

1. Let  $p$  be any real number. Solve the problem

$$y' + \frac{y}{t^p} = 0, \quad y(1) = 1.$$

**Note:** Your answer will look different when  $p = 1$  and otherwise.

2. Find the general solution of the equation

$$y'' + y = 0.$$

**Note:** There should be two arbitrary constants.

3. Find a condition on  $\lambda$  that ensures that every solution of the equation

$$y'' + \lambda y = 0,$$

is periodic with period  $2\pi$ , meaning  $y(t) = y(t + 2\pi)$  for all  $t$ .

4. Find a function  $f = f(t, y)$  so that the solution to the first-order ODE

$$y' = f(t, y), \quad y(0) = y_0$$

“breaks down in finite time” for some  $y_0$ , meaning there is a time  $T > 0$  so that  $\lim_{t \rightarrow T} y(t) = \infty$ . Explain your answer. Why does this not contradict the existence and uniqueness theorem?

**Hint:** Try  $f(t, y) = y^2$  and use “separation of variables”.

5. Find a function  $f = f(t, y)$  so that the initial-value problem

$$y' = f(t, y), \quad y(0) = 0$$

has not just one, but *two* solutions; that is, this ODE exhibits non-uniqueness. Why does your example not contradict the existence and uniqueness theorem?

**Hint:** Try  $f(t, y) = t\sqrt{y}$ . Then  $y(t) \equiv 0$  is a solution. Can you find another?

$$y' + \frac{y}{t^p} = 0$$

$$1) \frac{dy}{dt} + \frac{y}{t^p} = 0$$

$$\frac{dy}{dt} + \frac{y}{t^p} = 0$$

$$\left(\frac{dy}{y}\right) \frac{dy}{dt} = -\frac{y}{t^p} \left(\frac{dt}{y}\right)$$

$$-\int \frac{1}{y} dy = \int \frac{1}{t^p} dt$$

$$(-1)(-\ln|y|) = \left(\frac{t^{-p+1}}{-p+1} + c\right)(-1)$$

$$\ln|y| = \left(-\left(\frac{t^{-p+1}}{-p+1}\right) - c\right)$$

General Solution:  $y = A e^{\left(-\left(\frac{t^{-p+1}}{-p+1}\right)\right)}$

$$y(1) = A e^{\left(-\left(\frac{1^{-p+1}}{-p+1}\right)\right)} = 1$$

$$A = \frac{1}{e^{\left(-\left(\frac{1^{-p+1}}{-p+1}\right)\right)}} = e^{\left(-\left(\frac{1^{-p+1}}{-p+1}\right)\right)}$$

Particular Solution:  $y(t) = \frac{e^{\left(-\left(\frac{t^{-p+1}}{-p+1}\right)\right)}}{e^{\left(-\left(\frac{1^{-p+1}}{-p+1}\right)\right)}}$

$$2) y'' - y = 0 \quad y = e^{rt}$$

$$e^{rt}(r^2 - 1) = 0 \quad \text{solve the characteristic Eq}$$

$$r^2 - 1 = 0$$

$$(r+1)(r-1) = 0$$

$$r = -1, 1$$

$$\text{gen sol: } y = C_1 e^{-t} + C_2 e^t$$

$$3) \left\{ \begin{array}{l} \text{Condition on } \lambda \text{ so every solution of } y'' + \lambda y = 0 \\ \text{is periodic with period } 2\pi, \text{ meaning} \\ y(t) = y(t + 2\pi) \text{ for all } t \end{array} \right\}$$

The question is asking for a periodic solution with period  $2\pi$ . Thus implies the answer must repeat every  $2\pi$ . After solving the characteristic Equation the gen sol is  $y = e^0 (C_1 \sin(\sqrt{\lambda} t) + C_2 \cos(\sqrt{\lambda} t))$

$$- (0) \pm \sqrt{(0)^2 - 4(1)(\lambda)}$$

In order to maintain a period of  $2\pi$ ,  $\lambda$  must equal 1. This is because the parent function  $\cos x + \sin x$  inherently has the period  $2\pi$ . If  $\lambda$  was anything else it would shift the period of the parent function out of  $2\pi$ .

So the condition is  $\lambda = 1$  which means  $\sqrt{\lambda} = 1$

4.) A function  $f = (t, y)$ , that is a solution to the first order ODE

$\{y' = f(t, y), y(0) = y_0\}$  is the function  $y^2$ .

This is because, indeed, at some finite time when I separate the variables for  $y' = y^2$ , then apply its initial condition. I get a Particular Solution which if taken the limit of its variable,  $t$ , to some finite number  $\frac{1}{y_0}$ , I get a discontinuity to infinity ( $\infty$ ).

This can be shown mathematically as:

Solving as  
an IVP  $\Rightarrow$

$$y' = y^2$$

$$| y_0 = y(0) = \frac{1}{-c}$$

$$\left(\frac{dy}{y^2}\right) \frac{dy}{dt} = y^2 \left(\frac{dt}{y^2}\right) | (-c) y_0 = \frac{1}{-c} (-c)$$

$$\int \frac{1}{y^2} dy = \int dt \quad | \quad c = -\frac{1}{y_0}$$

$$\int y^{-2} dy = \int dt$$

particular solution:

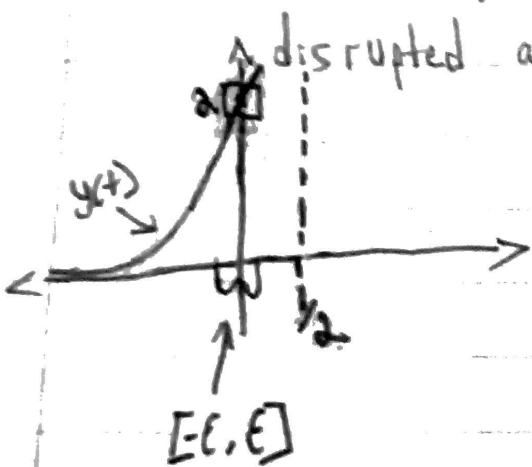
$$(-1) \frac{y^{-1}}{-1} = (t + c)(-1) \quad | \quad y(t) = \frac{1}{-t + \frac{1}{y_0}}$$

$$\left(\frac{y}{(-t-c)}\right) \frac{1}{y} = -t - c \left(\frac{y}{(-t-c)}\right) | \text{so if you} \quad \lim_{t \rightarrow \frac{1}{y_0} - t + \frac{1}{y_0}} \frac{1}{y_0} = \infty$$

$$y(0) = y_0, \quad y = \frac{1}{(-t - c)} = \text{gen sol} \quad |$$

The fact that the particular solution of  $y' = y^2$  acted in a way that diverges to infinity ( $\infty$ ) when approaching  $\frac{1}{y_0}$ , it does NOT mean it contradicts the existence & uniqueness theorem.

The theorem states that the function must be continuous within an interval  $[-\epsilon, \epsilon]$  "near" a point of solution which in our case is  $(0, y_0)$ , and given that the interval  $[-\epsilon, \epsilon]$  has the capability being very small the existence & uniqueness theorem is not



disrupted at all by  $y(t) = \frac{1}{-t + \frac{1}{y_0}}$

Ex:

Let's say

$$y_0 = 2$$

So point  $(0, 2)$  is a solution that's inside a boxed interval of the  $y(t)$  function, that's continuous!!!

The discontinuity did not disrupt the interval, so the theorem holds!!!

5.) The function  $f(t, y) = t\sqrt{y}$  is a function that satisfies the IVP  $\{y' = f(t, y), y(0) = 0\}$  & it contains two solutions that exhibit non-uniqueness. When you find the particular solution for  $y' = t\sqrt{y}$  you see it produces 2 solutions & more due to the function being a higher order polynomial to the 4<sup>th</sup> degree. So, your evaluation of the function at  $t=0, y(t)$  maps to 0. Thus the conditions stated are met. It doesn't contradict the existence & uniqueness theorem because it's non-uniqueness is not a criteria for contradiction it's just a statement of this is shown mathematically as: the function's behavior, for which the theorem identifies (non-uniqueness).

$$y' = t\sqrt{y}$$

$$y = \frac{t^4}{16} + \left(\frac{c}{2}\right)^2 : \text{Gen solution}$$

$$\left(\frac{dt}{\sqrt{y}}\right) \frac{dy}{dt} = t\sqrt{y} \left(\frac{dt}{\sqrt{y}}\right)$$

$$y(0) = \left(\frac{0^4}{4} + \frac{c}{2}\right)^2 = 0$$

$$\int \frac{1}{\sqrt{y}} dy = \int t dt$$

$$y(0) = \left(\frac{c}{2}\right)^2 = 0$$

$$\int y^{-\frac{1}{2}} dy = \frac{t^2}{2} + c$$

$$\sqrt{\left(\frac{c}{2}\right)^2} = \sqrt{0}$$

$$\left(\frac{1}{2}\right) \frac{y^{\frac{1}{2}}}{\frac{1}{2}} = \left(\frac{t^2}{2} + c\right) \left(\frac{1}{2}\right)$$

$$(2) \frac{c}{2} = 0 (2)$$

$$(\sqrt{y})^2 = \left(\frac{t^2}{4} + \frac{c}{2}\right)^2$$

$$[c=0]$$

$$\text{particular sol. } y = \frac{t^4}{16}$$

Another function that fits the conditions of the IVP is  $t^{-\frac{1}{3}} \sqrt[3]{y}$ . It basically produces similar behavior of  $t \sqrt{y}$  in that 2 solutions can be found within the ODE  $y' = t^{-\frac{1}{3}} \sqrt[3]{y}$ . In addition, both of its evaluations at  $t=0$  maps to  $y(t)=0$ .

Shown here mathematically:

$$y' = t^{-\frac{1}{3}} \sqrt[3]{y}$$

$$\left( \frac{dy}{dt} \right) \frac{dy}{dt} = t^{-\frac{1}{3}} \sqrt[3]{y} \left( \frac{dy}{dt} \right)$$

$$\int y^{\frac{1}{3}} dy = \int t^{-\frac{1}{3}} dt$$

$$\left( \frac{2}{3} \right) \frac{y^{\frac{2}{3}}}{\frac{2}{3}} = \left( \frac{t^{\frac{2}{3}}}{\frac{2}{3}} + C \right) \left( \frac{2}{3} \right)$$

$$\sqrt[3]{y^{\frac{2}{3}}} = \sqrt[3]{t^{\frac{2}{3}} + \frac{2C}{3}}$$

$$y^{\frac{2}{3}} = t^{\frac{2}{3}} + \sqrt[3]{\frac{2C}{3}}$$

$$\text{gen sol: } y(t) = \pm \sqrt{t^{\frac{2}{3}} + \sqrt[3]{\frac{2C}{3}}}$$

$$0 = y(0) = \pm \sqrt[3]{\frac{2C}{3}}$$

$$0 = y(0) = \pm \sqrt[6]{\frac{2C}{3}}$$

2-cases

$$(0)^6 = -\left( \sqrt[6]{\frac{2C}{3}} \right)^6$$

$$0 = -\frac{2C}{3}$$

$$\underline{\underline{C=0}}$$

$$(0)^6 = \left( \sqrt[6]{\frac{2C}{3}} \right)^6$$

$$0 = \frac{2C}{3}$$

$$\underline{\underline{C=0}}$$

particular sol:  $y(t) = \pm t$

$$y(t) = \pm \sqrt{t^2}$$