In problems 1-5 solve the initial value problem using Laplace transforms.

$f(t) = y$

1.) $y' - 4y = 0; \quad y(0) = 3$

$\mathcal{L}[f(t)] = f(s)$

$\mathcal{L}[f'(t)] = sf(s) - f(0)$

$\mathcal{L}[f''(t)] = s^2 f(s) - sf(0) - f'(0)$

$\mathcal{L}[y'] - \mathcal{L}[4y] = \mathcal{L}[0]$

$\mathcal{L}[y'] - 4\mathcal{L}[y] = 0$

$sf(s) - f(0) - 4(f(s)) = 0$

$sf(s) \cancel{-3} - 4f(s) = 0$

$\quad \cancel{+3} \qquad +3$

$sf(s) - 4f(s) = 3$

$$\frac{f(s)(s\cancel{-4})}{(s\cancel{-4})} = \boxed{\left(\frac{3}{s-4}\right)}$$

$x(t) = \mathcal{L}^{-1}\left[\frac{3}{s-4}\right] = 3\left[\mathcal{L}^{-1}\left[\frac{1}{s-4}\right]\right] = 3\left[e^{4t}\right]$

2.) $y'' + 9y = 0; \quad y(0) = 3, \quad y'(0) = 6 \qquad f(t) = y$

$(s^2 f(s) - s f(0) - f'(0)) + 9(f(s)) = \mathcal{L}[0]$

$s^2 f(s) - s(3) - 6 + 9 f(s) = 0$

$+6 \qquad +6$

$s^2 f(s) - 3s + 9 f(s) = 6$

$+3s \qquad +3s$

$s^2 f(s) + 9 f(s) = 6 + 3s$

$$\frac{f(s)\cancel{(s^2+9)}}{\cancel{(s^2+9)}} = \boxed{\frac{6+3s}{(s^2+9)}}$$

$$\mathcal{L}^{-1}\left[\frac{6+3s}{s^2+9}\right] = 6\sin(3t) + 3\cos(3t)$$

$f(t) = x$

3.) $x'' + 2x' - 3x = 0; \quad x(0) = 5, \; x'(0) = -3$

$s^2 f(s) - s f(0) - f'(0) + 2(s f(s) - f(0)) - 3(f(s)) = \mathcal{L}[0]$ | $\mathcal{L}[f(t)] = f(s)$

$s^2 f(s) - s(5) - (-3) + 2s f(s) - 2(5) - 3f(s) = 0$ | $\mathcal{L}[f'(t)] = s f(s) - f(0)$

$+5s \quad -3 \quad +10$ | $\mathcal{L}[f''(t)] = s^2 f(s) - s f(0) - f'(0)$

$s^2 f(s) + 2s f(s) - 3f(s) = 7 + 5s$

$$\frac{f(s)(s^2 + 2s - 3)}{(s^2 + 2s - 3)} = \boxed{\frac{7 + 5s}{(s^2 + 2s - 3)}}$$

$$(s+3)(s-1)\left(\frac{7 + 5s}{(s+3)(s-1)}\right) = \left(\frac{A}{(s+3)} + \frac{B}{(s-1)}\right)(s+3)(s-1)$$

$7 + 5s = A(s-1) + B(s+3)$

put $s = 1$ put $s = -3$

$\frac{12}{4} = \frac{4B}{4}$ $7 - 15 = -4A$

$\frac{-8}{-4} = \frac{-4A}{-4}$

$\boxed{B = \frac{12}{4}}$ $\boxed{\frac{8}{4} = A}$

$$\frac{8}{4}\mathcal{L}^{-1}\left[\frac{1}{(s+3)}\right] + \frac{12}{4}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right]$$

$$x(t) = \boxed{\frac{8}{4}\left[e^{-3t}\right] + \frac{12}{4}\left[e^{1t}\right]}$$

↑

Answer

4.) $x'' + 9x = \sin(2t)$; $x(0) = x'(0) = 0$

$$\mathcal{L}[x''] + 9\mathcal{L}[x] = \mathcal{L}[\sin(2t)]$$

$$(s^2 f(s) - sf(0) - f'(0)) + 9(f(s)) = \frac{2}{s^2+4}$$

$$s^2 f(s) - s(0) - (0) + 9(f(s)) = \frac{2}{s^2+4}$$

$$s^2 f(s) + 9f(s) = \frac{2}{s^2+4}$$

$$\frac{f(s)(\cancel{s^2+9})}{(\cancel{s^2+9})} = \frac{\left(\frac{2}{s^2+4}\right)}{s^2+9} = \boxed{\frac{2}{(s^2+4)(s^2+9)}}$$

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partial fractions

$$s^4 + 4s^2 + 9s^2 + 36$$

$$s^4 + 13s^2 + 36$$

$$2\left[\mathcal{L}^{-1}\left[\frac{1}{(s^2+2^2)(s^2+3^2)}\right]\right]$$

$\downarrow$

$$\frac{1}{(s^2+2^2)(s^2+3^2)} \Rightarrow \frac{As+B}{(s^2+2^2)} + \frac{Cs+D}{(s^2+3^2)}$$

$$2\left[\mathcal{L}^{-1}\left[\frac{\left(\frac{-3}{5}\right)s + \left(\frac{11}{3}\right)}{s^2+4} + \frac{\left(\frac{8}{5}\right)s + \left(\frac{-8}{3}\right)}{s^2+9}\right]\right]$$

$$2\left(\frac{11}{9}(\sin(2t)) - \frac{3}{5}\cos(2t) + \left(-\frac{8}{9}\sin(3t)\right) + \frac{8}{5}(\cos(3t))\right)$$

$\uparrow$

Answer

$$1 = As + B(s^2+3^2) + (Cs+D)(s^2+2^2)$$

$$1 = As(s^2+9) + B(s^2+9) + Cs(s^2+4) + D(s^2+4)$$

$$1 = As^3 + 9As + s^2B + 9B + Cs^3 + 4Cs + Ds^2 + 4D$$

$$1 = s^3(A+C) + s^2(B+D) + s(9A+4C) + (9B+4D)$$

Cleaner version

$$x(t) = \frac{11}{6}(\sin(2t)) - \frac{3}{5}\cos(2t) + \left(-\frac{8}{9}\sin 3t\right) + \left(\frac{8}{5}\cos 3t\right)$$

sys1

$1 = A + C \quad 1 = 9A + 4C$

$1 = B + D \quad 1 = 9B + 4D$

sys2

$1 - D = B$

$\boxed{\frac{11}{3} = B}$

$1 = B - \frac{8}{3}$

$1 = 9(1-D) + 4D$

$\frac{-8}{3} = \frac{3D}{3}$

$\boxed{D = \frac{-8}{3}}$

$1 = -\frac{3}{5} + C$

$\boxed{\frac{8}{5} = C}$

$\boxed{-\frac{3}{5} = A}$

$3 = -5A$

$1 = 9A + 4C$

$-9A \quad -9A$

$\frac{1-9A}{4} = \frac{4C}{4}$

$4(1) = \left(A + \left(\frac{1-9A}{4}\right)\right)$

$4 = 4A + (1-9$

$4 = 1 - 5A$

$-1 \quad -1$

5.) $x'' + 6x' + 8x = 8;\quad x(0) = x'(0) = 0$

$\mathcal{L}[x''] + 6\mathcal{L}[x'] + 8\mathcal{L}[x] = \mathcal{L}[8]$

$\mathcal{L}[f(t)] = f(s)$

$\mathcal{L}[f'(t)] = sf(s) - f(0)$

$\mathcal{L}[f''(t)] = s^2 f(s) - sf(0) - f'(0)$

$s^2 f(s) - sf(0) - f'(0) + 6(sf(s) - f(0)) + 8(f(s)) = \frac{8}{s}$

$s^2 f(s) - \cancel{s(0)} - \cancel{0} + 6sf(s) - \cancel{6(0)} + sf(s) = \frac{8}{s}$

$\frac{f(s)\cancel{(s^2 + 6s + s)}}{\cancel{(s^2 + 6s + s)}} = \frac{\left(\frac{8}{s}\right)}{(s^2 + 6s + s)} = \frac{8}{s(s^2 + 6s + s)} = \boxed{\frac{8}{s^3 + 6s^2 + s^2}}$

partial fractions

$8\mathcal{L}^{-1}\left[\frac{1}{s^3 + 6s^2 + s^2}\right]$

$\frac{1}{s^2(s+6+1)} = \frac{1}{s^2(s+7)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s+7)}$

$8\mathcal{L}^{-1}\left[-\frac{1}{49}\left[\frac{1}{s}\right] + \frac{1}{7}\left[\frac{1}{s^2}\right] + \frac{1}{49}\left[\frac{1}{s+7}\right]\right]$

$1 = A(s)(s+7) + B(s+7) + C(s^2)$

$x(t) = 8\left[-\frac{1}{49}[1] + \frac{1}{7}[t^1] + \frac{1}{49}[e^{-7t}]\right]$

put $s=0$

$\frac{1}{7} = \frac{7B}{7},\ \boxed{B = \frac{1}{7}}$

put $s=-7$

$\frac{1}{49} = \frac{49C}{49},\ \boxed{C = \frac{1}{49}}$

Put $s=1$

$1 = A(1)(1+7) + \frac{1}{7}(8) + \frac{1}{49}(1)$

$1 = 8A + \frac{8}{7} + \frac{1}{49}$

$-\frac{8}{7} - \frac{1}{49} \qquad -\frac{8}{7} - \frac{1}{49}$

$\frac{\left(-\frac{8}{49}\right)}{8} = \frac{8A}{8}$

$- \qquad = A$