(3.1) Interpolation and Polynomial Approximation

MATH 4701 Numerical Analysis

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This theorem covers a wide range of functions (continuous on $[a, b]$) and has an effective error bound. But it does not provide any computational way to find the polynomial $P(x)$ and it does not have any bounds on the degree of $P(x)$.

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Then for every $x \in [a, b]$, there is a number $\zeta = \zeta(x)$, between x_0 and x, such that

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Taylor's polynomial approximation, requires the values $f(x_0)$, $f'(x_0)$, ..., $f^{(n)}(x_0)$ to exists in order to be defined. It also requires $f \in C^n[a, b]$ and $f^{(n+1)}$ to exist on $[a, b]$ to have an effective error bound. It gives an explicit way of constructing polynomial $P_n(x)$ which only depends on the values f and its higher derivatives at $x = x_0$.

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Since the construction of interpolation polynomial only depends on the values of the function at finitely many points, one does not need an explicit formula for description of the function to set up the corresponding interpolation polynomial. One can determine the approximating polynomial using a table of values for the function or a list consisting of data evaluations at certain sample points.

Construction of Lagrange Interpolating Polynomials

For distinct values x_0, x_1, ..., x_n, and for the corresponding values (not necessarily distinct) y_0, y_1, ..., y_n we construct a polynomial $P(x)$ of degree at most n such that

$$P(x_i) = y_i \qquad \text{for all } 0 \leq i \leq n$$

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$$(x - x_0)(x - x_1)...(x - x_{k-1})(x - x_{k+1})...(x - x_n)$$

satisfies (b) and has degree n.

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$$\prod_{\substack{0 \leq j \leq n \\ j \neq k}} (x - x_j) = (x - x_0)(x - x_1)...(x - x_{k-1})(x - x_{k+1})...(x - x_n)$$

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$$P(x) = \sum_{0 \leq k \leq n} y_k L_k(x)$$

Example

Find a polynomial $P(x)$ of degree at most four such that $P(1) = 3$, $P(-1) = 5$, $P(2) = 0$, $P(-2) = -1$, and $P(3) = -6$.

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Uniqueness of Lagrange Interpolating Polynomials

For distinct values x_0, x_1, ..., x_n, and for the corresponding values y_0, y_1, ..., y_n there is a **unique** polynomial $P(x)$ of degree at most n such that

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Notice that in this expression x, $f(x)$, and $P(x)$ are all constants and the only variable is t.

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$f \in C^n[a,b]$ and $f^{(n+1)}(x)$ exists and the rest of the expression for $g(t)$ is a polynomial in t.

Error estimate for the Lagrange polynomial: Suppose $f \in C^n[a,b]$ and $f^{(n+1)}(x)$ exists for all $x \in (a,b)$ and x_0, x_1, x_2, ..., x_n are distinct numbers in $[a,b]$. Then for each $x \in [a,b]$, there is a number $\zeta(x) \in [x_0, x_1, ..., x_n, x]$ such that

$$f(x) - P(x) = \frac{f^{(n+1)}(\zeta(x))}{(n+1)!}(x-x_0)(x-x_1)...(x-x_n)$$

Proof: If $x = x_k$ for any $0 \leq k \leq n$ then $f(x_k) = P(x_k)$ and both sides of the above equation are equal to zero. Now if x is distinct from all x_k, $0 \leq k \leq n$ then we define $g : [a,b] \to \mathbb{R}$

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$f \in C^n[a,b]$ and $f^{(n+1)}(x)$ exists and the rest of the expression for $g(t)$ is a polynomial in t. Therefore, $g \in C^n[a,b]$ and $g^{(n+1)}(x)$ exists.

Error Bound for Lagrange Interpolating Polynomials

Error estimate for the Lagrange polynomial: Suppose $f \in C^n[a,b]$ and $f^{(n+1)}(x)$ exists for all $x \in (a,b)$ and x_0, x_1, x_2, ..., x_n are distinct numbers in $[a,b]$. Then for each $x \in [a,b]$, there is a number $\zeta(x) \in [x_0, x_1, ..., x_n, x]$ such that

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Error Bound for Lagrange Interpolating Polynomials

Error estimate for the Lagrange polynomial: Suppose $f \in C^n[a, b]$ and $f^{(n+1)}(x)$ exists for all $x \in (a, b)$ and x_0, x_1, x_2, ..., x_n are distinct numbers in $[a, b]$. Then for each $x \in [a, b]$, there is a number $\zeta(x) \in [x_0, x_1, ..., x_n, x]$ such that

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$g \in C^n[a, b]$ and $g^{(n+1)}(x)$ exists. This means for $n + 2$ distinct values x, x_0, x_1, ..., x_n, $g(t) = 0$.

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Therefore, by generalized Rolle's theorem, there is a number $\zeta = \zeta(x)$ in $[x_0, x_1, ..., x_n, x]$ such that $g^{(n+1)}(\zeta) = 0$.

$$g^{(n+1)}(t) = f^{(n+1)}(t) - P^{(n+1)}(t) - \frac{(n+1)![f(x) - P(x)]}{(x - x_0)(x - x_1)...(x - x_n)}$$

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For any polynomial $Q(x) = a_{n+1}x^{n+1} + a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$, $Q^{(n+1)}(x) = (n+1)! a_{n+1}$.

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$\frac{d^{n+1}}{dt^{n+1}}[f(x) - P(x)]\frac{(t-x_0)(t-x_1)...(t-x_n)}{(x-x_0)(x-x_1)...(x-x_n)} = \frac{(n+1)![f(x)-P(x)]}{(x-x_0)(x-x_1)...(x-x_n)}$

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Error estimate for the Lagrange polynomial: Suppose $f \in C^n[a, b]$ and $f^{(n+1)}(x)$ exists for all $x \in (a, b)$ and x_0, x_1, x_2, ..., x_n are distinct numbers in $[a, b]$. Then for each $x \in [a, b]$, there is a number $\zeta(x) \in [x_0, x_1, ..., x_n, x]$ such that

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$P(x)$ is a polynomial of degree at most n so $\frac{d^{n+1}}{dt^{n+1}}P(t) = 0$.

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Proof:

$$0 = f^{(n+1)}(\zeta) - \frac{(n+1)![f(x) - P(x)]}{(x - x_0)(x - x_1)...(x - x_n)}$$

which implies

$$f(x) - P(x) = \frac{f^{(n+1)}(\zeta(x))}{(n+1)!}(x - x_0)(x - x_1)...(x - x_n)$$

Example

Example:

- Find the Lagrange interpolating polynomial $P(x)$ for $f(x) = \sqrt{x}$. Let $x_0 = 1$, $x_1 = \frac{16}{9}$, $x_2 = \frac{9}{4}$, and $x_3 = 4$.

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$$L_3(x) = \frac{(x-1)(x-\frac{16}{9})(x-\frac{9}{4})}{(4-1)(4-\frac{16}{9})(4-\frac{9}{4})} = \frac{3}{35}(x - 1)(x - \frac{16}{9})(x - \frac{9}{4})$$

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$$f(x) = \sqrt{x} \Rightarrow y_0 = \sqrt{x_0} = 1,$$

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$$L_0(x) = \frac{(x-\frac{16}{9})(x-\frac{9}{4})(x-4)}{(1-\frac{16}{9})(1-\frac{9}{4})(1-4)} = -\frac{12}{35}(x - \frac{16}{9})(x - \frac{9}{4})(x - 4)$$

$$L_1(x) = \frac{(x-1)(x-\frac{9}{4})(x-4)}{(\frac{16}{9}-1)(\frac{16}{9}-\frac{9}{4})(\frac{16}{9}-4)} = \frac{729}{595}(x - 1)(x - \frac{9}{4})(x - 4)$$

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$\Rightarrow f^{(4)}(x) = -\frac{15}{16}x^{-\frac{7}{2}} \Rightarrow f(2) - P(2) = \frac{f^{(4)}(\zeta)}{216} = -\frac{5}{1152}\zeta^{-\frac{7}{2}}$

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  Notice that $P(2) = \frac{505}{357} \approx 1.414565826$ and
  $f(2) = \sqrt{2} \approx 1.414213562$. In particular,
  $f(2) - P(2) \approx -0.000352264$.