2.7-11. An airline always overbooks if possible. A particular plane has 95 seats on a flight in which a ticket sells for \$300. The airline sells 100 such tickets for this flight.

(a) If the probability of an individual not showing up is 0.05, assuming independence, what is the probability that the airline can accommodate all the passengers who do show up?

(b) If the airline must return the \$300 price plus a penalty of \$400 to each passenger that cannot get on the flight, what is the expected payout (penalty plus ticket refund) that the airline will pay?

a) Information:

airline sells 100 tickets

only 95 seats available

probability of individual not showing up: 0.05

We want to find:

P(all passengers showing up have a seat)

$= P(\underbrace{\text{more than 4 people do not show up}}_{\text{at most 95 show up}})$

Let X be the R.V. representing the number of individuals that do not show up.

Using independence, we know that $X \sim \text{Binomial}(\underbrace{100}_{n}, \underbrace{0.05}_{p})$

Note:

$X=0 \Rightarrow$ all 100 passengers show up

$X=1 \Rightarrow$ 99 " " "

$X=2 \Rightarrow$ 98 " " "

$X=3 \Rightarrow$ 97 passengers show up

$X=4 \Rightarrow$ 96 passengers show up

$X \geq 5 \Rightarrow$ at most 95 passengers show up

P (all passengers who show up can be seated)

$$= P(\underbrace{X > 4}_{X \geq 5}) = 1 - P(X \leq 4)$$

At this stage we use the approximation of the Binomial (n,p) by Poisson (λ) with $\lambda = np = 100(0.05) = 5$, that is

$$\binom{100}{x} \cdot (0.05)^x \cdot (0.95)^{100-x} \approx \frac{e^{-5} \cdot 5^x}{x!}$$

and so

$$P(X \geq 5) = P(X > 4) = 1 - P(X \leq 4) \approx 1 - 0.44 = 0.56$$

table for Poisson (5)

b) $X=0 \Rightarrow Y = 100 - X = 100$ passengers show up

$\Rightarrow$ 5 cannot be seated $\Rightarrow$ Payout is 700×5

$X=1 \Rightarrow$ 99 passengers show up $\Rightarrow$ 4 cannot be seated $\Rightarrow$ 700×4 payout

$\vdots$

$X=4 \Rightarrow$ 96 passengers show up $\Rightarrow$ 1 cannot be seated $\Rightarrow$ 700×1 payout

# of passengers that cannot be seated is $5-X$, for $X = 0,1,2,3,4$
and the payout is equal to $700(5-X)$, for $X=0,1,2,3,4$

$$E[\text{Payout}] = \sum_{x=0}^{4} 700(5-x) \cdot P(X=x) =$$

$$= 3500 \cdot \underbrace{P(X=0)}_{0.007} + 2800 \cdot \underbrace{P(X=1)}_{0.040-0.007} + 2100 \cdot P(X=2) + 1400\, P(X=3) + 700\, P(X=4)$$

All that remains is to compute $P(X=x)$, $x=0,1,2,3,4$

Possible strategies

Strategy 1 : Use binomial pmf

$$P(X=x) = \binom{100}{x}(0.05)^x \cdot (0.95)^{100-x}, \quad x=0,1,2,3,4$$ [not very convenient]

Strategy 2. Use Poisson distribution approximation

$$P(X=x) \approx \frac{e^{-5}\cdot 5^x}{x!}, \quad x=0,1,2,3,4$$

Strategy 3 : Use Poisson distribution approximation + tables for Poisson

$$P(X=0) = P(X\leq 0) = 0.007$$

$$P(X=1) = P(X\leq 1) - P(X\leq 0) = 0.040 - 0.007$$

For each $x = 1, 2, 3, 4$, we can find

$$P(X = x) = P(X \leq x) - P(X \leq x-1) = \ldots$$

read each from Poisson (5) table

3.1-18. Let $f(x) = 1/2$, $0 < x < 1$ or $2 < x < 3$, zero elsewhere, be the pdf of X.

(a) Sketch the graph of this pdf.

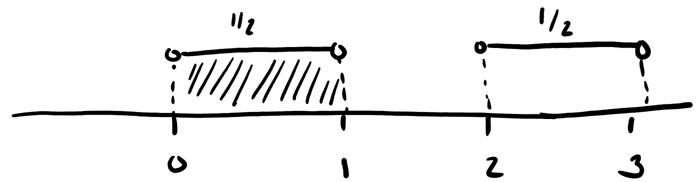
(b) Define the cdf of X and sketch its graph.

(c) Find $q_1 = \pi_{0.25}$.

(d) Find $m = \pi_{0.50}$. Is it unique?

(e) Find $q_3 = \pi_{0.75}$.

a)



$$f(x) = \begin{cases} 1/2 & \text{if } x \in (0,1) \text{ or } x \in (2,3) \\ 0, & \text{otherwise} \end{cases}$$

b) The cdf of X is

$F(x) = P(X \le x)$ → geometrically: this equals to the area under the graph of $f(x)$ to the left of x.

If $x < 0$, $F(x) = 0$

If $0 < x < 1$, then $F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx = \int_0^x \frac{1}{2}\,dt = \frac{x}{2}$

If $1 \le x \le 2$, $F(x) = \frac{1}{2}$ ⇝ $\int_{-\infty}^{x} f(t)\,dt = \underbrace{\int_0^1 f(t)\,dt}_{1/2} + \underbrace{\int_1^x f(t)\,dt}_{0} = 1/2$

If $2 < x < 3$, $\quad F(x) = \int_{-\infty}^{x} f(t)\,dt = \underbrace{\int_0^1 f(t)\,dt}_{1/2} + \underbrace{\int_1^2 f(t)\,dt}_{0} + \underbrace{\int_2^x f(t)\,dt}_{?}$

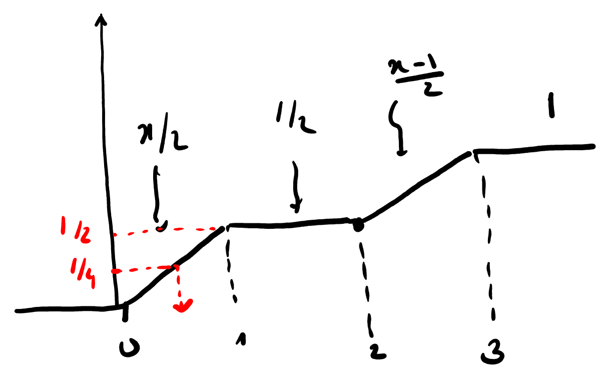
$$= \frac{1}{2} + \int_2^x \frac{1}{2}\,dt = \frac{1}{2} + \frac{x-2}{2}$$

$$= \frac{x-1}{2}$$

If $x \geq 3$, $\quad F(x) = 1$

Conclusion

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{x}{2} & \text{if } 0 < x < 1 \\ \frac{1}{2} & \text{if } 1 \leq x \leq 2 \\ \frac{x-1}{2} & \text{if } 2 < x < 3 \\ 1 & \text{if } x \geq 3 \end{cases}$$



c) To determine $q_1 = \pi_{0.25}$, we need to solve the equation $F(x) = 0.25$ for x which results in $\frac{x}{2} = 0.25$ or $x = 0.5$

We obtain $q_1 = \pi_{0.25} = 0.5$

d) To determine $m = \pi_{0.5}$, we need to solve

$$F(x) = 0.5 \quad \text{for } x$$

This equation has infinitely many solutions.

Any $x \in [1,2]$ satisfies this condition

Thus $m = \pi_{0.5}$ is not unique!

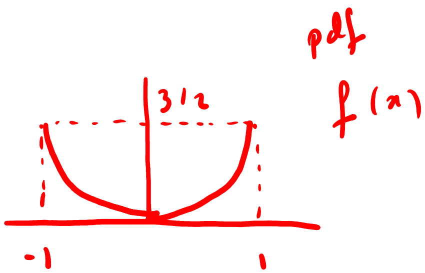
3.1-12. Sketch the graphs of the following pdfs and find and sketch the graphs of the cdfs associated with these distributions (note carefully the relationship between the shape of the graph of the pdf and the concavity of the graph of the cdf):

(a) $f(x) = \left(\frac{3}{2}\right)x^2, \qquad -1 < x < 1.$

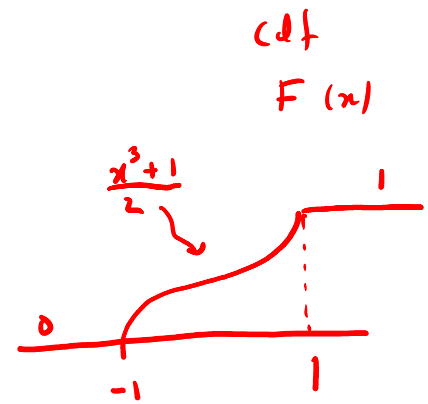
(b) $f(x) = \frac{1}{2}, \qquad -1 < x < 1.$

(c) $f(x) = \begin{cases} x+1, & -1 < x < 0, \\ 1-x, & 0 \le x < 1. \end{cases}$

a)



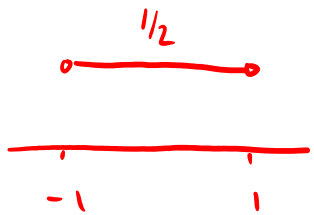
$$F(x) = \begin{cases} 0 & \text{if } x \le -1 \\ \frac{x^3+1}{2} & \text{if } -1 < x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$



$$F(x) = P(X \le x) = \int_{-1}^{x} f(t)\,dt = \int_{-1}^{x} \frac{3}{2} t^2\,dt = \left[\frac{t^3}{2}\right]_{t=-1}^{t=x} = \frac{x^3}{2} + \frac{1}{2}$$

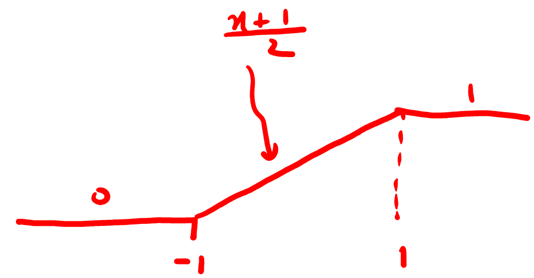
for $-1 < x < 1$

b)



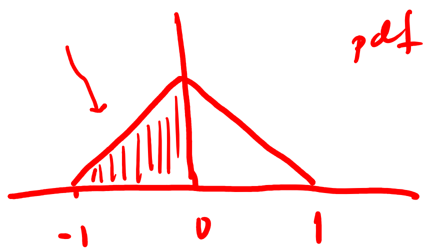
pdf $f(x)$

$$F(x) = \begin{cases} 0 & \text{if } x \le -1 \\ \frac{x+1}{2} & \text{if } -1 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$



If $-1 < x < 1$, then $F(x) = \int_{-1}^{x} f(t)\,dt = \int_{-1}^{x} \frac{1}{2}\,dt = \left[\frac{t}{2}\right]_{t=-1}^{t=x} = \frac{x+1}{2}$

c)



pdf $f(x)$

$$F(x) = \begin{cases} 0 & \text{if } x \le -1 \\ x + \frac{x^2}{2} + \frac{1}{2}, & -1 < x \le 0 \\ \frac{1}{2} + x - \frac{x^2}{2}, & 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

For $-1 < x < 0$:

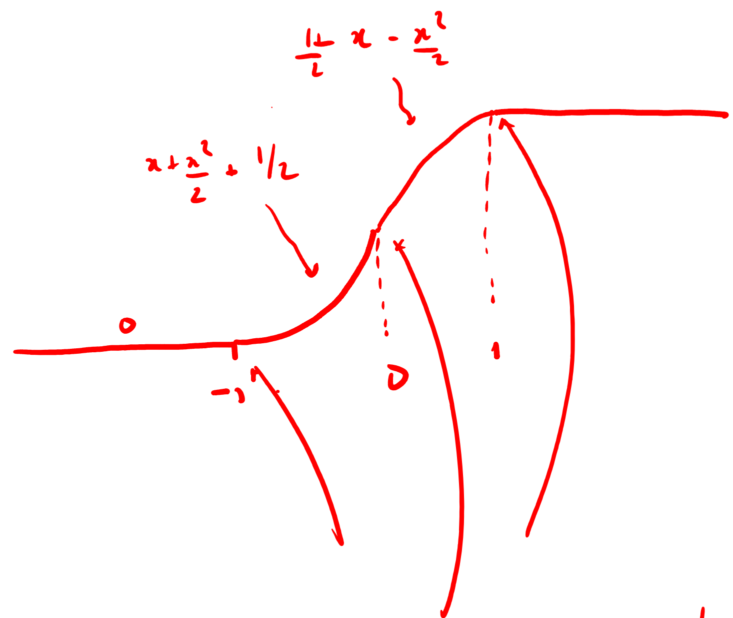
$$F(x) = \int_{-1}^{x} f(t)\,dt = \int_{-1}^{x} 1+t\,dt = \left[t + \frac{t^2}{2}\right]_{t=-1}^{t=x} = x + \frac{x^2}{2} + 1 - \frac{1}{2} = x + \frac{x^2}{2} + \frac{1}{2}$$

For $0 < x < 1$

$$F(x) = \underbrace{\int_{-1}^{0} f(t)\,dt}_{1/2} + \int_{0}^{x} f(t)\,dt = \frac{1}{2} + \int_{0}^{x} 1 - t\,dt =$$

$$= \frac{1}{2} + \left[t - \frac{t^2}{2} \right]_{t=0}^{t=x} =$$

$$= \frac{1}{2} + x - \frac{x^2}{2}$$



function is continuous and differentiable at $x = -1, 0, 1$

2.3-2. For each of the following distributions, find $\mu = E(X)$, $E[X(X-1)]$, and $\sigma^2 = E[X(X-1)] + E(X) - \mu^2$:

(a) $f(x) = \dfrac{3!}{x!(3-x)!}\left(\dfrac{1}{4}\right)^x\left(\dfrac{3}{4}\right)^{3-x}, \qquad x = 0, 1, 2, 3.$

(b) $f(x) = \dfrac{4!}{x!(4-x)!}\left(\dfrac{1}{2}\right)^4, \qquad x = 0, 1, 2, 3, 4.$

$8 + 36$

$44 + 48 = 94$

1 4 6 4 1

b) $E[X] = \sum_{x=0}^{4} x . f(x) = \sum_{x=1}^{4} x . f(x) = 1 . f(1) + 2 . f(2) + 3 . f(3) + 4 . f(4)$

$= 1 . 4 . \left(\frac{1}{2}\right)^4 + 2 . 6 . \left(\frac{1}{2}\right)^4 + 3 . 4 . \left(\frac{1}{2}\right)^4 + 4 . 1 . \left(\frac{1}{2}\right)^4$

we calculate

$= 32 . \left(\frac{1}{2}\right)^4 = \frac{32}{16} = 2$

$E[X(X-1)] = \sum_{x=0}^{4} x(x-1) . f(x) = \sum_{x=2}^{4} x(x-1) . f(x) = 2(2-1) . f(2) + 3(3-1) . f(3) + 4 . (4-1) f(4)$

$= 2 . 6 . \left(\frac{1}{2}\right)^4 + 6 . 4 \left(\frac{1}{2}\right)^4 + 12 . 1 . \left(\frac{1}{2}\right)^4$

$= \frac{48}{16} = 3$

$\sigma^2 = E[X(X-1)] + E[X] - \mu^2 = 3 + 2 - 2^2 = 1$

3.1-8. For each of the following functions, **(i)** find the constant c so that $f(x)$ is a pdf of a random variable X; **(ii)** find the cdf, $F(x) = P(X \le x)$; **(iii)** sketch graphs of the pdf $f(x)$ and the distribution function $F(x)$; and **(iv)** find μ, σ^2, and the index of skewness, γ:

(a) $f(x) = x^3/4, \qquad 0 < x < c.$

(b) $f(x) = (3/16)x^2, \qquad -c < x < c.$

(c) $f(x) = c/\sqrt{x}, \qquad 0 < x < 1$. Is this pdf bounded?

b)

i) To find c we require that $\int_{-\infty}^{\infty} f(x)\,dx = 1 \iff \int_{-c}^{c} \frac{3}{16} x^2\,dx = 1$

$$\iff \left.\frac{x^3}{16}\right]_{x=-c}^{x=c} = 1 \iff \frac{c^3}{16} + \frac{c^3}{16} = 1 \iff \frac{c^3}{8} = 1 \iff c^3 = 8$$

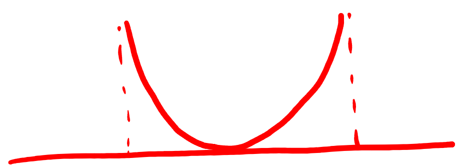
$\iff c = 2$

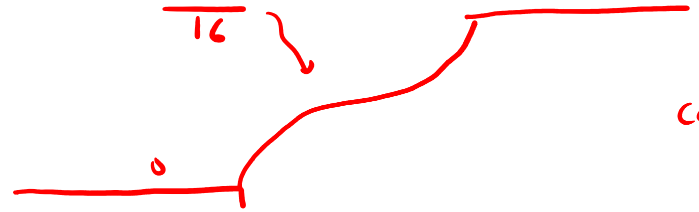
$$f(x) = \begin{cases} \frac{3}{16} x^2, & -2 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

ii)

and

iii)





For $-2 < x < 2$, $\quad F(x) = \int_{-2}^{x} f(t)\,dt = \int_{-2}^{x} \frac{3}{16} t^2\,dt = \frac{t^3}{16}\Big]_{t=-2}^{t=x}$

$$= \frac{x^3}{16} + \frac{8}{16} = \frac{x^3+8}{16}$$

$$F(x) = \begin{cases} 0, & \text{if } x < -2 \\ \frac{x^3+8}{16}, & \text{if } -2 < x < 2 \\ 1, & \text{if } x \geq 2 \end{cases}$$

iv)

$$\mu = E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_{-2}^{2} x \cdot \frac{3}{16} x^2\, dx = \int_{-2}^{2} \frac{3}{16} x^3\, dx =$$

$$= \left[\frac{3}{16} \cdot \frac{x^4}{4} \right]_{x=-2}^{x=2} = \frac{3}{16} \cdot 4 - \frac{3}{16} \cdot 4 = 0$$

$$\sigma^2 = E[X^2] - (\underbrace{E[X]}_{=0})^2 = E[X^2] = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx$$

$$= \int_{-2}^{2} x^2 \cdot \frac{3}{16} x^2\, dx = \int_{-2}^{2} \frac{3}{16} x^4\, dx = \left[\frac{3}{16} \cdot \frac{x^5}{5} \right]_{x=-2}^{x=2} = 2 \cdot \frac{3}{16} \cdot \frac{2^5}{5} = \frac{12}{5}$$

Since this distribution is symmetric, we have $\gamma = 0$.

2.6-10. In 2016, Red Rose tea randomly began placing one of ten English porcelain miniature figurines in a 100-bag box of the tea, selecting from ten figurines in the American Heritage Series.

(a) On the average, how many boxes of tea must be purchased by a customer to obtain a complete collection consisting of the ten different figurines?

(b) If the customer uses one tea bag per day, how long can a customer expect to take, on the average, to obtain a complete collection?

ten figurines

1 in each box of tea

a) Define the r.v.s:

X_1 = # of boxes to be purchased to observe the 1st figurine

X_2 = # " " " " " " " 2nd (distinct) figurine

⋮

X_{10} = # of boxes to be purchase to observe the 10th (distinct) figurine

$X_1 = 1$ [constant → we get the 1st figurine with the 1st box]

$X_2 \sim \text{Geometric}\left(\frac{9}{10}\right)$

$X_3 \sim \text{Geometric}\left(\frac{8}{10}\right)$

$\vdots$

$X_{10} \sim \text{Geometric}\left(\frac{1}{10}\right)$

We want to find $E\left[X_1 + X_2 + X_3 + \dots + X_{10}\right] = \overbrace{E[X_1]}^{1} + \overbrace{E[X_2]}^{10/9} + \overbrace{E[X_3]}^{10/8} + \dots + \overbrace{E[X_{10}]}^{10}$

$= 1 + \frac{10}{9} + \frac{10}{8} + \frac{10}{7} + \dots + \frac{10}{1}$

$= \dots$ calculate

b) $100 \times$ answer from item a)

2.7-12. A baseball team loses \$100,000 for each consecutive day it rains. Say X, the number of consecutive days it rains at the beginning of the season, has a Poisson distribution with mean 0.2. What is the expected loss before the opening game?

$X =$ r.v. representing the # of consecutive days it rains at the beginning of the season

We are told that $X \sim \text{Poisson}(0.2)$ $\quad \lambda = 0.2$

The expected loss is then $\underbrace{E[\underbrace{100\,000\, X}_{\text{loss}}]}_{\text{expected loss}} = 100000\, \underbrace{E[X]}_{\lambda = 0.2} = 100\,000\,(0.2)$

$= 20\,000$