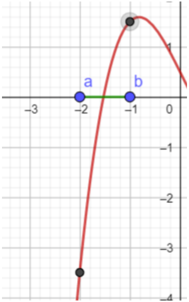
(2.1) The Bisection Method

MATH 4701 Numerical Analysis

Intermediate Value Theorem

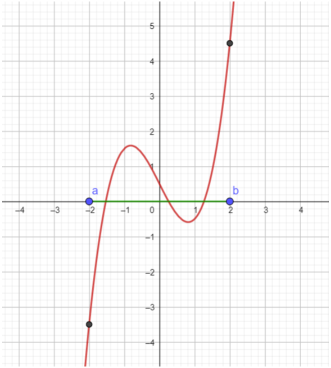
Intermediate value theorem (special case): If $f(x) : [a, b] \to \mathbb{R}$ is **continuous** and $f(a)f(b) < 0$ ($f(a)$ and $f(b)$ have opposite signs) then for some $a < \alpha < b$ we have $f(\alpha) = 0$.



Intermediate Value Theorem

Intermediate value theorem (special case): If $f(x) : [a, b] \to \mathbb{R}$ is **continuous** and $f(a)f(b) < 0$ ($f(a)$ and $f(b)$ have opposite signs) then for some $a < \alpha < b$ we have $f(\alpha) = 0$.

Notice that the theorem states the existence of one solution to $f(x) = 0$ while there may be multiple such solutions.



Intermediate Value Theorem

Intermediate value theorem (special case): If $f(x) : [a, b] \to \mathbb{R}$ is **continuous** and $f(a)f(b) < 0$ ($f(a)$ and $f(b)$ have opposite signs) then for some $a < \alpha < b$ we have $f(\alpha) = 0$.

Notice that the theorem states the existence of one solution to $f(x) = 0$ while there may be multiple such solutions.

While the theorem seems to only have theoretical implications on existence of a solution without giving any information on how that solution can be found, one can in fact approximate a solution $f(x) = 0$ using **bisection method** which is based on the intermediate value theorem.

Bisection Method

Suppose $f(x)$ is a **continuous** function which satisfies $f(x_0)f(x_1) < 0$ and we ask for a value $\epsilon > 0$ of maximum **error** we are willing to tolerate.

Bisection Method

Suppose $f(x)$ is a **continuous** function which satisfies $f(x_0)f(x_1) < 0$ and we ask for a value $\epsilon > 0$ of maximum **error** we are willing to tolerate.

We start by letting $A = x_0$ and $B = x_1$.

At each step of the algorithm, we let $x_{n+1} = \frac{A+B}{2}$ (midpoint of A and B).

Bisection Method

Suppose $f(x)$ is a **continuous** function which satisfies $f(x_0)f(x_1) < 0$ and we ask for a value $\epsilon > 0$ of maximum **error** we are willing to tolerate.

We start by letting $A = x_0$ and $B = x_1$.

At each step of the algorithm, we let $x_{n+1} = \frac{A+B}{2}$ (midpoint of A and B).

- If $f(x_{n+1})f(A) < 0$ then we let $B = x_{n+1}$ and we keep the value of A unchanged.

Suppose $f(x)$ is a **continuous** function which satisfies $f(x_0)f(x_1) < 0$ and we ask for a value $\epsilon > 0$ of maximum **error** we are willing to tolerate.

We start by letting $A = x_0$ and $B = x_1$.

At each step of the algorithm, we let $x_{n+1} = \frac{A+B}{2}$ (midpoint of A and B).

- If $f(x_{n+1})f(A) < 0$ then we let $B = x_{n+1}$ and we keep the value of A unchanged.
- If $f(x_{n+1})f(B) < 0$ then we let $A = x_{n+1}$ and we keep the value of B unchanged.

Bisection Method

Suppose $f(x)$ is a **continuous** function which satisfies $f(x_0)f(x_1) < 0$ and we ask for a value $\epsilon > 0$ of maximum **error** we are willing to tolerate.

We start by letting $A = x_0$ and $B = x_1$.

At each step of the algorithm, we let $x_{n+1} = \frac{A+B}{2}$ (midpoint of A and B).

- If $f(x_{n+1})f(A) < 0$ then we let $B = x_{n+1}$ and we keep the value of A unchanged.
- If $f(x_{n+1})f(B) < 0$ then we let $A = x_{n+1}$ and we keep the value of B unchanged.
- If $f(x_{n+1}) = 0$ we announce $\alpha = x_{n+1}$ as the solution of $f(x) = 0$ and we terminate the process.

Bisection Method

Suppose $f(x)$ is a **continuous** function which satisfies $f(x_0)f(x_1) < 0$ and we ask for a value $\epsilon > 0$ of maximum **error** we are willing to tolerate.

We start by letting $A = x_0$ and $B = x_1$.

At each step of the algorithm, we let $x_{n+1} = \frac{A+B}{2}$ (midpoint of A and B).

- If $f(x_{n+1})f(A) < 0$ then we let $B = x_{n+1}$ and we keep the value of A unchanged.
- If $f(x_{n+1})f(B) < 0$ then we let $A = x_{n+1}$ and we keep the value of B unchanged.
- If $f(x_{n+1}) = 0$ we announce $\alpha = x_{n+1}$ as the solution of $f(x) = 0$ and we terminate the process.

Notice that in every step $f(A)$ and $f(B)$ will remain two non zero numbers with opposite signs and the above three options are the only possibilities.

Bisection Method

We start by letting $A = x_0$ and $B = x_1$.

At each step of the algorithm, we let $x_{n+1} = \frac{A+B}{2}$ (midpoint of A and B).

- If $f(x_{n+1})f(A) < 0$ then we let $B = x_{n+1}$ and we keep the value of A unchanged.
- If $f(x_{n+1})f(B) < 0$ then we let $A = x_{n+1}$ and we keep the value of B unchanged.
- If $f(x_{n+1}) = 0$ we announce $\alpha = x_{n+1}$ as the solution of $f(x) = 0$ and we terminate the process.

Notice that in every step $f(A)$ and $f(B)$ will remain two non zero numbers with opposite signs and the above three options are the only possibilities.

Termination: We stop the algorithm and announce $\alpha = x_{n+1}$ as the solution of $f(x) = 0$ once $\frac{|x_1 - x_0|}{2^n} < \epsilon$.

Bisection Method

At each step of the algorithm, we let $x_{n+1} = \frac{A+B}{2}$ (midpoint of A and B).

- If $f(x_{n+1})f(A) < 0$ then we let $B = x_{n+1}$ and we keep the value of A unchanged.
- If $f(x_{n+1})f(B) < 0$ then we let $A = x_{n+1}$ and we keep the value of B unchanged.
- If $f(x_{n+1}) = 0$ we announce $\alpha = x_{n+1}$ as the solution of $f(x) = 0$ and we terminate the process.

Notice that in every step $f(A)$ and $f(B)$ will remain two non zero numbers with opposite signs and the above three options are the only possibilities.

Termination: We stop the algorithm and announce $\alpha = x_{n+1}$ as the solution of $f(x) = 0$ once $\frac{|x_1 - x_0|}{2^n} < \epsilon$. It is because x_{n+1} is the midpoint of an interval of length $\frac{|x_1 - x_0|}{2^{n-1}}$ where a solution of $f(x) = 0$ exists.

Examples

- Use the bisection method for $f(x) = \sin x + x^2 - 2$ with $x_0 = -2$ and $x_1 = -1$ to approximate a solution of $f(x) = 0$ accurate within an error of at most $\frac{1}{50}$.

Examples

- Use the bisection method for $f(x) = \sin x + x^2 - 2$ with $x_0 = -2$ and $x_1 = -1$ to approximate a solution of $f(x) = 0$ accurate within an error of at most $\frac{1}{50}$.

- If we use the bisection method for $f(x) = x^3 - 2x + 0.5$ with $x_0 = -2$ and $x_1 = 2$ what is the smallest index n for which we know x_n is within 0.001 of an actual solution of $f(x) = 0$.

Examples

- Use the bisection method for $f(x) = \sin x + x^2 - 2$ with $x_0 = -2$ and $x_1 = -1$ to approximate a solution of $f(x) = 0$ accurate within an error of at most $\frac{1}{50}$.

- If we use the bisection method for $f(x) = x^3 - 2x + 0.5$ with $x_0 = -2$ and $x_1 = 2$ what is the smallest index n for which we know x_n is within 0.001 of an actual solution of $f(x) = 0$.

- For $F(x) = (x+1)x^3(x-1)(x-2)^2(x-3)(x-5)$ if we start with $x_0 = -2$ and $x_1 = 7$ which of the solutions of $F(x) = 0$ will the bisection method find.