Math 3501 - Probability and Statistics I

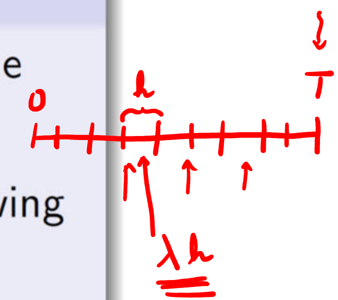
2.7 - The Poisson distribution

Poisson process

number of flights arriving at JFK over one hour
number of calls arriving at call center during 5 minutes
number of holes in a given mile of highway

Definition (informal)

Let the number of occurrences of some event in a given continuous interval be counted.



We have an (approximate) *Poisson process* with parameter $\lambda > 0$ if the following conditions are satisfied:

(a) The numbers of occurrences in nonoverlapping subintervals are independent.

(b) The probability of exactly one occurrence in a sufficiently short subinterval of length h is approximately λh.

(c) The probability of two or more occurrences in a sufficiently short subinterval is essentially zero.

Note: We refer to an approximate Poisson process above in order to replace some technical estimates by the words "approximately" in (b) and "essentially" in (c).

We will often simply say "Poisson process" and drop the word approximate.

Relation with the binomial distribution

parameter $\lambda > 0$

Suppose an experiment satisfies the three conditions for a Poisson process.

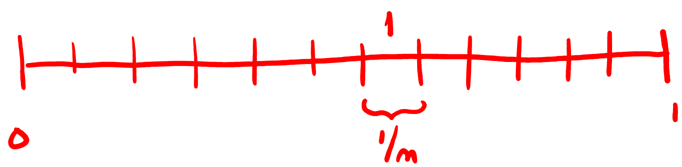
r.v

Let X denote the number of occurrences in an interval of length 1:

- "length 1" represents one unit of the quantity under consideration

  unit of relevance for the problem under consideration

Goal: find an approximation for $P(X = x)$, where x is a nonnegative integer.



divide the interval into m subintervals of equal length $1/m$

where m is much larger than x $(x << m)$

if the conditions for the Poisson process hold

then each subinterval has at most $\underbrace{\text{1 event of the Poisson process}}_{}$ with probability $\lambda \cdot \frac{1}{m} = \frac{\lambda}{m}$

"success in a Bernoulli trial")

The number of "successes" (occurrences on each subinterval of length $1/m$) is a $\boxed{\text{binomial}\left(m, \frac{\lambda}{m}\right)}$ r.v.

Strategy:

1) partition the unit interval into n subintervals of equal length $1/n$.

2) for sufficiently large n (i.e., much larger than x), approximate the probability that there are x occurrences in this unit interval by finding the probability that exactly x of these n subintervals each has one occurrence:

- the probability of one occurrence in any one subinterval of length $1/n$ is approximately $\lambda(1/n)$, by condition (b).
- the probability of two or more occurrences in any one subinterval is essentially zero, by condition (c).

3) By 1) and 2), for each subinterval, there is exactly one occurrence with a probability of approximately $\lambda(1/n)$.

4) Consider the occurrence or nonoccurrence in each subinterval as a Bernoulli trial.

- we have a sequence of n Bernoulli trials with probability of success p approximately equal to $\lambda(1/n)$, by condition (a).

$\frac{\lambda}{n}$

We obtain an approximation for $P(X=x)$ given by the binomial probability

↳ binomial $\left(n, \frac{\lambda}{n}\right)$

$$P(X=x) \approx \underbrace{\frac{n!}{x!(n-x)!}}_{\binom{n}{x}} \underbrace{\left(\frac{\lambda}{n}\right)^x}_{\cdot p^x} \underbrace{\left(1-\frac{\lambda}{n}\right)^{n-x}}_{(1-p)^{n-x}}$$

p.m.f of binomial

↑ prob. of success

Keeping λ fixed and allowing $n \to \infty$, one can check that

Fixed!!! → λ and x

$$\boxed{\lim_{n\to\infty} \frac{n!}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x} = \frac{\lambda^x e^{-\lambda}}{x!}}$$

$$\lim_{n\to\infty} \binom{n}{x}\cdot\left(\frac{\lambda}{n}\right)^x\cdot\left(1-\frac{\lambda}{n}\right)^{n-x} = \lim_{n\to\infty} \frac{n!}{(n-x)!\,x!}\,\frac{\lambda^x}{n^x}\left(1-\frac{\lambda}{n}\right)^n\cdot\left(1-\frac{\lambda}{n}\right)^{-x}$$

x terms

$$= \lim_{n\to\infty} \frac{n(n-1)(n-2)\cdots(n-x+1)}{n^x}\cdot\frac{\lambda^x}{x!}\left(1-\frac{\lambda}{n}\right)^n\cdot\left(1-\frac{\lambda}{n}\right)^{-x}$$

← x powers of n

$$= \lim_{n \to \infty} \left(\frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \ldots \cdot \frac{n-x+1}{n} \right) \cdot \frac{\lambda^x}{x!} \cdot \left(1 - \frac{\lambda}{n}\right)^n \cdot \left(1 - \frac{\lambda}{n}\right)^{-x}$$

$$= \lim_{n \to \infty} 1 \cdot \left(1 - \frac{1}{n}\right) \cdot \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{x-1}{n}\right) \cdot \frac{\lambda^x}{x!} \underbrace{\left(1 - \frac{\lambda}{n}\right)^n} \underbrace{\left(1 - \frac{\lambda}{n}\right)^{-x}}$$

$$= 1 \cdot 1 \cdot 1 \cdots 1 \cdot \frac{\lambda^x}{x!} \cdot e^{-\lambda} \cdot 1$$

$$= \boxed{\frac{\lambda^x e^{-\lambda}}{x!}}$$

we've used: $\lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^n = e^a$

Poisson distribution

A random variable X has a Poisson distribution if its pmf is of the form

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$$

where $\lambda > 0$.

Note that $f(x)$ has indeed the properties of a pmf since:

1) $f(x) \geq 0$ for every $x = 0, 1, 2, \ldots$

2) using the Maclaurin series expansion of e^{λ}, we find

replace x by λ
k by x

$$\sum_{x=0}^{\infty} f(x) = \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} e^{\lambda} = 1$$

Recall from Calculus 2:

MacLaurin series $\quad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

Moment generating function for Poisson distribution

Suppose X has a Poisson distribution (with parameter λ).

The mgf of X is given by

$$
\begin{aligned}
M(t) &= E\left[\mathrm{e}^{tX}\right] \\
&= \sum_{x=0}^{\infty} \mathrm{e}^{tx} f(x) \\
&= \sum_{x=0}^{\infty} \mathrm{e}^{tx} \frac{\lambda^x \mathrm{e}^{-\lambda}}{x!} \\
&= \mathrm{e}^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda \mathrm{e}^t)^x}{x!} \\
&= \mathrm{e}^{-\lambda} \mathrm{e}^{\lambda \mathrm{e}^t} \\
&= \mathrm{e}^{\lambda(\mathrm{e}^t - 1)}
\end{aligned}
$$

combine terms with powers of x

$e^{tx} . \lambda^x = (\lambda . e^t)^x$

Maclaurin series of $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

with x replaced λe^t and k replaced by x

for all $t \in \mathbb{R}$.

Poisson distribution mean and variance

Differentiating

$$\boxed{M(t) = \mathrm{e}^{\lambda(\mathrm{e}^t - 1)}}$$

chain rule

we obtain

$$\begin{aligned} M'(t) &= \lambda \mathrm{e}^t \mathrm{e}^{\lambda(\mathrm{e}^t - 1)} \\ M''(t) &= (\lambda \mathrm{e}^t)^2 \mathrm{e}^{\lambda(\mathrm{e}^t - 1)} + \lambda \mathrm{e}^t \mathrm{e}^{\lambda(\mathrm{e}^t - 1)} \end{aligned}$$

chain rule + product rule

Evaluating at $t = 0$, we find that

$$\mu = E(X) = M'(0) = \lambda$$

and

$$\begin{aligned} \sigma^2 = E\left(X^2\right) - [E(X)]^2 &= M''(0) - [M'(0)]^2 \\ &= \underbrace{(\lambda^2 + \lambda)} - \lambda^2 = \lambda \end{aligned}$$

Conclusion: If X has a Poisson distribution with parameters λ:

- $\mu = E[X] = \lambda$
- $\sigma^2 = \mathrm{Var}(X) = \lambda$

Interpretation: events in an approximate Poisson process occur at a mean rate of λ per unit.

Consequence: If events in an Poisson process occur at a mean rate of λ per unit, then the expected number of occurences in an interval of length t is λt and the number of occurrences in a time interval of length t has the Poisson pmf

$$\boxed{f(x) = \frac{(\lambda t)^x \mathrm{e}^{-\lambda t}}{x!}, \quad x = 0, 1, 2, \ldots\ .}$$

Poisson distribution with λ replaced by λt

to account for the change in the reference interval length!!!

Example

In the past, computer tapes have been used to backup computer files, and flaws occurred on these tapes. In a particular situation, flaws (bad records) on a used computer tape occurred on the average of one flaw per 1200 feet.

If one assumes a Poisson distribution, what is the distribution of the random variable X representing the number of flaws in a 4800-foot roll?

Find the probability that:

a) - there are no flaws in a 4800-foot roll.
b) - there are at most 2 flaws in a 4800-foot roll.

Given: one flaw per 1200 feet

{ event of Poisson process: occurrence of a flaw on the tape
{ reference interval: 1200 feet

Define X = # of flaws in a 4800 roll

The distribution of $\underbrace{X \text{ is Poisson}}$ (4) $\Rightarrow$ pmf of X is $f(x) = \dfrac{4^x e^{-4}}{x!}$, $x = 0, 1, 2, 3, \ldots$

$\lambda = 4$ because there are on average 4 flaws in a 4800-foot roll

a) $P(X=0) = f(0) = \dfrac{4^0 e^{-4}}{0!} = e^{-4} \approx 0.018$

b) $P(X \leq 2) = \displaystyle\sum_{x=0}^{2} f(x) = \underbrace{\sum_{x=0}^{2} \frac{4^x e^{-4}}{x!}}_{\text{3 terms to evaluate....}} \approx 0.238$ (from table)

Example

In a large city, telephone calls to 911 come on the average of two every 3 minutes.

If one assumes an approximate Poisson process, what is the probability of five or more calls arriving in a 9-minute period?

Define the r.v $X =$ # of phone calls arriving in a 9-minute period at the call center.

We know that X has a Poisson distribution with mean $\underbrace{\lambda = 6}$

Thus $X \sim \text{Poisson}(6)$. and the pmf of X is then

2 calls every 3 minutes
$\Downarrow$
6 calls on average every 9 minute

$$f(x) = \frac{6^x \cdot e^{-6}}{x!}, \quad x = 0, 1, 2, 3, \dots$$

$$P(X \geq 5) = 1 - P(\underbrace{X < 5}_{X \in \{0,1,2,3,4\}}) = 1 - P(X \leq 4) \approx 1 - 0.285 = 0.715$$

table

alternatively, evaluate the sum

$$\sum_{x=0}^{4} \frac{6^x e^{-6}}{x!} = \dots \leftarrow \text{five terms}$$

Poisson approximation for the binomial distribution

The Poisson distribution can also be used to approximate probabilities for a binomial distribution.

Recall: if X has a Poisson distribution with parameter λ, then for n large we have

approx. is good for n sufficiently large!

$$P(X = x) = \boxed{\frac{\lambda^x e^{-\lambda}}{x!} \approx \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}} ,$$

with probability of success $\underbrace{p = \lambda/n}_{\lambda = np}$ in the binomial probability.

Recall: if X has the binomial distribution $b(n, p)$ with large n and small p, then

$p = \frac{\lambda}{n}$

$$\binom{n}{x} p^x (1-p)^{n-x} \approx \frac{(np)^x e^{-np}}{x!}$$

approximation for binomial (n,p) by Poisson (np)

The Poisson approximation for the binomial distribution

$$\binom{n}{x} p^x (1-p)^{n-x} \approx \frac{(np)^x \mathrm{e}^{-np}}{x!}$$

is reasonably good if n is large.

- Since λ was a fixed constant in our earlier argument leading to the Poisson distribution pmf, p should be small, because $\lambda = np$.

In particular, the approximation is quite accurate if

- $n \geq 20$ and $p \leq 0.05$
- $n \geq 100$ and $p \leq 0.10$

The approximation is not bad in other situations violating these bounds somewhat (e.g., if $n = 50$ and $p = 0.12$).

Example

A manufacturer of Christmas tree light bulbs knows that 2% of its bulbs are defective.

Approximate the probability that a box of 100 of these bulbs contains at most three defective bulbs.

We're counting # of defective bulbs in a box of 100.

Define X = # of defective bulbs in such a box

$X \sim \text{binomial}(100, 0.02)$

$n = 100$ is # of bulbs in a box

$p = 0.02$ is the probability of a defective bulb

$$P(X \leq 3) = \sum_{x=0}^{3} \binom{100}{x} (0.02)^x \cdot (0.98)^{100-x} \approx \sum_{x=0}^{3} \frac{2^x e^{-2}}{x!} \approx 0.857$$

table for Poisson

approximate binomial $(100, 0.2)$ by Poisson (2)

n p

$2 = 100 \cdot (0.02)$