Math 3501 - Probability and Statistics I

3.3 - The normal distribution

the "most important" probability distribution

Reason: Central Limit Theorem (part of Chp 5)

Normal distribution describes how sample averages behave (under appropriate conditions)

Normal distribution

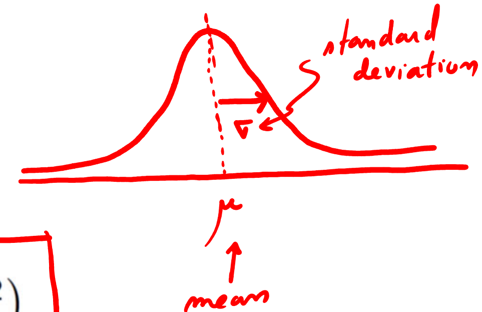
mu sigma

A random variable X has a normal distribution with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$ if its pdf is of the form

mean

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] , \quad x \in \mathbb{R} .$$

variance



Notation: We say that X is $N\left(\mu, \sigma^2\right)$ and denote it as $\boxed{X \sim N\left(\mu, \sigma^2\right)}$

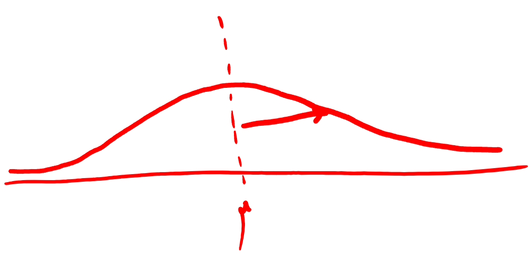
mean of X

variance of X

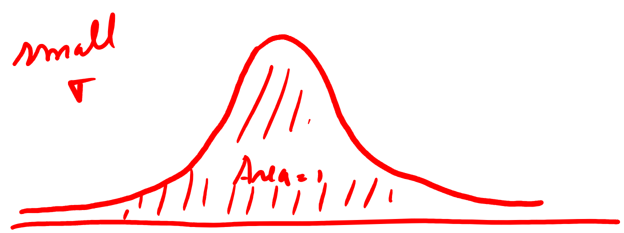
Note: It is possible to check that:

- $f(x) > 0$ for all $x \in \mathbb{R}$ ✓
- $\displaystyle\int_{\mathbb{R}} f(x)\,\mathrm{d}x = 1$

large σ



small σ



Normal distribution mgf, mean and variance

Suppose X has a normal distribution (with parameter μ and σ).

The mgf of X is given by

$$\int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx$$

variance

$$M(t) \overset{\text{def}}{=} E\left[e^{tX}\right] = \exp\left(\underbrace{\mu}_{\mu} t + \frac{\sigma^2 t^2}{2}\right) , \quad x \in \mathbb{R} .$$

Differentiating the mgf $M(t)$, we obtain

chain rule

$$M'(t) = (\mu + \sigma^2 t) \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

product + chain rule

$$M''(t) = \left[(\mu + \sigma^2 t)^2 + \sigma^2\right] \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) .$$

Evaluating at $t = 0$, we find that

$$E(X) = M'(0) = \mu$$

$$\mathrm{Var}(X) = E\left(X^2\right) - [E(X)]^2 = M''(0) - [M'(0)]^2 = (\mu^2 + \sigma^2) - \mu^2 = \sigma^2 .$$

Example

Suppose the random variable X has pdf given by

$$f(x) = \frac{1}{\sqrt{32\pi}} \exp\left[-\frac{(x+7)^2}{32}\right], \quad x \in \mathbb{R}.$$

Determine the mean and variance of X.

Observe that f is of the form

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in \mathbb{R}$$

with

$$\mu = -7$$

and

$$2\sigma^2 = 32 \Rightarrow \sigma^2 = 16$$

$X \sim N(-7, 16)$

and

$E[X] = -7$ and $Var(X) = 16$

Example

Suppose the random variable X has mgf given by

$$M(t) = \exp\left(5t + 12t^2\right) \ , \quad t \in \mathbb{R} \, .$$

Determine the mean and variance of X.

Observe that $M(t)$ is of the form

$$M(t) = \exp\left(\mu t + \frac{\sigma^2}{2} t^2\right) \ , \ t \in \mathbb{R}$$

with $\mu = 5$

and $\frac{\sigma^2}{2} = 12$

That is $\mu = 5$ and $\sigma^2 = 24$

$\Rightarrow$ $X \sim N(5, 24)$

and

$E[X] = 5$

$Var(X) = 24$

Standard normal distribution

Φ ← handwritten capital phi

We say that Z has a standard normal distribution if Z is $N(0, 1)$.

$f \to$ pdf

$F \to$ cdf

The pdf of Z is

lower case ϕ → $\phi(z)$ →
$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

and its cdf is

standard normal

$\phi \to$ pdf

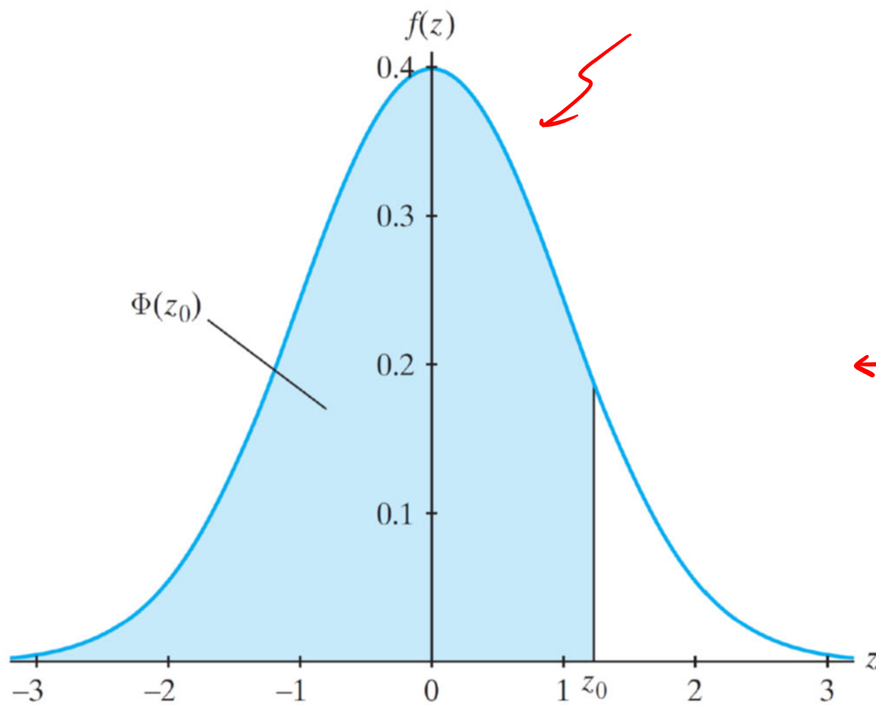
$\Phi \to$ cdf

capital phi →
$$\boxed{\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} \, \mathrm{d}w}$$

Notes:

- It is not possible to evaluate this integral by finding an antiderivative that can be expressed as an elementary function.
- Numerical approximations for integrals of this type have been tabulated. (Tables Va and Vb in textbook Appendix B)

The graph of the pdf of Z is a bell-shaped curved.



f(z) is symmetric w.r.t reflections on vertical axis

(f is an even function)

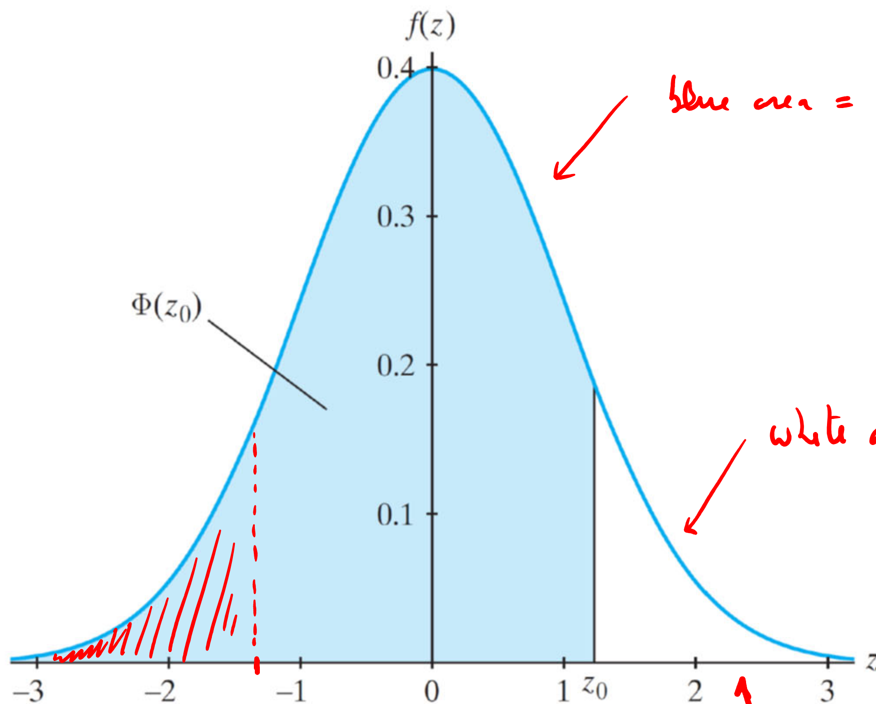
The shaded area equals $\underbrace{\Phi(z_0) = P(Z \leq z_0)}_{\text{cdf of } Z \sim N(0,1)}$.

Symmetry guarantees that for all real z, we have

$$\underbrace{\Phi(-z)} = P(Z \le -z) = P(Z > z) = 1 - P(Z \le z) = \underbrace{1 - \Phi(z)} .$$

$\phi(-z) = 1 - \phi(z)$

blue area = $\phi(z_0)$



white area = $1 - \phi(z_0)$

$-z_0$

$\phi(-z_0)$ = area in red

by symmetry

red area = white area

$\phi(-z_0) = 1 - \phi(z_0)$

Example

Suppose Z is $N(0,1)$.

Evaluate each of the following probabilities:

- $P(Z \leq 1.24)$
- $P(1.24 \leq Z \leq 2.37)$
- $P(-2.37 \leq Z \leq -1.24)$

Recall that the cdf of $Z \sim N(0,1)$

$\Phi(z) = P(Z \leq z) \leftarrow$ read from table!

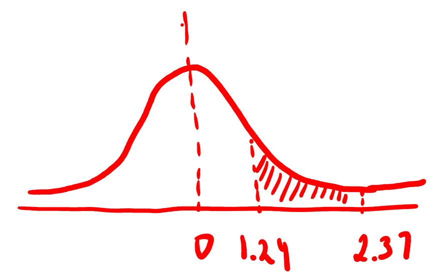
$P(Z \leq 1.24) = \Phi(1.24) = 0.8925$

$P(1.24 \leq Z \leq 2.37) = P(Z \leq 2.37) - P(Z < 1.24)$

$= P(Z \leq 2.37) - P(Z \leq 1.24)$

$= \Phi(2.37) - \Phi(1.24)$

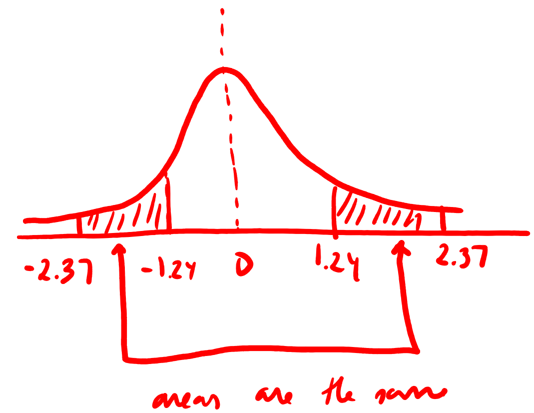
$= 0.9911 - 0.8925 = \ldots$ use a calculator



$P(-2.37 \le Z \le -1.24) = P(1.24 \le Z \le 2.37)$

$=$ previous item ....

Pay attention to any symmetry that can be used to simplify the evaluation!!!



Two more examples:

$P(Z \le -1.59) = \phi(-1.59) = 1 - \phi(\underbrace{1.59}_{\text{table}}) = 1 - 0.9441 =$ calculate

$P(Z > -0.42) = 1 - P(Z \le -0.42)$

$= 1 - \underbrace{\phi(-0.42)}_{1-\phi(0.42)} = 1 - (1 - \phi(0.42)) = 1 - 1 + \phi(0.42)$

$= \phi(0.42) = 0.6628$

table!

Example

Suppose Z is $N(0,1)$.

Find constants a and b such that

$$P(Z \leq a) = 0.9147 \quad \text{and} \quad P(Z \geq b) = 0.0526\ .$$

$$P(Z \leq a) = 0.9147 \iff \Phi(a) = 0.9147 \underset{\text{table}}{\iff} a = 1.37$$

$$\underbrace{P(Z \geq b) = 0.0526 \iff 1 - P(Z < b)}_{\text{complement}} = 0.0526 \iff \underbrace{1 - P(Z \leq b)}_{Z \text{ is a continuous r.v}} = 0.0526$$

$$\iff 1 - \Phi(b) = 0.0526 \iff$$

$$\iff \Phi(b) = 1 - 0.0526 \iff \Phi(b) = 0.9474 \quad \text{table!}$$

$$\iff b = 1.62$$

Percentiles for standard normal distribution

Suppose Z is $N(0,1)$, and let $\alpha \in (0,1)$ (usually $\alpha < 0.5$).

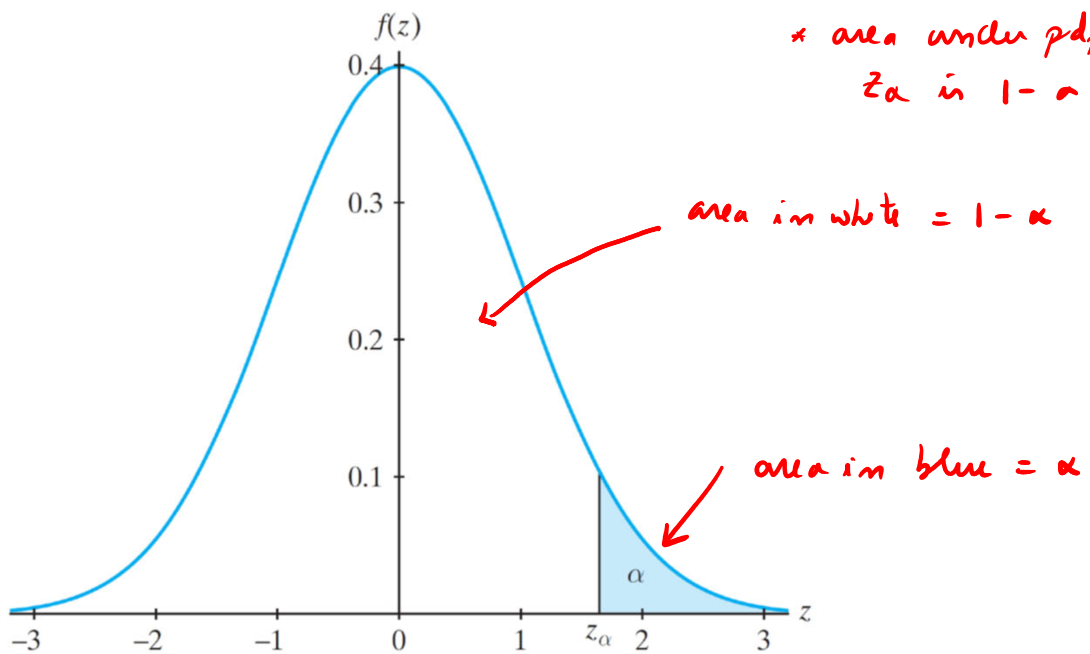
The $100(1-\alpha)$th percentile of its distribution is the number z_α such that

$$P[Z \geq z_\alpha] = \alpha \ ,$$

* area under pdf to the right of z_α is α

that is, the probability to the right of z_α is α.

* area under pdf to the left of z_α is $1-\alpha$



Due to the symmetry of the standard normal pdf, we have

$$P(Z \leq -z_\alpha) = P(Z \geq z_\alpha) = \alpha$$

and, since the subscript of z_α is the right-tail probability, we also have

same symmetry as used in $\phi(-z) = 1 - \phi(z)$ $\rightarrow$

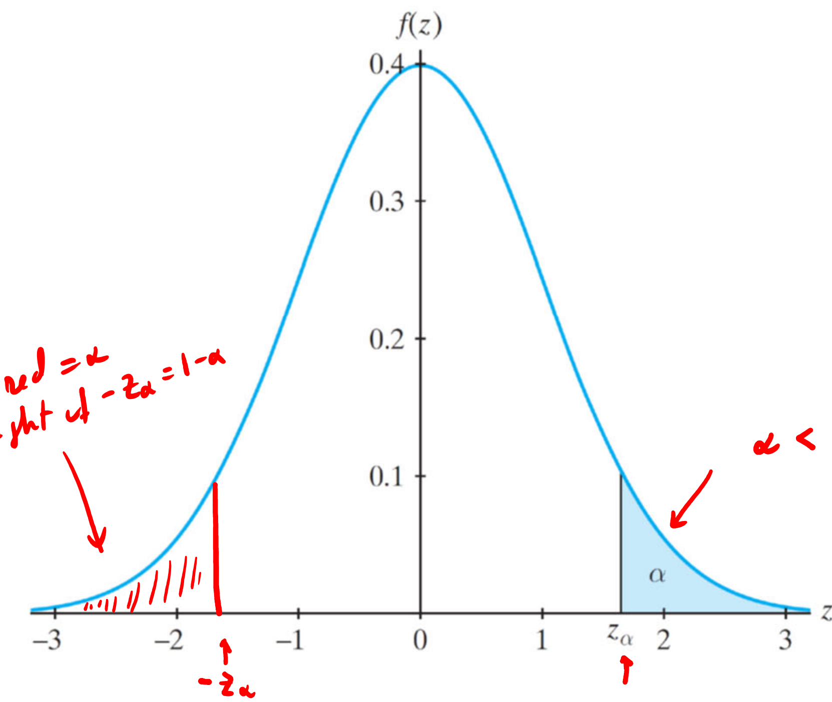
$$z_{1-\alpha} = -z_\alpha \; .$$

$\leftarrow$ if we need any z_α with $\alpha > 0.5$, use this property

area in red $= \alpha$

$\Rightarrow$ area to the right of $-z_\alpha = 1-\alpha$

$\alpha < 0.5$ means $z_\alpha > 0$



Example

Find:

- $z_{0.0125}$ =
- $z_{0.05}$ = 1.645 ← using row for z_α

note that $z_{0.0125} = z_{0.025/2} = 2.240$

look in the row

for $z_{\alpha/2}$ with $\alpha = 0.025$

Standard score

→ allows us to relate any r.v. $X \sim N(\mu, \sigma^2)$ with $Z \sim N(0,1)$

Theorem

If X is $N(\mu, \sigma^2)$, then

$$\boxed{Z = \frac{X - \mu}{\sigma} \sim N(0,1)}$$

is $N(0, 1)$.

Notes:

1. If X is normally distributed, then Z is normally distributed with zero mean and unit variance.

2. Z is often called the standard score associated with X.

3. Standard scores can be used to find probabilities associated with $X \sim N(\mu, \sigma^2)$:

$Z \sim N(0,1)$

$$P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Example

Suppose X is $N(3, 16)$. $\Rightarrow$ $\mu = 3$ and $\sigma^2 = 16$ (and so $\sigma = 4$)

Evaluate each of the following probabilities:

- $P(4 \leq X \leq 8)$
- $P(0 \leq X \leq 5)$

By the previous theorem, since $X \sim N(\mu, \sigma^2)$, then

$$Z = \frac{X-\mu}{\sigma} = \frac{X-3}{4} \sim N(0,1)$$

$$P(4 \leq X \leq 8) = P\left(\frac{4-3}{4} \leq \overbrace{\frac{X-3}{4}}^{Z} \leq \frac{8-3}{4}\right) = P\left(\frac{1}{4} \leq Z \leq \frac{5}{4}\right)$$

$$= P(0.25 \leq Z \leq 1.25) = \Phi(1.25) - \Phi(0.25)$$

$$= 0.8944 - 0.5987 = \cdots$$

$$P(0 \le X \le 5) = P\Big(\frac{0-3}{4} \le \underbrace{\frac{X-3}{4}}_{Z} \le \frac{5-3}{4}\Big) =$$

$$= P\Big(-\frac{3}{4} \le Z \le \frac{2}{4}\Big) = P(-0.75 \le Z \le 0.5)$$

$$= \phi(0.5) - \underbrace{\phi(-0.75)}_{}$$

$$= \phi(0.5) - \overbrace{(1 - \phi(0.75))}$$

$$= \phi(0.5) - 1 + \phi(0.75)$$

$$= 0.6915 - 1 + 0.7734 = \cdots$$

Example

If X is $N(25, 36)$, we find a constant c such that

$$P(|X - 25| \leq c) = 0.9544 \ .$$

Since $X \sim N(\underset{\substack{\uparrow \\ \mu}}{25}, \underset{\substack{\uparrow \\ \sigma^2}}{36})$, then $\mu = 25$ and $\sigma = 6$ (so that $\sigma^2 = 36$)

then $Z = \dfrac{X - \mu}{\sigma} = \dfrac{X - 25}{6} \sim N(0,1)$

Recall the absolute value inequality:

$|x| \leq a, \ a > 0$

is equivalent to

$-a \leq x \leq a$

Observe that $P(|X - 25| \leq c) = 0.9544$

$$\Rightarrow P(-c \leq X - 25 \leq c) = 0.9544$$

$$\Rightarrow P\Big(-\frac{c}{6} \leq \underbrace{\frac{X-25}{6}}_{Z} \leq \frac{c}{6}\Big) = 0.9544$$

$$\Leftrightarrow \quad P\left(-\frac{c}{6} \leq Z \leq \frac{c}{6}\right) = 0.9544$$

$$\Leftrightarrow \quad \phi\left(\frac{c}{6}\right) - \underbrace{\phi\left(-\frac{c}{6}\right)} = 0.9544$$

$$\Leftrightarrow \quad \phi\left(\frac{c}{6}\right) - \left(1 - \phi\left(\frac{c}{6}\right)\right) = 0.9544$$

$$\Leftrightarrow \quad \phi\left(\frac{c}{6}\right) - 1 + \phi\left(\frac{c}{6}\right) = 0.9544$$

$$\Leftrightarrow \quad 2\phi\left(\frac{c}{6}\right) = 1.9544 \quad \Leftrightarrow \quad \phi\left(\frac{c}{6}\right) = \frac{1.9544}{2} = 0.9772$$

$$\Leftrightarrow \quad \phi\left(\frac{c}{6}\right) = 0.9772 \underset{\text{table}}{\Leftrightarrow} \frac{c}{6} = 2 \quad \Leftrightarrow \quad \boxed{c = 12}$$

Math 3501 - Probability and Statistics I

4.1 - Bivariate distributions of the discrete type

Joint probability mass function

Definition (Joint probability mass function)

Let X and Y be two random variables defined on a discrete sample space, and let S denote the corresponding two-dimensional space of X and Y, the two random variables of the discrete type.

The *joint probability mass function* (abbreviated joint pmf) of X and Y, denoted $f(x, y)$, is defined as

$$f(x, y) = P(X = x, Y = y) .$$

Properties:

(a) $0 \leq f(x, y) \leq 1$ for all $(x, y) \in \mathbb{R}^2$

(b) $\displaystyle\sum_{(x,y)\in S} f(x, y) = 1$

(c) $P[(X, Y) \in A] = \displaystyle\sum_{(x,y)\in A} f(x, y)$, where $A \subset \mathbb{R}^2$

analogous to 1-dim case (Sec 2.1)

Example

Roll a pair of fair 6-faced dice. Let X denote the smaller and Y the larger outcome on the dice.

Determine the joint pmf of X and Y.

X = r.v. giving the smaller score

Y = r.v. " " larger score

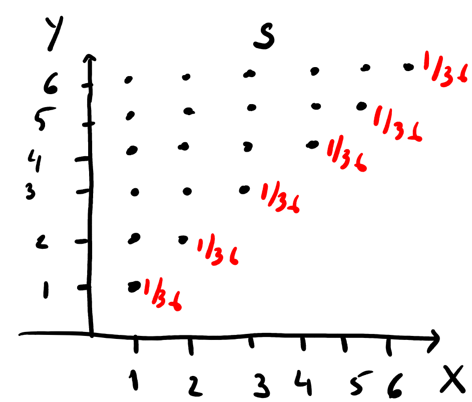
e.g.

if scores are $(2,3)$, $X = 2$ and $Y = 3$

if scores are $(4,4)$, $X = 4$ and $Y = 4$

The space of X and Y is

$$S = \{(i,j) : i,j \in \{1,2,\dots,6\} \text{ and } i \leq j\} \longrightarrow$$



$$f(2,3) = P(X=2, Y=3) = P(\{(2,3),(3,2)\}) = \frac{2}{36} = \frac{1}{18}$$

$$f(4,4) = P(X=4, Y=4) = P(\{(4,4)\}) = \frac{1}{36}$$

if $i < j \implies f(i,j) = \frac{2}{36}$

if $i = j \implies f(i,j) = \frac{1}{36}$

$$f(x,y) = \begin{cases} \frac{2}{36} & \text{if } i,j \in \{1,\dots,6\} \text{ and } i<j \\ \frac{1}{36} & \text{if } i,j \in \{1,\dots,6\} \text{ and } i=j \end{cases}$$