Systems of Non-linear Equations and Higher Dimensional Newton-Raphson Method

MATH 4701 Numerical Analysis

Example

Consider the system of two non-linear equations in two variables

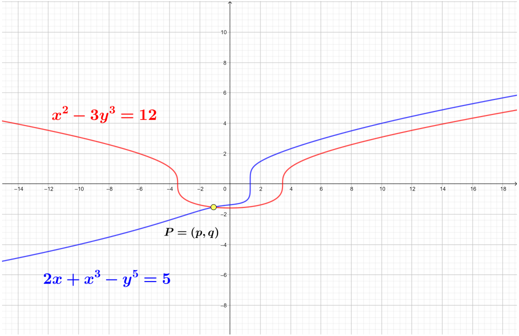
$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

Example

Consider the system of two non-linear equations in two variables

$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

A solution of this system can be regarded as a point of intersection of the curves $x^2 - 3y^3 = 12$ and $2x + x^3 - y^5 = 5$.



Example

Consider the system of two non-linear equations in two variables

$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

A solution of this system can be regarded as a point of intersection of the curves $x^2 - 3y^3 = 12$ and $2x + x^3 - y^5 = 5$.

Our approach to approximating a solution of this system of equations is based on the two variable version of the Newton-Raphson method.

Example

Consider the system of two non-linear equations in two variables

$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

A solution of this system can be regarded as a point of intersection of the curves $x^2 - 3y^3 = 12$ and $2x + x^3 - y^5 = 5$.

Our approach to approximating a solution of this system of equations is based on the two variable version of the Newton-Raphson method.

We define the two variable function $F : \mathbb{R}^2 \to \mathbb{R}^2$,
$F(x, y) = (x^2 - 3y^3 - 12, 2x + x^3 - y^5 - 5)$.

Example

Consider the system of two non-linear equations in two variables

$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

Our approach to approximating a solution of this system of equations is based on the two variable version of the Newton-Raphson method.

We define the two variable function $F : \mathbb{R}^2 \to \mathbb{R}^2$,
$F(x, y) = (x^2 - 3y^3 - 12, 2x + x^3 - y^5 - 5)$.

We notice that (p, q) satisfies the above system if and only if $F(p, q) = (0, 0)$.

Example

Consider the system of two non-linear equations in two variables

$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

We define the two variable function $F : \mathbb{R}^2 \to \mathbb{R}^2$,
$F(x, y) = (x^2 - 3y^3 - 12, 2x + x^3 - y^5 - 5)$.

We notice that (p, q) satisfies the above system if and only if $F(p, q) = (0, 0)$.

Suppose (p_n, q_n) is an approximation to (p, q). We seek to find a better approximation (p_{n+1}, q_{n+1}) by assuming that the linear approximation to $F(x, y)$ at (p_n, q_n) stays close to $F(x, y)$ where it takes the value $(0, 0)$ is nearly equal to the root (p, q) of $F(x, y)$ itself.

Example

Consider the system of two non-linear equations in two variables

$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

We define the two variable function $F : \mathbb{R}^2 \to \mathbb{R}^2$,
$F(x, y) = (x^2 - 3y^3 - 12, 2x + x^3 - y^5 - 5)$.

We notice that (p, q) satisfies the above system if and only if $F(p, q) = (0, 0)$.

Suppose (p_n, q_n) is an approximation to (p, q). We seek to find a better approximation (p_{n+1}, q_{n+1}) by assuming that the linear approximation to $F(x, y)$ at (p_n, q_n) stays close to $F(x, y)$ where it takes the value $(0, 0)$ is nearly equal to the root (p, q) of $F(x, y)$ itself.

If we let $P(x, y) = x^2 - 3y^3 - 12$ and $Q(x, y) = 2x + x^3 - y^5 - 5$ then

Example

Consider the system of two non-linear equations in two variables

$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

We define the two variable function $F : \mathbb{R}^2 \to \mathbb{R}^2$,
$F(x, y) = (x^2 - 3y^3 - 12, 2x + x^3 - y^5 - 5)$.

We notice that (p, q) satisfies the above system if and only if $F(p, q) = (0, 0)$.

If we let $P(x, y) = x^2 - 3y^3 - 12$ and $Q(x, y) = 2x + x^3 - y^5 - 5$ then the tangent plane approximation to $P(x, y)$ at (p_n, q_n) gives us

$$P(x, y) \approx P(p_n, q_n) + P_x(p_n, q_n)(x - p_n) + P_y(p_n, q_n)(y - q_n)$$

where $P_x(a, b) = \frac{\partial P}{\partial x}(a, b)$ and $P_y(a, b) = \frac{\partial P}{\partial y}(a, b)$ are the partial derivatives of P with respect to x and y.

Example

Consider the system of two non-linear equations in two variables

$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

We define the two variable function $F : \mathbb{R}^2 \to \mathbb{R}^2$,
$F(x, y) = (x^2 - 3y^3 - 12, 2x + x^3 - y^5 - 5)$.

If we let $P(x, y) = x^2 - 3y^3 - 12$ and $Q(x, y) = 2x + x^3 - y^5 - 5$ then

$P(x, y) \approx P(p_n, q_n) + P_x(p_n, q_n)(x - p_n) + P_y(p_n, q_n)(y - q_n)$

the tangent plane approximation to $Q(x, y)$ at (p_n, q_n) gives us

$Q(x, y) \approx Q(p_n, q_n) + Q_x(p_n, q_n)(x - p_n) + Q_y(p_n, q_n)(y - q_n)$

Example

Consider the system of two non-linear equations in two variables

$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

We define the two variable function $F : \mathbb{R}^2 \to \mathbb{R}^2$,
$F(x, y) = (x^2 - 3y^3 - 12, 2x + x^3 - y^5 - 5)$.

If we let $P(x, y) = x^2 - 3y^3 - 12$ and $Q(x, y) = 2x + x^3 - y^5 - 5$ then

$$P(x, y) \approx P(p_n, q_n) + P_x(p_n, q_n)(x - p_n) + P_y(p_n, q_n)(y - q_n)$$

$$Q(x, y) \approx Q(p_n, q_n) + Q_x(p_n, q_n)(x - p_n) + Q_y(p_n, q_n)(y - q_n)$$

These two equations can be written in the matrix form

$$\begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix} \approx \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix} + \begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix} \begin{bmatrix} x - p_n \\ y - q_n \end{bmatrix}$$

Example

Consider the system of two non-linear equations in two variables

$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

We define the two variable function $F : \mathbb{R}^2 \to \mathbb{R}^2$,
$F(x, y) = (x^2 - 3y^3 - 12, 2x + x^3 - y^5 - 5)$.

If we let $P(x, y) = x^2 - 3y^3 - 12$ and $Q(x, y) = 2x + x^3 - y^5 - 5$ then These two equations can be written in the matrix form

$$\begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix} \approx \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix} + \begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix} \begin{bmatrix} x - p_n \\ y - q_n \end{bmatrix}$$

Since we would like to find the root according to this linear approximation and assign their values to $x = p_{n+1}$ and $y = q_{n+1}$ we have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix} + \begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix} \begin{bmatrix} p_{n+1} - p_n \\ q_{n+1} - q_n \end{bmatrix}$$

Example

Consider the system of two non-linear equations in two variables

$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

We define the two variable function $F : \mathbb{R}^2 \to \mathbb{R}^2$,
$F(x, y) = (x^2 - 3y^3 - 12, 2x + x^3 - y^5 - 5)$.

If we let $P(x, y) = x^2 - 3y^3 - 12$ and $Q(x, y) = 2x + x^3 - y^5 - 5$ then Since we would like to find the root according to this linear approximation and assign their values to $x = p_{n+1}$ and $y = q_{n+1}$ we have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix} + \begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix} \begin{bmatrix} p_{n+1} - p_n \\ q_{n+1} - q_n \end{bmatrix}$$

$$\Rightarrow - \begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix} \begin{bmatrix} p_{n+1} - p_n \\ q_{n+1} - q_n \end{bmatrix} = \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix}$$

Example

Consider the system of two non-linear equations in two variables

$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

We define the two variable function $F : \mathbb{R}^2 \to \mathbb{R}^2$,
$F(x, y) = (x^2 - 3y^3 - 12, 2x + x^3 - y^5 - 5)$.

If we let $P(x, y) = x^2 - 3y^3 - 12$ and $Q(x, y) = 2x + x^3 - y^5 - 5$ then

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix} + \begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix} \begin{bmatrix} p_{n+1} - p_n \\ q_{n+1} - q_n \end{bmatrix}$$

$$\Rightarrow - \begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix} \begin{bmatrix} p_{n+1} - p_n \\ q_{n+1} - q_n \end{bmatrix} = \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix}$$

$$\Rightarrow - \begin{bmatrix} p_{n+1} - p_n \\ q_{n+1} - q_n \end{bmatrix} = \begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix}^{-1} \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix}$$

Example

Consider the system of two non-linear equations in two variables

$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

We define the two variable function $F : \mathbb{R}^2 \to \mathbb{R}^2$,
$F(x, y) = (x^2 - 3y^3 - 12, 2x + x^3 - y^5 - 5)$.

If we let $P(x, y) = x^2 - 3y^3 - 12$ and $Q(x, y) = 2x + x^3 - y^5 - 5$ then

$$\Rightarrow -\begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix} \begin{bmatrix} p_{n+1} - p_n \\ q_{n+1} - q_n \end{bmatrix} = \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix}$$

$$\Rightarrow -\begin{bmatrix} p_{n+1} - p_n \\ q_{n+1} - q_n \end{bmatrix} = \begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix}^{-1} \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} - \begin{bmatrix} p_n \\ q_n \end{bmatrix} = -\begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix}^{-1} \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix}$$

Example

Consider the system of two non-linear equations in two variables

$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

We define the two variable function $F : \mathbb{R}^2 \to \mathbb{R}^2$,
$F(x, y) = (x^2 - 3y^3 - 12, 2x + x^3 - y^5 - 5)$.

If we let $P(x, y) = x^2 - 3y^3 - 12$ and $Q(x, y) = 2x + x^3 - y^5 - 5$ then

$$\Rightarrow -\begin{bmatrix} p_{n+1} - p_n \\ q_{n+1} - q_n \end{bmatrix} = \begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix}^{-1} \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} - \begin{bmatrix} p_n \\ q_n \end{bmatrix} = -\begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix}^{-1} \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix}^{-1} \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix}$$

Example

Consider the system of two non-linear equations in two variables

$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

We define the two variable function $F : \mathbb{R}^2 \to \mathbb{R}^2$,
$F(x, y) = (x^2 - 3y^3 - 12, 2x + x^3 - y^5 - 5)$.

If we let $P(x, y) = x^2 - 3y^3 - 12$ and $Q(x, y) = 2x + x^3 - y^5 - 5$ then

$$\Rightarrow \begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix}^{-1} \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix}$$

Denoting $DF(a, b) = \begin{bmatrix} P_x(a, b) & P_y(a, b) \\ Q_x(a, b) & Q_y(a, b) \end{bmatrix}$ the above formula can be written as $(p_{n+1}, q_{n+1}) = (p_n, q_n) - [DF(p_n, q_n)]^{-1} F(p_n, q_n)$.

Example

$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

We define the two variable function $F : \mathbb{R}^2 \to \mathbb{R}^2$,
$F(x, y) = (x^2 - 3y^3 - 12, 2x + x^3 - y^5 - 5)$.

If we let $P(x, y) = x^2 - 3y^3 - 12$ and $Q(x, y) = 2x + x^3 - y^5 - 5$ then
$$\Rightarrow \begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix}^{-1} \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix}$$

Denoting $DF(a, b) = \begin{bmatrix} P_x(a, b) & P_y(a, b) \\ Q_x(a, b) & Q_y(a, b) \end{bmatrix}$ the above formula can be written as $(p_{n+1}, q_{n+1}) = (p_n, q_n) - [DF(p_n, q_n)]^{-1} F(p_n, q_n)$. It is important to remember that (p_{n+1}, q_{n+1}), (p_n, q_n), and $F(p_n, q_n)$ all should be written as column vectors for this formula to be compatible with the matrix notation.

Example

$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

We define the two variable function $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(x, y) = (x^2 - 3y^3 - 12, 2x + x^3 - y^5 - 5)$.

If we let $P(x, y) = x^2 - 3y^3 - 12$ and $Q(x, y) = 2x + x^3 - y^5 - 5$ then

$$\Rightarrow \begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix}^{-1} \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix}$$

Denoting $DF(a, b) = \begin{bmatrix} P_x(a, b) & P_y(a, b) \\ Q_x(a, b) & Q_y(a, b) \end{bmatrix}$ the above formula can be written as $(p_{n+1}, q_{n+1}) = (p_n, q_n) - [DF(p_n, q_n)]^{-1}F(p_n, q_n)$. If the derivative matrix $DF(x, y)$ is defined and is continuous within an open disk containing (p, q) and $DF(p, q)^{-1}$ exists then $det(DF(x, y)) \neq 0$ for all (x, y) and $DF(x, y)^{-1}$ exists for all points (x, y) sufficiently close to (p, q).

Example

$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

We define the two variable function $F : \mathbb{R}^2 \to \mathbb{R}^2$,
$F(x,y) = (x^2 - 3y^3 - 12, 2x + x^3 - y^5 - 5)$.

If we let $P(x,y) = x^2 - 3y^3 - 12$ and $Q(x,y) = 2x + x^3 - y^5 - 5$ then
$$\Rightarrow \begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix}^{-1} \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix}$$

Returning to the example above we have
$$DF(a,b) = \begin{bmatrix} P_x(a,b) & P_y(a,b) \\ Q_x(a,b) & Q_y(a,b) \end{bmatrix} = \begin{bmatrix} 2a & -9b^2 \\ 2 + 3a^2 & -5b^4 \end{bmatrix}$$

Example

$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

We define the two variable function $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(x, y) = (x^2 - 3y^3 - 12, 2x + x^3 - y^5 - 5)$.

If we let $P(x, y) = x^2 - 3y^3 - 12$ and $Q(x, y) = 2x + x^3 - y^5 - 5$ then

$$\Rightarrow \begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix}^{-1} \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix}$$

Returning to the example above we have

$$DF(a, b) = \begin{bmatrix} P_x(a, b) & P_y(a, b) \\ Q_x(a, b) & Q_y(a, b) \end{bmatrix} = \begin{bmatrix} 2a & -9b^2 \\ 2 + 3a^2 & -5b^4 \end{bmatrix}$$

Starting with $(p_0, q_0) = (-1, -2)$ we have

$$\begin{bmatrix} p_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} - \begin{bmatrix} 2p_0 & -9q_0^2 \\ 2 + 3p_0^2 & -5q_0^4 \end{bmatrix}^{-1} \begin{bmatrix} p_0^2 - 3q_0^3 - 12 \\ 2p_0 + p_0^3 - q_0^5 - 5 \end{bmatrix} =$$

Example

$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

We define the two variable function $F : \mathbb{R}^2 \to \mathbb{R}^2$,
$F(x, y) = (x^2 - 3y^3 - 12, 2x + x^3 - y^5 - 5)$.

If we let $P(x, y) = x^2 - 3y^3 - 12$ and $Q(x, y) = 2x + x^3 - y^5 - 5$ then

$$\Rightarrow \begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix}^{-1} \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix}$$

Starting with $(p_0, q_0) = (-1, -2)$ we have

$$\begin{bmatrix} p_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} - \begin{bmatrix} 2p_0 & -9q_0^2 \\ 2 + 3p_0^2 & -5q_0^4 \end{bmatrix}^{-1} \begin{bmatrix} p_0^2 - 3q_0^3 - 12 \\ 2p_0 + p_0^3 - q_0^5 - 5 \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{41}{85} \\ -\frac{567}{340} \end{bmatrix} \approx \begin{bmatrix} -0.4823529412 \\ -1.667647059 \end{bmatrix}$$

Example

$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

Starting with $(p_0, q_0) = (-1, -2)$ we have

$$\begin{bmatrix} p_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} - \begin{bmatrix} 2p_0 & -9q_0^2 \\ 2 + 3p_0^2 & -5q_0^4 \end{bmatrix}^{-1} \begin{bmatrix} p_0^2 - 3q_0^3 - 12 \\ 2p_0 + p_0^3 - q_0^5 - 5 \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{41}{85} \\ -\frac{567}{340} \end{bmatrix} \approx \begin{bmatrix} -0.4823529412 \\ -1.667647059 \end{bmatrix}$$

$$\begin{bmatrix} p_2 \\ q_2 \end{bmatrix} = \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} - \begin{bmatrix} 2p_1 & -9q_1^2 \\ 2 + 3p_1^2 & -5q_1^4 \end{bmatrix}^{-1} \begin{bmatrix} p_1^2 - 3q_1^3 - 12 \\ 2p_1 + p_1^3 - q_1^5 - 5 \end{bmatrix} \approx$$

Example

$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

Starting with $(p_0, q_0) = (-1, -2)$ we have

$$\begin{bmatrix} p_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} - \begin{bmatrix} 2p_0 & -9q_0^2 \\ 2 + 3p_0^2 & -5q_0^4 \end{bmatrix}^{-1} \begin{bmatrix} p_0^2 - 3q_0^3 - 12 \\ 2p_0 + p_0^3 - q_0^5 - 5 \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{41}{85} \\ -\frac{567}{340} \end{bmatrix} \approx \begin{bmatrix} -0.4823529412 \\ -1.667647059 \end{bmatrix}$$

$$\begin{bmatrix} p_2 \\ q_2 \end{bmatrix} = \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} - \begin{bmatrix} 2p_1 & -9q_1^2 \\ 2 + 3p_1^2 & -5q_1^4 \end{bmatrix}^{-1} \begin{bmatrix} p_1^2 - 3q_1^3 - 12 \\ 2p_1 + p_1^3 - q_1^5 - 5 \end{bmatrix} \approx$$

$$\approx \begin{bmatrix} -1.319235463 \\ -1.549649002 \end{bmatrix}$$

Example

$$\begin{cases} x^2 - 3y^3 = 12 \\ 2x + x^3 - y^5 = 5 \end{cases}$$

Starting with $(p_0, q_0) = (-1, -2)$ we have
$(p_1, q_1) \approx (-0.4823529412, -1.667647059)$,
$(p_2, q_2) \approx (-1.319235463, -1.549649002)$,
$(p_3, q_3) \approx (-1.113991441, -1.532858460)$,
$(p_4, q_4) \approx (-1.08991647255619, -1.53321765156882)$,
$(p_5, q_5) \approx (-1.08959809900892, -1.53322297388167)$,
$(p_6, q_6) \approx (-1.08959804385921, -1.53322297497739)$.

Two Variable Version of Newton-Raphson Method

Given the system of two non-linear equations in two variables

$$\begin{cases} P(x,y) = 0 \\ Q(x,y) = 0 \end{cases}$$

Two Variable Version of Newton-Raphson Method

Given the system of two non-linear equations in two variables

$$\begin{cases} P(x,y) = 0 \\ Q(x,y) = 0 \end{cases}$$

We define the two variable function $F : \mathbb{R}^2 \to \mathbb{R}^2$,
$F(x,y) = (P(x,y), Q(x,y))$.

Two Variable Version of Newton-Raphson Method

Given the system of two non-linear equations in two variables

$$\begin{cases} P(x,y) = 0 \\ Q(x,y) = 0 \end{cases}$$

We define the two variable function $F : \mathbb{R}^2 \to \mathbb{R}^2$,
$F(x,y) = (P(x,y), Q(x,y))$.

Then starting with an initial guess (p_0, q_0) sufficiently close to a solution (p,q) of the above system we can form the sequence

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \begin{bmatrix} P_x(p_n,q_n) & P_y(p_n,q_n) \\ Q_x(p_n,q_n) & Q_y(p_n,q_n) \end{bmatrix}^{-1} \begin{bmatrix} P(p_n,q_n) \\ Q(p_n,q_n) \end{bmatrix}$$

Two Variable Version of Newton-Raphson Method

Given the system of two non-linear equations in two variables

$$\begin{cases} P(x,y) = 0 \\ Q(x,y) = 0 \end{cases}$$

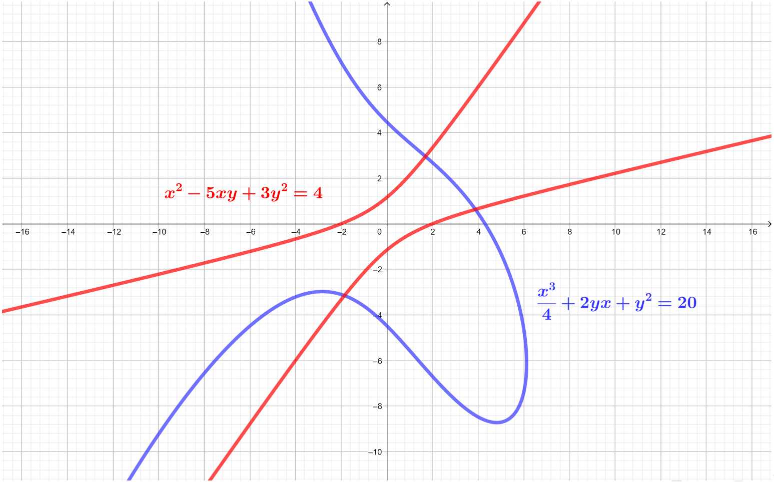
Then starting with an initial guess (p_0, q_0) sufficiently close to a solution (p, q) of the above system we can form the sequence

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix}^{-1} \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix}$$

If $\begin{bmatrix} P_x(p,q) & P_y(p,q) \\ Q_x(p,q) & Q_y(p,q) \end{bmatrix}$ is invertible (has non-zero determinant) and all the second order partial derivatives of $P(x,y)$ and $Q(x,y)$ exist and are continuous on an open ball centered at (p,q), then for any sufficiently close initial guess (p_0, q_0), the sequence (p_n, q_n) converges to (p, q).

Example

Using Newton-Raphson method, approximate a point of intersection of the hyperbola $x^2 - 5xy + 3y^2 = 4$ and the cubic curve $\frac{x^3}{4} + 2yx + y^2 = 20$. Start with initial guess $(p_0, q_0) = (-2, -4)$.



Example

Using Newton-Raphson method, approximate a point of intersection of the hyperbola $x^2 - 5xy + 3y^2 = 4$ and the cubic curve $\frac{x^3}{4} + 2yx + y^2 = 20$. Start with initial guess $(p_0, q_0) = (-2, -4)$.

$P(x, y) = x^2 - 5xy + 3y^2 - 4$, $Q(x, y) = \frac{x^3}{4} + 2yx + y^2 - 20$.

Example

Using Newton-Raphson method, approximate a point of intersection of the hyperbola $x^2 - 5xy + 3y^2 = 4$ and the cubic curve $\frac{x^3}{4} + 2yx + y^2 = 20$. Start with initial guess $(p_0, q_0) = (-2, -4)$.

$P(x, y) = x^2 - 5xy + 3y^2 - 4$, $Q(x, y) = \frac{x^3}{4} + 2yx + y^2 - 20$.

$P_x(x, y) = 2x - 5y$

Example

Using Newton-Raphson method, approximate a point of intersection of the hyperbola $x^2 - 5xy + 3y^2 = 4$ and the cubic curve $\frac{x^3}{4} + 2yx + y^2 = 20$. Start with initial guess $(p_0, q_0) = (-2, -4)$.

$P(x, y) = x^2 - 5xy + 3y^2 - 4,\ Q(x, y) = \frac{x^3}{4} + 2yx + y^2 - 20.$

$P_x(x, y) = 2x - 5y$
$P_y(x, y) = -5x + 6y$

Example

Using Newton-Raphson method, approximate a point of intersection of the hyperbola $x^2 - 5xy + 3y^2 = 4$ and the cubic curve $\frac{x^3}{4} + 2yx + y^2 = 20$. Start with initial guess $(p_0, q_0) = (-2, -4)$.

$P(x,y) = x^2 - 5xy + 3y^2 - 4$, $Q(x,y) = \frac{x^3}{4} + 2yx + y^2 - 20$.

$P_x(x,y) = 2x - 5y$
$P_y(x,y) = -5x + 6y$
$Q_x(x,y) = \frac{3x^2}{4} + 2y$

Example

Using Newton-Raphson method, approximate a point of intersection of the hyperbola $x^2 - 5xy + 3y^2 = 4$ and the cubic curve $\frac{x^3}{4} + 2yx + y^2 = 20$. Start with initial guess $(p_0, q_0) = (-2, -4)$.

$P(x, y) = x^2 - 5xy + 3y^2 - 4$, $Q(x, y) = \frac{x^3}{4} + 2yx + y^2 - 20$.

$P_x(x, y) = 2x - 5y$
$P_y(x, y) = -5x + 6y$
$Q_x(x, y) = \frac{3x^2}{4} + 2y$
$Q_y(x, y) = 2x + 2y$

Example

Using Newton-Raphson method, approximate a point of intersection of the hyperbola $x^2 - 5xy + 3y^2 = 4$ and the cubic curve $\frac{x^3}{4} + 2yx + y^2 = 20$. Start with initial guess $(p_0, q_0) = (-2, -4)$.

$P(x, y) = x^2 - 5xy + 3y^2 - 4,\ Q(x, y) = \frac{x^3}{4} + 2yx + y^2 - 20.$

$P_x(x, y) = 2x - 5y$
$P_y(x, y) = -5x + 6y$
$Q_x(x, y) = \frac{3x^2}{4} + 2y$
$Q_y(x, y) = 2x + 2y$

$$\begin{bmatrix} P_x(x, y) & P_y(x, y) \\ Q_x(x, y) & Q_y(x, y) \end{bmatrix}^{-1} = \begin{bmatrix} 2x - 5y & 5x + 6y \\ \frac{3x^2}{4} + 2y & 2x + 2y \end{bmatrix}^{-1}$$

Example

Using Newton-Raphson method, approximate a point of intersection of the hyperbola $x^2 - 5xy + 3y^2 = 4$ and the cubic curve $\frac{x^3}{4} + 2yx + y^2 = 20$. Start with initial guess $(p_0, q_0) = (-2, -4)$.

$P(x, y) = x^2 - 5xy + 3y^2 - 4$, $Q(x, y) = \frac{x^3}{4} + 2yx + y^2 - 20$.

$$P_x(x, y) = 2x - 5y$$
$$P_y(x, y) = -5x + 6y$$
$$Q_x(x, y) = \frac{3x^2}{4} + 2y$$
$$Q_y(x, y) = 2x + 2y$$
$$\begin{bmatrix} P_x(x,y) & P_y(x,y) \\ Q_x(x,y) & Q_y(x,y) \end{bmatrix}^{-1} = \begin{bmatrix} 2x - 5y & 5x + 6y \\ \frac{3x^2}{4} + 2y & 2x + 2y \end{bmatrix}^{-1} =$$
$$\frac{1}{(2x-5y)(2x+2y)-(5x+6y)(\frac{3x^2}{4}+2y)} \begin{bmatrix} 2x + 2y & -5x - 6y \\ -\frac{3x^2}{4} - 2y & 2x - 5y \end{bmatrix}$$

Using Newton-Raphson method, approximate a point of intersection of the hyperbola $x^2 - 5xy + 3y^2 = 4$ and the cubic curve $\frac{x^3}{4} + 2yx + y^2 = 20$. Start with initial guess $(p_0, q_0) = (-2, -4)$.

$P(x, y) = x^2 - 5xy + 3y^2 - 4$, $Q(x, y) = \frac{x^3}{4} + 2yx + y^2 - 20$.

$$\begin{bmatrix} P_x(x,y) & P_y(x,y) \\ Q_x(x,y) & Q_y(x,y) \end{bmatrix}^{-1} = \begin{bmatrix} 2x - 5y & 5x + 6y \\ \frac{3x^2}{4} + 2y & 2x + 2y \end{bmatrix}^{-1} =$$

$$\frac{1}{(2x-5y)(2x+2y)-(5x+6y)(\frac{3x^2}{4}+2y)} \begin{bmatrix} 2x + 2y & -5x - 6y \\ -\frac{3x^2}{4} - 2y & 2x - 5y \end{bmatrix}$$

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix}^{-1} \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix}$$

Example

Using Newton-Raphson method, approximate a point of intersection of the hyperbola $x^2 - 5xy + 3y^2 = 4$ and the cubic curve $\frac{x^3}{4} + 2yx + y^2 = 20$. Start with initial guess $(p_0, q_0) = (-2, -4)$.

$P(x, y) = x^2 - 5xy + 3y^2 - 4$, $Q(x, y) = \frac{x^3}{4} + 2yx + y^2 - 20$.

$$\begin{bmatrix} P_x(x,y) & P_y(x,y) \\ Q_x(x,y) & Q_y(x,y) \end{bmatrix}^{-1} = \begin{bmatrix} 2x-5y & 5x+6y \\ \frac{3x^2}{4}+2y & 2x+2y \end{bmatrix}^{-1} =$$

$$\frac{1}{(2x-5y)(2x+2y)-(5x+6y)(\frac{3x^2}{4}+2y)} \begin{bmatrix} 2x+2y & -5x-6y \\ -\frac{3x^2}{4}-2y & 2x-5y \end{bmatrix}$$

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \begin{bmatrix} P_x(p_n,q_n) & P_y(p_n,q_n) \\ Q_x(p_n,q_n) & Q_y(p_n,q_n) \end{bmatrix}^{-1} \begin{bmatrix} P(p_n,q_n) \\ Q(p_n,q_n) \end{bmatrix}$$

$$= \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \frac{1}{d_n} \begin{bmatrix} 2p_n+2q_n & -5p_n-6q_n \\ -\frac{3p_n^2}{4}-2q_n & 2p_n-5q_n \end{bmatrix} \begin{bmatrix} p_n^2-5p_nq_n+3q_n^2-4 \\ \frac{p_n^3}{4}+2q_np_n+q_n^2-20 \end{bmatrix}$$

where $d_n = (2p_n - 5q_n)(2p_n + 2q_n) - (5p_n + 6q_n)(\frac{3p_n^2}{4} + 2q_n)$

Example

Using Newton-Raphson method, approximate a point of intersection of the hyperbola $x^2 - 5xy + 3y^2 = 4$ and the cubic curve $\frac{x^3}{4} + 2yx + y^2 = 20$. Start with initial guess $(p_0, q_0) = (-2, -4)$.

$P(x, y) = x^2 - 5xy + 3y^2 - 4,\ Q(x, y) = \frac{x^3}{4} + 2yx + y^2 - 20.$

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix}^{-1} \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix}$$
$$= \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \frac{1}{d_n} \begin{bmatrix} 2p_n + 2q_n & -5p_n - 6q_n \\ -\frac{3p_n^2}{4} - 2q_n & 2p_n - 5q_n \end{bmatrix} \begin{bmatrix} p_n^2 - 5p_nq_n + 3q_n^2 - 4 \\ \frac{p_n^3}{4} + 2q_np_n + q_n^2 - 20 \end{bmatrix}$$

where $d_n = (2p_n - 5q_n)(2p_n + 2q_n) - (5p_n + 6q_n)(\frac{3p_n^2}{4} + 2q_n)$

$p_0 = -2,\ q_0 = -4 \Rightarrow p_1 \approx -1.325966851,\ q_1 \approx -3.447513812$

Example

Using Newton-Raphson method, approximate a point of intersection of the hyperbola $x^2 - 5xy + 3y^2 = 4$ and the cubic curve $\frac{x^3}{4} + 2yx + y^2 = 20$. Start with initial guess $(p_0, q_0) = (-2, -4)$.

$P(x, y) = x^2 - 5xy + 3y^2 - 4,\ Q(x, y) = \frac{x^3}{4} + 2yx + y^2 - 20.$

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix}^{-1} \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix}$$
$$= \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \frac{1}{d_n} \begin{bmatrix} 2p_n + 2q_n & -5p_n - 6q_n \\ -\frac{3p_n^2}{4} - 2q_n & 2p_n - 5q_n \end{bmatrix} \begin{bmatrix} p_n^2 - 5p_nq_n + 3q_n^2 - 4 \\ \frac{p_n^3}{4} + 2q_np_n + q_n^2 - 20 \end{bmatrix}$$

where $d_n = (2p_n - 5q_n)(2p_n + 2q_n) - (5p_n + 6q_n)(\frac{3p_n^2}{4} + 2q_n)$

$p_0 = -2,\ q_0 = -4 \Rightarrow p_1 \approx -1.325966851,\ q_1 \approx -3.447513812$
$p_2 \approx -1.62996858414052,\ q_2 \approx -3.22332353149285$

Example

Using Newton-Raphson method, approximate a point of intersection of the hyperbola $x^2 - 5xy + 3y^2 = 4$ and the cubic curve $\frac{x^3}{4} + 2yx + y^2 = 20$. Start with initial guess $(p_0, q_0) = (-2, -4)$.

$P(x,y) = x^2 - 5xy + 3y^2 - 4,\ Q(x,y) = \frac{x^3}{4} + 2yx + y^2 - 20.$

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \begin{bmatrix} P_x(p_n,q_n) & P_y(p_n,q_n) \\ Q_x(p_n,q_n) & Q_y(p_n,q_n) \end{bmatrix}^{-1} \begin{bmatrix} P(p_n,q_n) \\ Q(p_n,q_n) \end{bmatrix}$$

$$= \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \frac{1}{d_n} \begin{bmatrix} 2p_n + 2q_n & -5p_n - 6q_n \\ -\frac{3p_n^2}{4} - 2q_n & 2p_n - 5q_n \end{bmatrix} \begin{bmatrix} p_n^2 - 5p_nq_n + 3q_n^2 - 4 \\ \frac{p_n^3}{4} + 2q_np_n + q_n^2 - 20 \end{bmatrix}$$

where $d_n = (2p_n - 5q_n)(2p_n + 2q_n) - (5p_n + 6q_n)(\frac{3p_n^2}{4} + 2q_n)$

$p_0 = -2,\ q_0 = -4 \Rightarrow p_1 \approx -1.325966851,\ q_1 \approx -3.447513812$
$p_2 \approx -1.62996858414052,\ q_2 \approx -3.22332353149285$
$p_3 \approx -1.79017231562496,\ q_3 \approx -3.16886793134457$

Example

Using Newton-Raphson method, approximate a point of intersection of the hyperbola $x^2 - 5xy + 3y^2 = 4$ and the cubic curve $\frac{x^3}{4} + 2yx + y^2 = 20$. Start with initial guess $(p_0, q_0) = (-2, -4)$.

$P(x, y) = x^2 - 5xy + 3y^2 - 4$, $Q(x, y) = \frac{x^3}{4} + 2yx + y^2 - 20$.

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix}^{-1} \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix}$$
$$= \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \frac{1}{d_n} \begin{bmatrix} 2p_n + 2q_n & -5p_n - 6q_n \\ -\frac{3p_n^2}{4} - 2q_n & 2p_n - 5q_n \end{bmatrix} \begin{bmatrix} p_n^2 - 5p_nq_n + 3q_n^2 - 4 \\ \frac{p_n^3}{4} + 2q_np_n + q_n^2 - 20 \end{bmatrix}$$

where $d_n = (2p_n - 5q_n)(2p_n + 2q_n) - (5p_n + 6q_n)(\frac{3p_n^2}{4} + 2q_n)$

$p_0 = -2$, $q_0 = -4 \Rightarrow p_1 \approx -1.325966851$, $q_1 \approx -3.447513812$
$p_2 \approx -1.62996858414052$, $q_2 \approx -3.22332353149285$
$p_3 \approx -1.79017231562496$, $q_3 \approx -3.16886793134457$
$p_4 \approx -1.83718216734769$, $q_4 \approx -3.15494785449488$

Example

Using Newton-Raphson method, approximate a point of intersection of the hyperbola $x^2 - 5xy + 3y^2 = 4$ and the cubic curve $\frac{x^3}{4} + 2yx + y^2 = 20$. Start with initial guess $(p_0, q_0) = (-2, -4)$.

$P(x, y) = x^2 - 5xy + 3y^2 - 4$, $Q(x, y) = \frac{x^3}{4} + 2yx + y^2 - 20$.

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \begin{bmatrix} P_x(p_n, q_n) & P_y(p_n, q_n) \\ Q_x(p_n, q_n) & Q_y(p_n, q_n) \end{bmatrix}^{-1} \begin{bmatrix} P(p_n, q_n) \\ Q(p_n, q_n) \end{bmatrix}$$

$$= \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \frac{1}{d_n} \begin{bmatrix} 2p_n + 2q_n & -5p_n - 6q_n \\ -\frac{3p_n^2}{4} - 2q_n & 2p_n - 5q_n \end{bmatrix} \begin{bmatrix} p_n^2 - 5p_n q_n + 3q_n^2 - 4 \\ \frac{p_n^3}{4} + 2q_n p_n + q_n^2 - 20 \end{bmatrix}$$

where $d_n = (2p_n - 5q_n)(2p_n + 2q_n) - (5p_n + 6q_n)(\frac{3p_n^2}{4} + 2q_n)$

$p_0 = -2$, $q_0 = -4 \Rightarrow p_1 \approx -1.325966851$, $q_1 \approx -3.447513812$
$p_2 \approx -1.62996858414052$, $q_2 \approx -3.22332353149285$
$p_3 \approx -1.79017231562496$, $q_3 \approx -3.16886793134457$
$p_4 \approx -1.83718216734769$, $q_4 \approx -3.15494785449488$
$p_5 \approx -1.84891562719857$, $q_5 \approx -3.15091887184173$