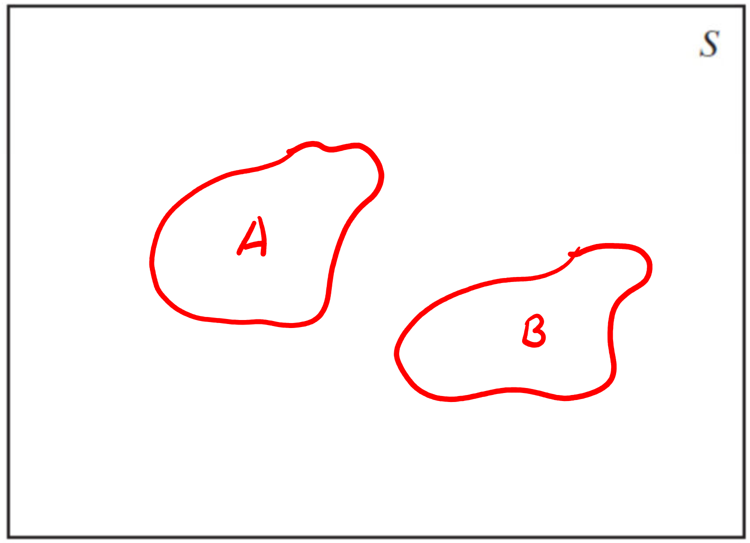
Math 3501 - Probability and Statistics I

1.1 - Properties of Probability

A and B are mutually exclusive events if $A \cap B = \emptyset$.



Let:

* A be the event "sum of the two die is 6"
* B be the event "sum of the two die is 7"

$A = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

$B = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

A and B are disjoint sets.

Since $A \cap B = \phi$ then A and B are mutually exclusive

$S = \{(i,j) : i,j \in \{1,2,3,4,5,6\}\}$

EXAMPLE:

Roll two 6-face die.

Sample space

$$S = \{ (1,1), (1,2), (1,3), (1,4), \ldots \\ (2,1), (2,2), \ldots \\ \vdots \\ (6,1), (6,2) \ldots (6,6) \}$$

$$\underbrace{|S| = 36}_{\text{cardinality of the set } S \text{ is } 36}$$

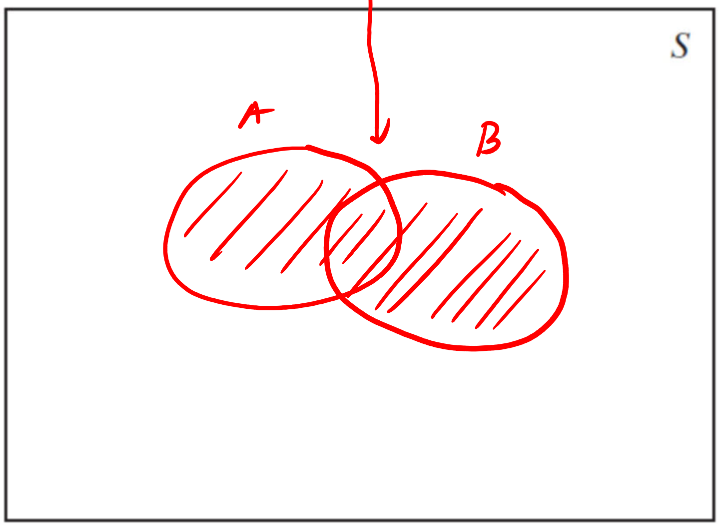
\# of elements of S

Coin toss $\{H\}$ $\{T\}$

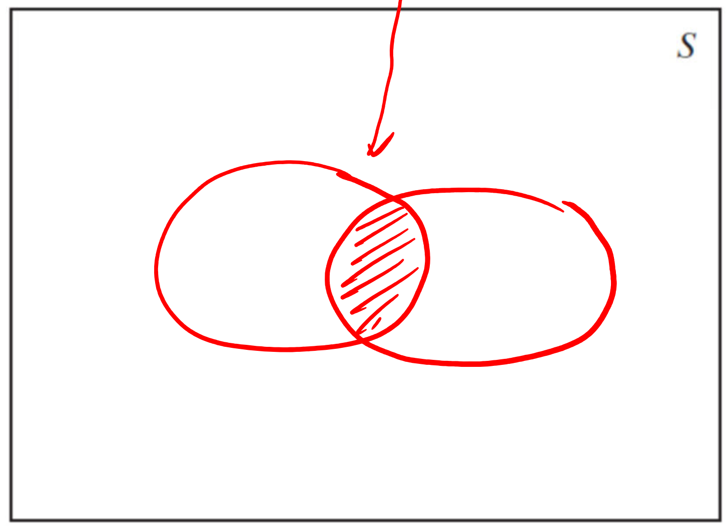
Roll a die $\{1\}, \{2\}, \{3\}, \ldots$

Commutative Laws

$$A \cup B = B \cup A$$

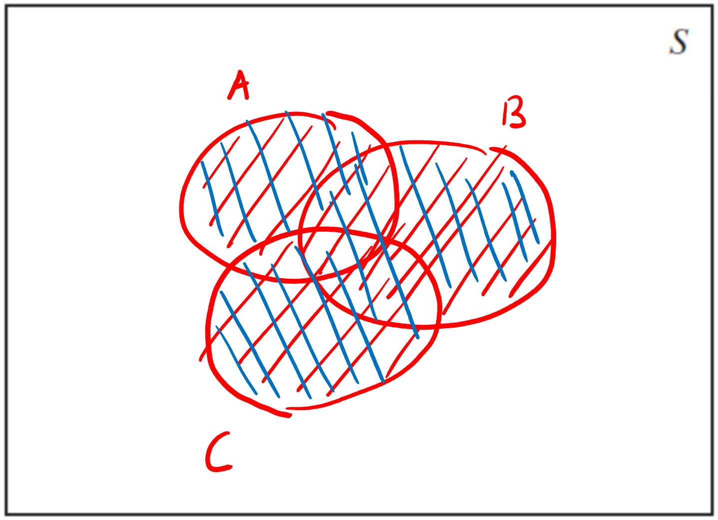


$$A \cap B = B \cap A$$

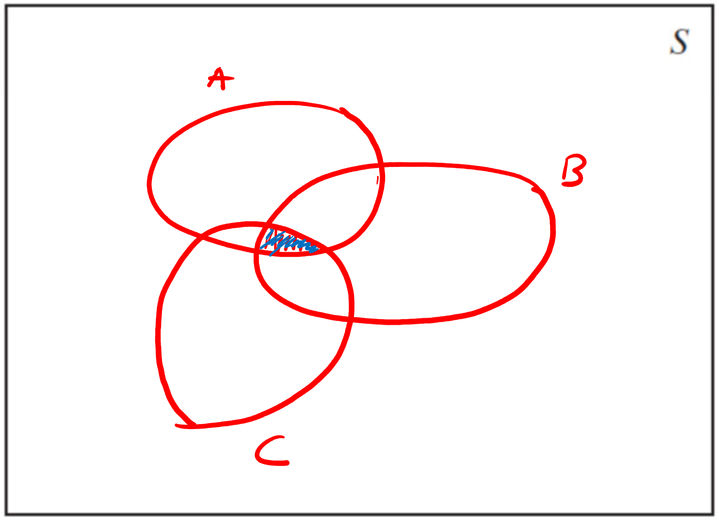


Associative Laws

$(A \cup B) \cup C = A \cup (B \cup C)$

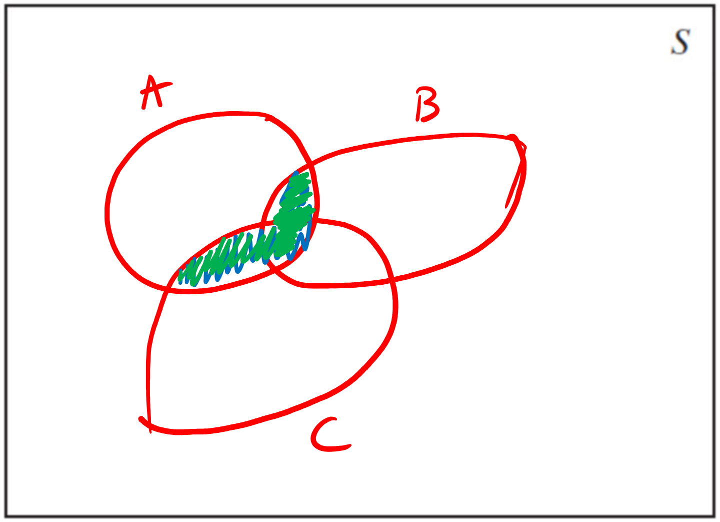


$(A \cap B) \cap C = A \cap (B \cap C)$

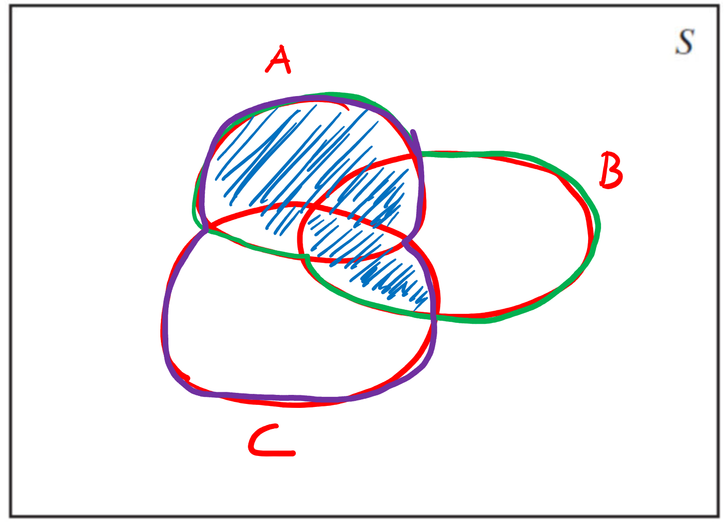


Distributive Laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

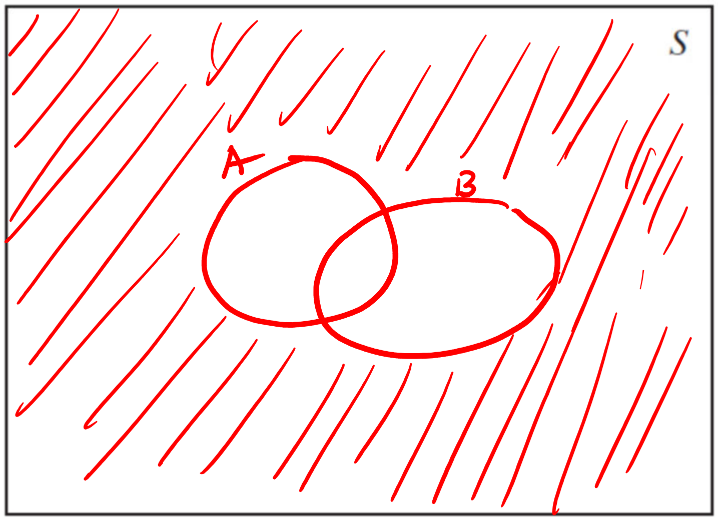


$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

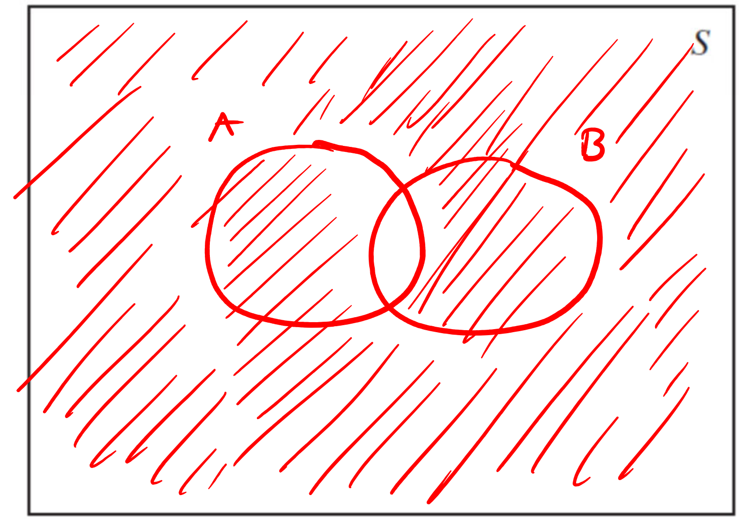


De Morgan's Laws

$$(A \cup B)' = A' \cap B'$$

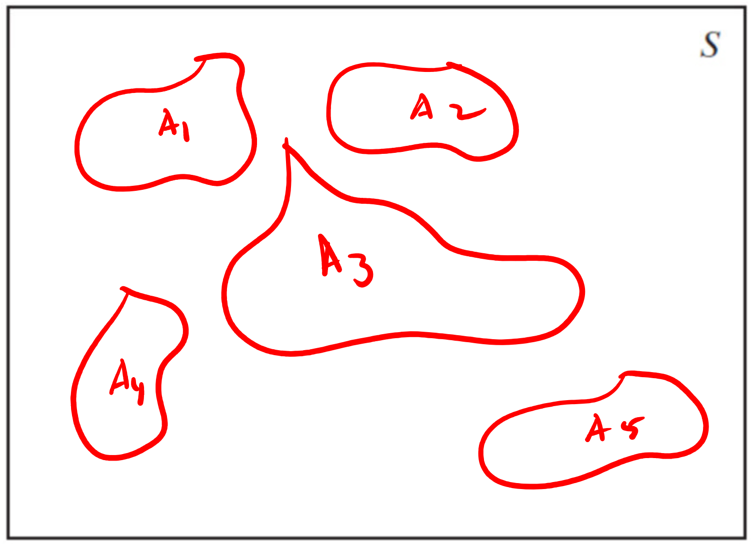


$$(A \cap B)' = A' \cup B'$$



Mutually exclusive events

$A_1, A_2, \ldots, A_k$ are *mutually exclusive events* if $A_i \cap A_j = \emptyset$ whenever $i \neq j$. for $i,j \in \{1, \ldots, k\}$



$A_1, A_2, \ldots, A_5$ are mutually exclusive

Exhaustive events

$A_1, A_2, \ldots, A_k$ are *exhaustive events* if $A_1 \cup A_2 \cup \cdots \cup A_k = S$

EXAMPLE:

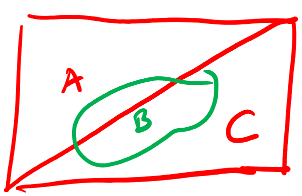
Roll two six-sided die

A = "sum is odd"

B = "sum is a multiple of 3"

C = "sum is even"

A, B, C are exhaustive



If $A_1, A_2, \ldots, A_k$ are mutually exclusive and exhaustive events, then

$$\rightarrow \quad A_i \cap A_j = \emptyset \quad \text{whenever} \quad i \neq j$$
$$\rightarrow \quad A_1 \cup A_2 \cup \cdots \cup A_k = S$$

EXAMPLE :

Roll two six-faced dice.

Consider the events

A_i = "Sum of the two dice is i"

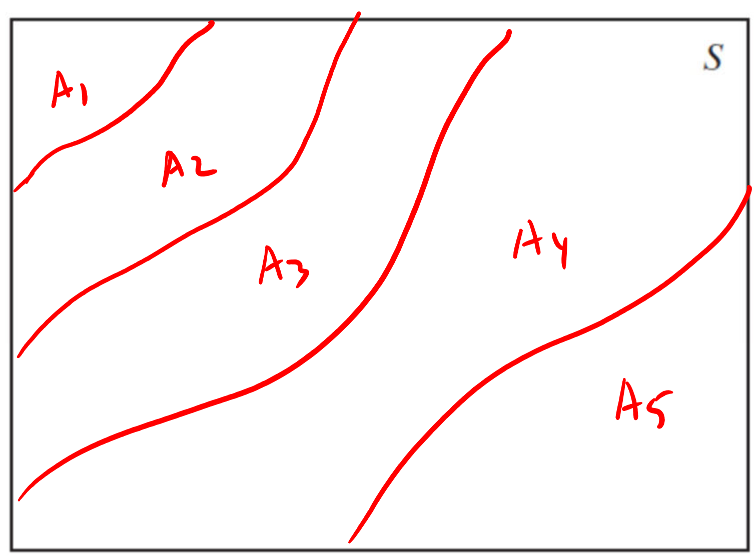
$i = 2, 3, \ldots, 12$

$A_2, A_3, A_4, \ldots, A_{12}$

are mutually exclusive

and

exhaustive events.



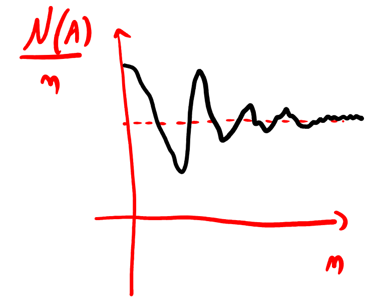
Empirical definition of probability

To access the probability of a certain event A occurring when performing a given random experiment:

1) Repeat the experiment a number of times: say, n times.

2) Count the number of times that the event A actually occurred throughout these n repetitions:

- this number is called the *frequency of event A* and is denoted by $\mathcal{N}(A)$.



3) Evaluate the ratio $\mathcal{N}(A)/n$, called the *relative frequency of event A* in these n repetitions of the experiment.

4) Define the probability p of the event A as the limit of the relative frequency of event A as the number of repetitions n increases to ∞.

Note:

- simulation can sometimes be used to assign probability empirically
- not always practical

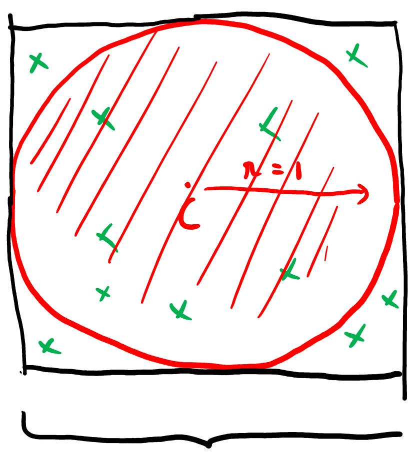
Note: sometimes intuition can be used to assign probability

Example

A circle of radius 1 is inscribed on a square.

If a point is selected at random from such square, assign a probability to the event that the point is also inside the circle.

S



side of square is $l = 2$

A is event
"point picked at random on the square lies inside the circle"

Area of square is $l^2 = 4$
Area of circle is $\pi r^2 = \pi$

We may think of the probability of A as being the ratio $\dfrac{\text{Area of circle}}{\text{Area of square}} = \dfrac{\pi}{4}$

Axiomatic definition of probability

We would like for the definition of probability to be consistent with any intuition gained from the empirical definition:

In particular, we observe that:

1) the relative frequency $\dfrac{\mathcal{N}(A)}{n}$ is always nonnegative;

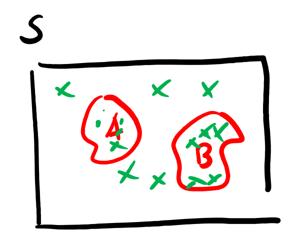
2) if $A = S$, then the outcome will always belong to S, and thus

$\mathcal{N}(S) = n$

$$\frac{\mathcal{N}(S)}{n} = 1 \, .$$

3) if A and B are two mutually exclusive events, then

$$\frac{\mathcal{N}(A \cup B)}{n} = \frac{\mathcal{N}(A)}{n} + \frac{\mathcal{N}(B)}{n} \, .$$



$\mathcal{N}(A \cup B) = \mathcal{N}(A) + \mathcal{N}(B)$

Definition (Axiomatic definition of probability)

Probability is a real-valued set function P that assigns, to each event A in the sample space S, a number $P(A)$, called *the probability of the event* A, such that the following properties are satisfied:

$A_1, A_2, A_3, \ldots$ mutually exclusive

(a) $\boxed{P(A) \geq 0}$

(b) $\boxed{P(S) = 1}$

(c) if $A_1, A_2, A_3, \ldots$ are events such that $A_i \cap A_j = \emptyset$ whenever $i \neq j$, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_k) = P(A_1) + P(A_2) + \cdots + P(A_k)$$

for each positive integer k, and

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots$$

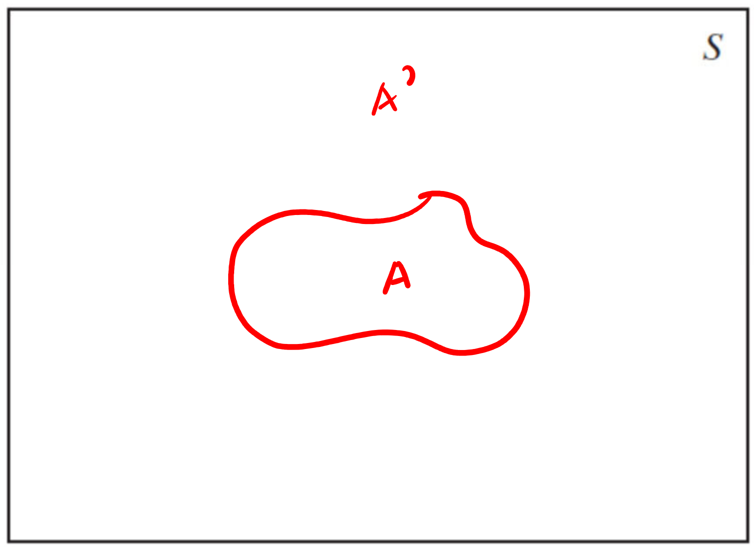
for an infinite, but countable, number of events.

Properties of probability

Theorem (very useful)

For each event A, it holds that

$$\boxed{P(A') = 1 - P(A)} \ .$$



Proof:

$$\underbrace{S = A \cup A' \quad \text{and} \quad A \cap A' = \phi}_{\text{A and A' are mutually exclusive and exhaustive}}$$

$$\underbrace{P(S)}_{1} = \underbrace{P(\overbrace{A \cup A'}^{S})}_{P(A) + P(A')}$$

$$\Rightarrow P(A') = 1 - P(A)$$

Example

Knowing that $P(A') = 0.65$, find $P(A)$.

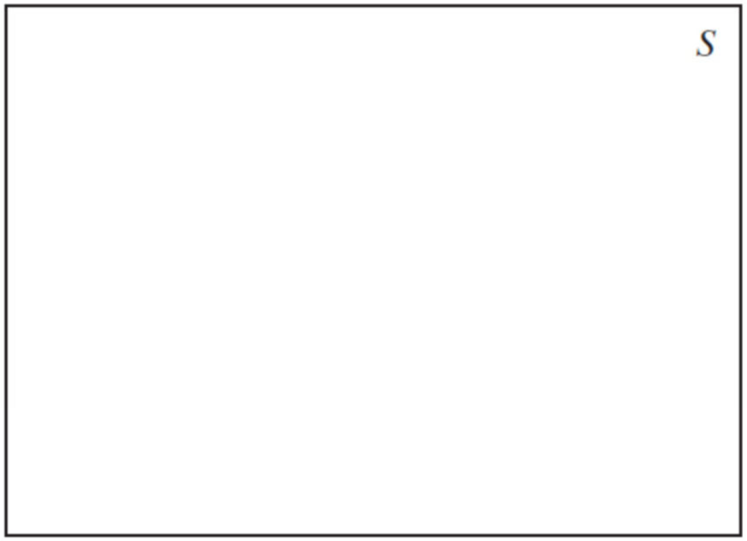
Since $P(A') = 1 - P(A)$

we get $0.65 = 1 - P(A)$

that is $P(A) = 1 - 0.65 = 0.35$

Theorem

$$P(\emptyset) = 0 \ .$$



Proof:

Note that

$$\phi = S'$$

$$P(\phi) = P(S')$$

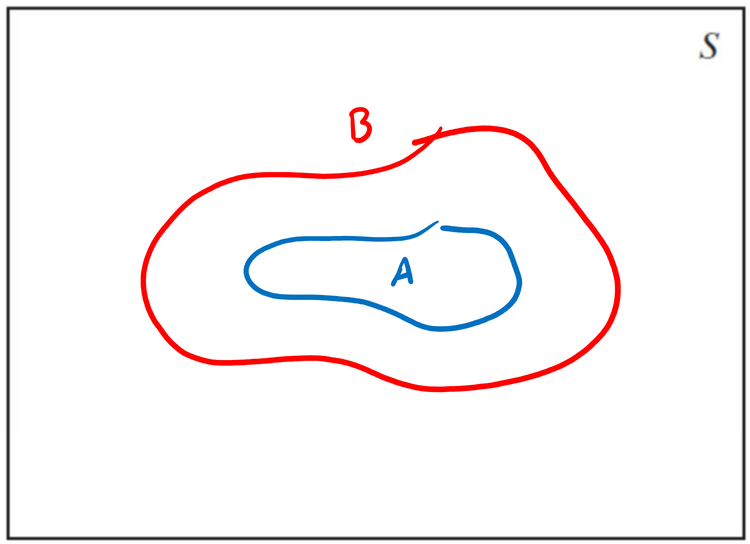
$$= 1 - \underbrace{P(S)}_{1}$$

$$= 1 - 1$$

$$= 0$$

Theorem

If events A and B are such that $A \subset B$, then $P(A) \leq P(B)$.



Theorem

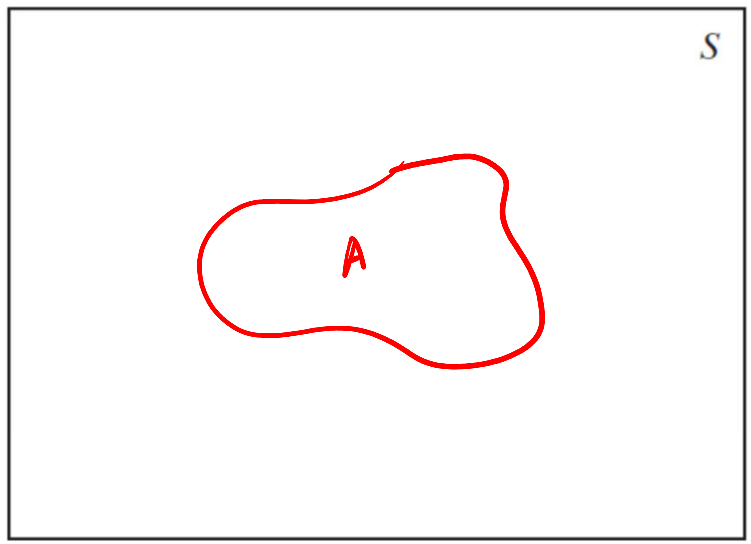
For each event A, we have $P(A) \leq 1$.

Proof :

Since $A \subset S$

$P(A) \leq \underbrace{P(S)}_{1}$

$P(A) \leq 1$



Theorem

If A and B are any two events, then

not necessarily mutually exclusive

subtract size of the intersection to avoid "counting" it twice"

$$P(A \cup B) = \underbrace{P(A)}_{\text{blue region}} + \underbrace{P(B)}_{\text{green region}} - P(A \cap B)\ .$$