# Math 3501 - Probability and Statistics I

## 3.2 - The exponential, gamma, and chi-square distributions

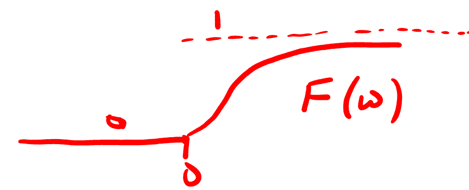
our focus in MATH 3501

special case of gamma distribution which will be covered in MATH 4501

# Waiting time until first occurrence of Poisson process

Let $W$ denote the waiting time until the first occurrence of a Poisson process with mean number of occurrences $\lambda$ in a unit interval. → $\lambda$ is the same parameter as in the Poisson distribution

Then $W$ is a continuous-type random variable and its cdf $F(w)$ is such that:




1. $\boxed{F(w) = 0 \text{ for } w < 0}$ (because the waiting time is nonnegative)
2. For $w \geq 0$, we have

$$F(w) = P(W \leq w) = 1 - P(W > w)$$
$$= 1 - P(\text{no occurrences in } [0, w])$$
$$= 1 - e^{-\lambda w},$$

$W > w$ means that we must wait longer than $w$ for the 1st occurrence of the Poisson process

where a Poisson distribution with mean $\lambda w$ was used for the last probability.

cdf of $W$:

$$F(w) = \begin{cases} 1 - e^{-\lambda w}, & w > 0 \\ 0, & w \leq 0 \end{cases}$$

# of occurrences in $[0, w]$ is Poisson $(\lambda w)$

pmf $f(x) = \dfrac{(\lambda w)^x e^{-\lambda w}}{x!}$, $x = 0, 1, 2, \ldots$

$f(0) = e^{-\lambda w}$

Thus, for $w > 0$, the pdf of $W$ is given by

$$f(w) = F'(w) = \begin{cases} \lambda e^{-\lambda w}, & w > 0 \\ 0, & w < 0 \end{cases}$$

$\lambda = 1/\theta$

# Exponential distribution

A random variable $X$ has an exponential distribution with parameter $\theta > 0$ if its pdf is of the form

$$\text{pdf} \quad f(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & x > 0 . \\ 0, & x \le 0 \end{cases} \qquad \text{cdf} \quad F(x) = \begin{cases} 1 - e^{-x/\theta}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

**Note**: the waiting time $W$ until the first occurrence in a Poisson process with parameter $\lambda$ has an exponential distribution with parameter

$$\theta = \frac{1}{\lambda} .$$

$\theta$ gives the mean waiting time between consecutive occurrences of a Poisson process

Example: if a call center receives 20 calls per hour on average

then: $\lambda = 20$ is the average number of calls per hour

while $\theta = \frac{1}{20}$ hours $=$ 3 minutes is the mean waiting time between consecutive calls

# Moment generating function for exponential distribution

Suppose $X$ has an exponential distribution (with parameter $\theta$).

The mgf of $X$ is given by

$$
\begin{aligned}
M(t) &= E\left[e^{tX}\right] \\
&= \int_0^\infty e^{tx} \overbrace{\frac{1}{\theta} e^{-x/\theta}}^{\text{pdf of } X} \, dx \\
&= \lim_{t\to\infty} \int_0^t \frac{1}{\theta} e^{-\overbrace{(1-\theta t)}^{>0} x/\theta} \, dx \\
&= \lim_{t\to\infty} \left[ -\frac{e^{-(1-\theta t)x/\theta}}{1-\theta t} \right]_{x=0}^{x=t} \\
&= \underbrace{\boxed{\frac{1}{1-\theta t}}}_{M(t)}
\end{aligned}
$$

improper integral converges if $1 - \theta t > 0$ that is

for all $t < \dfrac{1}{\theta}$.

# Exponential distribution mean and variance

Differentiating

$$M(t) = \frac{1}{1-\theta t}$$

we obtain

$$\begin{aligned} M'(t) &= \frac{\theta}{(1-\theta t)^2} \\ M''(t) &= \frac{2\theta^2}{(1-\theta t)^3} . \end{aligned}$$

Evaluating at $t = 0$, we find that

$$\mu = E(X) = M'(0) = \theta$$

and

$$\begin{aligned} \sigma^2 = \underbrace{E\left(X^2\right)}_{} - \underbrace{[E(X)]}_{}^2 &= M''(0) - [M'(0)]^2 \\ &= 2\theta^2 - \theta^2 = \theta^2 \end{aligned}$$

$\sim \theta = \frac{1}{\lambda}$

**Conclusion**: If $X$ has an exponential distribution with parameter $\theta$:

- $\mu = E[X] = \theta$
- $\sigma^2 = \mathrm{Var}(X) = \theta^2$

**Interpretation**: If $\lambda$ is the mean number of occurrences of a Poisson process in the unit interval, then

$$\boxed{\theta = \frac{1}{\lambda}}$$

is the mean waiting time for the first occurrence.

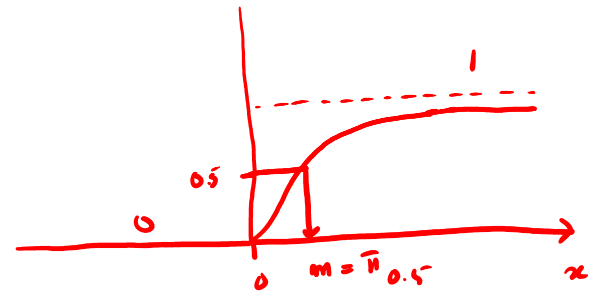
# Exponential distribution median

$\hookrightarrow$ 50th percentile $m = \pi_{0.5}$

Let $X$ have an exponential distribution with mean $\mu = \theta$.

The cdf of $X$ is

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - \mathrm{e}^{-x/\theta} & \text{if } x \geq 0 \end{cases}$$




The median of the distribution of $X$, $m$, is then found by solving

$$\boxed{F(m) = 0.5\ ,}$$

that is

$$\boxed{1 - \mathrm{e}^{-m/\theta} = 0.5\ ,}$$

$\Rightarrow e^{-m/\theta} = 0.5 \Rightarrow -m/\theta = \ln 0.5$

$\Rightarrow m/\theta = -\ln 0.5$

$\Rightarrow m/\theta = -\ln \frac{1}{2}$

$\Rightarrow m/\theta = \ln \left(\frac{1}{2}\right)^{-1}$

$\Rightarrow m/\theta = \ln 2$

$\Rightarrow m = \theta \ln 2$

yielding

$$\boxed{m = \theta \ln(2)\ .}$$

$\ln 2 < \ln e = 1$

$\ln 2 < 1$

**Note**: For the exponential distribution, we always have $\boxed{m < \mu.} = \theta$

exponential distribution median is less than its mean!!

## Example

Customers arrive in a certain shop according to an approximate Poisson process at a mean rate of 20 per hour.

What is the probability that the shopkeeper will have to wait more than 5 minutes for the arrival of the first customer?

We are told that the # of customers arriving at a store is a Poisson process with mean

$$\lambda = 20 \quad (\text{per hour})$$

on average $\qquad \lambda = \dfrac{20 \text{ clients}}{1 \text{ hour}} = \dfrac{20 \text{ clients}}{60 \text{ minutes}} = \dfrac{1}{3} \text{ client/minute}$

Let $W$ be the r.v. representing the waiting time for the $1^{st}$ client arrival.

Then, we know that $W \sim \text{Exponential}(\theta)$, where $\theta = \dfrac{1}{\lambda} = \underline{3 \text{ minutes}}$

We want to find $P(W > 5) = ??$

$$f(w) = \begin{cases} \frac{1}{3} e^{-w/3}, & w > 0 \\ 0, & \text{otherwise} \end{cases}$$

Two alternative ways to compute $P(W>5)$

evaluate to get $e^{-5/3}$

$\uparrow$

① $P(W>5) \to = \int_5^{\infty} f(w)\,dw = \int_5^{\infty} \frac{1}{3} e^{-w/3}\,dw = \dots$ (improper integral)

$\uparrow$

use pdf of W

$$= 1 - P(W \leq 5) = 1 - \int_0^5 f(w)\,dw = 1 - \int_0^5 \frac{1}{3} e^{-w/3}\,dw$$

$$= 1 - \left[ - e^{-w/3} \right]_{w=0}^{w=5} = 1 - \left( - e^{-5/3} - (-1) \right) =$$

use cdf of W

$\downarrow$

$$= 1 + e^{-5/3} - 1 = e^{-5/3}$$

② $P(W>5) = 1 - \underbrace{P(W \leq 5)}_{\text{cdf of W evaluated at } w=5} = 1 - F(5) = 1 - \left[1 - e^{-5/3}\right] = e^{-5/3}$

# Exponential distribution is memoryless

Let $X$ have an exponential distribution with mean $\theta > 0$. $\theta = \frac{1}{\lambda}$

HW exercise: Prove that $X \sim$ Exponential $(\theta)$ is memoryless !!

The probability distribution of $X$ is memoryless, that is, for every $x, y \geq 0$, we have

$$\underbrace{\boxed{P(X > x + y \mid X > x) = P(X > y)\ .}}_{\text{probability of waiting another } y \text{ minutes for the occurrence of the Poisson process does not depend on the time that we have already waited}}$$

← Memoryless property

**Notes**:

- Among continuous random variables with support $(0, \infty)$, the exponential distribution is the only with the memoryless property.
- Among discrete random variables with support $\mathbb{N}$, the geometric distribution is the only with the memoryless property.

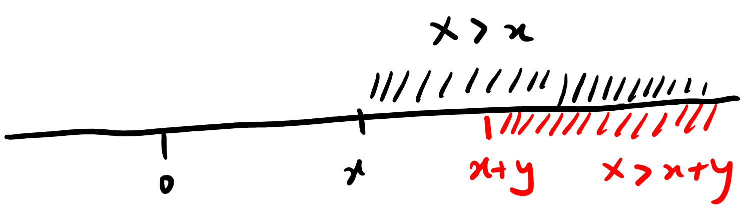
→ was a homework exercise!

Proof of memoryless property:

Take $x, y > 0$

$$P(\underbrace{X > x+y}_{A} \mid \underbrace{X > x}_{B}) = \frac{P(\overbrace{\{X > x+y\}}^{A} \cap \overbrace{\{X > x\}}^{B})}{P(X > x)} =$$

Recall $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$




$$= \frac{P(X > x+y)}{P(X > x)} = \frac{1 - P(X \le x+y)}{1 - P(X \le x)}$$

$$= \frac{1 - F(x+y)}{1 - F(x)} = \frac{1 - \left(1 - e^{-(x+y)/\theta}\right)}{1 - \left(1 - e^{-x/\theta}\right)}$$

$$= \frac{e^{-(x+y)/\theta}}{e^{-x/\theta}} = \frac{e^{-x/\theta} \cdot e^{-y/\theta}}{e^{-x/\theta}} =$$

$$= e^{-y/\theta} = 1 - \left(1 - e^{-y/\theta}\right) = 1 - F(y) = P(X > y)$$

Recall that cdf of $X \sim \text{Exponential}(\theta)$ is

$$F(x) = \begin{cases} 1 - e^{-x/\theta}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

## Example

Suppose that a certain type of electronic component has an exponential distribution with a mean life of 500 hours.

If the component has been in operation for 300 hours, find the conditional probability that it will last for another 600 hours.

Let $X$ be the r.v. representing the lifetime of such electronic component

We are told $X \sim \text{Exponential}(500)$ $\quad \theta = 500$ hours.

We are also told that $X > 300$ [from the info: component has operated for 300 hours already]

Find: $P(X > \underbrace{300+600}_{\text{another 600 hours}} \mid X > 300) \underbrace{=}_{\text{memoryless property}} P(X > 600) = 1 - P(X \leq 600)$

$= 1 - F(600)$

$= 1 - (1 - e^{-600/500})$

$= e^{-6/5}$

Recall that the cdf of $X \sim \text{Exponential}(500)$ is

$$F(x) = \begin{cases} 1 - e^{-x/500}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

# Gamma distribution

$\theta = \frac{1}{\lambda}$ is the mean waiting time between occurrences of the Poisson process

A random variable $X$ has a gamma distribution with parameters $\theta > 0$ and $\alpha > 0$ if its pdf is of the form

when $\alpha$ is a positive integer say $\alpha = 1, 2, 3, 4, \ldots$ then Gamma $(\alpha, \theta)$ gives the waiting time for occurrence number $\alpha$ of the Poisson process

$$\boxed{f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, & x > 0 . \\ 0, & x \leq 0 \end{cases}}$$

**Note**: the waiting time $W$ until the $\alpha$th occurrence in a Poisson process with parameter $\lambda$ has a gamma distribution with parameters $\alpha$ and

$$\theta = \frac{1}{\lambda} .$$

Observation: if $\alpha = 1$, then Gamma $(\alpha, \theta)$ is just Exponential $(\theta)$

that is, Exponential $(\theta)$ is Gamma $(\alpha, \theta)$ with $\alpha = 1$

## Gamma function

← special function

The gamma function is defined by

$$\boxed{\Gamma(t) = \int_0^\infty y^{t-1} e^{-y} \, dy \,, \quad t > 0 \,.}$$

**Properties**:

1. $\Gamma(t) > 0$ for every $t > 0$

think of $\Gamma(t+1)$ as being $t!$ for $t>0$

2. $\Gamma(1) = \int_0^\infty e^{-y} \, dy = 1$ ← try as exercise

key property → 3. If $t > 1$, integration by parts yields

$$\boxed{\Gamma(t) = (t-1)\Gamma(t-1)}$$

← try as an exercise

4. For every positive integer $n \in \mathbb{N}$, we have

$$\boxed{\Gamma(n) = (n-1)!}$$

← gamma function is the generalization of the notion of factorial to positive real numbers!!!

If $\alpha = m$ is a positive integer then the pdf of $\underbrace{\text{Gamma}(m, \theta)}$

represents the waiting time to the $m^{th}$ occurrence of a Poisson process with mean rate $\lambda = \frac{1}{\theta}$

is given by

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(m) \cdot \theta^m} \cdot x^{m-1} \cdot e^{-x/\theta} & , x > 0 \\ 0 & , x \leq 0 \end{cases}$$

$$= \begin{cases} \dfrac{x^{m-1} \, e^{-x/\theta}}{(m-1)! \, \theta^m} & , x > 0 \\ 0 & , x \leq 0 \end{cases}$$

# Moment generating function for gamma distribution

Suppose $X$ has a gamma distribution with parameters $\theta > 0$ and $\alpha > 0$.

The mgf of $X$ is given by

$$\boxed{M(t) = \frac{1}{(1-\theta t)^{\alpha}}, \qquad t < \frac{1}{\theta}} \leftarrow$$

on chp 5 we will find M(t) for integer values of $\alpha$

The mean and variance of $X$ are

- $\mu = E[X] = \underline{\alpha}\theta$ $\leftarrow$ $\underbrace{\text{\# of occurrences}}_{\alpha} \times \underbrace{\text{mean waiting time per occurrence}}_{\theta}$
- $\sigma^2 = \mathrm{Var}(X) = \underline{\alpha}\theta^2$

## Example

Suppose the number of customers per hour arriving at a shop follows a Poisson process with mean 30.

What is the probability that the shopkeeper will wait more than 5 minutes before both of the first two customers arrive?

# customer per hour arriving at shop follows a Poisson process with $\underbrace{\lambda = 30}$ clients/hour

on average 30 customer arrive at shoup each hour!

same as saying that one client arrives every 2 minutes

$\theta = 2$ minutes

Define the r.v. W to be the waiting time until the first two clients arrive.

$W \sim \text{Gamma}(\alpha, \theta)$ where

$\alpha = 2 \leftarrow$ # of clients we are counting

$\theta = 2 \leftarrow$ mean waiting time between arrivals

We want to find $P(W > 5) = ??$

Two possible strategies $\quad$ pdf $\quad f(w) = \frac{1}{2^2} \cdot x \cdot e^{-x/2}, \quad x > 0$

$$① \quad P(W > 5) = \int_5^\infty f(w)\, dw = \int_5^\infty \frac{1}{4} w\, e^{-w/2}\, dw = \dots \text{ improper integral} \dots$$

use pdf

$$= 1 - P(W \leq 5) = \int_0^5 f(w)\, dw = \int_0^5 \frac{1}{4} w\, e^{-w/2}\, dw = \text{integrate by parts}$$

$$= \dots$$

use Poisson distribution

$$② \quad P(\underbrace{W > 5}) = \sum_{k=0}^{1} \frac{\left(\frac{5}{2}\right)^x e^{-5/2}}{x!} = e^{-5/2} + \frac{5}{2} e^{-5/2} = \frac{7}{2} e^{-5/2}$$

$W>5$ means that it takes longer than 5 minutes for the first two clients to arrive

$\Downarrow$

$W>5$ means that at most 1 client arrives in the time interval $[0,5]$ $\Big\} \Rightarrow$ # of clients arriving over the interval $[0,5]$ is Poisson with parameter $\lambda = \frac{5}{2}$

on average 2.5 clients arrive each 5 minutes