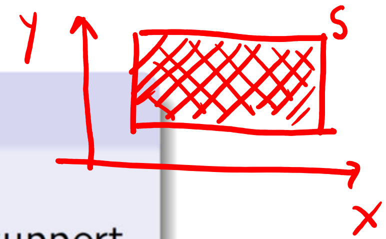
# Math 3501 - Probability and Statistics I

## 4.4 - Bivariate distributions of the continuous type

extends the concepts introduced in Secs 4.1, 4.2, and 4.3 to the case of continuous-type r.v.s.

# Joint probability density function




## Definition (Joint probability density function)

Let $X$ and $Y$ be two continuous random variables, and let $S$ denote the corresponding two-dimensional space of $X$ and $Y$ (also referred to as the support of $X$ and $Y$).

The *joint probability density function* (abbreviated joint pdf) of $X$ and $Y$ is an integrable real-valued function $f(x, y)$ with the following properties:

(a) $f(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}^2$ with $f(x, y) = 0$ when $(x, y) \notin S$.

f(x,y) is only positive on the support S of X and Y

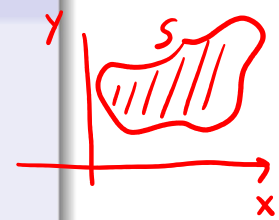
(b) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, \mathrm{d}x\mathrm{d}y = 1.$ $\Leftrightarrow$ $\iint_S f(x,y)\,dx\,dy = 1$

(c) $P[(X, Y) \in A] = \iint_A f(x, y) \, \mathrm{d}x\mathrm{d}y$ where $A \subset \mathbb{R}^2$.

# Marginal probability density function

## Definition (Marginal probability density function)

Let $X$ and $Y$ have the joint probability density function $f(x, y)$ with space (or support) $S$.




The probability density function of $X$, called the *marginal probability density function of X*, is defined by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\,\mathrm{d}y\ , \quad x \in S_X\ ,$$

where $S_X$ is the space of the random variable $X$

Similarly, the probability density function of $Y$, called the *marginal probability density function of Y*, is defined by

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\,\mathrm{d}x\ , \quad y \in S_Y\ ,$$

where $S_Y$ is the space of the random variable $X$

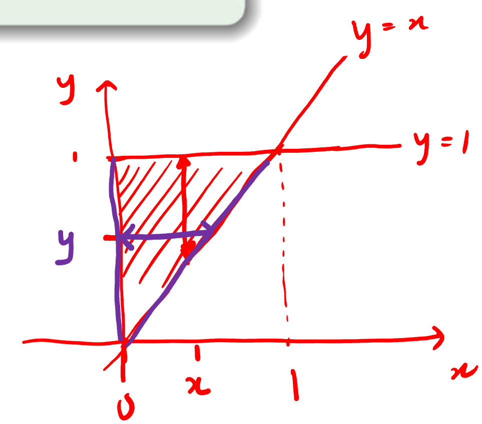
## Example (Continued)

Let $X$ and $Y$ have the joint pdf

$$f(x,y) = 2\,, \quad \underbrace{0 < x < y < 1}\,.$$

Determine the marginal pdfs of $X$ and $Y$.

$$S = \{ (x,y) \in \mathbb{R}^2 : \overbrace{x > 0,\ x < y,\ y < 1} \}$$




Marginal pdf of $X$ :

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\, dy = \int_x^1 2\, dy = \Big[ 2y \Big]_{y=x}^{y=1} = 2(1-x),\ x \in (0,1)$$

Marginal pdf of $Y$ :

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_0^y 2\, dx = \Big[ 2x \Big]_{x=0}^{x=y} = 2y, \quad y \in (0,1)$$

# Independence

## Definition (Independence)

Let $X$ and $Y$ be continuous random variables with joint probability density function $f(x, y)$ with space (or support) $S$.

We say that $X$ and $Y$ are independent if

$$f(x, y) = f_X(x) f_Y(y) \quad \text{for all } x \in S_x \text{ and } y \in S_y \ .$$

Otherwise, $X$ and $Y$ are said to be *dependent*.

## Example (Continued)

Let $X$ and $Y$ have the joint pdf

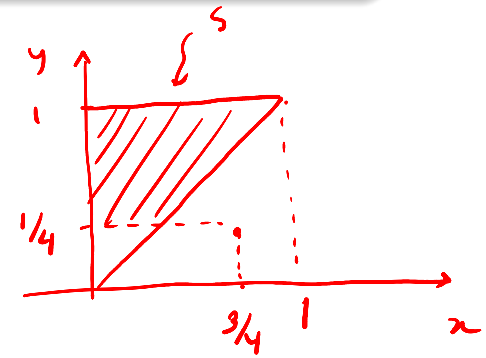
$$f(x,y) = 2\ , \quad 0 < x < y < 1\ .$$

Decide whether $X$ and $Y$ are independent.

Marginal pdfs:

$f_X(x) = 2(1-x)\ , \quad x \in (0,1)$

$f_Y(y) = 2y\ , \quad y \in (0,1)$




If $(x,y) = (3/4, 1/4)$, then $f(3/4, 1/4) = 0$ because $\left(\frac{3}{4}, \frac{1}{4}\right) \notin S$

**BUT** $f_X\left(\frac{3}{4}\right) = 2\left(1 - \frac{3}{4}\right) = \frac{1}{2}$ and $f_Y\left(\frac{1}{4}\right) = 2 \cdot \frac{1}{4} = \frac{1}{2}$ and so

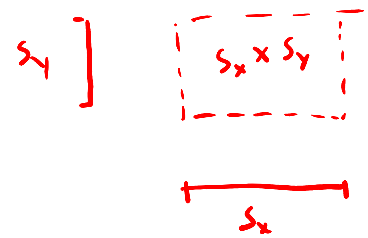
$$f\left(\frac{3}{4}, \frac{1}{4}\right) = 0 \neq f_X\left(\frac{3}{4}\right) \cdot f_Y\left(\frac{3}{4}\right) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

"Joint support" of X and Y

**Notes**:

- If the support $S$ of $X$ and $Y$ is not the product set

$$\{(x,y) : x \in S_X, y \in S_Y\} \ ,$$




  then $X$ and $Y$ must be dependent (no need to check any other condition).
- If the support $S$ of $X$ and $Y$ is equal to the product set

$$\{(x,y) : x \in S_X, y \in S_Y\} \ ,$$

  the condition

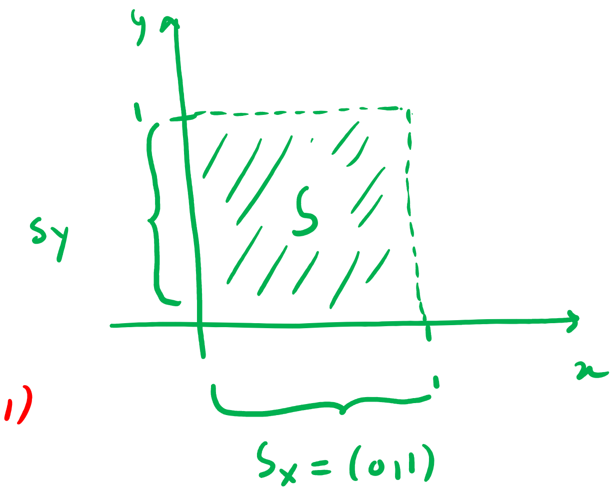
$$f(x,y) = f_X(x) f_Y(y) \quad \text{for all } x \in S_x \text{ and } y \in S_y \ .$$

  must still be checked to decide whether $X$ and $Y$ are independent.

EXAMPLE: Let $X$ and $Y$ be two random variables with joint pdf:

$$f(x,y) = 4xy, \qquad 0<x<1, \quad 0<y<1$$

Support of $X$ and $Y$ is the set $S = \{(x,y)\in \mathbb{R}^2 : 0<x<1,\ 0<y<1\} = \underbrace{(0,1)}_{S_x} \times \underbrace{(0,1)}_{S_y}$




Marginal pdf of $X$:

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 4xy\,dy = \left[2xy^2\right]_{y=0}^{y=1} = 2x, \quad x\in(0,1)$$

Marginal pdf of $Y$

$$f_y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 4xy\,dx = \left[2x^2y\right]_{x=0}^{x=1} = 2y, \quad y\in(0,1)$$

Since $f_x(x)\cdot f_y(y) = (2x)\cdot(2y) = 4xy = f(x,y)$ for all $x\in(0,1)$ and $y\in(0,1)$, then the r.v.s $X$ and $Y$ are independent.

# Mathematical expectation

## Definition (Mathematical expectation)

Let $X_1$ and $X_2$ be random variables of the continous type with the joint pdf $f(x_1, x_2)$ on the space $S$, and let $u(X_1, X_2)$ be a function of these two random variables.

The *mathematical expectation* (or *expected value*) of $u(X_1, X_2)$ is given by

$$\boxed{E[u(X_1, X_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x_1, x_2)\, f(x_1, x_2)\ \mathrm{d}x_1 \mathrm{d}x_2\ ,}$$

provided the double integral on the right is absolutely convergent.

**Note**: If $Y = u(X_1, X_2)$ is a random variable with pdf $g(y)$ on the space $S_Y$ it holds that

$$\underbrace{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x_1, x_2)\, f(x_1, x_2)\ \mathrm{d}x_1 \mathrm{d}x_2}_{E[u(x_1,x_2)]} = \underbrace{\int_{-\infty}^{\infty} y g(y)\ \mathrm{d}y}_{E[Y]}\ .$$

# Covariance and correlation coefficient

The notions of covariance and correlation coefficient, discussed earlier within the context of discrete random variables, carry over to the continuous case, with integrals replacing summations.

As in the discrete case, the *covariance of* $X$ *and* $Y$ is given by

$$\mathrm{Cov}(X,Y) = E\left[(X-\mu_X)(Y-\mu_Y)\right] = E[XY] - E[X]\,E[Y]$$

and the *correlation coefficient of* $X$ *and* $Y$ is given by

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}.$$

provided $\sigma_X, \sigma_Y > 0$

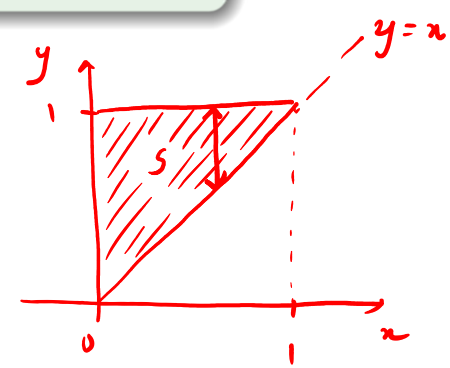
## Example (Continued)

Let $X$ and $Y$ have the joint pdf

$$f(x,y) = 2\ , \quad 0 < x < y < 1\ .$$

Compute the covariance of $X$ and $Y$.

$$\mathrm{Cov}(X,Y) = E[XY] - E[X] \cdot E[Y]$$




$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot f(x,y)\, dx\, dy = \int_0^1 \int_x^1 xy \cdot 2\, dy\, dx$$

$$= 2\int_0^1\int_x^1 xy\, dy\, dx = 2\int_0^1 \left[\frac{xy^2}{2}\right]_{y=x}^{y=1} dx = \int_0^1 x - x^3\, dx =$$

$$= \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{x=0}^{x=1} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\cdot f(x,y)\,dx\,dy = \int_0^1\int_x^1 x\cdot 2\,dy\,dx = \cdots$$

or, recalling that $f_X(x) = 2(1-x)$, $x \in (0,1)$, we can also do

$$E[X] = \int_{-\infty}^{\infty} x\cdot f_X(x)\,dx = \int_0^1 x\cdot 2(1-x)\,dx$$

$$= 2\int_0^1 x - x^2\,dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{x=0}^{x=1} = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$$

$$E[Y] = \int_{-\infty}^{\infty} y\cdot f_Y(y)\,dy = \int_0^1 y\cdot 2y\,dy = \left[\frac{2y^3}{3}\right]_{y=0}^{y=1} = \frac{2}{3}$$

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]\cdot E[Y] = \frac{1}{4} - \frac{1}{3}\cdot\frac{2}{3} = \cdots \neq 0$$

since $\mathrm{Cov}(X,Y) \neq 0$, we could conclude from this that $X$ and $Y$ are not independent.

# Conditional distributions

**Definition (Conditional probability density function)**

Let $X$ and $Y$ be continuous random variables with joint pdf $f(x,y)$ on the space $S$, and marginal pdfs $f_X(x)$ and $f_Y(y)$ with spaces $S_X$ and $S_Y$, respectively.

(a) The *conditional probability density function of $X$, given $Y = y$*, is given by

$$g(x \mid y) = \frac{f(x,y)}{f_Y(y)}, \quad \text{provided that } f_Y(y) > 0 \; . \qquad y \in S_y$$

(b) The *conditional probability density function of $Y$, given $X = x$*, is given by

$$h(y \mid x) = \frac{f(x,y)}{f_X(x)}, \quad \text{provided that } f_X(x) > 0 \; . \qquad x \in S_x$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad \begin{matrix} \leftarrow \text{"joint"} \\ \leftarrow \text{"marginal"} \end{matrix}$$

## Example (Continued)

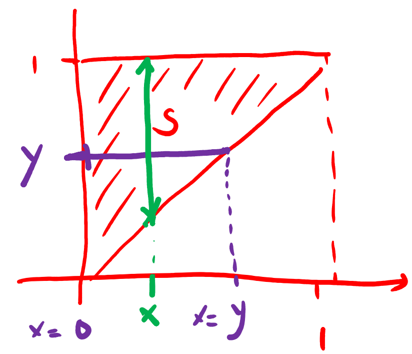
Let $X$ and $Y$ have the joint pdf

$$f(x,y) = 2\ , \quad 0 < x < y < 1\ .$$

Determine the conditional pdfs $g(x \mid y)$ and $h(y \mid x)$.

Recall that:

$$\left.\begin{aligned} f_X(x) &= 2(1-x)\ , & x &\in (0,1) \\ f_Y(y) &= 2y\ , & y &\in (0,1) \end{aligned}\right\} \text{ from a previous example}$$




Then, the conditional pdfs are given by:

$$g(x \mid y) = \frac{f(x,y)}{f_Y(y)} = \frac{2}{2y} = \frac{1}{y}\ , \quad x \in (0,y) \quad \text{for each } y \in (0,1)$$

$$h(y \mid x) = \frac{f(x,y)}{f_X(x)} = \frac{2}{2(1-x)} = \frac{1}{1-x}\ , \quad y \in (x,1) \quad \text{for each } x \in (0,1)$$

**Property**: the conditional pdfs $g(x \mid y)$ and $h(y \mid x)$ satisfy the conditions of a probability density function.

The claim above follows from observing that $0 \leq h(y \mid x)$ and

$$\int_{-\infty}^{\infty} h(y \mid x) \, \mathrm{d}y = \int_{-\infty}^{\infty} \frac{f(x,y)}{f_X(x)} \, \mathrm{d}y = \frac{1}{f_X(x)} \int_{-\infty}^{\infty} f(x,y) \, \mathrm{d}y = \frac{f_X(x)}{f_X(x)} = 1 \,,$$

with a similar reasoning applying to $g(x \mid y)$.

**Consequence**: we can compute conditional probabilities such as

$$P(a < Y < b \mid X = x) = \int_a^b h(y \mid x) \, \mathrm{d}y$$

and conditional expectations such as

$$E[u(Y) \mid X = x] = \int_{-\infty}^{\infty} u(y) h(y \mid x) \, \mathrm{d}y$$

as done with unconditional probabilities and expectations.

## Example (Continued)

Let $X$ and $Y$ have the joint pdf

$$f(x,y) = 2\ , \quad 0 < x < y < 1\ .$$

Determine the $P\left(\frac{3}{4} < Y < \frac{7}{8} \mid X = \frac{1}{4}\right)$.

Steps to complete solution:

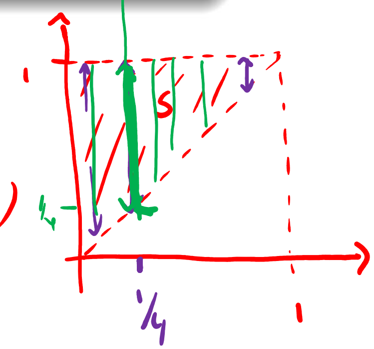
① find marginal pdf of $X$: $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

② find conditional pdf $h(y|x)$

$$h(y|x) = \frac{f(x,y)}{f_X(x)}$$

We need to use the conditional pdf $h(y|x)$.

We have seen that $h(y|x) = \dfrac{1}{1-x}$, $y \in (x,1)$, for each $x \in (0,1)$




and so (since we know that $X = 1/4$) we use

$$h(y \mid X = 1/4) = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}\ , \quad y \in \left(\tfrac{1}{4}, 1\right)$$

Thus $P\left(\frac{3}{4} < Y < \frac{7}{8} \mid X = \frac{1}{4}\right) = \int_{3/4}^{7/8} h(y \mid X = 1/4)\,dy = \int_{3/4}^{7/8} \frac{4}{3}\,dy = \left[\frac{4}{3}y\right]_{y=3/4}^{y=7/8} = \frac{4}{3}\left(\frac{7}{8} - \frac{3}{4}\right) = \frac{1}{6}$

EXAMPLE (continued) :

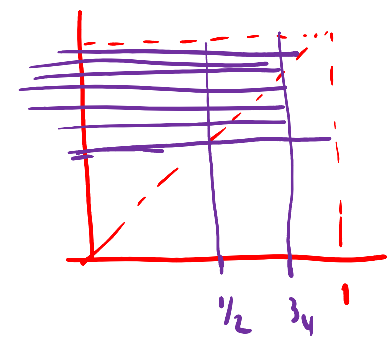
$$P\left(\tfrac{1}{2} < X < \tfrac{3}{4} \;\middle|\; \tfrac{1}{2} < Y < 1\right)$$

$$= P\left(\tfrac{1}{2} < X < \tfrac{3}{4} \;\middle|\; Y = y\right), \quad y \in \left(\tfrac{1}{2}, 1\right)$$

$$= \int_{1/2}^{3/4} g(x \mid y)\, dx, \quad y \in \left(\tfrac{1}{2}, 1\right)$$

$$= \int_{1/2}^{3/4} \frac{1}{y}\, dx, \quad y \in \left(\tfrac{1}{2}, 1\right)$$

$$= \left[\frac{x}{y}\right]_{x=1/2}^{x=3/4} = \frac{1}{4y}, \quad y \in \left(\tfrac{1}{2}, 1\right)$$



## Conditional mean and conditional variance

$E[X \mid Y=y]$

The conditional mean of $X$, given that $Y = y$, is defined as

$$\mu_{X|y} = E[X \mid y] = \int_{-\infty}^{\infty} x g(x \mid y)\,\mathrm{d}x\ ,$$

and the conditional variance of $X$, given that $Y = y$, is defined as

$$\sigma^2_{X|y} = \mathrm{Var}(X \mid y) = E\left[(X - E[X \mid y])^2 \mid y\right] = \int_{-\infty}^{\infty} (x - E[X \mid y])^2 g(x \mid y)\,\mathrm{d}x\ .$$

The latter may be computed using

$$\sigma^2_{X|y} = E\left[X^2 \mid y\right] - (E[X \mid y])^2\ .$$

both are conditional expectations
taken w.r.t $g(x|y)$

Similarly, the conditional mean of $Y$, given that $X = x$, is defined as

$E[Y \mid X = x]$

$$\mu_{Y|x} = E[Y \mid x] = \int_{-\infty}^{\infty} y h(y \mid x)\,\mathrm{d}y\ ,$$

and the conditional variance of $Y$, given that $X = x$, is defined as

$$\sigma^2_{Y|x} = \mathrm{Var}(Y \mid x) = E\left[(Y - E[Y \mid x])^2 \mid x\right] = \int_{-\infty}^{\infty} (y - E[Y \mid x])^2 h(y \mid x)\,\mathrm{d}y\ .$$

The latter may be computed using

$$\sigma^2_{Y|x} = \underbrace{E\left[Y^2 \mid x\right]} - \underbrace{(E[Y \mid x])^2}\ .$$

both conditional expectations taken w.r.t $h(y \mid x)$

## Conditional expectations

**Note**: If the conditioning is on a yet unobserved value of a random variable, the conditional expectation is a random variable itself.

In particular:

- $E[Y \mid X]$ is the random variable taking values $E[Y \mid x]$ for $x \in S_X$, with probabilities determined by the marginal pdf $f_X(x)$.
- $\text{Var}(Y \mid X)$ is the random variable taking values $Var(Y \mid x)$ for $x \in S_X$, with probabilities determined by the marginal pdf $f_X(x)$.

Similarly:

- $E[X \mid Y]$ is the random variable taking values $E[X \mid y]$ for $y \in S_Y$, with probabilities determined by the marginal pdf $f_Y(y)$.
- $\text{Var}(X \mid X)$ is the random variable taking values $Var(Y \mid x)$ for $x \in S_X$, with probabilities determined by the marginal pdf $f_Y(y)$.

# Laws of total probability

The laws of total probability for expectation and variance also hold in the continuous case:

**Theorem (Law of total probability for expectation)**

*Let $X$ and $Y$ be random variables such that $E(Y)$ exists. It holds that*

$$E[E[Y \mid X]] = E[Y] \; .$$

Same as in the discrete case (Sec 4.3)

**Theorem (Law of total probability for variance)**

*If $X$ and $Y$ are random variables, then*

$$E[\mathrm{Var}(Y \mid X)] + \mathrm{Var}(E[Y \mid X]) = \mathrm{Var}(Y)$$

*provided that all of the expectations and variances exist.*

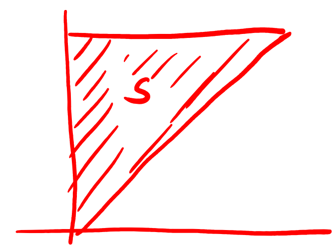
### Example (Continued)

Let $X$ and $Y$ have the joint pdf

$$f(x, y) = 2\ , \quad 0 < x < y < 1\ .$$

Determine the conditional mean $E[Y \mid x]$ and conditional variance $\mathrm{Var}(Y \mid x)$.

To determine $E[Y \mid X=x]$ and $\mathrm{Var}(Y \mid X=x)$, we need to use the conditional pdf $h(y|x)$.




We've seen that

$$\boxed{h(y|x) = \frac{1}{1-x}\ , \quad y \in (x,1) \text{ for each } x \in (0,1)}$$

Then

$$E[Y \mid X=x] = \int_{-\infty}^{\infty} y \cdot h(y|x)\, dy = \int_x^1 y \cdot \frac{1}{1-x}\, dy = \left[\frac{y^2}{2(1-x)}\right]_{y=x}^{y=1}$$

$$= \frac{1}{2(1-x)} - \frac{x^2}{2(1-x)} =$$

$$= \frac{1-x^2}{2(1-x)} = \frac{(1-x)(1+x)}{2(1-x)} = \frac{1+x}{2}$$

$E[Y \mid X = \underline{\underline{x}}] = \dfrac{1+x}{2}$ ← for a specific value of $X = x$

If we don't yet know the value that $X$ takes, then

$E[Y \mid X]$ is the r.v. $E[Y \mid X] = \dfrac{1+X}{2}$ is a random variable

with pdf given by $f_X(x) = 2(1-x), x \in (0,1)$

if we wanted to find $E[E[Y \mid X]] = E[Y]$ ← computed before!

↑ law of total probability

$$\mathrm{Var}(Y \mid X=x) = \underbrace{E[Y^2 \mid X=x]}_{?} - \Big(\underbrace{E[Y \mid X=x]}_{\frac{1+x}{2}}\Big)^2 = \frac{1+x+x^2}{3} - \left(\frac{1+x}{2}\right)^2$$

$= \cdots$ simplify!

$$E[Y^2 \mid X=x] = \int_{-\infty}^{\infty} y^2 \cdot h(y \mid x)\, dy = \int_x^1 y^2 \cdot \frac{1}{1-x}\, dy$$

$$h(y \mid x) = \frac{1}{1-x}, \quad y \in (x, 1)$$

$$= \frac{1}{1-x}\left[\frac{y^3}{3}\right]_{y=x}^{y=1} = \frac{1}{3}\,\frac{1}{1-x}\,(1-x^3) = \frac{1}{3}\,\frac{1}{\cancel{1-x}}\,\cancel{(1-x)}\,(1+x+x^2)$$

$$= \frac{1+x+x^2}{3}$$

# Math 3501 - Probability and Statistics I

## 5.1 - Functions of one random variable

# Discrete random variables

Let $X$ be a discrete random variable with pmf

$$f(x) = P(X = x)\ , \quad x \in S_X = \{c_1, c_2, c_3, \ldots\}$$

and let $u(x)$ be a real-valued function of a single real variable.

Define a new random variable

$$Y = u(X)\ .$$

and note that $Y$ is a discrete random variable with support

$$S_Y = \{d_1, d_2, d_3, \ldots\}\ , \quad \text{where } d_i = u(c_i) \text{ for each } i = 1, 2, \ldots\ .$$

The values $d_i$ are not necessarily distinct if $u$ is not one-to-one.

$$\begin{array}{lcccc} S_x & c_1 & c_2 & c_3 & c_4 \ldots \\ & \downarrow & \downarrow & \downarrow & \ldots \\ S_y & d_1 & d_2 & d_3 & \\ & \| & \| & \| & \\ & u(c_1) & u(c_2) & u(c_3) & \end{array}$$

The pmf of $Y = u(X)$ is given by

$$g(y) = P(Y = y) = P[u(X) = y] = \sum_{\{x:u(x)=y\}} f(x)\ , \qquad y \in S_Y\ .$$

**Note**: for each $y \in S_Y$, the value of $g(y)$ is found by summing probabilities over all values of $x$ for which $u(x)$ equals $y$.

$v$ is $u^{-1}$

**Special case**: If $y = u(x)$ is a one-to-one transformation with inverse $x = v(y)$, the pmf of $Y = u(X)$ is given by

$$g(y) = f(v(y))\ , \qquad y \in S_Y\ .$$

$$g(y) = P(Y = y) = P(u(X) = y) = P(X = u^{-1}(y)) = P(X = v(y)) = f(v(y))$$

## Example

Let $X$ have a discrete uniform distribution on the integers $-2, -1, 0, 1, 2, 3$, that is,

$$f(x) = 1/6\ , \quad x = -2, -1, \ldots, 3\ .$$

Find the pmf of $Y = X^2$.

$S_x = \{-2, -1, 0, 1, 2, 3\}$. Since $\underbrace{Y = X^2}$, $S_y = \{0, 1, 4, 9\}$

↑ not 1-to-1 because both -2 and 2 are mapped to 4 and -1 and 1 are mapped to 1

We need to find $g(y) = P(Y=y)$ for each $y \in S_y$

$g(0) = P(Y=0) = P(X^2=0) = P(X=0) = f(0) = \frac{1}{6}$

$g(1) = P(Y=1) = P(X^2=1) = \underbrace{P(X=-1 \text{ or } X=1) = P(X=-1) + P(X=1) = f(-1) + f(1)} = \frac{2}{6}$

$g(4) = P(Y=4) = P(X^2=4) = P(X=2 \text{ or } X=-2) = P(X=2)+P(X=-2) = f(2)+f(-2) = \frac{2}{6}$

$g(9) = P(Y=9) = P(X^2=9) = P(X=3) = f(3) = \frac{1}{6}$

because $-3$ is NOT in $S_X$

CONCLUSION: pmf of $Y$ is

$$g(y) = \begin{cases} \frac{1}{6} & \text{if } y=0 \text{ or } y=9 \\ \frac{2}{6} & \text{if } y=1 \text{ or } y=4 \end{cases}$$