Math 4501 - Probability and Statistics II

5.5 - Random functions associated with the normal distribution

(and more consequences of the m.g.f technique of Sec. 5.4)

Linear combination of normally distributed r.v.

Theorem (Review from 3501)

Let $X_1, X_2, \ldots, X_n$ be n independent normally distributed random variables with respective means $\mu_1, \mu_2, \ldots, \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$.

The linear combination

$$Y = \sum_{i=1}^{n} c_i X_i$$

has the normal distribution

$$N\left(\sum_{i=1}^{n} c_i \mu_i, \sum_{i=1}^{n} c_i^2 \sigma_i^2\right) .$$

In words: a linear combination (Y) of independent normally distributed r.v.s $(X_1, X_2, \ldots, X_n)$ is still normally distributed!!

Proof (another example of m.g.f technique):

Let $X_1, X_2, \ldots, X_n$ be independent r.v.s with respective distributions $N(\mu_i, \sigma_i^2)$

Recall, also, that if $X \sim N(\mu, \sigma^2)$, then its m.g.f is $M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$, $t \in \mathbb{R}$.

We want to find the m.g.f of $\boxed{Y = \sum_{i=1}^{n} c_i X_i}$, $c_1, \ldots, c_n \in \mathbb{R}$.

Option 1: (remember the main theorem from Sec. 5.4)

$e^{a_1} \cdot e^{a_2} \cdots e^{a_n} = e^{a_1 + a_2 + \cdots + a_n}$

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(c_i t) = \prod_{i=1}^{n} \exp\left(\mu_i (c_i t) + \frac{1}{2}\sigma_i^2 (c_i t)^2\right) = \exp\left(\sum_{i=1}^{n}\left(\mu_i c_i t + \frac{1}{2}\sigma_i^2 c_i^2 t^2\right)\right)$$

$$= \exp\left(\underbrace{\left(\sum_{i=1}^{n} c_i \mu_i\right)}_{\mu_Y} t + \frac{1}{2}\underbrace{\left(\sum_{i=1}^{n} c_i^2 \sigma_i^2\right)}_{\sigma_Y^2} t^2\right) \leftarrow \text{m.g.f of } N\left(\sum_{i=1}^{n} c_i \mu_i, \sum_{i=1}^{n} c_i^2 \sigma_i^2\right)$$

Option 2: apply the m.g.f technique directly

$$M_Y(t) \underset{\substack{\text{def}\\ \text{of } M_Y}}{=} E\left[e^{tY}\right] \underset{\substack{\text{def}\\ \text{of } Y}}{=} E\left[e^{t\sum_{i=1}^{n} c_i X_i}\right] = E\left[e^{c_1 t X_1 + c_2 t X_2 + \dots + c_n t X_n}\right]$$

$$\underset{\text{prop. of exp}}{=} E\left[e^{c_1 t X_1} \cdot e^{c_2 t X_2} \cdots e^{c_n t X_n}\right] = \underbrace{E\left[e^{c_1 t X_1}\right]}_{M_{X_1}(c_1 t)} \cdot \underbrace{E\left[e^{c_2 t X_2}\right]}_{M_{X_2}(c_2 t)} \cdots \underbrace{E\left[e^{c_n t X_n}\right]}_{M_{X_n}(c_n t)}$$

independent r vs because $X_1, \dots, X_n$ are independent

$$= e^{c_1 \mu_1 t + \frac{1}{2} c_1^2 \sigma_1^2 t^2} \cdots e^{c_n \mu_n t + \frac{1}{2} c_n^2 \sigma_n^2 t^2} = e^{\left(\sum_{i=1}^{n} c_i \mu_i\right) t + \frac{1}{2}\left(\sum_{i=1}^{n} c_i^2 \sigma_i^2\right) t^2}$$

same computation as before!

m.g.f of $Y \sim N\left(\sum_{i=1}^{n} c_i \mu_i, \sum_{i=1}^{n} c_i^2 \sigma_i^2\right)$

Example

Let X_1 and X_2 be independent normally distributed random variables with respective distributions $N(5, 2)$ and $N(2, 1)$.

Find $P(X_1 > X_2)$.

Let $X_1 \sim N(5,2)$ and $X_2 \sim N(2,1)$ independent r.v.s

Observe that $P(X_1 > X_2) = P(\underbrace{X_1 - X_2}_{Y} > 0) = ??$

Define $Y = X_1 - X_2$ and note that Y is $\overbrace{\text{normally distributed}}^{\text{previous theorem}}$ with

mean $\mu_Y = E[Y] = E[X_1 - X_2] = E[X_1] - E[X_2] = 5 - 2 = 3$

and variance $\sigma_Y^2 = Var(Y) = Var(X_1 - X_2) \underset{\uparrow\ \text{independence!}}{=} (1)^2 \cdot Var(X_1) + (-1)^2 \cdot Var(X_2) = 2 + 1 = 3$

Thus $Y \sim N(3, 3)$.

Then, we find that

$$P(X_1 > X_2) = P(\underbrace{X_1 - X_2}_{Y} > 0) = P(Y > 0) = P\left(\overbrace{\frac{Y-3}{\sqrt{3}}}^{Z} > \frac{0-3}{\sqrt{3}}\right)$$

$$= P(Z > -\sqrt{3})$$

$$= 1 - P(Z \leq -\sqrt{3})$$

$$= 1 - \phi(-\sqrt{3}) =$$

$$= 1 - (1 - \phi(\sqrt{3}))$$

$$= \phi(\sqrt{3})$$

$= \text{-----}$

use table to check value!

If $Y \sim N(\mu, \sigma^2)$

then $\underbrace{Z = \frac{Y-\mu}{\sigma} \sim N(0,1)}$

Since $\mu_Y = 3$ and $\sigma_Y^2 = 3 \Rightarrow Z = \frac{Y-3}{\sqrt{3}} \sim N(0,1)$

Sample mean of normally distributed r.v.

Corollary (review from 3501).

Let $X_1, X_2, \ldots, X_n$ be observations of a random sample of size n from the normal distribution $N(\mu, \sigma^2)$.

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The distribution of the sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is $N\left(\mu, \frac{\sigma^2}{n}\right)$

special case of the previous theorem with $c_1 = c_2 = \cdots = c_n = \frac{1}{n}$

and we already knew (result from 2 or 3 classes ago)

if $X_1, \ldots, X_n$ have mean μ and variance σ^2 (even if not normal!)

then $\mu_{\bar{X}} = \mu$ and $\sigma^2_{\bar{X}} = \frac{\sigma^2}{n}$

Example

Let $X_1, X_2, \ldots, X_{64}$ be a random sample from the $N(50, 16)$ distribution.

Compare $P(49 < \bar{X} < 51)$ with $\underbrace{P(49 < X_1 < 51)}$.

is the same as $P(49 < X_i < 51)$ for each $i = 1, \ldots, 64$

because $X_1, \ldots, X_{64}$ are all $N(50, 16)$

Let us compute $P(49 < X_i < 51)$ first!

Recall that if $X \sim N(\mu, \sigma^2)$ then $Z = \dfrac{X - \mu}{\sigma} \sim N(0,1)$

and so, since $X_i \sim N(\underset{\mu}{50}, \underset{\sigma^2 = 16 \Rightarrow \sigma = 4}{16})$, then $Z = \dfrac{X_i - 50}{4} \sim N(0,1)$

Then

$$P(49 < X_i < 51) = P\left(\frac{49-50}{4} < \overbrace{\frac{X_i - 50}{4}}^{Z} < \frac{51-50}{4} \right) =$$

$$= \underline{P(-0.25 < Z < 0.25)} = \phi(0.25) - \underbrace{\phi(-0.25)}_{\phi(-z) = 1 - \phi(z)}$$

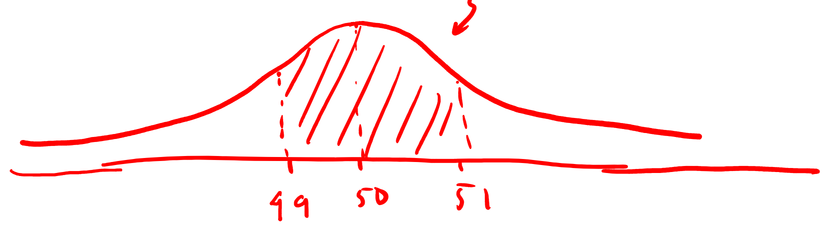
$$= \phi(0.25) - (1 - \phi(0.25)) =$$

$$= \phi(0.25) - 1 + \phi(0.25) =$$

$$= 2\phi(0.25) - 1 = 2 \cdot (0.5987) - 1 = 0.1974$$

$X_i \sim N(50, 16)$

shaded area is 0.1974



Computation for $\overline{X}$:

Since $X_1, \dots, X_{64} \sim N(50, 16) \implies \overline{X} \sim N\left(50, \frac{16}{64}\right) \implies \overline{X} \sim N\left(50, \frac{1}{4}\right)$

previous Corollary

$n = 64$

$X_1, \dots, X_n \sim N(\mu, \sigma^2)$ and independent $\implies \overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

$$\implies Z = \frac{\overline{X} - 50}{\sqrt{\frac{1}{4}}} \sim N(0,1) \implies Z = \frac{\overline{X} - 50}{1/2} \sim N(0,1)$$

If $X \sim N(\mu, \sigma^2) \implies Z = \dfrac{X - \mu}{\sigma} \sim N(0,1)$

Then: $P(49 < \overline{X} < 51) =$

$$= P\left(\frac{49-50}{1/2} < \frac{\overline{X} - 50}{1/2} < \frac{51-50}{1/2}\right)$$

$$= \phi(-2 < Z < 2) = \phi(2) - \phi(-2)$$

$$= 2\phi(2) - 1 = 2(0.9772) - 1 = 0.9544$$

CONCLUSIONS: $P(49 < X_i < 51) = \underline{0.1974}$

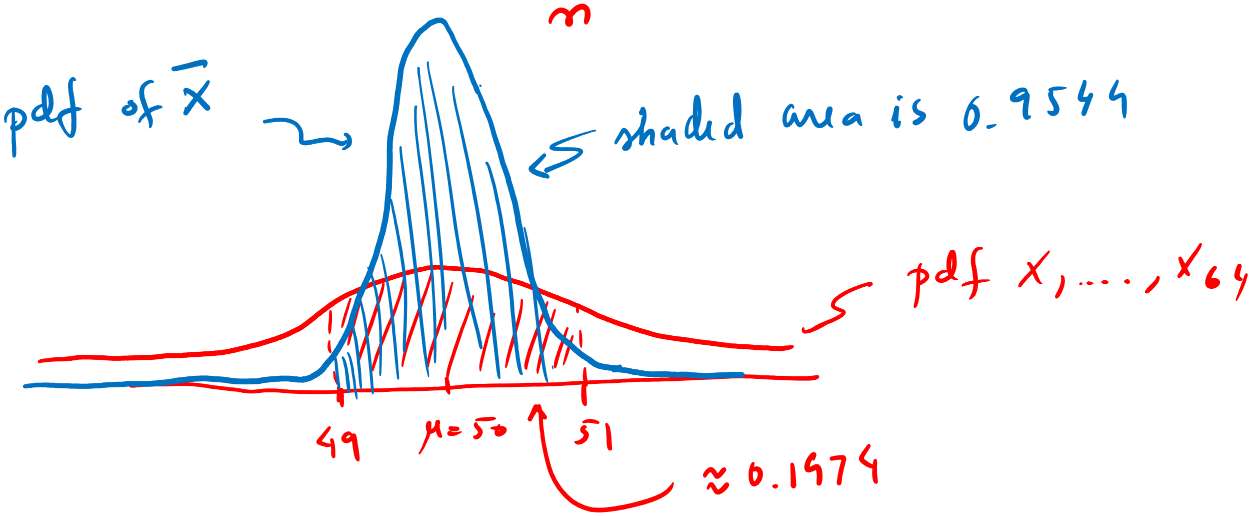
AND $P(49 < \overline{X} < 51) = \underline{0.9544}$

Why? mean of $\overline{X}$ = mean of $X_1, \dots, X_n$ $\leftarrow$

AND

Variance of $\overline{X} = \frac{\sigma^2}{n} <$ Variance $X_1, \dots, X_n = \sigma^2$

Illustration



Sample variance of normally distributed r.v.

Theorem (relation between χ^2 distribution and sampling)

Let $X_1, X_2, \ldots, X_n$ be observations of a random sample of size n from the normal distribution $N(\mu, \sigma^2)$.

previous corollary

The following hold: $\bar{x} = \frac{1}{n} \sum x_i \sim N(\mu, \frac{\sigma^2}{n})$

$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$

1) The sample mean $\bar{X}$ and the sample variance S^2 are independent random variables.

$(n-1)S^2 = \sum (x_i - \bar{x})^2$

2) The random variable

$$\frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{\sigma^2}$$

$\frac{(n-1)S^2}{\sigma^2} = \sum \frac{(x_i - \bar{x})^2}{\sigma^2}$

has a $\chi^2(n-1)$ distribution.

Proof : application of m.g.f - not as simple as the previous examples!

Let $X_1, \dots, X_n \sim N(\mu, \sigma^2)$ independent r.v.s.

We have seen before that :

(1) $Z_i = \dfrac{X_i - \mu}{\sigma} \sim N(0,1)$, $i = 1, \dots, n$, independent

(2) $(Z_i)^2 \sim \chi^2(1)$ independent

we want to see what happens when we replace μ by $\overline{X}$

(3) $W = \sum_{i=1}^{n} Z_i^2 = \sum_{i=1}^{n} \left(\dfrac{X_i - \mu}{\sigma} \right)^2 \sim \chi^2(n)$

Note that

$$W = \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma} \right)^2 = \sum_{i=1}^{n} \left(\frac{(x_i - \bar{x}) + (\bar{x} - \mu)}{\sigma^2} \right)^2$$

sum is zero

$$W = \underbrace{\boxed{\sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{\sigma^2}}}_{\frac{(n-1)s^2}{\sigma^2}} + \underbrace{2 \sum_{i=1}^{n} \frac{(x_i - \bar{x})(\bar{x} - \mu)}{\sigma^2}}_{=0} + \underbrace{\boxed{\sum_{i=1}^{n} \frac{(\bar{x} - \mu)^2}{\sigma^2}}}_{\text{follows } \chi^2(1)}$$

$W \sim \chi^2(n)$

2nd term $\frac{2}{\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x}) \underbrace{(\bar{x} - \mu)}_{\text{does not depend on } i} = \frac{2(\bar{x} - \mu)}{\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x}) = \frac{2(\bar{x} - \mu)}{\sigma^2} \left[n\bar{x} - n\bar{x} \right] = 0$

$\sum_{i=1}^{n} x_i = n\bar{x}$

$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$

Third term:

$$\sum_{i=1}^{n} \underbrace{\frac{(\bar{X}-\mu)^2}{\sigma^2}}_{\text{does not depend on } i} \overset{\text{summing the same value } n \text{ times}}{=} n \cdot \frac{(\bar{X}-\mu)^2}{\sigma^2} = \frac{\overbrace{(\bar{X}-\mu)^2}}{\sigma^2/n} = \Bigg(\underbrace{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}\Bigg)^2 \sim \chi^2(1)$$

since $X_1, \dots, X_n \sim N(\mu, \sigma^2)$

$\Rightarrow \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

$\Rightarrow Z = \dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

CONCLUSION (so far):

$$W = \frac{(n-1)S^2}{\sigma^2} + Z^2 \quad \text{where } W \sim \chi^2(n), \; Z^2 \sim \chi^2(1)$$

and by item 1 we know that S^2 and $\bar{X}$ are independent and so S^2 and Z^2 are also independent!

We can then compute m.g.f of W in terms of the m.g.f.s of $\frac{(n-1)S^2}{\sigma^2}$ and Z^2

$$M_W(t) \underset{\text{def}}{=} E\left[e^{tW}\right] = E\left[e^{t\left(\frac{(n-1)S^2}{\sigma^2} + Z^2\right)}\right] =$$

$$= E\left[e^{t\frac{(n-1)S^2}{\sigma^2}} \cdot e^{tZ^2}\right] =$$

independence of S^2 and Z^2

$$= \underbrace{E\left[e^{t\frac{(n-1)S^2}{\sigma^2}}\right]}_{M_{\frac{(n-1)S^2}{\sigma^2}}(t)} \cdot \underbrace{E\left[e^{tZ^2}\right]}_{M_{Z^2}(t)}$$

CONCLUSION: $$M_W(t) = M_{\frac{(n-1)S^2}{\sigma^2}}(t) \cdot M_{Z^2}(t)$$

Since $W \sim \chi^2(n) \Rightarrow M_W(t) = (1-2t)^{-\frac{n}{2}}, \quad t < \frac{1}{2}$

and since $Z^2 \sim \chi^2(1) \Rightarrow M_{Z^2}(t) = (1-2t)^{-1/2}, \quad t < \frac{1}{2}$

we get

$$\Rightarrow \underbrace{(1-2t)^{-\frac{n}{2}}}_{M_W(t)} = M_{\frac{(n-1)s^2}{\sigma^2}}(t) \cdot \underbrace{(1-2t)^{-1/2}}_{M_{Z^2}(t)}$$

$$\Rightarrow M_{\frac{(n-1)s^2}{\sigma^2}}(t) = \frac{(1-2t)^{-\frac{n}{2}}}{(1-2t)^{-1/2}} = (1-2t)^{-\frac{(n-1)}{2}} \leftarrow \text{mgf of } \chi^2(n-1)$$

$$\Longrightarrow \frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$$

Summary:

Let $X_1, X_2, \ldots, X_n$ be observations of a random sample of size n from the normal distribution $N\left(\mu, \sigma^2\right)$.

We have seen that:

1) the random variable $U = \sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^2}$ has a $\chi^2(n)$ distribution. (last class!)

2) The random variable $W = \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{\sigma^2}$ has a $\chi^2(n-1)$ distribution.

$$S^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$$

Example

Let X_1, X_2, X_3, X_4 be a random sample from the normal distribution $N(40, 200)$.

Then

$$U = \sum_{i=1}^{4} \frac{(X_i - 40)^2}{200} \quad \text{is} \quad \chi^2(4)$$

and

$$W = \sum_{i=1}^{4} \frac{(X_i - \bar{X})^2}{200} \quad \text{is} \quad \chi^2(3) \ .$$