Series Solutions Near Regular Singular Points: $r_1 - r_2$ is an Integer- HW Problems

In problems 1-4 find all possible Frobenius solutions. If a Frobenius solution does not exist, show why it doesn't.

1. $x^2y'' + xy' + (x^2 - 4)y = 0$
2. $x^2y'' + (6x + x^2)y' + xy = 0$
3. $x^2y'' - xy' + y = 0$
4. $xy'' + (4 - x)y' - y = 0.$

In problems 1-4 find all possible Frobenius solutions. If a Frobenius solution does not exist, show why it doesn't.

1.) $x^2y''+xy'+(x^2-4)y=0$

$$= y''+\frac{xy'}{x^2}+\frac{(x^2-4)}{x^2}y=0$$

$$p(x)=\cancel{x}\left(\frac{1}{\cancel{x}}\right) \qquad q(x)=x^2\left(\cancel{x^2}\left(\frac{(x^2-4)}{\cancel{x^2}}\right)\right)$$

$$p(x)=1 \qquad q(x)=-4$$

$$r(r-1)+r-4$$

$$r^2-r+r-4$$

$$(r^2-4)$$

$$r_1=2 \quad r_2=-2$$

$$r_1-r_2=4$$

$$x^2\sum_{n=0}^{\infty}(n+r-1)(n+r)c_n x^{n+r-2}-x\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}+(x^2-4)\sum_{n=0}^{\infty}c_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r-1)(n+r)c_n x^{n+r}-\sum_{n=0}^{\infty}(n+r)c_n x^{n+r}+\sum_{n=0}^{\infty}c_n x^{n+r+2}$$

$$-\sum_{n=0}^{\infty}4c_n x^{n+r}=0$$

Continued

$$\sum_{n=0}^{\infty}[(n+r-1)(n+r)-(n+r)-4]c_n x^{n+r}+\sum_{n=0}^{\infty}c_n x^{n+r+2}=0$$

$$[(n+r)^2-2(n+r)-4] \qquad \sum_{n=2}^{\infty}c_{n-2}x^{n+r}$$

$n=0,\ c_0=0$ & $n=1\ c_1=0$

$n\geq 2$

$$[(n+r)^2-2(n+r)-4]c_n+c_{n-2}=0$$

~~Start with Smaller root~~ $r=2$

$$c_n=-\frac{1}{(n+2)^2-2(n+2)-4}=-\frac{1}{n^2+2n+2n+4-2n-4-4}$$

$c_1=0$, all odds are zero $\quad =-\dfrac{1}{n^2+2n+4}$

$n=2 \quad c_2=-\frac{1}{12}c_0$

$n=4 \quad c_4=-\frac{1}{28}c_2=\frac{1}{336}c_0$

$n=6 \quad c_6=-\frac{1}{52}c_4=-\frac{1}{17472}c_0$

$$c_{2n}=\frac{(-1)^n c_0}{2^{2n-1}}$$

Since $c_1=0$, all odd c_ns $=0$ in addition to $n=4$ for $r=-2$

$-\dfrac{1}{n^2-6n+4}$

$c_2=-\frac{1}{2}c_0$

$n^2-6n+4(c_0)+c_2=0$,

but: $c_2=-\frac{1}{2}c_0\neq 0$

therefore -2 doesnt satisfy forbenius theorem & doesnt exist

<u>Solution</u>

$$y_1(x)=c_0x^2\left(1+\sum_{n=1}^{\infty}\left(\frac{(-1)^n x^{2n}}{12^{2n-1}}\right)\right)$$

2) $x^2y'' + (6x + x^2)y' + xy = 0$

$$y'' + \frac{(6x + x^2)}{x^2}y' + \frac{xy}{x^2} = 0$$

$$p(x) = x\left(\frac{6x + x^2}{x^2}\right) \quad q(x) = x^2\left(\frac{x}{x^2}\right)$$

$$p(x) = \frac{6x + x^2}{x} \quad q(x) = x$$

find their r-values at 0

$$p(0) = \frac{0}{0} \quad q(0) = 0$$

The point 0 is not Analytic

So, we can't proceed to set up our indicial equation, or find its R-values.
Therefore, there is no Frobenius Solution.

3.) $x^2y'' - xy' + y = 0$

$y'' - \frac{x}{x^2}y' + \frac{1}{x^2}y = 0$

$p(x) = x\left(\frac{-x}{x^2}\right)$ $q(x) = x^2\left(\frac{1}{x^2}\right)$

$p(0) = -1$ $q(0) = 1$

$r(r-1) - r + 1 = 0$

$r^2 - r - r + 1 = 0$

$r^2 - 2r + 1 = 0$

$(r-1)(r-1) = 0$

$r_1 = r_2 = 1$

There is only 1 frobenius Solution since $r_1 = r_2$

$y = y_1(\ln x) + \sum_{n=0}^{\infty} b_n x^{n+2}$

$y' = y_1\left(\frac{1}{x}\right) + y_1'(\ln x) + \sum_{n=1}^{\infty} (n+2)b_n x^{n+1}$

$y'' = y_1\left(-\frac{1}{x^2}\right) + y_1'\left(\frac{1}{x}\right) + y_1''(\ln x) + y_1'\left(\frac{1}{x}\right) + \sum_{n=2}^{\infty} (n+2)(n+1)b_n x^n$

$y_1 + y_1'x + y_1''x^2(\ln x) + y_1'x + \sum_{n=2}^{\infty}(n+2)(n+1)b_n x^{n+2} - y_1 - y_1'x\ln x$
$+ \sum_{n=1}^{\infty} (n+2)b_n x^n + y_1(\ln x) + \sum_{n=0}^{\infty} b_n x^{n+2}$

$\ln x\big[\underbrace{y_1''x^2 - y_1'x + y_1}_{\text{Just a Sol}}\big] + 2y_1'x + \sum_{n=2}^{\infty}(n+2)(n+1)b_n x^{n+2}$
$+ \sum_{n=1}^{\infty} (n+2)b_n x^n + \sum_{n=0}^{\infty} b_n x^{n+2}$

$2\left(\sum_{n=0}^{\infty} \frac{(-1)^n 2n x^{2n}}{2^{2n}(n!)^2}\right) + 3b_0x^2 + 6b_1x^3 + \sum_{n=0}^{\infty} b_n x^{n+2} = 0$

$2\left(\sum_{n=0}^{\infty} \frac{(-1)^n 2n x^{2n}}{2^{2n}(n!)^2}\right) + 3b_0x^2 + 6b_1x^3 + \sum_{n=2}^{\infty} n^2(b_n + b_{n-2})x^n = 0$

$n \geq 2$, $b_n = -\frac{b_{n-2}}{n^2}, \frac{1}{6n^2}\left[-(2n-2 + \frac{2(-1)^{n+1}2n}{2^{2n}(n!)^2}\right]$

$n=2: b_4 = \left[-\frac{1}{6} + \frac{-8}{64}\right] = -\frac{7}{24}$

$n=3: b_6 = \left[\frac{7}{24} + \frac{12}{2304}\right] = \frac{19}{64}$

Answer: $(J_0(x))\ln x + \frac{x^2}{6} - \frac{7}{24}x^4 + \frac{19}{64}x^6 - \cdots = 0$

$x^2\left(y_1\left(-\frac{1}{x^2}\right) + y_1'\left(\frac{1}{x}\right) + y_1''(\ln x) + y'\left(\frac{1}{x}\right) + \sum_{n=2}^{\infty}(n+2)(n+1)b_n x^n\right) - x\left(y_1\left(\frac{1}{x}\right) + y_1'(\ln x) + \sum_{n=1}^{\infty}(n+2)b_n x^{n+1}\right) + \left(y_1(\ln x) + \sum_{n=0}^{\infty} b_n x^{n+2}\right)$

4) $xy'' + 4(4-x)y' - y = 0$

$y'' + \frac{4(4-x)}{x} y' - \frac{y}{x} = 0$

$p(x) = \cancel{x}\left(\frac{4(4-x)}{\cancel{x}}\right) \quad q(x) = x^2\left(\frac{1}{x}\right)$

$p(x) = 4(4-x) \quad q(x) = x$

$p(0) = 16 \quad q(0) = 0$

$x\left(\sum_{n=0}^{\infty}(n+r)(n-r-1)c_n x^{n+r-2}\right) + 4(4-x)\left(\sum_{n=0}^{\infty} n+r c_n x^{n+r-1}\right)$

$(16-4x)$

$-\left(\sum_{n=0}^{\infty} c_n x^{n+r}\right) = 0$

$\sum_{n=0}^{\infty}(n+r)(n-r-1)c_n x^{n+r-1} + 16\sum_{n=0}^{\infty} n+r c_n x^{n+r-1}$

$-4\sum_{n=0}^{\infty} n+r c_n x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r}$

$r(r-1) + 16r = 0$

$r^2 - r + 16r = 0$

$r^2 + 15r = 0$

$r(r+15) = 0$

$r_1 = 0 \quad r_2 = -15$

$r_1 - r_2 = 15, \ r_1 - r_2 = N$

$\left[y = \sum_{n=0}^{\infty} c_n x^{n+r}\right]$

$\left[y' = \sum_{n=0}^{\infty} n+r c_n x^{n+r-1}\right]$

$\left[y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}\right]$

$\sum_{n=0}^{\infty} c_n \left[(n-r)(n-r-1) + (n+r)\right] x^{n+r-1} - \sum_{n=1}^{\infty}\left[4(n+r-1) - 1\right] c_{n-1} x^{n+r-1}$

$r_2 = -15$

$\cancel{c_n \frac{[(n+15)(n+15-1) + (n-15)]}{[(n+15)(n+15-1)] + (n-15)}} = c_{n-1}\frac{[4n+13]}{[(n+15)(n+14) + (n-15)]}$

$n=1: c_1 = c_0 \frac{17}{226} \qquad n \geq 1$

$n=2: c_2 = c_1 \frac{3}{37} = c_0 \frac{51}{8562}$

$n=3: c_3 = c_2 \frac{25}{294} c_0 \frac{1275}{2517228}$

Answer:

$y(x) = c_0\left(\frac{17}{226}\right) + c_1\left(\sum_{n=2}^{\infty}\frac{25(5n^2-5n+1)}{2517e^{5n-10}} x^n\right)$

The Gamma Function and Bessel Functions- HW Problems

1. Use the fact that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ to find $\Gamma\left(\frac{5}{2}\right)$.

2. Show that $\Gamma(x+5) = (x+4)(x+3)(x+2)(x+1)x\Gamma(x)$.

3. Find $J_3(x)$ in terms of $J_0(x)$ and $J_1(x)$.

4. Show that $y = xJ_1(x)$ is a solution to the differential equation $xy'' - y' - x^2 J_0'(x) = 0$.

   Hint: $J_1(x)$ satisfies Bessel's equation:

   $$x^2 J_1''(x) + xJ_1'(x) + (x^2 - 1)J_1(x) = 0,$$

   and $J_0'(x) = -J_1(x)$.

5. Write $\int x^4 J_0(x)dx$ in terms of Bessel functions and $\int J_0(x)dx$.

1.) Use the fact $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ to find $\Gamma(\frac{5}{2})$

Definition

$$\boxed{\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt, \quad x > 0}$$

remember

$$\Gamma(x+1) = x\Gamma(x)$$

$$\Gamma(\tfrac{5}{2}) = (\tfrac{5}{2}-1)\Gamma(\tfrac{5}{2}-1) = (\tfrac{3}{2})\underline{\Gamma(\tfrac{3}{2})}$$

$$\Gamma(\tfrac{3}{2}) = (\tfrac{3}{2}-1)\Gamma(\tfrac{3}{2}-1) = (\tfrac{1}{2})\underline{\Gamma(\tfrac{1}{2})}$$

Thus, $\Gamma(\frac{5}{2}) = (\frac{3}{2})(\frac{1}{2})\sqrt{\pi} = (\frac{3}{4})\sqrt{\pi}$

2.) show that $\Gamma(x+5) = (x+4)(x+3)(x+2)(x+1)x\Gamma(x)$

$$\Gamma(x+5) = ((x+5)-1)\underline{\Gamma((x+5)-1)}$$

$$\Gamma(x+4) = ((x+4)-1)\underline{\Gamma((x+4)-1)}$$

$$\Gamma(x+3) = ((x+3)-1)\underline{\Gamma((x+3)-1)}$$

$$\Gamma(x+2) = ((x+2)-1)\Gamma((x+2)-1)$$

$$\Gamma(x+1) = x\Gamma(x)$$

Therefore, $\Gamma(x+5) = (x+4)(x+3)(x+2)(x+1)x\Gamma(x)$

3.) Find $J_3(x)$ in terms of $J_0(x)$ & $J_1(x)$

use bessel function relationship

$$0 = J_{p-1}(x) - \left(\frac{2p}{x}\right)J_p(x) + J_{p+1}(x)$$

$$J_{p+1}(x) = \left(\frac{2p}{x}\right)J_p(x) - J_{p-1}(x)$$

$n=1$

$$J_2(x) = \left(\frac{2}{x}\right)J_1(x) - J_0(x)$$

$n=2$

$$J_3(x) = \left(\frac{4}{x}\right)J_2(x) - J_1(x)$$

$$J_3(x) = \left(\frac{4}{x}\right)\left(\left(\frac{2}{x}\right)J_1(x) - J_0(x)\right) - J_1(x)$$

4.) Show that $y = xJ_1(x)$ is a solution to the differential equation $xy'' - y' - x^2J_0'(x) = 0$

Hint

$J_1(x)$ satisfies Bessels equation: $x^2J_1''(x) + xJ_1'(x) + (x^2-1)J_1(x) = 0$, and $J_0'(x) = -J_1(x)$.

(Then sub into eq)

(first find y' & y'')

$$y' = J_1(x) + xJ_1'(x)$$

$$y'' = J_1'(x) + J_1'(x) + xJ_1''(x)$$

$$x(J_1'(x) + J_1'(x) + xJ''(x)) - (J_1(x) + xJ_1'(x)) - x^2(J_0'(x)) = 0$$

$$\cancel{xJ_1'(x)} + xJ_1'(x) + x^2J''(x) - J_1(x) - \cancel{xJ_1'(x)} - x^2(-J_1(x)) = 0$$

$$x^2J_1''(x) + xJ_1'(x) + (x^2-1)J_1(x) = 0$$

Answer simplifies to Bessel Eq

It takes this form, therefore a solution

5) Write $\int x^4 J_0(x)\,dx$ in terms of Bessel functions & $\int J_0(x)\,dx$

$$\int \underbrace{x}_{u}{}^{4} \underbrace{J_0(x)\,dx}_{dv} = xJ_1(x) - \int J_1(x)\, 4x^3\,dx$$

$$xJ_1(x) - 4\int \underbrace{J_1(x)}_{dv}\underbrace{x^3}_{u}\,dx$$

$du = 4x^3\,dx$

$v = J_1(x)$

$v = -J_0(x)$

$du = 3x^2\,dx$

$$xJ_1(x) - 4\left(-x^3J_0(x) - \int -J_0(x)\,3x^2\,dx\right)$$

$$xJ_1(x) - 4\left(-x^3J_0(x) + 3\int \underbrace{J_0(x)}_{dv}\underbrace{x^2}_{u}\,dx\right)$$

$v = J_1(x)$

$du = 2x\,dx$

$$xJ_1(x) - 4\left(-x^3J_0(x) + 3\left(x^2J_1(x) - \int J_1(x)\,2x\,dx\right)\right)$$

$$xJ_1(x) - 4\left(-x^3J_0(x) + 3\left(x^2J_1(x) - 2\int \underbrace{J_1(x)}_{dv}\underbrace{x}_{u}\,dx\right)\right)$$

$v = -J_0(x)$

$du = dx$

$$\int x^4 J_0(x)\,dx = xJ_1(x) - 4\left(-x^3J_0(x) + 3\left(x^2J_1(x) - 2\left(-xJ_0(x) + \int J_0(x)\,dx\right)\right)\right)$$

Laplace Transforms- HW Problems

In problems 1-5 use the definition of a Laplace transform to find the Laplace transform of the given function.

1. $f(t) = t^2 + t$
2. $f(t) = \cos^2(t)$
3. $f(t) = e^{(2t-1)}$
4. $f(t) = 2t \qquad 0 \le t \le 1$

   $\quad = 0 \qquad 1 < t$
5. $f(t) = 1 \qquad 0 \le t \le 2$

   $\quad = 0 \qquad 2 < t.$

In problems 6-10 use the Laplace transforms of functions developed in class and the linearity properties of the Laplace transform to find the Laplace transforms of the following functions.

6. $f(t) = 4 - 3t^2$
7. $f(t) = 3\cos(2t) + 2e^{3t}$
8. $f(t) = te^{2t}$ (Hint: Use the definition and integrate by parts first)
9. $f(t) = 4 + \sinh(2t)$
10. $f(t) = t^{\frac{5}{2}} + \sin(3t)$

In problems 11-17 use the Laplace transforms developed in class to find the inverse Laplace transforms of the following functions.

11. $F(s) = \frac{3}{s}$

12. $F(s) = \frac{1}{s^5}$

13. $F(s) = \frac{s}{s^2+9}$

14. $F(s) = \frac{3}{s-4}$

15. $F(s) = \frac{4-2s}{s^2+16}$

16. $F(s) = \frac{s-5}{9-s^2}$

17. $F(s) = -3s^{-1}e^{-4s}$

In problems 1-5 use the definition of a Laplace transforms to find the Laplace transform of a given function.

1.) $f(t) = t^2 + t$

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}[f(t)] = \mathcal{L}[t^2] + \mathcal{L}[t]$$

$$\mathcal{L}[t^2] = \int_0^\infty \underset{dv}{\underline{e^{-st}}}\ \underset{u}{\underline{t^2}}\,dt$$

$V = -\frac{1}{s} \cdot e^{-st}$

$du = 2t\,dt$

$$(t^2)\left(-\frac{1}{s} \cdot e^{-st}\right)\Big|_0^\infty - \int_0^\infty -\frac{1}{s} \cdot e^{-st}(2t)\,dt$$

$$\left[\infty^2\left(-\frac{1}{s} \cdot e^{-\infty}\right) - 0 \cdot \left(-\frac{1}{s} \cdot e^{0}\right)\right] + \frac{2}{s}\int_0^\infty \underset{dv}{\underline{e^{-st}}}\ \underset{u}{\underline{t}}\,dt$$

$du = dx$

$V = -\frac{1}{s} \cdot e^{-st}$

$$\mathcal{L}[t] = \int_0^\infty \underset{dv}{\underline{e^{-st}}}\ \underset{u}{\underline{t}}\,dt$$

$$+\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty - \int -\frac{1}{s}e^{-st}\,dt$$

$V = -\frac{1}{s} \cdot e^{-st}$ $du = dt$

0

$$= +\frac{1}{s}\int_0^\infty e^{-st}\,dt$$

$$(t)\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty - \int_0^\infty -\frac{1}{s} \cdot e^{-st}\,dt$$

$$\frac{2}{s}\left(\frac{1}{s}\left(-\frac{1}{s}e^{-\infty}\right) + \frac{1}{s}e^{0}\right)$$

$$0 + \int_0^\infty \frac{1}{s}e^{-st}\,dt$$

$$\frac{2}{s} \cdot \frac{1}{s} \cdot \frac{1}{s} = \frac{2}{s^3}$$

$$= \boxed{\mathcal{L}[t^2] = \frac{2}{s^3}}$$

$$= \frac{1}{s}\int_0^\infty e^{-st}\,dt$$

$$\frac{1}{s}\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty = \frac{1}{s}\left(0 + \frac{1}{s}e^{0}\right) = \boxed{\mathcal{L}[t] = \frac{1}{s^2}}$$

$$\boxed{\text{Answer } \mathcal{L}[t^2 + t] = \frac{2}{s^3} + \frac{1}{s^2} \quad s > 0}$$

2.) $f(t) = \cos^2(t)$

$$\mathcal{L}[\cos^2(t)] = \int_0^\infty e^{-st}((\cos^2(t))\,dt$$

$$\mathcal{L}\left[\frac{1+\cos(2t)}{2}\right] = \int_0^\infty e^{-st}\left(\frac{1+\cos(2t)}{2}\right)dt$$

Break it up $= \int_0^\infty e^{-st}\frac{1}{2}\,dt + \int_0^\infty e^{-st}\frac{\cos(2t)}{2}\,dt$

$$\frac{1}{2}\int_0^\infty e^{-st}\,dt = -\frac{1}{2s}(e^{-st})\Big|_0^\infty$$

$$-\frac{1}{2s}[-e^{-\infty} + e^0]\,dt$$

0

$$\mathcal{L}\left[\frac{1}{2}\right] = \frac{1}{2s}$$

$$\frac{1}{2}\int_0^\infty e^{-st}\cos(2t)\,dt$$

remember

$$\frac{1}{2}\lim_{N\to\infty}\left(\int_0^\infty \underbrace{e^{-st}}_{D} \cdot \underbrace{((\cos(2t))\,dt}_{I}\right)$$

	D	I
+	e^{-st}	$\cos(2t)$
−	$-se^{-st}$	$\frac{1}{2}\sin(2t)$
+	s^2e^{-st}	$-\frac{1}{4}\cos(2t)$

Make denom common $\frac{4}{4}$

$$1\int e^{-st}\cos(2t)\,dt = \frac{e^{-st}}{2}\sin 2t - \frac{se^{-st}}{4}\cos(2t) + \frac{s^2}{4}\int e^{-st}\cos(2t)\,dt$$

$$-\int \frac{s^2}{4}e^{-st}\cos 2t\,dt$$

$$\left[\frac{4+s^2}{4}\int e^{-st}\sin(2t)\,dt = \frac{e^{-st}}{2}\sin 2t - \frac{se^{-st}}{4}\cos 2t\right] + \int \frac{s^2}{4}e^{-st}\cos 2t\,dt$$

$$\int e^{-st}\sin(2t)\,dt = \frac{4}{4+s^2}\left[\frac{e^{-st}}{2}\sin 2t - \frac{se^{-st}}{4}\cos 2t\right]$$

continued

$$= \lim_{N \to \infty} \left(\frac{4}{4+s^2} e^{-st} \sin(2t) - \frac{s}{4+s^2} e^{-st} \cos 2t \right)_0^N$$

$$\lim_{N \to \infty} \left(\frac{4}{4+s^2} \overset{0}{e^{-sN}} \sin(2N) - \frac{s}{4+s^2} \overset{0}{e^{-sN}} (\cos 2N) \right)$$

$$- \left(\frac{4}{4+s^2} e^{0} (0) - \frac{s}{4+s^2} e^{-s(0)} (1) \right)$$

$$= 0 - \left(0 - \frac{s}{4+s^2} \right)$$

$$= \frac{1}{2} \left(\frac{s}{4+s^2} \right)$$

remember

$$\mathcal{L}[\cos^2(t)] = \underline{\frac{1}{2s} + \frac{1}{2}\left(\frac{s}{4+s^2} \right)} \quad s > 0$$

Answer

3.) $f(t) = e^{(2t-1)}$

$\mathcal{L}[e^{(2t-1)}] = \int_0^\infty e^{-st} e^{(2t-1)} dt$

$\int_0^\infty e^{(-st + (2t-1))} dt$

$\int_0^\infty e^{(-st+2t-1)} dt$

$\frac{1}{-s+2} \left(e^{(-st+2t-1)} \right) \Big|_0^\infty$

$\frac{1}{-(s-2)} \left(\lim_{t\to\infty} \left(e^{(-s(\infty)+2(\infty)-1)} - e^{(-1)} \right) \right)$

$= e^{(-1)} \left(\frac{1}{s-2} \right) \quad s \geq -2$

4.) $\boxed{\begin{array}{ll} f(t) = 2t & 0 \leq t \leq 1 \\ = 0 & 1 < t \end{array}}$

$\mathcal{L}[2t] = \int_0^1 \underbrace{e^{-st}}_{dv} \underbrace{2t}_{u} dt$

$v = -\frac{1}{s} e^{-st}, du = 2dt$

$2t(-\frac{1}{s}e^{-st}) - \int_0^1 -\frac{1}{s} e^{-st} 2dt$

$2t(-\frac{1}{s}e^{-st}) + \frac{2}{s}\int_0^1 e^{-st} dt$

$2(-\frac{1}{s}e^{-s}) + \frac{2}{s}(-\frac{1}{s}e^{-st})\Big|_0^1$

$+ \frac{2}{s}(-\frac{1}{s}e^{-s} + \frac{1}{s}e^0)$

$s > 0 \quad \boxed{\mathcal{L}[2t] = -\frac{2}{s^2}e^{-s} + \frac{2}{s^2} + 2(-\frac{1}{s}e^{-s})}$

5.) $$\boxed{\begin{aligned} f(t) &= 1 \quad 0 \leq t \leq 2 \\ &= 0 \quad 2 < t \end{aligned}}$$

$$\mathcal{L}[1] = \int_0^2 e^{-st} 1\,dt = \int_0^2 e^{-st}\,dt$$

$$\left(-\frac{1}{s} e^{-st}\right)\Big|_0^2$$

$$= -\frac{1}{s} e^{-s(2)} + \frac{1}{s} e^{-s(0)}$$

$$\boxed{\mathcal{L}[1] = -\frac{1}{s} e^{-2s} + \frac{1}{s}} \quad s > 0$$

In problems 6–18 use the Laplace transforms of functions developed in class & the linearity properties of the laplace transform to find the laplace transforms of the following functions:

6.) $f(t) = 4 - 3t^2$

$\mathcal{L}[4] - 3\mathcal{L}[t^2]$, first find $\mathcal{L}[4]$,

$\int_0^\infty e^{-st} 4\,dt$

$4\int_0^\infty e^{-st}\,dt$

$= 4\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty$

$4\left(0 + \frac{1}{s}e^{-s(0)}\right) = \boxed{\frac{4}{s}}$

Then find $\mathcal{L}[t^2]$

$\int_0^\infty \underbrace{e^{-st}}_{dv} \underbrace{t^2}_{u}\,dt$ $\boxed{\begin{array}{l} v = -\frac{1}{s}e^{-st} \\ du = 2t\,dt \end{array}}$

$t^2\left(-\frac{1}{s}e^{-st}\right) - \int_0^\infty -\frac{1}{s}e^{-st}\,2t\,dt$

0

$+\frac{2}{s}\int_0^\infty \underbrace{e^{-st}}_{dv} \underbrace{t}_{u}\,dt$ $\boxed{\begin{array}{l} v = -\frac{1}{s}e^{-st} \\ du = dt \end{array}}$

remember

$t\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}\,dt$

$t\left(-\frac{1}{s}e^{-st}\right) + \frac{1}{s}\int_0^\infty e^{-st}\,dt$

0

$\frac{2}{s}\cdot\frac{1}{s}\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty$

$\frac{2}{s}\cdot\frac{1}{s}\left(0 + \frac{1}{s}e^{0}\right) = \boxed{\frac{2}{s^3}}$

Answer

$$\boxed{\mathcal{L}[4 - 3t^2] = \frac{4}{s} - 3\left(\frac{2}{s^3}\right)}$$

S>0

7.) $f(t) = 3\cos(2t) + 2e^{3t}$

$3\mathcal{L}[\cos(2t)] + 2\mathcal{L}[e^{3t}]$

$\mathcal{L}[e^{3t}] = \int_0^\infty e^{-st} \cdot e^{3t}\,dt$

$\int_0^\infty e^{(-st+(3t))}\,dt$

$\left(\frac{1}{-s+3}\right) e^{(-st+(3t))}\Big|_0^\infty$

$-\left(\frac{1}{s-3}\right) e^{(-st+(3t))}\Big|_0^\infty$

$-\left(\frac{1}{s-3}\right)\left(e^{-\infty} - e^{0}\right)$

$\mathcal{L}[e^{3t}] = \boxed{\frac{1}{s-3}}$

$\mathcal{L}[\cos(2t)]$

$\int_0^\infty \underline{e^{-st}}\ \underline{\cos(2t)}\,dt$

	$\boxed{D}$	$\boxed{I}$
+	e^{-st}	$\cos(2t)$
−	$-se^{-st}$	$\frac{\sin(2t)}{2}$
+	$s^2 e^{-st}$	$-\frac{\cos(2t)}{4}$

Answer

$\mathcal{L}[3\cos(2t) + 2e^{3t}] = 3\left(\frac{4}{s^2+4}\right) + 2\left(\frac{1}{s-3}\right)$ S>0

$(e^{-st})\left(\frac{\sin(2t)}{2}\right) - (s\,e^{-st})\left(\frac{\cos(2t)}{4}\right) - \int (s^2 \cdot e^{-st})\left(\frac{\cos(2t)}{4}\right) dt$

$\int_0^\infty e^{-st}\cos(2t)\,dt = (e^{-st})\left(\frac{\sin(2t)}{2}\right) - \left(\frac{s}{4}e^{-st}\right)(\cos(2t)) - \frac{s^2}{4}\int_0^\infty e^{-st}\cos(2t)\,dt$

$+\frac{s^2}{4}\int_0^\infty e^{-st}\cos(2t)\,dt$ $\qquad + \frac{s^2}{4}\int_0^\infty e^{-st}\cos(2t)\,dt$

$\frac{s^2}{4} + \frac{4}{4}$ $\quad \frac{s^2+4}{4}\int_0^\infty e^{-st}\cos(2t)\,dt = (e^{-st})\left(\frac{\sin(2t)}{2}\right) - \left(\frac{s}{4}e^{-st}\right)(\cos(2t))$

$= \frac{4}{s^2+4}\left((e^{-st})\left(\frac{\sin(2t)}{2}\right) - \left(\frac{s}{4}e^{-st}\right)(\cos(2t))\right)\Big|_0^\infty$

$\frac{4}{s^2+4}\left(\left(\overset{0}{e^{-\infty}}\left(\frac{\sin(\infty)}{2}\right) - \left(\frac{1}{4}\overset{0}{e^{-\infty}}\right)(\cos(2t)\right) - \left(e^{0}\left(\frac{\overset{0}{\sin(0)}}{2}\right) - \left(\frac{s}{4}\right)(1)\right)\right)$

$\left(\frac{s}{4}\right)\left(\frac{4}{s^2+4}\right) = \boxed{\frac{s}{s^2+4} = \mathcal{L}[\cos(2t)]}$

8.) $f(t) = te^{2t}$

$$\mathcal{L}[te^{2t}] = \int_0^\infty e^{-st} \cdot te^{2t}\,dt$$

$$\int_0^\infty \underset{u}{\underline{t}}\,\underset{dv}{\underline{e^{(-st)+(2t)}}\,dt}$$

$du = dt$

$v = \left(\frac{1}{-s+2}\right)\left(e^{(-st)+(2t)}\right)$

$$(t)\left(\frac{e^{(-st)+(2t)}}{-s+2}\right)\Big|_0^\infty - \int_0^\infty \left(\frac{1}{-s+2}\right)\left(e^{(-st)+2t}\right)dt$$

$$(t)\left(\frac{e^{(-st)+(2t)}}{-s+2}\right)\Big|_0^\infty + \int_0^\infty \left(\frac{1}{s-2}\right)\left(e^{(-st)+2t}\right)dt$$

$$(t)\left(\frac{e^{(-st)+(2t)}}{-s+2}\right)\Big|_0^\infty + -\left(\frac{1}{(s-2)^2}\right)\left(e^{(-st)+2t}\right)\Big|_0^\infty$$

$$(-(\infty)(0)) - (0) + -\left(\frac{1}{(s-2)^2}\right)(0-1)$$

$$= \boxed{\frac{1}{(s-2)^2}} \quad s > 2$$

9.) $f(t) = 4 + \sinh(2t)$

$$\mathcal{L}[4] + \mathcal{L}[\sinh(2t)] = \mathcal{L}[4] + \left(\mathcal{L}\left[\frac{e^{kt} - e^{-kt}}{2}\right]\right)$$

$$\mathcal{L}[\sinh(2t)] = \int_0^\infty e^{-st} \sinh(2t)\,dt \qquad \frac{1}{2}\left(\mathcal{L}[e^{kt}] - \mathcal{L}[e^{-kt}]\right)$$

$$\mathcal{L}[e^{kt}] = \int_0^\infty e^{-st} e^{kt}\,dt \qquad \mathcal{L}[e^{-kt}] = \int_0^\infty e^{-st} e^{-kt}\,dt$$

$$\int_0^\infty e^{-st+(kt)}\,dt \qquad \int_0^\infty e^{-st-kt}\,dt$$

$$\left(\frac{1}{-s+k}\right) e^{-st+(kt)} \Big|_0^\infty \qquad \left(\frac{1}{-s-k}\right) e^{-st-kt} \Big|_0^\infty$$

$$\left(\frac{1}{s-k}\right) - \left(\frac{1}{s+k}\right)$$

$$\mathcal{L}[4] + \left(\frac{1}{s-k} - \frac{1}{s+k}\right)\frac{1}{2}$$

$$4\int e^{-st}\,dt$$

$$\frac{4}{s}$$

$$\boxed{\frac{4}{s} + \frac{1}{2}\left(\frac{1}{s-2} - \frac{1}{s+2}\right) = \mathcal{L}[4 + \sinh(2t)]}$$

$|\, s > 0$

or

$$\frac{1}{2}\left(\frac{s+2}{(s-2)(s+2)} - \frac{(s-2)}{(s-2)(s+2)}\right)$$

$$\frac{1}{2}\left(\frac{4}{s^2-4}\right)$$

$$\frac{2}{s^2-4}$$

Another Form

$$\frac{4}{s} + \left(\frac{2}{s^2-4}\right) = \mathcal{L}[4 + \sinh(2t)]$$

10.) $f(t) = t^{\frac{5}{2}} + \sin(3t)$

$\mathcal{L}[t^{\frac{5}{2}}] + \frac{1}{2i}\mathcal{L}[e^{ikt} - e^{-ikt}]$

$\int_0^\infty e^{-st} t^{\frac{5}{2}} dt$

$\mathcal{L}[t^a] = \frac{\Gamma(a+1)}{s^{a+1}}$

$\frac{\Gamma(\frac{5}{2}+1)}{s^{\frac{5}{2}+1}} = \frac{\Gamma(\frac{7}{2})}{s^{(\frac{7}{2})}}$

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$\mathcal{L}[e^{i3t}]$

$\int_0^\infty e^{-st} e^{i3t} dt$

$\int_0^\infty e^{-st+(i3t)} dt$

$\left(\frac{1}{-s+i3}\right) e^{-st+(i3t)}\Big|_0^\infty$

$-\left(\frac{1}{s-i3}\right) e^{-st+(i3t)}\Big|_0^\infty$

$= \frac{1}{s-ik}$

$\mathcal{L}[e^{-i3t}]$

$\int_0^\infty e^{-st} e^{-i3t} dt$

$\int_0^\infty e^{-st+(-i3t)} dt$

$\left(\frac{1}{-s-i3}\right) e^{-st+(-i3t)}\Big|_0^\infty$

$-\left(\frac{1}{s+ik}\right) e^{-st+(ikt)}\Big|_0^\infty$

$= \frac{1}{s+ik}$

$= \frac{1}{2i}\left(\left(\frac{1}{s-ik}\right) - \left(\frac{1}{s+ik}\right)\right)$

remember $= \boxed{s^2+k^2}$

$(s-i3)(s+i3) = s^2+9$

$= \frac{1}{2i}\left(\frac{1}{s-i3} - \frac{1}{s+i3}\right)$

$= \frac{\left(\frac{1}{s-i3} - \frac{1}{s+i3}\right)(-i)}{2i(-i)}$

$= \frac{-i\left(\frac{(2)3i}{s^2+3^2}\right)}{2}$

$= \frac{\frac{(2)3}{s^2+3^2}}{2} = \boxed{\frac{3}{s^2+9}}$

$\mathcal{L}[\sin(3t)] = \boxed{\frac{3}{s^2+9}}$

continued

$\Gamma(1+x) = x\Gamma(x)$, we want $\Gamma(\frac{7}{2})$, we know $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\Gamma(\frac{3}{2}) = \Gamma(1+\frac{1}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi} = \frac{\sqrt{\pi}}{2}$

$\Gamma(\frac{5}{2}) = \Gamma(1+\frac{3}{2}) = \frac{3}{2}\Gamma(\frac{3}{2}) = \frac{3}{2}\left(\frac{\sqrt{\pi}}{2}\right) = \frac{3\sqrt{\pi}}{4}$

$\Gamma(\frac{7}{2}) = \Gamma(1+\frac{5}{2}) = \frac{5}{2}\Gamma(\frac{5}{2}) = \frac{5}{2}\left(\frac{3\sqrt{\pi}}{4}\right) = \frac{15\sqrt{\pi}}{8}$

$$\boxed{\mathcal{L}[t^{\frac{5}{2}} + \sin(3t)] = \frac{\left(\frac{15\sqrt{\pi}}{8}\right)}{s^{(\frac{7}{2})}} + \frac{3}{s^2+9}} \quad s>0$$

Answer

In problems 11-17 use the Laplace transforms developed in class to find in inverse laplace transforms of the following functions.

11.) $F(s)=\frac{3}{s}$ $\mathcal{L}^{-1}\left[\frac{3}{s}\right]=\underline{3}$, $\mathcal{L}[a]=\frac{a}{s}$

12.) $F(s)=\frac{1}{s^5}$ $\mathcal{L}^{-1}\left[\frac{1}{s^5}\right]=\underline{\frac{1}{24}(t^4)}$, remember $\frac{n!}{s^{n+1}}=\mathcal{L}[t^n]$

13.) $F(s)=\frac{s}{s^2+9}$ $\mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right]=\underline{\cos(3t)}$, $\mathcal{L}[\cos kt]=\frac{s}{s^2+k^2}$

14.) $F(s)=\frac{3}{s-4}$ $\mathcal{L}^{-1}\left[\frac{3}{s-4}\right]=\underline{3(e^{4t})}$, $\mathcal{L}[e^{at}]=\frac{1}{s-a}$

15.) $F(s)=\frac{4-2s}{s^2+16}$ $\mathcal{L}^{-1}\left[\frac{4-2s}{s^2+16}\right]=\underline{(\sin(4t)-2\cos(4t))}$

$F(s)=\frac{4}{s^2+16}-\frac{2s}{s^2+16}$, $\mathcal{L}^{-1}\left[\frac{4}{s^2+16}\right]-2\mathcal{L}^{-1}\left[\frac{s}{s^2+16}\right]$

$\sin(4t)-2\cos(4t)$

16.) $F(s)=\frac{s-5}{9-s^2}$, $\mathcal{L}^{-1}\left[\frac{s}{9-s^2}\right]-\mathcal{L}^{-1}\left[\frac{5}{9-s^2}\right]$

$\underline{\cosh(t)-\frac{5}{3}(\sinh(t))}$

17.) $F(s)=-3s^{-1}e^{-4s}$, $\mathcal{L}^{-1}\left[-3\frac{e^{-4s}}{s}\right]=-3\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s}\right]$

$-3u(t-4)\mathcal{L}^{-1}\left[\frac{1}{s}\right]$

$u(t-a)f(t-a)\rightarrow$ sorta confusing

$=\underline{-3u(t-4)(1)}$