STUDENT'S t- distribution.

Let $Z \sim N(0,1)$, $U \sim \chi^2(n)$ be independent r.v.s

Then $$T = \frac{Z}{\sqrt{U/n}} \sim \underbrace{t(n)}$$

t-student distr. with n dofs.

VERY IMPORTANT / KEY EXAMPLE

$X_1, X_2, \dots, X_n$ random sample from a $N(\mu, \sigma^2)$ distribution, then:

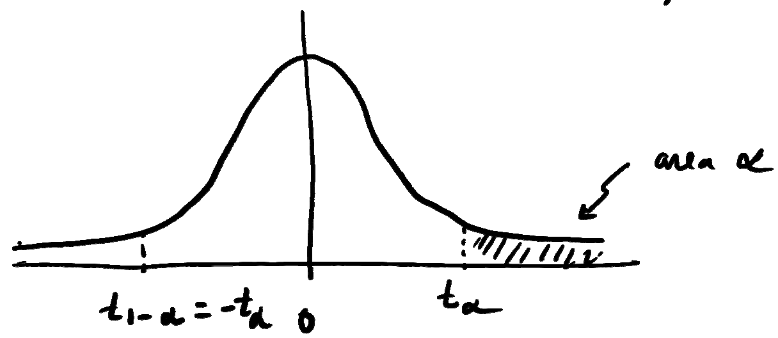
① $\bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right) \Rightarrow Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$

② $U = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$

Z and $\frac{(n-1)S^2}{\sigma^2}$ are independent r.v

③ $$T = \frac{Z}{\sqrt{U/(n-1)}} = \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{\sigma^2}/(n-1)}} \Rightarrow \boxed{T = \frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t(n-1)}$$

graph of t-distributions looks bell shaped



$P(T \geq t_\alpha) = \alpha$

$t_{1-\alpha} = -t_\alpha$

as $n \to \infty$ $\quad t(n) \xrightarrow{D} N(0,1)$

M.G.F TECHNIQUE:

Def of m.g.f (from 3501)

X is a r.v with pmf / pdf $f(x)$.

m.g.f of X is

$$\underbrace{\boxed{M_X(t) = E\left[e^{tX}\right]}}_{\text{from most of relevant families of r.v.s we can pick } M_X(t) \text{ from the formula sheet.}} = \overbrace{\begin{cases} \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\,dx, & X \text{ continuous} \\ \sum_{x} e^{tx} f(x), & X \text{ continuous} \end{cases}}$$

VERY IMPORTANT: $X_1, X_2, \ldots, X_n$ $\overbrace{\text{random sample}}^{\text{iid}}$ from a distribution with m.g.f $M_X(t)$

To find the m.g.f of $Y = \sum_{i=1}^{n} X_i$;

$$M_Y(t) \underset{\text{def of } M_Y}{=} E\left[e^{tY}\right] \underset{\text{def of } Y}{=} E\left[e^{t(X_1+X_2+\cdots+X_n)}\right] \underset{\text{prop. of exp}}{=} E\left[\underset{\text{independent}}{e^{tX_1} \cdot e^{tX_2} \cdots e^{tX_n}}\right] \overset{\text{independence of } X_1,\ldots,X_n}{=} E\left[e^{tX_1}\right] \cdot \ldots \cdot E\left[e^{tX_n}\right]$$

$$= M_X(t) \ldots M_X(t)$$

$$= \left(M_X(t)\right)^n$$

Sec. 5.4 Ex. 4

Let $X_1 \sim$ Poisson (μ_1), $X_2 \sim$ Poisson (μ_2), ..., $X_n \sim$ Poisson (μ_n), with $\mu_i > 0$ for $i = 1, 2, \dots, n$, be independent r.v.s.

To show that $Y = \sum_{i=1}^{n} X_i \sim \text{Poisson}(\mu_1 + \mu_2 + \dots + \mu_n)$ we can use the m.g.f technique:

$$M_Y(t) = E[e^{tY}] = E\left[e^{t\sum_{i=1}^{n} X_i}\right] = E\Big[\underbrace{e^{tX_1} \cdot e^{tX_2} \cdots e^{tX_n}}_{\prod_{i=1}^{n} e^{tX_i}}\Big]$$

independence $\sim$ $$= \underbrace{E[e^{tX_1}] \cdot E[e^{tX_2}] \cdots E[e^{tX_n}]}_{\prod_{i=1}^{n} E[e^{tX_i}]} = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t)$$

$$= e^{\mu_1(e^t-1)} \cdot e^{\mu_2(e^t-1)} \cdots e^{\mu_n(e^t-1)}$$

$$= \exp\left(\mu_1(e^t-1) + \mu_2(e^t-1) + \dots + \mu_n(e^t-1)\right) = e^{(\mu_1+\mu_2+\dots+\mu_n)(e^t-1)}$$

CONCLUSION

Since $M_Y(t) = \exp\left((\mu_1 + \mu_2 + \cdots + \mu_n)(e^t - 1) \right)$

is of the form $\underbrace{e^{\lambda(e^t-1)}}_{\text{m.g.f of Poisson}}$ with $\lambda = \mu_1 + \mu_2 + \cdots + \mu_n$

then $Y \sim \text{Poisson}(\mu_1 + \mu_2 + \cdots + \mu_n)$

6.3.3 $Y_1 < Y_2 < \cdots < Y_5$ $\underbrace{\text{order statistics}}_{\text{associated with } X_1, X_2, \ldots, X_5}$ of from exp. dist. with mean $\theta = 3$

b)

$$\text{pdf } f(x) = \frac{1}{3} e^{-x/3}, \quad x > 0$$

$$P(Y_4 < 5) = ??$$

$$\text{cdf } F(x) = 1 - e^{-x/3}, \quad x > 0$$

Define another r.v., call it N, representing the number of observations out of $\underbrace{X_1, X_2, \ldots, X_5}_{n=5 \text{ observations}}$ that have value less than 5.

$$N \sim \text{Binomial}(n, p) \quad \text{where} \begin{cases} n = 5 \\ p = P(X_i < 5) = F(5) = 1 - e^{-5/3} \end{cases}$$

$$P(Y_4 < 5) = P(N = 4 \text{ or } N = 5) = \sum_{k=4}^{5} \binom{5}{k} \left(1 - e^{-5/3}\right)^k \cdot \left(e^{-5/3}\right)^{5-k}$$

= evaluate ....

c) $P(1 < Y_1) = P(1 < \min\{X_1, X_2, \dots, X_5\})$

$= P(1 < X_1, 1 < X_2, \dots, 1 < X_5)$

independence

$= P(1 < X_1) \cdot P(1 < X_2) \cdots P(1 < X_5)$

all have the same distr.

$= \left(P(1 < X_1) \right)^5 = \left(P(X_1 > 1) \right)^5$

$= \left(1 - \underbrace{P(X_1 \leq 1)}_{F_{X_1}(1)} \right)^5 = \left(1 - (1 - e^{-1/3})\right)^5$

$= e^{-5/3}$

pdf: $f(x) = \frac{1}{3} e^{-x/3}, \quad \underline{x > 0}$

cdf = ?! Take $x > 0$:

$$F(x) = P(\underline{\underline{X \leqslant x}}) = \int_{-\infty}^{x} f(u)\,du = \int_{0}^{x} \frac{1}{3} e^{-u/3}\,du$$

def of cdf

X is continuous r.v.

$$= \left[- e^{-u/3} \right]_{u=0}^{u=x}$$

$$= - e^{-x/3} - \left(- e^{0} \right)$$

$$= 1 - e^{-x/3}$$

Sec. 5.5 Ex 16

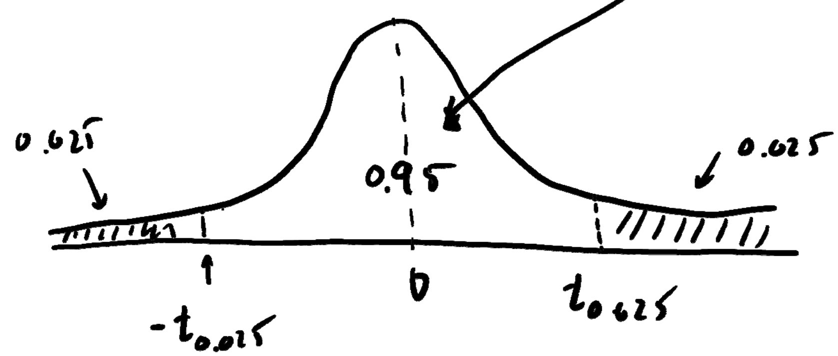
Let $\underline{\underline{n=9}}$ and $T = \dfrac{\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\dfrac{(n-1)S^2}{\sigma^2}\Big/(n-1)}} = \dfrac{\bar{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$

a) Let us find $t_{0.025}$ so that $\underbrace{P(-t_{0.025} \leq T \leq t_{0.025}) = 0.95}$

Check the table for

$t_{0.025}$ when $r = n-1 = 8$

to get $t_{0.025} = 2.306$



CONCLUSION: $P(-2.306 \leq T \leq 2.306) = 0.95$

b) $-t_{0.025} \leq T \leq t_{0.025}$ $\quad (t_{0.025} = 2.306)$

$$-t_{0.025} \leq \frac{\bar{X} - \mu}{S/\sqrt{n}} \leq t_{0.025} \qquad (n = 9)$$

Multiply by $\frac{S}{\sqrt{n}}$

$$-t_{0.025} \cdot \frac{S}{\sqrt{n}} \leq \bar{X} - \mu \leq t_{0.025} \cdot \frac{S}{\sqrt{n}}$$

subtract $\bar{X}$

$$-\bar{X} - t_{0.025}\frac{S}{\sqrt{n}} \leq -\mu \leq -\bar{X} + t_{0.025}\frac{S}{\sqrt{n}}$$

multiply by -1

$$\bar{X} - t_{0.025}\frac{S}{\sqrt{n}} \leq \mu \leq \bar{X} + t_{0.025}\frac{S}{\sqrt{n}}$$

3.3-12. If the moment-generating function of X is given by $M(t) = e^{500t+1250t^2}$, where $-\infty < t < \infty$, find $P[6{,}765 \le (\underbrace{X - 500}_{\mu})^2 \le 12{,}560]$.

Note that $M(t)$ is of the form $e^{\mu t + \frac{1}{2}\sigma^2 t^2}$ ← m.g.f of $N(\mu, \sigma^2)$ with

$\mu = 500$ and $\frac{1}{2}\sigma^2 = 1250$, that is $\sigma^2 = 2500$

This means that $X \sim N(500, 2500) \Rightarrow Z = \dfrac{X-\mu}{\sigma} \sim N(0,1)$

$$\Rightarrow U = Z^2 = \left(\frac{X-\mu}{\sigma}\right)^2 = \frac{(X-500)^2}{2500} \sim \chi^2(1)$$

Then, we find that $P(6{,}765 \le (X-500)^2 \le 12{,}560)$

$$= P\left(\frac{6{,}765}{2500} \le \underbrace{\frac{(X-500)^2}{2500}}_{U \sim \chi^2(1)} \le \frac{12{,}560}{2500}\right)$$

$$= P\left(2.706 \leq U \leq 5.024\right) = F(5.024) - F(2.706)$$

if we let F be the cdf of $\chi^2(1)$

$$= 0.975 - 0.9 = 0.075$$

table

$$X \sim N(\mu, \sigma^2) \Rightarrow Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

$$\Rightarrow U = Z^2 = \left(\frac{X-\mu}{\sigma}\right)^2 \sim \underline{\underline{\chi^2(1)}}$$

OVERVIEW OF F distn.

If $U \sim \chi^2(r_1)$ and $V \sim \chi^2(r_2)$ independent

then $$W = \frac{U/r_1}{V/r_2} \sim F(r_1, r_2)$$ take this as the def of F distribution

PROPERTY : If $W = \dfrac{U/r_1}{V/r_2} \sim F(r_1, r_2) \Rightarrow \dfrac{1}{W} = \dfrac{V/r_2}{U/r_1} \sim F(r_2, r_1)$

IMPORTANT APPLICATION / EXAMPLE

Let $X_1, \dots, X_n$ random sample from $N(\mu_x, \sigma_x^2)$ AND $Y_1, \dots, Y_m$ random sample from $N(\mu_y, \sigma_y^2)$ independent then

(1) $U = (n-1) S_x^2 / \sigma_x^2 \sim \chi^2(n-1)$

(2) $V = (m-1) S_y^2 / \sigma_y^2 \sim \chi^2(m-1)$

$$\Rightarrow W = \frac{\frac{(n-1)S_x^2}{\sigma_x^2} \Big/ (n-1)}{\frac{(m-1)S_y^2}{\sigma_y^2} \Big/ (m-1)} = \frac{S_x^2}{S_y^2} \cdot \frac{\sigma_y^2}{\sigma_x^2} \sim F(n-1, m-1)$$

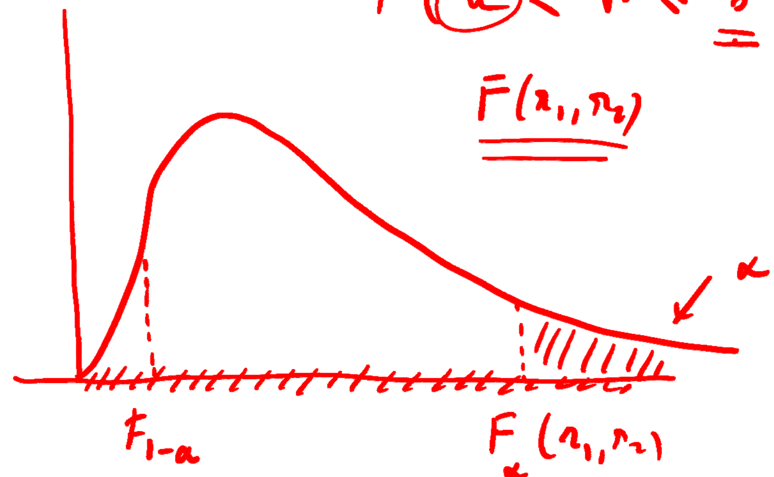
$$P(a \le W \le b)$$

$$F(n_1, n_2)$$

To find $F_{1-\alpha}(n_1, n_2)$

we do

$$F_{1-\alpha}(n_1, n_2) = \frac{1}{F_\alpha(n_2, n_1)}$$



$$\Downarrow$$

$$P(W > F_\alpha(n_1, n_2)) = \alpha$$

because

$$P(W \le F_{1-\alpha}(n_1, n_2)) = \alpha$$

$$P\left(\frac{1}{W} > \underbrace{\frac{1}{F_{1-\alpha}(n_1, n_2)}}\right) = \alpha$$

$$\frac{1}{F_{1-\alpha}(n_1, n_2)} = F_\alpha(n_2, n_1)$$