Math 3501 - Probability and Statistics I

1.1 - Properties of Probability

Example

Let A and B be events with that $P(A) = 0.4$, $P(B) = 0.3$, and $P(A \cap B) = 0.1$.

Find $P(A \cup B)$.

A and B are NOT mutually exclusive

Then, by the previous theorem

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.4 + 0.3 - 0.1$$

$$= 0.6$$

if A and B were mutually exclusive

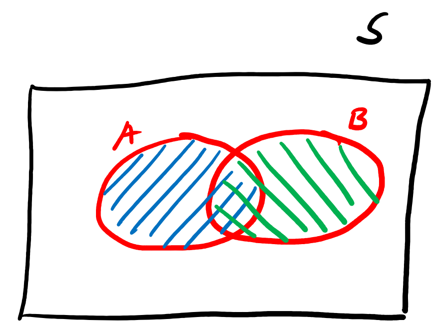
$\Rightarrow A \cap B = \phi \Rightarrow P(A \cap B) = P(\phi) = 0$

$\Rightarrow P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}_{=0}$

$\Rightarrow P(A \cup B) = P(A) + P(B)$ ✓

condition 3 in the axiomatic definition

EXAMPLE



A child takes two books on vacation:

→ she likes the 1^{st} book with prob 0.5

→ " " " 2^{nd} " " " 0.4

→ " " both books " " 0.3

Q_1: How likely is it that the child likes at least one book?

Q_2: " " " " " " " likes neither?

SOLUTION Define the following events:

A = "event child likes book 1"

B : "event child likes book 2"

We are given:

$P(A) = 0.5$

$P(B) = 0.4$

$P(A \cap B) = 0.3$

Q_1: $A \cup B$: event child likes either book 1 or book 2 or both ⇝ child likes at least one book!

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.5 + 0.4 - 0.3 = 0.6$$

Q_2 : event child likes neither book is

$$\overline{A} \cap \overline{B} = \overline{A \cup B}$$

not liking book 1

not liking book 2

De Morgan's

Conclusion: $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$

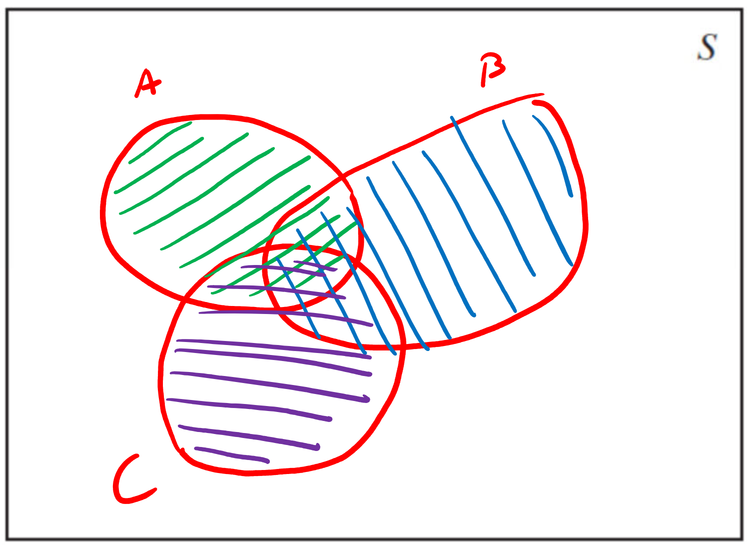
$= 1 - 0.6$

$= 0.4$

Theorem

If A, B, and C are any three events, then

$$\begin{aligned} P(A \cup B \cup C) = & P(A) + P(B) + P(C) \\ & - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \end{aligned}$$



Example

Let A, B, and C be events such that $P(A) = 0.4$, $P(B) = 0.35$, $P(C) = 0.3$, $P(A \cap B) = 0.2$, $P(A \cap C) = 0.25$, $P(B \cap C) = 0.20$, and $P(A \cap B \cap C) = 0.15$.

Find $P(A \cup B \cup C)$.

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$
$$- P(A \cap B) - P(A \cap C) - P(B \cap C)$$
$$+ P(A \cap B \cap C)$$
$$= 0.4 + 0.35 + 0.3 - 0.2 - 0.25 - 0.2 + 0.15$$
$$= \ldots$$

Outcome spaces having equally likely outcomes

Let P be a probability set function defined on the finite sample space

$$S = \{e_1, e_2, \ldots, e_m\} \ ,$$

$\longrightarrow$ $|S| = m$ } the set S has m elements

$N(S) = m$

where each e_i is a possible outcome of a random experiment.

The integer m is the total number of distinct ways in which the random experiment can terminate.

If each of the outcomes e_i, $i = 1, \ldots, m$, has the same probability of occurring, we say that the *m outcomes are equally likely*:

$$P(\{e_i\}) = \frac{1}{m}, \quad i = 1, 2, \ldots, m \ .$$

Example

- tossing a fair coin $S = \{H, T\}$ $P(\{H\}) = P(\{T\}) = \frac{1}{2}$
- rolling a fair six-sided die $S = \{1, 2, \ldots, 6\}$ $P(\{i\}) = \frac{1}{6}$, $i = 1, \ldots, 6$

Let

$$S = \{e_1, e_2, \ldots, e_m\}$$

be an outcome space endowed with a probability set function P assigning equal probability to all outcomes in S:

$$\boxed{P(\{e_i\}) = \frac{1}{m}, \quad i = 1, 2, \ldots, m\ .}$$

$A \subset S$

Let A be an event with h distinct outcomes: $|A| = h$ $[N(A) = h]$

1) we refer to h as the *number of ways that are favorable to the event* A.

2) the probability of A can be determined as

$$\boxed{P(A) = \frac{h}{m} = \frac{N(A)}{N(S)}}$$

where:

- $h = N(A)$ is the number of ways A can occur;
- $m = N(S)$ is the number of ways S can occur.

Example

Let a card be drawn at random from an ordinary deck of 52 playing cards.

What is the probability that an ace is drawn?

$|S| = 52$

A = "ace is drawn"

$|A| = 4$

$$P(A) = \frac{|A|}{|S|} = \frac{4}{52} = \frac{1}{13}$$

A standard deck of 52 cards has

4 suits $\begin{cases} \text{spades} & \text{(black)} \\ \text{clubs} & \text{(black)} \\ \text{hearts} & \text{(red)} \\ \text{diamonds} & \text{(red)} \end{cases}$

Each suit has 13 cards with values:

A (ace)
2
3
4
⋮
10
J → Jack
Q → Queen
K → King

EXAMPLE : Two die are rolled

what is the probability that the sum of the observed faces is 7

Solution: $S = \{ (i,j) : i,j \in \{1,\dots,6\} \}$

$|S| = 36$

$A =$ event "sum is 7" $= \{ (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) \}$

$|A| = 6$

$$P(A) = \frac{|A|}{|S|} = \frac{6}{36} = \frac{1}{6}$$

Math 3501 - Probability and Statistics I

1.2 - Methods of Enumeration

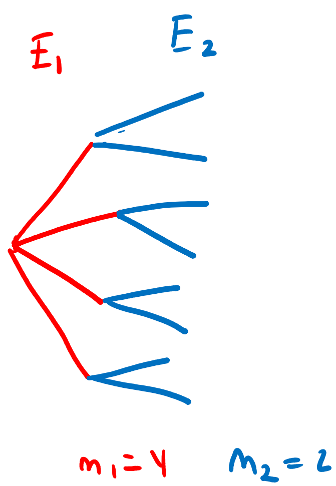
Methods of Enumeration

Goal: to develop counting techniques useful in determining the number of outcomes associated with the events of certain random experiments.

Multiplication Principle:

Suppose that an experiment E_1 has n_1 outcomes and, for each of these possible outcomes, an experiment E_2 has n_2 possible outcomes.

Then the composite experiment $E_1 E_2$ that consists of performing first E_1 and then E_2 has $n_1 \cdot n_2$ possible outcomes.



$$n_1 . n_2 = 4 . 2 = 8$$

Example

Random experiment: a fair coin is tossed and a fair die is rolled.

How many distinct possible outcomes does the experiment have?

$$S = \{ (i, j) : i \in \{H, T\}, \ j \in \{1, 2, \dots, 6\} \}$$

There are $n_1 = 2$ outcomes when tossing a coin

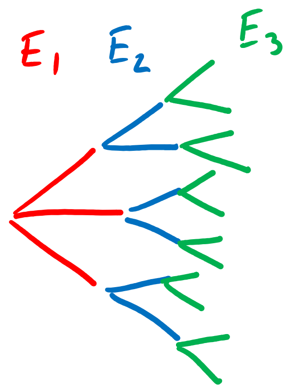
For each of these, there are $n_2 = 6$ possible outcomes when rolling a die

In total, we have $n_1 \cdot n_2 = 2 \cdot 6 = 12$ possible outcomes

Multiplication Principle:

Suppose that the experiment E_i has n_i, $i = 1, 2, \ldots, m$, possible outcomes after previous experiments have been performed.

Then the composite experiment $E_1 E_2 \cdots E_m$ that consists of performing E_1 then $E_2, \ldots,$ and finally E_m has $n_1 \cdot n_2 \cdots n_m$ possible outcomes.



possible outcomes is

$$n_1 . n_2 . n_3 = 3 . 2 . 2 = 12$$

$n_1 = 3$

$n_2 = 2$

$n_3 = 2$

Example

To build a sandwich: pick one out of six choices for bread; pick one out of three choices for meat; pick one of out five choices for cheese.

How many different types of sandwich can be built?

3 stages: E_1, E_2, E_3

E_1 : pick the bread $\rightarrow$ can occur in $m_1 = 6$ ways

E_2 : pick the meat $\rightarrow$ can occur in $m_2 = 3$ ways

E_3 : pick the cheese $\rightarrow$ " " " $m_2 = 5$ ways

total # sandwich that we can build is

$$m_1 . m_2 . m_3 = 6 . 3 . 5 = 90$$

Question: In how many ways can n positions be filled with n different objects?

Example $n = 5$ objects to fill 5 positions

$$\underbrace{\underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1}}_{5 \text{ positions}} = 5!$$

n objects in n positions

$$\underbrace{\underline{n} \cdot \underline{(n-1)} \cdot \underline{(n-2)} \cdot \underline{(n-3)} \cdots\cdots \underline{3} \cdot \underline{2} \cdot \underline{1}}_{n \text{ positions}} = n!$$

Factorial

Definition (Factorial)

Given $n \in \mathbb{N}$, the number n factorial, denoted $n!$, is defined as

$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

$$\boxed{n! = n(n-1)\cdots(2)(1)\ .}$$

Convention:

$$\boxed{0! = 1}$$

There is only one way to organize 0 objects in 0 positions

$1! = 1$

$2! = 2$

$3! = 6$

$4! = 24$

$5! = 120$

$6! = 720$

Permutations of n different objects

Definition

Each of the $n!$ arrangements (in a row) of n different objects is called a *permutation of the n objects*.

assigning n objects to n positions

Example

In how many distinct ways can the letters

$$a \quad b \quad c \quad d \quad e$$

five distinct letters

be arranged to form five character strings

- without any character repetition?
- allowing repeated characters?

Q_1: a b c e d d a b c e b a d e c

Permutation of 5 objects: $5!$

$\underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 5! = 120$

Q_2 $\underline{5} \cdot \underline{5} \cdot \underline{5} \cdot \underline{5} \cdot \underline{5} = 5^5$

aaabb accbb

Question: In how many ways can r positions be filled with objects selected from n different objects, where $r \leq n$?

n objects

$$\underbrace{\underline{n} \quad \underline{(n-1)} \quad \underline{(n-2)} \quad \cdots\cdots \quad \underline{} \quad \underline{n-r+1}}_{r \text{ positions}}$$

$n-(r-1)$

$$n(n-1)(n-2)\cdots(n-r+1) = \frac{\overbrace{n\cdot(n-1)\cdots(n-r+1)}\,.\overbrace{(n-r)(n-r-1)\cdots 3.2.1}^{\text{extra terms}}}{(n-r)(n-r-1)\cdots 3.2.1} = \frac{n!}{(n-r)!}$$

If $r = n$ (as in the previous case) : $\dfrac{n!}{(n-n)!} = \dfrac{n!}{0!} = \dfrac{n!}{1} = n!$

Permutations of n objects taken r at a time

Definition

The number of possible ordered arrangements of n objects to fill $r \leq n$ positions, denoted ${}_nP_r$ and called *permutation of n objects taken r at a time*, is given by

$$\boxed{{}_nP_r = \frac{n!}{(n-r)!}} \, .$$

Special case: ${}_nP_n = n!$

Example

How many three-letter code words can be formed, selecting from the 26 letters in the alphabet, in which all three letters are different?

Problem: in how many ways can we fill 3 positions using 26 distinct letters with no repetition.

$\underline{26} \cdot \underline{25} \cdot \underline{24}$ (by the multiplication principle)

or

$$ {}_{26}P_3 = \frac{26!}{(26-3)!} = \frac{26!}{23!} = \frac{26 \cdot 25 \cdot 24 \cdot (23!)}{(23)!} = 26 \cdot 25 \cdot 24 $$

Example

A club with 10 members has to select a president, a secretary, and a treasurer.

Knowing that no club member can simultaneously hold more than one office, in how many distinct ways can the offices be filled with club members?

Multiplication principle $\underline{10} \cdot \underline{9} \cdot \underline{8} = 10.9.8$

or $\quad {}_{10}P_3 = \dfrac{10!}{(10-3)!} = \dfrac{10!}{7!} = \dfrac{10.9.8.(7!)}{7!} = 10.9.8$

Problem:

Given a set with n objects, in how many ways can we select $r \leq n$ objects?

Factors to consider:

- is a selected object replaced before the next object is selected?
- is the order or the objects relevant?

are repetitions allowed ??

abc or aaa

??

Do we count abc acb bca

as distinct strings or not

Ordered sample

Definition

If r objects are selected from a set of n objects, and if the order of selection is noted, then the selected set of r objects is called an *ordered sample of size* r.

Sampling with replacement

Definition

Sampling with replacement occurs when an object is selected and then replaced before the next object is selected.

Remark

The number of possible ordered samples of size r *taken from a set of* n *objects when sampling with replacement is* n^r.

$$\underbrace{\underline{n} \quad \underline{n} \quad \underline{n} \quad \underline{n} \quad \ldots \quad \underline{n}}_{r \text{ position}} = n^r$$

Example

A coin is tossed 5 times.

Find the number of resulting $\underbrace{\text{possible ordered samples}}$.

T H H T T

H T T T T

$$\underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2} = 2^5$$