Math 3501 - Probability and Statistics I

1.3 - Conditional Probability

Conditional Probability

Some additional information

Definition

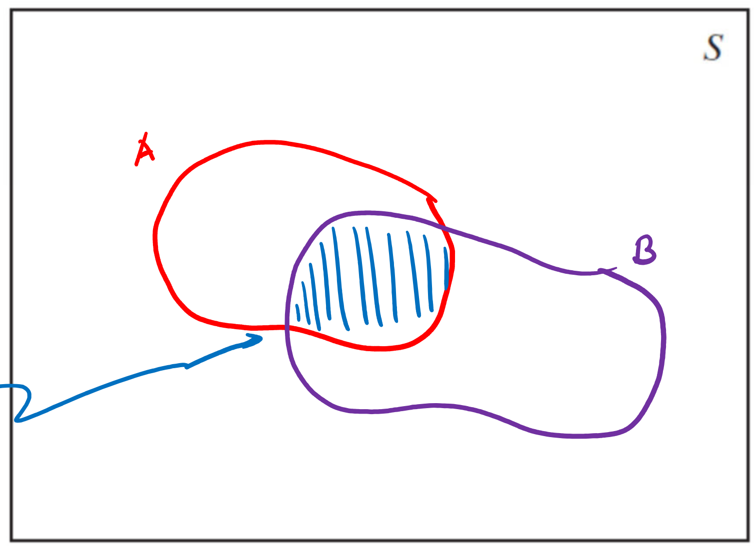
The conditional probability of an event A, given that event B has occurred, is defined by

$$\underbrace{P(A \mid B)}_{\text{probability of } A \text{ given } B} = \frac{P(A \cap B)}{P(B)}$$

relative size according to P of $A \cap B$ compared to B

provided that $P(B) > 0$.

P(A): relative size of A compared to S as measured by the function



Knowing that B occurs only the part of A shaded blue might occur

Conditional probability
↓
reassess how likely it is that A occurs knowing that B occurs
↕
how large is $A \cap B$ compared to B according to P

Remark

Conditional probability satisfies the axioms of probability.

$P(\cdot \mid B)$

$\cdot$ stands for "plug in" any subset of S we choose to

Namely, given an event B with $P(B) > 0$, we have:

(a) $P(A \mid B) \geq 0$ for any subset A of S

(b) $P(\cancel{B}S \mid B) = 1$

(c) if $A_1, A_2, A_3, \ldots$ *are mutually exclusive events, then*

$$P(A_1 \cup A_2 \cup \cdots \cup A_k \mid B) = P(A_1 \mid B) + P(A_2 \mid B) + \cdots + P(A_k \mid B)$$

for each positive integer k, and

$$P(A_1 \cup A_2 \cup \cdots \mid B) = P(A_1 \mid B) + P(A_2 \mid B) + \cdots$$

for an infinite, but countable, number of events.

OUTCOME: Given $B \subset S$ with $P(B) > 0$, we have that $P(\cdot \mid B)$ is by itself a probability!

Consequence:

Since $P(\cdot \mid B)$ satisfies the axioms of probability, then all the properties of probability discussed previously extend immediately to conditional probabilities:

1. $P(\bar{A} \mid B) = 1 - P(A \mid B)$ for any event A
2. $P(\phi \mid B) = 0$
3. if $A_1 \subseteq A_2$ then $P(A_1 \mid B) \leq P(A_2 \mid B)$
4. $P(A \mid B) \leq 1$ for any event A
5. $P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B) - P(A_1 \cap A_2 \mid B)$

Multiplication rule

The probability that two events, A and B, both occur is given by:

$$P(A \cap B) = P(A)P(B \mid A) \qquad \text{provided } P(A) > 0$$

$P(B|A) = \dfrac{P(A \cap B)}{P(A)}$

$\Rightarrow P(A \cap B) = P(A).P(B|A)$

or by

$$P(A \cap B) = P(B)P(A \mid B) \qquad \text{provided } P(B) > 0$$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$\Rightarrow P(A \cap B) = P(B).P(A|B)$

Example (continued)

An urn contains 10 balls, of which six are blue and four are red.

Find the probability that the sixth ball extracted is the third blue ball to be extracted.

assuming no replacement as in previous example

10 balls → 6 blue, 4 red

Experiment: extract 6 balls.

A_1: event "there are two blue balls among the first five balls extracted"

A_2: event "sixth ball to be extracted is blue"

Find $P(A_1 \cap A_2) = P(A_1) \cdot P(A_2 \mid A_1) = \dfrac{\binom{6}{2} \cdot \binom{4}{3}}{\binom{10}{5}} \cdot \dfrac{4}{5} = \dots$

multiplication rule

Intermediate steps:

$$P(A_1) = \frac{\binom{6}{2} \cdot \binom{4}{3}}{\binom{10}{5}}$$

$$P(A_2 \mid A_1) = \frac{4}{5}$$

QUESTION: How to solve the previous example if replacement was allowed?

Define A_1, A_2 in a similar way

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2 \mid A_1)$$

$$P(A_1) = \frac{\binom{5}{2} \cdot 6^2 \cdot 4^3}{10^5} = \binom{5}{2} \left(\frac{6}{10}\right)^2 \left(\frac{4}{10}\right)^3$$

related with binomial distr (we will study in chp 2)

$$P(A_2 \mid A_1) = \frac{6}{10}$$

(since we're replacing the balls, each time a ball is extracted, the urn has 10 balls, six of which are blue)

$$\left.\begin{matrix} R\,R\,R\,B\,B \\ B\,R\,R\,B\,B \end{matrix}\right\} \rightsquigarrow \frac{5!}{2!\,3!} = \binom{5}{2}$$

$$\underbrace{\underline{B}\ \underline{B}}_{6^2}\ \underbrace{\underline{R}\ \underline{R}\ \underline{R}}_{4^3}$$

Remark

The multiplication rule can be extended to three or more events.

Let A, B and C be three events and suppose that $P(A \cap B) > 0$.

Then, it holds that

$$P(A \cap B \cap C) = P(A)P(B \mid A)P(C \mid A \cap B)$$

For 4 events:

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = \overbrace{\underbrace{P(A_1) . P(A_2 | A_1)}_{P(A_1 \cap A_2)} . P(A_3 | A_1 \cap A_2)}^{P(A_1 \cap A_2 \cap A_3)} . P(A_4 | A_1 \cap A_2 \cap A_3)$$

Example

Five cards are to be dealt successively at random and without replacement from an ordinary deck of 52 playing cards.

Find the probability of receiving, in order, a 10, a jack, a queen, a king and an ace.

Define the events:

A_1 = "1st card is a 10"

A_2 = "2nd card is a jack"

A_3 = "3rd card is a queen"

A_4 = "4th card is a king" ← "queen"

A_5 = "5th card is an ace"

We want to find the probability of $A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5$.

Use the multiplication rule:

$$P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) = P(A_1) \cdot P(A_2 | A_1) \cdot P(A_3 | A_1 \cap A_2) \cdot P(A_4 | A_1 \cap A_2 \cap A_3) \cdot P(A_5 | A_1 \cap A_2 \cap A_3 \cap A_4)$$

$$= \frac{4}{52} \cdot \frac{4}{51} \cdot \frac{4}{50} \cdot \frac{4}{49} \cdot \frac{4}{48} = \dots$$

$$\frac{3}{49}$$

Two cards: $S = \{(i,j) : i$ in any card, j in all cards but the 1^{st} one$\}$ $\quad |S| = 52 \times 51$

$A_1 = \{(i,j) : i$ in a 10, j in any card$\}$ $\quad |A_1| = 4 \times 51$

$A_2 = \{(i,j) : i$ in any card, j in a jack$\}$ $\quad |A_2| = 4 \times 51$

$A_1 \cap A_2 = \{(i,j) : i$ in a 10, j in a jack$\}$ $\quad |A_1 \cap A_2| = 4 \times 4$

$$P(A_2 \mid A_1) = \frac{P(A_2 \cap A_1)}{P(A_1)} = \frac{\frac{16}{52 \times 51}}{\frac{4 \cdot 51}{52 \times 51}} = \frac{4 \cdot 4}{4 \cdot 51} = \frac{4}{51}$$

Math 3501 - Probability and Statistics I

1.4 - Independent Events

Independent Events

Two events A and B are independent if the occurrence of one of them does not affect the probability of the occurrence of the other.

More formally, the events A and B are independent if

$$P(B \mid A) = P(B) \qquad \text{provided } P(A) > 0$$

$$\frac{P(A \cap B)}{P(A)} = P(B)$$

or

$$P(A \mid B) = P(A) \qquad \text{provided } P(B) > 0$$

$$\frac{P(A \cap B)}{P(B)} = P(A)$$

both give

$$\boxed{P(A \cap B) = P(A) \cdot P(B)}$$

Definition

Events A and B are *independent* if and only if

$$\boxed{P(A \cap B) = P(A)P(B)}.$$

Otherwise, A and B are called *dependent* events.

Properties:

1) the definition always holds if $P(A) = 0$ or $P(B) = 0$.

events with zero probability are always independent from all other events.
In particular, ϕ is always independent from any other event

2) If A and B are independent, then so are the following pairs of events:

A and B' $\qquad$ A' and B $\qquad$ A' and B'

are independent

Example

Suppose we roll a fair die twice. $S = \{(i,j) : i,j \in \{1,\dots,6\}\}$ $|S| = 36$

Consider the following events:

- A: sum of the dice is 6 $A = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$ $|A| = 5$
- B: the first die equals 4 $B = \{(4,1),(4,2),\dots,(4,6)\}$ $|B| = 6$

Are the events A and B independent? $A \cap B = \{(4,2)\}$ $|A \cap B| = 1$

Since

i) $P(A) = \dfrac{|A|}{|S|} = \dfrac{5}{36}$

ii) $P(B) = \dfrac{|B|}{|S|} = \dfrac{6}{36}$

iii) $P(A \cap B) = \dfrac{|A \cap B|}{|S|} = \dfrac{1}{36}$

$P(A|B) \neq P(A)$

$P(B|A) \neq P(B)$

We have that $P(A) \cdot P(B) = \dfrac{5}{36} \cdot \dfrac{6}{36} \neq \dfrac{1}{36} = P(A \cap B)$

$\Rightarrow$ A and B are not independent

Example

Suppose we roll a fair die twice. $S = \{ (i,j) : \ i,j \in \{1,\dots,6\} \}$ $|S| = 6^2 = 36$

Consider the following events:

- A: sum of the dice is 7 $A = \{ (1,6), (2,5), \dots, (6,1) \}$ $|A| = 6$
- B: the first die equals 4 $B = \{ (4,1), (4,2), \dots, (4,6) \}$ $|B| = 6$

Are the events A and B independent? $A \cap B = \{ (4,3) \}$ $|A \cap B| = 1$

Note :

i) $P(A) = \frac{|A|}{|S|} = \frac{6}{36} = \frac{1}{6}$

ii) $P(B) = \frac{|B|}{|S|} = \frac{6}{36} = \frac{1}{6}$

iii) $P(A \cap B) = \frac{|A \cap B|}{|S|} = \frac{1}{36}$

$P(B | A) = P(B)$

$P(A | B) = P(A)$

Since $P(A) \cdot P(B) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = P(A \cap B)$, then A and B are independent events !!!

Example

Suppose we roll a fair die twice. $S = \{(i,j) : i,j \in \{1,\dots,6\}\}$ $|S| = 36$

Consider the following events:

- A: odd outcome in first roll $A = \{(i,j) : i \text{ odd}\}$ $|A| = 3 \cdot 6 = 18$ $P(A) = \frac{1}{2}$
- B: odd outcome in second roll $B = \{(i,j) : j \text{ odd}\}$ $|B| = 6 \cdot 3 = 18$ $P(B) = \frac{1}{2}$
- C: sum of the two rolls is odd $C = \{(i,j) : i+j \text{ odd}\}$ $|C| = 18$ $P(C) = \frac{1}{2}$

What can we say regarding independence of A, B and C?

A and B : $A \cap B = \{(i,j) : i \text{ odd}, j \text{ odd}\} \Rightarrow |A \cap B| = 3 \cdot 3 = 9 \Rightarrow P(A \cap B) = \frac{|A \cap B|}{|S|} = \frac{1}{4}$

$P(A) \cdot P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P(A \cap B) \Rightarrow$ A and B are independent

A and C : $A \cap C = \{(i,j) : i \text{ odd}, j \text{ even}\} \Rightarrow |A \cap C| = 3^2 = 9 \Rightarrow P(A \cap C) = \frac{|A \cap C|}{|S|} = \frac{1}{4}$

so that $i+j$ is odd

$P(A) \cdot P(C) = P(A \cap C) \Rightarrow$ A and C are independent

B and C: $B \cap C = \{ (i,j) : j \text{ is odd, } \underset{\substack{\uparrow \\ \text{so that } i+j \text{ is odd}}}{i \text{ is even}} \} \Rightarrow |B \cap C| = 3^2 = 9 \Rightarrow P(B \cap C) = \frac{1}{4}$

$P(B \cap C) = P(B) \cdot P(C) = \frac{1}{4} \Rightarrow$ B and C are independent

CONCLUSION (so far): A, B, and C are pairwise independent!

BUT $\quad A \cap B \cap C = \phi \quad$ [because if A and B both occur then since 1st and 2nd rolls are odd, their sum must be even, not odd]

and so $\quad P(A) \cdot P(B) \cdot P(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8} \neq 0 = P(A \cap B \cap C)$

independence of $\left\{ \begin{array}{l} \text{A and B} \\ \text{A and C} \\ \text{B and C} \end{array} \right.$ does not necessarily translate into independence of the triple A, B, and C!

Mutually independent events

Definition

Events A, B, and C are *mutually independent* if and only if the following two conditions hold:

(a) A, B, and C are *pairwise independent*; that is,

$$P(A \cap B) = P(A)P(B)\ , \quad P(A \cap C) = P(A)P(C)\ , \quad P(B \cap C) = P(B)P(C)$$

(b) $P(A \cap B \cap C) = P(A)P(B)P(C)$

Note: If there is no possibility of misunderstanding, the term "independent" is often used without the modifier "mutually" when several events are considered.

Remark

The definition of mutual independence can be extended to four or more events by requiring that each pair, triple, quartet, and so on, satisfy similar properties.

Namely, the events $A_1, A_2, \ldots, A_n$ are said to be independent if, for every subset $A_{1'}, A_{2'}, \ldots, A_{r'}$, with $r \leq n$, of these events, one has

$$P\left(A_{1'} \cap A_{2'} \cap \ldots \cap A_{r'}\right) = P\left(A_{1'}\right) P\left(A_{2'}\right) \cdots P\left(A_{r'}\right)$$

Finally, we define an infinite set of events to be independent if every finite subset of those events is independent.

Remark

If A, B, and C are mutually independent events, then A is independent of any event formed from B and C.

For instance, the following pairs of events are independent:

(a) A and $(B \cap C)$

(b) A and $(B \cup C)$

(c) A and $(B \cap C')$.

In addition, the events A', B', and C' are mutually independent.

Example

A redundant system has three components: C_1, C_2 and C_3.

In such system, if component C_1 fails, it is bypassed and component C_2 is used. If component C_2 fails, it is bypassed and component C_3 is used.

Suppose that the probability of failure of any one component is 0.1, and assume that the failures of these components are mutually independent events.

Find the probability that the system does not fail.

Define events:

A_i : event component C_i fails : $P(A_i) = 0.1$

$A_1 \cap A_2 \cap A_3 \rightsquigarrow$ event all components fail (system fails)

We want to find $P(\overline{A_1 \cap A_2 \cap A_3}) = 1 - P(A_1 \cap A_2 \cap A_3) \underset{\text{mutual independence}}{=} 1 - P(A_1).P(A_2).P(A_3) =$

$= 1 - (0.1)^3 = 0.999$