(2.4) Error Analysis for Iterative Methods

MATH 4701 Numerical Analysis

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For example, $p_n = \frac{1}{2^n}$ converges to zero linearly with $\lambda = \frac{1}{2}$.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$$

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But $\lambda = |g'(p)| \leq K < 1$ and $\lambda = |g'(p)| > 0$. Therefore, (p_n) converges to p linearly.

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The expression $R_{n+1}(x) = \frac{f^{(n+1)}(\zeta)(x-x_0)^{n+1}}{(n+1)!}$ is called the **remainder term** and represents the **error** for using $P_n(x)$ to calculate the value of $f(x)$.

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$$f^{(4)}(x) = (\tfrac{1}{2})(-\tfrac{1}{2})(-\tfrac{3}{2})(-\tfrac{5}{2})x^{-\frac{7}{2}} \Rightarrow f^{(4)}(1) = -\frac{15}{16}.$$

Example

For $f(x) = \sqrt{x}$,

(1) Find the fourth degree Taylor polynomial $P_4(x)$ for $f(x)$ centered at (about) $c = 1$.

$$f(x) = x^{\frac{1}{2}} \Rightarrow f(1) = \sqrt{1} = 1$$
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$$P_4(x) = 1 + \frac{1}{2}(x-1) - \frac{\frac{1}{4}}{2}(x-1)^2 + \frac{\frac{3}{8}}{6}(x-1)^3 - \frac{\frac{15}{16}}{24}(x-1)^4$$

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&P_4(x) = 1 + \tfrac{1}{2}(x-1) - \tfrac{1}{8}(x-1)^2 + \tfrac{1}{16}(x-1)^3 + \tfrac{5}{128}(x-1)^4
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(2) Find the maximum and minimum possible error when we use $P_4(x)$ to approximate $f(0.5) = \sqrt{\frac{1}{2}}$.

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$f^{(5)}(x) = (\frac{105}{32})x^{-\frac{9}{2}}$. Therefore, for $0.5 \leq \zeta \leq 1$, we have

$$R_5(0.5) = \frac{f^{(5)}(\zeta)(0.5-1)^5}{5!}$$

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$h(\zeta) = -\frac{7}{8192\zeta^{\frac{9}{2}}}$ is an **increasing** function.

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Therefore, the error in calculating $f(0.5) = \sqrt{\frac{1}{2}} \approx 0.7071067810$ using $P_4(0.5) = 1 + \frac{1}{2}(0.5-1) - \frac{\frac{1}{4}}{2}(0.5-1)^2 + \frac{\frac{3}{8}}{6}(0.5-1)^3 - \frac{\frac{15}{16}}{24}(0.5-1)^4$

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Therefore, the error in calculating $f(0.5) = \sqrt{\frac{1}{2}} \approx 0.7071067810$ using $P_4(0.5) \approx 0.7084960938$ is between $-\frac{7}{8192(1)^{\frac{9}{2}}} \approx -0.00085449$ and $-\frac{7}{8192(0.5)^{\frac{9}{2}}} \approx -0.019335$.

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(In fact, $f(0.5) - P_4(0.4) \approx -0.0013893128$)

Quadratic Convergence of Newton's Method

Theorem: Suppose $g \in C^2[a, b]$ and $g(p) = p$ for some $p \in (a, b)$.

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Therefore, if p_0 is in $[p - \delta, p + \delta]$ all the values of p_n are in $[p - \delta, p + \delta]$ as well.

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Since $g(p) = p$ and $g'(p) = 0$

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In particular, since all the values of p_n are in $[p - \delta, p + \delta]$, there is ζ_n between p_n and p such that

$$p_{n+1} = g(p_n) = p + \frac{g''(\zeta_n)}{2}(p_n - p)^2 \Rightarrow \frac{p_{n+1} - p}{(p_n - p)^2} = \frac{g''(\zeta_n)}{2}$$

Quadratic Convergence of Newton's Method

Theorem: Suppose $g \in C^2[a, b]$ and $g(p) = p$ for some $p \in (a, b)$. **Additionally, assume $g'(p) = 0$.** Then there is $\delta > 0$ such that for all $p_0 \in [p - \delta, p + \delta]$, the sequence (p_n) defined by $p_{n+1} = g(p_n)$ converges to p **at least quadratically**.

Proof:

By **Taylor's Theorem** for all $x \in [p - \delta, p + \delta]$, there is ζ_x between x and p such that

$$g(x) = g(p) + g'(p)(x - p) + \frac{g''(\zeta_x)}{2}(x - p)^2 = p + \frac{g''(\zeta_x)}{2}(x - p)^2$$

In particular, since all the values of p_n are in $[p - \delta, p + \delta]$, there is ζ_n between p_n and p such that

$$p_{n+1} = g(p_n) = p + \frac{g''(\zeta_n)}{2}(p_n - p)^2 \Rightarrow \frac{p_{n+1} - p}{(p_n - p)^2} = \frac{g''(\zeta_n)}{2}$$

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Since ζ_n is between p_n and p, $\lim_{n\to\infty} p_n = p$ and $g''(x)$ is continuous on $[p - \delta, p + \delta]$ which contains p and all values of p_n.

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Proof:

$$\Rightarrow \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \lim_{n\to\infty} \frac{|g''(\zeta_n)|}{2} = \frac{|g''(p)|}{2}$$

Since ζ_n is between p_n and p, $\lim_{n\to\infty} p_n = p$ and $g''(x)$ is continuous on $[p-\delta, p+\delta]$ which contains p and all values of p_n.
This means (p_n) converges to p quadratically if $g''(p) \neq 0$ and of higher convergence order if $g''(p) = 0$.

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In attempting to create a fixed point method that helps us find root$p$ of $f(x) = 0$, we can try to work with $g(x) = x + \phi(x)f(x)$, where $\phi(x)f(x)$ as a functional multiple of $f(x)$ will also have a root at p and adding x will change the root finding problem to a fixed point problem.

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Then $g'(x) = 1 + \phi'(x)f(x) + \phi(x)f'(x)$

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Therefore, $g'(p) = 0 \Leftrightarrow 1 + \phi(p)f'(p) = 0 \Leftrightarrow \phi(p) = \frac{-1}{f'(p)}$.

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In attempting to create a fixed point method that helps us find root p of $f(x) = 0$, we can try to work with $g(x) = x + \phi(x)f(x)$, where $\phi(x)f(x)$ as a functional multiple of $f(x)$ will also have a root at p and adding x will change the root finding problem to a fixed point problem.

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Therefore, $g'(p) = 0 \Leftrightarrow 1 + \phi(p)f'(p) = 0 \Leftrightarrow \phi(p) = \frac{-1}{f'(p)}$.

In particular, if we take $\phi(x) = \frac{-1}{f'(x)}$ we can accomplish $g'(p) = 0$ for $g(x) = x - \frac{f(x)}{f'(x)}$ which gives us Newton's method.

Quadratic Convergence of Newton's Method

Theorem: Suppose $g \in C^2[a, b]$ and $g(p) = p$ for some $p \in (a, b)$. **Additionally, assume $g'(p) = 0$.** Then there is $\delta > 0$ such that for all $p_0 \in [p - \delta, p + \delta]$, the sequence (p_n) defined by $p_{n+1} = g(p_n)$ converges to p **at least quadratically**.

If we want to use the theorem above, we would need for f to be in $C^3[a, b]$ (so that $g \in C^2[a, b]$).

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If we want to use the theorem above, we would need for f to be in $C^3[a, b]$ (so that $g \in C^2[a, b]$). We can, however, show that Newton's method converges at least quadratically only using the conditions stated in the convergence of Newton's method theorem and only assuming $f \in C^2[a, b]$.

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If $f \in C^1[a,b]$ and $f''(x)$ exists, by **Taylor's theorem**,

$$f(x) = f(p) + f'(p)(x-p) + f''(\zeta)\frac{(x-p)^2}{2}$$

for some $\delta > 0$ and all x in $[p-\delta, p+\delta]$ and some ζ between x and p.

Quadratic Convergence of Newton's Method

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for some $\delta > 0$ and all x in $[p - \delta, p + \delta]$ and some ζ between x and p.

We have already seen for a choice of $\delta > 0$ all points p_n are in $[p - \delta, p + \delta]$ and letting the center $c = p_n$ and $x = p$, we get

$$f(p) = f(p_n) + f'(p_n)(p - p_n) + f''(\zeta_n)\frac{(p - p_n)^2}{2}$$

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for some ζ_n between p_n and p.
Since $f(p) = 0$ we have

$$0 = f(p_n) + f'(p_n)(p - p_n) + f''(\zeta_n)\frac{(p - p_n)^2}{2}$$

for some ζ_n between p_n and p.

Quadratic Convergence of Newton's Method

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Dividing both sides by $f'(p_n) \neq 0$ we have

$$0 = \frac{f(p_n)}{f'(p_n)} + p - p_n + \frac{f''(\zeta_n)}{f'(p_n)}\frac{(p - p_n)^2}{2}$$

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for some ζ_n between p_n and p.

Since $p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$ we get

$$p_n - \frac{f(p_n)}{f'(p_n)} = p + \frac{f''(\zeta_n)}{f'(p_n)} \frac{(p - p_n)^2}{2}$$

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$$\Rightarrow \frac{p_{n+1} - p}{(p - p_n)^2} = \frac{f''(\zeta_n)}{2f'(p_n)}$$

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Since $p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$ we get

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$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p - p_n|^2} = \lim_{n\to\infty} \frac{|f''(\zeta_n)|}{2|f'(p_n)|}$$

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If $f''(x)$ is continuous on $[p - \delta, p + \delta]$, since ζ_n between p_n and p and p and p_n are in $[p - \delta, p + \delta]$.

Quadratic Convergence of Newton's Method

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p - p_n|^2} = \lim_{n\to\infty} \frac{|f''(\zeta_n)|}{2|f'(p_n)|} = \frac{|f''(p)|}{2|f'(p)|}$$

If $f''(x)$ is continuous on $[p-\delta, p+\delta]$, since ζ_n between p_n and p and p and p_n are in $[p-\delta, p+\delta]$.

This means Newton's method converges quadratically if $f''(p) \neq 0$ and it converges faster than quadratically if $f''(p) = 0$.

Multiple Roots

A solution p of $f(x) = 0$ is a **zero of multiplicity** m if $f(x) = (x - p)^m q(x)$ for a function $q(x)$ satisfying $\lim_{x \to p} q(x) \neq 0$.

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p is called a **simple zero** of $f(x)$ if p is a zero of **multiplicity one**.

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Theorem: For $f \in C^1[a,b]$, $p \in (a,b)$ is a simple zero of $f(x)$ if and only if $f(p) = 0$ and $f'(p) \neq 0$.

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Proof: If p is a simple zero of $f(x)$ then $f(x) = (x-p)q(x)$ such that $\lim_{x \to p} q(x) \neq 0$.

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Using product rule we get $f'(x) = q(x) + (x-p)q'(x)$ for all $x \neq p$ and since $f \in C^1[a, b]$,
$f'(p) = \lim_{x \to p} f'(x) = q(p) + (p-p)q'(p) = q(p) \neq 0$.

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Using product rule we get $f'(x) = q(x) + (x-p)q'(x)$ for all $x \neq p$ and since $f \in C^1[a,b]$,
$f'(p) = \lim_{x \to p} f'(x) = q(p) + (p-p)q'(p) = q(p) \neq 0$. Therefore, a simple zero p satisfies $f(p) = 0$ and $f'(p) \neq 0$.

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Now assume $f(p) = 0$ and $f'(p) \neq 0$.

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Proof:
Now assume $f(p) = 0$ and $f'(p) \neq 0$. Then for all $x \in [a, b]$, $f(x) = f(p) + f'(\zeta(x))(x - p)$ where $\zeta(x)$ is between p and x.

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Now assume $f(p) = 0$ and $f'(p) \neq 0$. Then for all $x \in [a, b]$, $f(x) = f(p) + f'(\zeta(x))(x-p)$ where $\zeta(x)$ is between p and x. Letting $q(x) = f'(\zeta(x))$, using $f(p) = 0$ we get $f(x) = q(x)(x-p)$.

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Now assume $f(p) = 0$ and $f'(p) \neq 0$. Then for all $x \in [a, b]$, $f(x) = f(p) + f'(\zeta(x))(x - p)$ where $\zeta(x)$ is between p and x. Letting $q(x) = f'(\zeta(x))$, using $f(p) = 0$ we get $f(x) = q(x)(x - p)$. Moreover, $\lim_{x \to p} q(x) = \lim_{x \to p} f'(\zeta(x))$

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Now assume $f(p) = 0$ and $f'(p) \neq 0$. Then for all $x \in [a, b]$, $f(x) = f(p) + f'(\zeta(x))(x - p)$ where $\zeta(x)$ is between p and x. Letting $q(x) = f'(\zeta(x))$, using $f(p) = 0$ we get $f(x) = q(x)(x - p)$. Moreover, $\lim_{x \to p} q(x) = \lim_{x \to p} f'(\zeta(x)) = f'(p) \neq 0$.

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and $f^{(3)}(x) = e^x \Rightarrow f^{(3)}(0) = 1 \neq 0$.

Therefore, $p = 0$ is a zero of **multiplicity three** for $f(x)$.

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