Math 3501 - Probability and Statistics I

2.1 - Random variables of discrete type

Note: A sample space S may be difficult to describe if the elements of S are not numbers.

Possible solution: Assign to each element $s \in S$ an appropriate real number x.

Example

A coin is tossed and the outcome is observed.

$H \to 0$

$T \to 1$

The outcome space is $S = \{H, T\}$.

Let X be a function defined on S such that

$$X(H) = 0 \quad \text{and} \quad X(T) = 1 .$$

This is a real-valued function that has the outcome space S as its domain and the set of real numbers $\{x : x = 0, 1\}$ as its range.

Random variable

Definition

Given a random experiment with an outcome space S, a *random variable* is a real-valued function with domain S.

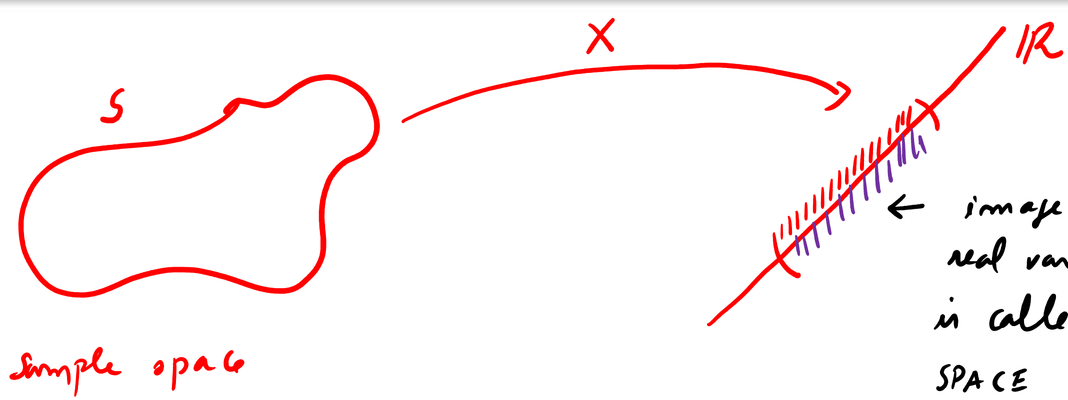
$X : S \longrightarrow \mathbb{R}$

The *space of* X is the set of real numbers

$$\{x : X(s) = x, s \in S\} \;,$$

← range of X

where $s \in S$ means that the element s belongs to the set S.



← image of the real variable X is called the SPACE OF X

like the sample space as observed by the r.v. X

Remark

1) There are many ways of defining a function X on S, so one might wonder which function to use?

2) While observing a random experiment, the experimenter may take some type of measurement (or measurements):

- such measurement may be regarded as the outcome of a random variable.

3) It may be that the set S has elements that are real numbers:

- we can define $X(s) = s$, so that X is the identity function and the space of X is also S.

e.g. Roll a die: $S = \{1, 2, \dots, 6\}$ $\rightarrow$ if all we want is to record the observed face, then we may define $X(s) = s$

Example

Random experiment: roll a fair die.

Sample space $S = \{1, 2, 3, \ldots, 6\}$

Possible random variables

$X(s) = s$ ← will have output equal to the value of observed face

$Y(s) = \begin{cases} 1 & \text{if } s \text{ is odd} \\ 0 & \text{if } s \text{ is even} \end{cases}$ → distinguishes only between even and odd scores.

Roll two fair dice

$S = \{ (i,j) : i,j \in \{1, \dots, 6\} \}$

Possible r.v.s :

$X_1(s) = i + j$ $\leftarrow$ records the sum of the two faces

$X_2(s) = \max\{i,j\}$ $\leftarrow$ record the maximum observed value.

$X_3(s) = i$ $\leftarrow$ records only the value observed on 1^{st} die

Remark

Let X denote a random variable with space S.

we will use the same symbol for the space of X and the sample space of the random experiment

Suppose that we know how the probability is distributed over the various subsets A of S, that is, we know

$$P(X \in A),$$

we want to determine this quantity and study some consequences!

for each subset $A \subset S$.

When often speak of the distribution of the random variable X, meaning the distribution of probability associated with the space S of X.

goal of Chps 2 + 3 : study the probability distributions of some special r.v.s.

Discrete outcome space

Definition

An outcome space S is said to be *discrete* if it contains a countable number of points, that is, either:

i) S contains a finite number of points; or

ii) the points of S can be put into a one-to-one correspondence with $\mathbb{N}$.

$$\left.\begin{array}{l} S = \{ b_1, b_2, \ldots, b_n \} \quad \text{(finite)} \\ \text{or} \\ S = \{ b_1, b_2, b_3, b_4, \ldots \} \quad \text{(infinite countable)} \end{array}\right\} \text{countable}$$

$\downarrow$ there is a 1-to-1 map between S and $\mathbb{N}$

Discrete random variable

Definition

A random variable defined on a discrete outcome space S is called a *discrete random variable*.

The probability distribution of a discrete random variable is said to be of the discrete type.

Note: A discrete random variable can assume at most a countable number of values.

if sample space S is discrete → finite S / → infinite contable

THEN ⟹ space of X [which is $X(S)$] must also be discrete

Probability mass function

Definition

Let X be a discrete random variable with space S.

space of X

The *probability mass function* (pmf) of X , denoted $f(x)$ or $f_X(x)$, is given by

$$\boxed{f(x) = P(X = x) \qquad x \in \mathbb{R}\ .}$$

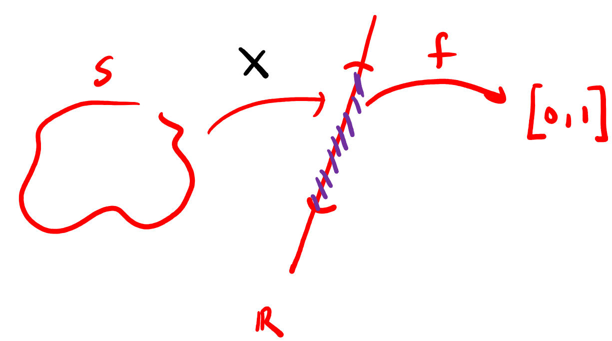
Properties:

range of the r.v. X

(a) $f(x) > 0$ for $x \in S$ and $f(x) = 0$ for $x \notin S$, that is S is the support of f

(b) $\boxed{\sum_{x \in S} f(x) = 1}$

(c) $\boxed{P(X \in A) = \sum_{x \in A} f(x)\ , \quad \text{where } A \subset \mathbb{R}}$



Cumulative distribution function

Definition

Let X be a discrete random variable with space S.

The *cumulative distribution function* (cdf) of X, denoted $F(x)$ or $F_X(x)$, is defined as

$$\boxed{F(x) = P(X \leq x) \, , \qquad x \in \mathbb{R} \, .}$$

Note: The cdf of X is sometimes referred to as the distribution function of the random variable X.

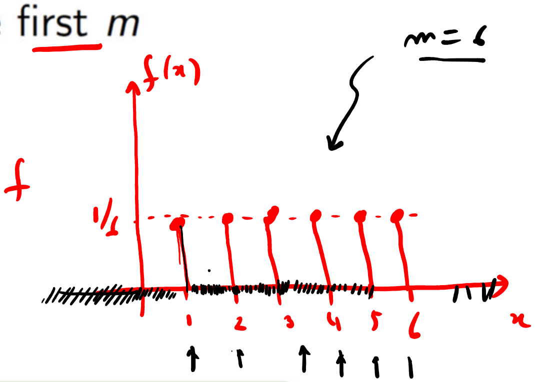
This definition is general in the sense that applies to all types of r.v.s

Discrete uniform distribution

Note: When a pmf is constant on its support, and the support is finite we say that the distribution is *uniform* (over that space).

The random variable X has a discrete uniform distribution over the first m positive integers if its pmf is

$$f(x) = \frac{1}{m}, \quad x = 1, 2, 3, \ldots, m \ . \quad \Big\} \text{ pmf}$$



Example

A fair six-faced die has a discrete uniform distribution over the first 6 positive integers.

$P(X \leq 1)$

Discrete uniform distribution

Let X be a random variable with a discrete uniform distribution over the first m positive integers.

$x = 1, 2, 3, \dots, m$

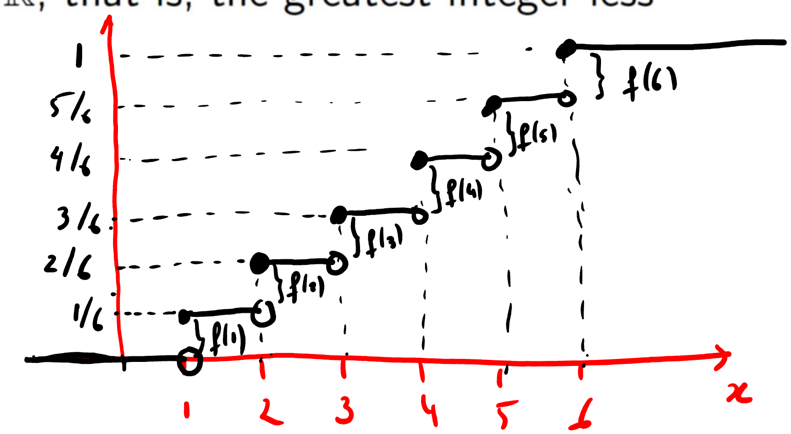
The cdf of X is given by

$$\underbrace{F(x) = P(X \le x)} = \begin{cases} 0\,, & x < 1 \\ \frac{\lfloor x \rfloor}{m}\,, & 1 \le x < m \\ 1\,, & m \le x \end{cases},$$

where we have denoted by $\lfloor x \rfloor$ the floor of $x \in \mathbb{R}$, that is, the greatest integer less than or equal to x.

For the fair six-faced die:

$$F(x) = P(X \le x) = \begin{cases} 0\,, & \text{if } x < 1 \\ 1/6\,, & \text{if } 1 \le x < 2 \\ 2/6\,, & \text{if } 2 \le x < 3 \\ 3/6\,, & \text{if } 3 \le x < 4 \\ 4/6\,, & \text{if } 4 \le x < 5 \\ 5/6\,, & \text{if } 5 \le x < 6 \\ 1\,, & \text{if } x \ge 6 \end{cases}$$



Example

Roll a fair four-sided die twice, and let $\underline{\underline{X}}$ be the maximum of the two outcomes. Determine the pmf of X.

Sample space $S = \{(i,j) : i,j \in \{1,2,3,4\}\}$ has 16 elements (all equally likely)

The random variable is $X(i,j) = \max\{i,j\}$

Space of X is the set of values X can take, i.e., $S_X = \{1,2,3,4\}$

The space of X $(S_X = \{1,2,3,4\})$ will be the support of the pmf of X

We only need to determine $f(x) = P\{X = x\}$ for $x = 1,2,3,4$ because $f(x) = 0$ otherwise.

If $x=1$, $f(1) = P\{X=1\} = P\{(1,1)\} = \frac{1}{16}$

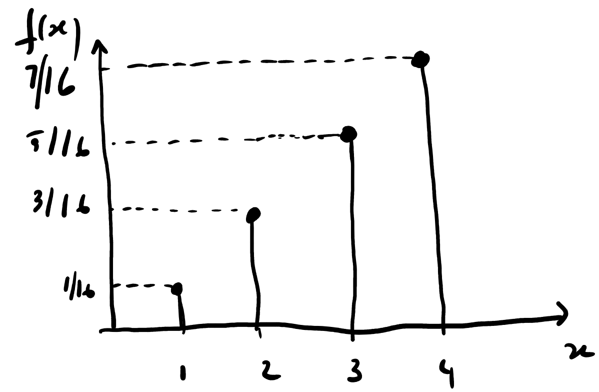
If $x=2$, $f(2) = P\{X=2\} = P\{(1,2),(2,2),(2,1)\} = \frac{3}{16}$

If $x = 3$, then $f(3) = P\{X = 3\} = P\{(1,3), (2,3), (3,3), (3,2), (3,1)\} = \frac{5}{16}$

If $x = 4$, then $f(4) = P\{X = 4\} = P\{(1,4), (2,4), (3,4), (4,4), (4,3), (4,2), (4,1)\} = \frac{7}{16}$

The pmf of X is then

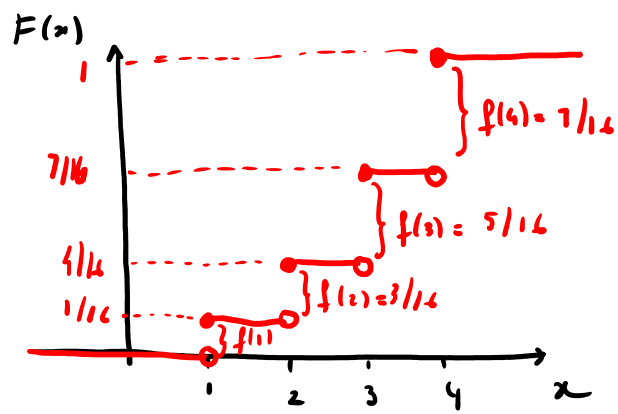
$$f(x) = \begin{cases} 1/16 & \text{if } x = 1 \\ 3/16 & \text{if } x = 2 \\ 5/16 & \text{if } x = 3 \\ 7/16 & \text{if } x = 4 \\ 0 & \text{otherwise} \end{cases}$$



$P\{X \leq 1\}$

The cdf of X is

$$F(x) = P\{X \leq x\} = \begin{cases} 0 & \text{if } x < 1 \\ 1/16 & \text{if } 1 \leq x < 2 \\ 4/16 & \text{if } 2 \leq x < 3 \\ 9/16 & \text{if } 3 \leq x < 4 \\ 1 & \text{if } x \geq 4 \end{cases}$$

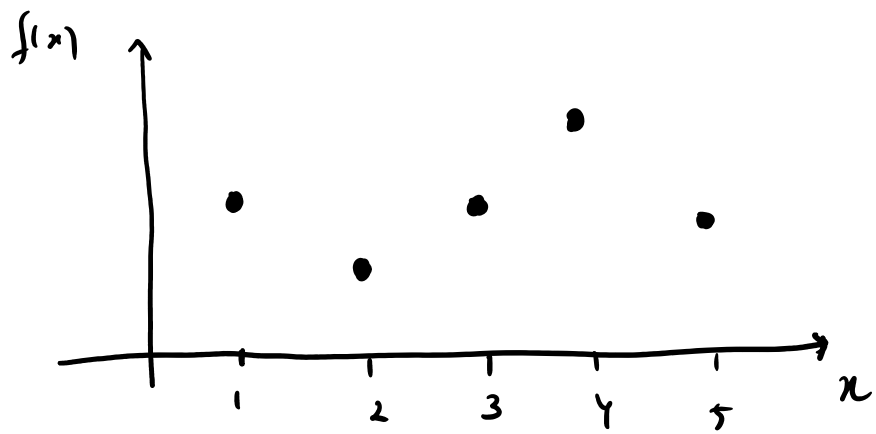


Graphical representations

Note: the graph of a pmf is the set of points

$$\{[x, f(x)] : x \in S\} \ ,$$

where S is the space of X.

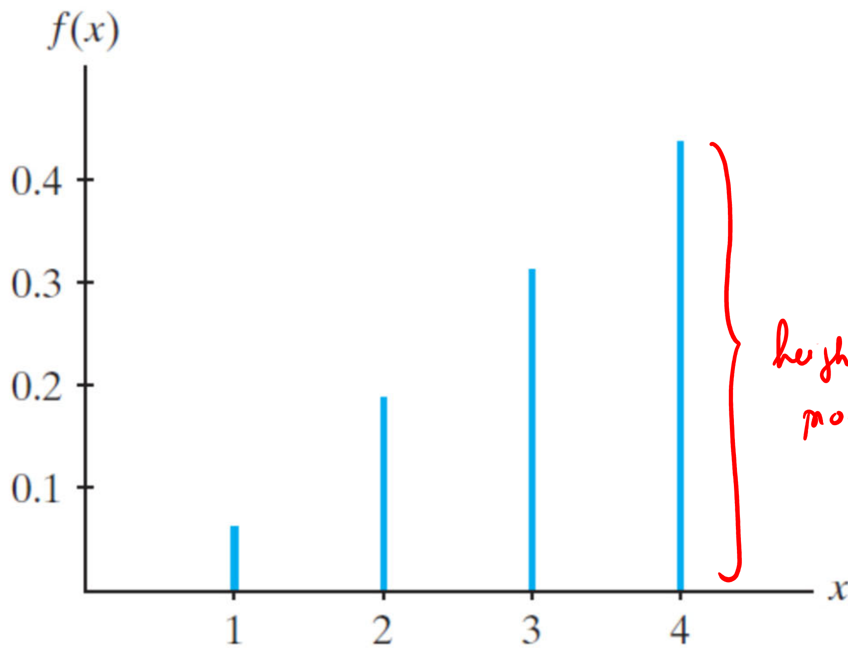


Alternative graphical representations:

- line graph
- probability histogram

Line graph

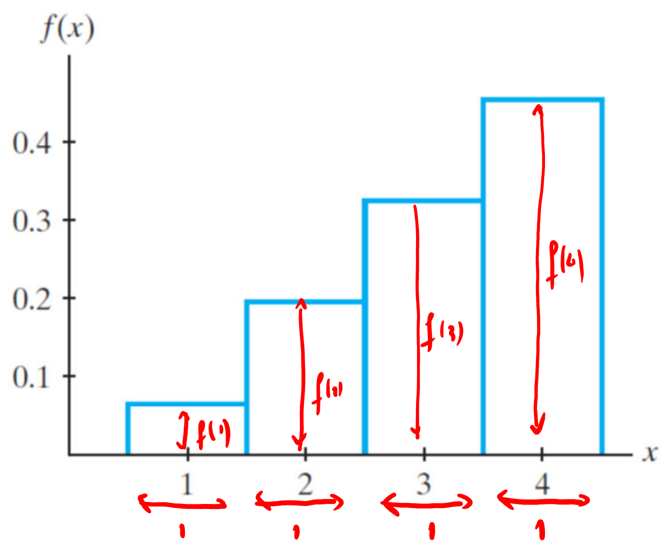
A *line graph* of the pmf $f(x)$ of the random variable X is a graph having a vertical line segment drawn from $(x, 0)$ to $(x, f(x))$ at each $x \in S$, the space of X.



height of each line equals the probability of the corresponding value of x

Probability histogram

Whenever X can assume only integer values, a *probability histogram* of the pmf $f(x)$ is a graphical representation that has a rectangle of height $f(x)$ and a base of length 1 centered at x for each $x \in S$, the space of X.



Note:

- the area of each rectangle is equal to the respective probability $f(x)$
- the total area of a probability histogram is 1.