MATH 4701 Numerical Analysis

Problem Set #4

(1) **a:** Find the general solution of the linear difference equation $u_k - 3u_{k-1} - 4u_{k-2} = 0$.

b: Write the first 6 entries in the solution of the linear difference equation $u_k - 3u_{k-1} - 4u_{k-2} = 0$ with initial conditions $u_0 = 1$ and $u_1 = -2$.

c: Find an explicit (non-recursive) formula for the solution of the linear difference equation $u_k - 2u_{k-1} - 3u_{k-2} = 0$ with initial conditions $u_0 = 1$ and $u_1 = -2$.

a: The characteristic polynomial for this equation is $\lambda^2 - 3\lambda - 4$ which has two roots $\lambda_1 = 4$ and $\lambda_2 = -1$. Therefore the two solutions $u_n = 4^n$ and $v_n = (-1)^n$ generate the two-dimensional solution space. Consequently, any solution of this difference equation is in the form $x_n = \alpha 4^n + \beta(-1)^n$.

b: $u_k - 3u_{k-1} - 4u_{k-2} \Rightarrow u_k = 3u_{k-1} + 4u_{k-2}$. So we have $u_0 = 1$, $u_1 = -2$, $u_2 = 3u_1 + 4u_0 = -2$, $u_3 = 3u_2 + 4u_1 = -14$, $u_4 = 3u_3 + 4u_2 = -50$, $u_5 = 3u_4 + 4u_3 = -206$.

c: The characteristic polynomial for this equation is $x^2 - 2x - 3$ which has two roots $x_1 = 3$ and $x_2 = -1$. Therefore the two solutions $v_n = 3^n$ and $v'_n = (-1)^n$ generate the two-dimensional solution space. Consequently, any solution of this difference equation is in the form $u_n = \alpha 3^n + \beta(-1)^n$.

$$\begin{cases} \alpha 3^0 + \beta(-1)^0 = u_0 = 1 \\ \alpha 3^1 + \beta(-1)^1 = u_1 = -2 \end{cases} \Rightarrow \begin{cases} \alpha + \beta = 1 \\ 3\alpha - \beta = -2 \end{cases}$$

$$\Rightarrow 4\alpha = -1 \Rightarrow \alpha = -\frac{1}{4} \Rightarrow \beta = 1 - \alpha = \frac{5}{4}$$

$$\Rightarrow u_n = -\frac{1}{4}3^n + \frac{5}{4}(-1)^n = -\frac{3^n}{4}3^n + \frac{(-1)^n 5}{4} = \frac{-3^n + (-1)^n 5}{4}$$

(For the equation in part (b) we have

$$\begin{cases} \alpha 4^0 + \beta(-1)^0 = u_0 = 1 \\ \alpha 4^1 + \beta(-1)^1 = u_1 = -2 \end{cases} \Rightarrow \begin{cases} \alpha + \beta = 1 \\ 4\alpha - \beta = -2 \end{cases}$$

$$\Rightarrow 5\alpha = -1 \Rightarrow \alpha = -\frac{1}{5} \Rightarrow \beta = 1 - \alpha = \frac{6}{5}$$

$$\Rightarrow u_n = -\frac{1}{5}4^n + \frac{6}{5}(-1)^n = -\frac{4^n}{5} + \frac{(-1)^n 6}{5} = \frac{-4^n + (-1)^n 6}{5}$$

(2) **a:** Find the general solution of the linear difference equation $u_k - 3u_{k-1} + 3u_{k-2} - u_{k-3} = 0$.

b: Write the first 6 entries in the solution of the linear difference equation $u_k - 3u_{k-1} + 3u_{k-2} - u_{k-3} = 0$ with initial conditions $u_0 = -1$, $u_1 = 0$, $u_2 = 1$.

c: Find an explicit (non-recursive) formula for the solution of the linear difference equation $u_k - 3u_{k-1} + 3u_{k-2} - u_{k-3} = 0$ with initial conditions $u_0 = -1$, $u_1 = 0$, $u_2 = 1$.

a: The characteristic polynomial for this equation is $\lambda^3 - 3\lambda^2 + 3\lambda - 1$ which has one root of multiplicity three $\lambda = 1$. Therefore the three solutions $u_n = 1^n = 1$, $v_n = n(1)^n = n$, and $w_n = n^2(1)^n = n^2$ generate the two-dimensional solution space. Consequently, any solution of this difference equation is in the form $x_n = \alpha + \beta n + \gamma n^2$.

b: $u_k - 3u_{k-1} - 4u_{k-2} \Rightarrow u_k = 3u_{k-1} - 3u_{k-2} + u_{k-3}$. So we have $u_0 = -1$, $u_1 = 0$, $u_2 = 1$, $u_3 = 3u_2 - 3u_1 + u_0 = 2$, $u_4 = 3u_3 - 3u_2 + u_1 = 3$, $u_5 = 3u_4 - 3u_3 + u_2 = 4$.

c:

$$\begin{cases} \alpha + \beta(0) + \gamma(0)^2 = u_0 = -1 \Rightarrow \alpha = -1 \\ \alpha + \beta(1) + \gamma(1)^2 = u_1 = 0 \Rightarrow \beta + \gamma = 1 \alpha + \beta(2) + \gamma(2)^2 = u_1 = 1 \Rightarrow 2\beta + 4\gamma = 2 \end{cases}$$

$\Rightarrow \beta + 2\gamma = 1 \Rightarrow \gamma = 0$ and $\beta = 1$. Therefore, $u_n = n - 1$.

(3) Use Bernoulli's method to estimate the root of polynomial $P(x) = x^3 - 5x^2 - 4x + 7$ with largest absolute value accurate to four decimals.

$P(x)$ is the characteristic polynomial for the difference equation $x_k - 5x_{k-1} - 4x_{k-2} + 7x_{k-3} = 0$. We first find suitable initial conditions for this equation to make sure the coefficient of the root of $P(x)$ we are looking for is not going to be zero.

$a_3x_1 + (1)a_2 = 0 \Rightarrow x_1 - 5 = 0 \Rightarrow x_1 = 5$

$a_3x_2 + a_2x_1 + 2a_1 = 0 \Rightarrow x_2 - 5x_1 + 2(-4) = 0 \Rightarrow x_2 = 25 + 8 = 33$

$a_3x_3 + a_2x_2 + a_1x_1 + 3a_0 = 0 \Rightarrow x_3 - 5x_2 - 4x_1 + 3(7) = 0 \Rightarrow x_3 = 165 + 20 - 21 = 164$

$u_1 = \frac{x_2}{x_1} = \frac{33}{5} = 6.6$
$u_2 = \frac{x_3}{x_2} = \frac{164}{33} = 4.9697$

$x_4 - 5x_3 - 4x_2 + 7x_1 = 0 \Rightarrow x_4 = 5(164) + 4(33) - 7(5) = 917$
$u_3 = \frac{x_4}{x_3} = \frac{917}{164} = 5.59146$

$x_5 - 5x_4 - 4x_3 + 7x_2 = 0 \Rightarrow x_5 = 5(917) + 4(164) - 7(33) = 5010$
$u_4 = \frac{x_5}{x_4} = \frac{5010}{917} = 5.46347$

$x_6 - 5x_5 - 4x_4 + 7x_3 = 0 \Rightarrow x_6 = 5(5010) + 4(917) - 7(164) = 27570$
$u_5 = \frac{x_6}{x_5} = \frac{27570}{5010} = 5.50299$

$x_7 - 5x_6 - 4x_5 + 7x_4 = 0 \Rightarrow x_7 = 5(27570) + 4(5010) - 7(917) = 151471$
$u_6 = \frac{x_7}{x_6} = \frac{151471}{27570} = 5.49405$

$x_8 - 5x_7 - 4x_6 + 7x_5 = 0 \Rightarrow x_8 = 5(151471) + 4(27570) - 7(5010) = 832565$
$u_7 = \frac{x_8}{x_7} = \frac{832565}{151471} = 5.49653$

$x_9 - 5x_8 - 4x_7 + 7x_6 = 0 \Rightarrow x_9 = 5(832565) + 4(151471) - 7(27570) = 4575719$
$u_8 = \frac{x_9}{x_8} = \frac{4575719}{832565} = 5.49593$

$x_{10} - 5x_9 - 4x_8 + 7x_7 = 0 \Rightarrow x_{10} = 5(4575719) + 4(832565) - 7(151471) = 25148558$
$u_9 = \frac{x_{10}}{x_9} = \frac{25148558}{4575719} = 5.49609$

$x_{11} - 5x_{10} - 4x_9 + 7x_8 = 0 \Rightarrow x_{11} = 5(25148558) + 4(4575719) - 7(832565) = 138217711$
$u_{10} = \frac{x_{11}}{x_{10}} = \frac{138217711}{25148558} = 5.49605$

Therefore, the root of $P(x) = x^3 - 5x^2 - 4x + 7$ with largest absolute value with accuracy of within 10^{-4} is 5.4960.

(4) **a:** Suppose the roots of polynomial $P(x) = x^3 + a_2x^2 + a_1x + a_0$ are α, β, and γ. Find $S_1 = \alpha + \beta + \gamma$, $S_2 = \alpha^2 + \beta^2 + \gamma^2$, and $S_3 = \alpha^3 + \beta^3 + \gamma^3$ in terms of a_0, a_1, and a_2.

b: Show that if $P(x)$ has three distinct real roots and the initial values of the linear difference equation $x_k + a_2x_{k-1} + a_1x_{k-2} + a_0x_{k-3} = 0$ are chosen using $x_1 + a_2 = 0$, $x_2 + a_2x_1 + 2a_1 = 0$, $x_3 + a_2x_2 + a_1x_1 + 3a_0 = 0$, then the general solution of the equation is $x_k = \alpha^k + \beta^k + \gamma^k$.

a: We know $a_2 = -(\alpha+\beta+\gamma) = -S_1$, $a_1 = \alpha\beta+\alpha\gamma+\beta\gamma$, and $a_0 = -\alpha\beta\gamma$. Therefore, $\mathbf{S_1 = -a_2}$. Moreover,

$a_2^2 = (\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma = S_2 + 2a_1 \Rightarrow \mathbf{S_2 = a_2^2 - 2a_1}$

$-a_2^3 = (\alpha + \beta + \gamma)^3 = \alpha^3 + \beta^3 + \gamma^3 + 3\alpha\beta^2 + 3\alpha\gamma^2 + 3\beta\alpha^2 + 3\beta\gamma^2 + 3\gamma\alpha^2 + 3\gamma\beta^2 + 6\alpha\beta\gamma =$

$= S_3 + 3(\beta\alpha^2 + \gamma\alpha^2 + \alpha\beta\gamma) + 3(\alpha\beta^2 + \gamma\beta^2 + \alpha\beta\gamma) + 3(\alpha\gamma^2 + \beta\gamma^2 + \alpha\beta\gamma) - 3\alpha\beta\gamma =$

$= S_3 + 3\alpha a_1 + 3\beta a_1 + 3\gamma a_1 + 3a_0 = S_3 + 3a_1(\alpha + \beta + \gamma) + 3a_0 = S_3 - 3a_1a_2 + 3a_0 \Rightarrow \mathbf{S_3 = -a_2^3 + 3a_1a_2 - 3a_0}$.

b: The linear difference equation $x_k + a_2 x_{k-1} + a_1 x_{k-2} + a_0 x_{k-3}$ has characteristic polynomial $P(x) = x^3 + a_2 x^2 + a_1 x + a_0$ which has three distinct real roots α, β, and γ. Therefore, the general solution for this difference equation can be represented as $x_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n$. If we generate the initial conditions using the given relations we have

$$x_1 + a_2 = 0 \Rightarrow x_1 = -a_2 = S_1$$

$$x_2 + a_2 x_1 + 2a_1 = 0 \Rightarrow x_2 = a_2^2 - 2a_1 = S_2$$

$$x_3 + a_2 x_2 + a_1 x_1 + 3a_0 = 0 \Rightarrow x_3 = -a_2(a_2^2 - 2a_1) - a_1(-a_2) - 3a_0 = -a_2^3 + 3a_1 a_2 - 3a_0 = S_3$$

This yields the following system of linear equations in three variables c_1, c_2, and c_3:

$$\begin{cases} c_1\alpha + c_2\beta + c_3\gamma = x_1 = S_1 = \alpha + \beta + \gamma \\ c_1\alpha^2 + c_2\beta^2 + c_3\gamma^2 = x_2 = S_2 = \alpha^2 + \beta^2 + \gamma^2 \\ c_1\alpha^3 + c_2\beta^3 + c_3\gamma^3 = x_3 = S_3 = \alpha^3 + \beta^3 + \gamma^3 \end{cases}$$

which has the obvious solution $c_1 = c_2 = c_3 = 1$. Therefore, the general solution for this difference equation can be represented as $x_k = \alpha^k + \beta^k + \gamma^k$.

(5) **a:** Use three steps of the root squaring method to estimate the absolute values of the three roots of polynomial $P(x) = x^3 + 3x^2 - 5x + 8$. Which ones of these absolute values belong to a real root of $P(x)$?

b: Use Newton's Method with an initial value of the approximate roots from part **a** to find real roots of $P(x)$ within 10^{-5} accuracy.

a:

$$-P(x)P(-x) = -(x^3 + 3x^2 - 5x + 8)(-x^3 + 3x^2 + 5x + 8) = x^6 - 19x^4 - 23x^2 - 64 \Rightarrow$$

$$\Rightarrow P_1(x) = x^3 - 19x^2 - 23x - 64$$

$$-P(x)_1 P_1(-x) = -(x^3 - 19x^2 - 23x - 64)(-x^3 - 19x^2 + 23x - 64) = x^6 - 407x^4 - 1903x^2 - 4096 \Rightarrow$$

$$\Rightarrow P_2(x) = x^3 - 407x^2 - 1903x - 4096$$

$$-P(x)_2 P_2(-x) = -(x^3 - 407x^2 - 1903x - 4096)(-x^3 - 407x^2 + 1903x - 4096) =$$

$$= x^6 - 169455x^4 + 287265x^2 - 16777216 \Rightarrow P_3(x) = x^3 - 169455x^2 + 287265x - 16777216$$

$$\Rightarrow \alpha^8 + \beta^8 + \gamma^8 = 169455 \Rightarrow |\alpha| \approx \sqrt[8]{\alpha^8 + \beta^8 + \gamma^8} = \sqrt[8]{169455} \approx 4.50435$$

$\Rightarrow \alpha^8\beta^8 + \alpha^8\gamma^8 + \beta^8\gamma^8 = 287265 \Rightarrow |\alpha\beta| \approx \sqrt[8]{\alpha^8\beta^8 + \alpha^8\gamma^8 + \beta^8\gamma^8} = \sqrt[8]{287265} \approx 4.81155$

$\Rightarrow |\beta| \approx \frac{4.81155}{4.50435} = 1.0682$

$|\alpha\beta\gamma| = 8 \Rightarrow |\gamma| = \frac{8}{|\alpha\beta|} \approx \frac{8}{4.81155} = 1.66267$
$P(4.50435) = 137.735$ and $P(-4.50435) = -0.0001624 \Rightarrow \alpha = -4.50435$

$P(1.0682) = 12.5764$ and $P(-1.0682) = 20.0103 \Rightarrow \beta$ is not a real root.

$\Rightarrow \gamma$ is not a real root either since non-real roots appear in pairs.

(Also since we got $|\beta| < |\gamma|$ as opposed to the assumption that β has the second largest absolute value, we know both roots should have equal absolute values since the calculation for $|\alpha\beta|$ was done assuming $|\beta| > |\gamma|$ which in this case is false.)

b: $\Gamma(x) = x - \frac{P(x)}{P'(x)} = x - \frac{x^3+3x^2-5x+8}{3x^2+6x-5}$
$x_1 = -4.50435$
$x_2 = \Gamma(x_1) = -4.504340$
$x_3 = \Gamma(x_2) = -4.504340$
Therefore, α to within 10^{-5} error is equal to -4.504340.

(6) Use the Multi-variable Newton-Raphson Method to estimate a point of intersection of the ellipse $3x^2 + 4xy + 2y^2 = 12$ with the hyperbola $x^2 - 5xy + 2y^2 = 9$ within an accuracy of 10^{-1} using an initial guess of $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

With $F(x, y) = (3x^2 + 4xy + 2y^2 - 12, x^2 - 5xy + 2y^2 - 9)$, we have

$$DF(x, y) = \begin{bmatrix} 6x + 4y & 4x + 4y \\ 2x - 5y & -5x + 4y \end{bmatrix}$$

We use the recursive formula

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - DF(x_n, y_n)^{-1} \begin{bmatrix} 3x_n^2 + 4x_ny_n + 2y_n^2 - 12 \\ x_n^2 - 5x_ny_n + 2y_n^2 - 9 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 6(0) + 4(2) & 4(0) + 4(2) \\ 2(0) - 5(2) & -5(0) + 4(2) \end{bmatrix}^{-1} \begin{bmatrix} 3(0)^2 + 4(0)(2) + 2(2)^2 - 12 \\ (0)^2 - 5(0)(2) + 2(2)^2 - 9 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ 2 \end{bmatrix} - \frac{1}{144} \begin{bmatrix} 8 & -8 \\ 10 & 8 \end{bmatrix} \begin{bmatrix} -4 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} - \begin{bmatrix} -\frac{24}{144} \\ -\frac{48}{144} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ \frac{7}{3} \end{bmatrix} = \begin{bmatrix} 0.166667 \\ 2.33333 \end{bmatrix}$$

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ \frac{7}{3} \end{bmatrix} - \begin{bmatrix} 6(\frac{1}{6})+4(\frac{7}{3}) & 4(\frac{1}{6})+4(\frac{7}{3}) \\ 2(\frac{1}{6})-5(\frac{7}{3}) & -5(\frac{1}{6})+4(\frac{7}{3}) \end{bmatrix}^{-1} \begin{bmatrix} 3(\frac{1}{6})^2+4(\frac{1}{6})(\frac{7}{3})+2(\frac{7}{3})^2-12 \\ (\frac{1}{6})^2-5(\frac{1}{6})(\frac{7}{3})+2(\frac{7}{3})^2-9 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{6} \\ \frac{7}{3} \end{bmatrix} - \begin{bmatrix} \frac{31}{3} & 10 \\ -\frac{34}{3} & \frac{17}{2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{19}{36} \\ -\frac{1}{36} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ \frac{7}{3} \end{bmatrix} - \frac{6}{1207} \begin{bmatrix} \frac{17}{2} & -10 \\ \frac{34}{3} & \frac{31}{3} \end{bmatrix} \begin{bmatrix} \frac{19}{36} \\ -\frac{1}{36} \end{bmatrix} \approx \begin{bmatrix} 0.142985 \\ 2.30503 \end{bmatrix}$$

So within an error of at most 10^{-1} the intersection point is $(0.14, 2.3)$.

(7) Use synthetic division to show that $x = -2$ is a root of $P(x) = x^6 + 10x^5 + 37x^4 + 56x^3 + 8x^2 - 64x - 48$ and find the multiplicity of this root.

$$\begin{array}{r|rrrrrrr}
-2 & 1 & 10 & 37 & 56 & 8 & -64 & -48 \\
 & & -2 & -16 & -42 & -28 & 40 & 48 \\
\hline
-2 & 1 & 8 & 21 & 14 & -20 & -24 & |\ 0 \\
 & & -2 & -12 & -18 & -8 & 24 & \\
\hline
-2 & 1 & 6 & 9 & -4 & -12 & |\ 0 & \\
 & & -2 & -8 & -2 & 12 & & \\
\hline
-2 & 1 & 4 & 1 & -6 & |\ 0 & & \\
 & & -2 & -4 & 6 & & & \\
\hline
-2 & 1 & 2 & -3 & |\ 0 & & & \\
 & & -2 & 0 & & & & \\
\hline
-2 & 1 & 0 & |\ -3 & & & &
\end{array}$$

Since in four steps of the synthetic division, the remainder was equal to zero, $x = -2$ is a root of multiplicity four for $P(x)$.

(8) Let $P(x) = x^6 - 10x^5 + 49x^4 - 94x^3 - 40x^2 + 176x - 103$. Using synthetic division evaluate $P(3 - 4\mathbf{i})$.

To find $P(3-4\mathbf{i})$ we divide $P(x)$ by the quadratic polynomial with conjugate roots $3 - 4\mathbf{i}$ and $3 + 4\mathbf{i}$ which is $x^2 - 6x + 25$.

6	−25	1	−10	49	−94	−40	176	−103
				−25	100	0	−150	100
			6	−24	0	36	−24	
		1	−4	0	6	−4	|2	−3

Therefore, the remainder is $2x-3$ and $P(3-4\mathbf{i}) = 2(3-4\mathbf{i}) - 3 = 3 - 8\mathbf{i}$.

(9) Use two steps of the Bairstow's Method with initial guess of $\begin{bmatrix} p_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}$ (corresponding with quadratic polynomial $x^2 - x + \frac{1}{2}$) to estimate a quadratic factor $x^2 + px + q$ of $P(x) = x^4 - 2x^2 + 2x - 1$.

$-p$	$-q$	1	0	-2	2	-1
				$-q$	pq	$-qp^2 + q^2 + 2q$
			$-p$	p^2	$-p^3 + pq + 2p$	
		1	$-p$	$p^2 - q - 2$	$-p^3 + 2pq + 2p + 2$	$-qp^2 + q^2 + 2q - 1$

To have a quadratic factor is equivalent to the system of equations:

$$\begin{cases} -p^3 + 2pq + 2p + 2 = 0 \\ -qp^2 + q^2 + 2q - 1 = 0 \end{cases}$$

When $F(p,q) = (-p^3 + 2pq + 2p + 2, -qp^2 + q^2 + 2q - 1)$, we have

$$DF(p,q) = \begin{bmatrix} -3p^2 + 2q + 2 & 2p \\ -2qp & -p^2 + 2q + 2 \end{bmatrix}$$

Then

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \frac{\begin{bmatrix} -p_n^2 + 2q_n + 2 & -2p_n \\ 2q_n p_n & -3p_n^2 + 2q_n + 2 \end{bmatrix} \begin{bmatrix} -p_n^3 + 2p_n q_n + 2p_n + 2 \\ -q_n p_n^2 + q_n^2 + 2q_n - 1 \end{bmatrix}}{(-3p_n^2 + 2q_n + 2)(-p_n^2 + 2q_n + 2) - (-2p_n)(2q_n p_n)}$$

In particular,

$$\begin{bmatrix} p_2 \\ q_2 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix} - \frac{\begin{bmatrix} -1 + 1 + 2 & 2 \\ -1 & -3 + 1 + 2 \end{bmatrix} \begin{bmatrix} 1 - 1 - 2 + 2 \\ -\frac{1}{2} + \frac{1}{4} + 1 - 1 \end{bmatrix}}{(-1 + 1 + 2)(-3 + 1 + 2) - (2)(-1)} =$$

$$= \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix} - \frac{\begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix}}{2} = \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix} - \begin{bmatrix} -\frac{1}{4} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{2} \end{bmatrix}$$

And

$$\begin{bmatrix} p_3 \\ q_3 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{2} \end{bmatrix} - \frac{\begin{bmatrix} -\frac{9}{16} + 1 + 2 & \frac{3}{2} \\ -\frac{3}{4} & -\frac{27}{16} + 1 + 2 \end{bmatrix} \begin{bmatrix} \frac{27}{64} - \frac{3}{4} - \frac{3}{2} + 2 \\ -\frac{9}{32} + \frac{1}{4} + 1 - 1 \end{bmatrix}}{(-\frac{9}{16} + 1 + 2)(-\frac{27}{16} + 1 + 2) - (\frac{3}{2})(-\frac{3}{4})}$$

$$= \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{2} \end{bmatrix} - \frac{256 \begin{bmatrix} \frac{39}{16} & \frac{3}{2} \\ -\frac{3}{4} & \frac{21}{16} \end{bmatrix} \begin{bmatrix} \frac{11}{64} \\ -\frac{1}{32} \end{bmatrix}}{1107} = \begin{bmatrix} -\frac{617}{738} \\ \frac{199}{369} \end{bmatrix}$$

Therefore, after two iterations of the Bairstow's Method we get $x^2 - \frac{617}{738}x + \frac{199}{369}$ as an estimate to a quadratic factor of $P(x)$.