(2.3) Newton's Method and Its Extensions

MATH 4701 Numerical Analysis

Newton's Method

Newton's method is among the most commonly used numerical methods for finding roots of functions.

Newton's Method

Newton's method is among the most commonly used numerical methods for finding roots of functions.

Newton's method is an iterative method based on the first degree (linear) Taylor approximation of the function.

Newton's Method

Newton's method is among the most commonly used numerical methods for finding roots of functions.

Newton's method is an iterative method based on the first degree (linear) Taylor approximation of the function.

The method is applicable to functions that are in $C^2[a, b]$ such that the root p we are trying to find is inside (a, b).

Newton's Method

Newton's method is among the most commonly used numerical methods for finding roots of functions.

Newton's method is an iterative method based on the first degree (linear) Taylor approximation of the function.

The method is applicable to functions that are in $C^2[a, b]$ such that the root p we are trying to find is inside (a, b).

The notation $f \in C^n[a, b]$ means f has up to the nth derivative defined and continuous on the interval $[a, b]$.

Newton's Method

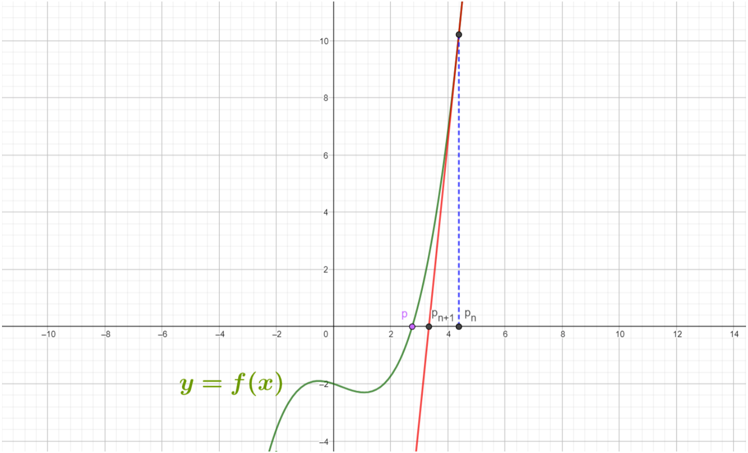
Newton's method is among the most commonly used numerical methods for finding roots of functions.

Newton's method is an iterative method based on the first degree (linear) Taylor approximation of the function.

The method is applicable to functions that are in $C^2[a, b]$ such that the root p we are trying to find is inside (a, b).

The notation $f \in C^n[a, b]$ means f has up to the nth derivative defined and continuous on the interval $[a, b]$. In case of Newton's method, we require the second derivative to exist and be continuous on $[a, b]$.

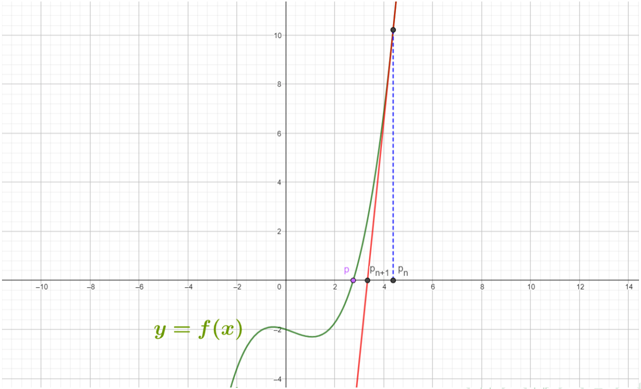
In Newton's method we start with an initial guess p_0 for the root of $f(x) = 0$ and at each step we find a better approximation p_{n+1} using the only root of the tangent line to $f(x)$ at p_n.



Newton's Method

In Newton's method we start with an initial guess p_0 for the root of $f(x) = 0$ and at each step we find a better approximation p_{n+1} using the only root of the tangent line to $f(x)$ at p_n.

The tangent line to $f(x)$ at the point $(p_n, f(p_n))$ is defined by the equation $y - f(p_n) = f'(p_n)(x - p_n)$.



Newton's Method

In Newton's method we start with an initial guess p_0 for the root of $f(x) = 0$ and at each step we find a better approximation p_{n+1} using the only root of the tangent line to $f(x)$ at p_n.

The tangent line to $f(x)$ at the point $(p_n, f(p_n))$ is defined by the equation $y - f(p_n) = f'(p_n)(x - p_n)$.

To find the only root of the linear function described by this line we let $y = 0$ and solve for x to get the value of p_{n+1}:

Newton's Method

In Newton's method we start with an initial guess p_0 for the root of $f(x) = 0$ and at each step we find a better approximation p_{n+1} using the only root of the tangent line to $f(x)$ at p_n.

The tangent line to $f(x)$ at the point $(p_n, f(p_n))$ is defined by the equation $y - f(p_n) = f'(p_n)(x - p_n)$.

To find the only root of the linear function described by this line we let $y = 0$ and solve for x to get the value of p_{n+1}:

$$(0 - f(p_n)) = f'(p_n)(x - p_n)$$

Newton's Method

In Newton's method we start with an initial guess p_0 for the root of $f(x) = 0$ and at each step we find a better approximation p_{n+1} using the only root of the tangent line to $f(x)$ at p_n.

The tangent line to $f(x)$ at the point $(p_n, f(p_n))$ is defined by the equation $y - f(p_n) = f'(p_n)(x - p_n)$.

To find the only root of the linear function described by this line we let $y = 0$ and solve for x to get the value of p_{n+1}:

$$(0 - f(p_n)) = f'(p_n)(x - p_n) \Rightarrow -\frac{f(p_n)}{f'(p_n)} = x - p_n$$

Newton's Method

In Newton's method we start with an initial guess p_0 for the root of $f(x) = 0$ and at each step we find a better approximation p_{n+1} using the only root of the tangent line to $f(x)$ at p_n.

The tangent line to $f(x)$ at the point $(p_n, f(p_n))$ is defined by the equation $y - f(p_n) = f'(p_n)(x - p_n)$.

To find the only root of the linear function described by this line we let $y = 0$ and solve for x to get the value of p_{n+1}:

$$(0-f(p_n)) = f'(p_n)(x-p_n) \Rightarrow -\frac{f(p_n)}{f'(p_n)} = x-p_n \Rightarrow p_{n+1} = x = p_n - \frac{f(p_n)}{f'(p_n)}$$

Newton's Method

In Newton's method we start with an initial guess p_0 for the root of $f(x) = 0$ and at each step we find a better approximation p_{n+1} using the only root of the tangent line to $f(x)$ at p_n.

The tangent line to $f(x)$ at the point $(p_n, f(p_n))$ is defined by the equation $y - f(p_n) = f'(p_n)(x - p_n)$.

To find the only root of the linear function described by this line we let $y = 0$ and solve for x to get the value of p_{n+1}:

$$(0-f(p_n)) = f'(p_n)(x-p_n) \Rightarrow -\frac{f(p_n)}{f'(p_n)} = x-p_n \Rightarrow p_{n+1} = x = p_n-\frac{f(p_n)}{f'(p_n)}$$

As a result, Newton's method, converts the problem of finding a root for $f(x)$ to finding a fixed point of $g(x) = x - \frac{f(x)}{f'(x)}$.

Example

Suppose $f(x) = \frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2$. Starting with $p_0 = 6$ approximate a root of $f(x)$ using Newton's method accurate to within an error of less than 0.0000001.

Example

Suppose $f(x) = \frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2$. Starting with $p_0 = 6$ approximate a root of $f(x)$ using Newton's method accurate to within an error of less than 0.0000001.

$g(x) = x - \frac{f(x)}{f'(x)}$

Example

Suppose $f(x) = \frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2$. Starting with $p_0 = 6$ approximate a root of $f(x)$ using Newton's method accurate to within an error of less than 0.0000001.

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{\frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2}{\frac{3x^2}{5} - \frac{x}{3} - \frac{1}{3}}.$$

Example

Suppose $f(x) = \frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2$. Starting with $p_0 = 6$ approximate a root of $f(x)$ using Newton's method accurate to within an error of less than 0.0000001.

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{\frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2}{\frac{3x^2}{5} - \frac{x}{3} - \frac{1}{3}}.$$

$p_1 = g(p_0) \approx 4.276816609$

Example

Suppose $f(x) = \frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2$. Starting with $p_0 = 6$ approximate a root of $f(x)$ using Newton's method accurate to within an error of less than 0.0000001.

$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{\frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2}{\frac{3x^2}{5} - \frac{x}{3} - \frac{1}{3}}$.

$p_1 = g(p_0) \approx 4.276816609$

$p_2 = g(p_1) \approx 3.281623803$

Example

Suppose $f(x) = \frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2$. Starting with $p_0 = 6$ approximate a root of $f(x)$ using Newton's method accurate to within an error of less than 0.0000001.

$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{\frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2}{\frac{3x^2}{5} - \frac{x}{3} - \frac{1}{3}}$.

$p_1 = g(p_0) \approx 4.276816609$

$p_2 = g(p_1) \approx 3.281623803$

$p_3 = g(p_2) \approx 2.848730902$

Example

Suppose $f(x) = \frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2$. Starting with $p_0 = 6$ approximate a root of $f(x)$ using Newton's method accurate to within an error of less than 0.0000001.

$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{\frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2}{\frac{3x^2}{5} - \frac{x}{3} - \frac{1}{3}}$.

$p_1 = g(p_0) \approx 4.276816609$

$p_2 = g(p_1) \approx 3.281623803$

$p_3 = g(p_2) \approx 2.848730902$

$p_4 = g(p_3) \approx 2.759077035$

Example

Suppose $f(x) = \frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2$. Starting with $p_0 = 6$ approximate a root of $f(x)$ using Newton's method accurate to within an error of less than 0.0000001.

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{\frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2}{\frac{3x^2}{5} - \frac{x}{3} - \frac{1}{3}}.$$

$p_1 = g(p_0) \approx 4.276816609$

$p_2 = g(p_1) \approx 3.281623803$

$p_3 = g(p_2) \approx 2.848730902$

$p_4 = g(p_3) \approx 2.759077035$

$p_5 = g(p_4) \approx 2.755379685$

Example

Suppose $f(x) = \frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2$. Starting with $p_0 = 6$ approximate a root of $f(x)$ using Newton's method accurate to within an error of less than 0.0000001.

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{\frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2}{\frac{3x^2}{5} - \frac{x}{3} - \frac{1}{3}}.$$

$p_1 = g(p_0) \approx 4.276816609$

$p_2 = g(p_1) \approx 3.281623803$

$p_3 = g(p_2) \approx 2.848730902$

$p_4 = g(p_3) \approx 2.759077035$

$p_5 = g(p_4) \approx 2.755379685$

$p_6 = g(p_5) \approx 2.755373527$

Example

Suppose $f(x) = \frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2$. Starting with $p_0 = 6$ approximate a root of $f(x)$ using Newton's method accurate to within an error of less than 0.0000001.

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{\frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2}{\frac{3x^2}{5} - \frac{x}{3} - \frac{1}{3}}.$$

$p_1 = g(p_0) \approx 4.276816609$

$p_2 = g(p_1) \approx 3.281623803$

$p_3 = g(p_2) \approx 2.848730902$

$p_4 = g(p_3) \approx 2.759077035$

$p_5 = g(p_4) \approx 2.755379685$

$p_6 = g(p_5) \approx 2.755373527$

$p_7 = g(p_6) \approx 2.755373527$

Example

Suppose $f(x) = \frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2$. Starting with $p_0 = 6$ approximate a root of $f(x)$ using Newton's method accurate to within an error of less than 0.0000001.

$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{\frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2}{\frac{3x^2}{5} - \frac{x}{3} - \frac{1}{3}}$.

$p_3 = g(p_2) \approx 2.848730902$

$p_4 = g(p_3) \approx 2.759077035$

$p_5 = g(p_4) \approx 2.755379685$

$p_6 = g(p_5) \approx 2.755373527$

$p_7 = g(p_6) \approx 2.755373527$

Since p_6 and p_7 share their first seven decimal places, we stop here and announce the root of $f(x) = 0$ as $p = 2.755373527$ to within an error of at most 0.0000001.

Convergence of Newton's Method

Theorem: Let $f \in C^2[a, b]$. Suppose $p \in (a, b)$ satisfies $f(p) = 0$ and $f'(p) \neq 0$.

Convergence of Newton's Method

Theorem: Let $f \in C^2[a, b]$. Suppose $p \in (a, b)$ satisfies $f(p) = 0$ and $f'(p) \neq 0$. Then **there is** a $\delta > 0$ such that **for all** $p_0 \in [p - \delta, p + \delta]$, the sequence (p_n) generated by the Newton's method with initial guess p_0 is convergent to p.

Convergence of Newton's Method

Theorem: Let $f \in C^2[a, b]$. Suppose $p \in (a, b)$ satisfies $f(p) = 0$ and $f'(p) \neq 0$. Then **there is** a $\delta > 0$ such that **for all** $p_0 \in [p - \delta, p + \delta]$, the sequence (p_n) generated by the Newton's method with initial guess p_0 is convergent to p.

Proof: Since $f'(p) \neq 0$ and $f'(x)$ is continuous on $[a, b]$ there is a $\delta_1 > 0$ such that for $[p - \delta_1, p + \delta_1] \subset [a, b]$

Convergence of Newton's Method

Theorem: Let $f \in C^2[a,b]$. Suppose $p \in (a,b)$ satisfies $f(p) = 0$ and $f'(p) \neq 0$. Then **there is** a $\delta > 0$ such that **for all** $p_0 \in [p-\delta, p+\delta]$, the sequence (p_n) generated by the Newton's method with initial guess p_0 is convergent to p.

Proof: Since $f'(p) \neq 0$ and $f'(x)$ is continuous on $[a,b]$ there is a $\delta_1 > 0$ such that for $[p-\delta_1, p+\delta_1] \subset [a,b]$ and $f'(x) \neq 0$ for all $x \in [p-\delta_1, p+\delta_1]$.

Convergence of Newton's Method

Theorem: Let $f \in C^2[a, b]$. Suppose $p \in (a, b)$ satisfies $f(p) = 0$ and $f'(p) \neq 0$. Then **there is** a $\delta > 0$ such that **for all** $p_0 \in [p - \delta, p + \delta]$, the sequence (p_n) generated by the Newton's method with initial guess p_0 is convergent to p.

Proof: Since $f'(p) \neq 0$ and $f'(x)$ is continuous on $[a, b]$ there is a $\delta_1 > 0$ such that for $[p - \delta_1, p + \delta_1] \subset [a, b]$ and $f'(x) \neq 0$ for all $x \in [p - \delta_1, p + \delta_1]$.

The function $g(x) = x - \frac{f(x)}{f'(x)}$ is defined and is continuous on $[p - \delta_1, p + \delta_1]$.

Convergence of Newton's Method

Theorem: Let $f \in C^2[a, b]$. Suppose $p \in (a, b)$ satisfies $f(p) = 0$ and $f'(p) \neq 0$. Then **there is** a $\delta > 0$ such that **for all** $p_0 \in [p - \delta, p + \delta]$, the sequence (p_n) generated by the Newton's method with initial guess p_0 is convergent to p.

Proof: Since $f'(p) \neq 0$ and $f'(x)$ is continuous on $[a, b]$ there is a $\delta_1 > 0$ such that for $[p - \delta_1, p + \delta_1] \subset [a, b]$ and $f'(x) \neq 0$ for all $x \in [p - \delta_1, p + \delta_1]$.

The function $g(x) = x - \frac{f(x)}{f'(x)}$ is defined and is continuous on $[p - \delta_1, p + \delta_1]$.

$$g'(x) = 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2}$$

Convergence of Newton's Method

Theorem: Let $f \in C^2[a, b]$. Suppose $p \in (a, b)$ satisfies $f(p) = 0$ and $f'(p) \neq 0$. Then **there is** a $\delta > 0$ such that **for all** $p_0 \in [p - \delta, p + \delta]$, the sequence (p_n) generated by the Newton's method with initial guess p_0 is convergent to p.

Proof: Since $f'(p) \neq 0$ and $f'(x)$ is continuous on $[a, b]$ there is a $\delta_1 > 0$ such that for $[p - \delta_1, p + \delta_1] \subset [a, b]$ and $f'(x) \neq 0$ for all $x \in [p - \delta_1, p + \delta_1]$.

The function $g(x) = x - \frac{f(x)}{f'(x)}$ is defined and is continuous on $[p - \delta_1, p + \delta_1]$.

$$g'(x) = 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2}$$

Since $f \in C^2[a, b]$, $g'(x)$ is continuous on $[p - \delta_1, p + \delta_1]$.

Convergence of Newton's Method

Theorem: Let $f \in C^2[a, b]$. Suppose $p \in (a, b)$ satisfies $f(p) = 0$ and $f'(p) \neq 0$. Then **there is** a $\delta > 0$ such that **for all** $p_0 \in [p - \delta, p + \delta]$, the sequence (p_n) generated by the Newton's method with initial guess p_0 is convergent to p.

Proof: Since $f'(p) \neq 0$ and $f'(x)$ is continuous on $[a, b]$ there is a $\delta_1 > 0$ such that for $[p - \delta_1, p + \delta_1] \subset [a, b]$ and $f'(x) \neq 0$ for all $x \in [p - \delta_1, p + \delta_1]$.

The function $g(x) = x - \frac{f(x)}{f'(x)}$ is defined and is continuous on $[p - \delta_1, p + \delta_1]$.

$$g'(x) = 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2}$$

Since $f \in C^2[a, b]$, $g'(x)$ is continuous on $[p - \delta_1, p + \delta_1]$. In other words, $g \in C^1[p - \delta_1, p + \delta_1]$.

Convergence of Newton's Method

Theorem: Let $f \in C^2[a, b]$. Suppose $p \in (a, b)$ satisfies $f(p) = 0$ and $f'(p) \neq 0$. Then **there is** a $\delta > 0$ such that **for all** $p_0 \in [p - \delta, p + \delta]$, the sequence (p_n) generated by the Newton's method with initial guess p_0 is convergent to p.

Proof: Since $f'(p) \neq 0$ and $f'(x)$ is continuous on $[a, b]$ there is a $\delta_1 > 0$ such that for $[p - \delta_1, p + \delta_1] \subset [a, b]$ and $f'(x) \neq 0$ for all $x \in [p - \delta_1, p + \delta_1]$.

The function $g(x) = x - \frac{f(x)}{f'(x)}$ is defined and is continuous on $[p - \delta_1, p + \delta_1]$.

$$g'(x) = 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2}$$

Since $f \in C^2[a, b]$, $g'(x)$ is continuous on $[p - \delta_1, p + \delta_1]$. In other words, $g \in C^1[p - \delta_1, p + \delta_1]$. Moreover, $g'(p) = \frac{f(p)f''(p)}{[f'(p)]^2} = 0$ because $f(p) = 0$ and $f'(p) \neq 0$.

Convergence of Newton's Method

Theorem: Let $f \in C^2[a, b]$. Suppose $p \in (a, b)$ satisfies $f(p) = 0$ and $f'(p) \neq 0$. Then **there is** a $\delta > 0$ such that **for all** $p_0 \in [p - \delta, p + \delta]$, the sequence (p_n) generated by the Newton's method with initial guess p_0 is convergent to p.

Proof: Since $f'(p) \neq 0$ and $f'(x)$ is continuous on $[a, b]$ there is a $\delta_1 > 0$ such that for $[p - \delta_1, p + \delta_1] \subset [a, b]$ and $f'(x) \neq 0$ for all $x \in [p - \delta_1, p + \delta_1]$.

The function $g(x) = x - \frac{f(x)}{f'(x)}$ satisfies $g \in C^1[p - \delta_1, p + \delta_1]$ and $g'(p) = 0$ and $g(p) = p - \frac{f(p)}{f'(p)} = p$.

Convergence of Newton's Method

Theorem: Let $f \in C^2[a, b]$. Suppose $p \in (a, b)$ satisfies $f(p) = 0$ and $f'(p) \neq 0$. Then **there is** a $\delta > 0$ such that **for all** $p_0 \in [p - \delta, p + \delta]$, the sequence (p_n) generated by the Newton's method with initial guess p_0 is convergent to p.

Proof: Since $f'(p) \neq 0$ and $f'(x)$ is continuous on $[a, b]$ there is a $\delta_1 > 0$ such that for $[p - \delta_1, p + \delta_1] \subset [a, b]$ and $f'(x) \neq 0$ for all $x \in [p - \delta_1, p + \delta_1]$.

The function $g(x) = x - \frac{f(x)}{f'(x)}$ satisfies $g \in C^1[p - \delta_1, p + \delta_1]$ and $g'(p) = 0$ and $g(p) = p - \frac{f(p)}{f'(p)} = p$.

Since g' is a continuous function that is equal to zero at $x = p$, given any $0 < K < 1$, there is $0 < \delta < \delta_1$ such that for all $x \in [p - \delta, p + \delta]$, $|g'(x)| < K$.

Convergence of Newton's Method

Theorem: Let $f \in C^2[a, b]$. Suppose $p \in (a, b)$ satisfies $f(p) = 0$ and $f'(p) \neq 0$. Then **there is** a $\delta > 0$ such that **for all** $p_0 \in [p - \delta, p + \delta]$, the sequence (p_n) generated by the Newton's method with initial guess p_0 is convergent to p.

Proof: Since $f'(p) \neq 0$ and $f'(x)$ is continuous on $[a, b]$ there is a $\delta_1 > 0$ such that for $[p - \delta_1, p + \delta_1] \subset [a, b]$ and $f'(x) \neq 0$ for all $x \in [p - \delta_1, p + \delta_1]$.

The function $g(x) = x - \frac{f(x)}{f'(x)}$ satisfies $g \in C^1[p - \delta_1, p + \delta_1]$ and $g'(p) = 0$ and $g(p) = p - \frac{f(p)}{f'(p)} = p$.

Since g' is a continuous function that is equal to zero at $x = p$, given any $0 < K < 1$, there is $0 < \delta < \delta_1$ such that for all $x \in [p - \delta, p + \delta]$, $|g'(x)| < K$.

Now if $x \in [p - \delta, p + \delta]$, then $|g(x) - p| = |g(x) - g(p)|$

Convergence of Newton's Method

Theorem: Let $f \in C^2[a,b]$. Suppose $p \in (a,b)$ satisfies $f(p) = 0$ and $f'(p) \neq 0$. Then **there is** a $\delta > 0$ such that **for all** $p_0 \in [p-\delta, p+\delta]$, the sequence (p_n) generated by the Newton's method with initial guess p_0 is convergent to p.

Proof: Since $f'(p) \neq 0$ and $f'(x)$ is continuous on $[a,b]$ there is a $\delta_1 > 0$ such that for $[p-\delta_1, p+\delta_1] \subset [a,b]$ and $f'(x) \neq 0$ for all $x \in [p-\delta_1, p+\delta_1]$.

The function $g(x) = x - \frac{f(x)}{f'(x)}$ satisfies $g \in C^1[p-\delta_1, p+\delta_1]$ and $g'(p) = 0$ and $g(p) = p - \frac{f(p)}{f'(p)} = p$.

Since g' is a continuous function that is equal to zero at $x = p$, given any $0 < K < 1$, there is $0 < \delta < \delta_1$ such that for all $x \in [p-\delta, p+\delta]$, $|g'(x)| < K$.

Now if $x \in [p-\delta, p+\delta]$, then $|g(x) - p| = |g(x) - g(p)| \leq K|x-p|$

Convergence of Newton's Method

Theorem: Let $f \in C^2[a, b]$. Suppose $p \in (a, b)$ satisfies $f(p) = 0$ and $f'(p) \neq 0$. Then **there is** a $\delta > 0$ such that **for all** $p_0 \in [p - \delta, p + \delta]$, the sequence (p_n) generated by the Newton's method with initial guess p_0 is convergent to p.

Proof: Since $f'(p) \neq 0$ and $f'(x)$ is continuous on $[a, b]$ there is a $\delta_1 > 0$ such that for $[p - \delta_1, p + \delta_1] \subset [a, b]$ and $f'(x) \neq 0$ for all $x \in [p - \delta_1, p + \delta_1]$.

The function $g(x) = x - \frac{f(x)}{f'(x)}$ satisfies $g \in C^1[p - \delta_1, p + \delta_1]$ and $g'(p) = 0$ and $g(p) = p - \frac{f(p)}{f'(p)} = p$.

Since g' is a continuous function that is equal to zero at $x = p$, given any $0 < K < 1$, there is $0 < \delta < \delta_1$ such that for all $x \in [p - \delta, p + \delta]$, $|g'(x)| < K$.

Now if $x \in [p - \delta, p + \delta]$, then
$|g(x) - p| = |g(x) - g(p)| \leq K|x - p| \leq K\delta < \delta$

Convergence of Newton's Method

Theorem: Let $f \in C^2[a, b]$. Suppose $p \in (a, b)$ satisfies $f(p) = 0$ and $f'(p) \neq 0$. Then **there is** a $\delta > 0$ such that **for all** $p_0 \in [p - \delta, p + \delta]$, the sequence (p_n) generated by the Newton's method with initial guess p_0 is convergent to p.

Proof: Since $f'(p) \neq 0$ and $f'(x)$ is continuous on $[a, b]$ there is a $\delta_1 > 0$ such that for $[p - \delta_1, p + \delta_1] \subset [a, b]$ and $f'(x) \neq 0$ for all $x \in [p - \delta_1, p + \delta_1]$.

The function $g(x) = x - \frac{f(x)}{f'(x)}$ satisfies $g \in C^1[p - \delta_1, p + \delta_1]$ and $g'(p) = 0$ and $g(p) = p - \frac{f(p)}{f'(p)} = p$.

Since g' is a continuous function that is equal to zero at $x = p$, given any $0 < K < 1$, there is $0 < \delta < \delta_1$ such that for all $x \in [p - \delta, p + \delta]$, $|g'(x)| < K$.

Now if $x \in [p - \delta, p + \delta]$, then
$|g(x) - p| = |g(x) - g(p)| \leq K|x - p| \leq K\delta < \delta \Rightarrow g(x) \in [p - \delta, p + \delta]$.

Convergence of Newton's Method

Theorem: Let $f \in C^2[a,b]$. Suppose $p \in (a,b)$ satisfies $f(p) = 0$ and $f'(p) \neq 0$. Then **there is** a $\delta > 0$ such that **for all** $p_0 \in [p-\delta, p+\delta]$, the sequence (p_n) generated by the Newton's method with initial guess p_0 is convergent to p.

Proof: Since $f'(p) \neq 0$ and $f'(x)$ is continuous on $[a,b]$ there is a $\delta_1 > 0$ such that for $[p-\delta_1, p+\delta_1] \subset [a,b]$ and $f'(x) \neq 0$ for all $x \in [p-\delta_1, p+\delta_1]$.

The function $g(x) = x - \frac{f(x)}{f'(x)}$ satisfies $g \in C^1[p-\delta_1, p+\delta_1]$ and $g'(p) = 0$ and $g(p) = p - \frac{f(p)}{f'(p)} = p$.

Now if $x \in [p-\delta, p+\delta]$, then
$|g(x) - p| = |g(x) - g(p)| \leq K|x-p| \leq K\delta < \delta \Rightarrow g(x) \in [p-\delta, p+\delta]$.
Therefore, $g([p-\delta, p+\delta]) \subset [p-\delta, p+\delta]$.

Convergence of Newton's Method

Theorem: Let $f \in C^2[a, b]$. Suppose $p \in (a, b)$ satisfies $f(p) = 0$ and $f'(p) \neq 0$. Then **there is** a $\delta > 0$ such that **for all** $p_0 \in [p - \delta, p + \delta]$, the sequence (p_n) generated by the Newton's method with initial guess p_0 is convergent to p.

Proof: Since $f'(p) \neq 0$ and $f'(x)$ is continuous on $[a, b]$ there is a $\delta_1 > 0$ such that for $[p - \delta_1, p + \delta_1] \subset [a, b]$ and $f'(x) \neq 0$ for all $x \in [p - \delta_1, p + \delta_1]$.

The function $g(x) = x - \frac{f(x)}{f'(x)}$ satisfies $g \in C^1[p - \delta_1, p + \delta_1]$ and $g'(p) = 0$ and $g(p) = p - \frac{f(p)}{f'(p)} = p$.

Therefore, $g([p - \delta, p + \delta]) \subset [p - \delta, p + \delta]$.

Since $g \in C^1[p - \delta, p + \delta]$, $g([p - \delta, p + \delta]) \subset [p - \delta, p + \delta]$, and $|g'(x)| \leq K < 1$ for all $x \in [p - \delta, p + \delta]$, by the fixed point iteration theorem the sequence $p_{n+1} = g(p_n)$ with any $p_0 \in [p - \delta, p + \delta]$ is convergent to the fixed point p.

Secant Method

Another method that uses a line that shares with the given function two pieces of information to generate better estimates of a root of that function is the **secant method**.

Secant Method

Another method that uses a line that shares with the given function two pieces of information to generate better estimates of a root of that function is the **secant method**.

In the secant method, instead of using the slope and one point to form the approximating line, we use two points.

Secant Method

Another method that uses a line that shares with the given function two pieces of information to generate better estimates of a root of that function is the **secant method**.

In the secant method, instead of using the slope and one point to form the approximating line, we use two points.

The advantage of using the secant method is that it does not need any evaluation of the derivative of the function.

Secant Method

Another method that uses a line that shares with the given function two pieces of information to generate better estimates of a root of that function is the **secant method**.

In the secant method, instead of using the slope and one point to form the approximating line, we use two points.

The advantage of using the secant method is that it does not need any evaluation of the derivative of the function. The convergence of the method however requires the same conditions as needed for the convergence of Newton's method.

Secant Method

Another method that uses a line that shares with the given function two pieces of information to generate better estimates of a root of that function is the **secant method**.

In the secant method, instead of using the slope and one point to form the approximating line, we use two points.

The advantage of using the secant method is that it does not need any evaluation of the derivative of the function. The convergence of the method however requires the same conditions as needed for the convergence of Newton's method.

Similar to the **bisection** algorithm, in each step of the secant method we use two previous approximations for the root to generate a new one which replaces one of the two.

Secant Method

In the secant method, instead of using the slope and one point to form the approximating line, we use two points.

The advantage of using the secant method is that it does not need any evaluation of the derivative of the function. The convergence of the method however requires the same conditions as needed for the convergence of Newton's method.

Similar to the **bisection** algorithm, in each step of the secant method we use two previous approximations for the root to generate a new one which replaces one of the two.

Unlike bisection algorithm, in secant method the actual values of the function are used to generate the new estimate.

Secant Method

The advantage of using the secant method is that it does not need any evaluation of the derivative of the function. The convergence of the method however requires the same conditions as needed for the convergence of Newton's method.

Similar to the **bisection** algorithm, in each step of the secant method we use two previous approximations for the root to generate a new one which replaces one of the two.

Unlike bisection algorithm, in secant method the actual values of the function are used to generate the new estimate.

The secant method starts with two initial guesses p_0 and p_1 which are sufficiently close to the root p of $f(x)$.

Secant Method

The advantage of using the secant method is that it does not need any evaluation of the derivative of the function. The convergence of the method however requires the same conditions as needed for the convergence of Newton's method.

Similar to the **bisection** algorithm, in each step of the secant method we use two previous approximations for the root to generate a new one which replaces one of the two.

Unlike bisection algorithm, in secant method the actual values of the function are used to generate the new estimate.

The secant method starts with two initial guesses p_0 and p_1 which are sufficiently close to the root p of $f(x)$.

We then produce the new estimate p_2 as the x-intercept of the line through both $(p_0, f(p_0))$ and $(p_1, f(p_1))$.

Secant Method

Similar to the **bisection** algorithm, in each step of the secant method we use two previous approximations for the root to generate a new one which replaces one of the two.

Unlike bisection algorithm, in secant method the actual values of the function are used to generate the new estimate.

The secant method starts with two initial guesses p_0 and p_1 which are sufficiently close to the root p of $f(x)$.

We then produce the new estimate p_2 as the x-intercept of the line through both $(p_0, f(p_0))$ and $(p_1, f(p_1))$.

In every step, using the previously generated estimates p_{n-1} and p_n, we produce the new estimate p_{n+1} as the x-intercept of the line through both $(p_{n-1}, f(p_{n-1}))$ and $(p_n, f(p_n))$.

Secant Method

The secant method starts with two initial guesses p_0 and p_1 which are sufficiently close to the root p of $f(x)$.

We then produce the new estimate p_2 as the x-intercept of the line through both $(p_0, f(p_0))$ and $(p_1, f(p_1))$.

In every step, using the previously generated estimates p_{n-1} and p_n, we produce the new estimate p_{n+1} as the x-intercept of the line through both $(p_{n-1}, f(p_{n-1}))$ and $(p_n, f(p_n))$.

Since the three points $(p_{n-1}, f(p_{n-1}))$, $(p_n, f(p_n))$, and $(p_{n+1}, 0)$ are co-linear we have

$$\frac{f(p_n) - f(p_{n-1})}{p_n - p_{n-1}} = \frac{0 - f(p_n)}{p_{n+1} - p_n}$$

Secant Method

The secant method starts with two initial guesses p_0 and p_1 which are sufficiently close to the root p of $f(x)$.

We then produce the new estimate p_2 as the x-intercept of the line through both $(p_0, f(p_0))$ and $(p_1, f(p_1))$.

In every step, using the previously generated estimates p_{n-1} and p_n, we produce the new estimate p_{n+1} as the x-intercept of the line through both $(p_{n-1}, f(p_{n-1}))$ and $(p_n, f(p_n))$.

Since the three points $(p_{n-1}, f(p_{n-1}))$, $(p_n, f(p_n))$, and $(p_{n+1}, 0)$ are co-linear we have

$$\frac{f(p_n) - f(p_{n-1})}{p_n - p_{n-1}} = \frac{0 - f(p_n)}{p_{n+1} - p_n}$$

$$\Rightarrow (p_{n+1} - p_n)(f(p_n) - f(p_{n-1})) = -f(p_n)(p_n - p_{n-1})$$

Secant Method

In every step, using the previously generated estimates p_{n-1} and p_n, we produce the new estimate p_{n+1} as the x-intercept of the line through both $(p_{n-1}, f(p_{n-1}))$ and $(p_n, f(p_n))$.

Since the three points $(p_{n-1}, f(p_{n-1}))$, $(p_n, f(p_n))$, and $(p_{n+1}, 0)$ are co-linear we have

$$\frac{f(p_n) - f(p_{n-1})}{p_n - p_{n-1}} = \frac{0 - f(p_n)}{p_{n+1} - p_n}$$

$$\Rightarrow (p_{n+1} - p_n)(f(p_n) - f(p_{n-1})) = -f(p_n)(p_n - p_{n-1})$$

$$\Rightarrow p_{n+1}(f(p_n) - f(p_{n-1})) - p_n f(p_n) + p_n f(p_{n-1}) = -f(p_n)p_n + f(p_n)p_{n-1}$$

Secant Method

In every step, using the previously generated estimates p_{n-1} and p_n, we produce the new estimate p_{n+1} as the x-intercept of the line through both $(p_{n-1}, f(p_{n-1}))$ and $(p_n, f(p_n))$.

Since the three points $(p_{n-1}, f(p_{n-1}))$, $(p_n, f(p_n))$, and $(p_{n+1}, 0)$ are co-linear we have

$$\frac{f(p_n) - f(p_{n-1})}{p_n - p_{n-1}} = \frac{0 - f(p_n)}{p_{n+1} - p_n}$$

$$\Rightarrow p_{n+1}(f(p_n) - f(p_{n-1})) - p_n f(p_n) + p_n f(p_{n-1}) = -f(p_n)p_n + f(p_n)p_{n-1}$$

$$\Rightarrow p_{n+1}(f(p_n) - f(p_{n-1})) + p_n f(p_{n-1}) = f(p_n)p_{n-1}$$

Secant Method

In every step, using the previously generated estimates p_{n-1} and p_n, we produce the new estimate p_{n+1} as the x-intercept of the line through both $(p_{n-1}, f(p_{n-1}))$ and $(p_n, f(p_n))$.

Since the three points $(p_{n-1}, f(p_{n-1}))$, $(p_n, f(p_n))$, and $(p_{n+1}, 0)$ are co-linear we have

$$\frac{f(p_n) - f(p_{n-1})}{p_n - p_{n-1}} = \frac{0 - f(p_n)}{p_{n+1} - p_n}$$

$$\Rightarrow p_{n+1}(f(p_n) - f(p_{n-1})) - p_n f(p_n) + p_n f(p_{n-1}) = -f(p_n)p_n + f(p_n)p_{n-1}$$

$$\Rightarrow p_{n+1}(f(p_n) - f(p_{n-1})) + p_n f(p_{n-1}) = f(p_n)p_{n-1}$$

$$\Rightarrow p_{n+1}(f(p_n) - f(p_{n-1})) = f(p_n)p_{n-1} - p_n f(p_{n-1})$$

Secant Method

In every step, using the previously generated estimates p_{n-1} and p_n, we produce the new estimate p_{n+1} as the x-intercept of the line through both $(p_{n-1}, f(p_{n-1}))$ and $(p_n, f(p_n))$.

Since the three points $(p_{n-1}, f(p_{n-1}))$, $(p_n, f(p_n))$, and $(p_{n+1}, 0)$ are co-linear we have

$$\frac{f(p_n) - f(p_{n-1})}{p_n - p_{n-1}} = \frac{0 - f(p_n)}{p_{n+1} - p_n}$$

$$\Rightarrow p_{n+1}(f(p_n) - f(p_{n-1})) - p_n f(p_n) + p_n f(p_{n-1}) = -f(p_n)p_n + f(p_n)p_{n-1}$$

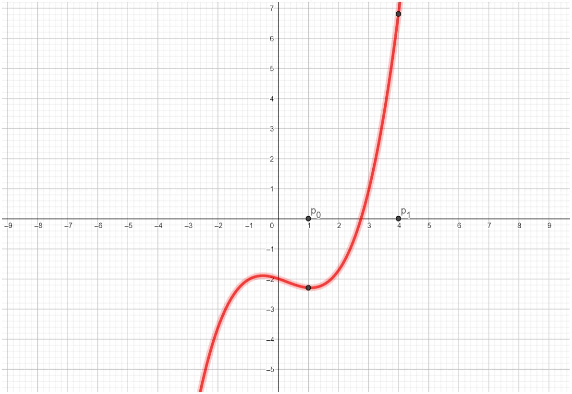
$$\Rightarrow p_{n+1}(f(p_n) - f(p_{n-1})) + p_n f(p_{n-1}) = f(p_n)p_{n-1}$$

$$\Rightarrow p_{n+1}(f(p_n) - f(p_{n-1})) = f(p_n)p_{n-1} - p_n f(p_{n-1})$$

$$\Rightarrow p_{n+1} = \frac{f(p_n)p_{n-1} - p_n f(p_{n-1})}{f(p_n) - f(p_{n-1})}$$

Example

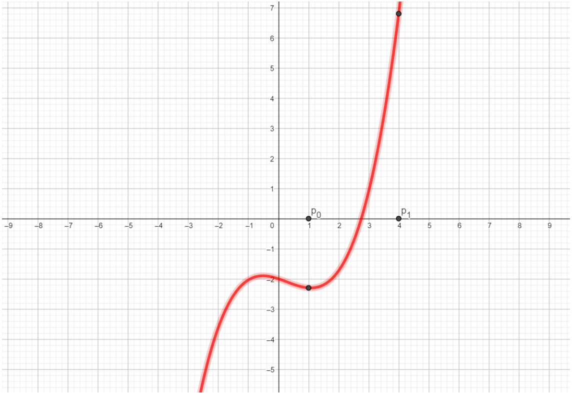
Suppose $f(x) = \frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2$. Starting with $p_0 = 1$ and $p_1 = 4$ approximate a root of $f(x)$ using three steps of the **secant method** to generate p_2, p_3, and p_4.



Example

Suppose $f(x) = \frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2$. Starting with $p_0 = 1$ and $p_1 = 4$ approximate a root of $f(x)$ using three steps of the **secant method** to generate p_2, p_3, and p_4.

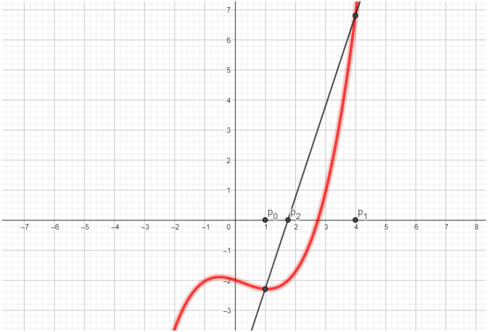
$$p_{n+1} = \frac{f(p_n)p_{n-1} - p_n f(p_{n-1})}{f(p_n) - f(p_{n-1})}$$



Suppose $f(x) = \frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2$. Starting with $p_0 = 1$ and $p_1 = 4$ approximate a root of $f(x)$ using three steps of the **secant method** to generate p_2, p_3, and p_4.

$$p_{n+1} = \frac{f(p_n)p_{n-1} - p_n f(p_{n-1})}{f(p_n) - f(p_{n-1})}$$

$$p_2 = \frac{f(p_1)p_0 - p_1 f(p_0)}{f(p_1) - f(p_0)} \approx 1.758241758$$

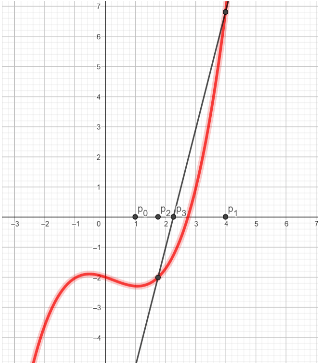


Example

Suppose $f(x) = \frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2$. Starting with $p_0 = 1$ and $p_1 = 4$ approximate a root of $f(x)$ using three steps of the **secant method** to generate p_2, p_3, and p_4.

$$p_{n+1} = \frac{f(p_n)p_{n-1} - p_n f(p_{n-1})}{f(p_n) - f(p_{n-1})}$$

$$p_2 = \frac{f(p_1)p_0 - p_1 f(p_0)}{f(p_1) - f(p_0)} \approx 1.758241758 \qquad p_3 = \frac{f(p_2)p_1 - p_2 f(p_1)}{f(p_2) - f(p_1)} \approx 2.270528038$$



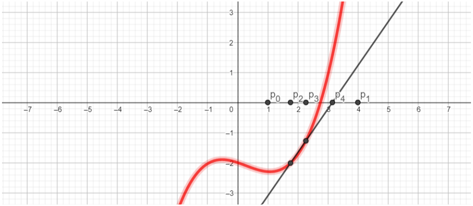
Example

Suppose $f(x) = \frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2$. Starting with $p_0 = 1$ and $p_1 = 4$ approximate a root of $f(x)$ using three steps of the **secant method** to generate p_2, p_3, and p_4.

$$p_{n+1} = \frac{f(p_n)p_{n-1} - p_n f(p_{n-1})}{f(p_n) - f(p_{n-1})}$$

$$p_2 = \frac{f(p_1)p_0 - p_1 f(p_0)}{f(p_1) - f(p_0)} \approx 1.758241758 \qquad p_3 = \frac{f(p_2)p_1 - p_2 f(p_1)}{f(p_2) - f(p_1)} \approx 2.270528038$$

$$p_4 = \frac{f(p_3)p_2 - p_3 f(p_2)}{f(p_3) - f(p_2)} \approx 3.154125867$$



The Method of False Position (Regula Falsi)

The method of false position is a variation on the secant method in which we make sure the root we are trying to find remains in between every two consecutive points we generate in the sequence of estimates.

The Method of False Position (Regula Falsi)

The method of false position is a variation on the secant method in which we make sure the root we are trying to find remains in between every two consecutive points we generate in the sequence of estimates.

In this sense we can consider the method of false position to be a hybrid of the secant method and the bisection method.

The Method of False Position (Regula Falsi)

The method of false position is a variation on the secant method in which we make sure the root we are trying to find remains in between every two consecutive points we generate in the sequence of estimates.

In this sense we can consider the method of false position to be a hybrid of the secant method and the bisection method.

We start with two initial guesses p_0 and p_1 such that $f(p_0)f(p_1) < 0$.

The Method of False Position (Regula Falsi)

The method of false position is a variation on the secant method in which we make sure the root we are trying to find remains in between every two consecutive points we generate in the sequence of estimates.

In this sense we can consider the method of false position to be a hybrid of the secant method and the bisection method.

We start with two initial guesses p_0 and p_1 such that $f(p_0)f(p_1) < 0$.

At the beginning we let $A = p_0$ and $B = p_1$ as we did in bisection method.

The Method of False Position (Regula Falsi)

The method of false position is a variation on the secant method in which we make sure the root we are trying to find remains in between every two consecutive points we generate in the sequence of estimates.

In this sense we can consider the method of false position to be a hybrid of the secant method and the bisection method.

We start with two initial guesses p_0 and p_1 such that $f(p_0)f(p_1) < 0$.

At the beginning we let $A = p_0$ and $B = p_1$ as we did in bisection method.

At each step we let after p_0, ..., p_n have been found and A and B are assigned, we let $p_{n+1} = \frac{f(B)A - Bf(A)}{f(B) - f(A)}$ as we do in the secant method.

The Method of False Position (Regula Falsi)

In this sense we can consider the method of false position to be a hybrid of the secant method and the bisection method.

We start with two initial guesses p_0 and p_1 such that $f(p_0)f(p_1) < 0$.

At the beginning we let $A = p_0$ and $B = p_1$ as we did in bisection method.

At each step we let after p_0, ..., p_n have been found and A and B are assigned, we let $p_{n+1} = \frac{f(B)A - Bf(A)}{f(B) - f(A)}$ as we do in the secant method.

- If $f(p_{n+1})f(A) < 0$ then we let $B = p_{n+1}$ and we keep the value of A unchanged.

The Method of False Position (Regula Falsi)

We start with two initial guesses p_0 and p_1 such that $f(p_0)f(p_1) < 0$.

At the beginning we let $A = p_0$ and $B = p_1$ as we did in bisection method.

At each step we let after p_0, ..., p_n have been found and A and B are assigned, we let $p_{n+1} = \frac{f(B)A - Bf(A)}{f(B) - f(A)}$ as we do in the secant method.

- If $f(p_{n+1})f(A) < 0$ then we let $B = p_{n+1}$ and we keep the value of A unchanged.
- If $f(p_{n+1})f(B) < 0$ then we let $A = p_{n+1}$ and we keep the value of B unchanged.

The Method of False Position (Regula Falsi)

We start with two initial guesses p_0 and p_1 such that $f(p_0)f(p_1) < 0$.

At the beginning we let $A = p_0$ and $B = p_1$ as we did in bisection method.

At each step we let after p_0, ..., p_n have been found and A and B are assigned, we let $p_{n+1} = \frac{f(B)A - Bf(A)}{f(B) - f(A)}$ as we do in the secant method.

- If $f(p_{n+1})f(A) < 0$ then we let $B = p_{n+1}$ and we keep the value of A unchanged.
- If $f(p_{n+1})f(B) < 0$ then we let $A = p_{n+1}$ and we keep the value of B unchanged.
- If $f(p_{n+1}) = 0$ we announce $\alpha = p_{n+1}$ as the solution of $f(x) = 0$ and we terminate the process.

The Method of False Position (Regula Falsi)

We start with two initial guesses p_0 and p_1 such that $f(p_0)f(p_1) < 0$.

At the beginning we let $A = p_0$ and $B = p_1$ as we did in bisection method.

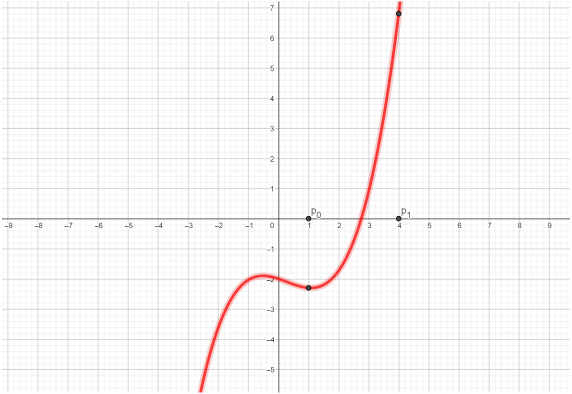
At each step we let after p_0, ..., p_n have been found and A and B are assigned, we let $p_{n+1} = \frac{f(B)A - Bf(A)}{f(B) - f(A)}$ as we do in the secant method.

- If $f(p_{n+1})f(A) < 0$ then we let $B = p_{n+1}$ and we keep the value of A unchanged.
- If $f(p_{n+1})f(B) < 0$ then we let $A = p_{n+1}$ and we keep the value of B unchanged.
- If $f(p_{n+1}) = 0$ we announce $\alpha = p_{n+1}$ as the solution of $f(x) = 0$ and we terminate the process.

Notice that in every step $f(A)$ and $f(B)$ will remain two non zero numbers with opposite signs and the above three options are the only possibilities.

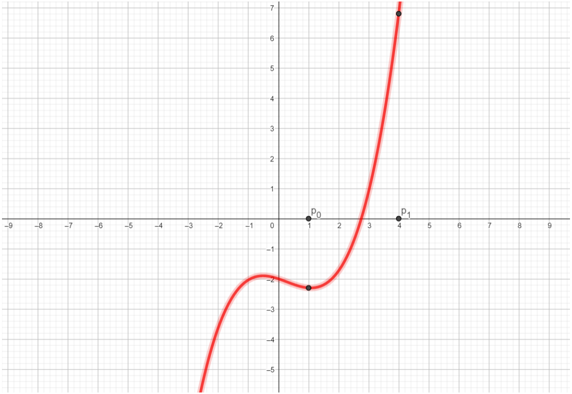
Example

Suppose $f(x) = \frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2$. Starting with $p_0 = 1$ and $p_1 = 4$ approximate a root of $f(x)$ using three steps of the **method of false position** to generate p_2, p_3, and p_4.



Suppose $f(x) = \frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2$. Starting with $p_0 = 1$ and $p_1 = 4$ approximate a root of $f(x)$ using three steps of the **method of false position** to generate p_2, p_3, and p_4.

$f(p_0) = -\frac{23}{10} < 0$ and $f(p_1) = \frac{34}{5} > 0$

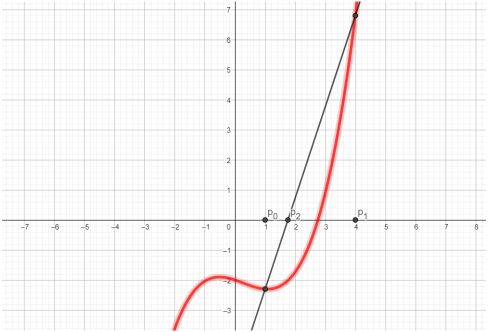


Example

Suppose $f(x) = \frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2$. Starting with $p_0 = 1$ and $p_1 = 4$ approximate a root of $f(x)$ using three steps of the **method of false position** to generate p_2, p_3, and p_4.

$f(p_0) = -\frac{23}{10} < 0$ and $f(p_1) = \frac{34}{5} > 0$

$p_2 = \frac{f(p_1)p_0 - p_1 f(p_0)}{f(p_1) - f(p_0)} \approx 1.758241758 \qquad f(p_2) \approx -2.0142 < 0$



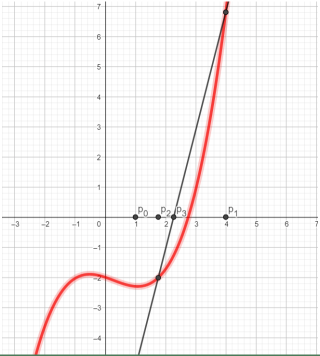
Example

Suppose $f(x) = \frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2$. Starting with $p_0 = 1$ and $p_1 = 4$ approximate a root of $f(x)$ using three steps of the **method of false position** to generate p_2, p_3, and p_4.

$f(p_0) = -\frac{23}{10} < 0$ and $f(p_1) = \frac{34}{5} > 0$

$p_2 = \frac{f(p_1)p_0 - p_1 f(p_0)}{f(p_1) - f(p_0)} \approx 1.758241758 \qquad f(p_2) \approx -2.0142 < 0$

$p_3 = \frac{f(p_2)p_1 - p_2 f(p_1)}{f(p_2) - f(p_1)} \approx 2.270528038 \qquad f(p_3) \approx -1.2750 < 0$



Example

Suppose $f(x) = \frac{x^3}{5} - \frac{x^2}{6} - \frac{x}{3} - 2$. Starting with $p_0 = 1$ and $p_1 = 4$ approximate a root of $f(x)$ using three steps of the **method of false position** to generate p_2, p_3, and p_4.

$f(p_0) = -\frac{23}{10} < 0$ and $f(p_1) = \frac{34}{5} > 0$

$p_2 = \frac{f(p_1)p_0 - p_1 f(p_0)}{f(p_1) - f(p_0)} \approx 1.758241758 \qquad f(p_2) \approx -2.0142 < 0$

$p_3 = \frac{f(p_2)p_1 - p_2 f(p_1)}{f(p_2) - f(p_1)} \approx 2.270528038 \qquad f(p_3) \approx -1.2750 < 0$

$p_4 = \frac{f(p_3)p_1 - p_3 f(p_1)}{f(p_3) - f(p_1)} \approx 2.543604255$