Sec. 7.2 Ex. 9

a) Let $X_1, \dots, X_m$ and $Y_1, \dots, Y_m$, with $m = 10$, be the percentages for males, pre-program and post-program, respectively.

Define a new sequence of random variables

$$D_i = X_i - Y_i \quad , \quad i = 1, 2, \dots, m \quad (m = 10)$$

and suppose that $D_i \sim N(\mu_D, \sigma_D^2)$ where $\mu_D = \mu_X - \mu_Y$ and σ_D^2 are both unknown.

Recall that:

(1) $\bar{D} \sim N\left(\mu_D, \frac{\sigma_D^2}{m}\right)$ and so $Z = \dfrac{\bar{D} - \mu_D}{\sigma_D / \sqrt{m}} \sim N(0,1)$

(2) $U = \dfrac{(m-1) S_D^2}{\sigma_D^2} \sim \chi^2(m-1)$

(3) $\bar{D}$ and S_D^2 are independent

(4) $T = \dfrac{Z}{\sqrt{U/m-1}} = \dfrac{\bar{D} - \mu_D}{S_D / \sqrt{m}} \sim t(m-1)$

Thus, since $P\left(-t_{\alpha/2}(n-1) \leq T \leq t_{\alpha/2}(n-1)\right) = 1-\alpha$, we obtain that $P\left(-t_{\alpha/2}(n-1) \leq \frac{\bar{D} - \mu_D}{S_D/\sqrt{n}} \leq t_{\alpha/2}(n-1)\right) = 1-\alpha$

Solving the inequalities for μ gives $P\left(\bar{D} - t_{\alpha/2}(n-1) \cdot \frac{S_D}{\sqrt{n}} \leq \mu_D \leq \bar{D} + t_{\alpha/2}(n-1) \cdot \frac{S_D}{\sqrt{n}}\right) = 1-\alpha$.

Since we want a 90% confidence interval for $\mu_D = \mu_X - \mu_Y$ we need to pick $\alpha = 0.1$ (so that $1-\alpha = 0.9$). Moreover, we need to evaluate

$$\bar{d} = \frac{1}{n} \sum_{i=1}^{n} d_i \approx 0.447$$

and

$$S_D = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (d_i - \bar{d})^2} \approx 1.7296$$

Hence, the endpoints of the 90% confidence interval for $\mu_D = \mu_X - \mu_Y$ are

$$\bar{d} \pm t_{\alpha/2}(n-1) \cdot \frac{S_D}{\sqrt{n}} = 0.447 \pm \overbrace{(1.833)}^{t_{0.05}(8)} \cdot \frac{(1.7296)}{\sqrt{10}}$$

Completing the evaluations, we obtain the interval $[-0.556, 1.450]$