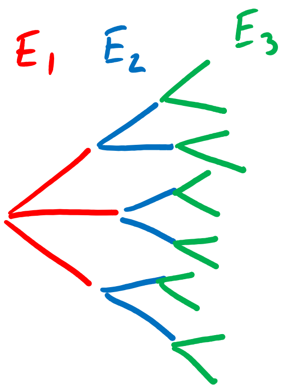
Math 3501 - Probability and Statistics I

1.2 - Methods of Enumeration

Multiplication Principle:

Suppose that the experiment E_i has n_i, $i = 1, 2, \ldots, m$, possible outcomes after previous experiments have been performed.

Then the composite experiment $E_1 E_2 \cdots E_m$ that consists of performing E_1 then $E_2, \ldots,$ and finally E_m has $n_1 \cdot n_2 \cdots n_m$ possible outcomes.



\# possible outcomes is

$$n_1 \cdot n_2 \cdot n_3 = 3 \cdot 2 \cdot 2 = 12$$

$n_1 = 3$

$n_2 = 2$

$n_3 = 2$

Question: In how many ways can n positions be filled with n different objects?

Example $n = 5$ objects to fill 5 positions

$$\underbrace{\underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1}}_{5 \text{ positions}} = 5!$$

n objects in n positions

$$\underbrace{\underline{n} \cdot \underline{(n-1)} \cdot \underline{(n-2)} \cdot \underline{(n-3)} \cdots\cdots \underline{3} \cdot \underline{2} \cdot \underline{1}}_{n \text{ positions}} = n!$$

Factorial

Definition (Factorial)

Given $n \in \mathbb{N}$, the number n factorial, denoted $n!$, is defined as

$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

$$\boxed{n! = n(n-1)\cdots(2)(1)\ .}$$

Convention:

$$\boxed{0! = 1}$$

There is only one way to organize 0 objects in 0 positions

$1! = 1$

$2! = 2$

$3! = 6$

$4! = 24$

$5! = 120$

$6! = 720$

Permutations of n different objects

Definition

Each of the $n!$ arrangements (in a row) of n different objects is called a *permutation of the n objects*.

assigning n objects to n positions

Question: In how many ways can r positions be filled with objects selected from n different objects, where $r \leq n$?

n objects

$$\underbrace{\underline{n}\quad \underline{(n-1)}\quad \underline{(n-2)}\quad \cdots\quad \underline{}\quad \underline{n-r+1}}_{r \text{ positions}}$$

$n-(r-1)$

without replacement!

$$n(n-1)(n-2)\cdots(n-r+1) = \frac{\overbrace{n\cdot(n-1)\cdots(n-r+1)}\,\overbrace{(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}^{\text{extra terms}}}{(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1} = \frac{n!}{(n-r)!}$$

If $r = n$ (as in the previous case): $\dfrac{n!}{(n-n)!} = \dfrac{n!}{0!} = \dfrac{n!}{1} = n!$

Permutations of n objects taken r at a time

Definition

The number of possible ordered arrangements of n objects to fill $r \leq n$ positions, denoted ${}_nP_r$ and called *permutation of n objects taken r at a time*, is given by

$$ {}_nP_r = \frac{n!}{(n-r)!} . $$

Special case: ${}_nP_n = n!$

Problem:

Given a set with n objects, in how many ways can we select $r \leq n$ objects?

Factors to consider:

- is a selected object replaced before the next object is selected?
- is the order or the objects relevant?

are repetitions allowed ??

abc or aaa

??

Do we count abc acb bca

as distinct strings or not

Ordered sample

Definition

If r objects are selected from a set of n objects, and if the order of selection is noted, then the selected set of r objects is called an *ordered sample of size* r.

$\{A, B, C, D, E\}$ $n = 5$

A C D

D C A

A D C

all counted individually

↓

there are distinct ordered samples!

Sampling with replacement

Definition

Sampling with replacement occurs when an object is selected and then replaced before the next object is selected.

{ A B C D E }

with replacement :

A A B

B C C

sequences are available when sampling with replacement

all need to be counted

Remark

The number of possible ordered samples of size r taken from a set of n objects when sampling with replacement is n^r.

$$\underbrace{\underline{n}\quad \underline{n}\quad \underline{n}\quad \underline{n}\quad \cdots\cdots\quad \underline{n}}_{r \text{ position}} = n^r$$

Sampling without replacement

Definition

Sampling without replacement occurs when an object is not replaced after it has been selected.

when sampling $r = 3$ objects from a set $\{A, B, C, D, E\}$ with $n = 5$ elements

we get sequences:

A B E

C E B

earlier discussion

$\underline{5} \cdot \underline{4} \cdot \underline{3}$

$$\boxed{{}_5P_3 = 5 \cdot 4 \cdot 3}$$

Remark

The number of possible ordered samples of size r taken from a set of n objects without replacement is

$$_nP_r = \frac{n!}{(n-r)!},$$

the number of permutations of n objects taken r at a time.

SUMMARY : So far:

ordered sequences of r objects taken from a set with n objects

with replacement : n^r

without replacement : $_nP_r = \frac{n!}{(n-r)!}$

Example

Find the number of ordered samples of five cards that can be drawn without replacement from a standard deck of 52 playing cards.

A B C
$\neq$
C B A

$${}_{52}P_5 \leftarrow \underline{52} \cdot \underline{51} \cdot \underline{50} \cdot \underline{49} \cdot \underline{48}$$ without replacement

If, instead, we sample with replacement, the answer would be

$$52^5 \leftarrow \underline{52} \cdot \underline{52} \cdot \underline{52} \cdot \underline{52} \cdot \underline{52}$$ with replacement

$$52^5 > {}_{52}P_5$$

Note:

- we must have $r \leq n$ when sampling without replacement
- r can exceed n when sampling with replacement

Question: Find the number of (unordered) subsets of size r that can be selected from a set of n different objects.

$\{A, B, C, D, E\}$ ← set with $n = 5$ distinct objects

Count # of subsets with $r = 3$ objects

Key observations:

① $\{A, B, C\}$ $\{B, C, A\}$ $\{A, C, B\}$ all of these count as the same object

all of these are the same set

② $\{A, A, B\}$ → a set with just two elements

↳ $\{A, B\}$

to construct a subset with 3 elements, we need to draw 3 distinct elements from the original set

GOAL: Count # of unordered subsets / sequences without replacement

subsets of 3 elements out of a set with $n=5$ elements

set we're sampling from $\rightarrow \{A, B, C, D, E\}$ $\rightarrow$ $\{A, B, C\}$, $\{A, D, E\}$, $\{A, C, D\}$, ⋮ $\Big\}$ how many

ordered sequences of 3 elements out of a set with 5 elements without replacement is

$$_5P_3 = \underline{5} \cdot \underline{4} \cdot \underline{3} \leftarrow \underbrace{ABC,\ ACB,\ CBA}_{\text{all counted separately}}$$

ordered sequences of 3 elements can we make : $3!$

unordered sequences without replacement is

$$\boxed{\frac{_5P_3}{3!} = \frac{5!}{3!\,(5-3)!}}$$

what we are looking

Let C be the number of subsets of r elements out of a set with n elements

[with $r \leq n$]

number of ordered sequences of r objects sampled without replacement from

this set with n objects is

$$\underbrace{{}_nP_r}_{\substack{\#\text{ ordered}\\\text{sequences}}} = \underbrace{C}_{\substack{\#\text{ unordered}\\\text{sequences}}} \cdot \underbrace{r!}_{\substack{\#\text{ of distinct}\\\text{orders under}\\\text{which we can place}\\r\text{ distinct objects}}} \Longrightarrow \boxed{C = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}}$$

binomial coefficient!

Combination of n objects taken r at a time

Definition

Given a set of n different objects, the number of (unordered) subsets of size $r \leq n$, denoted ${}_nC_r$ or $\binom{n}{r}$, and called a *combination of n objects taken r at a time*, is

$$\boxed{{}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}} .$$

Note: the quantity defined above

$$\binom{n}{r} = {}_nC_r$$

may also be regarded as the number of ways in which r objects can be selected without replacement from n objects when the order of selection is disregarded.

Note: The numbers

$$ {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} \, . $$

are often called *binomial coefficients* since they arise in the binomial expansion

$$ (a+b)^n = \sum_{r=0}^{n} \binom{n}{r} b^r a^{n-r} \, . $$

$$ \begin{aligned} (a+b)^0 &= 1 \\ (a+b)^1 &= 1.a + 1.b \\ (a+b)^2 &= 1.a^2 + 2ab + 1.b^2 \\ (a+b)^3 &= 1.a^3 + 3a^2b + 3ab^2 + 1.b^3 \\ (a+b)^4 &= 1.a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1.b^4 \end{aligned} $$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Note: The binomial coefficients

$$ {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}. $$

are the entries of Pascal's triangle:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

$n = 2$

$r = 1$

$$\binom{2}{1} = \underbrace{\binom{1}{0}}_{1} + \underbrace{\binom{1}{1}}_{1} = 2$$

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

$\xrightarrow{\text{sum}} 2^0 = 1$

$\xrightarrow{\text{sum}} 2^1 = 2$

$\xrightarrow{\text{sum}} 2^2 = 4$

$\xrightarrow{\text{sum}} 2^3 = 8$

$$
\begin{array}{ccccc}
 & & \binom{0}{0} & & \\
 & \binom{1}{0} & & \binom{1}{1} & \\
\binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} \\
\binom{3}{0} & \binom{3}{1} & & \binom{3}{2} & \binom{3}{3} \\
\binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \binom{4}{4}
\end{array}
$$

Properties: $\underbrace{\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}}$ and $\boxed{\binom{n}{r} = \binom{n}{n-r}}$ ← symmetry

Take a set with n objects.

Question how many subsets does it have ?

Answer : 2^n

Reasoning:

$\binom{n}{r}$: # of subsets of size r from a set with n elements

Total number of subsets is then $\boxed{\sum_{r=0}^{n} \binom{n}{r} = ?}$

But, we know that $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$ (Binomial expansion)

Set $a=1, b=1$, we get is $\boxed{2^n = \sum_{r=0}^{n} \binom{n}{r}}$

$\{a, b\}$ $\quad n=2$

$\underbrace{\phi}_{\binom{2}{0}}, \underbrace{\{a\}, \{b\}}_{\binom{2}{1}}, \underbrace{\{a,b\}}_{\binom{2}{2}}$: 4 subsets in total

$4 = 2^2$

$\{a, b, c\}$

Example

Find the number of possible five-card hands drawn from a deck of 52 playing cards.

order does not matter :

ace spades + 2 hearts + 3 clubs + K spades + J clubs

same hand

2 hearts + 3 club + ace spades + K spades + J clubs

To get a 5-card hand, cards are not replaced :

Goal :

Find # of unordered sequences of 5 elements out of a set with 52 elements sampled without replacement :

$$\leadsto \binom{52}{5} = \frac{52!}{5!\,(52-5)!}$$

EXAMPLE (continued)

As before, we take a 5-card hand from a regular 52-card deck.

Find probability of

(1) hand has only spades

(2) hand has 3 kings and 2 Queens

(3) two kings, 2 Queens, 1 Jack

Define sample space:

$S = \{ \text{all 5-card hands} \}$

We have seen $|S| = \binom{52}{5}$

NOTE: All 5-card hands are equally likely

Answer to 1:

Define event A = "drawing a hand consisting of 5 spades"

spades to draw

of spades available

$$P(A) = \frac{|A|}{|S|} = \frac{\binom{13}{5} \cdot \binom{39}{0}}{\binom{52}{5}} =$$

$|A|$: # outcomes favorable to A

total number of outcomes

Answer to 2:

B : event "draw a 5-card hand with 3 Ks and 2 Qs"

$$P(B) = \frac{|B|}{|S|} = \frac{\binom{4}{3} \cdot \binom{4}{2} \cdot \binom{44}{0}}{\binom{52}{5}}$$

Answer to 3:

C : event "draw a 5-card hand with 2 Ks, 2 Qs, 1 J"

$$P(C) = \frac{|C|}{|S|} = \frac{\binom{4}{2} \cdot \binom{4}{2} \cdot \binom{4}{1} \cdot \binom{40}{0}}{\binom{52}{5}}$$

Problem: Suppose that a set contains n objects of two types: r of one type and $n - r$ of the other type.

Find the number of distinguishable arrangements of the n objects.

Two types r Os and $n-r$ Xs

all the Os look the same
all the Xs look the same

Imagine we could distinguish Os and Xs

$O_1 \; X_1 \; X_2 \; O_2 \; O_3 \; O_4 \; X_3 \; X_5$

we would have → 8! arrangements

If we can't

O X X O O O X X

$$\frac{8!}{4!\,4!}$$

8! ← # of permutations of 8 objects

4! ← # of permutations of Os

4! ← # permutations of Xs

If we have n objects

$\underline{r}$ of type 1

$n-r$ of type 2

$$\underbrace{\_\ \_\ \_\ \_\ \_\ \cdots\ \_}_{n \text{ positions to fill}}$$

$$\# \text{ of distinguishable permutations} = \frac{n!}{r!\,(n-r)!} = \binom{n}{r}$$

$r!$: to remove all permutations among type 1 objects

$(n-r)!$: to remove all permutations of type 2 objects

Distinguishable permutation

Definition

Each of the ${}_nC_r$ permutations of n objects, r of one type and $n - r$ of another type, is called a *distinguishable permutation*.

Coin toss

H H T T T H

H T T T H H

⋮

Example

A coin is flipped 8 times and the sequence of heads and tails is observed.

Find the number of possible 8-tuplets that result in 3 heads and 5 tails.

H T T H H T T T

$$\binom{8}{5} = \binom{8}{3} = \frac{8!}{3!\,5!}$$