Gräffe's Root Squaring Method

MATH 4701 Numerical Analysis

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For all $n \geq 0$ we have

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$$P_2(x) = x^3 - (\alpha_1^4 + \alpha_2^4 + \alpha_3^4)x^2 + (\alpha_1^4\alpha_2^4 + \alpha_1^4\alpha_3^4 + \alpha_2^4\alpha_3^4)x - (\alpha_1^4\alpha_2^4\alpha_3^4)$$

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Example

Let $P(x) = x^3 - 4x^2 + 3x + 1$. Suppose α_1, α_2, and α_3 are the three roots of $P(x)$ and $|\alpha_1| > |\alpha_2| > |\alpha_3|$. In particular,

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Since $|\alpha_1| > |\alpha_2| \geq |\alpha_3|$, $\lim_{n\to\infty} \left|\frac{\alpha_2}{\alpha_1}\right|^{2^n} = 0$ and $\lim_{n\to\infty} \left|\frac{\alpha_3}{\alpha_1}\right|^{2^n} = 0$.

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$$= \lim_{n\to\infty} |\alpha_2| \sqrt[2^n]{\left|\frac{1+(\frac{\alpha_3}{\alpha_2})^{2^n}+(\frac{\alpha_3}{\alpha_1})^{2^n}}{1+(\frac{\alpha_2}{\alpha_1})^{2^n}+(\frac{\alpha_3}{\alpha_1})^{2^n}}\right|} = |\alpha_2| \sqrt[2^n]{\left|\frac{1+0+0}{1+0+0}\right|} = |\alpha_2|$$

Example

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Gräffe's Root Squaring Method

Suppose $P(x) = x^N + a_{N-1}x^{N-1} + ... + a_1 x + a_0$ is a polynomial of degree N with real roots α_1, α_2, ..., α_N such that $|\alpha_1| > |\alpha_2| > ... > |\alpha_N|$.

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Then $\sigma_1^{(n)} = \sum_{1 \leq i \leq N} \alpha_i^{2^n}$, $\sigma_2^{(n)} = \sum_{1 \leq i < j \leq N} \alpha_i^{2^n} \alpha_j^{2^n}$, Suppose $P(x) = x^N + a_{N-1}x^{N-1} + ... + a_1x + a_0$ is a polynomial of degree N with real roots α_1, α_2, ..., α_N such that $|\alpha_1| > |\alpha_2| > ... > |\alpha_N|$. Letting $P_0(x) = P(x)$ and defining $P_{n+1}(w) = (-1)^N P(x)P(-x)$ where $w = x^2$ we have

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Moreover, $\lim_{n\to\infty} \sqrt[2^n]{|\sigma_1^{(n)}|} = |\alpha_1|$ and $\lim_{n\to\infty} \sqrt[2^n]{\frac{|\sigma_k^{(n)}|}{|\sigma_{k-1}^{(n)}|}} = |\alpha_k|$ for

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Example

Approximate the roots of $P(x) = x^4 + 6x^3 + 7x^2 - 7x - 4$ using four steps of the root squaring method.

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$P_1(x) = x^4 - 22x^3 + 125x^2 - 105x + 16$

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Approximate the roots of $P(x) = x^4 + 6x^3 + 7x^2 - 7x - 4$ using four steps of the root squaring method.

$P_1(x) = x^4 - 22x^3 + 125x^2 - 105x + 16$

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$P(2.829731735) \approx 232.3141638$ and $P(-2.829731735) \approx 0.02550304$
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$P(0.8822991819) \approx 0.000020487$ and $P(-0.8822991819) \approx 4.110279912$
therefore, $\alpha_3 \approx 0.8822991819$.

Example

Approximate the roots of $P(x) = x^4 + 6x^3 + 7x^2 - 7x - 4$ using four steps of the root squaring method.

$P_4(x) = x^4 - 831056762x^3 + 140460733053509049x^2 - 1894126393761729x + 4294967296$

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Example

Approximate the roots of $P(x) = x^4 + 6x^3 + 7x^2 - 7x - 4$ using four steps of the root squaring method.

$P_4(x) = x^4 - 831056762x^3 + 14046073305350949x^2 - 1894126393761729x + 4294967296$

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$\alpha_3 \approx 0.8822991819.$
$|\alpha_4| \approx \sqrt[16]{|\frac{\sigma_4^{(4)}}{\sigma_3^{(4)}}|} = \sqrt[16]{\frac{4294967296}{1894126393761729}} \approx 0.4438353932$
$P(0.4438353932) \approx -5.164527333$ and
$P(-0.4438353932) \approx -4.672 \times 10^{-6}$ therefore.

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Fundamental Theorem of Symmetric Polynomials

Suppose

$$P(x) = (x - \alpha_1)(x - \alpha_2)...(x - \alpha_N) = x^N + \sigma_1 x^{N-1} + ... + \sigma_{N-1} x + \sigma_N$$

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Fundamental Theorem of Symmetric Polynomials: Every **symmetric** polynomial in variables α_1, α_2, ..., α_N can be written in a unique way as a polynomial in σ_1, σ_2, ..., σ_N.

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A polynomial in variables x_1, x_2, ..., x_m is symmetric if it is left unchanged after the variables x_1, x_2, ..., x_m are permuted for every permutation of the m variables.

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For example, for $N = 3$, $\sigma_1 = \alpha_1 + \alpha_2 + \alpha_3$, $\sigma_2 = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1$, and $\sigma_3 = \alpha_1\alpha_2\alpha_3$.

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Then $\alpha_1^2 + \alpha_2^2 + \alpha_3^2$ is a symmetric polynomial in the three variables α_1, α_2, and α_3.

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$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = (\alpha_1 + \alpha_2 + \alpha_3)^2 - 2(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1) = \sigma_1^2 - 2\sigma_2$.