(2.5) Accelerating Convergence

MATH 4701 Numerical Analysis

Aitken's Δ^2 Method

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Aitken's Δ^2 method assumes that the sequence $(\widehat{p_n})$ defined by $\widehat{p_n} = p_n - \frac{(p_{n+1}-p_n)^2}{p_n+p_{n+2}-2p_{n+1}}$ converges to p faster than the original sequence (p_n).

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Theorem: Suppose $\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = \lambda$ such that $0 < \lambda < 1$. Then $\lim_{n\to\infty} \frac{|\widehat{p_n}-p|}{|p_n-p|} = 0$.

Examples

If $p_n = \frac{1}{2^n}$, then

$$\widehat{p_n} = p_n - \frac{(p_{n+1} - p_n)^2}{p_n + p_{n+2} - 2p_{n+1}}$$

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So for $p_n = \frac{1}{2^n}$ which converges to zero linearly, the sequence $(\widehat{p_n})$ is constant zero sequence.

Examples

Suppose $g(x) = \frac{x^2+2}{2x}$. Let $p_0 = 4$ and $p_{n+1} = g(p_n)$. Find the first five entries in the sequence $(\widehat{p_n})$.

Δ Notation

If (a_n) is a sequence of numbers, Δa_n is the sequence of **forward differences** of consecutive terms of (a_n) defined by

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Therefore, Aitken's Δ^2 accelerated sequence can be described by

$$\widehat{p_n} = p_n - \frac{(p_{n+1} - p_n)^2}{p_n + p_{n+2} - 2p_{n+1}} = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n}$$

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Staring with an initial guess p_0 we find $p_1 = g(p_0)$ and $p_2 = g(p_1)$.

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At this point, using these three entries we find the Aitken's Δ^2 entry $\widehat{p_0}$ and we call it $p_0^{(1)}$.

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At this point, using these three entries we find the Aitken's Δ^2 entry $\widehat{p_0}$ and we call it $p_0^{(1)}$.

We then use $p_0^{(1)}$ as an initial guess to generate $p_1^{(1)} = g(p_0^{(1)})$ and $p_2^{(1)} = g(p_1^{(1)})$.

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In every step, once $p_0^{(n)}$ is generated we use it as an initial guess to generate $p_1^{(n)} = g(p_0^{(n)})$ and $p_2^{(n)} = g(p_1^{(n)})$.

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Example: Suppose $g(x) = \frac{x^2+2}{2x}$. Let $p_0 = 4$ and use the Steffensen's method to approximate a fixed point of $g(x)$ accurate to an error of at most 10^{-6}.