MATH 4701 Numerical Analysis

On Exam

Problem Set #3

(1) For $f(x) = \sqrt{3+x}$, find the fourth degree Taylor polynomial $P_4(x)$ for $f(x)$ centered at $c = 1$. Find the maximum and minimum possible error when we use $P_4(x)$ to approximate $f(1.2)$ and $f(0.9)$.

$f(x) = \sqrt{3+x} = (3+x)^{\frac{1}{2}} \Rightarrow f(1) = \sqrt{4} = 2$
$f'(x) = (\frac{1}{2})(3+x)^{-\frac{1}{2}} \Rightarrow f'(1) = \frac{1}{4}$
$f''(x) = (\frac{1}{2})(-\frac{1}{2})(3+x)^{-\frac{3}{2}} \Rightarrow f''(1) = -\frac{1}{2^5} = -\frac{1}{32}$
$f^{(3)}(x) = (\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(3+x)^{-\frac{5}{2}} \Rightarrow f^{(3)}(1) = \frac{3}{2^8} = \frac{3}{256}$
$f^{(4)}(x) = (\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(3+x)^{-\frac{7}{2}} \Rightarrow f^{(4)}(1) = -\frac{15}{2^{11}} = -\frac{15}{2048}$

4th degree taylor series Computation

$$\Rightarrow P_4(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f^{(3)}(1)}{6}(x-1)^3 + \frac{f^{(4)}(1)}{24}(x-1)^4 =$$

$$= 2 + \frac{1}{4}(x-1) - \frac{1}{64}(x-1)^2 + \frac{1}{512}(x-1)^3 - \frac{5}{16384}(x-1)^4$$

By Taylor's Theorem we know $f(x) - P_4(x) = \frac{f^{(5)}(\zeta)}{5!}(x-1)^5$ where ζ is a number between $c = 1$ and x.

Given $f^{(5)}(x) = (\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})(3+x)^{-\frac{9}{2}} = \frac{105}{32\sqrt{(3+x)^9}}$ which is decreasing in x for $x > -3$, we have

center

$$\frac{f^{(5)}(1.2)}{5!}(1.2-1)^5 \leq f(1.2) - P_4(1.2) \leq \frac{f^{(5)}(1)}{5!}(1.2-1)^5 \Rightarrow$$

$$\Rightarrow \frac{\frac{105}{32\sqrt{(4.2)^9}}}{120}(0.2)^5 \leq f(1.2) - P_4(1.2) \leq \frac{\frac{105}{32\sqrt{(4)^9}}}{120}(0.2)^5 \Rightarrow$$

$$\Rightarrow 1.372101 \times 10^{-8} \leq f(1.2) - P_4(1.2) \leq 1.708985 \times 10^{-8}$$

Similarly,

center

$$\frac{f^{(5)}(0.9)}{5!}(0.9-1)^5 \leq f(0.9) - P_4(0.9) \leq \frac{f^{(5)}(1)}{5!}(0.9-1)^5 \Rightarrow$$

$$\Rightarrow \frac{\frac{105}{32\sqrt{(3.9)^9}}}{120}(-0.1)^5 \leq f(0.9) - P_4(0.9) \leq \frac{\frac{105}{32\sqrt{(4)^9}}}{120}(-0.1)^5 \Rightarrow$$

$$\Rightarrow -5.985044 \times 10^{-10} \leq f(0.9) - P_4(0.9) \leq -5.340576 \times 10^{-10}$$

(2) For $g(x) = \sin x$, let $P_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$ be the fifth degree Taylor polynomial for $g(x)$ centered at $c = 0$. Find the maximum possible absolute error when we use $P_5(x)$ to approximate $g(x)$ on the interval $[-\pi, \pi]$.

Since the sixth degree term of the Taylor series for $g(x)$ centered at $c = 0$ is zero $P_5(x)$ is in fact the sixth degree Taylor polynomial for $g(x)$ at $c = 0$. In particular, $g(x) - P_5(x) = \frac{g^{(7)}(\zeta)}{7!}x^7 = \frac{-\cos(\zeta)}{5040}x^7$. $(g^{(7)}(\zeta) = g^{(3)}(\zeta) = -\cos(\zeta))$. Since $|\cos(\zeta)| \leq 1$ and $|x| \leq \pi$ we have

$$\left|g(x) - P_5(x)\right| = \frac{|-\cos(\zeta)|}{5040}|x|^7 \leq \frac{\pi^7}{5040} \approx 0.59926453$$

(3) For $h(x) = e^x$, find a positive integer n for which the nth degree Taylor polynomial $P_n(x)$ for $h(x)$ centered at $c = 0$ approximates $h(x)$ accurate to within an absolute error of less than 10^{-8} for all values of x satisfying $|x| < 1$.

Notice that for all $n \geq 1$, $h^{(n)}(\zeta) = e^\zeta$. Therefore,

$$h(x) - P_n(x) = \frac{h^{(n+1)}(\zeta)}{(n+1)!}x^{n+1} = \left(\frac{e^\zeta}{(n+1)!}x^{n+1}\right)$$

In particular, for $|x| < 1$ we have

$$\left(\left|h(x) - P_n(x)\right| \leq \frac{e^1}{(n+1)!}(1)^{n+1}\right) = \left(\frac{e}{(n+1)!}\right)$$

Which means to make sure $\left|h(x) - P_n(x)\right| < 10^{-8}$ we can make sure $\frac{e}{(n+1)!} < 10^{-8}$ or $(n+1)! > e \times 10^8$. Experimenting with values of $n!$ we have $12! = 479001600 > 271828182.8 \approx e \times 10^8$ which means $P_{11}(x)$ approximates $h(x)$ accurate to within an absolute error of less than 10^{-8} for all values of x satisfying $|x| < 1$.

(4) (a) Show that for any positive integer k, the sequence $p_n = \frac{1}{n^k}$ converges sub-linearly to $p = 0$.

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^k}}{\frac{1}{n^k}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^k = \left(\lim_{n\to\infty} \frac{n}{n+1}\right)^k = 1$$

$|P_n - P|^1$ means linear

Therefore, $p_n = \frac{1}{n^k}$ converges sub-linearly to $p = 0$, with asymptotic error constant 1.

if $|P_n - P|^2$

(Quadratic Convergence)

(b) Show that the sequence $p_n = \frac{1}{10^{2^n}}$ converges quadratically to $p = 0$.

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \lim_{n\to\infty} \frac{\frac{1}{10^{2^{n+1}}}}{\left(\frac{1}{10^{2^n}}\right)^2} = \lim_{n\to\infty} \frac{\frac{1}{\left(10^{2^n}\right)^2}}{\left(\frac{1}{10^{2^n}}\right)^2} = 1$$

Therefore, $p_n = \frac{1}{10^{2^n}}$ converges quadratically to $p = 0$, with asymptotic error constant 1.

(5) (a) Use Newton's method to find a solution accurate to within 10^{-5} to the equation $\cos(x+\sqrt{2})+x(\frac{x}{2}+\sqrt{2})=0$. Use initial guess $p_0=-1.5$.

newtons Method converge to 10^{-5}

Given $f(x)=\cos(x+\sqrt{2})+x(\frac{x}{2}+\sqrt{2})$ we let $g(x)=x-\frac{f(x)}{f'(x)}=x-\frac{\cos(x+\sqrt{2})+x(\frac{x}{2}+\sqrt{2})}{-\sin(x+\sqrt{2})+(\frac{x}{2}+\sqrt{2})+\frac{x}{2}}=x-\frac{\cos(x+\sqrt{2})+x(\frac{x}{2}+\sqrt{2})}{-\sin(x+\sqrt{2})+x+\sqrt{2}}$. We have: $p_1=g(p_0)\approx-1.478542125$, $p_2=g(p_1)\approx-1.462445763$, $p_3=g(p_2)\approx-1.450359310$, $p_4=g(p_3)\approx-1.441211005$, $p_5=g(p_4)\approx-1.434502563$, $p_6=g(p_5)\approx-1.428755189$, $p_7=g(p_6)\approx-1.424852674$, $p_8=g(p_7)\approx-1.419870361$, $p_9=g(p_8)\approx-1.419870361$.

modified Newton's method

(b) Use the modified Newton's method to find a solution accurate to within 10^{-5} to the same equation. Use initial guess $p_0=-1.5$.

$h(x)=x-\frac{f(x)f'(x)}{[f'(x)]^2-f(x)f''(x)}=x-\frac{(\cos(x+\sqrt{2})+x(\frac{x}{2}+\sqrt{2}))(-\sin(x+\sqrt{2})+x+\sqrt{2})}{(-\sin(x+\sqrt{2})+x+\sqrt{2})^2-(\cos(x+\sqrt{2})+x(\frac{x}{2}+\sqrt{2}))(-\cos(x+\sqrt{2})+1)}$.
Then $p_1=h(p_0)\approx-1.414096386$, $p_2=h(p_1)\approx-1.414052908$, $p_3=h(p_2)\approx-1.414052908$.

(6) The sequence $p_0=0.5$, $p_n=\frac{2-e^{p_{n-1}}+p_{n-1}^2}{3}$, $n\geq 1$ is linearly convergent. Generate the terms up to $\hat{p_5}$ of the sequence $\{\hat{p_n}\}$ using Aitken's Δ^2 method.

Aitken's method

With $g(x)=\frac{2-e^x+x^2}{3}$ and $p_0=0.5$ we have

$p_1=g(p_0)\approx 0.2004262430$

$p_2=g(p_1)\approx 0.2727490650$

$\hat{p_0}=p_0-\frac{(p_1-p_0)^2}{p_2-2p_1+p_0}=0.5-\frac{(0.2004262430-0.5)^2}{0.2727490650-2(0.2004262430)+0.5}\approx 0.2586844275$

$p_3=g(p_2)\approx 0.2536071565$

$\hat{p_1}=p_1-\frac{(p_2-p_1)^2}{p_3-2p_2+p_1}=0.2004262430-\frac{(0.2727490650-0.2004262430)^2}{0.2536071565-2(0.2727490650)+0.2004262430}\approx 0.2576132106$

$p_4=g(p_3)\approx 0.2585503763$

$\hat{p_2}=p_2-\frac{(p_3-p_2)^2}{p_4-2p_3+p_2}=0.2727490650-\frac{(0.2536071565-0.2727490650)^2}{0.2585503763-2(0.2536071565)+0.2727490650}\approx 0.2575358323$

$p_5=g(p_4)\approx 0.2572656364$

$\hat{p_3}=p_3-\frac{(p_4-p_3)^2}{p_5-2p_4+p_3}=0.2536071565-\frac{(0.2585503763-0.2536071565)^2}{0.2572656364-2(0.2585503763)+0.2536071565}\approx 0.2575306601$

$p_6=g(p_5)\approx 0.2575989852$

$\hat{p_4}=p_4-\frac{(p_5-p_4)^2}{p_6-2p_5+p_4}=0.2585503763-\frac{(0.2572656364-0.2585503763)^2}{0.2575989852-2(0.2572656364)+0.2585503763}\approx 0.2575303107$

$p_7=g(p_6)\approx 0.2575124545$

$\hat{p_5}=p_5-\frac{(p_6-p_5)^2}{p_7-2p_6+p_5}=0.2572656364-\frac{(0.2575989852-0.2572656364)^2}{0.2575124545-2(0.2575989852)+0.2572656364}\approx 0.2575302871$

(7) The sequence $p_0 = 0.5$, $p_n = \cos(p_{n-1})$, $n \geq 1$ is linearly convergent. Generate the terms up to $\hat{p_5}$ of the sequence $\{\hat{p_n}\}$ using Aitken's Δ^2 method.

(Same)

With $h(x) = \cos x$ and $p_0 = 0.5$ we have
$p_1 = h(p_0) \approx 0.8775825619$
$p_2 = h(p_1) \approx 0.6390124942$
$\hat{p_0} = p_0 - \frac{(p_1-p_0)^2}{p_2-2p_1+p_0} = 0.5 - \frac{(0.8775825619-0.5)^2}{0.6390124942-2(0.8775825619)+0.5} \approx 0.7313851863$
$p_3 = h(p_2) \approx 0.8026851007$
$\hat{p_1} = p_1 - \frac{(p_2-p_1)^2}{p_3-2p_2+p_1} = 0.8775825619 - \frac{(0.6390124942-0.8775825619)^2}{0.8026851007-2(0.6390124942)+0.8775825619} \approx 0.7360866919$
$p_4 = h(p_3) \approx 0.6947780268$
$\hat{p_2} = p_2 - \frac{(p_3-p_2)^2}{p_4-2p_3+p_2} = 0.6390124942 - \frac{(0.8026851007-0.6390124942)^2}{0.6947780268-2(0.8026851007)+0.6390124942} \approx 0.7376528716$
$p_5 = h(p_4) \approx 0.7681958313$
$\hat{p_3} = p_3 - \frac{(p_4-p_3)^2}{p_5-2p_4+p_3} = 0.8026851007 - \frac{(0.6947780268-0.8026851007)^2}{0.7681958313-2(0.6947780268)+0.8026851007} \approx 0.7384692207$
$p_6 = h(p_5) \approx 0.7191654459$
$\hat{p_4} = p_4 - \frac{(p_5-p_4)^2}{p_6-2p_5+p_4} = 0.6947780268 - \frac{(0.7681958313-0.6947780268)^2}{0.7191654459-2(0.7681958313)+0.6947780268} \approx 0.7387980650$
$p_7 = h(p_6) \approx 0.7523557594$
$\hat{p_5} = p_5 - \frac{(p_6-p_5)^2}{p_7-2p_6+p_5} = 0.7681958313 - \frac{(0.7191654459-0.7681958313)^2}{0.7523557594-2(0.7191654459)+0.7681958313} \approx 0.7389577109$

(8) Use Steffensen's method to approximate the solution of the fixed point equation $x = 2^{-x}$, accurate to within 10^{-4}. Use initial guess $p_0 = 1$.

(Steffensens update only 2 updates)

With $f(x) = 2^{-x}$ and $p_0^{(0)} = 1$ we have
$p_1^{(0)} = f(p_0^{(0)}) = 0.5$
$p_2^{(0)} = f(p_1^{(0)}) = 0.7071067812$
$p_0^{(1)} = p_0^{(0)} - \frac{(p_1^{(0)}-p_0^{(0)})^2}{p_2^{(0)}-2p_1^{(0)}+p_0^{(0)}} = 1 - \frac{(0.5-1)^2}{0.7071067812-2(0.5)+1} = 0.6464466095$
$p_1^{(1)} = f(p_0^{(1)}) = 0.6388518841$
$p_2^{(1)} = f(p_1^{(1)}) = 0.6422238357$
$p_0^{(2)} = p_0^{(1)} - \frac{(p_1^{(1)}-p_0^{(1)})^2}{p_2^{(1)}-2p_1^{(1)}+p_0^{(1)}} = 0.6464466095 - \frac{(0.6388518841-0.6464466095)^2}{0.6422238357-2(0.6388518841)+0.6464466095} = 0.6411870534$
$p_1^{(2)} = f(p_0^{(2)}) = 0.6411851628$
We stop here because the last two numbers are within 10^{-4} of each other.

Steffensen updates

[3 updates]

(9) Use Steffensen's method to approximate the solution of the fixed point equation $x = \frac{2-e^x+x^2}{3}$, accurate to within 10^{-4}. Use initial guess $p_0 = 0$.

With $g(x) = \frac{2-e^x+x^2}{3}$ and $p_0^{(0)} = 0$ we have
$p_1^{(0)} = g(p_0^{(0)}) = \frac{1}{3}$
$p_2^{(0)} = g(p_1^{(0)}) = 0.2384995620$
$p_0^{(1)} = p_0^{(0)} - \frac{(p_1^{(0)}-p_0^{(0)})^2}{p_2^{(0)}-2p_1^{(0)}+p_0^{(0)}} = 0 - \frac{(\frac{1}{3}-0)^2}{0.2384995620-2(\frac{1}{3})+0} = 0.2595040812$
$p_1^{(1)} = g(p_0^{(1)}) = 0.2570184314$
$p_2^{(1)} = g(p_1^{(1)}) = 0.2576631714$
$p_0^{(2)} = p_0^{(1)} - \frac{(p_1^{(1)}-p_0^{(1)})^2}{p_2^{(1)}-2p_1^{(1)}+p_0^{(1)}} = 0.2595040812 - \frac{(0.2570184314-0.2595040812)^2}{0.2576631714-2(0.2570184314)+0.2595040812} =$ 0.2575303797
$p_1^{(2)} = g(p_0^{(2)}) = 0.2575302608$
$p_2^{(2)} = g(p_1^{(2)}) = 0.2575302918$ (**We can stop here**)
$p_0^{(3)} = p_0^{(2)} - \frac{(p_1^{(2)}-p_0^{(2)})^2}{p_2^{(2)}-2p_1^{(2)}+p_0^{(2)}} = 0.2575303797 - \frac{(0.2575302608-0.2575303797)^2}{0.2575302918-2(0.2575302608)+0.2575303797} =$ 0.2575302854