

In Problems 1 & 3 find the Laplace transform of the given function.

1.) $x(t) = t^2 e^{3t}$

$$\int_0^\infty e^{-st} e^{3t} t^2 dt$$

$$\left(\frac{2}{s-3} \right) \left(\frac{1}{s-3} \right) \left(\frac{1}{(s-3)^2} \right) \left(e^{-st + (3t)} \right) \Big|_0^\infty \Rightarrow (1)$$

$$\int_0^\infty \frac{e^{-st + (3t)}}{u} \frac{t^2}{u} dt \quad v = -\frac{1}{(s-3)} e^{-st + (3t)}$$

$$du = 2t dt$$

$$= \frac{1}{(s-3)(s-3)(s-3)}$$

$$+ \left. t^2 \left(-\frac{1}{(s-3)^2} e^{-st + (3t)} \right) \right|_0^\infty - \int_0^\infty -\frac{1}{(s-3)} e^{-st + (3t)} 2t dt$$

$d[t^2 e^{3t}] = \frac{2}{(s-3)^3}$
Answer

$$\frac{1}{s-3} \int_0^\infty \frac{e^{-st + (3t)}}{u} \frac{t}{u} dt \quad v = -\frac{1}{(s-3)} e^{-st + (3t)}$$

$$du = dt$$

$$\frac{1}{s-3} \left(t \left(-\frac{1}{(s-3)} e^{-st + (3t)} \right) \Big|_0^\infty + \int_0^\infty \frac{1}{(s-3)} e^{-st + (3t)} dt \right)$$

$$\left(\frac{1}{s-3} \right) \left(\frac{1}{s-3} \right) \int_0^\infty e^{-st + (3t)} dt$$

In Problems 1 & 3 find the Laplace transform of the given function. 2.) $f(t) = e^{\frac{t}{2}} \cos(5t)$

$$\int_0^\infty e^{-st} e^{\frac{t}{2}} \cos(5t) dt$$

$$\int_0^\infty \frac{e^{-st + (\frac{t}{2})}}{D} \frac{(os(5t))}{I} dt$$

| | |
|--|------------------------|
| $e^{-st + (\frac{t}{2})}$ | $\cos(5t)$ |
| $(-s + \frac{1}{2}) e^{-st + (\frac{t}{2})}$ | $\frac{\sin(5t)}{5}$ |
| $(-s + \frac{1}{2})^2 e^{-st + (\frac{t}{2})}$ | $-\frac{\cos(5t)}{25}$ |

$$\begin{aligned} & \frac{25}{25 + 6s + \frac{1}{4}} \left[\left(e^{-st + (\frac{t}{2})} \left(\frac{\sin(5t)}{5} \right) + (s + \frac{1}{2}) e^{-st + (\frac{t}{2})} \frac{\cos(5t)}{25} \right) \Big|_0^\infty \right] \\ & \quad \times \left(-\frac{(s + \frac{1}{2})}{25} \right) \\ & \quad \boxed{-\frac{(s + \frac{1}{2})}{25 + 6s + \frac{1}{4}}} = L[e^{\frac{t}{2}} \cos(5t)] \end{aligned}$$

∴ Answer

$$\int_0^\infty e^{-st + (\frac{t}{2})} \cos(5t) dt = e^{-st + (\frac{t}{2})} \left(\frac{\sin(5t)}{5} \right) + (6s + \frac{1}{2}) e^{-st + (\frac{t}{2})} \frac{\cos(5t)}{25} \Big|_0^\infty$$

$$1 + \frac{(-s + \frac{1}{2})^2}{25} \Big|_0^\infty = \frac{25 + 6s + \frac{1}{4}}{25} \xrightarrow{\text{Reciprocal}} \frac{-(-s + \frac{1}{2})^2}{25} \Big|_0^\infty$$

$$\frac{25 + 6s + \frac{1}{4}}{25} \Big|_0^\infty + \frac{(-s + \frac{1}{2})^2}{25} \Big|_0^\infty$$

$$3.) X(t) = e^{5t} \sin(2t)$$

$$\int_0^{\infty} e^{-st} (e^{st} \sin(2t)) dt$$

$$\int_0^{\infty} \frac{e^{-st+st}}{D} \frac{\sin(2t)}{I} dt$$

| | D | I |
|---|--------------------|-----------------------|
| + | e^{-st+5t} | $\sin(2t)$ |
| - | $(s+5)e^{-st+5t}$ | $-\frac{\cos(2t)}{2}$ |
| + | $(-s+5)e^{-st+5t}$ | $-\frac{\sin(2t)}{4}$ |

$$\begin{aligned} & \int_0^{\infty} e^{-st+(5t)} \sin(2t) dt = \\ & -e^{-st+(5t)} \left(\frac{\cos(2t)}{2} \right) + (-s+5) e^{-st+(5t)} \left(\frac{\sin(2t)}{4} \right) - \int_0^{\infty} (-s+5)^2 e^{-st+(5t)} \left(\frac{\sin(2t)}{4} \right) dt \\ & -\frac{(-s+5)^2}{4} \int_0^{\infty} e^{-st+(5t)} \sin(2t) dt \end{aligned}$$

Adding to other side of Eq : $1 + \frac{(-s+5)^2}{4} = \frac{4+(-s+5)^2}{4}$

Multiply reciprocal to other side & Evaluate

$$\begin{aligned} & \frac{4}{4+(-s+5)^2} \left(-e^{-st+(5t)} \left(\frac{\cos(2t)}{2} \right) + (-s+5) e^{-st+(5t)} \left(\frac{\sin(2t)}{4} \right) \right) \Big|_0^{\infty} \\ & \frac{4}{4+(-s+5)^2} \left(-\left(-1 \left(\frac{1}{2} \right) \right) \right) = \boxed{\frac{2}{4+(-s+5)^2}} \text{ Answer} \end{aligned}$$

In problems 1-3 find the inverse Laplace Transform of the given function.
use partial fractions when appropriate.

4.) $Q(s) = \frac{1}{s^2 + 6s + 9}$

use partial fractions

$$\frac{1}{(s+3)^2} = \frac{A}{(s+3)} + \frac{B}{(s+3)^2} = \frac{0}{s+3} + \frac{1}{(s+3)^2}$$

$$1 = A(s+3) + B$$

$$s = -3$$

$$s = 1$$

$$\boxed{1=6}$$

$$\begin{matrix} 1 = 4A + 1 \\ -1 \end{matrix}$$

$$\frac{0}{4} = \frac{4A}{4}$$

$$\boxed{0=4}$$

funny (sorta redundant)

$$\begin{aligned} L^{-1}\left[\frac{1}{s^2 + 6s + 9}\right] &= L^{-1}\left[0 + \frac{1}{(s+3)^2}\right] \\ &= 0 + e^{-3}t^1 \end{aligned}$$

$$= e^{-3}t^1 \leftarrow \underline{\text{Answer}}$$

In problems 1-3 find the inverse Laplace Transform of the given function.
Use partial fractions when appropriate.

$$5.) \mathcal{F}(s) = \frac{s+3}{s^2+4s+8}$$

$$\downarrow$$

$$\left(\frac{s+3}{(s^2+4s+4)+4} \right)$$

$$\mathcal{L}^{-1} \left[\frac{s+3}{(s+2)^2+4} \right] = e^{-3t} (\cos 2t +$$

$$6.) \mathcal{F}(s) = \frac{6s^2-s-6}{s^3-s^2-6s}$$

$$\frac{6s^2-s-6}{s(s^2-s-6)} = \frac{A}{s} + \frac{B}{(s-3)} + \frac{C}{(s+2)}$$

$$6s^2-s-6 = A(s-3)(s+2) + B(s)(s+2) + C(s)(s-3)$$

$$s=3$$

$$\frac{45}{15} = 15b$$

$$s=-2$$

$$\frac{20}{10} = 10c$$

$$s=0$$

$$\frac{-6}{-6} = -1$$

$$\mathcal{L}^{-1} \left[\left(\frac{s}{s^2-4} \right) + (-1) \left(\frac{1}{s} \right) \right]$$

$$= (\cosh(kt)) + (-1)$$

Answer

$$7.) \mathcal{F}(s) = \frac{4}{s^3-4s}$$

$$\frac{4}{s(s^2-4)} = \frac{As+b}{s^2-4} + \frac{C}{s}$$

$$4 = (As+b)s + C(s^2-4)$$

$$4 = As^2 + bs + C(s^2-4)$$

$$4 = s^2(A+C) + bs - 4C$$

$$0 = A+C \quad A = 1$$

$$4 = -4C \quad C = -1$$

$$0 = B$$

$$\mathcal{L}^{-1} \left[1 \left(\frac{1}{s} \right) + 3 \left(\frac{1}{s-3} \right) + 2 \left(\frac{1}{s+2} \right) \right]$$

$$= (1) + 3(e^{3t}) + 2(e^{-2t})$$

Answer

In problems 1-3 find the inverse Laplace Transform of the given function.
Use partial fractions when appropriate.

$$8.) F(s) = \frac{1}{s^4 - 2s^2 + 1}$$

$$U = s^2, \frac{1}{U^2 - 2U + 1}$$

$$= \frac{1}{(U-1)(U-1)}$$

$$= \frac{1}{(s^2-1)(s^2-1)}$$

$$= \frac{1}{(s+1)(s-1)(s+1)(s-1)}$$

$$= \frac{1}{(s+1)^2(s-1)^2}$$

$$= \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s-1)} + \frac{D}{(s-1)^2}$$

$$1 = A(s+1)(s-1)^2 + B(s-1)^2 + C(s-1)(s+1)^2 + D(s+1)^2$$

$$\begin{matrix} s^2 + 2s + 1 \\ (s-1)(s-1) \end{matrix}$$

$$s^2 - 2s + 1$$

$$s = 1,$$

$$\frac{1}{4} = \frac{4}{4}$$

$$D = \frac{1}{4}$$

$$B = \frac{1}{4}$$

$$\mathcal{L}^{-1} \left[\frac{1}{4} \left(\frac{1}{s+1} \right) + \frac{1}{4} \left(\frac{1}{(s+1)^2} \right) - \frac{1}{4} \left(\frac{1}{s-1} \right) + \frac{1}{4} \left(\frac{1}{(s-1)^2} \right) \right]$$

$$= \underbrace{\frac{1}{4} (e^{-t})}_{\text{Answer}} + \underbrace{\frac{1}{4} (e^{-t} t)}_{-} - \underbrace{\frac{1}{4} (e^{t})}_{-} + \underbrace{\frac{1}{4} (e^{t} t)}_{-} - \frac{3}{4} = -A$$

$$\therefore 0 = A + C$$

$$\therefore 0 = -A + B + C + D$$

$$A = \frac{1}{4}$$

$$-A = C$$

$$\therefore 0 = -A + (\frac{1}{4})$$

$$-\frac{1}{4} = -A + C$$

$$-\frac{1}{4} = 2C$$

$$\begin{aligned} 1 &= As^3 - As - As^2 + A + s^2B - 2Bs + B + s^3C - Cs \\ &\quad + s^2C - C + s^2D + 2Ds + D \end{aligned}$$

$$-\frac{1}{4} = C$$

In problems 9-11 solve the initial value problems with Laplace transforms.

$$9.) x'' + 2x' + 6x = 6, \quad x(0) = x'(0) = 0$$

$$x = f(t)$$

$$\mathcal{L}[f(t)] = f(s)$$

$$\mathcal{L}[f'(t)] = s f(s) - f(0)$$

$$\mathcal{L}[f''(t)] = s^2 f(s) - s f(0) - f'(0)$$

$$[s^2 f(s) - s f(0) - f'(0)] + 2[s f(s) - f(0)] + 6[f(s)] = \mathcal{L}[6]$$

$$s^2 f(s) - s f(0) - f'(0) + 2[s f(s) - 0] + 6[f(s)] = \frac{6}{s}$$

$$s^2 f(s) + 2s f(s) + 6 f(s) = \frac{6}{s}$$

$$\frac{(s^2 + 2s + 6)f(s)}{s^2 + 2s + 6} = \frac{6}{s}$$

$$= \frac{6}{s(s^2 + 2s + 6)} = \frac{A}{s} + \frac{bs + c}{(s^2 + 2s + 6)}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s} + \frac{-s-2}{s^2+2s+6}\right]$$

$$\mathcal{L}^{-1}\left[\frac{1}{s} + \frac{-s-2}{(s+1)^2+5}\right]$$

$$\mathcal{L}^{-1}\left[\frac{1}{s} + \frac{-K(s+1)+1}{(s+1)^2+5}\right]$$

$$\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s+1}{(s+1)^2+5} - \frac{1}{(s+1)^2+5}\right]$$

$$0 = A + b$$

$$\begin{array}{|c|} \hline s=0 \\ \hline A=1 \\ \hline \end{array}$$

$$0 = 2A + c$$

$$\begin{array}{|c|} \hline B=-1 \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline C=-2 \\ \hline \end{array}$$

$$6 = As^2 + 2As + 6A + bs^2 + cs$$

$$6 = A(s^2 + 2s + 6) + bs + c(s)$$

In problems 9-11 solve the initial value problems with Laplace transforms.

10.) $y'' + 4y' + 8y = e^{-t}$, $y(0) = y'(0) = 0$
 $y = f(t)$

$$\mathcal{L}[f(t)] = f(s)$$

$$\mathcal{L}[f'(t)] = sf(s) - f(0)$$

$$\mathcal{L}[f''(t)] = s^2 f(s) - sf(0) - f'(0)$$

$$[s^2 f(s) - sf(0) - f'(0)] + 4[sf(s) - f(0)] + 8[f(s)] = \mathcal{L}[e^{-t}]$$

$$s^2 f(s) - 0 - 0 + 4sf(s) - 0 + 8f(s) = \frac{1}{s+1}$$

$$s^2 f(s) + 4sf(s) + 8f(s) = \frac{1}{s+1}$$

$$f(s)[s^2 + 4s + 8] = \frac{1}{s+1}$$

$$f(s) = \frac{1}{(s+1)(s^2 + 4s + 8)} = \frac{1}{(s+1)} + \frac{bs+c}{(s^2 + 4s + 8)}$$

$$0 = A + B$$

$$0 = 4A + B + C$$

$$\begin{matrix} 1 \\ s = -1 \end{matrix}$$

$$\begin{matrix} \frac{1}{3} = \frac{3A}{3} \\ A = \frac{1}{3} \end{matrix}$$

$$\boxed{-\frac{1}{3} = B}$$

$$\frac{4}{3} - \frac{1}{3} + C$$

$$\boxed{-1 = C}$$

$$0 = \frac{2}{3} + C$$

$$\mathcal{L}^{-1} \left[\frac{1}{3} \left(\frac{1}{s+1} \right) + \left(\frac{-\frac{1}{3}s - 1}{s^2 + 4s + 8} \right) \right]$$

$$\frac{-\frac{1}{3}(s+3)}{(s+2)^2 + 4} = \frac{-\frac{1}{3}(s+2+1)}{(s+2)^2 + 4}$$

$$\mathcal{L}^{-1} \left[\frac{1}{3} \left(\frac{1}{s+1} \right) + \frac{\frac{1}{3}(s+2) - \frac{1}{3}}{(s+2)^2 + 4} \right]$$

$$\mathcal{L}^{-1} \left[\frac{1}{3} \left(\frac{1}{s+1} \right) - \frac{\frac{1}{3}(s+2+1)}{(s+2)^2 + 4} \right]$$

$$\mathcal{L}^{-1} \left[\frac{1}{3} \left(\frac{1}{s+1} \right) - \frac{\frac{1}{3}(s+2)}{(s+2)^2 + 4} - \frac{1}{3} \left(\frac{1}{(s+2)^2 + 4} \right) \right]$$

$$\begin{matrix} \frac{1}{3}(e^{-t}) - \frac{1}{3}(e^{-2t}(\cos 2t)) - \frac{1}{3}(e^{-2t} \sin 2t) \end{matrix}$$

$$1 = As^2 + 4As + 8A + bs^2 + bs + Cs + C$$

\sum Answer

In problems 9-11 solve the initial value problems with Laplace transforms.

$$11.) y'' + 2y' + 2y = 2(\cos(t) + \sin(t)), y(0) = y'(0) = 0 \quad L^{-1}\left[\frac{\frac{2}{5}s + \frac{4}{5}}{(s^2+1)} + \frac{-\frac{2}{5}s - \frac{6}{5}}{(s+1)^2+1}\right] \xrightarrow{\text{continued}} \rightarrow$$

$$Y = f(t)$$

$$\mathcal{L}[f(t)] = f(s)$$

$$\mathcal{L}[f'(t)] = sf(s) - f(0)$$

$$\mathcal{L}[f''(t)] = s^2 f(s) - sf(0) - f'(0)$$

$$[s^2 f(s) - sf(0) - f'(0)] + 2[sf(s) - f(0)] + 2[f(s)] = \mathcal{L}[2(\cos(t) + \sin(t))]$$

$$s^2 f(s) - 0 - 0 + 2sf(s) + 2f(s) = 2\left(\frac{s}{s^2+1}\right) + \left(\frac{1}{s^2+1}\right)$$

$$f(s)(s^2 + 2s + 2) = \frac{2s}{s^2+1} + \frac{1}{s^2+1}$$

$$f(s) = \frac{2s}{(s^2+1)(s^2+2s+2)} + \frac{1}{(s^2+1)(s^2+2s+2)}$$

$$\frac{2s}{(s^2+1)(s^2+2s+2)} = \frac{As+B}{(s^2+1)} + \frac{(s+D)}{(s^2+2s+2)}$$

$$2s = As + b(s^2 + 2s + 2) + (s + D)(s^2 + 1)$$

$$2s = As^3 + 2As^2 + 2As + Bs^2 + B + s^3 + Cs + Ds^2 + D$$

$$L^{-1}\left[\frac{\frac{2}{5}s + \frac{4}{5}}{(s^2+1)} + \frac{-\frac{2}{5}s - \frac{6}{5}}{(s+1)^2+1}\right] \xrightarrow{\text{continued}} \frac{-\frac{1}{5}(s+1) - \frac{6}{5}}{(s+1)^2+1}$$

$$L^{-1}\left[\frac{\frac{2}{5}s}{s^2+1} + \frac{4}{5}\left(\frac{1}{s^2+1}\right) - \frac{2}{5}\left(\frac{s+1}{(s+1)^2+1}\right) - \frac{6}{5}\left(\frac{1}{(s+1)^2+1}\right)\right] \\ = \left\{ \frac{2}{5}(\cos t) + \frac{4}{5}(\sin t) - \frac{2}{5}(e^{-t}\cos t) - \frac{6}{5}(e^{-t}\sin t) \right\}$$

(Half the answer)

$$\frac{2}{5} = A$$

$$\frac{4}{5} = 2A$$

$$-\frac{8}{5} = D$$

$$2 = 2A + 2B - A$$

$$0 = 2A + B - 2B$$

$$C = -\frac{2}{5}$$

$$B = \frac{4}{5}$$

$$\therefore 0 = 2B + D$$

$$2 = A + 2B$$

$$S: 2 = 2A + 2B + C$$

$$0 = 2A - B$$

$$S^2: 0 = A + C$$

$$2 - 2B = A$$

$$S^3: 0 = 2A + B + D$$

$$0 = 2(2 - 2B) - B$$

$$-\frac{9}{5} = -\frac{5B}{5}$$

$$0 = 4 - 7B - B$$

In problems 9-11 solve the initial value problems with Laplace transforms.

11.) $y'' + 2y' + 2y = 2(\cos(t) + \sin(t)), y(0) = y'(0) = 0$ Now we're Answering the rest.

$$y = f(t)$$

$$\mathcal{L}[f(t)] = f(s)$$

$$\mathcal{L}[f'(t)] = sf(s) - f(0)$$

$$\mathcal{L}[f''(t)] = s^2 f(s) - sf(0) - f'(0)$$

$$[s^2 f(s) - sf(0) - f'(0)] + 2[sf(s) - f(0)] + 2[f(s)] = \mathcal{L}[2(\cos(t) + \sin(t))]$$

$$s^2 f(s) - 0 - 0 + 2sf(s) + 2f(s) = 2\left(\frac{s}{s^2+1}\right) + \left(\frac{1}{s^2+1}\right)$$

$$f(s)(s^2 + 2s + 2) = \frac{2s}{s^2+1} + \frac{1}{s^2+1}$$

Full
Answer

$$f(s) = \frac{2s}{(s^2+1)(s^2+2s+2)} + \frac{1}{(s^2+1)(s^2+2s+2)}$$

$$1 = \frac{As+B}{(s^2+1)} + \frac{(s+1)}{(s^2+2s+2)}$$

$$1 = As^3 + 2As^2 + 2As + Bs^2 + 2Bs + 2 + (s^3 + s^2 + s + 1)$$

$$\mathcal{L}^{-1}\left[\frac{-\frac{2}{5}s + \frac{1}{5}}{(s^2+1)} + \frac{\frac{2}{5}s + \frac{3}{5}}{(s+1)^2+1}\right]$$

$$\mathcal{L}^{-1}\left[-\frac{2}{5}\left(\frac{s}{s^2+1}\right) + \frac{1}{5}\left(\frac{1}{s^2+1}\right) + \frac{\frac{2}{5}(s+1) + \frac{1}{5}}{(s+1)^2+1}\right] \leftarrow \text{all about factoring and extracting 1}$$

$$\mathcal{L}^{-1}\left[-\frac{2}{5}\left(\frac{s}{s^2+1}\right) + \frac{1}{5}\left(\frac{1}{s^2+1}\right) + \frac{\frac{2}{5}\left(\frac{s+1}{(s+1)^2+1}\right) + \frac{1}{5}\left(\frac{1}{(s+1)^2+1}\right)}{(s+1)^2+1}\right]$$

$$\text{Second part of the answer. } \left\{ \frac{-2}{5}(\cos t) + \frac{1}{5}(\sin t) + \frac{1}{5}(e^t \cos t) + \frac{1}{5}(e^t \sin t)\right\}$$

$$\boxed{\frac{3}{5}=1}$$

$$0 = 2A + B + 1$$

$$0 = 2A + B + (1 - 2B)$$

$$0 = 2A + 2B - A$$

$$\boxed{C = \frac{2}{5}}$$

$$\boxed{A = -\frac{1}{5}}$$

$$\therefore 0 = 2A + B + 1$$

$$\therefore 1 = 2B + 1$$

$$\therefore 0 = 2A + 2B + C$$

$$\therefore 0 = A + C$$

$$-1 = 2A - B$$

$$0 = A + 2B$$

$$-2B = A$$

$$-1 = 2(-2B) - B$$

$$\boxed{B = \frac{1}{5}}$$

{Full Answer for
Question (11)}

$$\left[\frac{2}{5}(\cos t) + \frac{4}{5}(\sin t) - \frac{1}{5}(e^t \cos t) - \frac{6}{5}(e^t \sin t) - \frac{2}{5}(\cos t) + \frac{1}{5}(\sin t) + \frac{2}{5}(e^t \cos t) + \frac{1}{5}(e^t \sin t) \right]$$

{Simplified
version}

$$\underline{\underline{\sin t - (e^t \sin t)}}$$