Math 3501 - Probability and Statistics I

1.2 - Methods of Enumeration

Combination of n objects taken r at a time

Definition

Given a set of n different objects, the number of (unordered) subsets of size $r \leq n$, denoted ${}_nC_r$ or $\left(\begin{array}{c} n \\ r \end{array} \right)$, and called a *combination of n objects taken r at a time*, is

$$ {}_nC_r = \left(\begin{array}{c} n \\ r \end{array} \right) = \frac{n!}{r!(n-r)!} \ . $$

Note: the quantity defined above

$$ \left(\begin{array}{c} n \\ r \end{array} \right) = {}_nC_r $$

may also be regarded as the number of ways in which r objects can be selected without replacement from n objects when the order of selection is disregarded.

Distinguishable permutation

Definition

Each of the ${}_nC_r$ permutations of n objects, r of one type and $n-r$ of another type, is called a *distinguishable permutation*.

n objects → r type 1, $n-r$ type 2

cannot distinguish elements of same type

$\underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_}_{n \text{ positions}}$

$$\frac{n!}{r!\,(n-r)!} = {}_nC_r = \binom{n}{r}$$

$$\frac{n!}{n_1!\,n_2!}$$

$r!$ $(n-r)!$

EXAMPLE 12 flowers

→ 4 red

→ 8 blue

How many ways can we arrange flowers in a row

$$\underbrace{\underline{R}\ \underline{\ \ }\ \underline{R}\ \underline{R}\ \ldots\ \underline{R}}_{12 \text{ positions}}$$

$$\binom{12}{8} = \binom{12}{4} = \frac{12!}{4!\,8!}$$

Multinomial coefficients

Suppose that in a set of n objects, n_1 are similar, n_2 are similar, ..., n_s are similar, where

$$n_1 + n_2 + \cdots + n_s = n \; .$$

s distinct types of objects

Then the number of distinguishable permutations of the n objects is

multinomial coefficient

$$\binom{n}{n_1, n_2, \ldots, n_s} = \frac{n!}{n_1! n_2! \cdots n_s!}$$

$$\frac{n!}{r!\,(n-r)!}$$

The quantity above is sometimes called a *multinomial coefficient* as it appears in the expansion of

$$(a_1 + a_2 + \cdots + a_s)^n \; .$$

$S = 2$

$n_1 + n_2 = n$

$n_2 = n - n_1$

BC: $n_1 = r$, $n_2 = n - r$

Example

If 10 people are to be divided into 3 committees of respective sizes 3, 3, and 4, how many divisions are possible?

$n = 10$ → Committee 1 → $n_1 = 3$

" 2 → $n_2 = 3$

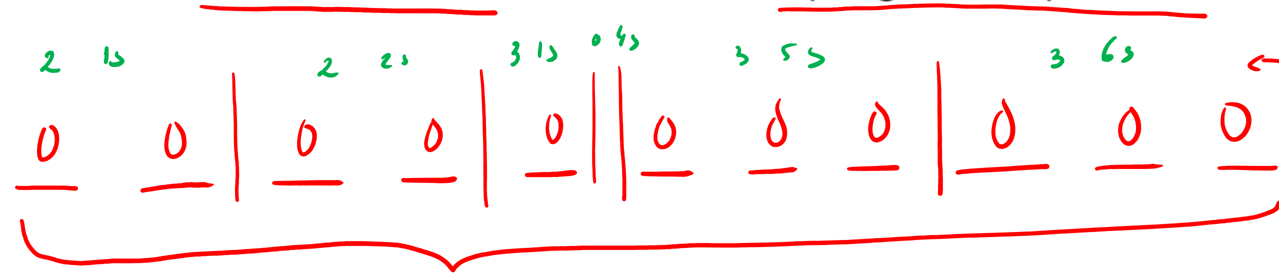
3 → $n_3 = 4$

$$\binom{10}{3,3,4} = \frac{10!}{3!\cdot 3!\cdot 4!}$$

Question: Find the number of possible samples of size r that can be selected out of n objects when the order is irrelevant and when sampling with replacement.

# of objects in the set from which we sample

$m-1$ dividers to split objects into m group



Sample of size r

Total # of such sequences is $\dfrac{(r+m-1)!}{r!\,(m-1)!} = {}_{r+m-1}C_{r} = \dbinom{r+m-1}{r}$

Roll 6-faced die → possible outcomes are $\{1, 2, \ldots, 6\}$ $\qquad \underline{\underline{m = 6}}$

10 times $\qquad r = 10$

How many possible sequences of values can we get without regard for the ordering

counted as the same sequence as

$1\ 6\ 6\ 3\ 3\ 4\ 5\ 5\ 3\ 3 \rightarrow 1\ 3\ 3\ 3\ 3\ 4\ 5\ 5\ 6\ 6$

one 1 $\quad$ one 2 $\quad$ 3 three times $\quad$ 5 twice $\quad$ 6 twice

$$\underbrace{0\,|\,|\,0\,0\,0\,0\,|\,0\,|\,0\,0\,|\,0\,0}_{r = 10 \text{ positions to be filled}}$$

1 once
2 does not show
3 4 times
4 once
5 twice
6 twice

Use $m-1 = 6-1 = 5$ vertical bars to split the sequence of 0s into $m = 6$ classes

Total number of ways is
$$\frac{\overset{(r+m-1)!}{15!}}{\underset{r!}{10!}\ \underset{(m-1)!}{5!}}$$

Remark

The number of unordered samples of size r that can be selected out of n objects when sampling with replacement is

$$\boxed{{}_{n-1+r}C_r = \binom{n-1+r}{r} = \frac{(n-1+r)!}{r!(n-1)!}}$$

SUMMARY: Sample r objects
from a set with n objects

$\ast$ ordered sample
- with replacement : n^r
- without replacement : ${}_nP_r = \dfrac{n!}{(n-r)!}$ ←

$\ast$ unordered sample
- with replacement : ${}_{r+n-1}C_r = \binom{r+n-1}{r} = \dfrac{(r+n-1)!}{r!\,(n-1)!}$
- without replacement : ${}_nC_r = \binom{n}{r} = \dfrac{n!}{(n-r)!\,r!}$

Math 3501 - Probability and Statistics I

1.3 - Conditional Probability

Conditional Probability

Definition

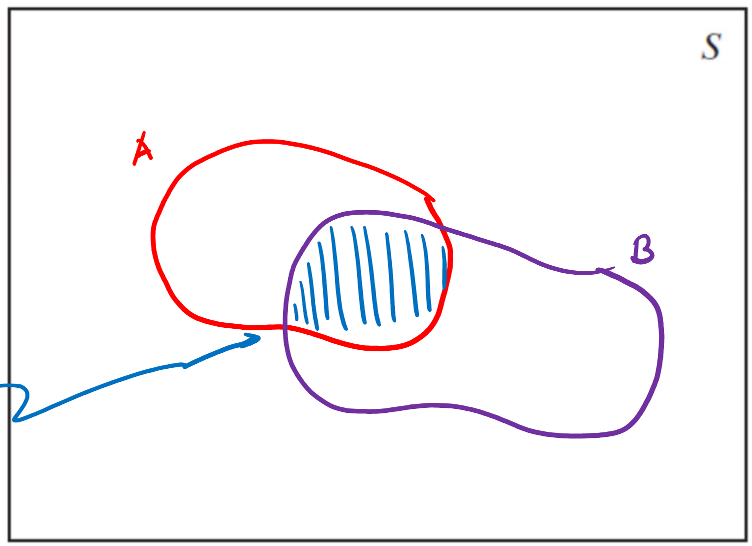
The conditional probability of an event A, given that $\overbrace{\text{event } B \text{ has occurred}}^{\text{some additional information}}$, is defined by

$$\underbrace{P(A \mid B)}_{\text{probability of } A \text{ given } B} = \frac{P(A \cap B)}{P(B)}$$

relative size according to P of $A \cap B$ compared to B

provided that $P(B) > 0$.

$P(A)$: relative size of A compared to S as measured by the function

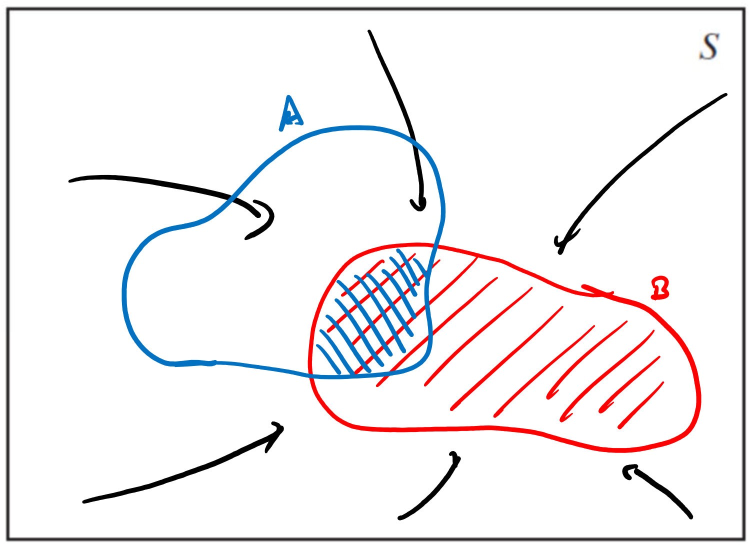


Knowing that B occurs only the part of A shaded blue might occur

Conditional probability
↓
reassess how likely it is that A occurs knowing that B occurs
↓
how large is $A \cap B$ compared to B according to P

Interpretation:

- the information "event B has occurred" may be regarded as specifying a new, smaller, sample space
- to determine $P(A \mid B)$, we calculate the probability of the part of A that is contained in B, and normalize it by $P(B)$



$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Remark

Conditional probability satisfies the axioms of probability.

$P(\cdot \mid B)$

· stands for "plug in" any subset of S we choose to

Namely, given an event B with $P(B) > 0$, we have:

(a) $P(A \mid B) \geq 0$ for any subset A of S

(b) $P(\cancel{B}S \mid B) = 1$

(c) *if $A_1, A_2, A_3, \ldots$ are mutually exclusive events, then*

$$P(A_1 \cup A_2 \cup \cdots \cup A_k \mid B) = P(A_1 \mid B) + P(A_2 \mid B) + \cdots + P(A_k \mid B)$$

for each positive integer k, and

$$P(A_1 \cup A_2 \cup \cdots \mid B) = P(A_1 \mid B) + P(A_2 \mid B) + \cdots$$

for an infinite, but countable, number of events.

OUTCOME: Given $B \subset S$ with $P(B) > 0$, we have that $P(\cdot \mid B)$ is by itself a probability!

Consequence:

Since $P(\cdot \mid B)$ satisfies the axioms of probability, then all the properties of probability discussed previously extend immediately to conditional probabilities:

1. $P(\bar{A} \mid B) = 1 - P(A \mid B)$ for any event A
2. $P(\phi \mid B) = 0$
3. if $A_1 \subseteq A_2$ then $P(A_1 \mid B) \leq P(A_2 \mid B)$
4. $P(A \mid B) \leq 1$ for any event A
5. $P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B) - P(A_1 \cap A_2 \mid B)$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

$$P(C \mid D) = \frac{P(C \cap D)}{P(D)}$$

Example

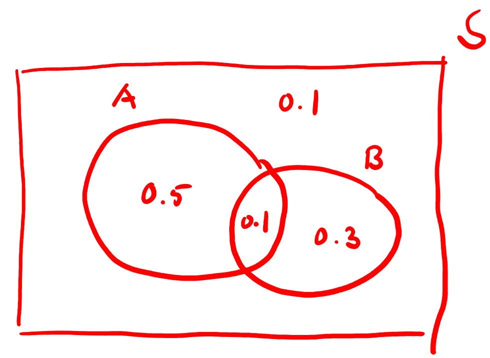
Suppose that $P(A) = 0.6$, $P(B) = 0.4$, and $P(A \cap B) = 0.1$.

Given that the outcome of the experiment belongs to B, what is the probability that A occurs?

0.25

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.1}{0.4} = \frac{1}{4}$$

different from $P(A) = 0.6$



Example

Suppose that two fair dice are rolled and that each of the 36 possible outcomes is equally likely to occur.

Suppose further that we observe that the first die is a 3. B

Given this information, what is the probability that the sum of the two dice is 5? A

Experiment: Roll two fair dice $\quad S = \{(i,j) : i,j \in \{1,2,\dots,6\}\} \quad |S| = 36$

we are told that

B : event "first die came up 3" has occurred

and we want to find $P(A|B)$ for the event A : "sum is 5"

In terms of sets, we have $\quad A = \{(1,4),(2,3),(3,2),(4,1)\}$ and so $P(A) = \frac{|A|}{|S|} = \frac{4}{36} = \frac{1}{9}$

$\to B = \{(3,1),(3,2),\dots,(3,6)\}$ and so $P(B) = \frac{|B|}{|S|} = \frac{6}{36} = \frac{1}{6}$

$A \cap B = \{(3,2)\}$ and so $P(A\cap B) = \frac{|A\cap B|}{|S|} = \frac{1}{36}$

Conclusion : $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{1/36}{1/6} = \frac{1}{6}$ $\leftarrow$ different from $P(A) = \frac{1}{9}$

Multiplication rule

The probability that two events, A and B, both occur is given by:

$$P(A \cap B) = P(A)P(B \mid A) \qquad \text{provided } P(A) > 0 \longrightarrow P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$\Rightarrow P(A \cap B) = P(A).P(B|A)$$

or by

$$P(A \cap B) = P(B)P(A \mid B) \qquad \text{provided } P(B) > 0 \longrightarrow P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow P(A \cap B) = P(B).P(A|B)$$

Example

An urn contains 10 balls, of which six are blue and four are red.

Two balls are extracted, in sequence, with no replacement.

Find the probability that the first ball extracted is blue and the second is red.

10 balls → 6 blue, 4 red → Experiment: extract two balls in sequence with no replacement

Define the events: A_1 : 1st ball is blue, A_2 : 2nd ball is red } we want to find $P(A_1 \cap A_2)$

$$\underbrace{P(A_1 \cap A_2) = P(A_1) \cdot P(A_2 \mid A_1)}_{\text{Multiplication rule}} = \underbrace{\frac{6}{10}}_{P(A_1)} \cdot \underbrace{\frac{4}{9}}_{P(A_2 \mid A_1)} = \frac{24}{90} = \cdots$$

1st extraction: 6 blue balls out of 10

2nd extraction: 5 blue balls (and 4 red) out of 9

Example (continued)

An urn contains 10 balls, of which six are blue and four are red.

assuming no replacement as in previous example

Find the probability that the sixth ball extracted is the third blue ball to be extracted.

10 balls → 6 blue, 4 red

Experiment : extract 6 balls.

A_1 : event "there are two blue balls among the first five balls extracted"

A_2 : event "sixth ball to be extracted is blue"

Find $P(A_1 \cap A_2) = P(A_1) \cdot P(A_2 \mid A_1) = \dfrac{\binom{6}{2} \cdot \binom{4}{3}}{\binom{10}{5}} \cdot \dfrac{4}{5} = \cdots$

multiplication rule

Intermediate steps : $P(A_1) = \dfrac{\binom{6}{2} \cdot \binom{4}{3}}{\binom{10}{5}}$

$P(A_2 \mid A_1) = \dfrac{4}{5}$