Linear Difference Equations and Bernoulli's Method

MATH 4701 Numerical Analysis

Linear Difference Equations

A **linear difference equation** of **order** n is an equation of the form

$$a_n u_k + a_{n-1} u_{k-1} + ... + a_1 u_{k-n+1} + a_0 u_{k-n} = 0$$

where a_0, a_1, ..., a_n are $n + 1$ constants and $a_n \neq 0$.

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A solution of the equation is uniquely determined by any choice of n (the order of the equation) **initial conditions** u_0, u_1, ..., u_{n-1}.

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With initial conditions $u_0 = 1$, $u_1 = 1$ we get the solution

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...$$

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With initial conditions $u_0 = -1$, $u_1 = 4$ we get the solution

$$-1, 4, 3, 7, 10, 17, 27, 44, 71, 115, 186, ...$$

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is a linear difference equation of **order three**.

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With initial conditions $u_0 = -1$, $u_1 = 1$, $u_2 = -1$ we get the solution

$$-1, 1, -1, -6, -22, -69, \dots$$

Linear Subspace of Solutions of a Linear Difference Eq.

The space of sequences $\mathcal{S}$ of numbers can be regarded as a **vector space**.

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All entries of a sequence can be multiplied by a number to define the scalar product of the vectors with numbers.

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All entries of a sequence can be multiplied by a number to define the scalar product of the vectors with numbers. For example if $\mathbf{u} = (\frac{1}{n}) = (1, \frac{1}{2}, \frac{1}{3}, ...)$, then $2\mathbf{u} = (\frac{2}{n}) = (2, 1, \frac{2}{3}, ...)$.

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The vector addition is defined by adding the two sequences term by term. For example if $\mathbf{u}_1 = (2^n) = (2, 4, 8, 16, ...)$ and $\mathbf{u}_2 = ((-1)^n) = (-1, 1, -1, 1, ...)$, then $\mathbf{u}_1 + \mathbf{u}_2 = (2^n + (-1)^n) = (1, 5, 7, 17, ...)$.

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  where $\mathbf{O}$ is the **constant zero sequence**.

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  where $\mathbf{O}$ is the **constant zero sequence**.
- For every vector $\mathbf{u}$, $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{O}$
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- For every vector $\mathbf{u}$, $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{O}$ where $-\mathbf{u}$ is the sequence with each entry equal to the opposite of the corresponding entry of $\mathbf{u}$. For example if $\mathbf{u}=(\frac{1}{n})=(1,\frac{1}{2},\frac{1}{3},\ldots)$, then $-\mathbf{u}=(-\frac{1}{n})=(-1,-\frac{1}{2},-\frac{1}{3},\ldots)$.

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- For every scalar α and vectors $\mathbf{u}$ and $\mathbf{v}$, $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$

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- For every scalar α and vectors $\mathbf{u}$ and $\mathbf{v}$, $\alpha(\mathbf{u}+\mathbf{v})=\alpha\mathbf{u}+\alpha\mathbf{v}$
- For every scalars α and β and every vector $\mathbf{u}$, $(\alpha+\beta)\mathbf{u}=\alpha\mathbf{u}+\beta\mathbf{u}$

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- For every scalars α and β and every vector $\mathbf{u}$, $\quad (\alpha+\beta)\mathbf{u}=\alpha\mathbf{u}+\beta\mathbf{u}$
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- For every scalar α and vectors **u** and **v**, $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$
- For every scalars α and β and every vector **u**, $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$
- For every scalars α and β and every vector **u**, $(\alpha\beta)\mathbf{u} = \alpha(\beta\mathbf{u})$
- For every vector **u**, $1\mathbf{u} = \mathbf{u}$

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A subset of $\mathcal{S}$ is called a **linear subspace** of $\mathcal{S}$ if with the scalar product and vector addition of $\mathcal{S}$ it also forms a vector space.

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To check if a subset $\mathcal{A}$ of $\mathcal{S}$ is a linear subspace it is sufficient to verify that it is **non-empty** and for every scalar α and vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{A}$, $\alpha\mathbf{u}$ and $\mathbf{u} + \mathbf{v}$ are also in $\mathcal{A}$.

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For any linear difference equation

$a_n u_k + a_{n-1} u_{k-1} + \dots + a_1 u_{k-n+1} + a_0 u_{k-n} = 0$

if (x_i) and (y_i) are two solutions of the equation and α is any number then
$a_n x_k + a_{n-1} x_{k-1} + \dots + a_1 x_{k-n+1} + a_0 x_{k-n} = 0$ and
$a_n y_k + a_{n-1} y_{k-1} + \dots + a_1 y_{k-n+1} + a_0 y_{k-n} = 0$

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$a_n y_k + a_{n-1} y_{k-1} + \ldots + a_1 y_{k-n+1} + a_0 y_{k-n} = 0$

Multiplying the first equation by α and regrouping we get

$$a_n(\alpha x_k) + a_{n-1}(\alpha x_{k-1})\ldots + a_1(\alpha x_{k-n+1}) + a_0(\alpha x_{k-n}) = 0$$

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Adding the two equations and regrouping we get

$$a_n(x_k + y_k) + a_{n-1}(x_{k-1} + y_{k-1}) ... + a_0(x_{k-n} + y_{k-n}) = 0$$

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Adding the two equations and regrouping we get

$$a_n(x_k + y_k) + a_{n-1}(x_{k-1} + y_{k-1}) ... + a_0(x_{k-n} + y_{k-n}) = 0$$

This means for solutions (x_i) and (y_i) of the equation both $\alpha(x_i)$ and $(x_i) + (y_i)$ are also solutions of the difference equation.

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if (x_i) and (y_i) are two solutions of the equation and α is any number then This means for solutions (x_i) and (y_i) of the equation both $\alpha(x_i)$ and $(x_i) + (y_i)$ are also solutions of the difference equation.

Moreover, since with any choice of n initial conditions a solution can be generated, the set of solutions to the difference equation is not empty.

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if (x_i) and (y_i) are two solutions of the equation and α is any number then This means for solutions (x_i) and (y_i) of the equation both $\alpha(x_i)$ and $(x_i) + (y_i)$ are also solutions of the difference equation.

Moreover, since with any choice of n initial conditions a solution can be generated, the set of solutions to the difference equation is not empty.

Hence, **the set of solutions of a linear difference equation forms a linear subspace of $\mathcal{S}$.**

A Finite Basis For the Subspace of Solutions

For every linear difference equation of order n

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a polynomial of degree n

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

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If characteristic polynomial $P(x)$ has n distinct roots λ_1, λ_2, ..., λ_n, then the n sequences $\mathbf{u}_1 = (\lambda_1^i)_{i\in\mathbb{N}}$, $\mathbf{u}_2 = (\lambda_2^i)_{i\in\mathbb{N}}$, ..., $\mathbf{u}_n = (\lambda_n^i)_{i\in\mathbb{N}}$ form a basis for the linear subspace of all solutions of the difference equation.

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For example, the solution with initial conditions $u_0 = 1$ and $u_1 = 2$ is the sequence

$$1, 2, 4, 8, 16, 32, ...$$

which is $\mathbf{u}_1$ itself and can be obtained by letting $\alpha = 1$ and $\beta = 0$ in $u_i = \alpha 2^i + \beta(-3)^i$.

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$\lambda_2 - \lambda_1 = \frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2} = -\sqrt{5}$

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$1 - \lambda_1 = 1 - \frac{1+\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2} = \lambda_2$

A Finite Basis For the Subspace of Solutions

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Therefore, from the first equation, we have $\alpha = 1 - \beta = \frac{\lambda_1}{\sqrt{5}}$.

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$\Rightarrow \beta = -\frac{\lambda_2}{\sqrt{5}}$
$\alpha = 1 - \beta = \frac{\lambda_1}{\sqrt{5}}$.
This means the particular solution with initial conditions $u_0 = 1$ and $u_1 = 1$can be expressed by the explicit formula

$$u_i = \alpha\lambda_1^i + \beta\lambda_2^i = \frac{\lambda_1}{\sqrt{5}}\lambda_1^i - \frac{\lambda_2}{\sqrt{5}}\lambda_2^i$$

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$$u_i = \frac{\lambda_1^{i+1} - \lambda_2^{i+1}}{\sqrt{5}}$$

For example $u_7 = \frac{\lambda_1^8 - \lambda_2^8}{\sqrt{5}} = 21$ which is the 7th entry in the Fibonacci sequence.

A Finite Basis For the Subspace of Solutions

Example (3): Find an explicit description for terms of the solution of the linear difference equation

$$u_k - 7u_{k-2} - 6u_{k-3} = 0$$

with initial conditions $u_0 = 2$, $u_1 = -7$, $u_2 = -3$.

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Multiple Roots of Characteristic Polynomial

When characteristic polynomial $P(x)$ of a linear difference equation has a root δ of multiplicity m then this root generates m basis solutions in the form $\mathbf{x}_k = (i^k \delta^i)$ for $k = 0, 1, ..., m-1$ (assuming $0^0 = 1$).

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$$u_k + u_{k-1} - 5u_{k-2} - u_{k-3} + 8u_{k-4} - 4u_{k-5} = 0$$

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Bernoulli's Method

Suppose $P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ is a polynomial with n distinct real roots α_1, α_2, ..., α_n such that

$$|\alpha_1| > |\alpha_2| \geq |\alpha_3| \geq .... \geq |\alpha_n|$$

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The linear difference equation

$$a_n u_k + a_{n-1} u_{k-1} + \dots + a_1 u_{k-n+1} + a_0 u_{k-n} = 0$$

has characteristic polynomial $P(x)$.
Then the limit of the ratio of consecutive terms of the sequence (u_i) is

$$\lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \frac{c_1\alpha_1^{k+1} + c_2\alpha_2^{k+1} + \dots + c_n\alpha_n^{k+1}}{c_1\alpha_1^{k} + c_2\alpha_2^{k} + \dots + c_n\alpha_n^{k}}$$

$$= \lim_{k\to\infty} \frac{c_1\alpha_1^{k+1}(1 + \frac{c_2}{c_1}(\frac{\alpha_2}{\alpha_1})^{k+1} + \dots + \frac{c_n}{c_1}(\frac{\alpha_n}{\alpha_1})^{k+1})}{c_1\alpha_1^{k}(1 + \frac{c_2}{c_1}(\frac{\alpha_2}{\alpha_1})^{k} + \dots + \frac{c_n}{c_1}(\frac{\alpha_n}{\alpha_1})^{k})}$$

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Assuming $\mathbf{c_1 \neq 0}$.

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For all $j = 2, ..., n$, since $|\alpha_1| > |\alpha_j|$, we have $\lim_{k\to\infty}(\frac{\alpha_j}{\alpha_1})^k = 0$.

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Therefore, if a randomly generated solution of the difference equation associated with the polynomial $P(x)$ has non-zero c_1 coefficient, for large values of k, $\frac{u_{k+1}}{u_k}$ is an approximation of the root of $P(x)$ with largest absolute value.

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At every step we also check the ratio of the last two numbers

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(Multiplying both sides by x^3)

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In this example, the actual root of smallest absolute value is $\alpha_3 \approx -0.2469796037$.

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It can be shown that even if $P(x)$ has non-real complex solutions or multiple solutions the Bernoulli method converges to the root of largest absolute value as long as there is only one root of largest absolute value which is simple (multiplicity one) and real.

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When there are non-real roots of largest absolute value, the algorithm requires significant modifications and even with those modifications it will be preferred to use alternative methods.