(2.2) Fixed-Point Iteration

MATH 4701 Numerical Analysis

Fixed Points and Solving Equations

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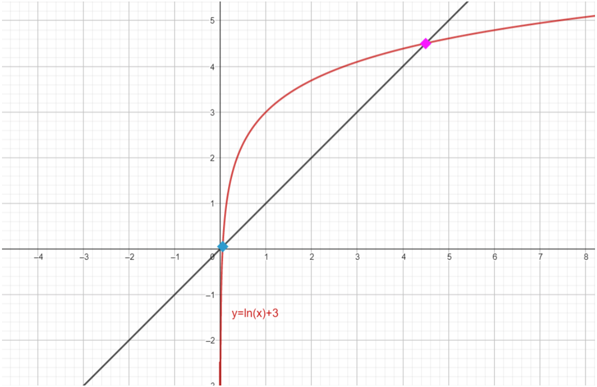

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On the other hand, we can define $g(x) = \frac{2}{x}$ and find fixed points of $g(x)$ $(\frac{2}{x} = x \Leftrightarrow x^2 - 2 = 0)$.

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On the other hand, we can define $g(x) = \frac{2}{x}$ and find fixed points of $g(x)$ ($\frac{2}{x} = x \Leftrightarrow x^2 - 2 = 0$).

Or we can define $g(x) = \frac{x^2+2}{2x}$ and find fixed points of $g(x)$ ($\frac{x^2+2}{2x} = x \Leftrightarrow x^2 + 2 = 2x^2 \Leftrightarrow x^2 - 2 = 0$).

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For example, for $g(x) = x^2 + x - 2$ with $x_0 = 4$, we get

$$x_0 = 4 \qquad x_1 = g(x_0) = 18 \qquad x_2 = g(x_1) = 340 \qquad x_3 = g(x_2) = 115938$$

diverges to $+\infty$.

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For $g(x) = \frac{2}{x}$ with $x_0 = 4$, we get

$$x_0 = 4 \qquad x_1 = g(x_0) = 0.5 \qquad x_2 = g(x_1) = 4 \qquad x_3 = g(x_2) = 0.5$$

forms a divergent periodic cycle.

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When this sequence converges, its limit is a fixed point of g.

But it is possible that this sequence does not converge or it converges slowly.

But for $g(x) = \frac{x^2+2}{2x}$ with $x_0 = 4$, we get

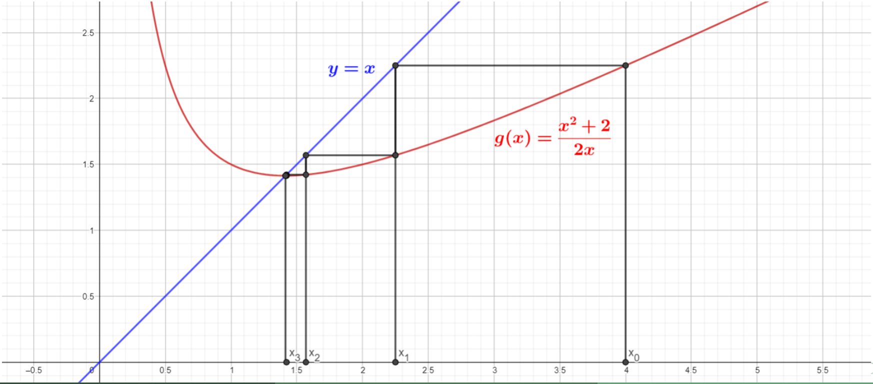
$$x_0 = 4 \qquad x_1 = g(x_0) = \frac{9}{4} = 2.25 \qquad x_2 = g(x_1) = \frac{113}{72} = 1.569\bar{4}$$

$$x_3 = g(x_2) = \frac{23137}{16272} \approx 1.42189036 \qquad x_4 = g(x_3) \approx 1.414234286$$

Converges to $\sqrt{2} \approx 1.41421356$ which is a fixed point of $g(x) = \frac{x^2+2}{2x}$ and a solution of $f(x) = x^2 - 2 = 0$.

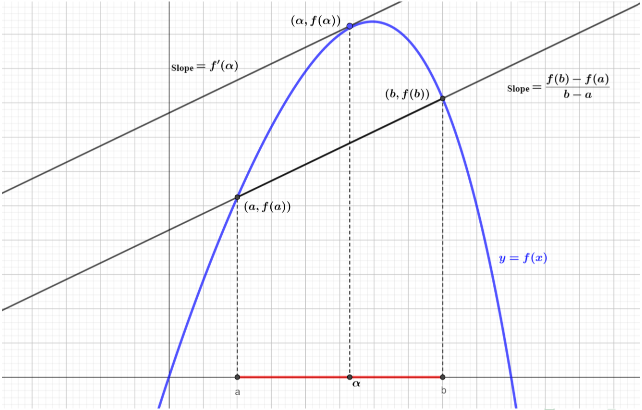
For a function $g : [a, b] \to \mathbb{R}$, and an **initial guess** p_0, we can consider the sequence of iterations of g defined by $x_0 = p_0$ and $x_{n+1} = g(x_n)$.

The sequence can be geometrically traced by starting at the value of any x_n on the x-axis, drawing a vertical line until we hit the graph of g, from that point drawing a horizontal line to hit the line $y = x$, and finally drawing a vertical line until we hit the x-axis at the next number x_{n+1}.



Mean Value Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there is $\alpha \in (a, b)$ such that

$$f'(\alpha) = \frac{f(b) - f(a)}{b - a}$$



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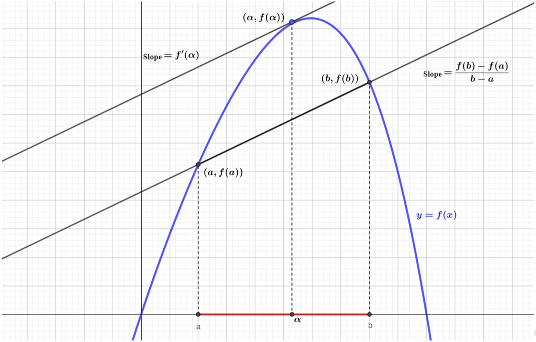
In other words, on the graph of the function, there is a point in between the end-points where the slope of the tangent line is equal to the slope of the secant line through the two end-points.



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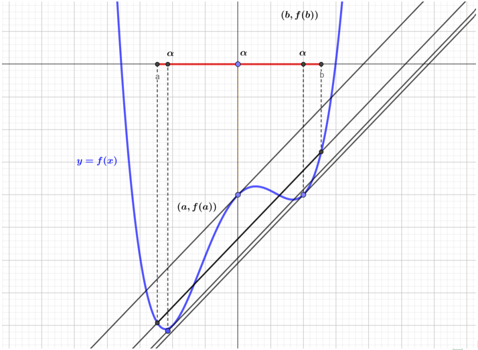
Notice that the value α may not be unique and the theorem only claims the existence of at least one such value.



Mean Value Theorem and Estimate on Distance

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In addition to being one of the most important theoretical theorems that is used in proving numerous other theorems in Calculus, the Mean Value Theorem allows you to estimate how much the distance between two numbers changes after you apply the function f to both of them.

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An Important Consequence of the Mean Value Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), and if there is a positive number K such that for all $x \in (a, b)$, $|f'(x)| \leq K$, then for any x and y in $[a, b]$,

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Notice that by the Mean Value Theorem, there is α between x and y such that

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$$\Rightarrow \frac{|f(x) - f(y)|}{|x - y|} = |f'(\alpha)| \leq K \Rightarrow |f(x) - f(y)| \leq K|x - y|$$

Existence and Uniqueness Theorem for Fixed Points:

(1) If $g : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ ($g \in C[a, b]$) and $g(x) \in [a, b]$ for all $x \in [a, b]$ ($g([a, b]) \subset [a, b]$) then g has at least one fixed point in $[a, b]$.

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(2) If in addition to conditions of part (1) we also assume g is differentiable on (a, b) and if there is a positive number $K < 1$ such that for all $x \in (a, b)$, $|g'(x)| \leq K$, then g has **exactly one fixed point** inside $[a, b]$.

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To see why the assertion of this part is true, notice that if p and q are both fixed points of g and they are both in $[a, b]$, by the **Mean Value Theorem**,
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$|p - q| = |g(p) - g(q)| \leq K|p - q| \Rightarrow 1 \leq K$ which contradicts the assumption on K being strictly less than 1.

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Proof: Since $g([a, b]) \subset [a, b]$, all the entries p_n in the sequence are defined and they are all in $[a, b]$.

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Proof: Since $g([a, b]) \subset [a, b]$, all the entries p_n in the sequence are defined and they are all in $[a, b]$. Moreover, by the theorem we just proved g has a unique fixed point p in $[a, b]$.

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We have seen that $g(x)$ is continuous on $[1, 8]$, differentiable on $(1, 8)$, $g([1, 8]) \subset [1, 8]$ and $|g'(x)| \leq \frac{1}{2}$ for all $x \in [1, 8]$.

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Since $K = \frac{1}{2} < 1$ by the two theorems we discussed, g has a unique fixed point p in $[1, 8]$ and the sequence of iterations $g(p_{n+1}) = g(p_n)$ converges to p for any choice of p_0 in $[1, 8]$.

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  Distance between p and p_0 which is within $[a, b]$ is less than the larger value of the distance between p and the two end-points of $[a, b]$.

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(Triangle Inequality)

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&\leq K^{m-1}|p_1 - p_0| + K^{m-2}|p_1 - p_0| + \ldots + K^n|p_1 - p_0| \\
&= (K^{m-1} + K^{m-2} + \ldots + K^n)|p_1 - p_0|
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Suppose a function g satisfies the conditions of the fixed point iteration theorem on the interval $[a, b]$.

- For all $n \geq 1$, $|p_n - p| \leq K^n \max\{p_0 - a, b - p_0\}$
- For all $n \geq 1$, $|p_n - p| \leq \frac{K^n}{1-K}|p_1 - p_0|$

  where K is a **positive constant strictly less than** 1 such that $|g'(x)| \leq K$ for all x in $[a, b]$.

For all $m > n \geq 1$

$$|p_m - p_n| = |p_m - p_{m-1} + p_{m-1} - p_{m-2} + \dots + p_{n+1} - p_n|$$

$$\begin{aligned}
&\leq |p_m - p_{m-1}| + |p_{m-1} + p_{m-2}| + \dots + |p_{n+1} - p_n| \\
&\leq K^{m-1}|p_1 - p_0| + K^{m-2}|p_1 - p_0| + \dots + K^n|p_1 - p_0| \\
&= (K^{m-1} + K^{m-2} + \dots + K^n)|p_1 - p_0| \leq (\textstyle\sum_{i=n}^{\infty} K^i)|p_1 - p_0|
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$\sum_{i=0}^{\infty} \alpha r^i = \frac{\alpha}{1-r}$ when $|r| < 1$.

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For all $m > n \geq 1$ $|p_m - p_n| \leq \frac{K^n}{1-K}|p_1 - p_0|$ and $\lim_{m \longrightarrow \infty} p_m = p$ implies $|p - p_n| = |p_n - p| \leq \frac{K^n}{1-K}|p_1 - p_0|$.

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The second inequality is more useful when the original interval $[a, b]$ has a large length and when we might have already started close to the actual fixed point. The second inequality can also be used with replacing the initial guess with a number in the sequence that is already close to the fixed point. By restarting the iteration at such number, we can have a better estimate on how many more iterations are needed to improve the accuracy of our estimate of the fixed point.

Example

For $g(x) = \frac{x^2+2}{2x}$ on the interval $[1, 8]$, if we start with $p_0 = 3$, how large should n be so that the nth iterate p_n is within 0.0001 of the fixed point p of $g(x)$?

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$$\frac{5}{2^n} < 0.0001 \Leftrightarrow \ln(50000) < n \ln(2) \Leftrightarrow n > \frac{\ln(50000)}{\ln(2)} \approx 15.610$$

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Therefore, using the first estimate, we conclude that p_{16} will be within 0.0001 of the fixed point p.

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Using the second estimate we get

$$|p_n - p| \leq \frac{K^n}{1-K}|p_1 - p_0| = \frac{(\frac{1}{2})^n}{1-\frac{1}{2}}|3 - \frac{11}{6}|$$

$p_1 = g(p_0) = g(3) = \frac{11}{6}$

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For $g(x) = \frac{x^2+2}{2x}$ on the interval $[1, 8]$, if we start with $p_0 = 3$, how large should n be so that the nth iterate p_n is within 0.0001 of the fixed point p of $g(x)$?

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$$|p_n - p| \leq \frac{K^n}{1-K}|p_1 - p_0| = \frac{(\frac{1}{2})^n}{1-\frac{1}{2}}|3 - \frac{11}{6}| = \frac{7}{2^n \times 3}$$

So to make sure $|p_n - p| < 0.0001$, we can choose n large enough so that

$$\frac{7}{2^n \times 3} < 0.0001$$

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So to make sure $|p_n - p| < 0.0001$, we can choose n large enough so that

$$\frac{7}{2^n \times 3} < 0.0001 \Leftrightarrow \frac{70000}{3} < 2^n$$

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For $g(x) = \frac{x^2+2}{2x}$ on the interval $[1, 8]$, if we start with $p_0 = 3$, how large should n be so that the nth iterate p_n is within 0.0001 of the fixed point p of $g(x)$?

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So to make sure $|p_n - p| < 0.0001$, we can choose n large enough so that

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So to make sure $|p_n - p| < 0.0001$, we can choose n large enough so that

$$\frac{7}{2^n \times 3} < 0.0001 \Leftrightarrow \ln(\frac{70000}{3}) < n\ln(2) \Leftrightarrow n > \frac{\ln(\frac{70000}{3})}{\ln(2)} \approx 14.511$$

Therefore, using the second estimate, we conclude that p_{15} will be within 0.0001 of the fixed point p.