Math 4501 - Probability and Statistics II

8.8 - Hypothesis testing: likelihood ratio tests

Overview

We will consider a general test-construction method that is applicable when either or both the null and alternative hypotheses are composite.

Assumption: the functional form of the pdf/pmf $f(x;\theta)$ is known, but depends on one or more unknown parameters θ.

Notation: The total parameter space will be denoted Ω, the set of all possible values of the parameter θ given by either H_0 or H_1, and the hypotheses will be stated as

$$H_0 : \theta \in \omega \quad \text{against} \quad H_1 : \theta \in \omega' \ ,$$

where ω is a subset of Ω and ω' is the complement of ω with respect to Ω.

Intuition:

- the test will be constructed with the use of a ratio of likelihood functions that have been maximized in ω and Ω, respectively.
- natural generalization of the ratio appearing in the Neyman-Pearson lemma when the two hypotheses were simple.

Likelihood ratio

Definition (Likelihood ratio)

The *likelihood ratio* is the quotient

$$\lambda = \frac{L(\widehat{\omega})}{L(\widehat{\Omega})} ,$$

where

$$L(\widehat{\omega}) = \max_{\theta \in \omega} L(\theta) \qquad \text{and} \qquad L(\widehat{\Omega}) = \max_{\theta \in \Omega} L(\theta) .$$

Property: $0 \leq \lambda \leq 1$

- being the quotient of two nonnegative real numbers, $\lambda \geq 0$.
- since $\omega \subset \Omega$, $L(\widehat{\omega}) \leq L(\widehat{\Omega})$ and $\lambda \leq 1$.

Interpretation: $L(\widehat{\omega})$ being much smaller than $L(\widehat{\Omega})$ suggests that the data $x_1, x_2, \ldots, x_n$ do not support the hypothesis $H_0 : \theta \in \omega$:

- λ small suggests rejection of H_0
- λ close to 1 suggests support of the null hypothesis H_0.

Definition (Likelihood ratio test)

To test

$$H_0 : \theta \in \omega \qquad \text{against} \qquad H_1 : \theta \in \omega' \ ,$$

the *critical region for the likelihood ratio test* is the set of points in the sample space for which

$$\lambda = \frac{L(\widehat{\omega})}{L(\widehat{\Omega})} \leq k,$$

where $0 < k < 1$ and k is selected so that the test has a desired significance level α.

Example

Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal distribution $N(\mu, 5)$. Construct the critical region for the likelihood ratio test for testing

$$H_0 : \mu = 162 \qquad \text{against} \qquad H_1 : \mu \neq 162$$

with significance level α.

We want to test $H_0 : \mu \in \omega$ vs $H_1 : \mu \in \omega'$, where

$$\omega = \{\underbrace{162}_{\mu_0}\} \quad \text{and} \quad \underbrace{\Omega = \mathbb{R}}_{\text{full parameter space}} \quad (\text{and so } \omega' = \mathbb{R} \setminus \{162\})$$

Let

$$L(\mu) = \prod_{i=1}^{n} f(x_i; \mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \cdot 5}} \cdot \exp\left(-(x_i - \mu)^2 / 2\cdot(5) \right)$$

$$= (10\pi)^{-n/2} \exp\left(-\frac{1}{10} \sum_{i=1}^{n} (x_i - \mu)^2 \right)$$

and observe that:

(1) $L(\hat{\omega}) = \max_{\mu \in \omega} L(\mu) = L(162) = (10)^{-n/2} \exp\left(-\frac{1}{10}\sum_{i=1}^{n}(x_i - 162)^2\right)$

because $\omega = \{162\}$ has a single element

(2) $L(\hat{\Omega}) = \max_{\mu \in \Omega} L(\mu) = L(\bar{x}) = (10)^{-n/2} \exp\left(-\frac{1}{10}\sum_{i=1}^{n}(x_i - \bar{x})^2\right)$

because $\bar{x}$ is the MLE of μ

Hence, the likelihood ratio is

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \frac{(10)^{-n/2} \exp\left(-\frac{1}{10}\sum_{i=1}^{n}(x_i - 162)^2\right)}{(10)^{-n/2} \exp\left(-\frac{1}{10}\sum_{i=1}^{n}(x_i - \bar{x})^2\right)} = \frac{\exp\left(-\frac{1}{10}\sum_{i=1}^{n}(x_i - 162)^2\right)}{\exp\left(-\frac{1}{10}\sum_{i=1}^{n}(x_i - \bar{x})^2\right)}$$

We observe that

$$\sum_{i=1}^{m}(x_i-162)^2 = \sum_{i=1}^{m}(x_i-\bar{x}+\bar{x}-162)^2 =$$

$$= \sum_{i=1}^{m}(x_i-\bar{x})^2 + 2(\bar{x}-162)\underbrace{\sum_{i=1}^{m}(x_i-\bar{x})}_{=0} + m(\bar{x}-162)^2$$

$$= \sum_{i=1}^{m}(x_i-\bar{x})^2 + m(\bar{x}-162)^2$$

and so

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \frac{\exp\left(-\frac{1}{10}\sum_{i=1}^{m}(x_i-\bar{x})^2 - \frac{m}{10}(\bar{x}-162)^2\right)}{\exp\left(-\frac{1}{10}\sum_{i=1}^{m}(x_i-\bar{x})^2\right)} = \exp\left(-\frac{m}{10}(\bar{x}-162)^2\right)$$

Observe that:

(1) for $\bar{x}$ close to 162, λ is close to 1

(2) for $\bar{x}$ far from 162, λ is close to 0.

Indeed, the critical region for the likelihood ratio test is given by $\lambda \leq k$, that is

$$\exp\left(-\frac{n}{10}(\bar{x}-162)^2\right) \leq k$$

this is equivalent to

$$-\frac{n}{10}(\bar{x}-162)^2 \leq \ln k$$

$$\Leftrightarrow \quad (\bar{x}-162)^2 \geq \frac{-10\ln k}{n}$$

$$\Leftrightarrow \quad |\bar{x}-162| \geq \sqrt{-10\ln k/n}$$

or even

$$\frac{|\bar{x}-162|}{\sqrt{5/n}} \geq \sqrt{-2\ln k} = c$$

Since $X_1, X_2, \ldots, X_n \sim N(\mu, 5)$, then $\dfrac{\bar{X} - \mu}{\sqrt{5/n}} \sim N(0,1)$ and so,

under $H_0 : \mu = 162$, we have that $Z = \dfrac{\bar{X} - 162}{\sqrt{5/n}} \sim N(0,1)$

Letting $c = z_{\alpha/2}$, we obtain the critical region

$$C = \left\{ \bar{x} : \frac{|\bar{x} - 162|}{\sqrt{5/n}} \geq z_{\alpha/2} \right\}$$ $\leftarrow$ NOTE: This is the same critical region as in Section 8.1

Test for mean: normal population with known variance

Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal population $N\left(\mu, \sigma^2\right)$, where μ is unknown but σ^2 is known.

The likelihood ratio test for

$$H_0 : \mu = \mu_0 \qquad \text{against} \qquad H_1 : \mu \neq \mu_0$$

is the same as the one discussed previously (in section 8.1).

We want to test $H_0 : \mu \in \omega$ vs $H_1 : \mu \in \omega'$, where

$\omega = \{\mu_0\}$ and $\Omega = \mathbb{R}$ (and so $\omega' = \mathbb{R} \setminus \{\mu_0\}$)

Let
$$L(\mu) = \prod_{i=1}^{n} f(x_i; \mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-(x_i - \mu)^2 / 2\sigma^2\right)$$

$$= \left(2\pi\sigma^2\right)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2\right)$$

and observe that:

(1) $L(\hat{\omega}) = \max_{\mu \in \omega} L(\mu) = L(\mu_0) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu_0)^2\right)$

because $\omega = \{\mu_0\}$ has
a single element

(2) $L(\hat{\Omega}) = \max_{\mu \in \Omega} L(\mu) = L(\bar{x}) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \bar{x})^2\right)$

because $\bar{x}$ is the
MLE of μ

Hence, the likelihood ratio is

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \frac{(2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu_0)^2\right)}{(2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \bar{x})^2\right)} = \frac{\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu_0)^2\right)}{\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \bar{x})^2\right)}$$

We observe that

$$\sum_{i=1}^{m} (x_i - \mu_0)^2 = \sum_{i=1}^{m} (x_i - \bar{x} + \bar{x} - \mu_0)^2 =$$

$$= \sum_{i=1}^{m} (x_i - \bar{x})^2 + 2(\bar{x} - \mu_0) \underbrace{\sum_{i=1}^{m} (x_i - \bar{x})}_{=0} + m(\bar{x} - \mu_0)^2$$

$$= \sum_{i=1}^{m} (x_i - \bar{x})^2 + m(\bar{x} - \mu_0)^2$$

and so

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \frac{\exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{m} (x_i - \bar{x})^2 - m(\bar{x} - \mu_0)^2\right)}{\exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{m} (x_i - \bar{x})^2\right)} = \exp\left(-\frac{m}{2\sigma^2}(\bar{x} - \mu_0)^2\right)$$

Observe that:

(1) for $\bar{x}$ close to μ_0, λ is close to 1

(2) for $\bar{x}$ far from μ_0, λ is close to 0.

Indeed, the critical region for the likelihood ratio test is given by $\lambda \leq k$, that is

$$\exp\left(-\frac{n}{2\sigma^2}(\bar{x}-\mu_0)^2\right) \leq k$$

this is equivalent to

$$-\frac{n}{2\sigma^2}(\bar{x}-\mu_0)^2 \leq \ln k$$

$$\Leftrightarrow \quad (\bar{x}-\mu_0)^2 \geq \frac{-2\sigma^2 \ln k}{n}$$

$$\Leftrightarrow \quad |\bar{x}-\mu_0| \geq \sqrt{-2\sigma^2 \ln k/n}$$

or even

$$\frac{|\bar{x}-\mu_0|}{\sqrt{\sigma^2/n}} \geq \sqrt{-2\ln k} = c$$

Since $X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)$, then $\dfrac{\bar{X} - \mu}{\sqrt{\sigma^2/n}} \sim N(0,1)$ and so,

under $H_0: \mu = \mu_0$, we have that $Z = \dfrac{\bar{X} - \mu_0}{\sqrt{\sigma^2/n}} \sim N(0,1)$

Letting $c = z_{\alpha/2}$, we obtain the critical region

$$C = \left\{ (x_1, \ldots, x_n) : \frac{|\bar{x} - \mu_0|}{\sqrt{\sigma^2/n}} \geq z_{\alpha/2} \right\}$$

← NOTE: This is the same critical region as in Section 8.1

Test for mean: normal population with unknown variance

Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal population $N(\mu, \sigma^2)$, where both μ and σ^2 are unknown.

The likelihood ratio test for

$$H_0 : \mu = \mu_0 \qquad \text{against} \qquad H_1 : \mu \neq \mu_0$$

is the same as the one discussed previously (in section 8.1).

We want to test $H_0 : \mu \in \omega$ vs $H_1 : \mu \in \omega'$, where

$\omega = \{(\mu, \sigma^2) : \mu = \mu_0 \text{ and } \sigma^2 > 0\}$ and $\Omega = \{(\mu, \sigma^2) : \mu \in \mathbb{R} \text{ and } \sigma^2 > 0\}$

Let $L(\mu, \sigma^2) = \prod_{i=1}^{n} f(x_i; \mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-(x_i - \mu)^2 / 2\sigma^2\right)$

$$= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2\right)$$

and observe that:

(1) $$L(\hat{\omega}) = \max_{(\mu,\sigma^2)\in\omega} L(\mu,\sigma^2) = L(\mu_0,\widehat{\sigma^2}) = \left(\frac{1}{\frac{2\pi}{m}\sum_{i=1}^{m}(x_i-\mu_0)^2}\right)^{m/2} \cdot \exp\left(-\frac{\sum_{i=1}^{m}(x_i-\mu_0)^2}{\frac{2}{m}\sum_{i=1}^{m}(x_i-\mu_0)^2}\right)$$

because when $\mu=\mu_0$, such as in ω,
the MLE for σ^2 is $\widehat{\sigma^2} = \frac{1}{m}\sum_{i=1}^{m}(x_i-\mu_0)^2$

$$= \left(\frac{m\,e^{-1}}{2\pi\sum_{i=1}^{m}(x_i-\mu_0)^2}\right)^{m/2}$$

(2) $$L(\hat{\Omega}) = \max_{\in\Omega} L(\mu,\sigma^2) = L(\bar{x},\widehat{\sigma^2}) = \left(\frac{1}{\frac{2\pi}{m}\sum_{i=1}^{m}(x_i-\bar{x})^2}\right)^{m/2} \cdot \exp\left(-\frac{\sum_{i=1}^{m}(x_i-\bar{x})^2}{\frac{2}{m}\sum_{i=1}^{m}(x_i-\bar{x})^2}\right)$$

because when the MLE for μ and σ^2 are
$\hat{\mu}=\bar{x}$ and $\widehat{\sigma^2} = \frac{1}{m}\sum_{i=1}^{m}(x_i-\bar{x})^2$

$$= \left(\frac{m\,e^{-1}}{2\pi\sum_{i=1}^{m}(x_i-\bar{x})^2}\right)^{m/2}$$

Hence, the likelihood ratio is

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \frac{\left(\frac{m\,e^{-1}}{2\pi\sum_{i=1}^{m}(x_i-\mu_0)^2}\right)^{m/2}}{\left(\frac{m\,e^{-1}}{2\pi\sum_{i=1}^{m}(x_i-\bar{x})^2}\right)^{m/2}} = \left[\frac{\sum_{i=1}^{m}(x_i-\bar{x})^2}{\sum_{i=1}^{m}(x_i-\mu_0)^2}\right]^{m/2}$$

We observe that

$$\sum_{i=1}^{m}(x_i-\mu_0)^2 = \sum_{i=1}^{m}(x_i-\bar{x}+\bar{x}-\mu_0)^2 =$$

$$= \sum_{i=1}^{m}(x_i-\bar{x})^2 + 2(\bar{x}-\mu_0)\underbrace{\sum_{i=1}^{m}(x_i-\bar{x})}_{=0} + m(\bar{x}-\mu_0)^2$$

$$= \sum_{i=1}^{m}(x_i-\bar{x})^2 + m(\bar{x}-\mu_0)^2$$

and so

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \left[\frac{\sum_{i=1}^{m}(x_i-\bar{x})^2}{\sum_{i=1}^{m}(x_i-\bar{x})^2 + m(\bar{x}-\mu_0)^2}\right]^{m/2} = \left[\frac{1}{1+\frac{m(\bar{x}-\mu_0)^2}{\sum_{i=1}^{m}(x_i-\bar{x})^2}}\right]^{m/2}$$

Observe that :

(1) for $\bar{x}$ close to μ_0 , λ is close to 1

(2) for $\bar{x}$ far from μ_0 , λ is close to 0.

Indeed, the critical region for the likelihood ratio test is given by $\lambda \le k$, that is

$$\frac{1}{1 + \frac{n(\bar{x}-\mu_0)^2}{\sum_{i=1}^{n} (x_i-\bar{x})^2}} \le k^{2/n}$$

this is equivalent to

$$\frac{n(\bar{x}-\mu_0)^2}{\frac{1}{n-1}\sum_{i=1}^{n} (x_i-\bar{x})^2} \geq (n-1)\left(k^{-2/n} - 1\right)$$

or even

$$\frac{|\bar{x}-\mu_0|}{s/\sqrt{n}} \geq \sqrt{(n-1)\left(k^{-2/n}-1\right)} = c$$

Since $X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)$, then $\dfrac{\bar{X} - \mu}{\sqrt{\sigma^2/n}} \sim N(0,1)$ and so,

under $H_0: \mu = \mu_0$, we have that $Z = \dfrac{\bar{X} - \mu_0}{\sqrt{\sigma^2/n}} \sim N(0,1)$

Moreover, $U = \dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$ and $Z \sim N(0,1)$ are independent, and so

$$T = \frac{Z}{\sqrt{U/(n-1)}} = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t(n-1)$$

Letting $c = t_{\alpha/2}(n-1)$ we obtain the critical region:

$$C = \left\{ (x_1, \ldots, x_n) : \frac{|\bar{x} - \mu_0|}{S/\sqrt{n}} \geq t_{\alpha/2}(n-1) \right\} \leftarrow$$

Same critical region as obtained in Sec 8.1 for this test

Test for variance

Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal population $N\left(\mu, \sigma^2\right)$, where both μ andσ^2 are unknown.

We will determine the likelihood ratio test for

$$H_0 : \sigma^2 = \sigma_0^2 \qquad \text{against} \qquad H_1 : \sigma^2 \neq \sigma_0^2 \ .$$

We want to test $H_0 : \in \omega$ vs $H_1 : \in \omega'$, where

$\omega = \{ (\mu, \sigma^2) : \mu \in \mathbb{R} \, , \ \sigma^2 = \sigma_0^2 \}$ and $\Omega = \{ (\mu, \sigma^2) : \mu \in \mathbb{R}$ and $\sigma^2 > 0 \}$

Let $L(\mu, \sigma^2) = \prod_{i=1}^{n} f(x_i ; \mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-(x_i - \mu)^2 / 2\sigma^2 \right)$

$$= \left(2\pi\sigma^2\right)^{-n/2} \exp\left(- \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 \right)$$

and observe that :

(1) $$L(\hat{\omega}) = \max_{(\mu,\sigma^2)\in\omega} L(\mu,\sigma^2) = L(\bar{x},\sigma_0^2) = \left(2\pi\sigma_0^2\right)^{-m/2} \exp\left(-\frac{1}{2\sigma_0^2}\sum_{i=1}^{m}(x_i-\bar{x})^2\right)$$

because when $\sigma^2=\sigma_0^2$, such as in ω, the MLE for μ is $\hat{\mu}=\bar{x}$

(2) $$L(\hat{\Omega}) = \max_{\in\Omega} L(\mu,\sigma^2) = L(\bar{x},\widehat{\sigma^2}) = \left(\frac{1}{\frac{2\pi}{m}\sum_{i=1}^{m}(x_i-\bar{x})^2}\right)^{m/2} \cdot \exp\left(-\frac{\sum_{i=1}^{m}(x_i-\bar{x})^2}{\frac{2}{m}\sum_{i=1}^{m}(x_i-\bar{x})^2}\right)$$

$$= \left(\frac{m e^{-1}}{2\pi\sum_{i=1}^{m}(x_i-\bar{x})^2}\right)^{m/2}$$

because the MLE for μ and σ^2 are $\hat{\mu}=\bar{x}$ and $\widehat{\sigma^2}=\frac{1}{m}\sum_{i=1}^{m}(x_i-\bar{x})^2$

Hence, the likelihood ratio is

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \frac{\left(2\pi\sigma_0^2\right)^{-m/2}\cdot\exp\left(-\frac{1}{2\sigma_0^2}\sum_{i=1}^{m}(x_i-\bar{x})^2\right)}{\left(\frac{m e^{-1}}{2\pi\sum_{i=1}^{m}(x_i-\bar{x})^2}\right)^{m/2}} = \Bigg(\underbrace{\frac{\sum_{i=1}^{m}(x_i-\bar{x})^2}{m\sigma_0^2}}_{\frac{q}{m}}\Bigg)^{m/2}\cdot\exp\Bigg(\underbrace{-\frac{\sum_{i=1}^{m}(x_i-\bar{x})^2}{2\sigma_0^2}}_{-\frac{q}{2}}+\frac{m}{2}\Bigg)$$

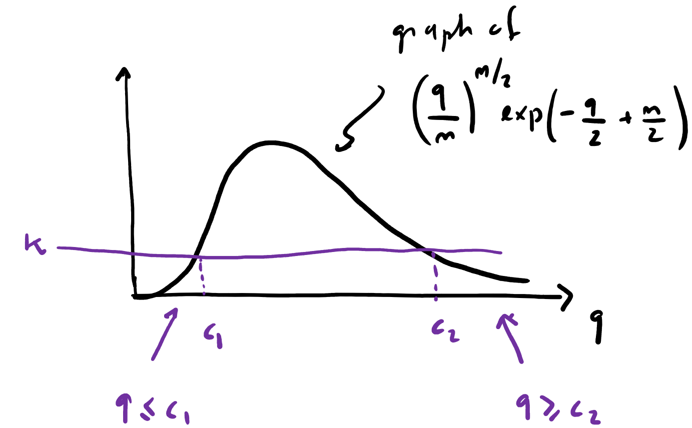
Set $q = \dfrac{\sum_{i=1}^{m}(x_i-\bar{x})^2}{\sigma_0^2}$ so that $\lambda = \left(\dfrac{q}{m}\right)^{m/2}\exp\left(-\dfrac{q}{2}+\dfrac{m}{2}\right)$

The critical region for the likelihood ratio test is given by $\lambda \leq K$, that is

$$\left(\frac{q}{m}\right)^{m/2} \exp\left(-\frac{q}{2} + \frac{m}{2}\right) \leq K$$

We cannot solve this inequality explicitly for q in terms of the remaining parameters, but it is possible to see that a solution must be of the form

$$q \leq c_1 \quad \text{or} \quad q \geq c_2 \, ,$$



where c_1 and c_2 are to be picked to achieve the desired significance level α.

However, these will not necessarily place probability $\alpha/2$ in each of the two regions $q \leq c_1$ and $q \geq c_2$

(due to a lack of symmetry in the graph to the right / pdf of the χ^2 distribution)

Since $Q = \frac{(n-1)S^2}{\sigma_0^2} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{\sigma_0^2} \sim \chi^2(n-1)$ under $H_0 : \sigma^2 = \sigma_0^2$,

it is common to modify the test by taking

$$c_1 = \chi^2_{1-\alpha/2}(n-1) \quad \text{and} \quad c_2 = \chi^2_{\alpha/2}(n-1)$$

yielding the critical region:

$$q \leq \chi^2_{1-\alpha/2}(n-1) \quad \text{or} \quad q \geq \chi^2_{\alpha/2}(n-1)$$

obtained previously (section 8.3).

Relation with sufficient statistics

Note: As was also true of best critical regions and uniformly most powerful critical regions, likelihood ratio tests are based on sufficient statistics when they exist.