(3.4) Hermite Interpolation

MATH 4701 Numerical Analysis

Hermite Interpolation

Suppose x_0, x_1, ..., x_n are $n+1$ **distinct** numbers, and y_0, y_1, ...,y_n and z_0, z_1, ...,z_n are arbitrary numbers.

Hermite Interpolation

Suppose x_0, x_1, ..., x_n are $n+1$ **distinct** numbers, and y_0, y_1, ...,y_n and z_0, z_1, ...,z_n are arbitrary numbers.

We would like to find a polynomial $H(x)$ of degree at most $2n+1$ such that

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Although we already know how to find such polynomial using divided differences, we want to find a description of such polynomial in a way similar to that of Lagrange interpolating polynomial as it turns out to be useful later.

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For each $0 \le k \le n$, we find a polynomial $H_k(x)$ of degree $2n+1$ satisfying

$$\begin{cases} H_k(x_k) = 1 & \\ H_k(x_j) = 0 & \text{for all } 0 \le j \le n,\ j \ne k \\ H_k'(x_j) = 0 & \text{for all } 0 \le j \le n \end{cases}$$

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For each $0 \le k \le n$, we find a polynomial $\hat{H}_k(x)$ of degree $2n+1$ satisfying

$$\begin{cases} \hat{H}_k(x_j) = 0 & \text{for all } 0 \le j \le n \\ \hat{H}_k{}'(x_j) = 0 & \text{for all } 0 \le j \le n,\ j \ne k \\ \hat{H}_k{}'(x_k) = 1 & \end{cases}$$

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Then $H(x) = y_0 H_0(x) + \ldots + y_n H_n(x) + z_0 \hat{H}_0(x) + \ldots + z_n \hat{H}_n(x)$ is a polynomial of degree at most $2n+1$ satisfying the conditions (H) above.

Construction of $H_k(x)$

For each $0 \leq k \leq n$, we would like to find the polynomial $H_k(x)$ of degree $2n+1$ satisfying

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Polynomial $L_k(x) = \prod_{\substack{1 \leq j \leq n \\ j \neq k}} \frac{(x-x_j)}{(x_k-x_j)} = \frac{(x-x_0)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)}$, of degree n, satisfies $L_k(x_k) = 1$ and $L_k(x_j) = 0$ for all $0 \leq j \leq n$, $j \neq k$.

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Polynomial $L_k^2(x) = \prod_{\substack{1 \le j \le n \\ j \ne k}} \frac{(x-x_j)^2}{(x_k-x_j)^2} = \frac{(x-x_0)^2\dots(x-x_{k-1})^2(x-x_{k+1})^2\dots(x-x_n)^2}{(x_k-x_0)^2\dots(x_k-x_{k-1})^2(x_k-x_{k+1})^2\dots(x_k-x_n)^2}$, of degree $2n$, satisfies $L_k^2(x_k) = 1$ and $L_k^2(x_j) = 0$ and $(L_k^2)'(x_j) = 0$ for all $0 \le j \le n$, $j \ne k$ (all x_j, $j \ne k$ are double roots of $L_k^2(x)$).

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Additionally, $H_k'(x_k) = AL_k^2(x_k) + (A(x_k-x_k)+1)2L_k(x_k)L_k'(x_k)$

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Polynomial $H_k(x) = (A(x-x_k)+1)L_k^2(x)$, of degree $2n+1$, satisfies $H_k(x_k) = (A(x_k-x_k)+1)L_k^2(x_k) = 1$ and $H_k(x_j) = (A(x_j-x_k)+1)L_k^2(x_j) = 0$.

$H_k'(x) = AL_k^2(x) + (A(x-x_k)+1)2L_k(x)L_k'(x)$.

For all $0 \leq j \leq n$, $j \neq k$, $H_k'(x_j) = AL_k^2(x_j) + (A(x_j-x_k)+1)2L_k(x_j)L_k'(x_j) = 0$

Additionally, $H_k'(x_k) = AL_k^2(x_k) + (A(x_k-x_k)+1)2L_k(x_k)L_k'(x_k) = A(1)^2 + (A(0)+1)2(1)L_k'(x_k) = A + 2L_k'(x_k)$. Therefore, to make sure $H_k'(x_k) = 0$, we choose $A = -2L_k'(x_k)$.

Construction of $H_k(x)$

For each $0 \le k \le n$, we would like to find the polynomial $H_k(x)$ of degree $2n+1$ satisfying

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Polynomial $H_k(x) = (A(x - x_k) + 1)L_k^2(x)$, of degree $2n+1$, satisfies $H_k(x_k) = (A(x_k - x_k) + 1)L_k^2(x_k) = 1$ and $H_k(x_j) = (A(x_j - x_k) + 1)L_k^2(x_j) = 0$.

$H_k'(x) = AL_k^2(x) + (A(x - x_k) + 1)2L_k(x)L_k'(x)$.

For all $0 \le j \le n$, $j \ne k$, $H_k'(x_j) = AL_k^2(x_j) + (A(x_j - x_k) + 1)2L_k(x_j)L_k'(x_j) = 0$

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Therefore, $H_k(x) = (-2L_k'(x_k)(x - x_k) + 1)L_k^2(x)$, where $L_k(x) = \prod_{\substack{1 \le j \le n \\ j \ne k}} \frac{(x - x_j)}{(x_k - x_j)}$ is the polynomial of degree $2n+1$ satisfying the conditions $\blacksquare$.

Construction of $\hat{H}_k(x)$

For each $0 \leq k \leq n$, we would like to find the polynomial $\hat{H}_k(x)$ of degree $2n+1$ satisfying

$$\begin{cases} \hat{H}_k(x_j) = 0 & \text{for all } 0 \leq j \leq n \\ \hat{H}_k{}'(x_j) = 0 & \text{for all } 0 \leq j \leq n,\ j \neq k \\ \hat{H}_k{}'(x_k) = 1 & \end{cases}$$

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$$\blacksquare \begin{cases} \hat{H}_k(x_j) = 0 & \text{for all } 0 \leq j \leq n \\ \hat{H}_k{}'(x_j) = 0 & \text{for all } 0 \leq j \leq n,\ j \neq k \\ \hat{H}_k{}'(x_k) = 1 & \end{cases}$$

$\hat{H}_k(x) = (x - x_k)L_k^2(x)$, where $L_k(x) = \prod_{\substack{1 \leq j \leq n \\ j \neq k}} \frac{(x-x_j)}{(x_k-x_j)}$ is the polynomial of degree $2n+1$ satisfying the conditions $\blacksquare$.

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Example

Find the Hermite interpolation polynomial for the following data:

x	1.1	1.3	1.5
$f(x)$	0.45	0.27	0.07
$f'(x)$	-0.89	-0.96	-1

Error Estimate For Hermite Interpolation

Suppose $f \in C^{2n+1}[a, b]$ and $f^{(2n+2)}(x)$ exists for all $x \in (a, b)$ and x_0, x_1, x_2, ..., x_n are distinct numbers in $[a, b]$.

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$$H_n(x_k) = f(x_k) \quad \text{and} \quad H_n'(x_k) = f'(x_k) \qquad \text{for all } 0 \le k \le n.$$

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Then for each $x \in [a, b]$, there is a number $\zeta(x) \in [x_0, x_1, ..., x_n, x]$ such that

$$f(x) - H_n(x) = \frac{f^{(2n+2)}(\zeta(x))}{(2n+2)!}(x-x_0)^2(x-x_1)^2...(x-x_n)^2$$