Sec. 6.6 Ex. 2

$X_1, X_2, \dots, X_n$ random sample from $\underbrace{\text{binomial}}_{\text{this is the same as Bernoulli}(p)}$ $b(1,p)$

We are told that:

i) $\bar{X}$ is an unbiased estimator of p.

ii) $Var(\bar{X}) = \dfrac{p(1-p)}{n}$

a) The Cramer - Rao inequality guarantees that (under mild conditions):

$$Var(\hat{p}) \geq \frac{1}{I(p)}$$

for every unbiased estimator $\hat{p}$ of p, where $I(p)$ is the Fisher information:

$$I(p) = n\, E\left[\left(\frac{\partial}{\partial p} \ln f(X,p)\right)^2\right] = -n\, E\left[\frac{\partial^2}{\partial p^2} \ln f(X,p)\right]$$

Since $f(x,p) = p^x (1-p)^{1-x}$, $x = 0,1$, then

$$\ln f(x,p) = x \ln p + (1-x) \ln(1-p)$$

and so

$$\frac{\partial}{\partial p} \ln f(x,p) = \frac{x}{p} - \frac{1-x}{1-p} \quad \text{and} \quad \frac{\partial^2 \ln f(x,p)}{\partial p^2} = -\frac{x}{p^2} - \frac{1-x}{(1-p)^2}$$

Hence, we get that

$$E\left[\frac{\partial^2}{\partial p^2} \ln f(X,p)\right] = E\left[-\frac{X}{p^2} - \frac{1-X}{(1-p)^2}\right]$$

linearity of expected value

$$= -\frac{E[X]}{p^2} - \frac{1-E[X]}{(1-p)^2}$$

$$= -\frac{p}{p^2} - \frac{1-p}{(1-p)^2}$$

$$= -\frac{1}{p} - \frac{1}{1-p} =$$

$$= \frac{-(1-p)-p}{p(1-p)} = -\frac{1}{p(1-p)}.$$

Thus, we conclude that

$$I(p) = -n\, E\left[\frac{\partial^2}{\partial p^2} \ln f(X,p)\right] = -n\, E\left[-\frac{1}{p(1-p)}\right] = \frac{n}{p(1-p)}$$

and so the Cramer-Rao lower bound is

$$\frac{1}{I(p)} = \frac{1}{\frac{n}{p(1-p)}} = \frac{p(1-p)}{n}$$

b) The efficiency of $\bar{X}$ is $e(\bar{X}) = \dfrac{1/I(p)}{Var(\bar{X})} = \dfrac{\frac{p(1-p)}{n}}{\frac{p(1-p)}{n}} = 1$