Sec 1.1 Ex 2

Define events:

A : event "customer insures more than one car"

B : event "customer insures a sports car"

Given the information

$P(A) = 0.85$ $P(B) = 0.23$ $P(\underbrace{A \cap B}) = 0.17$

$A \cap B$: event "customer insures more than one car, including a sports car"

Asked to find:

$P(\overline{A} \cap \overline{B}) = ?$

$\overline{A}$: customer insures only one car

$\overline{B}$: car insured is not a sports car

Note that $\overline{A} \cap \overline{B} = \overline{A \cup B}$ (by De Morgan's law)

and so $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - \underbrace{P(A \cup B)}_{?}$

But then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.85 + 0.23 - 0.17 = 0.91$

Thus, $P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B) = 1 - 0.91 = 0.09$

$$P(\underbrace{\bar{A} \cap \bar{B} \cap \bar{C}}_{\overline{A \cup B \cup C}}) = P(\overline{A \cup B \cup C})$$

$$= 1 - P(A \cup B \cup C)$$

$$= 1 - \big[P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \big]$$

given $P(A)$, $P(B)$, $P(A \cap B)$

Find $P(A \cap \bar{B})$

Note that

$$A = (A \cap B) \cup (A \cap \bar{B})$$

mutually exclusive because $B, \bar{B}$ are mutually exclusive

$$P(A) = P(A \cap B) + P(A \cap \bar{B})$$

Sec 1.2 Ex 7

1.2-7. In a state lottery, four digits are drawn at random one at a time with replacement from 0 to 9. Suppose that you win if any permutation of your selected integers is drawn. Give the probability of winning if you select

(a) 6, 7, 8, 9.

(b) 6, 7, 8, 8.

(c) 7, 7, 8, 8.

(d) 7, 8, 8, 8.

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$S = \{$ all 4 digit sequences made with digits 0 to 9 with replacement $\}$

a) There 4! ways of reordering 6, 7, 8, 9

A : event "6, 7, 8, 9" are drawn

$$P(A) = \frac{|A|}{|S|} = \frac{4!}{10^4}$$

b) B : event "6, 7, 8, 8" are drawn $\quad |B| = \frac{4!}{2!} \Rightarrow P(B) = \frac{|B|}{|S|} = \frac{4!/2!}{10^4}$

c) C : event "7, 7, 8, 8" are drawn $\quad |C| = \frac{4!}{2!\,2!} \Rightarrow P(C) = \frac{|C|}{|S|} = \frac{\frac{4!}{2!\,2!}}{10^4}$

d) D : event "7, 8, 8, 8" are drawn $\quad |D| = \frac{4!}{3!} \Rightarrow P(D) = \frac{|D|}{|S|} = \frac{4!/3!}{10^4}$

2.1-3. For each of the following, determine the constant c so that $f(x)$ satisfies the conditions of being a pmf for a random variable X, and then depict each pmf as a bar graph:

(a) $f(x) = x/c, \quad x = 1, 2, 3, 4.$

(b) $f(x) = cx, \quad x = 1, 2, 3, \ldots, 10.$

(c) $f(x) = c(1/4)^x, \quad x = 1, 2, 3, \ldots.$

(d) $f(x) = c(x+1)^2, \quad x = 0, 1, 2, 3.$

(e) $f(x) = x/c, \quad x = 1, 2, 3, \ldots, n.$

(f) $f(x) = \dfrac{c}{(x+1)(x+2)}, \quad x = 0, 1, 2, 3, \ldots.$

HINT: In part (f), write $f(x) = c[1/(x+1) - 1/(x+2)]$.

Recall: $\displaystyle\sum_{x=1}^{n} x = \underbrace{\frac{n(n+1)}{2}}_{\text{sum of } 1^{st}\ n \text{ positive integers}}$

e) $f(x) = \dfrac{x}{c}, \quad x = 1, 2, 3, \ldots, n$

For f to be pmf we need to ensure that

① $f(x) > 0$ for all $x = 1, 2, 3, \ldots, n$

② $\displaystyle\sum_{x=1}^{n} f(x) = 1$

Condition ① tells us that c must be positive

Condition ② gives the exact value of c: $\displaystyle\sum_{x=1}^{n} f(x) = 1 \iff \sum_{x=1}^{n} \frac{x}{c} = 1 \iff \frac{1}{c} \overbrace{\sum_{x=1}^{n} x}^{\frac{n(n+1)}{2}} = 1$

$\iff \dfrac{1}{c} \dfrac{n(n+1)}{2} = 1 \iff c = \dfrac{n(n+1)}{2}$

f) $f(x) = \dfrac{c}{(x+1)(x+2)}$, $x = 0, 1, 2, \dots$

Hint $f(x) = c\left(\dfrac{1}{x+1} - \dfrac{1}{x+2}\right)$

For f to be a pmf it must satisfy:

(1) $f(x) > 0$ for all $x = 0, 1, 2, \dots$

(2) $\displaystyle\sum_{x=0}^{\infty} f(x) = 1$

telescoping sum

$\displaystyle\sum_{m=0}^{\infty} a_m - a_{m+1} = ?$

$\displaystyle\lim_{N\to\infty} \sum_{m=0}^{N} a_m - a_{m+1} =$

$= \displaystyle\lim_{N\to\infty} (a_0 - a_1) + (a_1 - a_2) + \dots + (a_N - a_{N+1})$

$= \displaystyle\lim_{N\to\infty} a_0 - a_{N+1}$

Condition 1 implies that c must be positive

Condition 2 given that $\displaystyle\sum_{m=0}^{\infty} f(m) = 1$ $\Rightarrow c \displaystyle\sum_{m=0}^{\infty} \left[\frac{1}{m+1} - \frac{1}{m+2}\right] = 1$

a_m

$\Rightarrow c \displaystyle\lim_{N\to\infty} \left(\frac{1}{0+1} - \underbrace{\frac{1}{N+2}}_{\text{limit is } 0}\right) = 1$

$\Rightarrow c \cdot 1 = 1 \Rightarrow c = 1$

1.2-4. The "eating club" is hosting a make-your-own sundae at which the following are provided:

Ice Cream Flavors	Toppings
Chocolate	Caramel
Cookies-'n'-cream	Hot fudge
Strawberry	Marshmallow
Vanilla	M&Ms
	Nuts
	Strawberries

(a) How many sundaes are possible using one flavor of ice cream and three different toppings?

(b) How many sundaes are possible using one flavor of ice cream and from zero to six toppings?

(c) How many different combinations of flavors of three scoops of ice cream are possible if it is permissible to make all three scoops the same flavor?

a) $\binom{4}{1} \cdot \binom{6}{3}$ ← ordering does not matter

there is no repetition

$\binom{4}{1}$: # of possible flavors

$\binom{6}{3}$: 3 different toppings out of 6

$$\frac{6!}{3!\,3!} = \binom{6}{3}$$

b) $\binom{4}{1} \cdot 2^6$

$$\binom{6}{0} + \binom{6}{1} + \dots + \binom{6}{6} = 2^6$$

$$\underset{2}{\text{Y/N}} \cdot \underset{2}{\text{Y/N}} \cdot \underset{2}{\text{Y/N}} \cdot \underset{2}{\_} \cdots \underset{2}{\_} \cdot \underset{2}{\text{Y/N}} = 2^6$$

c) Allow repetitions of flavors and order does not matter!

|O O|O|

O | O O | |

1 chocolate, 2 cookies, 0 strawberry, 0 vanilla

no chocolate and 1 of each of the others: | O | O | O

NOT on this problem: how many ways to pick 3 scoops of distinct flavors

$\binom{4}{3}$ → 0 repetition of 4 flavors

pick 3 distinct ones

set with n distinct element / elements

draw an unordered sample of size r with replacement

$\underbrace{\text{with replacement}}$ — can have repetitions

Roll a 6-face dice 10 times $\quad n = 6 \quad r = 10$

zeros ← $r + \overset{\text{bars}}{\widetilde{n-1}}$ symbols

$$0\ 0 \mid 0\ 0\ 0 \mid 0 \mid\mid 0\ 0\ 0 \mid 0$$

$$\_\ \_ \mid \_\ \_\ \_ \mid \_ \mid\mid \_\ \_\ \_ \mid \_$$

how many distinguishable permutations do we have

$$\frac{(r+n-1)!}{r!\,(n-1)!}$$

sequence $\quad$ 2 1s, 3 2s, 1 3s, 0 4s, 3 5s, 1 6s

$$\binom{r+n-1}{r}$$

# of unordered sequences with r elements extracted with replacement from a set with n elements

1.2-13. A bridge hand is found by taking 13 cards at random and without replacement from a deck of 52 playing cards. Find the probability of drawing each of the following hands.

(a) One in which there are five spades, four hearts, three diamonds, and one club.

(b) One in which there are five spades, four hearts, two diamonds, and two clubs.

(c) One in which there are five spades, four hearts, one diamond, and three clubs.

(d) Suppose you are dealt five cards of one suit, four cards of another. Would the probability of having the other suits split 3 and 1 be greater than the probability of having them split 2 and 2?

$S = \{\text{all bridge hands}\}$

$|S| = \binom{52}{13}$

without replacement means that cards previously drawn are not put back into the deck of cards.

a) A: "5 spades, 4 hearts, 3 diamonds, 1 club"

$$|A| = \binom{13}{5}\cdot\binom{13}{4}\cdot\binom{13}{3}\cdot\binom{13}{1}$$

$$P(A) = \frac{|A|}{|S|} = \frac{\binom{13}{5}\binom{13}{4}\binom{13}{3}\cdot\binom{13}{1}}{\binom{52}{13}}$$

b) B: "5 spades, 4 hearts, 2 diamonds, 2 clubs"

$|B| = \binom{13}{5}\cdot\binom{13}{4}\cdot\binom{13}{2}\cdot\binom{13}{2}$ and so $P(B) = \frac{|B|}{|S|} = \frac{\binom{13}{5}\binom{13}{4}\binom{13}{2}\cdot\binom{13}{2}}{|S|}$

c) C: "5 spaces, 4 hearts, 1 diamond, 3 clubs" $|C| = \binom{13}{5}\binom{13}{4}\binom{13}{1}\cdot\binom{13}{3} = |A|$

$P(C) = P(A)$ found in item a)

d) 3-1 split has probability $P(A) + P(C) = 2P(A)$ while 2-2 split has probability $P(B)$

We can evaluate these to find that $P(B) < 2P(A)$ 2-2 split is less likely than a 3-1 split

$$(A \cap B) \cup (A \cap \bar{B}) = A$$

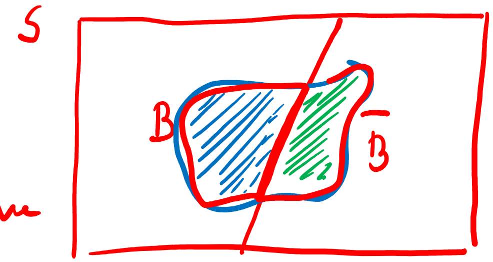
$$P(A) = P\big((A \cap B) \cup (A \cap \bar{B})\big)$$

$$P(A) = P(A \cap B) + P(A \cap \bar{B})$$

only because $(A \cap B)$ and $(A \cap \bar{B})$ are mutually exclusive

$$= P(B) . P(A|B) + P(\bar{B}) . P(A|\bar{B})$$

law of total prob.



$P(A \cap B) = ?$

MULT. RULE

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(B) . P(A|B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

if A and B not mutually exclusive

A, B mutually exclusive : $A \cap B = \phi$ ←

A, B independent $P(A \cap B) = P(A) \cdot P(B)$

A, B, C mutually independent

$$\begin{cases} P(A \cap B) = P(A) \cdot P(B) \\ P(A \cap C) = P(A) \cdot P(C) \\ P(B \cap C) = P(B) \cdot P(C) \\ P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C) \end{cases}$$