Sec. 8.3 Ex. 9

Let $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$ be independent.

a) We want to test

$$H_0: \sigma_x^2 = \sigma_y^2 \quad \text{vs} \quad H_1: \sigma_x^2 > \sigma_y^2$$

at significance level $\alpha = 0.01$ using a sample of

i.e. $n_x = n_y = 31$

The test statistic for this hypothesis test

$$F = \frac{\dfrac{(n_x-1)S_x^2}{\sigma_x^2} \Big/ (n_x-1)}{\dfrac{(n_y-1)S_y^2}{\sigma_y^2} \Big/ (n_y-1)} = \frac{S_x^2}{S_y^2} \cdot \frac{\sigma_y^2}{\sigma_x^2} \overset{\text{under } H_0:\ \sigma_x^2 = \sigma_y^2}{=} \frac{S_x^2}{S_y^2} \sim F(\overbrace{30}^{n_x-1}, \overbrace{30}^{n_y-1})$$

The critical region for this test is 

$$f \geq F_{\alpha}(n_x-1, n_y-1) = F_{0.01}(30,30) = 2.39 ,$$

where f is the observed value of the test statistic F.

Since we are given that $\frac{S_x^2}{S_y^2} = 3.2053$, then

$$f = 3.2053 > 2.39 = F_{0.01}(30,30)$$

and we reject H_0 at significance level $\alpha = 0.01$

b) 2.39 is $F_{0.01}(30,30)$

c) To construct a two sided 95% confidence interval for σ_x^2/σ_y^2, we note that

$$F = \frac{\frac{(n_y-1)S_y^2}{\sigma_y^2} \Big/ (n_y-1)}{\frac{(n_x-1)S_x^2}{\sigma_x^2} \Big/ (n_x-1)} = \frac{\sigma_x^2}{\sigma_y^2} \cdot \frac{S_y^2}{S_x^2} \sim F(n_y-1, n_x-1)$$

and so, since

$$P\left(F_{1-\alpha/2}(n_y-1, n_x-1) < F < F_{\alpha/2}(n_y-1, n_x-1)\right) = 1-\alpha$$

we get

$$P\left(F_{1-\alpha/2}(n_y-1, n_x-1) < \frac{\sigma_x^2}{\sigma_y^2} \cdot \frac{S_y^2}{S_x^2} < F_{\alpha/2}(n_y-1, n_x-1)\right) = 1-\alpha$$

and solving for σ_x^2/σ_y^2, we obtain

$$P\left(\underbrace{F_{1-\alpha/2}(n_y-1, n_x-1)}_{\text{equal to } \frac{1}{F_{\alpha/2}(n_x-1, n_y-1)}} \cdot \frac{S_x^2}{S_y^2} < \frac{\sigma_x^2}{\sigma_y^2} < F_{\alpha/2}(n_y-1, n_x-1) \frac{S_x^2}{S_y^2}\right) = 1-\alpha$$

Hence, the two sided confidence interval is

$$\left[\frac{1}{F_{\alpha/2}(n_x-1, n_y-1)} \cdot \frac{S_x^2}{S_y^2} \,,\; F_{\alpha/2}(n_y-1, n_x-1) \frac{S_x^2}{S_y^2} \right]$$

Using the values given in item a) and $\alpha = 0.05$

$$\left[\frac{1}{\underbrace{F_{0.025}(30,30)}_{1/2.07}} \cdot (3.2053) \,,\; \underbrace{F_{0.025}(30,30)}_{2.07} \cdot (3.2053)\right] \approx [1.548, 6.635]$$