

Vibrating Springs- HW Problems

1. A mass of 1000g is attached to a spring which is stretched 25cm by a force of 9N. At time $t = 0$ the mass is moved stretching the spring 2m to the right, and setting it in motion with an intial velocity of 6m/sec to the left. Find the position of the mass at time t , $x(t)$. Write $x(t)$ in the form $x(t) = C\cos(\omega t - \alpha)$. Find the amplitude and period of the motion.

In problems 2-5 a mass is attached to a spring, with spring constant k , and a dashpot, with damping constant c . The initial velocity of the mass is v_0 and the initial position is x_0 . Find the position function of the mass, $x(t)$, and identify if the motion is overdamped, critically damped, or underdamped. If the motion is underdamped, write the position in the form $x(t) = Ce^{-\rho t}(\cos(\omega t - \alpha))$. Also find the undamped position function (ie where $c = 0$), $u(t) = C\cos(\omega t - \alpha)$.

2. $m = 2, k = 50, c = 12, x_0 = -1, v_0 = -1$
3. $m = 2, k = 6, c = 8, x_0 = 3, v_0 = 2$
4. $m = 1, k = 9, c = 6, x_0 = -2, v_0 = -6$
5. $m = 1, k = 50, c = 10, x_0 = 0, v_0 = 10.$

1.) A mass of 1000g is attached to a spring

which is stretched 25 cm by force of 9N.

At time t=0 the mass is moved stretching the

spring 2m to the right, and setting it in motion
with an initial velocity of 6m/sec to the left.

Find the position of the mass at time t , $x(t)$. Write

$x(t)$ in the form $x(t) = C \cos(\omega t - \alpha)$. Find the amplitude

& period of the motion. $F = 9N$ $X = .25m$

$$\textcircled{1} \quad k = \frac{F}{x} = \frac{9N}{.25m} = 36 \text{ N/m}$$

$$1000g = 1 \text{ kg}$$

$$\textcircled{2} \quad w_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{36 \text{ N}}{1 \text{ kg}}} = 6 \sqrt{\frac{\text{N}}{\text{kg}}}$$

$$\textcircled{3} \quad \frac{v(0)}{-Aw_0} = -\frac{A w_0 \sin(\theta)}{-Aw_0}$$

↓

⑤

$$x(t) = 2m \cos(6t + \frac{\pi}{6})$$

$$\textcircled{4} \quad \frac{-(-6 \text{ m/s})}{(2m * 6 \sqrt{\frac{\text{N}}{\text{kg}}})} = \boxed{1.5}$$

$$\alpha = \tan^{-1}(1.5) = \boxed{\frac{\pi}{6}}$$

$$x(t) = 2 \cos(6t + \frac{\pi}{6})$$

$$\text{period} = \frac{2\pi}{6} = \frac{\pi}{3}$$

$$\text{Amplitude} = |2| = 2$$

In problems 2-5 a mass is attached to a spring with spring constant k , & a dashpot, with damping constant c . The initial velocity of the mass is v_0 and the initial position is x_0 . Find the position function of the mass, $x(t)$, & identify if the motion is overdamped, critically damped, or underdamped. If the motion is underdamped, write the position in the form $x(t) = e^{-\frac{ct}{2m}}(\cos(\omega t - \alpha))$. Also find the undamped position function (i.e. where $c=0$), $u(t) = (\cos(\omega t - \alpha))$.

answer 2

2.) $m=2, k=50, c=12, x_0=-1, v_0=-1$

$$c^2 - 4km = 12^2 - 4(50)(2) = -256$$

$$x(t) = \sqrt{\frac{65}{16}} e^{-6t} \cos(25t - 1.0516);$$

motion is underdamped

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{50}{2}} = 25$$

$$mx'' + cx' + kx = 0$$

$$< \frac{\pi}{2} \quad | \quad 12x'' + 12x' + 50x = 0$$

$$\alpha = \tan^{-1}\left(\frac{b}{a}\right) = 1.0516 \quad | \quad (2r^2 + 12r + 50) = 0$$

$$\begin{cases} -1 = X'(0) = -6A + 4B \\ -1 = X(0) = A \end{cases}$$

$$r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(50)}}{2(2)}$$

$$(-) \sqrt{A^2 + B^2}$$

$$-1 = -6(-1) + 4B$$

$$\frac{-7}{4} = \frac{4B}{4}$$

$$B = -\frac{7}{4}$$

$$A = -1$$

$$r = -6 \pm \frac{16}{4} i$$

$$r = -6 \pm 4i$$

$$x(t) = e^{-6t}(A \cos 4t + B \sin 4t)$$

$$x'(t) = -6(e^{-6t})(A \cos 4t + B \sin 4t) + e^{-6t}(-4A \sin 4t + 4B \cos 4t)$$

(undamped)
 $C=0$

$$x(t) = mx'' + kx = 0$$

$$= 2x'' + 50x = 0$$

$$2r^2 + 50 = 0$$

$$\frac{2r^2}{2} = -\frac{50}{2}$$

$$\sqrt{r^2} = \sqrt{-25}$$

2nd Answer

$$r = \pm 5i$$

$$v(t) = \sqrt{\frac{2k}{m}} \cos(2St - 1.3734)$$

$$v(t) = A \sin St + B \cos St$$

$$v'(t) = SA \cos St - SB \sin St$$

$$-1 = v(0) = B$$

$$-1 = v'(0) = SA$$

$$\begin{array}{l} -1 = B \\ \hline \frac{-1}{S} = A \end{array}$$

$$\sqrt{\frac{k}{m}} = 25$$

$$C = \sqrt{A^2 + B^2} = \sqrt{\frac{25}{25}}$$

$$\alpha = \tan^{-1}\left(\frac{B}{A}\right) = 1.3734$$

$$3) M=2, K=6, C=8, X_0=3, V_0=2$$

$$C^2 - 4KM = 8^2 - 4(6)(2) = 16$$

(It's over damped motion)

$$MX'' + CX' + KX = 0$$

$$2X'' + 8X' + 6X = 0$$

$$(2r^2 + 8r + 6 = 0)$$

$$r = \frac{-8 \pm \sqrt{4(2)(6)}}{2(2)}$$

$$-2 \pm 4\sqrt{3}$$

↓ 1st answer

$$\begin{aligned} X(t) &= A e^{(-2-4\sqrt{3})t} + B e^{(-2+4\sqrt{3})t} \\ X'(t) &= (-2-4\sqrt{3})A e^{(-2-4\sqrt{3})t} + (-2+4\sqrt{3})B e^{(-2+4\sqrt{3})t} \end{aligned}$$

$$\boxed{\begin{aligned} X(t) &= \left(\frac{7\sqrt{3}+13}{11}\right)e^{(-2-4\sqrt{3})t} + \left(\frac{-7\sqrt{3}+20}{11}\right)e^{(-2+4\sqrt{3})t} \\ 2 &= V_0 = (-2-4\sqrt{3})A + (-2+4\sqrt{3})B \end{aligned}}$$

$$\boxed{\begin{aligned} A &= \left(\frac{7\sqrt{3}+13}{11}\right) \\ B &= \left(\frac{-7\sqrt{3}+20}{11}\right) \end{aligned}}$$

$$(3-B) = A$$

$$2 = (-2-4\sqrt{3})(3-B) + (-2+4\sqrt{3})B$$

$$2 = (12\sqrt{3}-6) - (-2-4\sqrt{3})B + (-2+4\sqrt{3})B$$

$$\frac{-12\sqrt{3}+8}{-8\sqrt{3}-4} = \frac{-8\sqrt{3}-4B}{-8\sqrt{3}-4} \Leftrightarrow 2 = (12\sqrt{3}-6) - 8\sqrt{3}-4B$$

$(\zeta = 0)$
(undamped)

$$x(t) = m x'' + k x = 0$$

$$= 2x'' + 6x = 0$$

$$\frac{2r^2 + 6}{2} = 0$$
$$r^2 + 3 = 0$$

$$\sqrt{r^2} = \sqrt{-3}$$

$$r = \pm \sqrt{-3} i$$

$$v(t) = A(\cos \sqrt{3}t) + B(\sin \sqrt{3}t)$$

$$v'(t) = -\sqrt{3}A(\sin \sqrt{3}t) + \sqrt{3}B(\cos \sqrt{3}t)$$

$$x = x_0 = A$$

$$\frac{2}{\sqrt{3}} = V_0 = \frac{\sqrt{3}B}{\sqrt{3}}$$

$$(= \sqrt{A^2 + B^2} = \sqrt{\frac{21}{3}})$$

$$v(t) = \sqrt{\frac{21}{3}} \cos\left(\sqrt{3}t - 36.74^\circ\right)$$

$$B = \frac{2}{\sqrt{3}}$$
$$A = 3$$
$$C = \sqrt{\frac{21}{3}}$$

$$\alpha = \tan^{-1}\left(\frac{B}{A}\right) = 36.74^\circ$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{3}$$

↑ 2nd answer!

4)

$$m=1, k=9, c=6, x_0=-2, v_0=-6$$

$$c^2 - 4km = 6^2 - 4(9)(1) = 0$$

Critically damped

$$mx'' + cx' + kx = 0$$

$$x'' + 6x' + 9x = 0$$

$$r^2 + 6r + 9 = 0$$

$$(r+3)(r+3) = 0$$

$$r = -3$$

$$x(t) = C_1 e^{-3t} + (C_2 t e^{-3t}) = e^{-3t}(C_1 + C_2 t)$$

$$x'(t) = e^{-3t}(C_2) + -3e^{-3t}(C_1 + C_2 t)$$

$$x_0 = -2 = C_1 + C_2$$

$$v_0 = -6 = C_2 - 3(C_1)$$

$$x(t) = e^{-3t}(2t + 1)$$

$$C_2 = -2 - C_1$$

$$-6 = (-2 - C_1) - 3C_1$$

1st answer

$$\begin{array}{rcl} -6 & = & -2 - 2C_1 \\ +2 & +2 & \end{array}$$

$$\frac{-4}{-2} = -2 - C_1$$

$$-2 = C_1$$

$$-6 = C_2 - 3(-2)$$

$$C_2 = 12$$

($\zeta = 0$
Undamped)

$$x(t) = m x'' + k x = 0$$

$$= b x'' + q x = 0$$

$$\omega^2 = -q$$

$$\sqrt{\omega^2} = \pm \sqrt{-q}$$

$$\omega = \pm 3$$

$$x(t) = A \cos(3t) + B \sin(3t)$$

$$v(t) = -3A \sin(3t) + 3B \cos(3t)$$

$$x_0 = -2 = A$$

$$v_0 = -6 = 3B$$

$$A = -2 \quad \omega = \sqrt{\frac{k}{m}} = 3$$

$$B = -2 \quad C = \sqrt{A^2 + B^2} = \sqrt{8}$$

$$\alpha = \tan^{-1}\left(\frac{B}{A}\right) = -0.7853$$

$$v(t) = \sqrt{8} \cos(3t - 0.7853)$$

$$5) m=1, K=50, t=10, X_0=0, V_0=10$$

$$\zeta^2 - 4KM = 10^2 - 4(50)(1) = -100$$

Motion is underdamped

$$mx'' + cx' + kx = 0$$

$$x'' + 10x' + 50x = 0$$

$$r^2 + 10r + 50 = 0$$

$$r = \frac{-(10) \pm \sqrt{(10)^2 - 4(1)(50)}}{2(1)}$$

$$r = (-5 \pm 5i)$$

$$v(t) = e^{-5t} (A \cos(5t) + B \sin(5t))$$

$$v'(t) = -5e^{-5t} (A \cos(5t) + B \sin(5t))$$

$$+ e^{-5t} (-5A \sin(5t) + 5B \cos(5t))$$

$$X_0 = 0 = A$$

$$V_0 = 10 = -5A + 5B$$

$$x(t) = 2e^{-5t} (\cos(5t))$$

↑ 1st Answer

$A = 0$	$\omega = \sqrt{\frac{K}{M}} = \sqrt{50}$
$B = 2$	$c = \sqrt{A^2 + B^2} = 2$
$\alpha = \tan^{-1}\left(\frac{B}{A}\right) = 0$	

$(\theta = c)$
(undamped)

$$x(t) = mx'' + kx = 0$$

$$x'' + 50x = 0$$

$$r^2 + 50 = 0$$

$$-50 \quad -50$$

$$\sqrt{r^2} = \sqrt{50}$$

$$r = \pm \sqrt{50};$$

$$u(t) = A \cos \sqrt{50}t + B \sin \sqrt{50}t$$

$$v'(t) = -\sqrt{50}A \sin \sqrt{50}t + \sqrt{50}B \cos \sqrt{50}t$$

$$x_0 = 0 = A$$

$$v_0 = 10 = \sqrt{50}B$$

$$u(t) = \sqrt{2} (\cos(\sqrt{50}t))$$

2nd answer

$A = 0$	$w = \sqrt{\frac{k}{m}} = \sqrt{50}$
$B = \frac{10}{\sqrt{50}}$	$C = \sqrt{A^2 + B^2} = \sqrt{2}$
$\alpha = \tan^{-1}\left(\frac{B}{A}\right) = 0$	

**Nonhomogeneous Equations: The Methods of Undetermined
Coefficients and Variation of Parameters- HW Problems**

In problems 1-5 find a particular solution y_p of the given equation.

1. $y'' - y' - 2y = 4x^2$
2. $y'' + 4y' + 13y = 20e^{-x}$
3. $y' - 3y = -4xe^x$
4. $y'' - y' - 2y = -10 \cos(x)$
5. $y^{(3)} + y'' = 12x + 4.$

In problems 6-8 solve the initial value problem.

6. $y'' + y' - 2y = -10 \sin(x); \quad y(0) = 4, \quad y'(0) = 3.$
7. $y'' - 2y' + 2y = 2x + 1; \quad y(0) = 3, \quad y'(0) = 1.$
8. $y^{(4)} - y^{(3)} = 24x - 6;$
 $y(0) = 4, \quad y'(0) = 6, \quad y''(0) = 6, \quad y^{(3)} = -10.$

In problems 9-11 use the method of variation of parameters to find a particular solution to the differential equation.

9. $y'' - \frac{3}{x}y' + \frac{3}{x^2}y = \frac{1}{x}; \quad \text{where } y_1 = x \text{ and } y_2 = x^3 \text{ are solutions to}$
 $y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 0.$
10. $y'' - 2y' + y = \frac{e^x}{x}$
11. $y'' + 4y = \sin(2x).$

(In problems 1-5 find a particular solution y_p of the given equation)

1) $y'' - y' - 2y = 4x^2$

$f(x) = 4x^2$

$y_p = Ax^2$

$y'_p = 2Ax$

$y''_p = 2A$

$(2A) - (2Ax) - 2(Ax^2) = 4x^2$

$\cancel{2A}$

$(2A) - 2A(x) - 2A(x^2) = 4x^2$

$\frac{2A(1-x-x^2)}{(1-x-x^2)} = \frac{4x^2}{(1-x-x^2)}$

$\left(\frac{1}{2}\right) 2A = \frac{4x^2}{(1-x-x^2)} \left(\frac{1}{2}\right)$

$A = \frac{2x^2}{(1-x-x^2)}$

$y_p = \boxed{\frac{2x^4}{(1-x-x^2)}}$

$$2) y'' + 4y' + 13y = 20e^{-x}$$

$$f(x) = 20e^{-x}$$

$$y_p = Ae^{-x}$$

$$y'_p = -Ae^{-x}$$

$$y''_p = Ae^{-x}$$

$$(Ae^{-x}) + 4(-Ae^{-x}) + 13(Ae^{-x}) = 20e^{-x}$$

$$A(e^{-x} + 4e^{-x} + 13e^{-x}) = 20e^{-x}$$

$$3) y' - 3y = -4xe^x \quad | A(10e^{-x}) = \frac{20e^{-x}}{10e^{-x}}$$

$$f(x) = -4xe^x$$

$$y_p = Axe^x \quad | A = 2$$

$$y'_p = Ae^x + Axe^x \quad | \boxed{y_p = 2e^{-x}}$$

$$(Ae^x + Axe^x) - 3(Axe^x) = -4xe^x$$

$$Ae^x - 2A(Axe^x) = -4xe^x$$

$$\frac{(Ae^x)(1-2x)}{(1-2x)} = \frac{-4xe^x}{(1-2x)}$$

$$\left(\frac{1}{2}\right)(Ae^x) = \frac{4xe^x}{(1-2x)} \quad (+)$$

$$\boxed{y_p = \left(\frac{4x}{1-2x}\right)e^x}$$

$$A = \frac{4x}{1-2x}$$

$$4) y'' - y' - 2y = -10 \cos x$$

$$\left| \begin{array}{l} y_p = A \cos x + B \sin x \\ y'_p = -A \sin x + B \cos x \\ y''_p = -A \cos x - B \sin x \end{array} \right|$$

$$\underline{(-A \cos x - B \sin x)} - \underline{(-A \sin x + B \cos x)} - 2\underline{(A \cos x + B \sin x)} = -10 \cos x$$

$$(-A + B - 2A) \cos x + (-B - A - 2B) \sin x = -10 \cos x + 0 \sin x$$

$$\begin{aligned} -A + B - 2A &= -10 \\ -B - A - 2B &= 0 \end{aligned} \Rightarrow \begin{cases} B - 3A = -10 \\ -3B - A = 0 \end{cases}$$

$$B = -10 + 3A$$

$$-3(-10 + 3A) - A = 0$$

$$\boxed{y_p = 3 \cos x + 1 \sin x} \quad \begin{aligned} -3B - 3 &= 0 & 30 - 9A - A &= 0 \\ +3 +3 & & 30 - 10A &= 0 \end{aligned}$$

$$\frac{-3B}{3} = \frac{3}{-3}$$

$$\boxed{B = -1}$$

$$\frac{30}{10} = \frac{10A}{10}$$

$$\boxed{A = 3}$$

$$5) y''' + y'' = 12x + 4$$

$$y_p = Ax + B$$

$$y'_p = A$$

$$y''_p = 0$$

$$y'''_p = 0$$

$$0 = 12x + 4$$

$$\frac{4}{12} = x$$

$$y_p = A\left(\frac{4}{12}\right) + B$$

In problems 6-8 solve the initial value problem

$$6) y'' + y' - 2y = -10\sin(x); \quad y(0) = 4, \quad y'(0) = 3$$

$$y_p = A\sin(x) + B\cos(x)$$

$$y'_p = A\cos(x) - B\sin(x)$$

$$y''_p = -A\sin(x) - B\cos(x)$$

$$(-A\sin(x) - B\cos(x)) + (A\cos(x) - B\sin(x)) - 2(A\sin x + B\cos x) = -10\sin x$$

$$(-A - B - 2A)\sin x + (-B + A + B)\cos x = -10\sin x + 0\cos x$$

$$y_p = 10\cos(x)$$

$$\begin{cases} -B - 3A = -10 \\ A = 0 \end{cases}$$

$$\begin{aligned} A &= 0 \\ B &= 10 \end{aligned}$$

$$y'' + y' - 2y = 0$$

$$r^2 + r - 2 = 0 = (r+2)(r-1)$$

$$\frac{-(1) \pm \sqrt{(1)^2 - 4(1)(-2)}}{2(1)}$$

$$r = (-2, 1)$$

$$y_1 = C_1 e^{-2x} + C_2 e^{1x}$$

$$y(x) = C_1 e^{-2x} + C_2 e^{1x} + 10 \cos x$$

$$4 = y(0) = C_1 + C_2 + 10 \Rightarrow -6 = C_1 + C_2$$

$$3 = y'(0) = -2C_1 + C_2 - 10 \Rightarrow 13 = -2C_1 + C_2$$

$$y(x) = \left(\frac{17}{3}\right) e^{-2x} + \left(\frac{1}{3}\right) e^{1x} + 10 \cos x \quad C_1 = -6 - C_2$$

$$13 = -2(-6 - C_2) + C_2$$

$$13 = 12 + 3C_2$$

$$\frac{1}{3} = 3C_2$$

$$-\frac{1}{3} = C_1 + \frac{1}{3} \quad \frac{1}{3} = 3C_2$$

$$C_1 = \frac{1}{3}$$

$$\frac{1}{3} = C_2$$

$$7) y'' - 2y' + 2y = 2x + 1; \quad y(0) = 3, \quad y'(0) = 1$$

$$y_p = Ax + B$$

$$y'_p = A$$

$$y''_p = 0$$

$$-2(A) + 2(Ax + B) = 2x + 1$$

$$(-2A + 2B) + (2Ax) = 2x + 1$$

$$y_p = x + 2$$

$$\frac{2A}{2} = \frac{2}{2}$$

$$A = 1$$

$$y'' - 2y' + 2y = 0$$

$$\frac{-2}{+3} + \frac{2B}{+3} = \frac{1}{+3}$$

$$r^2 - 2r - 2 = 0$$

$$\frac{-2B}{2} = \frac{4}{2}$$

$$\frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = (1 \pm i)$$

$$B = 2$$

$$y_c = e^{ix} (\cos x + \sin x)$$

$$y = x + 2 + e^{ix} (\cos x + \sin x)$$

$$8) \quad y^{(4)} - y^{(3)} = 2x + b$$

$$r^4 - r^3 = 0$$

$$r^3(r-1) = 0$$

$$r^3 = 0 \quad r = 1$$

$$y_c = C_1 + C_2 x + C_3 x^2 + C_4 e^x$$

$$y_p = Ax + b$$

$$y'_p = A$$

$$y''_p = 0$$

$$y'''_p = 0$$

$$y^{(4)}_p = 0$$

$$y = -2 + 5x - 5x^2 + 6e^x$$

$$y_p = 0$$

$$C_1 = -2$$

$$y = C_1 + C_2 x + C_3 x^2 + C_4 e^x$$

$$C_2 = 5$$

$$C_3 = -5$$

$$\frac{b}{-b} = C_2 + C_3 x + b$$

$$0 = C_2 + 2C_3 x \quad \frac{-b}{2} = \frac{2C_3}{2}$$

$$C_2 = -2C_3 x$$

In problems 9-11 use the method of Variation of Parameters to find a particular solution to the differential equation.

9.) $y'' - \frac{3}{x}y' + \frac{3}{x^2}y = \frac{1}{x}$; where $y_1 = x$ & $y_2 = x^3$
 are solutions to $y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 0$

$$y_c = C_1 x + C_2 x^3$$

$$\begin{vmatrix} y_1 = x & y_2 = x^3 \\ y'_1 = 1 & y'_2 = 3x^2 \end{vmatrix}$$

$$W(y_1, y_2) = \det \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}$$

$$= 3x^3 - x^3 = \underline{\underline{2x^3}}$$

$$y_p = -x \left\{ \frac{x^3(\frac{1}{x})}{2x^3} dx + x^3 \right\} \frac{x(\frac{1}{x})}{2x^3} dx$$

$$y_p = -\frac{x}{2} \int \frac{1}{x} dx + \frac{x^3}{2} \int \frac{1}{x^3} dx$$

$$y_p = -\frac{x}{2} (\ln|x|) + \left(\frac{x^3}{2} \right) \frac{x^{-2}}{-2}$$

$$10.) y'' - 2y' + y = \frac{e^x}{x}$$

$$r^2 - 2r + 1 = 0$$

$$(r-1)(r-1)$$

$$r=1$$

$$y_c = C_1 e^x + C_2 x e^x$$

$$y_1 = e^x \quad y_2 = x e^x$$

$$y'_1 = e^x \quad y'_2 = e^x + x e^x$$

$$W(y_1, y_2) = \det \begin{vmatrix} e^x & x e^x \\ e^x & e^x + x e^x \end{vmatrix}$$

$$\cancel{(e^x)^2 + x(e^x)^2 - x(e^x)^2} \\ = (e^x)^2$$

$$y_p = -e^x \int \frac{(x e^x) \left(\frac{e^x}{x}\right)}{(e^x)^2} dx + x e^x \int \frac{\left(\frac{e^x}{x}\right) \left(\frac{e^x}{x}\right)}{(e^x)^2} dx$$

$$-e^x \int 1 dx + x e^x \int \frac{1}{e^x} dx$$

$$-e^x \int 1 dx + x e^x \int \frac{1}{x} dx$$

$$\boxed{y_p = -e^x(x) + x e^x(\ln(x))}$$

$$11) y'' + 4y = \sin(2x)$$

$$y'' + 4y = 0$$

$$\begin{array}{r} r^2 + 4 = 0 \\ -4 \quad -4 \end{array}$$

$$\sqrt{r^2} = \sqrt{-4}$$

$$r = \pm 2i$$

$$y_c = C_1 \sin(2x) + C_2 \cos(2x)$$

$$y_1 = \sin 2x \quad y_2 = \cos 2x$$

$$y'_1 = 2\cos 2x \quad y'_2 = -2\sin 2x$$

$$W(y_1, y_2) = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix}$$

$$= -2\sin^2(2x) - (2\cos^2(2x))$$

$$\sin^2 2x + \cos^2 2x = 1 \quad = -2(\sin^2(2x) + \cos^2(2x))$$

$$y_p = -\frac{\sin 2x}{2} \int \frac{\cos 2x (\sin 2x) dx}{(\sin^2 2x + \cos^2 2x)} + \frac{\cos 2x}{2} \int \frac{\sin 2x (\sin 2x) dx}{(\sin^2 2x + \cos^2 2x)}$$

$$y_p = \frac{\sin 2x}{2} \int (\cos 2x (\sin 2x) dx) + \frac{\cos 2x}{2} \int \sin 2x (\sin 2x) dx$$

$$u = \sin 2x$$

$$du = 2 \cos 2x$$

$$y_p = \frac{\sin 2x}{2} \int \frac{\cos 2x}{2 \cos 2x} u du + \frac{\cos 2x}{-2} \int \frac{1 - \cos(4x)}{2} dx$$

$$= \frac{\sin 2x}{4} \int u du + \frac{\cos 2x}{-4} \int 1 - \cos(4x) dx$$

$$= \frac{\sin 2x}{4} \int \frac{u^2}{2} du + \frac{\cos 2x}{-4} \int 1 dx - \left(\frac{\cos 2x}{-4} \right) \int \cos(4x) dx$$

$$y_p = \left(\frac{\sin 2x}{4} \right) \left(\frac{(\sin 2x)^2}{2} \right) + \left(\frac{\cos 2x}{-4} \right) (x) - \frac{\cos 2x}{-16} (\sin(4x))$$

Electrical Circuits- HW Problems

In problems 1 and 2 assume we have an RL circuit (ie there is no capacitor) where at time $t = 0$, $I(0) = 0$. Assume that $L = 5H$ and $R = 20\text{ohms}$.

1. If $E(t) = 60$ find $I(t)$. Find $\lim_{t \rightarrow \infty} I(t)$.
2. If $E(t) = 60 \sin(30t)$ find $I(t)$.

3. Assume we have an RC circuit (ie, $L = 0$) where the resistance is $R = 10\text{ohms}$ and the capacitance is $C = 0.01\text{Farads}$. Suppose the charge at time $t = 0$ is $Q(0) = 10\text{Coulombs}$ and $E(t) = 50\sin(20t)$. Find $Q(t)$ and the current $I(t)$.

In problems 4 and 5 we have an RLC circuit where $R, L, C, E(t), I(0)$, and $Q(0)$ are given. Find $I(t)$.

4. $L = 2H$, $R = 12\text{ohms}$, $C = 0.02F$, $E(t) = 100V$, $I(0) = 0$, $Q(0) = 4$
5. $L = 2H$, $R = 22\text{ohms}$, $C = \frac{1}{60}F$, $E(t) = 10e^{-t}$, $I(0) = 0$, $Q(0) = 0$.

In problem 1 & 2 assume we have an RL circuit (i.e there is no generator) where at time $t=0$, $I(0)=0$. Assume that $L=5H$ and $R=20\Omega$ ohms.

1.) If $E(t)=60$, find $I(t)$. Find $\lim_{t \rightarrow \infty} I(t)$.

12.) If $E(t)=60\sin(30t)$ find $I(t)$
 \therefore (Complementary solution is the same)

$$LI' + RI = E(t) : \quad E(t) = 60$$

$$5I' + 20I = 60 : \quad I_p = A \quad I' + 4I = 12$$

$$\frac{5(I' + 4I)}{5} = \frac{60}{5} : \quad I'_p = 0 \quad 0 + 4(A) = 12$$

$$I' + 4I = 12 : \quad I_p(t) = 3 \quad A = 3$$

$$\frac{dI}{dt} = -4I \quad \left(\frac{dI}{I}\right) : \quad I(t) = C_1 e^{-4t} + 3$$

$$\int \frac{1}{I} dI = \int -4 dt : \quad I(0) = 0 = C_1 e^{(-4 \cdot 0)} + 3$$

$$\ln|I| = -4t + C_1 : \quad C_1 = -3$$

$$\ln|I| = (-4t + C_1) : \quad I(t) = -3e^{-4t} + 3$$

$$I_c(t) = C_1 e^{-4t} : \quad \lim_{t \rightarrow \infty} -3e^{-4t} + 3 = 3$$

$$I_c(t) = C_1 e^{-4t}$$

$$E(t) = 60\sin(30t)$$

$$I_p = A \sin(30t) + B \cos(30t)$$

$$I'_p = 30A \cos(30t) - 30B \sin(30t)$$

$$I(t) = C_1 e^{-4t} + 3 + (20k \cos(30t) - 20B \sin(30t)) + 4(45 \sin(30t) + B \cos(30t)) \\ = 60 \sin(30t) \\ = 60 \sin(30t)$$



continued

$$\begin{cases} 30A + 4B = 0 \\ -30B + 4A = 60 \end{cases}$$

Solve A + B

$$B = -\frac{30A}{4}$$

$$B = \left(-\frac{450}{229} \right)$$

$$-30\left(-\frac{30A}{4}\right) + 4A = 60$$

$$\left(\frac{900}{4}\right)A + 4A = 60$$

$$A = \frac{60}{229}$$

$$I_p(t) = \frac{60}{229} \sin(30t) - \frac{450}{229} \cos(30t)$$

$$I_c(t) = C_1 e^{(-4t)}$$

$$I(t) = C_1 e^{(-4t)} + \frac{60}{229} \sin(30t) - \frac{450}{229} \cos(30t)$$

$$I(0) = 0 = C_1 - \frac{450}{229} = C_1 = \frac{450}{229}$$

Ans

$$I(t) = \left(\frac{450}{229}\right) e^{(-4t)} + \frac{60}{229} \sin(30t) - \frac{450}{229} \cos(30t)$$

3.) Assume we have an RC circuit ($i_C, L=0$) where the resistance is $R=10 \text{ Ohms}$ & the capacitance is $C=.01 \text{ Farads}$. Suppose the charge at time $t=0$ is $Q(0)=10 \text{ Coulombs}$ & $E(t)=50 \sin(20t)$. Find $Q(t)$ & the current $I(t)$.

$$R \frac{dQ}{dt} + \frac{1}{C} Q = E(t)$$

$$R=10, C=.01$$

$$E(t)=50 \sin(20t)$$

$$10 \frac{dQ}{dt} + \frac{1}{0.01} Q = 50 \sin(20t)$$

$$\frac{10 \left(\frac{dQ}{dt} + 10Q \right)}{10} = \frac{50 \sin(20t)}{10}$$

$$Q' + 10Q = 5 \sin(20t); \quad \text{Solve for } B$$

$$Q' + 10Q = 0$$

$$-10Q$$

$$Q' = -10Q$$

$$\left(\frac{dt}{Q}\right) \frac{dQ}{dt} = -10Q \left(\frac{dt}{Q}\right)$$

$$\frac{dQ}{Q} = -10dt$$

$$\int \frac{1}{Q} dQ = -10 \int dt$$

$$\ln|Q| = (-10t) + C_1$$

$$e^{\ln|Q|} = e^{(-10t) + C_1}$$

$$Q_c(t) = C_1 e^{-10t}$$

$$(20A \cos(20t) - 20B \sin(20t)) + 10(A \sin(20t) + B \cos(20t))$$

$$\underline{(20A+B)\cos(20t) + (-20B+10A)\sin(20t)} = 5 \sin(20t)$$

$$\begin{cases} -20B + 10A = 5 \\ 20A + B = 0 \end{cases}$$

Solve for A

$$B = -20A$$

$$-20(-20A) + 10A = 5$$

$$400A + 10A = 5$$

$$A = \frac{1}{410}$$

$$\frac{10A}{410} = \frac{5}{410}$$

$$B = \frac{-10}{41}$$

$$Q_c(t) = \frac{1}{41} 5 \sin(20t) - \frac{10}{41} \cos(20t)$$

In problems 9 & 15 we have an RLC circuit where R , L , C , $E(t)$, $I(0)$, & $Q(0)$ are given. Find $I(t)$. ; $I(0) = 0 \Rightarrow C_1 = 0$

$$4.) L = 2H, R = 12 \text{ ohms}, C = 0.02F, I'(0)$$

$$E(t) = 100V, I(0) = 0, Q(0) = 4, L I'(0) + R I(0) + \frac{1}{C} Q(0) = E(t)$$

$$2I'' + 12I' + \frac{1}{0.02}I = E'(t), 2(12) + 0 + \frac{4}{0.02} = 100$$

$$2I'' + 12I' + \frac{1}{0.02}I = 0$$

$$2I'' + 12I' + 50I = 0$$

$$2(I'' + 6I' + 25I) = 0$$

$$I'' + 6I' + 25I = 0$$

$$r^2 + 6r + 25 = 0$$

$$r = \frac{-6 \pm \sqrt{(6)^2 - 4(1)(25)}}{2(1)} = -3 \pm 4i$$

$$I(t) = e^{-3t}(C_1 \cos(4t) + C_2 \sin(4t))$$

$$I'(t) = -3e^{-3t}(C_1 \cos(4t) + C_2 \sin(4t)) + e^{-3t}(-4C_1 \sin(4t) + 4C_2 \cos(4t))$$

$$(1) 2(I'(0)) = \frac{2498}{25}\left(\frac{1}{2}\right)$$

$$I'(0) = \frac{1249}{25}$$

$$\frac{1249}{25} = I'(0) = -3C_1 + 4C_2$$

$$(2) \frac{1249}{25} = 4C_2\left(\frac{1}{4}\right)$$

$$C_2 = \frac{1249}{100}$$

$$I(t) = e^{-3t}\left(\frac{1249}{100}\right) \sin(4t)$$

In problems 9 & 5 we have an RLC circuit where $R, L, C, E(t), I(0)$, & $Q(0)$ are given. Find $I(t)$.

5.) $L = 2H, R = 2\Omega$, $C = \frac{1}{60} F$

$$E(t) = 10e^t V, I(0) = 0, Q(0) = 0$$

$$LI'' + RI' + \frac{1}{C}I = E'(t)$$

$$2I'' + 2I' + 60I = -10e^t$$

$$I(0) = 0 = I_1 + I_2 \quad \underline{I_1 = I_2}$$

$$\frac{2I'(0)}{2} = \frac{10e^t}{2}$$

$$I'(0) = \underline{5} = -5I_1 - 6I_2$$

$$\frac{2(I'' + II + 30I)}{2} = -10e^t$$

$$I'' + II + 30I = -5e^t$$

$$r^2 + 11r + 30 = 0$$

$$(r+5)(r+6) = 0$$

$$r = -5 \quad r = -6$$

$$I(t) = I_1 e^{-5t} + I_2 e^{-6t}$$

$$I'(t) = -5I_1 e^{-5t} - 6I_2 e^{-6t}$$

$$5 = -5I_2 - 6I_2$$

$$\boxed{\frac{5}{-11} = -\cancel{5} \frac{I_2}{-11}}$$

$$\boxed{I(t) = \left(\frac{5}{-11}\right) e^{-5t} + \left(\frac{5}{-11}\right) e^{-6t}}$$