Math 3501 - Probability and Statistics I

1.4 - Independent Events

Independent Events

Two events A and B are independent if the occurrence of one of them does not affect the probability of the occurrence of the other.

More formally, the events A and B are independent if

$$P(B \mid A) = P(B) \qquad \text{provided } P(A) > 0$$

or

$$P(A \mid B) = P(A) \qquad \text{provided } P(B) > 0$$

$$\frac{P(A \cap B)}{P(A)} = P(B)$$

$$\frac{P(A \cap B)}{P(B)} = P(A)$$

both give

$$P(A \cap B) = P(A) . P(B)$$

Definition

Events A and B are *independent* if and only if

$$\boxed{P(A \cap B) = P(A)P(B)}.$$

Otherwise, A and B are called *dependent* events.

Properties:

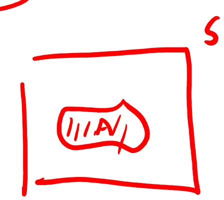
1) the definition always holds if $P(A) = 0$ or $P(B) = 0$.

events with zero probability are always independent from all other events.
In particular, ϕ is always independent from any other event

2) If A and B are independent, then so are the following pairs of events:

A and B' $\qquad$ A' and B $\qquad$ A' and B'

are independent



Mutually independent events

Definition

Events A, B, and C are *mutually independent* if and only if the following two conditions hold:

(a) A, B, and C are *pairwise independent*; that is,

$$P(A \cap B) = P(A)P(B)\ , \quad P(A \cap C) = P(A)P(C)\ , \quad P(B \cap C) = P(B)P(C)$$

(b) $P(A \cap B \cap C) = P(A)P(B)P(C)$

Note: If there is no possibility of misunderstanding, the term "independent" is often used without the modifier "mutually" when several events are considered.

Remark

The definition of mutual independence can be extended to four or more events by requiring that each pair, triple, quartet, and so on, satisfy similar properties.

Namely, the events $A_1, A_2, \ldots, A_n$ *are said to be independent if, for every subset* $\underbrace{A_{1'}, A_{2'}, \ldots, A_{r'}}$, *with* $r \leq n$, *of these events, one has*

$$\underbrace{P(A_{1'} \cap A_{2'} \cap \ldots \cap A_{r'}) = P(A_{1'})\, P(A_{2'}) \cdots P(A_{r'})}$$

Finally, we define an infinite set of events to be independent if every finite subset of those events is independent.

Sequences of independent trials

We will often consider random experiments consisting of a sequence of n trials that are mutually independent:

If the outcomes of distinct trials have no mutual influence, then events associated with a distinct trials may be assumed to be independent:

Specifically, if the event A_i is associated with the ith trial, $i = 1, 2, \ldots, n$, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) P(A_2) \cdots P(A_n)$$

Example

Suppose that on five consecutive days an "instant winner" lottery ticket is purchased and the probability of winning is 1/5 on each day.

Assuming independent trials, find the probability of purchasing:

a) - winning tickets on the first two days and loosing tickets on the other days
b) - winning tickets on the first and last day and loosing tickets on the other days
c) - exactly two winning tickets in the five days

Define events:

A_i : "purchasing a winning ticket on day i", $i = 1,2,3,4,5$

We are told: $A_1, A_2, \ldots, A_5$ are mutually independent (independent trials)

$P(A_i) = \frac{1}{5}$ for $i = 1,2,\ldots,5$ ← Given info

a) $$P(A_1 \cap A_2 \cap \overline{A_3} \cap \overline{A_4} \cap \overline{A_5}) \underset{\text{independence}}{=} P(A_1) \cdot P(A_2) \cdot P(\overline{A_3}) \cdot P(\overline{A_4}) \cdot P(\overline{A_5})$$

$$= P(A_1) \cdot P(A_2) \cdot (1 - P(A_3)) \cdot (1 - P(A_4)) \cdot (1 - P(A_5))$$

$$= \frac{1}{5} \cdot \frac{1}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} = \left(\frac{1}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^3$$

b) $P(A_1 \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4} \cap A_5) = P(A_1) \cdot P(\overline{A_2}) \cdot P(\overline{A_3}) \cdot P(\overline{A_4}) \cdot P(A_5)$

independence

$= \frac{1}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{1}{5} = \left(\frac{1}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^3$

c) Many ways we can buy exactly 2 winning tickets in 5 fives

W W L L L ← item a)

W L L L W ← item b

L W L W L ← other option

more options

each of them orders has the same probability

$\left(\frac{1}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^3$

in how many constind ways can we rearrange 2 Ws and 3 Ls?

$$\underbrace{\frac{5!}{2!\,3!} = \binom{5}{2} = \binom{5}{3}}_{\#\text{ of distinguishable permutations!}}$$

Define event B:

B: "purchasing exactly 2 winning tickets out of 5"

$$P(B) = \binom{5}{2} \left(\frac{1}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^3$$

"related with binomial distribution"

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1.5 - Bayes' Theorem

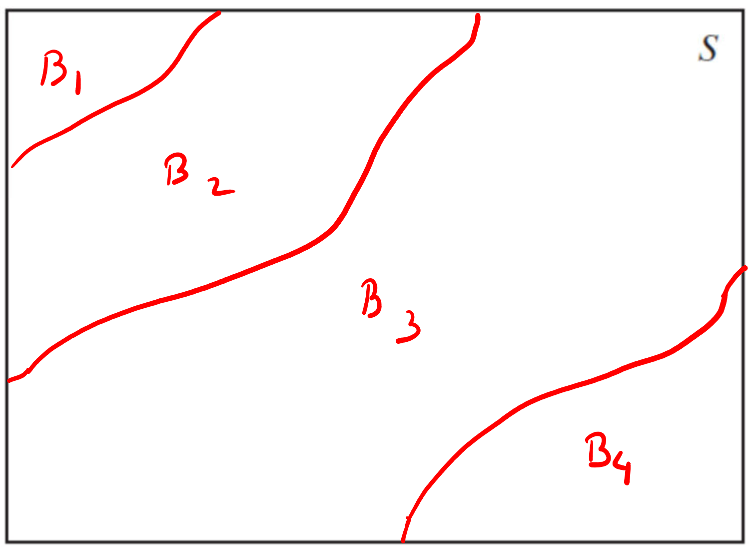
Partition

Let the events $B_1, B_2, \ldots, B_m$ form a partition of the sample space S, that is:

1) $B_1, B_2, \ldots, B_m$ are mutually exclusive: $B_i \cap B_j = \emptyset$ whenever $i \neq j$

2) $B_1, B_2, \ldots, B_m$ are exhaustive: $S = B_1 \cup B_2 \cup \cdots \cup B_m$

3) the events $B_1, B_2, \ldots, B_m$ are all nonempty



Suppose also that all elements of the partition $B_1, B_2, \dots, B_m$ are such that

$$P(B_i) > 0 \qquad i = 1, \dots, m \,.$$

$B_1, \dots, B_m$ are NOT just non empty, they have positive probabilities

Then, any event A may be written as the union of m mutually exclusive events:

$$A = (B_1 \cap A) \cup (B_2 \cap A) \cup \cdots \cup (B_m \cap A)$$

$$\underbrace{B_1 \cap A,\ B_2 \cap A,\ B_3 \cap A,\ B_4 \cap A}_{\text{mutually exclusive}}$$

$$\bigcup_{i=1}^{m} (B_i \cap A)$$

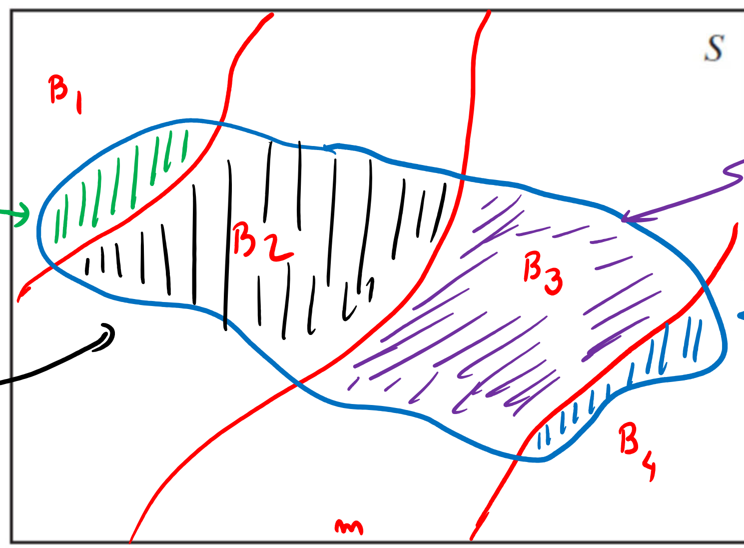
A does not need to intersect all of the B_is

it may happen that

$A \cap B_i = \phi$ for some $i = 1, \dots, m$

$B_1 \cap A$

$P(B_i \cap A) = P(B_i) \cdot P(A | B_i)$



$B_3 \cap A$

$B_4 \cap A$

$B_2 \cap A$

$$P(A) = P\left(\bigcup_{i=1}^{m} B_i \cap A\right) \overset{\text{AXIOM 3}}{=} \sum_{i=1}^{m} P(B_i \cap A) = \sum_{i=1}^{m} P(B_i)\, P(A | B_i)$$

LAW OF TOTAL PROBABILITY

Proposition (Law of total probability)

Let $B_1, B_2, \ldots, B_m$ be a partition of the sample space S with the property that $P(B_i) > 0$ for each $i = 1, \ldots, m$.

Then, for any event A, we have

multiplication rule

axiom 3

$$P(A) = \sum_{i=1}^{m} P\left(B_i \cap A\right) = \sum_{i=1}^{m} P\left(B_i\right) P\left(A \mid B_i\right) .$$

Theorem (Bayes' Theorem)

Let $B_1, B_2, \ldots, B_m$ be a partition of the sample space S with the property that $P(B_i) > 0$ for each $i = 1, \ldots, m$.

For any event A such that $P(A) > 0$, we have

$$P(B_k \mid A) = \frac{P(B_k)\, P(A \mid B_k)}{\sum_{i=1}^{m} P(B_i)\, P(A \mid B_i)}, \qquad k = 1, 2, \ldots, m\ .$$

Notes: For $k = 1, 2, \ldots, m$:

- $P(B_k)$ are often called the *prior probabilities* of the events B_k
- $P(B_k \mid A)$ are often called the *posterior probabilities* of the events B_k

Proof:

$$P(B_k \mid A) = \frac{P(B_k \cap A)}{P(A)} = \frac{P(B_k) \cdot P(A \mid B_k)}{\sum_{i=1}^{m} P(B_i)\, P(A \mid B_i)}$$

multiplication rule

law of total probability

Example

A laboratory blood test is 95 percent effective in detecting a certain disease when it is, in fact, present.

However, the test also yields "false positive" results for 1 percent of the healthy persons tested, that is, if a healthy person is tested, then, with probability 0.01, the test result is positive.

If 0.5 percent of the population actually has the disease, what is the probability that a person has the disease given that the test result is positive?

Partition: B_1 : "having disease" B_2 : "not having disease" $\rightsquigarrow B_2 = \overline{B_1}$

A : "Test result is positive"

B_1, B_2 are
* mutually exclusive
* exhaustive

Asked: $P(B_1 | A) = ??$

Given info $P(B_1) = 0.5\% = 0.005 \Rightarrow P(B_2) = P(\overline{B_1}) = 1 - P(B_1) = 0.995$

$P(A | B_1) = 0.95$ $P(A | B_2) = 0.01$

def cond prob

$$P(B_1 | A) = \frac{P(B_1 \cap A)}{P(A)} \overset{\text{mult rule}}{=} \frac{P(B_1) \cdot P(A|B_1)}{P(A)} = \frac{(0.005)(0.95)}{(0.005)(0.95) + (0.995)(0.01)} = \cdots$$

law of Total probability

$$P(A) = P\big((A \cap B_1) \cup (A \cap B_2)\big) = P(A \cap B_1) + P(A \cap B_2)$$

mutually exclusive

multiplication rule

$$= P(B_1) \cdot P(A|B_1) + P(B_2) \cdot P(A|B_2)$$

$$= (0.005)(0.95) + (0.995)(0.01)$$

Law of Total probability

Example

In a certain factory, machines I, II, and III all produce springs of the same length.

Of their production, machines I, II, and III respectively produce 2%, 1% and 3% defective springs.

Of the total production of springs in the factory, machine I produces 35%, machine II produces 25%, and machine III produces 40%.

If one spring is selected at random from the total springs produced in a day:

a) - find the probability that it is defective
b) - knowing that the selected spring is defective, find the probability that it was produced by machine III $\leftarrow B_3$

Define:

B_i : event "spring is produced by machine i" $\quad i = 1, 2, 3 \quad (I, II, III)$

A : event "spring is defective"

B_1, B_2, B_3 are mutually exclusive and exhaustive $\begin{cases} P(B_1) = 0.35 \\ P(B_2) = 0.25 \\ P(B_3) = 0.40 \end{cases}$

We also know that $\begin{cases} P(A \mid B_1) = 0.02 \\ P(A \mid B_2) = 0.01 \\ P(A \mid B_3) = 0.03 \end{cases}$

Axiom 3

a) $P(A) = P\big((A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3)\big) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3)$

mutually exclusive

(mult. rule)

$= P(B_1) . P(A|B_1) + P(B_2) . P(A|B_2) + P(B_3) . P(A|B_3)$

law total probability

$= (0.35)(0.02) + (0.25)(0.01) + (0.4)(0.03)$

$\approx \ldots$

cond. prob

b) $P(B_3 | A) = \dfrac{P(B_3 \cap A)}{P(A)} = \dfrac{P(B_3) . P(A|B_3)}{P(A)} = \dfrac{(0.4)(0.03)}{(0.35)(0.02) + (0.25)(0.01) + (0.4)(0.03)}$

Known already from item a)

$\approx \ldots$

Bayes' Theorem