Math 4501 - Probability and Statistics II

8.1 - Hypothesis tests: one mean

Setup: tests of hypotheses about one mean

Let X be a random variable with mean μ and variance σ^2.

We are interested in testing the null hypothesis

$$\boxed{H_0 : \mu = \mu_0}$$ ← simple null hypothesis

against one of the following three composite alternative hypotheses:

(i) that μ has increased: $H_1 : \mu > \mu_0$

(ii) that μ has decreased: $H_1 : \mu < \mu_0$

} one-sided

(iii) that μ has changed, but it is not known whether it has increased or decreased, leading to the two-sided alternative hypothesis: $H_1 : \mu \neq \mu_0$ } two-sided

One random sample $X_1, \ldots, X_n$ is taken from the distribution of X.

The sample mean, $\bar{x}$, is found, as well as the sample variance s^2 in case the variance σ^2 is not known.

- A decision concerning rejection of H_0, is related with how close $\bar{x}$ is to μ_0.

Normal distribution with known variance

Let X be a random variable with a normal distribution $N\left(\mu, \sigma^2\right)$.

unknown known

Suppose σ^2 is known.

Under the null hypothesis $H_0 : \mu = \mu_0$, the test statistic

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \quad \text{is } N(0,1)$$

Under H_0

$X_1, \ldots, X_n \sim N(\mu_0, \sigma^2)$

$\bar{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)$

$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$

$\alpha = P(\text{Error type I}) = P(\text{Rej } H_0 \mid \underbrace{H_0 \text{ true}}_{\mu = \mu_0})$

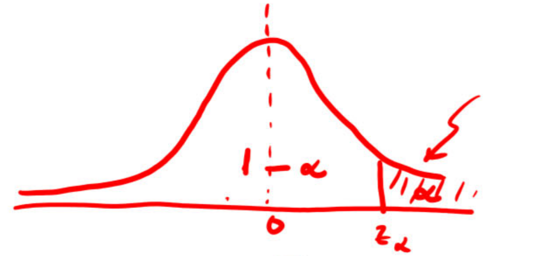
$\uparrow$

significance level

Let $\bar{x}$ be the observed value of the sample mean, and let

$$\boxed{z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}}$$

be the corresponding value of the test statistic.



$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$

(i) When testing $H_0 : \mu = \mu_0$ against $H_1 : \mu > \mu_0$, we reject H_0 if

$$\boxed{z \geq z_\alpha} \qquad \text{or, equivalently,} \qquad \boxed{\bar{x} \geq \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}}$$

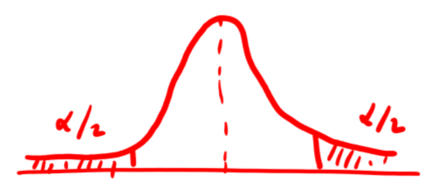
z close to 0 iff $\bar{x}$ close to μ_0

(ii) When testing $H_0 : \mu = \mu_0$ against $H_1 : \mu < \mu_0$, we reject H_0 if

$$z \leq -z_\alpha \qquad \text{or, equivalently,} \qquad \boxed{\bar{x} \leq \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}}}$$

(iii) When testing $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$, we reject H_0 if

$$|z| \geq z_{\alpha/2} \qquad \text{or, equivalently,} \qquad |\bar{x} - \mu_0| \geq z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$



$z \geq z_{\alpha/2}$ or $z \leq -z_{\alpha/2}$

$\bar{x} \geq \mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ or $\bar{x} \leq \mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$

Example

variance is known!

Assume that X is $N(\mu, \sigma^2)$ with $\sigma = 1.2$. $(\sigma^2 = 1.2^2)$

Test the hypotheses

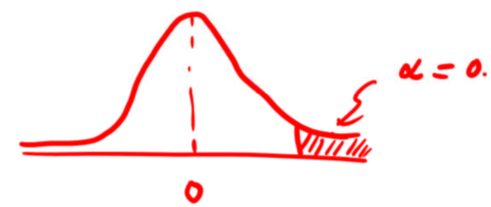
$$\boxed{\begin{aligned} H_0 &: \mu = 4 \\ H_1 &: \mu > 4 \end{aligned}} \leftarrow \text{one-sided}$$

at significance level $\alpha = 0.1$ knowing that a random sample with 16 observations yielded a sample mean $\bar{x} = 4.3$.

$n = 16$

Since the sample is taken from a normal distribution with known variance, the test critical region is

$$N(0,1) \rightarrow \boxed{z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}} \geq z_\alpha$$



Since $n = 16$, $\bar{x} = 4.3$ and $z_{0.1} = 1.282$, we find that

$$\underbrace{\boxed{z = \frac{4.3 - 4.0}{1.2/\sqrt{16}} = \frac{0.3}{0.3} = 1}}_{z} < \overbrace{1.282}^{z_{0.1}} = z_{0.1}$$

and we do not reject H_0 at significance level $\alpha = 0.1$.

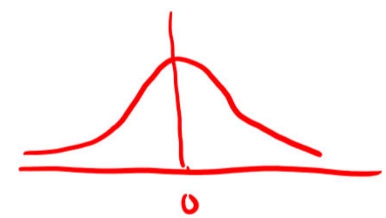
Normal distribution with unknown variance

Let X be a random variable with a normal distribution $N\left(\mu, \sigma^2\right)$.

Suppose σ^2 is unknown.

Under the null hypothesis $H_0 : \mu = \mu_0$, the test statistic is

$$\boxed{T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \quad \text{is } t(n-1)}$$



$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$U = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$

Z and U are independent

$$T = \frac{Z}{\sqrt{U/(n-1)}} \sim t(n-1)$$

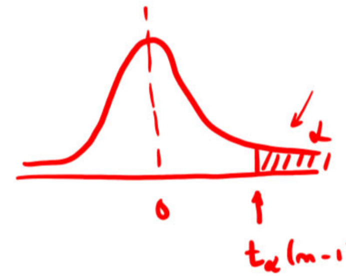
Let $\bar{x}$ and s be, respectively, the observed values of the sample mean and sample standard deviation, and let

$$\boxed{t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}}\ , \longrightarrow \text{follows } t(n-1)$$

be the corresponding value of the test statistic.

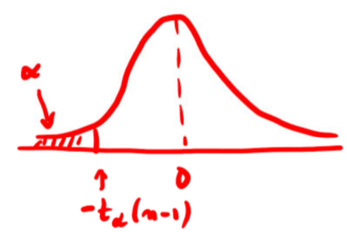
(i) When testing $H_0 : \mu = \mu_0$ against $H_1 : \mu > \mu_0$, we reject H_0 if

$$t \geq t_\alpha(n-1) \qquad \text{or, equivalently,} \qquad \bar{x} \geq \mu_0 + [t_\alpha(n-1)]\frac{s}{\sqrt{n}}$$



(ii) When testing $H_0 : \mu = \mu_0$ against $H_1 : \mu < \mu_0$, we reject H_0 if

$$t \leq -t_\alpha(n-1) \qquad \text{or, equivalently,} \qquad \bar{x} \leq \mu_0 - [t_\alpha(n-1)]\frac{s}{\sqrt{n}}$$



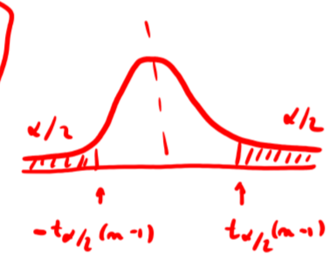
(iii) When testing $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$, we reject H_0 if

$$|t| \geq t_{\alpha/2}(n-1) \qquad \text{or, equivalently,} \qquad |\bar{x} - \mu_0| \geq [t_{\alpha/2}(n-1)]\frac{s}{\sqrt{n}}$$

$\hookrightarrow$ $t \geq t_{\alpha/2}(n-1)$ or $t \leq -t_{\alpha/2}(n-1)$

$\bar{x} \geq \mu_0 + t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}}$

or $\bar{x} \leq \mu_0 - t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}}$



Example

Assume that X is $N(\mu, \sigma^2)$. ← we are not given σ^2 or σ

Test the hypotheses

$\mu_0 = 4$

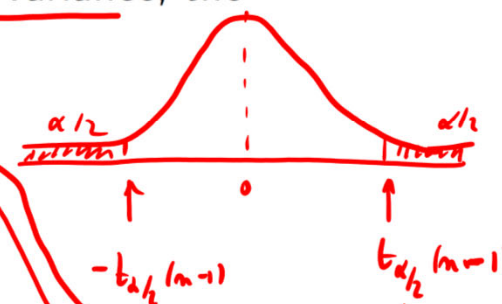
$$\boxed{\begin{aligned} H_0 &: \mu = 4 \\ H_1 &: \mu \neq 4 \end{aligned}}$$

← alternative is 2-sided

at significance level $\alpha = 0.1$ knowing that a random sample with 9 observations yielded a sample mean $\bar{x} = 4.3$ and a sample standard deviation $s = 1.2$. → $n = 9$

Since the sample is taken from a normal distribution with unknown variance, the test critical region is

$$|t| = \frac{|\bar{x} - \mu_0|}{s/\sqrt{n}} \geq t_{\alpha/2}(n-1)$$



Since $n = 9$, $\bar{x} = 4.3$, $s = 1.2$ and $t_{0.05}(8) = 1.86$, we find that

$$\boxed{|t| = \frac{|4.3 - 4.0|}{1.2/\sqrt{9}} = \frac{0.3}{0.4} = 0.75 < 1.86 = t_{0.05}(8)}$$

and we do not reject H_0 at significance level $\alpha = 0.1$.

Alternatively, we could have computed the p-value. $\longrightarrow$ Rej H_0 if p-value $< \alpha$

Recalling that under the null hypothesis $H_0 : \mu = \underset{\mu_0}{4}$, the test statistic

$$\boxed{T = \frac{\bar{X} - \mu_0}{\sqrt{S^2/n}} = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \quad \text{is } t(n-1)}$$

and that the observed value of T (when $n = 9$, $\bar{x} = 4.3$, $s = 1.2$) is:

$$\boxed{t = \frac{4.3 - 4.0}{1.2/\sqrt{9}} = \frac{0.3}{0.4} = 0.75} \longleftarrow \text{observed value for the sample collected.}$$

observing a value even more extreme

we find that

$$p = P(\underbrace{|T| \geq 0.75}) = 2P(T \geq 0.75) = 0.474$$

Since $p = 0.474 > 0.10 = \alpha$, we do not reject H_0 at the significance level $\alpha = 0.1$.

Example

Assume that X is $N(\mu, \sigma^2)$. ← σ^2 is unknown

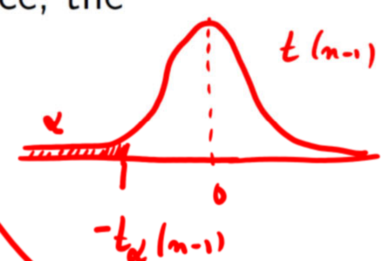
Test the hypotheses

$$\begin{cases} H_0 : \mu = 500 \\ H_1 : \mu < 500 \end{cases}$$ ← one sided

at significance level $\alpha = 0.01$ knowing that a random sample with $\overbrace{\text{25 observations}}^{n=25}$ yielded a sample mean $\bar{x} = 308.8$ and a sample standard deviation $s = 115.15$.

Since the sample is taken from a normal distribution with unknown variance, the test critical region is

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \leq -t_\alpha(n-1)$$



Since $n = 25$, $\bar{x} = 308.8$, $s = 115.15$ and $t_{0.01}(24) = 2.492$, we find that

n-1

$$t = \frac{308.8 - 500}{115.15/\sqrt{25}} = -8.30 < -2.492 = t_{0.01}(24)$$

and we reject H_0 at significance level $\alpha = 0.01$.

Math 4501 - Probability and Statistics II

8.2 - Hypothesis testing: equality of means

similar to 7.2

Overview

We will study hypothesis tests for equality of two means in the following cases:

- Paired sampling ← non-independent samples → reduces to Sec 8.1
- Independent sampling:
  - two independent random variables with normal distributions of known variances
  - two independent random variables with normal distributions of unknown but equal variances

We will also study approximate hypothesis tests for equality of two means:

- two independent random variables with normal distributions of unknown and unequal variances
- using large samples

Dependent random variables: paired t-test

Let X and Y be dependent and suppose we want to test the null hypothesis

$$H_0 : \mu_X = \mu_Y$$

against one of alternative hypothesis

one-sided

two-sided

$$H_1 : \mu_X > \mu_Y \qquad \text{or} \qquad H_1 : \mu_X < \mu_Y \qquad \text{or} \qquad H_1 : \mu_X \neq \mu_Y$$

Define the random variable $W = X - Y$ and rewrite the hypotheses above as

$E[W] = E[X-Y] = E[X] - E[Y]$
$= \mu_X - \mu_Y$

$$H_0 : \mu_W = 0$$

and

$$H_1 : \mu_W > 0 \qquad \text{or} \qquad H_1 : \mu_W < 0 \qquad \text{or} \qquad H_1 : \mu_W \neq 0$$

Under the assumption that W is approximately $N(\mu_W, \sigma_W^2)$, we can use the one-sample t-test discussed earlier for the mean of a normal distribution with unknown variance.

no info about σ_W^2
$\Downarrow$
same as last case of sec. 8.1 $\Leftarrow$ σ_W^2 is unknown!

Example

$m = 24$

Twenty-four students in the 9th and 10th grades were put on an rope-jumping program because someone thought that such a program would increase their speed in the 40-yard dash.

The following data give the difference in time that it took each student to run the 40-yard dash, with positive numbers indicating a faster time after the program

positive numbers mean "improvement"

0.28	0.01	0.13	0.33	−0.03	0.07	−0.18	−0.14
−0.33	0.01	0.22	0.29	−0.08	0.23	0.08	0.04
−0.30	−0.08	0.09	0.70	0.33	−0.34	0.50	0.06

→ compute $\overline{W}$ ← sample mean and S_W ← sample variance!

Let W equal the difference in time to run the 40-yard dash: the "before-program time" minus the "after-program time". → positive values mean improvement

Assuming that the distribution of W is approximately $N\left(\mu_W, \sigma_W^2\right)$, test whether the program had an impact using a significance level of $\alpha = 0.05$.

We want to test

$$\underbrace{H_0 : \mu_W = 0}_{\text{no succes}} \qquad \text{vs} \qquad \underbrace{H_1 : \mu_W > 0}_{\text{improvement observed}}$$

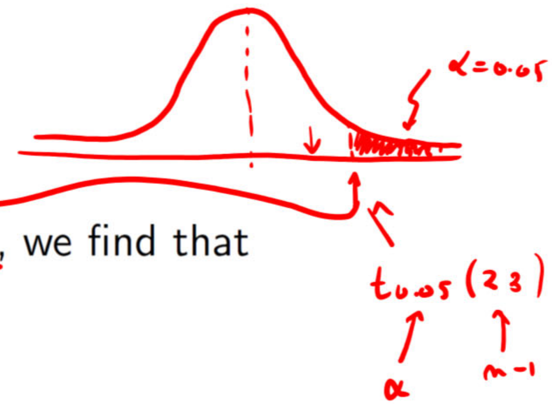
Under the assumption that W is approximately normal (with unknown variance), the test critical region is

$$Z = \frac{\bar{W} - 0}{\sigma_W/\sqrt{n}} \sim N(0,1)$$

$$V = \frac{(n-1)S_W^2}{\sigma_W^2} \sim \chi^2(n-1)$$

$$\Rightarrow T = \frac{Z}{\sqrt{V/(n-1)}}$$

$$\left\{ t = \frac{\bar{w} - 0}{s_w/\sqrt{n}} \geq t_\alpha(n-1) \right\}$$



Noting that $n = 24$, and computing $\bar{w} = 0.0788$ and $s_w = 0.2549$, we find that

$$t = \frac{0.0788 - 0}{0.2549/\sqrt{24}} = 1.514 < 1.714 = t_{0.05}(23)$$

and we do not reject H_0 at significance level $\alpha = 0.05$.

Setup

Let X and Y be independent random variables with respective means μ_X and μ_Y and variances σ_X^2 and σ_Y^2.

We are interested in testing the null hypothesis

$$H_0 : \mu_X = \mu_Y$$

against one of the following three composite alternative hypothesis:

$$H_1 : \mu_X > \mu_Y \quad \text{or} \quad H_1 : \mu_X < \mu_Y \quad \text{or} \quad H_1 : \mu_X \neq \mu_Y$$

Two independent random samples are taken, one from each distribution:

- $X_1, \ldots, X_n$ from the distribution of X (sample size n)
- $Y_1, \ldots, Y_m$ from the distribution of Y (sample size m)

The corresponding sample means, $\bar{x}$ and $\bar{y}$, are found, as well as the corresponding sample variances s_x^2 and s_y^2 if the true variances σ_X^2 and σ_Y^2 are not known .

- A decision concerning rejection of H_0, is related with how close $\bar{x} - \bar{y}$ is to 0.

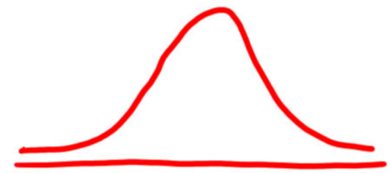
Normal distributions with known variances

Let X and Y be independent random variables with normal distributions $N\left(\mu_X, \sigma_X^2\right)$ and $N\left(\mu_Y, \sigma_Y^2\right)$, respectively.

Suppose σ_X^2 and σ_Y^2 are known.

Under the null hypothesis $H_0 : \mu_X = \mu_Y$, the test statistic

$$Z = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \quad \text{is } N(0, 1)$$



$$\left.\begin{array}{l} \bar{X} \sim N\left(\mu_X, \frac{\sigma_X^2}{n}\right) \\ \bar{Y} \sim N\left(\mu_Y, \frac{\sigma_Y^2}{m}\right) \end{array}\right\} \quad \bar{X} - \bar{Y} \sim N\left(\underbrace{\overbrace{\mu_X - \mu_Y}^{0}}, \overbrace{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}\right) \Rightarrow Z = \frac{\bar{X} - \bar{Y} - 0}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}}$$

under $H_0 : \mu_X = \mu_Y$ we have $\mu_X - \mu_Y = 0$

Let $\bar{x}$ and $\bar{y}$ be the observed values of the respective sample means, and let

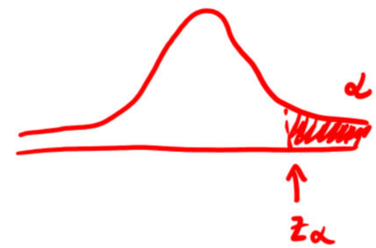
$$z = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}}$$

be the corresponding value of the test statistic.

$\mu_x - \mu_y > 0$

(i) When testing $H_0 : \mu_X = \mu_Y$ against $H_1 : \mu_X > \mu_Y$, we reject H_0 if

$$z \geq z_\alpha \quad \text{or, equivalently,} \quad \bar{x} - \bar{y} \geq z_\alpha \sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}$$

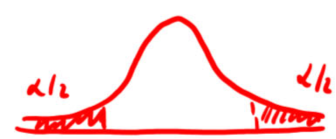


(ii) When testing $H_0 : \mu_X = \mu_Y$ against $H_1 : \mu_X < \mu_Y$, we reject H_0 if

$$z \leq -z_\alpha \quad \text{or, equivalently,} \quad \bar{x} - \bar{y} \leq -z_\alpha \sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}$$

(iii) When testing $H_0 : \mu_X = \mu_Y$ against $H_1 : \mu_X \neq \mu_Y$, we reject H_0 if

$$|z| \geq z_{\alpha/2} \quad \text{or, equivalently,} \quad |\bar{x} - \bar{y}| \geq z_{\alpha/2} \sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}$$



Example

$\sigma_X^2 = 1.6$ $\sigma_Y^2 = 3.75$

Assume that X is $N(\mu_X, \underbrace{1.6})$ and Y is $N(\mu_Y, \underbrace{3.75})$.

Both variances σ_X^2 and σ_Y^2 are known!!

Test the hypotheses

independence of X and Y is required!

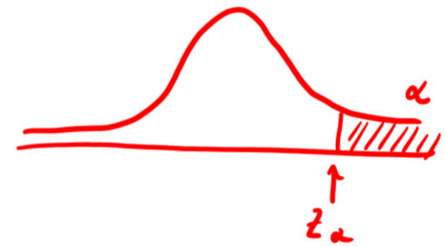
$$H_0 : \mu_X = \mu_Y$$

$$H_1 : \mu_X > \mu_Y$$

at significance level $\alpha = 0.01$ knowing that a random sample from the distribution of X with $n = 16$ observations yielded a sample mean $\bar{x} = 1.05$, while a random sample from the distribution of Y with $m = 25$ observations yielded a sample mean $\bar{y} = 0.85$.

Since the samples are taken from normal distributions with known variances, the test critical region is

$$z = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \geq z_\alpha$$



Since $n = 16$, $m = 25$, $\bar{x} = 1.05$, $\sigma_x^2 = 1.6$, $\bar{y} = 0.85$, $\sigma_y^2 = 3.75$, and $z_{0.01} = 2.327$, we find that

$\alpha = 0.01$

$$z = \frac{1.05 - 0.85}{\sqrt{1.6/16 + 3.75/25}} = 0.4$$

and so

$$z = 0.4 < 2.327 = z_{0.01}$$

and we do not reject H_0 at significance level $\alpha = 0.01$.

Normal distributions with unknown equal variances

Let X and Y be independent random variables with normal distributions $N\left(\mu_X, \sigma_X^2\right)$ and $N\left(\mu_Y, \sigma_Y^2\right)$, respectively.

Suppose the variances σ_X^2 and σ_Y^2 are unknown, but $\sigma_X^2 = \sigma_Y^2$.

Let

$$S_P = \sqrt{\frac{(n-1)S_X^2 + (m-1)S_Y^2}{n+m-2}}$$

estimator for the common variance using the two samples

$$\underbrace{S_p^2 = \frac{(n-1)S_X^2 + (m-1)S_Y^2}{n+m-2}}$$

be the pooled estimator for the common variance of X and Y.

Under the null hypothesis $H_0 : \mu_X = \mu_Y$, the test statistic

$$T = \frac{Z}{\sqrt{V/(n+m-2)}}$$

$$T = \frac{\bar{X} - \bar{Y}}{S_P\sqrt{1/n + 1/m}} \text{ is } t(r)$$

with

$$r = n + m - 2\,.$$

Under $H_0: \mu_X - \mu_Y = 0$

$$Z = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma^2}{n} + \frac{\sigma^2}{m}}}$$

$$U = \frac{(n+m-2)\,S_p^2}{\sigma^2}$$

$\chi^2(n+m-2)$

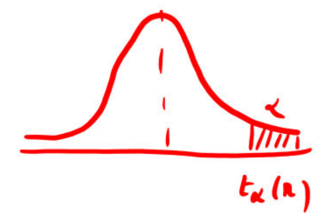
Let $\bar{x}$ and $\bar{y}$ be the respective sample means, s_x^2 and s_y^2 the respective sample variances, and s_p the corresponding pooled estimate for the standard deviation, and let

$$t = \frac{\bar{x} - \bar{y}}{s_p\sqrt{1/n + 1/m}}$$

be the value of the test statistic.

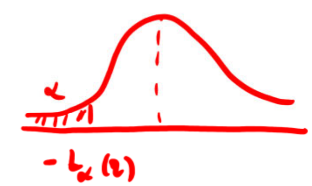
(i) When testing $H_0 : \mu_X = \mu_Y$ against $H_1 : \mu_X > \mu_Y$, we reject H_0 if

$$t \geq t_\alpha(r) \qquad \text{or, equivalently,} \qquad \bar{x} - \bar{y} \geq [t_\alpha(r)]s_p\sqrt{1/n + 1/m}$$



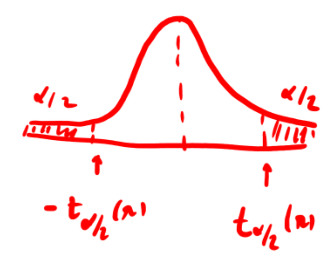
(ii) When testing $H_0 : \mu_X = \mu_Y$ against $H_1 : \mu_X < \mu_Y$, we reject H_0 if

$$t \leq -t_\alpha(r) \qquad \text{or, equivalently,} \qquad \bar{x} - \bar{y} \leq -[t_\alpha(r)]s_p\sqrt{1/n + 1/m}$$



(iii) When testing $H_0 : \mu_X = \mu_Y$ against $H_1 : \mu_X \neq \mu_Y$, we reject H_0 if

$$|t| \geq t_{\alpha/2}(r) \qquad \text{or, equivalently,} \qquad |\bar{x} - \bar{y}| \geq [t_{\alpha/2}(r)]s_p\sqrt{1/n + 1/m}$$



Example

Assume that X is $N(\mu_X, \sigma^2)$ and Y is $N(\mu_Y, \sigma^2)$.

We are not given either of σ_x^2 or σ_Y^2 **BUT** we are told that both equal σ^2 that is $\sigma_x^2 = \sigma_Y^2 = \sigma^2$ But unknown!

Test the hypotheses

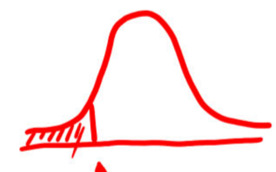
$$\begin{cases} H_0 : \mu_X = \mu_Y \\ H_1 : \mu_X < \mu_Y \end{cases}$$

at significance level $\alpha = 0.05$ knowing that a random sample from the distribution of X with $n = 11$ observations yielded a sample mean $\bar{x} = 1.03$ and a sample variance $s_x^2 = 0.24$, while a random sample from the distribution of Y with $m = 13$ observations yielded a sample mean $\bar{y} = 1.66$ and a sample variance $s_y^2 = 0.35$.

Since the samples are taken from normal distributions with unknown variance but equal variances, the test critical region is

$$\boxed{t = \frac{\bar{x} - \bar{y}}{s_p\sqrt{1/n + 1/m}}} \leq -t_\alpha(r)$$

$$s_p = \sqrt{\frac{(n-1)s_x^2 + (m-1)s_y^2}{n+m-2}}$$



Since $n = 11$, $m = 13$, $\overbrace{r = n + m - 2 = 22}$, $\bar{x} = 1.03$, $s_x^2 = 0.24$, $\bar{y} = 1.66$, $s_y^2 = 0.35$, and $t_{0.05}(22) = 1.717$, we find that

$$t = \frac{1.03 - 1.66}{\sqrt{\underbrace{\{[10(0.24) + 12(0.35)]/(11 + 13 - 2)\}}_{s_p^2}\underbrace{(1/11 + 1/13)}_{1/n + 1/m}}} = -2.81$$

and so

$$t = -2.81 < -1.717 = -t_{0.05}(22)$$

and we do reject H_0 at significance level $\alpha = 0.05$.

Normal distributions with unknown unequal variances

Let X and Y be independent random variables with normal distributions $N\left(\mu_X, \sigma_X^2\right)$ and $N\left(\mu_Y, \sigma_Y^2\right)$, respectively.

Suppose σ_X^2 and σ_Y^2 are unknown and not necessarily equal.

Welch's t-test: under the null hypothesis $H_0 : \mu_X = \mu_Y$, the test statistic

$$W = \frac{\bar{X} - \bar{Y}}{\sqrt{S_X^2/n + S_Y^2/m}} \quad \text{is approximately} \quad t\left(r\right) \; ,$$

where r is the integer part of

$$\frac{\left(\frac{s_x^2}{n} + \frac{s_y^2}{m}\right)^2}{\frac{1}{n-1}\left(\frac{s_x^2}{n}\right)^2 + \frac{1}{m-1}\left(\frac{s_y^2}{m}\right)^2} \, .$$

holds even if n and m are not large

Let $\bar{x}$ and $\bar{y}$ be the respective sample means, s_x^2 and s_y^2 the respective sample variances, and let

$$\boxed{w = \frac{\bar{x} - \bar{y}}{\sqrt{s_x^2/n + s_y^2/m}}}$$

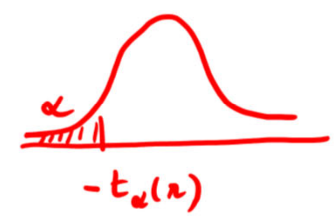
be the value of the test statistic.

(i) When testing $H_0 : \mu_X = \mu_Y$ against $H_1 : \mu_X > \mu_Y$, we reject H_0 if

$$w \geq t_\alpha(r) \qquad \text{or, equivalently,} \qquad \bar{x} - \bar{y} \geq [t_\alpha(r)]\sqrt{s_x^2/n + s_y^2/m}$$

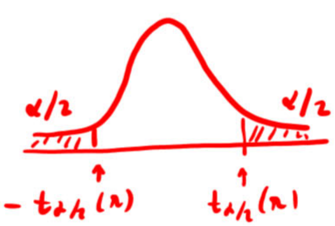
(ii) When testing $H_0 : \mu_X = \mu_Y$ against $H_1 : \mu_X < \mu_Y$, we reject H_0 if

$$w \leq -t_\alpha(r) \qquad \text{or, equivalently,} \qquad \bar{x} - \bar{y} \leq -[t_\alpha(r)]\sqrt{s_x^2/n + s_y^2/m}$$



(iii) When testing $H_0 : \mu_X = \mu_Y$ against $H_1 : \mu_X \neq \mu_Y$, we reject H_0 if

$$|w| \geq t_{\alpha/2}(r) \qquad \text{or, equivalently,} \qquad |\bar{x} - \bar{y}| \geq [t_{\alpha/2}(r)]\sqrt{s_x^2/n + s_y^2/m}$$



Example

Assume that X is $N(\mu_X, \sigma_X^2)$ and Y is $N(\mu_Y, \sigma_Y^2)$.

Test the hypotheses

$$\begin{cases} H_0 : \mu_X = \mu_Y \\ H_1 : \mu_X \neq \mu_Y \end{cases}$$

at significance level $\alpha = 0.01$ knowing that a random sample from the distribution of X with $n = 8$ observations yielded a sample mean $\bar{x} = 1.05$ and a sample variance $s_x^2 = 0.8$, while a random sample from the distribution of Y with $m = 16$ observations yielded a sample mean $\bar{y} = 0.85$ and a sample variance $s_y^2 = 2.4$.

Since the samples are taken from normal distributions with unknown (eventually unequal) variances, the approximate test critical region is

sample sizes $n = 8$ and $m = 16$ not large!

Welch's t-test $\Rightarrow$

$$|w| = \frac{|\bar{x} - \bar{y}|}{\sqrt{\frac{s_x^2}{n} + \frac{s_y^2}{m}}} \geq t_{\alpha/2}(r) \ ,$$

where r is the integer part of

$$\frac{\left(\frac{s_x^2}{n} + \frac{s_y^2}{m}\right)^2}{\frac{1}{n-1}\left(\frac{s_x^2}{n}\right)^2 + \frac{1}{m-1}\left(\frac{s_y^2}{m}\right)^2} \ .$$

same quantity as in Sec. 7.2

Since $n = 8$, $m = 16$, $s_x^2 = 0.8$ and $s_y^2 = 2.4$, we find that

$$\frac{\left(\frac{s_x^2}{n} + \frac{s_y^2}{m}\right)^2}{\frac{1}{n-1}\left(\frac{s_x^2}{n}\right)^2 + \frac{1}{m-1}\left(\frac{s_y^2}{m}\right)^2} \approx 21.34 \ .$$

and we take $\boxed{r = 21.}$

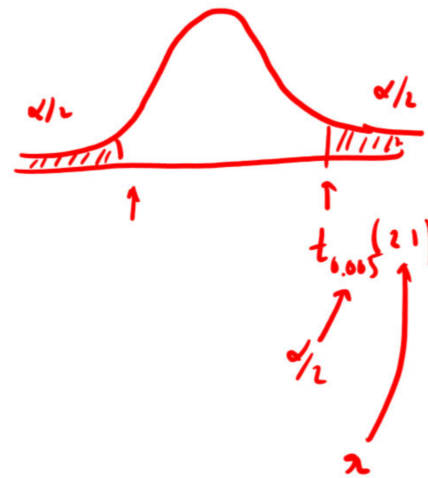
d.o.f. of t-distribution

Recalling that:

- $n = 8$, $\bar{x} = 1.05$, and $s_x^2 = 0.8$
- $m = 16$, $\bar{y} = 0.85$, and $s_y^2 = 2.4$

we find that

$$|w| = \frac{|\bar{x} - \bar{y}|}{\sqrt{\frac{s_x^2}{n} + \frac{s_y^2}{m}}} = \frac{|1.05 - 0.85|}{\sqrt{0.8/8 + 2.4/16}} = 0.4\ .$$



Noting that $t_{\alpha/2}(r) = t_{0.005}(21) = 2.832$, we conclude that

$$|w| = 0.4 < 2.832 = t_{0.005}(21)$$

and we do not reject H_0 at significance level $\alpha = 0.01$.

Approximate tests for large samples

Let X and Y be independent random variables with respective means μ_X and μ_Y and respective variances σ_X^2 and σ_Y^2.

Suppose σ_X^2 and σ_Y^2 are both unknown.

Furthermore, suppose we have:

- a large random sample $X_1, \ldots, X_n$, with $n \geq 30$, from the distribution of X.
- a large random sample $Y_1, \ldots, Y_m$, with $m \geq 30$, from the distribution of Y.

samples are large and distributions are NOT necessarily normal!

Under the null hypothesis $H_0 : \mu_X = \mu_Y$, the test statistic

$$Z = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{S_X^2}{n} + \frac{S_Y^2}{m}}} \quad \text{is approximately } N(0, 1)$$

Note: if the variances σ_X^2 and σ_Y^2 were known, we would use these instead of the sample variances s_x^2 and s_y^2.

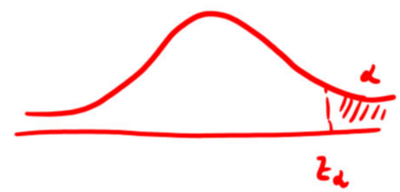
Let $\bar{x}$ and $\bar{y}$ be the respective sample means, s_x^2 and s_y^2 the respective sample variances, and let

$$z = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{s_x^2}{n} + \frac{s_y^2}{m}}}$$

be the corresponding value of the test statistic.

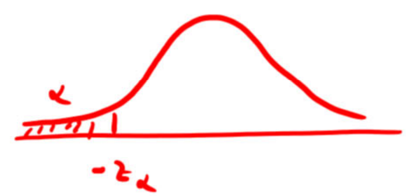
(i) When testing $H_0 : \mu_X = \mu_Y$ against $H_1 : \mu_X > \mu_Y$, we reject H_0 if

$$z \geq z_\alpha \qquad \text{or, equivalently,} \qquad \bar{x} - \bar{y} \geq z_\alpha \sqrt{\frac{s_x^2}{n} + \frac{s_y^2}{m}}$$



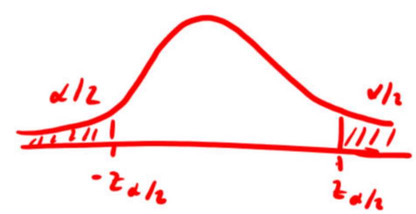
(ii) When testing $H_0 : \mu_X = \mu_Y$ against $H_1 : \mu_X < \mu_Y$, we reject H_0 if

$$z \leq -z_\alpha \qquad \text{or, equivalently,} \qquad \bar{x} - \bar{y} \leq -z_\alpha \sqrt{\frac{s_x^2}{n} + \frac{s_y^2}{m}}$$



(iii) When testing $H_0 : \mu_X = \mu_Y$ against $H_1 : \mu_X \neq \mu_Y$, we reject H_0 if

$$|z| \geq z_{\alpha/2} \qquad \text{or, equivalently,} \qquad |\bar{x} - \bar{y}| \geq z_{\alpha/2} \sqrt{\frac{s_x^2}{n} + \frac{s_y^2}{m}}$$



Example

Let X and Y be independent random variables with respective means μ_X and μ_Y and variances σ_X^2 and σ_Y^2.

Test the hypotheses

$$H_0 : \mu_X = \mu_Y$$
$$H_1 : \mu_X < \mu_Y$$

at significance level $\alpha = 0.05$ knowing that a random sample from the distribution of X with $n = 36$ observations yielded a sample mean $\bar{x} = 1.05$ and a sample variance $s_x^2 = 3.6$, while a random sample from the distribution of Y with $m = 64$ observations yielded a sample mean $\bar{y} = 2.05$ and a sample variance $s_y^2 = 9.6$.

Since both samples are large ($n = 36 \geq 30$ and $m = 64 \geq 30$), can construct an approximate hypothesis test. The test critical region is

$$z = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{s_X^2}{n} + \frac{s_Y^2}{m}}} \leq -z_\alpha$$

Since $n = 36$, $m = 64$, $\bar{x} = 1.05$, $s_x^2 = 3.6$, $\bar{y} = 2.05$, $s_y^2 = 9.6$, and $z_{0.05} = 1.645$, we find that

$$z = \frac{1.05 - 2.05}{\sqrt{3.6/36 + 9.6/64}} = -2$$

and so

$$z = -2 < -1.645 = -z_{0.05}$$

and we reject H_0 at significance level $\alpha = 0.05$.