6}{s}$$

$$= \frac{6}{s(s^2 + 2s + 6)} = \frac{A}{s} + \frac{bs + c}{(s^2 + 2s + 6)}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s} + \frac{-s-2}{s^2+2s+6}\right]$$

$$\mathcal{L}^{-1}\left[\frac{1}{s} + \frac{-s-2}{(s+1)^2+5}\right]$$

$$\mathcal{L}^{-1}\left[\frac{1}{s} + \frac{-K(s+1)+1}{(s+1)^2+5}\right]$$

$$\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s+1}{(s+1)^2+5} - \frac{1}{(s+1)^2+5}\right]$$

$$0 = A + b$$

$$\begin{array}{|c|} \hline s=0 \\ \hline A=1 \\ \hline \end{array}$$

$$0 = 2A + c$$

$$\begin{array}{|c|} \hline B=-1 \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline C=-2 \\ \hline \end{array}$$

$$6 = As^2 + 2As + 6A + bs^2 + cs$$

$$6 = A(s^2 + 2s + 6) + bs + c(s)$$

In problems 9-11 solve the initial value problems with Laplace transforms.

10.)  $y'' + 4y' + 8y = e^{-t}$ ,  $y(0) = y'(0) = 0$   
 $y = f(t)$

$$\mathcal{L}[f(t)] = f(s)$$

$$\mathcal{L}[f'(t)] = sf(s) - f(0)$$

$$\mathcal{L}[f''(t)] = s^2 f(s) - sf(0) - f'(0)$$

$$[s^2 f(s) - sf(0) - f'(0)] + 4[sf(s) - f(0)] + 8[f(s)] = \mathcal{L}[e^{-t}]$$

$$s^2 f(s) - 0 - 0 + 4sf(s) - 0 + 8f(s) = \frac{1}{s+1}$$

$$s^2 f(s) + 4sf(s) + 8f(s) = \frac{1}{s+1}$$

$$f(s)[s^2 + 4s + 8] = \frac{1}{s+1}$$

$$f(s) = \frac{1}{(s+1)(s^2 + 4s + 8)} = \frac{1}{(s+1)} + \frac{bs+c}{(s^2 + 4s + 8)}$$

$$0 = A + B$$

$$0 = 4A + B + C$$

$$\begin{matrix} 1 \\ s = -1 \end{matrix}$$

$$\begin{matrix} \frac{1}{3} = \frac{3A}{3} \\ A = \frac{1}{3} \end{matrix}$$

$$\boxed{-\frac{1}{3} = B}$$

$$\frac{4}{3} - \frac{1}{3} + C$$

$$\boxed{-1 = C}$$

$$0 = \frac{2}{3} + C$$

$$\mathcal{L}^{-1} \left[ \frac{1}{3} \left( \frac{1}{s+1} \right) + \left( \frac{-\frac{1}{3}s - 1}{s^2 + 4s + 8} \right) \right]$$

$$\begin{aligned} \frac{-\frac{1}{3}(s+3)}{(s+2)^2 + 4} &= -\frac{1}{3}(s+2+1) \\ &\downarrow \\ &- \frac{1}{3}(s+2) - \frac{1}{3} \end{aligned}$$

$$\mathcal{L}^{-1} \left[ \frac{1}{3} \left( \frac{1}{s+1} \right) + \frac{\frac{1}{3}(s+2) + \frac{1}{3}}{(s+2)^2 + 4} \right]$$

$$\mathcal{L}^{-1} \left[ \frac{1}{3} \left( \frac{1}{s+1} \right) - \frac{\frac{1}{3}(s+2)+1}{3((s+2)^2+4)} \right]$$

$$\mathcal{L}^{-1} \left[ \frac{1}{3} \left( \frac{1}{s+1} \right) - \frac{1}{3} \left( \frac{s+2}{(s+2)^2+4} \right) - \frac{1}{6} \left( \frac{2}{(s+2)^2+4} \right) \right]$$

$$1 = A(s^2 + 4s + 8) + Bs + C(s+1) \quad \text{for } t = 0: \frac{1}{3}(e^{-t}) - \frac{1}{3}(e^{-t}(\cos 2t) + \frac{1}{6}e^{-t} \sin 2t)$$

$$1 = As^2 + 4As + 8A + Bs^2 + Bs + Cs + C$$

$\uparrow$   
Answer  
 $\sum$

In problems 9-11 solve the initial value problems with Laplace transforms.

$$11.) y'' + 2y' + 2y = 2(\cos(t) + \sin(t)), y(0) = y'(0) = 0 \quad L^{-1}\left[\frac{\frac{2}{5}s + \frac{4}{5}}{(s^2+1)} + \frac{-\frac{2}{5}s - \frac{6}{5}}{(s+1)^2+1}\right] \xrightarrow{\text{continued}} \rightarrow$$

$$Y = f(t)$$

$$\mathcal{L}[f(t)] = f(s)$$

$$\mathcal{L}[f'(t)] = sf(s) - f(0)$$

$$\mathcal{L}[f''(t)] = s^2 f(s) - sf(0) - f'(0)$$

$$[s^2 f(s) - sf(0) - f'(0)] + 2[sf(s) - f(0)] + 2[f(s)] = \mathcal{L}[2(\cos(t) + \sin(t))]$$

$$s^2 f(s) - 0 - 0 + 2sf(s) + 2f(s) = 2\left(\frac{s}{s^2+1}\right) + \left(\frac{1}{s^2+1}\right)$$

$$f(s)(s^2 + 2s + 2) = \frac{2s}{s^2+1} + \frac{1}{s^2+1}$$

$$f(s) = \frac{2s}{(s^2+1)(s^2+2s+2)} + \frac{1}{(s^2+1)(s^2+2s+2)}$$

$$\frac{2s}{(s^2+1)(s^2+2s+2)} = \frac{As+B}{(s^2+1)} + \frac{(s+D)}{(s^2+2s+2)}$$

$$2s = As + b(s^2 + 2s + 2) + (s + D)(s^2 + 1)$$

$$2s = As^3 + 2As^2 + 2As + Bs^2 + B + s^3 + Cs + Ds^2 + D$$

$$L^{-1}\left[\frac{\frac{2}{5}s + \frac{4}{5}}{(s^2+1)} + \frac{-\frac{2}{5}s - \frac{6}{5}}{(s+1)^2+1}\right] \xrightarrow{\text{continued}} \frac{-\frac{1}{5}(s+1) - \frac{6}{5}}{(s+1)^2+1}$$

$$L^{-1}\left[\frac{\frac{2}{5}s}{s^2+1} + \frac{4}{5}\left(\frac{1}{s^2+1}\right) - \frac{2}{5}\left(\frac{s+1}{(s+1)^2+1}\right) - \frac{6}{5}\left(\frac{1}{(s+1)^2+1}\right)\right] \\ = \left\{ \frac{2}{5}(\cos t) + \frac{4}{5}(\sin t) - \frac{2}{5}(e^{-t}\cos t) - \frac{6}{5}(e^{-t}\sin t) \right\}$$

(Half the answer)

$$\frac{2}{5} = A$$

$$\frac{4}{5} = 2A$$

$$-\frac{8}{5} = D$$

$$2 = 2A + 2B - A$$

$$0 = 2A + B - 2B$$

$$C = -\frac{2}{5}$$

$$B = \frac{4}{5}$$

$$\therefore 0 = 2B + D$$

$$2 = A + 2B$$

$$S: 2 = 2A + 2B + C$$

$$0 = 2A - B$$

$$S^2: 0 = A + C$$

$$2 - 2B = A$$

$$S^3: 0 = 2A + B + D$$

$$0 = 2(2 - 2B) - B$$

$$-\frac{9}{5} = -\frac{5B}{5}$$

$$0 = 4 - 7B - B$$

In problems 9-11 solve the initial value problems with Laplace transforms.

11.)  $y'' + 2y' + 2y = 2(\cos(t) + \sin(t)), y(0) = y'(0) = 0$  Now we're Answering the rest.

$$y = f(t)$$

$$\mathcal{L}[f(t)] = f(s)$$

$$\mathcal{L}[f'(t)] = sf(s) - f(0)$$

$$\mathcal{L}[f''(t)] = s^2 f(s) - sf(0) - f'(0)$$

$$[s^2 f(s) - sf(0) - f'(0)] + 2[sf(s) - f(0)] + 2[f(s)] = \mathcal{L}[2(\cos(t) + \sin(t))]$$

$$s^2 f(s) - 0 - 0 + 2sf(s) + 2f(s) = 2\left(\frac{s}{s^2+1}\right) + \left(\frac{1}{s^2+1}\right)$$

$$f(s)(s^2 + 2s + 2) = \frac{2s}{s^2+1} + \frac{1}{s^2+1}$$

Full  
Answer

$$f(s) = \frac{2s}{(s^2+1)(s^2+2s+2)} + \frac{1}{(s^2+1)(s^2+2s+2)}$$

$$1 = \frac{As+B}{(s^2+1)} + \frac{(s+1)}{(s^2+2s+2)}$$

$$1 = As^3 + 2As^2 + 2As + Bs^2 + 2Bs + 2 + (s^3 + s^2 + s + 1)$$

$$\mathcal{L}^{-1}\left[\frac{-\frac{2}{5}s + \frac{1}{5}}{(s^2+1)} + \frac{\frac{2}{5}s + \frac{3}{5}}{(s+1)^2+1}\right]$$

$$\mathcal{L}^{-1}\left[-\frac{2}{5}\left(\frac{s}{s^2+1}\right) + \frac{1}{5}\left(\frac{1}{s^2+1}\right) + \frac{\frac{2}{5}(s+1) + \frac{1}{5}}{(s+1)^2+1}\right] \leftarrow \text{all about factoring and extracting 1}$$

$$\mathcal{L}^{-1}\left[-\frac{2}{5}\left(\frac{s}{s^2+1}\right) + \frac{1}{5}\left(\frac{1}{s^2+1}\right) + \frac{\frac{2}{5}\left(\frac{s+1}{(s+1)^2+1}\right) + \frac{1}{5}\left(\frac{1}{(s+1)^2+1}\right)}{(s+1)^2+1}\right]$$

$$\text{Second part of the answer. } \left\{ \frac{-2}{5}(\cos t) + \frac{1}{5}(\sin t) + \frac{1}{5}(e^t \cos t) + \frac{1}{5}(e^t \sin t)\right\}$$

$$\boxed{\frac{3}{5} = 1}$$

$$0 = 2A + B + 1$$

$$0 = 2A + B + (1 - 2B)$$

$$0 = 2A + 2B - A$$

$$\boxed{C = \frac{2}{5}}$$

$$\boxed{A = -\frac{1}{5}}$$

$$\therefore 0 = 2A + B + 1$$

$$\therefore 1 = 2B + 1$$

$$\therefore 0 = 2A + 2B + C$$

$$\therefore 0 = A + C$$

$$-1 = 2A - B$$

$$0 = A + 2B$$

$$-2B = A$$

$$-1 = 2(-2B) - B$$

$$\boxed{B = \frac{1}{5}}$$

{Full Answer for  
Question (11)}

$$x(t) = \left[ \frac{1}{5}(\cos t) + \frac{4}{5}(\sin t) - \frac{1}{5}(e^t \cos t) - \frac{6}{5}(e^t \sin t) - \frac{1}{5}(\cos t) + \frac{1}{5}(\sin t) + \frac{1}{5}(e^t \cos t) + \frac{1}{5}(e^t \sin t) \right]$$

{ Simplified  
version }

$$x(t) = \underbrace{\sin t}_{\text{---}} - (e^t \sin t)$$

## The Convolution Theorem/Derivatives & Integrals of Transforms- HW Problems

In problems 1-4 calculate  $f(t) * g(t)$ .

1.  $f(t) = 1, \quad g(t) = e^t$
2.  $f(t) = t, \quad g(t) = \sin(t)$
3.  $f(t) = t^2, \quad g(t) = e^t$
4.  $f(t) = t, \quad g(t) = t$

5a. Show that if  $f(t) = e^{at}$  and  $g(t) = e^{bt}$ , where  $a, b$  are constants, then

$$(f * g)(t) = \frac{1}{a-b} (e^{at} - e^{bt}).$$

b. By direct calculation of Laplace transforms, show that

$$\mathcal{L}((f * g)(t)) = (\mathcal{L}(f(t)))(\mathcal{L}(g(t)))$$

for  $f(t) = e^{at}$  and  $g(t) = e^{bt}$  (where  $\mathcal{L}$  is the Laplace transform).

In problems 6-8 use the convolution theorem to find the inverse Laplace transform of the given function.

6.  $F(s) = \frac{3}{s^2-1}$
7.  $F(s) = \frac{1}{(s-1)^2}$
8.  $F(s) = \frac{4}{s(s^2+4)}$

In problems 9-11 find the Laplace transform of the given function.

$$9. \quad f(t) = t \cos(3t)$$

$$10. \quad f(t) = t^2 \sin(3t)$$

$$11. \quad f(t) = \frac{e^{t-1}}{t}$$

In problems 12-14 find the inverse Laplace transform of the given function.

$$12. \quad F(s) = \ln\left(\frac{s+3}{s-3}\right)$$

$$13. \quad F(s) = \ln\left(\frac{s^2+4}{s+4}\right)$$

$$14. \quad F(s) = \frac{2s}{(s^2+1)^2}$$

15. Use the Laplace transform to transform  $ty'' + (t - 2)y' + y = 0$  to find a solution where  $y(0) = 0$  but  $y(t) \not\equiv 0$ .

In problems 1-4 calculate  $f(t) * g(t)$ .

1.)  $f(t) = 1 \quad g(t) = e^t$

$$= \int_0^t f(w)g(t-w)dw$$

↑

Assuming  
they're two  
piecewise functions

$$\int_0^t e^{t-w} dw$$

$$= \int_0^t e^u dw$$

$$u = t-w$$

$$\frac{du}{-1} = -1 dw$$

$$= - \int_0^t e^u du$$

$$= -(e^{t-w}) \Big|_0^t$$

$$= -e^{t-t} - (-e^{t-0})$$

$$f(t) = -1 + e^t \quad \text{Answer}$$

2.)  $f(t) = t, g(t) = \sin(t)$

$$\int_0^t \frac{w}{u} \frac{\sin(t-w)}{dv} dw$$

$$dv = dw \quad v = -\underline{\cos(t-w)}$$

$$v = \cos(t-w)$$

$$w(\cos(t-w)) \Big|_0^t - \int_0^t \cos(t-w) dw \quad \text{Answer}$$

$$+ (\cos(0)) - 0 - [0 - \sin(t)] = f(t) = \boxed{t - \sin t}$$

In problems 1-4 calculate  $f(t) * g(t)$ .

3.)  $f(t) = t^2, g(t) = e^t$

$$\int_0^t w^2 e^{t-w} dw$$

$$e^t \int \frac{w^2}{u} \frac{e^{-w}}{dv} dw \quad v = -e^{-w}$$
$$du = 2w dw$$

$$-w^2 e^{-w} \Big|_0^t - \int_0^t -e^{-w} 2w dw$$

$$\underline{-t^2 e^t} + 2 \int_0^t \frac{e^{-w}}{u} \frac{w dw}{dv} \quad v = -e^{-w}$$
$$du = dw$$

$$-w e^{-w} \Big|_0^t - \int_0^t -e^{-w} dw$$

$$-w e^{-w} \Big|_0^t + \int_0^t e^{-w} dw$$

↑

$$\underline{-2t e^t} \quad 2(-e^{-w}) \Big|_0^t$$

$$\underline{-2e^t + 2}$$

4.)  $f(t) = t, g(t) = t$

$$\int_0^t w(t-w) dw$$

$$\int_0^t w^2 - w^2 dw$$

$$+ \int_0^t w dw - \int_0^t w^2 dw$$

$$+ \left( \frac{w^2}{2} \right) \Big|_0^t - \frac{w^3}{3} \Big|_0^t$$

$$f(t) = \underline{\underline{+ \left( \frac{t^2}{2} \right) - \frac{t^3}{3}}} \leftarrow \text{Answer}$$

Answer:  $e^t \left[ -t^2 e^t - 2t e^t - 2e^t + 2 \right]$

5a) Show that if  $f(t) = e^{at}$  &  $g(t) = e^{bt}$ , where  $a, b$  are constants,  
then  $(f \cdot g)(t) = \frac{1}{a-b}(e^{at} - e^{bt})$ .

$$\int_0^t e^{aw} e^{bw(t-w)} dw$$

$$\int_0^t e^{aw} e^{bt-bw} dw$$

$$e^{bt} \int_0^t e^{aw-bw} dw$$

$$e^{bt} \int_0^t e^{aw-bw} dw$$

$$e^{bt} \left[ \frac{1}{a-b} (e^{at-bt} - e^0) \right]$$

Exp 1:

$$\mathcal{L}(e^{at}) \mathcal{L}(e^{bt}) = \boxed{\left( \frac{1}{s-a} \right) \left( \frac{1}{s-b} \right)}$$

Exp 2:

$$\mathcal{L}((f \cdot g)(t)) = \frac{1}{a-b} \mathcal{L}[e^{at} - e^{bt}]$$

$$\uparrow$$

$$\frac{1}{a-b} (e^{at} - e^{bt}) \quad \boxed{\frac{1}{a-b} \left( \frac{1}{s-a} - \frac{1}{s-b} \right)}$$

Let's prove their Equal by Simplifying Both Expressions

$$\text{Exp 1: } \left( \frac{1}{s-a} \right) \left( \frac{1}{s-b} \right) = \frac{s-b-s+a}{(s-a)(s-b)} = \frac{-b+a}{a-b} = \frac{1}{(s-a)(s-b)}$$

Exp 1:   $\left( \frac{1}{s-a} \right) \left( \frac{1}{s-b} \right) = \frac{1}{(s-a)(s-b)}$

Therefore, both Exp 1 & L are = .

5b) By direct calculation of Laplace transforms show that  
 $\mathcal{L}((f \cdot g)(t)) = (\mathcal{L}(f(t)) \mathcal{L}(g(t)))$   
 for  $f(t) = e^{at}$  &  $g(t) = e^{bt}$  (where L is the Laplace transform).

In problems 6-8 use the convolution theorem to find the inverse Laplace transform of the given function.

$$6.) F(s) = \frac{3}{s^2 - 1}$$

$$= \frac{3}{(s+1)(s-1)}$$

$$3L^{-1}\left[\frac{1}{s+1}\right] * L^{-1}\left[\frac{1}{s-1}\right]$$

$$3e^{-t} * e^t$$

$$\begin{matrix} \uparrow & \uparrow \\ f(t) & g(t) \end{matrix}$$

$$3 \int_0^t e^w e^{t-w} dw$$

$$3 \int_0^t e^{t-w} dw$$

$$-\frac{3}{2}(e^{t-2w}) \Big|_0^t$$

$$f(t) = -\frac{3}{2}e^{-t} + \frac{3}{2}e^t$$

$$7.) F(s) = \frac{1}{(s-1)^2}$$

$$L^{-1}\left[\frac{1}{s-1}\right] * L^{-1}\left[\frac{1}{s-1}\right]$$

$$\begin{matrix} e^{it} & * & e^{it} \\ \uparrow & & \uparrow \\ f(t) & & g(t) \end{matrix}$$

$$\int_0^t e^w e^{t-w} dw$$

$$e^t \int_0^t 1 dw$$

$$e^t [w] \Big|_0^t$$

$$f(t) = \underline{e^t[t]}$$

$$8.) F(s) = \frac{4}{s(s^2+4)}$$

$$L^{-1}\left[\frac{2}{s}\right] * L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$\begin{matrix} 2 & * & \sin 2t \\ \uparrow & & \uparrow \\ f(t) & & g(t) \end{matrix}$$

$$\int_0^t 2 \sin 2(t-w) dw$$

$$2 \int_0^t \sin 2t - 2w dw$$

$$\cos 2t - 2w \Big|_0^t$$

$$f(t) = \underline{\underline{(1 - \cos 2t)}}$$

In problems 9-11 find the Laplace transform of the given function.

9.)  $f(t) = t \cos(3t)$

$$-\frac{d}{ds} \left( \frac{s}{s^2 + 9} \right)$$

$$= -\frac{(s^2 + 9) \cdot 1 - s(2s)}{(s^2 + 9)^2}$$

$$= -\frac{(s^2 + 9)}{(s^2 + 9)^2}$$

$$f(s) = \boxed{\frac{s^2 - 9}{(s^2 + 9)^2}}$$

10.)  $f(t) = t^2 \sin(3t)$

$$(-1)^t \cdot F^{(2)}(s)$$

$$F(s) = \frac{3}{(s^2 + 9)}$$

$$F^{(1)}(s) = \frac{(0)(s^2 + 9) - (3)(2s)}{(s^2 + 9)^2}$$

$$F^{(1)}(s) = \frac{-6s}{(s^2 + 9)^2}$$

$$F^{(2)}(s) = \frac{(-6(s^2 + 9)^2) - (-6s)(2(s^2 + 9) \cdot 2s)}{(s^2 + 9)^4}$$

↓

$$f(s) = \boxed{\frac{-6(s^2 + 9)^2 + 12s(s^2 + 9) \cdot 2s}{(s^2 + 9)^4}}$$

In problems 9-11 find the Laplace transform of the given function.

11.)  $f(t) = \frac{e^t - 1}{t}$

use  $\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty f(u)du$

$$\mathcal{L}[e^t] - \mathcal{L}[1]$$

$$\left[ \frac{1}{s-1} - \frac{1}{s} \right] = \frac{s}{(s-1)s} - \frac{(s-1)}{s(s-1)} = \frac{s-s+1}{s(s-1)} = \frac{1}{s(s-1)}$$

$$\int_s^\infty \frac{1}{u(u-1)} du \rightarrow = \frac{A}{u} + \frac{B}{(u-1)}$$

$$\int_s^\infty \frac{1}{u} + \frac{1}{(u-1)} du \quad 1 = A(u-1) + B(u)$$

$$1 = Au - A + Bu$$

$$-\ln(u) + \ln(u-1) \Big|_s^\infty \quad 1 = -A \quad \boxed{\begin{array}{l} B=1 \\ A=-1 \end{array}}$$

$$0 - (-\ln(s) + \ln(s-1)) \quad 0 = A + B \quad \boxed{A=-1} \\ \downarrow \quad -A = B$$

Answer:  $f(s) = \ln(s) - \ln(s-1)$

In problems 12-14 find the inverse Laplace transform function.

$$\mathcal{L}^{-1}[F(s)] = -\frac{1}{t} \mathcal{L}'[F'(s)]$$

12.)  $F(s) = \ln\left(\frac{s+3}{s-3}\right)$

$$\frac{d}{ds}(\ln(s+3)) - \frac{d}{ds}(\ln(s-3))$$

$$-\frac{1}{t} \mathcal{L}'\left(\frac{1}{s+3}\right) + \frac{1}{t} \mathcal{L}'\left(\frac{1}{s-3}\right)$$

$$x(t) = -\frac{1}{t} [e^{-3t} - e^{3t}]$$

13.)  $F(s) = \ln\left(\frac{s^2+4}{s+4}\right)$

$$\frac{d}{ds}(\ln(s^2+4)) - \frac{d}{ds}(\ln(s+4))$$

$$-\frac{1}{t} \mathcal{L}'\left(\frac{2s}{s^2+4}\right) + \frac{1}{t} \mathcal{L}'\left(\frac{1}{s+4}\right)$$

$$x(t) = -\frac{1}{t} [\cos 2t + e^{-4t}]$$

In questions 12-14 find the inverse laplace transform of the given functions.

$$14.) \mathcal{F}(s) = \frac{2s}{(s^2+1)^2}$$

$$\mathcal{L}^{-1}\left(\frac{2}{s^2+1}\right) = \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right)$$

$$2\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right)$$

$$2\sin(t) + \cos(t)$$

$$2\int_0^+ \sin(w)\cos(t-w)dw \quad \text{use trig formula}$$

$$2\int_0^+ \sin(w)[\cos(t-\cos w + \sin t + \sin w)]dw$$

$$2\int_0^+ \sin(w)\cos(t)(\cos w)dw + 2\int_0^+ \sin^2 w \sin(t)dw$$

$$2\cos(t)\int_0^+ \sin(w)\cos(w)dw + 2\sin(t)\int_0^+ \sin^2 w dw \quad \text{use another trig identity (half angle identity)}$$

$$+ 2\sin(t)\int_0^+ 1 - \frac{\cos 2w}{2} dw$$

$$2\cos(t)\int_0^+ \sin(w)\cos(w)dw + 2\sin(t)\int_0^+ \frac{1}{2} dw - 2\sin(t)\int_0^+ \cos 2w dw = -\sin(t)\sin(2t)$$

$$2(\cos(t)) \int_0^+ \sin(w)\cos(w)dw \quad u = \sin w$$

$$2(\cos(t)) \int_0^+ \frac{u \cos(u) du}{\cos(u)} \quad \frac{du}{\cos u} = dw$$

$$2(\cos(t)) \int_0^+ u du$$

$$\frac{u^2}{2}$$

$$2(\cos(t)) \left( \frac{\sin^2 w}{2} \right) \Big|_0^+ = \underline{\cos(t) \sin^2(t)}$$

$$2\sin(t) \int_0^+ \frac{1}{2} dw$$

$$\sin(t)[w] \Big|_0^+ = \underline{\sin(t)(t)}$$

$$-2\sin(t) \int_0^+ \cos 2w dw$$

$$-2\sin(t) \left[ \frac{\sin 2w}{2} \right] \Big|_0^+$$

$$-\sin(t) [\sin 2w] \Big|_0^+$$

$$= -\underline{\sin(t) \sin(2t)}$$

Answer:

$$X(t) = \cos(t) \sin^2(t) + \sin(t)(t) - \sin(t) \sin(2t)$$

15.) Use the Laplace transform to transform  $+y'' + (-2)y' + y = 0$  to find a solution where  $y(0) = 0$  but  $y(t) \neq 0$ .  $\underline{Y=f(t)}$  {We want to find our non-trivial solution}

$$\mathcal{L}[+y'' + (-2)y' + y] = \mathcal{L}[0]$$

$$\mathcal{L}[+y''] + \mathcal{L}[+y'] - 2\mathcal{L}[y'] + \mathcal{L}[y] = 0$$

$$\mathcal{L}[y''] = s^2 f(s) - f(0) - f'(0) = s^2 f(s) - f'(0)$$

$$\mathcal{L}[+y'] = -(2sf(s) + s^2 f'(s)) \leftarrow \text{product rule}$$

Apply  $\rightarrow$

$$\mathcal{L}[+^n f(t)] = -i^n F^{(n)}(s)$$

$$\mathcal{L}[y'] = s f(s) - f(0)$$

$$\mathcal{L}[+y] = -(f(s) + s f'(s))$$

$$[-2sf(s) - s^2 f'(s)] + [-f(s) - sf'(s)] - 2[sf(s) - 0] + f(s) = 0$$

$$f(s)[-2s - 1 - 2s + 1] + f'(s)[-s^2 - s] = 0$$

$$f(s)[-4s] + f'(s)[-s^2 - s] = 0$$

$$\left(\frac{1}{s^2 - s}\right) \frac{f(s)[-4s]}{f(s)} = \frac{f'(s)[s^2 - s]}{f(s)} \left(\frac{1}{s^2 - s}\right)$$

$$-\frac{4s}{s^2 - s} = \frac{f'(s)}{f(s)}$$

$$\frac{-4s}{s(s-1)} = \frac{-4}{s-1}$$

$$\int \frac{f'(s)}{f(s)} ds = -4 \int \frac{1}{s-1} ds$$

$$\ln|f(s)| = -4 \ln|s-1| + C$$

$$f(s) = e^{-4 \ln|s-1| + C}$$

$$e^{-4 \ln|s-1| + C} = |s-1|^{-4} + C$$

$$\text{using } \frac{n!}{(s-a)^{n+1}} = \frac{a^n}{e^a} t^n \left[ \frac{t^n}{(s-a)^n} \right]$$

$$X(t) = t^3 e^t$$

Answer

## Piecewise Continuous Functions- HW Problems

In problems 1-4 find the inverse Laplace transform of the given function.

$$1. \quad F(s) = s^{-4}e^{-s}$$

$$2. \quad F(s) = \frac{e^{-3s}}{s-3}$$

$$3. \quad F(s) = \frac{e^{-2s}}{s^2+4}$$

$$4. \quad F(s) = \frac{se^{-s}}{s^2+9}$$

In problems 5-7 find the Laplace transform of the given function.

$$5. \quad f(t) = \begin{cases} 3 & 0 \leq t \leq 4 \\ 0 & t > 4 \end{cases}$$

$$6. \quad f(t) = \begin{cases} t & 0 \leq t < 2 \\ 0 & t \geq 2 \end{cases}$$

$$7. \quad f(t) = \begin{cases} \sin(2t) & 0 \leq t \leq 2\pi \\ 0 & t > 2\pi. \end{cases}$$

In problems 8-10 solve the initial value problem.

8.  $x'' + 9x = f(t)$ , where

$$f(t) = 1 \quad 0 \leq t < 2$$

$$= 0, \quad t \geq 2$$

and  $x(0) = x'(0) = 0$ .



Must be a typo since there are  
only 8-9 questions

9.  $x'' + 4x = f(t)$ , where

$$f(t) = \sin(t) \quad 0 \leq t \leq 2\pi$$

$$= 0 \quad t > 2\pi$$

and  $x(0) = x'(0) = 0$ .

.

In Problems 1-4 find the Inverse Laplace transform of the given function.

1.)  $F(s) = s^{-4} e^{-s}$

{ Rule:  $\mathcal{L}^{-1}(e^{-as} G(s)) = u(t-a) g(t-a)$  }

3.)

$$F(s) = \frac{e^{-2s}}{s^2 + 4}$$

$$\mathcal{L}^{-1}(F(s)) = u(t-4) \frac{1}{s} (t-4)^3$$

$$\mathcal{L}^{-1}(F(s)) = u(t-3) e^{3(t-3)}$$

$$g(s) = \frac{1}{s-3}$$

$x(t) = u(t-3) e^{3t-9}$   
↑  
ANSWER

$$4.) F(s) = \frac{s e^{-s}}{s^2 + 9}$$

$$\mathcal{L}^{-1}(F(s)) = u(t-1) (os 3(t-1))$$

$$g(s) = \frac{s}{s^2 + 9} = (os 3t)$$

Answer:  $u(t-1) (os(3t-3))$

$$\frac{1}{6} \mathcal{L}^{-1}\left(\frac{6}{s^4}\right) = \frac{1}{6} (t^3)$$

$$x(t) = u(t-1) \frac{(t-1)^3}{6}$$

↑  
ANSWER

$$\mathcal{L}^{-1}(F(s)) = u(t-2) \frac{1}{2} \sin 2(t-2)$$

$$g(s) = \frac{1}{s^2 + 4}$$

$$\frac{1}{2} \mathcal{L}^{-1}\left(\frac{2}{s^2 + 4}\right) = \frac{1}{2} (\sin 2t)$$

$$x(t) = u(t-2) \frac{\sin(2t-4)}{2}$$

↑  
ANSWER

In problems 5-7 find the Laplace transform of the given function.

$$5.) \left\{ \begin{array}{ll} f(t) = 3 & 0 \leq t \leq 4 \\ & \\ & = 0 \quad t > 4 \end{array} \right\}$$

$$f(t) = 3[u(t) - u(t-4)] + 0[u(t-4)]$$

$$f(t) = 3u(t) - 3u(t-4)$$

$$\begin{aligned} L[f(t)] &= \frac{3}{s} - 3e^{-4s}L[1] \\ &= \frac{3}{s} - e^{-4s}\left[\frac{1}{s}\right] \end{aligned}$$

$$6.) \left\{ \begin{array}{ll} f(t) = t & 0 \leq t \leq 2\pi \\ & \\ & = 0 \quad t > 2\pi \end{array} \right\}$$

$$f(t) = t[u(t) - u(t-2\pi)] + 0[u(t-2\pi)]$$

$$f(t) = tu(t) - t u(t-2\pi)$$

$$\begin{aligned} L[f(t)] &= e^{-s}L[t] - e^{-s}L[t-2\pi] \\ &= e^{-s} \cdot \frac{1}{s^2} - e^{-s} \left[ \frac{1}{s^2} - \frac{2\pi}{s} \right] \end{aligned}$$

$$7.) \left\{ \begin{array}{ll} f(t) = \sin(2t) & 0 \leq t \leq 2\pi \\ & \\ & = 0 \quad t > 2\pi \end{array} \right\} = f(t) = \sin(2t)[u(t) - u(t-2\pi)] + 0[u(t-2\pi)]$$
$$f(t) = \sin(2t)u(t) - \sin(2t)u(t-2\pi)$$

$$L[f(t)] = e^0 \cdot \frac{2}{s^2+4} - e^{-2\pi s} \cdot \frac{2}{s^2+4}$$

In Problems 8-10 Solve the initial Value Problems

<sup>↑</sup> must be a typo because there is only 8-9 problems

8.)  $x'' + 9x = f(t)$ , where  $\begin{cases} x = f(t) \\ f(t) = 1 & 0 \leq t < 2 \\ = 0 & t \geq 2 \end{cases}$

and  $x(0) = x'(0) = 0$

$$\mathcal{L}[f(t)] = \mathcal{L}[f''(t)] = \frac{s^2 f(s) - s f(0) - f'(0)}{s^2 + 9}$$

$$[s^2 f(s) - s f(0) - f'(0)] + 9 [f(s)] = \mathcal{L}[f(t)]$$

$$\mathcal{L}[f(t)] = \mathcal{L}[(1)u(t) - u(t-2) + 0(u(t-2))]$$

$$\mathcal{L}[u(t) - u(t-2)]$$

$$= \frac{1}{s} - e^{-2s} \frac{1}{s}$$

$$s^2 f(s) + 9f(s) = \frac{1}{s} - \frac{e^{-2s}}{s}$$

$$s^2 + 9(f(s)) = \frac{1}{s} - \frac{e^{-2s}}{s}$$

$$\mathcal{L}^{-1}\left[f(s) = \frac{1}{s(s^2+9)} - \frac{e^{-2s}}{s(s^2+9)}\right]$$

↓ continued

9.)  $sx'' + 4x = f(t)$ , where  $\begin{cases} f(t) = \sin t & 0 \leq t < 2\pi \\ = 0 & t > 2\pi \end{cases}$

and  $x(0) = x'(0) = 0$

$$[s^2 f(s) - s f(0) - f'(0)] + 4 [f(s)] = \mathcal{L}[f(t)]$$

$$\mathcal{L}[f(t)] = \mathcal{L}[\sin t(u(t) - u(t-2\pi)) + 0(u(t-2\pi))]$$

$$= 0 \cdot \frac{1}{s^2+1} - e^{2\pi s} \cdot \frac{1}{s^2+1} = \frac{1}{s^2+1} - \frac{e^{2\pi s}}{s^2+1}$$

$$s^2 f(s) + 4f(s) = \frac{1}{s^2+1} - \frac{e^{2\pi s}}{s^2+1}$$

$$(s^2 + 4)f(s) = \frac{1}{s^2+1} - \frac{e^{2\pi s}}{s^2+1}$$

$$\mathcal{L}^{-1}\left[f(s) = \frac{1}{s^2+1(s^2+4)} - \frac{e^{2\pi s}}{s^2+1(s^2+4)}\right]$$

↓ continued

In Problems 8-10 Solve the initial Value Problems

↑ must be a typo because there is only 8-9 problems

8.) continued

$$\mathcal{L}^{-1}[f(s) = \frac{1}{s(s^2+9)} - \frac{e^{-2s}}{s(s^2+9)}]$$

$$\frac{1}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+D}{s^2+9}$$

$$1 = A(s^2+9) + Bs + D(s)$$

$$1 = s^2A + 9A + Bs^2 + Ds$$

$$\frac{1}{s} = \frac{9A}{s} \quad \boxed{A = \frac{1}{9}}$$

$$0 = A + B \quad \boxed{B = -\frac{1}{9}}$$

$$f(A) = \frac{1}{9} \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{1}{9} \mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right] - \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s(s^2+9)}\right]$$

$$f(t) = \frac{1}{9} [u(t)] - \frac{1}{9} [\cos 3t] - u(t-2) \left[ \frac{1}{9} [u(t-2)] - \frac{1}{9} [\cos 3(t-2)] \right]$$

↑ answer

9.) continued

$$\mathcal{L}^{-1}[f(s) = \frac{1}{s^2+1(s^2+4)} - \frac{e^{2\pi s}}{s^2+1(s^2+4)}]$$

$$\frac{1}{s^2+1(s^2+4)} = \frac{As+B}{s^2+1} + \frac{(s+D)}{s^2+4}$$

$$1 = As + B(s^2+4) + (s+D)(s^2+1)$$

$$1 = As^3 + 4As + Bs^2 + 4B + (s^3 + s + Ds^2 + D)$$

$$\therefore 1 = 4B + D$$

$$\therefore 0 = A + C$$

$$\therefore 0 = B + D$$

$$0 = 9A + C$$

$$\frac{1}{3} = \frac{3B}{3}$$

$$\boxed{B = \frac{1}{3}}$$
  
$$\boxed{D = -\frac{1}{3}}$$

$$f(t) = \frac{1}{3} + \frac{-\frac{1}{3}}{s^2+4} - \mathcal{L}^{-1}\left[\frac{e^{2\pi s}}{s^2+1(s^2+4)}\right]$$

$$f(t) = \frac{1}{3} [sint] - \frac{1}{6} [sin 2t]$$

$$-u(t+2\pi) \left[ \frac{1}{3} [sint+2\pi] - \frac{1}{6} [sin 2(t+2\pi)] \right]$$

Ansver ↑