MATH 4701 Numerical Analysis

Problem Set #2

(1) Let $f(x) = (x+2)(x+1)^2x(x-1)^3(x-2)$.

(a) To which zero of $f(x)$ does Bisection Method converge when applied on $[-1.5, 2.5]$?

(b) To which zero of $f(x)$ does Bisection Method converge when applied on $[-0.5, 2.4]$?

First we determine the sign of $f(x)$ on various intervals of x. Since $\lim_{x\to\infty} f(x) = +\infty$, $f(x) > 0$ for $x > 2$ and the sign of $f(x)$ stays the same when moving from the right to the left we pass a root of even multiplicity and the sign of $f(x)$ changes when moving from the right to the left we pass a root of odd multiplicity. Therefore, $f(x) > 0$ for $x > 2$, $0 < x < 1$, and $x < -2$ and $f(x) < 0$ for $1 < x < 2$ and $-2 < x < 0$. Now we determine the answer to the two questions:

Bisection

$[f(x)f(y) < 0]$

(a) $f(x_1) = f(-1.5) < 0$ and $f(x_2) = f(2.5) > 0$.
$x_3 = \frac{2.5+(-1.5)}{2} = \frac{1}{2} \Rightarrow f(x_3) > 0$.
$x_4 = \frac{x_1+x_3}{2} = \frac{-1.5+0.5}{2} = -\frac{1}{2} \Rightarrow f(x_4) < 0$.
$x_5 = \frac{x_3+x_4}{2} = \frac{-0.5+0.5}{2} = 0$.
So the algorithm ends at the root $x = 0$.

Same idea as H.V.

(b) $f(x_1) = f(-0.5) < 0$ and $f(x_2) = f(2.4) > 0$.
$x_3 = \frac{2.4+(-0.5)}{2} = 0.95 \Rightarrow f(x_3) > 0$.
$x_4 = \frac{x_1+x_3}{2} = \frac{-0.5+0.95}{2} = 0.225 \Rightarrow f(x_4) > 0$.
$x_5 = \frac{x_1+x_4}{2} = \frac{-0.5+0.225}{2} = -0.1375 \Rightarrow f(x_5) < 0$.
$x_6 = \frac{x_4+x_5}{2} = \frac{0.225+(-0.1375)}{2} = 0.0.04375 \Rightarrow f(x_6) > 0$.
Since the only root of $f(x)$ between x_4 and x_5 is $x = 0$ the algorithm converges to the root $x = 0$.

(2) Let $f(x) = x^3 - 2x + 1$. To solve $f(x) = 0$, the following four fixed point problems are proposed. Derive each fixed point method and compute p_1, p_2, p_3, and p_4. Which methods seem to be appropriate?

Fixed Point-Iteration

(a) $x = \frac{1}{2}(x^3+1)$, $p_0 = \frac{1}{2}$

$g(x) =$

$$x = \frac{1}{2}(x^3+1) \Leftrightarrow 2x = x^3 + 1 \Leftrightarrow x^3 - 2x + 1 = 0$$

$$g_1(x) = \frac{1}{2}(x^3+1),\ p_0 = \frac{1}{2},\ p_1 = g_1(p_0) = \frac{9}{16} = 0.5625,\ p_2 = g_1(p_1) = \frac{4825}{8192} \approx 0.58899,$$

$$p_3 = g_1(p_2) \approx 0.60216,\ p_4 = g_1(p_3) \approx 0.60917$$

(We're approaching approx value when $f(x)=0$)

$p_1 =$

(b) $x = \frac{2}{x} - \frac{1}{x^2}$, $p_0 = \frac{1}{2}$

As long as $x \neq 0$ we can multiply both sides by x^2 to show

$$x = \frac{2}{x} - \frac{1}{x^2} \Leftrightarrow x^3 = 2x - 1 \Leftrightarrow x^3 - 2x + 1 = 0$$

$$g_2(x) = \frac{2}{x} - \frac{1}{x^2},\ p_0 = \frac{1}{2},\ p_1 = g_2(p_0) = 4 - 4 = 0,\ p_2 = g_2(p_1) = \frac{2}{0} - \frac{1}{0}$$

which is not defined so the iteration stops after the first application of g_2.

(c) $x = \sqrt{2 - \frac{1}{x}}$, $p_0 = \frac{1}{2}$

As long as $x \neq 0$ and $2 - \frac{1}{x} \geq 0$ and $x > 0$ ($x \geq \frac{1}{2}$ or $x < 0$) we have

$$x = \sqrt{2 - \frac{1}{x}} \Leftrightarrow x^2 = 2 - \frac{1}{x} \Leftrightarrow x^3 = 2x - 1 \Leftrightarrow x^3 - 2x + 1 = 0$$

$$g_3(x) = \sqrt{2 - \frac{1}{x}},\ p_0 = \frac{1}{2},\ p_1 = g_3(p_0) = \sqrt{2-2} = 0,\ p_2 = g_3(p_1) = \sqrt{2 - \frac{1}{0}}$$

which is not defined so the iteration stops after the first application of g_3.

(d) $x = -\sqrt[3]{1-2x}$, $p_0 = \frac{1}{2}$

$-\sqrt[3]{1-2x}$ is defined for all real numbers x and

$$x = -\sqrt[3]{1-2x} \Leftrightarrow x^3 = -(1-2x) \Leftrightarrow x^3 - 2x + 1 = 0$$

$$g_4(x) = -\sqrt[3]{1-2x},\ p_0 = \frac{1}{2},\ p_1 = g_4(p_0) = -\sqrt[3]{1-1} = 0,\ p_2 = g_4(p_1) = -\sqrt[3]{1-0} = -1,$$

$$p_3 = g_4(p_2) = -\sqrt[3]{1+2} \approx -1.44225,\ p_4 = g_4(p_3) \approx -\sqrt[3]{1 - 2(-1.44225)} \approx -1.57197$$

$f(x) = x^3 - 2x + 1$ has three real roots $-2 < \alpha < -1$, $0 < \beta < 1$, and $\gamma = 1$. Methods **a** and **d** seem to be appropriate, respectively to find the roots β and α.

(3) Let $f(x) = 2 + \sin x$

(a) Show that $f(x)$ satisfies the conditions of the fixed point theorem on the interval $[2,3]$.

For $\frac{\pi}{2} < 2 \leq x \leq 3 < \pi$ we have $0 < \sin x < 1$ and $2 < 2 + \sin x < 3$. Therefore $f([2,3]) \subset [2,3]$. Moreover, $f'(x) = \cos x$ which is negative and decreasing on the interval $[\frac{\pi}{2}, \pi]$.* As a result, $|f'(x)|$ gains its absolute maximum value over the interval $[2,3]$ at $x = 3$ and this maximum value is $|\cos 3| \approx 0.98999 < 1$.* Therefore, $f(x)$ satisfies the conditions of the fixed point theorem on the interval $[2,3]$.

(b) Estimate the number of iterations needed to obtain an approximation of the fixed point of $f(x)$ on $[2,3]$ accurate to within 10^{-5}.

a b

max value

Without a choice of p_0 we can use the estimate $K = f'(x)$ $f(x) = 2 + \sin x$

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\} \leq k^n(b-a) = |\cos 3|^n(3-2) = |\cos 3|^n$$

$f'(x) = \cos x$

In order to have $|p_n - p| < 10^{-5}$ it is enough to have $|\cos 3|^n < 10^{-5}$ or $n \ln(|\cos 3|) < -5\ln(10)$ or $n > \frac{-5\ln(10)}{\ln(|\cos 3|)} \approx 1144.66$ which means we need to perform 1145 iterations to obtain an approximation of the fixed point of $f(x)$ on $[2,3]$ accurate to within 10^{-5}. The convergence is very slow because k is very close to 1. In practice we don't need as many iterations because all the points in the iteration sequence stay below the maximum value 3 of the interval.

$(\cos(3))$

$|\cos(3)|^n(3-2) < 10^{-5}$

Then Solve for n

(c) Perform the first few iterations up to the number of iterations you found in the previous part.

$p_0 = 2.5$, $p_1 = f(p_0) = 2 + \sin(2.5) \approx 2.598472$, $p_2 = f(p_1) \approx 2.51681$, $p_3 = f(p_2) \approx 2.58492$, $p_4 = f(p_3) \approx 2.52836$, $p_5 = f(p_4) \approx 2.57551$, $p_6 = f(p_5) \approx 2.53633$, $p_7 = f(p_6) \approx 2.56898$, $p_8 = f(p_7) \approx 2.54183$, $p_9 = f(p_8) \approx 2.56445$, $p_{10} = f(p_9) \approx 2.54563$, $p_{11} = f(p_{10}) \approx 2.56131$, $p_{11} = f(p_{10}) \approx 2.54826$, $p_{12} = f(p_{11}) \approx 2.55913$, $p_{13} = f(p_{12}) \approx 2.55008$, $p_{14} = f(p_{13}) \approx 2.55762$, $p_{15} = f(p_{14}) \approx 2.55134$, $p_{16} = f(p_{15}) \approx 2.55657$, $p_{17} = f(p_{16}) \approx 2.55222$, $p_{17} = f(p_{16}) \approx 2.55222$, $p_{18} = f(p_{17}) \approx 2.55584$, $p_{19} = f(p_{18}) \approx 2.55283$, $p_{20} = f(p_{19}) \approx 2.55533$, $p_{21} = f(p_{20}) \approx 2.55325$, $p_{22} = f(p_{21}) \approx 2.55498$, $p_{23} = f(p_{22}) \approx 2.55354$, $p_{24} = f(p_{23}) \approx 2.55474$, $p_{25} = f(p_{24}) \approx 2.55374$, $p_{26} = f(p_{25}) \approx 2.55458$, $p_{27} = f(p_{26}) \approx 2.55387$, $p_{28} = f(p_{27}) \approx 2.55446$, $p_{29} = f(p_{28}) \approx 2.55397$, $p_{30} = f(p_{29}) \approx 2.55438$, $p_{31} = f(p_{30}) \approx 2.55404$, $p_{32} = f(p_{31}) \approx 2.55433$, $p_{33} = f(p_{32}) \approx 2.55408$, $p_{34} = f(p_{33}) \approx 2.55429$, $p_{35} = f(p_{34}) \approx 2.55412$, $p_{36} = f(p_{35}) \approx 2.55426$, $p_{37} = f(p_{36}) \approx 2.55414$, $p_{38} = f(p_{37}) \approx 2.55424$, $p_{39} = f(p_{38}) \approx 2.55416$, $p_{40} = f(p_{39}) \approx 2.55423$, $p_{41} = f(p_{40}) \approx 2.55417$, $p_{42} = f(p_{41}) \approx 2.55422$, $p_{43} = f(p_{42}) \approx 2.55418$, $p_{44} = f(p_{43}) \approx 2.554209$, $p_{45} = f(p_{44}) \approx 2.554185$, $p_{46} = f(p_{45}) \approx 2.554205$, $p_{47} = f(p_{46}) \approx 2.554188$, $p_{48} = f(p_{47}) \approx 2.554203$, $p_{49} = f(p_{48}) \approx 2.554190$, $p_{50} = f(p_{49}) \approx 2.554200$, , $p_{51} = f(p_{50}) \approx 2.554193$, $p_{52} = f(p_{51}) \approx 2.554198$.

Which means 2.55419 is within 10^{-5} of the fixed point p of $f(x)$.

(4) Let $g(x) = \sin x - e^{-x}$.

a: Write three steps of the secant method starting with $x_1 = 0$ and $x_2 = 1$ to find the estimates x_3, x_4, and x_5 to a root of $g(x)$.

Secant Method

$x_1 = 0$, $y_1 = g(x_1) = -1$

$x_2 = 1$, $y_2 = g(x_2) \approx 0.473592$

$x_3 = \frac{y_2 x_1 - y_1 x_2}{y_2 - y_1} = \frac{(0.473592)0 - (-1)(1)}{0.473592 - (-1)} \approx 0.678614$
$\Rightarrow y_3 = g(0.678614) \approx 0.120395$

$x_4 = \frac{y_3 x_2 - y_2 x_3}{y_3 - y_2} = \frac{(0.120395)(1) - (0.473592)(0.678614)}{0.120395 - 0.473592} \approx 0.569062$
$\Rightarrow y_4 = g(0.569062) \approx -0.027214$

$x_5 = \frac{y_4 x_3 - y_3 x_4}{y_4 - y_3} = \frac{(-0.027214)(0.678614) - (0.120395)(0.569062)}{-0.027214 - 0.120395} \approx 0.589260$
$\Rightarrow y_5 = g(0.589260) \approx 0.001008$

b: Write three steps of the method of false position starting with $x_1 = 0$ and $x_2 = 1$ to find the estimates x_3, x_4, and x_5 to a root of $g(x)$.

(False Position)

$x_1 = 0$, $y_1 = g(x_1) = -1$

$x_2 = 1$, $y_2 = g(x_2) \approx 0.473592$

$x_3 = \frac{y_2 x_1 - y_1 x_2}{y_2 - y_1} = \frac{(0.473592)0 - (-1)(1)}{0.473592 - (-1)} \approx 0.678614$
$\Rightarrow y_3 = g(0.678614) \approx 0.120395$

Since $y_3 > 0$ and $y_1 < 0$ we use: $x_4 = \frac{y_3 x_1 - y_1 x_3}{y_3 - y_1} = \frac{(0.120395)(0) - (-1)(0.678614)}{0.120395 - (-1)} \approx 0.605692$
$\Rightarrow y_4 = g(0.605692) \approx 0.023634$

Since $y_4 > 0$ and $y_1 < 0$ we use: $x_5 = \frac{y_4 x_1 - y_1 x_4}{y_4 - y_1} = \frac{(0.023634)(0) - (-1)(0.605692)}{0.023634 - (-1)} \approx 0.591708$
$\Rightarrow y_5 = g(0.591708) \approx 0.004398$

c: Apply Newton's method to evaluate the terms of the sequence (x_n) until the first 5 digits of two consecutive x_n's are the same. Use $x_1 = 0$.

(Newton's Method)

$h(x) = x - \frac{g(x)}{g'(x)} = x - \frac{\sin x - e^{-x}}{\cos x + e^{-x}}$
$x_1 = 0$
$x_2 = h(x_1) = 0.5$
$x_3 = h(x_2) \approx 0.585644$
$x_4 = h(x_3) \approx 0.588529$
$x_5 = h(x_4) \approx 0.588532$
$x_6 = h(x_5) \approx 0.588532$

d: Apply Newton's method to evaluate the terms of the sequence (x_n) until the first 5 digits of two consecutive x_n's are the same. Use $x_1 = 3$.

$x_1 = 3$
$x_2 = h(x_1) \approx 3.097141$
$x_3 = h(x_2) \approx 3.096364$
$x_4 = h(x_3) \approx 3.096364$

(5) Let $H(x) = x^5 - 2x^3 + 2x - 2$. H has one positive root γ between 1 and 2.

a: Write all steps of the bisection method for finding the root γ within precision of $\frac{1}{10}$.

b: Write three steps of the secant method starting with $x_1 = 1$ and $x_2 = 2$ to find the estimates x_3, x_4, and x_5 to γ.

c: Write three steps of the method of false position starting with $x_1 = 1$ and $x_2 = 2$ to find the estimates x_3, x_4, and x_5 to γ.

d: Apply Newton's method to evaluate the terms of the sequence (x_n) until the first 5 digits of two consecutive x_n's are the same. Use $x_1 = 1$.

a: $A = 1$, $B = 2$, . $H(1) = 1 - 2 + 2 - 2 = -1 < 0$, $H(2) = 32 - 16 + 4 - 2 = 18 > 0$

$C = \frac{3}{2}$, $H(\frac{3}{2}) = \frac{59}{32} > 0 \Rightarrow$ We let $B = \frac{3}{2}$

$C = \frac{1+\frac{3}{2}}{2} = \frac{5}{4}$, $H(\frac{5}{4}) = -\frac{363}{1024} < 0$. We let $A = \frac{5}{4}$.
$C = \frac{\frac{5}{4}+\frac{3}{2}}{2} = \frac{11}{8}$, $H(\frac{11}{8}) = \frac{15259}{32768} > 0$. We let $B = \frac{11}{8}$.
$C = \frac{\frac{5}{4}+\frac{11}{8}}{2} = \frac{21}{16}$, $H(\frac{21}{16}) = -\frac{2171}{1048576} < 0$. We let $A = \frac{21}{16}$.
Since $B - A = \frac{1}{16} < 0.1$ we stop and announce the root γ as $\frac{A+B}{2} = \frac{\frac{21}{16}+\frac{11}{8}}{2} = \frac{43}{32} = 1.34375$

b: $x_1 = 1,\ y_1 = H(x_1) = -1$

$x_2 = 2,\ y_2 = H(x_1) = 18$

$x_3 = \frac{y_2}{y_2 - y_1} x_1 + \frac{y_1}{y_1 - y_2} x_2 = \frac{18}{18+1} + \frac{-2}{-1-18} = \frac{20}{19}$
$\Rightarrow y_3 = H(\frac{20}{19}) = -.935083$

$x_4 = \frac{y_3}{y_3 - y_2} x_2 + \frac{y_2}{y_2 - y_3} x_3 = \frac{-.935083}{-.935083-18}(2) + \frac{18}{18+.935083}(\frac{20}{19}) = 1.099416$
$\Rightarrow y_4 = H(1.099416) = -0.852691$

$x_5 = \frac{y_4}{y_4 - y_3} x_3 + \frac{y_3}{y_3 - y_4} x_4 = \frac{-0.852691}{-0.852691-(-.935083)}(\frac{20}{19}) + \frac{-.935083}{-.935083-(-0.852691)}(1.099416) \approx 1.583597$
$\Rightarrow y_5 = H(1.583597) \approx 3.183754550$

c: $x_1 = 1,\ y_1 = H(x_1) = -1$

$x_2 = 2,\ y_2 = H(x_1) = 18$

$x_3 = \frac{y_2}{y_2 - y_1} x_1 + \frac{y_1}{y_1 - y_2} x_2 = \frac{18}{18+1} + \frac{-2}{-1-18} = \frac{20}{19}$
$\Rightarrow y_3 = H(\frac{20}{19}) = -.935083$

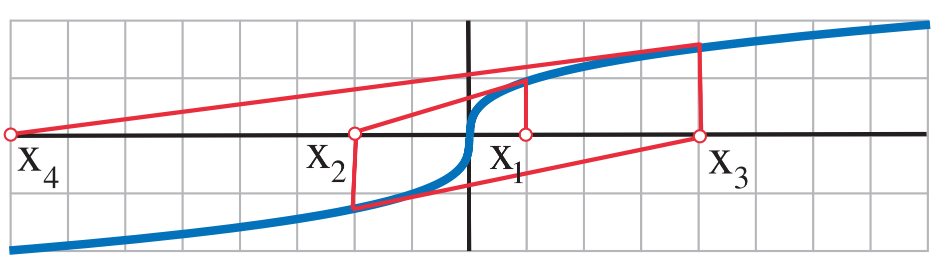
since $y_3 < 0$ and $y_2 > 0$ we have
$x_4 = \frac{y_3}{y_3 - y_2} x_2 + \frac{y_2}{y_2 - y_3} x_3 = \frac{-.935083}{-.935083-18}(2) + \frac{18}{18+.935083}(\frac{20}{19}) = 1.099416$
$\Rightarrow y_4 = H(1.099416) = -0.852691$

Since $y_4 < 0$ and $y_2 > 0$ we have
$x_5 = \frac{y_4}{y_4 - y_2} x_2 + \frac{y_2}{y_2 - y_4} x_4 = \frac{-0.852691}{-0.852691-18}(2) + \frac{18}{18-0.852691}(1.099416) = 1.140149$
$\Rightarrow y_5 = H(1.140149) = -0.757279$

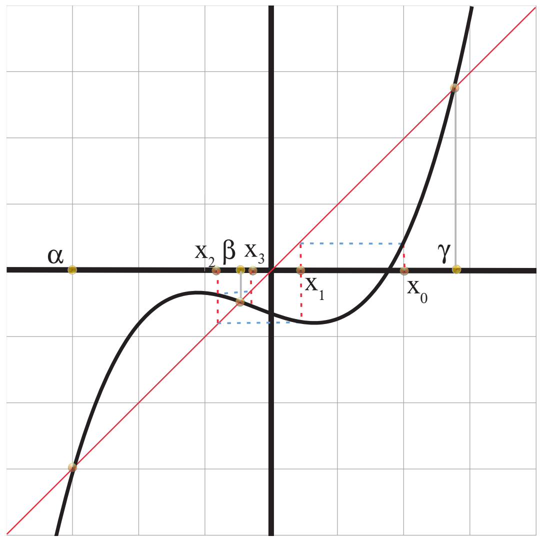
d: $\Gamma(x) = x - \frac{H(x)}{H'(x)} = x - \frac{x^5 - 2x^3 + 2x - 2}{5x^4 - 6x^2 + 2}$
$x_1 = 1$
$x_2 = \Gamma(x_1) = 2$
$x_3 = \Gamma(x_2) = 1.689655$
$x_4 = \Gamma(x_3) = 1.474877$
$x_5 = \Gamma(x_4) = 1.354943$
$x_6 = \Gamma(x_5) = 1.316456$
$x_7 = \Gamma(x_6) = 1.312847$
$x_8 = \Gamma(x_7) = 1.312818$
$x_9 = \Gamma(x_8) = 1.312818$

(6) Explain why Newton's method fails when applied to the equation $\sqrt[3]{x} = 0$ with any initial approximation $x_1 \neq 0$. Illustrate your explanation with a sketch.

With Newton's method for the function $f(x) = \sqrt[3]{x}$ we must start with a number x_1 and generate the sequence of iterates of x_1 under the function $F(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^{\frac{1}{3}}}{\frac{1}{3}x^{-\frac{2}{3}}} = x - 3x = -2x$. If $x_1 \neq 0$, the sequence x_n generated by Newton's method is equal to $x_n = (-2)^n x_1$ which is divergent.

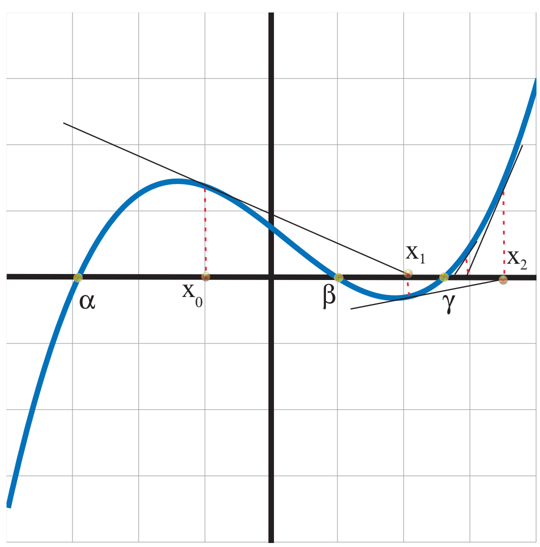


(7) The picture below shows graph of a function f with three **fixed points** α, β, and γ. Starting with the marked number x_0, trace three steps of function iteration method on the graph to obtain the location of x_1 x_2, and x_3. Which one of the three fixed points is the sequence going to converge to?



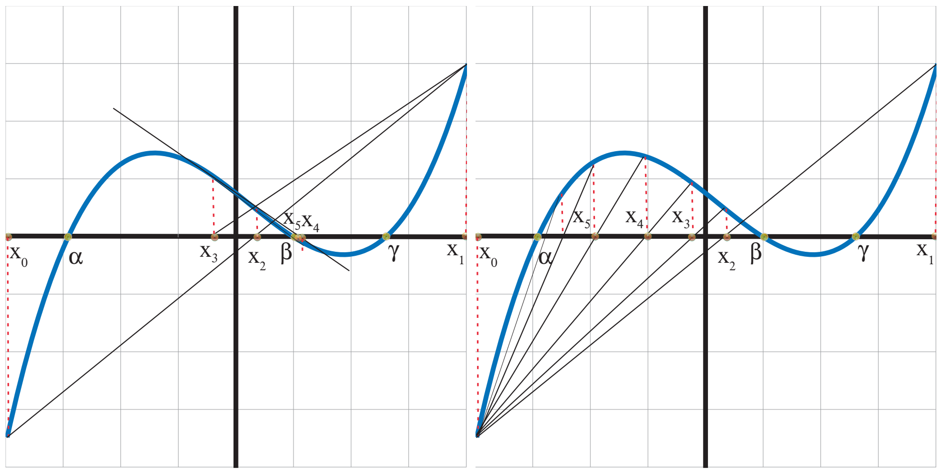
The sequence converges to the fixed point β.

(8) The picture below shows graph of a function f with three roots α, β, and γ. Starting with the marked number x_0, trace two steps of Newton's method on the graph to obtain the location of x_1 and x_2. Which one of the three roots is the sequence going to converge to?



Newton's Method will converge to the root γ.

(9) The picture below shows graph of a function f with three roots α, β, and γ. Starting with the marked numbers x_0 and x_1, trace four steps of Secant method and method of false position on the graph to obtain the location of x_2, x_3, x_4, and x_5. Which one of the three roots is the sequence going to converge to?



Left: Secant Method will converge to the root β. Right: Method of false position will converge to the root α.