Problem 4: Lorentz Invariance

(*Lorentz invariance of the wave equation*) Thinking of the coordinates of space-time as 4-vectors (x, y, z, t), let Γ be the diagonal matrix with the diagonal entries $1, 1, 1, -1$. Another matrix L is called a *Lorentz transformation* if L has an inverse and $L^{-1} = \Gamma\,{}^tL\,\Gamma$, where ${}^tL$ is the transpose.

(a) If L and M are Lorentz, show that LM and L^{-1} also are.

(b) Show that L is Lorentz if and only if $m(L\mathbf{v}) = m(\mathbf{v})$ for all 4-vectors $\mathbf{v} = (x, y, z, t)$, where $m(\mathbf{v}) = x^2 + y^2 + z^2 - t^2$ is called the *Lorentz metric*.

(c) If $u(x, y, z, t)$ is any function and L is Lorentz, let $U(x, y, z, t) = u(L(x, y, z, t))$. Show that

$$U_{xx} + U_{yy} + U_{zz} - U_{tt} = u_{xx} + u_{yy} + u_{zz} - u_{tt}.$$

(d) Explain the meaning of a Lorentz transformation in more geometrical terms. (*Hint*: Consider the level sets of $m(\mathbf{v})$.)

(a) let L & M be Lorentz.

$L^{-1} = \Gamma\,{}^tL\,\Gamma$

$M^{-1} = \Gamma\,{}^tM\,\Gamma$

$\Gamma = {}^t\Gamma = \Gamma^{-1}$

the inverse of LM is

$(LM)^{-1} = M^{-1}L^{-1}$

$= \Gamma\,{}^tM\,\underbrace{\Gamma\,\Gamma^{-1}}_{I}\,{}^tL\,\Gamma$

$= \Gamma\,{}^tM\,{}^tL\,\Gamma = \Gamma\,{}^t(LM)\,\Gamma$

is a linear transformation

LM is Lorentz

the inverse of L^{-1} is

$(L^{-1})^{-1} = L = {}^t(\Gamma^{-1}L^{-1}\Gamma^{-1})$

$= {}^t(\Gamma L^{-1}\Gamma) = {}^t\Gamma\,{}^tL^{-1}\,{}^t\Gamma$

$= \Gamma\,{}^tL^{-1}\,\Gamma$

L^{-1} is Lorentz

(b) Assume L is Lorentz

$m(v) = x^2 + y^2 + z^2 - t^2 = {}^tv\,\Gamma\,v.$

we can derive $\Gamma = {}^tL\,\Gamma\,L$

$m(v) = {}^tv\,{}^tL\,\Gamma\,L\,v = {}^t(Lv)\,\Gamma\,(Lv) = m(Lv)$

now assume that there exists a matrix L such for every vector v $m(v) = m(Lv)$

${}^tv\,\Gamma\,v = {}^tv\,{}^tL\,\Gamma\,L\,v$

for all vectors v

$\Gamma = {}^tL\,\Gamma\,L$ which is = part (1) & is Lorentz.

(c) let $u(x,y,z,t)$ be an arbitrary function & L Lorentz. let U be

$U(x,y,z,t) = u(L(x,y,z,t))$

denote $v = (v_1, v_2, v_3, v_4)$ then

$$U_{xx} + U_{yy} + U_{zz} - U_{tt} = \sum_{i=1}^{4}\sum_{j=1}^{4} \Gamma_{ij} \frac{d}{di}\frac{d}{dj} u(Lv) = \sum_{i=1}^{4}\sum_{j=1}^{4} \Gamma_{ij} \sum_{k=1}^{4}\sum_{l=1}^{4} \frac{d^2u(Lv)}{dv_k\,dv_l}\frac{d}{dv_i}(Lv)_k \frac{d}{dv_j}(Lv)_l =$$

$$\sum_{i=1}^{4}\sum_{j=1}^{4} \Gamma_{ij} \sum_{k=1}^{4}\sum_{l=1}^{4} \frac{d^2u(Lv)}{dv_k\,dv_l}\left(\sum_{n=1}^{4} L_{kn} \underbrace{\frac{dv_n}{dv_i}}_{\delta_{ni}}\right)\left(\sum_{n=1}^{4} L_{ln} \underbrace{\frac{dv_n}{dv_j}}_{\delta_{nj}}\right) = \sum_{i=1}^{4}\sum_{j=1}^{4} \Gamma_{ij} \sum_{k=1}^{4}\sum_{l=1}^{4} \frac{d^2u(Lv)}{dv_k\,dv_l} L_{ki}L_{lj} =$$

$$\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4}\sum_{l=1}^{4} L_{ki}\Gamma_{ij}L_{lj}\left(\frac{d^2u(Lv)}{dv_k\,dv_l}\right) \overset{\Gamma = L\Gamma{}^tL}{=} \sum_{k=1}^{4}\sum_{l=1}^{4} \Gamma_{kl}\left(\frac{d^2U(v)}{dv_k\,dv_l}\right) = \boxed{U_{xx} + U_{yy} + U_{zz} - U_{tt}}$$

- so the equation is invariant under lorentz transformations

(d) the lorentz metric of a vector $v = (x, y, z, t)$ is $m(v) = |(x, y, z, it)|^2$, (linear transformation made at constant velocity) represents magnitude in space. This transformation is an imaginary rotation in space

Problem 4: Lorentz Invariance

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$\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4}\sum_{l=1}^{4} L_{ki}\Gamma_{ij}L_{lj} \left(\frac{d^2 u(Lv)}{dv_k dv_l}\right) \overset{\Gamma = L\Gamma^{t}L}{=}$

$\sum_{k=1}^{4}\sum_{l=1}^{4} \Gamma_{kl}{}^{t}\left(\frac{d^2 U(v)}{dv_k dv_n}\right) = \boxed{U_{xx} + U_{yy} + U_{zz} - U_{tt}}$

. So the equation is invariant under lorentz transformations

(d) the lorentz metric of a vector $v = (x,y,z,t)$ is $m(v) = |(x,y,z,it)|^2$. (linear transformation move at constant velocity rotation in space) this represents magnitude in space. this transformation is an

Problem 1

We well know that the smallest eigenvalue for the Neumann BCs is $\tilde{\lambda}_1 = 0$ (with the constant eigenfunction). Show that $\tilde{\lambda}_2 > 0$. This is the same as saying that zero is a simple eigenvalue, that is, of multiplicity 1.

$$\frac{\|\nabla v_i\|^2}{\|v_i\|^2} = 0 \Rightarrow \nabla v_i = 0, \; i = 1, 2, \ldots$$

by orthognality this is true. But further explanation is needed.

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[illegible]

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Problem 2

Derive the equations of electrostatics from the Maxwell equations by assuming that $\partial E/\partial t = \partial B/\partial t \equiv 0$.

1) Derive the continuity equation $d\rho/dt + \nabla \cdot J = 0$ from the inhomongeneous Maxwell equations.

$$\frac{d\rho}{dt} = -\nabla \cdot J$$

$$\frac{d}{dt} \nabla \cdot E = 4\pi \frac{d\rho}{dt}$$

$$\frac{dE}{dt} = c\nabla \times B - 4\pi J$$

$$\nabla \cdot c\nabla \times B - 4\pi \nabla \cdot J = 4\pi \frac{d\rho}{dt}.$$

$$c\nabla \cdot \nabla \times B = 0$$

$$\frac{d\rho}{dt} = -\nabla \cdot J.$$

Problem 7

(a) Show that if $\lambda \neq 2k+1$, any solution of Hermite's ODE is a power series but not a polynomial.

(b) Deduce that in this case no solution of Hermite's ODE can satisfy the condition at infinity. (*Hint*: Use the recursion relation (18) to find the behavior of a_k as $k \to \infty$. Compare with the power series expansion of e^{x^2}. Deduce that $u(x,t)$ behaves like e^{x^2} as $|x| \to \infty$.)

(a) Let λ be an arbitrary number non odd

$$v'' - 2xv' + (\lambda - 1)v = 0, \text{ using the series method}$$

$$v(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$v'' - 2xv' + (\lambda-1)v = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - 2\sum_{n=0}^{\infty} n a_n x^n + (\lambda-1)\sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} a_n(\lambda - 2n - 1)x^n = \sum_{n=0}^{\infty}\left[a_{n+2}(n+2)(n+1) + a_n(\lambda - 2n - 1)\right]x^n$$

$$v'' - 2xv' + (\lambda-1)v = 0$$

$$a_{n+2} = \underbrace{\frac{\overbrace{\lambda - 2n - 1}^{\neq 0}}{(n+2)(n+1)}}_{\neq 0} a_n.$$

the solution is trivial

1 is a solution to a power series

(b)

$$v(x) = a_0 + a_1 x + \sum_{n=0}^{\infty} \frac{\lambda - 2n - 1}{(n+2)(n+1)} a_n x^{n+2}$$

$$a_{n+2} = \frac{\lambda - 2n - 1}{(n+2)(n+1)} a_n = \frac{\lambda - 2n - 1}{(n+2)(n+1)} \cdot \frac{\lambda - 2(n-1) - 1}{(n)(n-1)} a_{n-2}$$

$$a_n = \frac{(\lambda - 2(n-1) - 1)\cdot \ldots \cdot(\lambda - 1)}{n!} a_0$$

$$a_n = \frac{(\lambda - 2(n-1) - 1)\cdot \ldots \cdot(\lambda - 3)}{n!} a_1$$

Either way the sum doesn't converge except in the trivial case.

Problem 4

Let $S(x,t)$ be the source function (Riemann function) for the one-dimensional wave equation. Calculate $\partial S/\partial t$ and find the PDE and initial conditions that it satisfies.

this given function

$$S(x,t) \begin{cases} \frac{1}{2c}, & |x| < ct \\ 0, & |x| > ct \end{cases}$$

$$(S,\phi) = \frac{1}{2c}\int_{-ct}^{ct} \phi(x)\,dx$$

$$\left(\frac{dS}{dt},\phi\right) = \frac{1}{2c}\left(c\phi(ct) + c\phi(-ct)\right)$$

$$= \frac{1}{2}\left(\phi(ct) + \phi(-ct)\right)$$

$$\frac{dS}{dt} = \frac{1}{2}\left(\delta(x-ct) + \delta(x+ct)\right)$$

$$A = \frac{dS}{dt}$$

$$A_{tt} = \frac{c^2}{2}\left(\delta''(x-ct) + \delta''(x+ct)\right)$$

$$A_{xx} = \frac{1}{2}\left(\delta''(x-ct) + \delta''(x+ct)\right)$$

$$A_{tt} = c^2 A_{xx}$$

$$A(x,0) = \frac{1}{2}\left(\delta(x) + \delta(x)\right) = \delta(x)$$

$$A_t(x,0) = S_{tt}(x,0) = c^2 \Delta S(x,0) = 0$$

$\frac{dS}{dt}$ satisfies this problem

$$\begin{cases} \left(\frac{dS}{dt}\right)_{tt} = c^2\left(\frac{dS}{dt}\right)_{xx} \\ \left(\frac{dS}{dt}\right)(x,0) = \delta(x) \\ \left(\frac{dS}{dt}\right)_t(x,0) = 0 \end{cases}$$

~~Problem 7~~

~~Use the Laplace transform to solve $u_t = ku_{xx}$ in $(0,l)$, with $u_x(0,t) = 0$, $u_x(l,t) = 0$, and $u(x,0) = 1 + \cos(2\pi x/l)$.~~

5.) use the laplace transform to solve $u_{tt} = c^2 u_{xx}$ for $0 < x < l$, $u(0,t) = u(l,t) = 0$, $u(x,0) = \sin(\pi x/l)$, & $u_t(x,0) = -\sin(\pi x/l)$.

$$s^2 L(x,s) - s\,u(x,0) - u_t(x,0) = c^2 L_{xx}(x,s)$$

$$c^2 L_{xx}(x,s) = s^2 L(x,s) + (1-s)\sin\left(\frac{\pi x}{l}\right)$$

$$L(0,s) = L(l,s) = 0$$

$$L(x) = A\sin\left(\frac{\pi x}{l}\right)$$

$$-\frac{c^2\pi^2}{l^2} A = s^2 A + 1 - s$$

$$A = \frac{s-1}{s^2 + \left(\frac{c^2\pi^2}{l^2}\right)}$$

$$L(x,s) = \frac{s-1}{s^2 + \left(\frac{c^2\pi^2}{l^2}\right)} \sin\left(\frac{\pi x}{l}\right)$$

$$\left(\cos\left(\frac{ct\pi}{l}\right) - \frac{l}{c\pi}\sin\left(\frac{ct\pi}{l}\right)\right)$$

$$\boxed{u(x,t) = \left(\cos\left(\frac{ct\pi}{l}\right) - \frac{l}{c\pi}\sin\left(\frac{ct\pi}{l}\right)\right)\sin\left(\frac{x\pi}{l}\right)}$$

Problem 4

Solve the wave equation in three dimensions with the initial data $\phi \equiv 0$, $\psi(x, y, z) = x^2 + y^2 + z^2$. (*Hint:* Use (5).)

Plug in the initial conditions too the d'Alembert's formula

$$U(x_0, y_0, z_0, t_0) = \frac{1}{4\pi c^2 t_0}\iint_S \psi(x)\,dS + \frac{d}{dt_0}\left(\frac{1}{4\pi c^2 t_0}\iint_S \phi(x)\,dS\right)$$

$$\frac{1}{4\pi c^2 t_0}\iint_S x^2 + y^2 + z^2\,dS + \frac{d}{dt_0}\left(\frac{1}{4\pi c^2 t_0}\iint_S 0\,dS\right)$$

$$\frac{1}{4\pi c^2 t_0}\iint_S x^2 + y^2 + z^2\,dS$$

$$\frac{t_0}{4\pi}\int_0^{\pi}\int_0^{2\pi}[x_0^2 + y_0^2 + z_0^2 + c^2t_0^2 + 2ct_0(x_0\sin\theta\cos\phi + y_0\sin\theta\sin\phi + z_0\cos\theta]\sin\theta\,d\phi\,d\theta$$

$$= \frac{t_0(x_0^2 + y_0^2 + z_0^2 + c^2t_0^2)}{4\pi}\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\,d\theta + \frac{ct_0^2 z_0}{2\pi}\int_0^{2\pi}d\theta\int_0^{\pi}\underbrace{\cos\theta\sin\theta\,d\theta}_{\underbrace{=\frac{1}{2}\sin 2\theta}_{=0}}$$

$$= \frac{t_0(x_0^2 + y_0^2 + z_0^2 + c^2t_0^2)}{\cancel{4\pi}}\cdot\cancel{2\pi\cdot 2}$$

$$= t_0(x_0^2 + y_0^2 + z_0^2 + c^2t_0^2)$$

↑

Solution to wave with initial conditions

Problem 14

Solve the equation $-u_{xx} - u_{yy} + k^2u = 0$ in the disk $\{x^2 + y^2 < a^2\}$ with $u \equiv 1$ on the boundary circle. Write your answer in terms of the Bessel functions $J_s(iz)$ of imaginary argument.

(The boundary condition is radically symmetric)

$$u_{rr} + \frac{1}{r}u_r = k^2 u$$

$$u(r) = v(ikr)$$

$$-k^2 v_{rr} - \frac{k^2}{r} v_r - k^2 v = 0$$

$$v_{rr} + \frac{1}{r} v_r + v = 0$$

The Bessel equation of order zero

$$u(r) = v(ikr) = C J_0(ikr)$$

where C is a constant & BC is satisfied

$$\boxed{u(r) = \frac{J_0(ikr)}{J_0(ika)}}$$

Problem 5: Vector Space Dimensions

Find the dimension of each of the following vector spaces.

(a) The space of all the solutions of $u'' + x^2 u = 0$.

(b) The eigenspace with eigenvalue $(2\pi/l)^2$ of the operator $-d^2/dt^2$ on the interval $(-l, l)$ with the periodic boundary conditions.

(c) The space of harmonic functions in the unit disk with the homogeneous Neumann BCs.

(d) The eigenspace with eigenvalue $\lambda = 25\pi^2$ of $-\Delta$ in the unit square $(0,1)^2$ with the homogeneous Neumann BCs on all four sides.

(e) The space of all the solutions of $u_{tt} = c^2 u_{xx}$ in $-\infty < x < \infty$, $-\infty < t < \infty$.

(a)

both linearly independent

The space of the solutions is 2 because its a second order ode $u'' + x^2 u = 0$

(b)

$$\begin{cases} -\frac{d^2}{dt} v = \frac{4\pi^2}{l^2} v \\ v(-l) = v(l) \\ v'(-l) = v'(l) \end{cases}$$

The solution is:

$v(x) = A\cos\left(\frac{2\pi x}{l}\right) + B\sin\left(\frac{2\pi x}{l}\right)$ the dimension of the space of all solutions is 2 with A,B arbitrary

(c)

$$\begin{cases} -\Delta u = 0, D \\ \frac{du}{dn} = 0, \text{bdy } D \end{cases}$$

equiv in polar form

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, & r \le 1, 0 \le \theta \le 2\pi \\ u_r(1,\theta) = 0 \end{cases}$$

So solution $u(r,\theta) = A$

$0 \; (\forall n \in \mathbb{N})$

$B_n = A_n = 0$

Space dimension is 1

D is unit disk $u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$, $0 = \sum_{n=1}^{\infty} n(A_n \cos n\theta + B_n \sin n\theta)$

(d)

$$\begin{cases} -\Delta v = 25\pi^2 v, \; 0 < x, y < 1 \\ v_x(0,y) = v_x(1,y) = v_y(x,0) = v_y(x,1) \end{cases}$$

this is the dimension of 4

$\{\cos(5\pi x), \cos(5\pi y), \cos(3\pi x)\cos(4\pi y), \cos(4\pi x)\cos(3\pi y)\}$

So the solution space is

such $k^2 + l^2 = 25$, $l \in \mathbb{N} \cup \{0\}$

$v_{kl}(x) = A_{kl}\cos(k\pi x)\cos(l\pi y)$,

The solution is the sum of functions

$v(x,y) = X(x)Y(y) \rightsquigarrow \frac{X''}{X} + \frac{Y''}{Y} = -25\pi^2 \rightsquigarrow X'(0) = X'(1) = Y'(0) = Y'(1)$

(e)

$-u_{tt} = c^2 u_{xx}$, $-\infty < x, t < \infty$

Dimension: ∞

this solution is via d'Alembert formula & initial conditions (the solutions dimension is ∞

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So solution $u(r,\theta) = A$

$0 \; (\forall n \in \mathbb{N})$ $= B_n$ $= A_n$

space dimension is 1

D is unit disk $u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n(A_n\cos n\theta + B_n\sin n\theta)$, $0 = \sum_{n=1}^{\infty} n(A_n\cos n\theta + B_n\sin n\theta)$

(d)

$$\begin{cases} -\Delta v = 25\pi^2 v, \quad 0 < x, y < 1 \\ v_x(0,y) = v_x(1,y) = v_y(x,0) = v_y(x,1) \end{cases}$$

this is the dimension of 4

$\{\cos(5\pi x), \cos(5\pi y), \cos(3\pi x)\cos(4\pi y), \cos(4\pi x)\cos(3\pi y)\}$

So the solution space is

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(e)

$-u_{tt} = c^2 u_{xx}$, $-\infty < x, t < \infty$

Dimension: ∞

this solution is via d'Alembert formula initial conditions (the solutions dimension is ∞

Problem 4: Convolution Properties

Prove the following properties of the convolution

(a) $f * g = g * f$

(b) $(f * g)' = f' * g = f * g'$, where $'$ denotes the derivative in one variable

(c) $f * (g * h) = (f * g) * h$.

(a)

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy$$

$$= \int_{\infty}^{-\infty} f(z)\, g(x-z)\, (-dz)$$

$$= \int_{-\infty}^{\infty} g(x-z)\, f(z)\, dz$$

$$= \boxed{g * f(x)}$$

hence convolution is commutative

(b)

$$(f * g)'(x) = \frac{d}{dx}\left(\int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy\right)$$

$$= \int_{-\infty}^{\infty} f'(x-y)\, g(y)\, dy$$

$$= f' * g$$

moreover partially integrate

$$\boxed{\begin{array}{ll} g(y) & g'(y) \\ f'(x-y) & -f(x-y) \end{array}}$$

$$(f*g)'(x) = f(x-y)\, g(y)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} f(x-y)\, g'(y)\, dy = \int_{\infty}^{-\infty} f'(z)\, g'(x-z)\, dz = \boxed{f' * g'}$$

$$f * (g * h)(x) = f * \left(\int_{-\infty}^{\infty} g(x-y)\, h(y)\, dy\right) = \int_{-\infty}^{\infty} f(x-z)\left(\int_{-\infty}^{\infty} g(z-y)\, h(y)\, dy\right) dz$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-z)\, g(z-y)\, h(y)\, dy\, dz = \int_{-\infty}^{\infty} h(y) \int_{-\infty}^{\infty} f((x-y)-(z-y))\, g(z-y)\, dz\, dy$$

$$= \int_{-\infty}^{\infty} h(y) \int_{-\infty}^{\infty} f((x-y)-p)\, g(p)\, dp\, dy$$

$$= \int_{-\infty}^{\infty} (f * g)(x-y)\, h(y)\, dy = \boxed{(f * g) * h(x)}$$

Problem 4: Convolution Properties

Prove the following properties of the convolution.

(a) $f * g = g * f$.

(b) $(f * g)' = f' * g = f * g'$, where $'$ denotes the derivative in one variable.

(c) $f * (g * h) = (f * g) * h$.

(a)

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy$$
$$= \int_{\infty}^{-\infty} f(z)\, g(x-z)\,(-dz)$$
$$= \int_{-\infty}^{\infty} g(x-z)\, f(z)\, dz$$
$$= \boxed{g * f(x)}$$

hence convolution is commutative

(b) $$(f * g)'(x) = \frac{d}{dx}\left(\int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy\right.$$
$$= \int_{-\infty}^{\infty} f'(x-y)\, g(y)\, dy$$
$$= f' * g$$

moreover partially integrate

$$\boxed{\begin{array}{ll} g(y) & g'(y) \\ f'(x-y) & -f(x-y) \end{array}}$$

$$(f*g)'(x) = f(x-y)\, g(y)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} f(x-y)\, g'(y)\, dy = \int_{\infty}^{-\infty} f'(z)\, g'(x-z)\, dz = \boxed{f' * g'}$$

$$f*(g*h)(x) = f*\left(\int_{-\infty}^{\infty} g(x-y)\, h(y)\, dy\right) = \int_{-\infty}^{\infty} f(x-z)\left(\int_{-\infty}^{\infty} g(z-y)\, h(y)\, dy\right) dz$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-z)\, g(z-y)\, h(y)\, dy\, dz = \int_{-\infty}^{\infty} h(y) \int_{-\infty}^{\infty} f((x-y)-(z-y))\, g(z-y)\, dz\, dy$$
$$= \int_{-\infty}^{\infty} h(y) \int_{-\infty}^{\infty} f((x-y)-p)\, g(p)\, dp\, dy$$
$$= \int_{-\infty}^{\infty} (f*g)(x-y)\, h(y)\, dy = \boxed{(f*g) * h(x)}$$

Problem 5

(a) Show that the lowest eigenvalue λ_1 of $-\Delta$ with the *Robin* boundary condition $\partial u/\partial n + a(\mathbf{x})u = 0$ is given by

$$\lambda_1 = \min \left\{ \frac{\iiint_D |\nabla w|^2 \, d\mathbf{x} + \iint_{\text{bdy } D} aw^2 \, dS}{\iiint_D w^2 \, d\mathbf{x}} \right\}$$

among all the C^2 functions $w(\mathbf{x})$ for which $w \not\equiv 0$.

(b) Show that λ_1 increases as $a(\mathbf{x})$ increases.

(a)

$$m = \frac{\int_D |\nabla u|^2 + \int_{\partial D} au^2}{\int_D u^2} \leq \frac{\int_D |\nabla w|^2 + \int_{\partial D} aw^2}{\int_D w^2}$$

$$f(\varepsilon) = \frac{\int_D |\nabla(u+\varepsilon v)|^2 + \int_{\partial D} a(u+\varepsilon v)^2}{\int_D (u+\varepsilon v)^2}$$

$$f'(0) = 2 \cdot \frac{\left(\int_D \nabla u \cdot \nabla v + \int_{\partial D} auv\right)\int_D u^2 - \left(\int_D |\nabla u|^2 + \int_{\partial D} au^2\right)\int_D uv}{\int_D u^2}$$

$$0 = \left(\int_D \nabla u \cdot \nabla v + \int_{\partial D} auv\right)\int_D u^2 - \left(\int_D |\nabla u|^2 + \int_{\partial D} au^2\right)\int_D uv$$

$$\int_D \nabla u \cdot \nabla v + \int_{\partial D} auv = \frac{\int_D |\nabla u|^2 + \int_{\partial D} au^2}{\int_D u^2} \int_D uv = m \int_D uv.$$

$$\int_{\partial D} auv = -\int_{\partial D} (\nabla u \cdot n) v = -\int_D \nabla u \nabla v - \int_D (\Delta u) v$$

$$-\int_D (\Delta u) v = m \int_D uv.$$

$$\int_D (\Delta u + mu) v = 0$$

$$m \leq \frac{\int_D |\nabla v_j|^2 + \int_{\partial D} av_j^2}{\int_D v_j^2} = \frac{\int_D |\nabla v_j|^2 - \int_{\partial D} (\nabla v_j \cdot n) v_j}{\int_D v_j^2}$$

$$= \frac{-\int_D \Delta v_j v_j}{\int_D v_j^2}$$

$$= \frac{\lambda_j \int_D v_j^2}{\int_D v_j^2} = \lambda_j$$

(b)

Since $a \leq \tilde{a}$ implies $\int_{\partial D} aw^2 \leq \int_{\partial D} \tilde{a} w^2$

thus

$$\min \left\{ \frac{\int_D |\nabla w|^2 + \int_{\partial D} aw^2}{\int_D w^2} \right\} \leq \min \left\{ \frac{\int_D |\nabla w|^2 + \int_{\partial D} \tilde{a} w^2}{\int_D w^2} \right\}$$

Problem 2

Use the completeness to show that the solutions of the wave equation in any domain with a standard set of BCs satisfy the usual expansion $u(\mathbf{x}, t) = \sum_{n=1}^{\infty}[A_n \cos(\sqrt{\lambda_n}ct) + B_n \sin(\sqrt{\lambda_n}ct)]v_n(\mathbf{x})$. In particular, show that the series converges in the L^2 sense.

$$u_{tt} = c^2 \Delta u$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t) v_n(t)$$

$$u_{tt}(x,t) = \sum_{n=1}^{\infty} d_n(t) v_n(t)$$

$$\Delta u(x,t) = \sum_{n=1}^{\infty} e_n(t) v_n(x)$$

$$c_n(t) = \frac{1}{(v_n, v_n)} \iiint_D u(x,t) v_n(x)\, dx$$

$$d_n(t) = \frac{1}{(v_n, v_n)} \iiint_D u(x,t) v_n(x)\, dx = c_n''(t)$$

$$e_n(t) = \frac{1}{(v_n, v_n)} \iiint_D \Delta u(x,t) v_n(x)\, dx$$

$$e_n(t) = \frac{1}{(v_n, v_n)} \iiint_D u(x,t) \Delta v_n(x)\, dx$$

the right side is equal to

$$- \frac{\lambda}{(v_n, v_n)} \iiint_D u(x,t) v_n(x)\, dx = -\lambda_n c_n(t)$$

$$0 = u_{tt} - c^2 \Delta u = \sum_{n=1}^{\infty} \left(d_n(t) - c^2 e_n(t) v_n(x)\right)$$

$$d_n(t) = c^2 e_n(t)$$

$$c''_n(t) = -c^2 \lambda_n c_n(t)$$

$$c_n = A_n \cos(c\sqrt{\lambda_n} t) + B_n \sin(c\sqrt{\lambda_n} t)$$

(b) The Max temp is reached such that x is

$$\frac{du}{dx}(x)=0$$

u being function from (a)

So $x \in (\frac{l}{2}, l)$ because the rod is subjected to heat on Interval

$$0=-\frac{H}{8}(-4(l-x)-(l-4x)) \Rightarrow \frac{5l}{8}-x=0$$

hottest temp reached: $x=\frac{5l}{8}$

& the temp is $u(\frac{5l}{8})=\frac{9Hl^2}{128}$

4. A rod occupying the interval $0 \leq x \leq l$ is subject to the heat source $f(x)=0$ for $0<x<\frac{l}{2}$, and $f(x)=H$ for $\frac{l}{2}<x<l$ where $H>0$. The rod has physical constants $c=\rho=\kappa=1$, and its ends are kept at zero temperature.

(a) Find the steady-state temperature of the rod.

(b) Which point is the hottest, and what is the temperature there?

(a) At steady state temperature is a constant in respect to t

$$u_t=0$$

the temp satisfies $u_t=\Delta u+f(x,t)$

the rod is 1 dimension : $\frac{d^2}{dx^2}+f(x)=0$

the solution for $0<x<\frac{1}{2}$ is: $u(x)=Ax+B$

& for $\frac{1}{2}<x<1$ is $u(x)=-\frac{1}{2}Hx^2+Cx+D$

let's find A, B, C, D, mind the end kept at 0

$0=u(0,t)=B, 0=u(1,t)=-\frac{1}{2}Hl^2+Cl+D$

temp & Heat flux is continuous

$\frac{Al}{2}=\lim_{x\to\frac{l}{2}} u(x)=\lim_{x\to\frac{l}{2}} u(x)=\frac{l}{8}(3Hl-4C)$ so, $u(x)=\begin{cases} \frac{Hl}{8}x, & 0<x<\frac{l}{2} \\ -\frac{H}{8}(l-4x)(l-x), & \frac{l}{2}<x<l \end{cases}$

(b) The Max temp is reached such that x is

$$\frac{du}{dx}(x)=0$$

u being function from (a)

So $x \in (\frac{l}{2}, l)$ because the rod is subjected to heat on Interval

$$0 = -\frac{H}{8}(-4(l-x)-(l-4x)) \Rightarrow \frac{5l}{8}-x=0$$

hottest temp reached: $x=\frac{5l}{8}$

& the temp is $u(\frac{5l}{8}) = \frac{9Hl^2}{128}$

4. **A rod occupying the interval $0 \le x \le l$ is subject to the heat source $f(x) = 0$ for $0 < x < \frac{l}{2}$, and $f(x) = H$ for $\frac{l}{2} < x < l$ where $H > 0$. The rod has physical constants $c = \rho = \kappa = 1$, and its ends are kept at zero temperature.**
   (a) Find the steady-state temperature of the rod.
   (b) Which point is the hottest, and what is the temperature there?

(a) At steady state temperature is a constant in respect to t

$$u_t = 0$$

the temp satisfies $u_t = \Delta u + f(x,t)$

the rod is 1 dimension: $\frac{d^2}{dx^2} + f(x) = 0$

the solution for $0<x<\frac{l}{2}$ is: $u(x) = Ax + B$

& for $\frac{l}{2}<x<l$ is $u(x) = -\frac{1}{2}Hx^2 + Cx + D$

let's find A, B, C, D, mind the end kept at 0

$0 = u(0,t) = B$, $0 = u(l,t) = -\frac{1}{2}Hl^2 + Cl + D$

temp & Heat flux is continuous

$$\frac{Al}{2} = \lim_{x\to\frac{l}{2}^-} u(x) = \lim_{x\to\frac{l}{2}^+} u(x) = \frac{1}{8}(3Hl-4C)$$

so, $u(x) = \begin{cases} \frac{Hl}{8}x, & 0<x<\frac{l}{2} \\ -\frac{H}{8}(l-4x)(l-x), & \frac{l}{2}<x<l \end{cases}$

6b) The function given the conditions is

$$u(x) = \begin{cases} \dfrac{10\cdot 1c\rho}{3\cdot 1c\rho + 2\cdot 2c\rho}\, x, & 0<x<3 \\ 10 - \dfrac{10\cdot 2c\rho}{3\cdot 1c\rho + 2\cdot 2c\rho}(5-x), & 3<x<5 \end{cases}$$

the final form is: $u(x) = \begin{cases} \frac{10}{7} x, & 0<x<3 \\ 10 + \frac{10}{7}(2x-3), & 3<x<5 \end{cases}$

$$\rightarrow u(x) = \begin{cases} \dfrac{\kappa_2 T}{\kappa_2 L_1 + \kappa_1 L_2}\, x, & 0<x<L_1 \\ T - \dfrac{\kappa_1 T}{\kappa_2 L_1 + \kappa_1 L_2}(L_1+L_2-x), & L_1<x<L_1+L_2 \end{cases}$$

6. Two homogeneous rods have the same cross section, specific heat c, and density ρ but different heat conductivities κ_1 and κ_2 and lengths L_1 and L_2. Let $k_j = \kappa_j/c\rho$ be their diffusion constants. They are welded together so that the temperature u and the heat flux κu_x at the weld are continuous. The left-hand rod has its left end maintained at temperature zero. The right-hand rod has its right end maintained at temperature T degrees.
   (a) Find the *equilibrium* temperature distribution in the composite rod.
   (b) Sketch it as a function of x in case $k_1 = 2,\ k_2 = 1,\ L_1 = 3,\ L_2 = 2$, and $T = 10$. (This exercise requires a lot of elementary algebra, but it's worth it.)

(a) At the equilibrium temp is a constant in respect to t,

so $u_t = 0$

The Temp satisfies since the bar has different physical properties

$$c\rho u_t = \begin{cases} \nabla\cdot(\kappa_1 \nabla u), & 0<x<L_1 \\ \nabla\cdot(\kappa_2 \nabla u), & L_1<x<L_1+L_2 \end{cases}$$

moreover, the rod is 1 dimensional & $u_t = 0$ at equilibrium

$$0 = \begin{cases} \dfrac{d^2 u}{dx^2}, & 0<x<L_1 \\ \dfrac{d^2 u}{dx^2}, & L_1<x<L_1+L_2 \end{cases}$$

$-\kappa_1 A = \lim_{x\to L_1^-} \frac{du}{dx} = \lim_{x\to L_1^+} \frac{du}{dx} = -\kappa_2 C$

The temp & heat flux must be continuous

$AL_1 + B = \lim_{x\to L_1^-} u(x) = \lim_{x\to L_1^+} u(x) = CL_1 + D$

lets find A, B, C, D

Solve the two ode

$0 = u(0,t) = B,\ T = u(L_1+L_2) = C(L_1+L_2) + D$

$$u(x) = \begin{cases} Ax + B, & 0<x<L_1 \\ Cx + D, & L_1<x<L_1+L_2 \end{cases} = [redacted]$$

(b) integrate ODE from 0 to l

$$\int_0^l u''(x) + u'(x)\,dx = \int_0^l f(x)\,dx$$

$$u'(l) + u(l) - u'(0) - u(0) = \int_0^l f(x)\,dx$$

using the boundary conditions

$$0 = u'(l) + u(l) - \tfrac{1}{2}(u'(l) + u(l)) - \tfrac{1}{2}(u'(l) + u(l)) = \int_0^l f(x)$$

So $\int_0^l f(x)\,dx = 0$

so the avg is zero

2. Consider the problem

$$u''(x) + u'(x) = f(x)$$
$$u'(0) = u(0) = \tfrac{1}{2}[u'(l) + u(l)],$$

with $f(x)$ a given function.

(a) Is the solution *unique*? Explain.

(b) Does a solution necessarily *exist*, or is there a condition that $f(x)$ must satisfy for existence? Explain.

(a)

Assume the solution exists

let u be a solution to the given ode

define v as $v(x) = u(x) + e^{-x} - 2$

So $v \neq u$

~~[illegible]~~ let's show v is also a solution to the ode

1. $v''(x) + v'(x) = u''(x) + e^{-x} + u'(x) - e^{-x}$
$= u''(x) + u'(x)$
$= f(x)$

2. $v'(0) = u'(0) - 1$
$v(0) = u(0) + 1 - 2 = u(0) - 1$
$\frac{1}{2}(v'(l) + v(l)) = \frac{1}{2}(u'(l) - e^{-l} + u(l) + e^{-l} - 2) =$ ~~[illegible]~~ $\frac{1}{2}(u'(l) + u(l)) - 1$

So, the solution is not unique

5. Consider the equation

$$u_x + yu_y = 0$$

with the boundary condition $u(x,0) = \phi(x)$.

(a) For $\phi(x) \equiv x$, show that no solution exists.

(b) For $\phi(x) \equiv 1$, show that there are many solutions.

(a) $u_x + yu_y = 0$

$\frac{dy}{dx} = y \Rightarrow ye^{-x} = C$

$u(x,y) = f(ye^{-x})$ where $x \in \mathbb{R}$, this doesnt exist because $x = f(0)$ for all $x \in \mathbb{R}$

(b) $u(x,0) = x$

so this exists

$1 = u(x,0) = f(0)$

for all $x \in \mathbb{R}$

This obviously does exist a trivial function is $f(x) = 1$ for all $x \in \mathbb{R}$

6. Solve the equation $u_x + 2xy^2u_y = 0$.

$$u_x + 2xy^2 u_y = 0$$

using characteristics.

$$\frac{dy}{dx} = \frac{2xy^2}{1}$$

$$\frac{dy}{y^2} = 2x\,dx$$

$$-\frac{1}{y} = x^2 + c$$

$$u(x,y) = f\left(\frac{1}{y} + x^2\right)$$

$y = \frac{3V_g}{2c}$

f_y

5

1. What is the type of each of the following equations?

(a) $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0.$

(b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0.$

(a) $a_{12}^2 - a_{11}a_{22} = 9/4 - (1)(1)$

$= 5/4 > 0$

Then the equation (a) is hyperbolic

$u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0$

$\Leftarrow$ Compare this equation with

$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0$

(b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$

Compare this equation with

$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0$

$\Downarrow$

Since $a_{12}^2 - a_{11}a_{22} = 9 - (9)(1)$

$= 0$

thus the equation is parabolic.

1. What is the type of each of the following equations?

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(b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0.$

(a) $a_{12}^2 - a_{11}a_{22} = 9/4 - (1)(1)$

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$u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0$

$\Leftarrow$ Compare this equation with

$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0$

Then the equation (a) is hyperbolic

(b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$

Compare this equation with

$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0$

$\Downarrow$

Since $a_{12}^2 - a_{11}a_{22} = 9 - (9)(1)$

$= 0$

thus the equation is parabolic.

4. What is the *type* of the equation

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0?$$

Show by direct substitution that $u(x, y) = f(y + 2x) + xg(y + 2x)$ is a solution for arbitrary functions f and g.

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0$$

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0$$

here $a_{11} = 1, a_{12} = -2, a_{22} = 4$

$$a_{12}^2 - a_{11}a_{22} = 4 - 4$$
$$= 0$$

Thus equation 1 is parabolic.

$u(x,y) = f(y+2x) + xg(y+2x)$

$u_x = 2f_x(y+2x) + 2xg_x(y+2x) + g(y+2x)$

$u_{xx} = 4f_{xx}(y+2x) + 4xg_{xx}(y+2x) + 2g_x(y+2x) + g_x(y+2x)$

$u_y = f_y(y+2x) + xg_y(y+2x)$

$u_{yy} = f_{yy}(y+2x) + xg_{yy}(y+2x)$

$u_{xy} = 2f_{xy}(y+2x) + 2xg_{xy}(y+2x) + g_y(y+2x)$

thus

$u_{xx} - 4u_{xy} + 4u_{yy} = 0$

1. Solve $u_{tt} = c^2 u_{xx}$, $u(x,0) = e^x$, $u_t(x,0) = \sin x$.

The problem gives
$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = e^x = \phi(x) \\ u_t(x,0) = \sin x = \psi(x) \end{cases}$$

let's use D'Alembert's formula

$$u(x,t) = \frac{1}{2}\left[e^{x+ct} + e^{x-ct}\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(s)\,ds$$

$$= \frac{e^x}{2}\left(e^{ct} + e^{-ct}\right) - \frac{1}{2c}\left(\cos(x+ct) - \cos(x-ct)\right)$$

$$u(x,t) = e^x \cosh(ct) + \frac{1}{c}\sin x \sin ct$$

we used trigonometric identity to simplify the expression.

3. The midpoint of a piano string of tension T, density ρ, and length l is hit by a hammer whose head diameter is $2a$. A flea is sitting at a distance $l/4$ from one end. (Assume that $a < l/4$; otherwise, poor flea!) How long does it take for the disturbance to reach the flea?

The speed of the wave is given by

$$c^2 = \frac{T}{\rho}$$

~~B~~ it is constant hence the distance is given by

$$x = vt$$

The wave that is going to reach the flea is the one starting in $l/2 + a$ traveling to the right thus the distance is

$$x = \frac{3l}{4} - \left(\frac{l}{4} + a\right) = \frac{l}{4} - a$$

Moreover, the wave speed is $c = \sqrt{\frac{T}{\rho}}$

So we have everything to know when the disturbance will reach the flea

$$t = \frac{x}{v} = \frac{\frac{l}{4} - a}{\sqrt{\frac{T}{\rho}}}, \text{ so } t = \left(\frac{l}{4} - a\right)\sqrt{\frac{\rho}{T}}$$

(c) find solution for v if the initial conditions are

$$v(r,0) = r u(r,0) = r\phi(r)$$
$$v_t(r,0) = r u_t(r,0) = r\psi(r)$$
$$v(r,t) = \frac{1}{2}\left((r+ct)\phi(r+ct) + (r-ct)\phi(r-ct)\right) + \frac{1}{2c}\int_{r-ct}^{r+ct} s\psi(s)\,ds$$

expressing u using v

$$u(r,t) = \frac{1}{2r}\left((r+ct)\phi(r+ct) + (r-ct)\phi(r-ct)\right) + \frac{1}{2rc}\int_{r-ct}^{r+ct} s\psi(s)\,ds$$

8. A *spherical wave* is a solution of the three-dimensional wave equation of the form $u(r, t)$, where r is the distance to the origin (the spherical coordinate). The wave equation takes the form

$$u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right) \quad \text{(“spherical wave equation”).}$$

(a) Change variables $v = ru$ to get the equation for v: $v_{tt} = c^2 v_{rr}$.
(b) Solve for v using (3) and thereby solve the spherical wave equation.
(c) Use (8) to solve it with initial conditions $u(r, 0) = \phi(r)$, $u_t(r, 0) = \psi(r)$, taking both $\phi(r)$ and $\psi(r)$ to be even functions of r.

(a) Let function v be defined as

$$v = ru$$

It's easy to find the expression

$$v_{tt} = r u_{tt}$$

(b)

$$v_{rr} = (v_r)_r = (u + r u_r)_r = 2(u_r) + r u_{rr}$$

From (3) solution is $v(r,t) = f(r+ct) + g(r-ct)$ for arbitrary functions f & g.

[redacted] Combine the results to find the solution

$$v_{tt} = r u_{tt} = r c^2\left(u_{rr} + \frac{2}{r}u_r\right) = c^2(r u_{rr} + 2u_r)$$

$\boxed{v_{tt} = c^2 v_{rr}}$ final solution

5 - $\frac{\pi}{5}$

lve $u_{xx} - 3u_{xt} - 4u_{tt} = 0$, $u(x,0) = x^2$, $u_t(x,0) = e^x$. (*Hint:* Factor the operator as we did for the wave equation.)

We are given equation $u(x,y) = \left\{ \begin{array}{l} u_{xx} - 3u_{xt} - 4u_{tt} = 0 \\ u(x,0) = x^2 \\ u_t(x,0) = e^x \end{array} \right\}$

The factored operator is:

$$(\partial_x + \partial_t)(\partial_x - 4\partial_t)_u = 0$$

$$V = u_x - 4u_t$$

(once we factor the operator we proceed in this fashion)

after applying the first factored operator we get new equation:

$$V_x + V_t = 0$$

we know the equation general solution is:

$$V = h(x-t)$$

from here we have

$$V = u_x - 4u_t = h(x-t)$$

this is a transport equation

it's a non-homongeneous equation with 1 partial derivative, so it can be solved as for each variable

$$u(x,t) = f(x-t) + g(4x-t)$$

Next we take care of the initial conditions, so

$$f(x) + g(4x) = x^2$$

$$-f'(x) + g'(4x) = e^x$$

from above it follows

$$f'(x) = \frac{2x}{5} - \frac{4e^x}{5}$$

$$f(x) = \frac{x^2}{5} - \frac{4e^x}{5}$$

$$g'(s) = \frac{s}{10} + \frac{e^{\frac{s}{4}}}{5}$$

$$g(s) = \frac{s^2}{20} + \frac{4e^{\frac{s}{4}}}{5}$$

$$u(x,t) = \frac{(x-t)^2}{5} - \frac{4e^{x-t}}{5} + \frac{(4x+t)^2}{20} + \frac{4e^{\frac{4x+t}{4}}}{5}$$

(simplified)

$$\frac{4}{5}\left[e^{\frac{x+t}{4}} - e^{x-t}\right] + x^2 + \frac{t^2}{4}$$

$= k\ 3.$

lve $u_{xx} - 3u_{xt} - 4u_{tt} = 0$, $u(x,0) = x^2$, $u_t(x,0) = e^x$. (*Hint:* Factor the operator as we did for the wave equation.)

We are given equation $u(x,y) = \left\{ \begin{array}{l} u_{xx} - 3u_{xt} - 4u_{tt} = 0 \\ u(x,0) = x^2 \\ u_t(x,0) = e^x \end{array} \right\}$

The factored operator is:

$$(\partial_x + \partial_t)(\partial_x - 4\partial_t)u = 0$$

$$v = u_x - 4u_t$$

(once we factor the operator we proceed in this fashion)

after applying the first factored operator we get new equation:

$$v_x + v_t = 0$$

we know the equation general solution is:

$$v = h(x-t)$$

from here we have

$$v = u_x - 4u_t = h(x-t)$$

this is a transport equation

it's a non-homongeneous equation with 1 partial derivative, so it can be solved as for each variable

$$u(x,t) = f(x-t) + g(4x-t)$$

Next we take care of the initial conditions, so

$$f(x) + g(4x) = x^2$$

$$-f'(x) + g'(4x) = e^x$$

from above it follows

$$f'(x) = \frac{2x}{5} - \frac{4e^x}{5}$$

$$f(x) = \frac{x^2}{5} - \frac{4e^x}{5}$$

$$g'(s) = \frac{s}{10} + \frac{e^{\frac{s}{4}}}{5}$$

$$g(s) = \frac{s^2}{20} + \frac{4e^{\frac{s}{4}}}{5}$$

$$u(x,t) = \frac{(x-t)^2}{5} - \frac{4e^{x-t}}{5} + \frac{(4x+t)^2}{20} + \frac{4e^{\frac{x+t}{4}}}{5}$$

(simplified)

$$\frac{4}{5}\left[e^{\frac{x+t}{4}} - e^{x-t}\right] + x^2 + \frac{t^2}{4}$$

6. Prove that, among all possible dimensions, only in three dimensions can one have distortionless spherical wave propagation with attenuation. This means the following. A spherical wave in n-dimensional space satisfies the PDE

$$u_{tt} = c^2\left(u_{rr} + \frac{n-1}{r}u_r\right),$$

where r is the spherical coordinate. Consider such a wave that has the special form $u(r, t) = \alpha(r)f(t - \beta(r))$, where $\alpha(r)$ is called the

attenuation and $\beta(r)$ the delay. The question is whether such solutions exist for "arbitrary" functions f.

(a) Plug the special form into the PDE to get an ODE for f.
(b) Set the coefficients of f'', f', and f equal to zero.
(c) Solve the ODEs to see that $n = 1$ or $n = 3$ (unless $u \equiv 0$).
(d) If $n = 1$, show that $\alpha(r)$ is a constant (so that "there is no attenuation").

(T. Morley, *American Mathematical Monthly*, Vol. 27, pp. 69–71, 1985)

(a) find the following expressions

$u_t = \alpha f'$

$u_{tt} = \alpha f''$

$u_r = \alpha' f - \alpha\beta' f'$

$u_{rr} = (u_r)_r = \alpha'' f - 2\alpha'\beta' f - \alpha\beta'' f' + \alpha(\beta')^2 f''$

substitute expressions into this given EQ: $\alpha f'' = u_{tt}$

$= c^2(\alpha'' f - 2\alpha'\beta f - \alpha\beta'' f' + \alpha(\beta')^2 f'' + \frac{n-1}{r}(\alpha' f - \alpha\beta' f'))$

outcome

(5) $0 = c^2(\alpha'' + \frac{n-1}{r}\alpha')f - c^2(2\alpha'\beta' + \alpha\beta'' + \frac{n-1}{r}\alpha\beta')f' + \alpha(c^2\beta'^2 - 1)f''$

(b) set coefficient from (a) = 0

$c^2(\alpha'' + \frac{n-1}{r}\alpha') = 0$

$-c^2(2\alpha'\beta' + \alpha\beta'' + \frac{n-1}{r}\alpha\beta') = 0$

$\alpha(c^2\beta'^2 - 1) = 0$

(c) first solve (1), it is reduced to

$a'(r) = a(r)$

$\alpha'' + \frac{n-1}{r}\alpha' = 0$

$\frac{a'}{a} = \frac{1-n}{r}$

$\alpha' = a = c_1 r^{1-n}$

$\alpha = \frac{c_1}{2-n}r^{2-n} + c_2$

$\beta' = \pm\frac{1}{c} \Rightarrow \beta = \pm\frac{1}{c}r + c_3$

$0 = (2\alpha' + \frac{n-1}{r}\alpha)\beta' + \alpha\beta''$

$= (2c_1 r^{1-n} + \frac{c_1(n-1)}{2-n}r^{1-n} + \frac{c_2(n-1)}{r})\frac{1}{c} = 0$

$2c_1 r^{1-n} + \frac{c_1(n-1)}{2-n}r^{1-n} + \frac{c_2(n-1)}{r}$

$\frac{3-n}{2-n}c_1 r^{1-n} + \frac{c_2(n-1)}{r}$

(d) If $n=1$ then in (5)

$\frac{3-n}{2-n}c_1 = 0 \Rightarrow c_1 = 0$

thus $\alpha(r) = c_2$ is a constant

$= V\frac{3_4}{2c}$

$u(r$

4. Here is a direct relationship between the wave and diffusion equations. Let $u(x, t)$ solve the wave equation on the whole line with bounded second derivatives. Let

$$v(x,t) = \frac{c}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-s^2c^2/4kt}\, u(x,s)\, ds.$$

(a) Show that $v(x, t)$ solves the diffusion equation!
(b) Show that $\lim_{t\to 0} v(x,t) = u(x,0)$.

(*Hint:* (a) Write the formula as $v(x,t) = \int_{-\infty}^{\infty} H(s,t)u(x,s)\,ds$, where $H(x, t)$ solves the diffusion equation with constant k/c^2 for $t > 0$. Then differentiate $v(x, t)$ using Section A.3. (b) Use the fact that $H(s, t)$ is essentially the source function of the diffusion equation with the spatial variable s.)

(a) $H(x,t) = \frac{c}{\sqrt{4\pi k t}}\, e^{-\left(\frac{x^2c^2}{4kt}\right)}$ (step 1)

$H_t = \frac{c}{\sqrt{4\pi k}}\left(-\frac{1}{2t\sqrt{t}}\, e^{-\left(\frac{x^2c^2}{4kt}\right)} + \frac{x^2c^2}{4kt^2\sqrt{t}}\, e^{-\left(\frac{x^2c^2}{4kt}\right)}\right)$

$= \frac{c}{4t\sqrt{\pi t k}}\, e^{-\left(\frac{x^2c^2}{4kt}\right)}\left(-1 + \frac{x^2c^2}{2kt}\right)$

$H_x = -\frac{c^3 x}{4kt\sqrt{\pi k t}}\, e^{-\left(\frac{x^2c^2}{4kt}\right)}$

$H_{xx} = \frac{c^3}{4kt\sqrt{\pi k t}}\, e^{-\left(\frac{x^2c^2}{4kt}\right)}\left(-1 + \frac{x^2c^2}{2kt}\right)$

So, $H_t = \frac{k}{c^2} H_{xx}$

H solves the diffusion equation with constant $\frac{k}{c^2}$

step 2) Now prove $v(x,t) = \int_{-\infty}^{\infty} H(s,t)u(x,s)\,ds$

$v_t = \int_{-\infty}^{\infty} H_t(s,t)u(x,s)\,ds$

$= \frac{k}{c^2}\int_{-\infty}^{\infty} H_{xx}(s,t)u(x,s)\,ds$

$= \frac{k}{c^2}\left(H_x(s,t)u(s,t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H_x(s,t)u(x,s)\,ds\right)$ partial integration

$= -\frac{k}{c^2}\int_{-\infty}^{\infty} H_x(s,t)u_x(x,s)\,ds$

$= -\frac{k}{c^2}\left(H(s,t)u_x(x,s)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H(s,t)u_{xx}(x,s)\,ds\right)$

$= \frac{k}{c^2}\int_{-\infty}^{\infty} H(s,t)u_{xx}(x,s)\,ds$

$v_{xx} = \int_{-\infty}^{\infty} H(s,t)u_{xx}(x,s)\,ds$

thus, $v_t = \frac{k}{c^2} v_{xx}$

$= v_{3a}$

(step 3) $H(x,0) = \delta(x)$, where δ is the Dirac function.

$\lim_{t\to0} v(x,t) = \lim_{t\to0}\int_{-\infty}^{\infty} H(s,t)u(x,s)\,ds$

$= \int_{-\infty}^{\infty} \lim_{t\to0} H(s,t)u(x,s)\,ds$

$= \int_{-\infty}^{\infty}\left[\lim_{t\to0} H(s,t)\right]u(x,s)\,ds$

$= \int_{-\infty}^{\infty} \delta(s)u(x,s)\,ds$

$= u(x,0)$

Solve $u_t = ku_{xx}$; $u(x,0) = e^{-x}$; $u(0,t) = 0$ on the half-line $0 < x < \infty$.

from the diffusion equation we see that
$u(x,0) = \phi(x)$ for $t=0$

we see that

$$\phi(x) = e^{-x}$$

now we find the solution to the diffusion equation $u(x,t)$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left(e^{-\left(\frac{(x-y)^2}{4kt}\right)} - e^{-\left(\frac{(x+y)^2}{4kt}\right)} \right) \phi(y)\,dy$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left(e^{-\left(\frac{(x-y)^2}{4kt}\right)} - e^{-\left(\frac{(x+y)^2}{4kt}\right)} \right) e^{-y}\,dy$$

$$u(x,y) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-\left(\frac{(x-y)^2}{4kt}\right)-y} - \frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-\left(\frac{(x+y)^2}{4kt}\right)-y}\,d$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{\frac{4kt(kt-x)}{4kt}} e^{-\left(\frac{y+(2kt-x)}{\sqrt{4kt}}\right)^2} dy$$

$$= e^{kt-x} \frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-\left(\frac{y+(2kt-x)}{\sqrt{4kt}}\right)^2} dy$$

now we need the following substitution

$$z = \frac{y+(2kt-x)}{\sqrt{4kt}} \quad \text{then } dz = \frac{1}{\sqrt{4kt}}\,dy$$

$$dy = \sqrt{4kt}\,dz$$

find the lower limit of z (make $y=0$)

$$z = \frac{2kt-x}{\sqrt{4kt}}$$

now find upper limit of y

$$\frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-\frac{(x-y)^2}{4kt}-y}\,dy = e^{kt-x} \frac{1}{\sqrt{4\pi kt}} \int_z^\infty e^{-z^2} \sqrt{4kt}\,dz$$

$$= e^{kt-x} \frac{1}{\sqrt{\pi}} \int_z^\infty e^{-z^2}\,dz$$

(Now put the integral in terms of y)

$$\frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-\left(\frac{(x-y)^2}{4kt}\right)-y}\,dy$$

$$= \frac{e^{kt-x}}{\sqrt{\pi}} \int_z^\infty e^{-y^2}\,dy$$

$$= \frac{e^{kt-x}}{\sqrt{\pi}} \left(\int_0^\infty e^{-y^2}\,dy - \int_0^z e^{-y^2}\,dy \right)$$

$$= \frac{e^{kt-x}}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \operatorname{Erf}\left(\frac{2kt-x}{\sqrt{4kt}}\right) \right)$$

$$= \frac{e^{kt-x}}{\sqrt{\pi}} \left(1 - \operatorname{Erf}\left(\frac{2kt-x}{\sqrt{4kt}}\right) \right)$$

thus, $u(x,t) = \dfrac{e^{kt-x}}{2} \left(\left(1 - \operatorname{Erf}\left(\frac{2kt-x}{\sqrt{4kt}}\right) \right) - \left(1 - \operatorname{Erf}\left(\frac{2kt+x}{\sqrt{4kt}}\right) \right) \right)$

Consider the following problem with a Robin boundary condition:

DE: $u_t = ku_{xx}$ on the half-line $0 < x < \infty$ (and $0 < t < \infty$)

IC: $u(x, 0) = x$ for $t = 0$ and $0 < x < \infty$ (*)

BC: $u_x(0, t) - 2u(0, t) = 0$ for $x = 0$.

The purpose of this exercise is to verify the solution formula for (*). Let $f(x) = x$ for $x > 0$, let $f(x) = x + 1 - e^{2x}$ for $x < 0$, and let

$$v(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\,dy.$$

(a) What PDE and initial condition does $v(x,t)$ satisfy for $-\infty < x < \infty$?

(b) Let $w = v_x - 2v$. What PDE and initial condition does $w(x, t)$ satisfy for $-\infty < x < \infty$?

(c) Show that $f'(x) - 2f(x)$ is an odd function (for $x \neq 0$).

(d) Use Exercise 2.4.11 to show that w is an odd function of x.

(e) Deduce that $v(x, t)$ satisfies (*) for $x > 0$. Assuming uniqueness, deduce that the solution of (*) is given by

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\,dy.$$

(a) $u_t = k u_{xx}$ for $0<x<\infty$, $0<t<\infty$

$u(x,0) = x$ for $0<x<\infty$

$u_x(0,t) - 2u(0,t) = 0$

$f(x) = \begin{cases} x & \text{for } x> \\ x+1-e^{2x} & \text{for } x<0 \end{cases}$

$V(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} f(y)\,dy$

(b)

$w = v_x - 2v$

$u_t = k u_{xx}$

$u_x(0,t) - 2u(0,t) = 0$

$w(x,t)$ for $-\infty<x<\infty$

$w_t = k w_{xx}$

$w(x,0) = f'(x) - 2f(x)$

- Consider the following problem with a Robin boundary condition:

DE:	$u_t = ku_{xx}$	on the half-line $0 < x < \infty$ (and $0 < t < \infty$)	
IC:	$u(x,0) = x$	for $t = 0$ and $0 < x < \infty$	$(*)$
BC:	$u_x(0,t) - 2u(0,t) = 0$	for $x = 0$.	

The purpose of this exercise is to verify the solution formula for (*). Let $f(x) = x$ for $x > 0$, let $f(x) = x + 1 - e^{2x}$ for $x < 0$, and let

$$v(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\,dy.$$

(a) What PDE and initial condition does $v(x,t)$ satisfy for $-\infty < x < \infty$?
(b) Let $w = v_x - 2v$. What PDE and initial condition does $w(x,t)$ satisfy for $-\infty < x < \infty$?
(c) Show that $f'(x) - 2f(x)$ is an odd function (for $x \neq 0$).
(d) Use Exercise 2.4.11 to show that w is an odd function of x.

(e) Deduce that $v(x, t)$ satisfies (*) for $x > 0$. Assuming uniqueness, deduce that the solution of (*) is given by

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\,dy.$$

(a) $u_t = k u_{xx}$ for $0<x<\infty$, $0<t<\infty$

$u(x,0) = x$ for $0<x<\infty$

$u_x(0,t) - 2u(0,t) = 0$

$f(x) = \begin{cases} x & \text{for } x> \\ x+1-e^{2x} & \text{for } x<0 \end{cases}$

$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} f(y)\,dy$

(b)

$w = v_x - 2v$

$u_t = k u_{xx}$

$u_x(0,t) - 2u(0,t) = 0$

$w(x,t)$ for $-\infty<x<\infty$

$w_t = k w_{xx}$

$w(x,0) = f'(x) - 2f(x)$

2. The longitudinal vibrations of a semi-infinite flexible rod satisfy the wave equation $u_{tt} = c^2 u_{xx}$ for $x > 0$. Assume that the end $x = 0$ is free ($u_x = 0$); it is initially at rest but has a constant initial velocity V for $a < x < 2a$ and has zero initial velocity elsewhere. Plot u versus x at the times $t = 0, a/c, 3a/2c, 2a/c$, and $3a/c$.

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{external}(s)\,ds$$

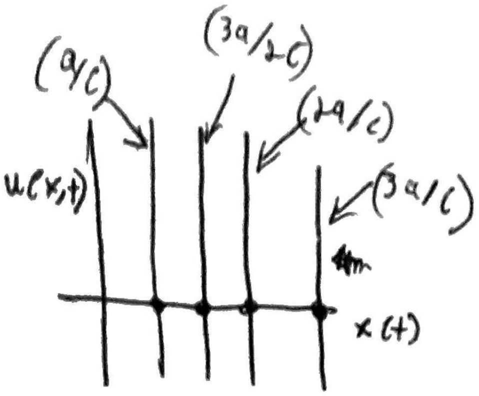
$$\psi_{external}(s) = V\begin{cases} a<s<2a \\ -2a<s<-a \end{cases}$$

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{external}(s)\,ds$$

$$= \frac{1}{2c}\int_{x-ct}^{x+ct} V\,ds$$

$$= \frac{V}{2c}\left[x+ct-x+ct\right]$$

$$u(x,t) = Vt$$



when $t=0$ we use $u(x,t) = x(t)\,g(t)$ & $V = a = c = 1 \Rightarrow \frac{Va}{c} = 1$

where $x(t) = 1$ & $y(t) = t$

thus $u(x,t) = Vt$

now when $t = 0$

$u(x,t) = V(0) = 0$

$u(x,t) = 0$

when $t = \frac{a}{c}$

$u(x,t) = \frac{a}{c}t$ where $x(t) = 1$ & $y(t) = t$

$u(x,t) = Vt$

$= V\left(\frac{a}{c}\right)$

$u(x,t) = \frac{Va}{c}$

when $t = \frac{3a}{c}$

$= V\,\frac{3a}{c}$

$u(t,x) = \frac{3Va}{c}$

3 seperate lines

when $t = \frac{2a}{c}$

$= V\left(\frac{2a}{c}\right)$

$u(x,t) = \frac{2Va}{c}$

when $t = \frac{3a}{2c}$

$= V\,\frac{3a}{2c}$

$u(x,t) = \frac{3Va}{2c}$

10. Solve $\cancel{u_{tt} = 9u_{xx}}$ ~~in~~ $\cancel{0 < x < \pi/2,\ u(x,0) = \cos x,\ u_t(x,0) = 0,}$
$\cancel{u_x(0,t) = 0,\ u(\pi/2,t) = 0.}$

If were given
$$\begin{cases} u_{tt} = v_{xx} & -\infty < x < \infty,\ 0 < t \\ v(x,0) = \phi_{ext}(x) \\ u_t(x,0) = \psi_{ext}(x) \end{cases}$$

$$\phi_{ext}(x) = \cos(x),\ \psi_{ext}(x) = 0$$

v is given by d'Alembert's formula

$$v(x,t) = \frac{1}{2}\left(\phi_{ext}(x+3t) + \phi_{ext}(x-3t)\right) + \frac{1}{2\cdot 3}\int_{x-3t}^{x+3t} \psi_{ext}(s)\,ds$$

$$= \frac{1}{2}\left(\cos(x+3t) + \cos(x-3t)\right) + 0$$

$$= \frac{1}{2}(\cos x \cos 3t - \sin x \sin 3t + \cos x \cos 3t + \sin x \sin 3t)$$

$$= \frac{1}{2}(\cos x \cos 3t + \cos 3\cancel{x} \cos 3t)$$

$$= \cos x \cos 3t$$

$$u(x,t) = \cos x \cos 3t,\quad 0 < x < \frac{\pi}{2}$$

2. Solve the completely inhomogeneous diffusion problem on the half-line

$$v_t - kv_{xx} = f(x,t) \quad \text{for } 0 < x < \infty, \quad 0 < t < \infty$$
$$v(0,t) = h(t) \qquad v(x,0) = \phi(x).$$

15 step problem

$$w(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\left(\frac{(x-y)^2}{4kt}\right)} \psi_{odd}(y)\,dy$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{0} e^{-\left(\frac{(x-y)^2}{4kt}\right)} \psi_{odd}(y)\,dy + \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty}$$

$$\vdots$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left(e^{-\left(\frac{(x-y)^2}{4kt}\right)} - e^{-\left(\frac{(x+y)^2}{4kt}\right)}\right) \psi(y)\,dy$$

use Duhamel's principle

$$P(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)} g_{odd}(y,s)\,dy\,ds$$

$$\int_0^t \int_{-\infty}^{0} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\left(\frac{(x+y)^2}{4k(t-s)}\right)} g_{odd}(y,s)\,dy\,ds + \int_0^t \int_0^\infty \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)} g_{odd}(y,s)\,dy\,ds$$

$$\boxed{\int_0^t \int_0^\infty \left(e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)} - e^{-\left(\frac{(x+y)^2}{4k(t-s)}\right)}\right) \frac{1}{\sqrt{4\pi k(t-s)}} g(y,s)\,dy\,ds}$$

$v(x,t)$

$$\boxed{= h(t) + \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left(e^{-} - e^{-}\right) \phi(y)\,dy}$$

$$\boxed{+ \int_0^t \int_0^\infty \left(e^{-} - e^{-}\right) \ldots (y,s)\,dy\,ds}$$

$$\boxed{+ \int_0^t \int_0^\infty \frac{d}{ds}\left(e^{-} - e^{-}\right) \ldots\,dy\,ds}$$

us v as the sum of w & p

$$V(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left(e^{-\left(\frac{(x-y)^2}{4kt}\right)} - e^{-\left(\frac{(x+y)^2}{4kt}\right)}\right)(\phi(y) - h(0))$$

$$+ \int_0^t \int_0^\infty e^{-\left(\frac{(x+y)^2}{4k(t-s)}\right)} \frac{1}{\sqrt{4\pi k(t-s)}} (f(y,s) - h'(s))\,dy\,ds$$

partial integration . . . .

$$w(x,t)=\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\psi_{odd}(y)dy$$

$$=\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{0} e^{-\left(\frac{(x-y)^2}{4kt}\right)}\psi_{odd}(y)dy+\frac{1}{\sqrt{4\pi kt}}\int_0^{\infty} e^{-\left(\frac{(x-y)^2}{4kt}\right)}\psi_{odd}(y)dy$$

$$=\frac{1}{\sqrt{4\pi kt}}\int_0^{\infty} e^{-\left(\frac{(x+y)^2}{4kt}\right)}\psi_{odd}(-y)+\frac{1}{\sqrt{4\pi kt}}\int_0^{\infty} e^{-\left(\frac{(x-y)^2}{4kt}\right)}\psi_{odd}(y)dy$$

$$=-\frac{1}{\sqrt{4\pi kt}}\int_0^{\infty} e^{-\left(\frac{(x+y)^2}{4kt}\right)}\psi(y)dy+\frac{1}{\sqrt{4\pi kt}}\int_0^{\infty} e^{-\left(\frac{(x-y)^2}{4kt}\right)}\psi(y)dy$$

$$=\frac{1}{\sqrt{4\pi kt}}\int_0^{\infty}\left(e^{-\left(\frac{(x-y)^2}{4kt}\right)}-e^{-\left(\frac{(x+y)^2}{4kt}\right)}\right)\psi(y)dy$$

now find l solution using Duhamel's principle

$$l(x,t)=\int_0^t\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi k(t-s)}}e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)}g_{odd}(y,s)dyds$$

$$=\int_0^t\int_{-\infty}^{0}\frac{1}{\sqrt{4\pi k(t-s)}}e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)}g_{odd}(y,s)dyds+\int_0^t\int_0^{\infty}\frac{1}{\sqrt{4\pi k(t-s)}}e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)}g_{odd}$$

$$=\int_0^t\int_0^{\infty}\frac{1}{\sqrt{4\pi k(t-s)}}e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)}g_{odd}(-y,s)dyds+\int_0^t\int_0^{\infty}\frac{1}{\sqrt{4\pi k(t-s)}}e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)}g_{odd}(y,s$$

$$=-\int_0^t\int_0^{\infty}\frac{1}{\sqrt{4\pi k(t-s)}}e^{-\left(\frac{(x+y)^2}{4k(t-s)}\right)}g(y,s)dyds+\int_0^t\int_0^{\infty}\frac{1}{\sqrt{4\pi k(t-s)}}e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)}g(y,s)$$

$$=\int_0^t\int_0^{\infty}\left(e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)}-e^{-\left(\frac{(x+y)^2}{4k(t-s)}\right)}\right)\frac{1}{\sqrt{4\pi k(t-s)}}g(y,s)dyds$$

now that we have our v as the sum of w & l the solution for V is restricted

$$V(x,t)=\frac{1}{\sqrt{4\pi kt}}\int_0^{\infty}\left(e^{-\left(\frac{(x-y)^2}{4kt}\right)}-e^{-\left(\frac{(x+y)^2}{4kt}\right)}\right)(\phi(y)-h(0))dy$$

$$+\int_0^t\int_0^{\infty}\left(e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)}-e^{-\left(\frac{(x+y)^2}{4k(t-s)}\right)}\right)\frac{1}{\sqrt{4\pi k(t-s)}}(f(y,s)-h'(s))dyds=\underline{\text{Partial Integ}}$$

Answer: $V(x,t)=H(x)+\frac{1}{\sqrt{4\pi kt}}\int_0^{\infty}\left(e^{-\left(\frac{(x-y)^2}{4kt}\right)}-e^{-\left(\frac{(x+y)^2}{4kt}\right)}\right)\phi(y)dy$

$$\cancel{\frac{\ }{kt}\int_0^{\infty}\left(e^{-\left(\frac{(x-y)^2}{4kt}\right)}-e^{\left(\frac{(x+y)^2}{4kt}\right)}\right)}h(0)dy+\int_0^t\int_0^{\infty}\left(e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)}-e^{-\left(\frac{(x+y)^2}{4k(t-s)}\right)}\right)\frac{1}{\sqrt{4\pi k(t-s)}}f(y,s)dyds$$

$$\cancel{\int_0^{\infty}\left(e^{-\left(\frac{(x-y)^2}{4kt}\right)}-e^{-\left(\frac{(x+y)^2}{4kt}\right)}\right)}h(0)dy+\int_0^t\int_0^{\infty}\frac{d}{ds}\left(\left(e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)}-e^{-\left(\frac{(x+y)^2}{4k(t-s)}\right)}\right)\frac{1}{\sqrt{4\pi k(t-s)}}\right)h(s)dyds$$

$$\omega(x,t) = \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \psi_{odd}(y)\,dy$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{0} e^{-\left(\frac{(x-y)^2}{4kt}\right)} \psi_{odd}(y)\,dy + \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-\left(\frac{(x-y)^2}{4kt}\right)} \psi_{odd}(y)\,dy$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-\left(\frac{(x+y)^2}{4kt}\right)} \psi_{odd}(-y) + \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-\left(\frac{(x-y)^2}{4kt}\right)} \psi_{odd}(y)\,dy$$

$$= -\frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-\left(\frac{(x+y)^2}{4kt}\right)} \psi(y)\,dy + \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-\left(\frac{(x-y)^2}{4kt}\right)} \psi(y)\,dy$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} \left(e^{-\left(\frac{(x-y)^2}{4kt}\right)} - e^{-\left(\frac{(x+y)^2}{4kt}\right)}\right) \psi(y)\,dy$$

now find I solution using Duhamel's principle

$$I(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)} g_{odd}(y,s)\,dy\,ds$$

$$= \int_0^t \int_{-\infty}^{0} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)} g_{odd}(y,s)\,dy\,ds + \int_0^t \int_0^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)} g_{odd}$$

$$= \int_0^t \int_0^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\left(\frac{(x+y)^2}{4k(t-s)}\right)} g_{odd}(-y,s)\,dy\,ds + \int_0^t \int_0^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)} g_{odd}(y,s$$

$$= -\int_0^t \int_0^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\left(\frac{(x+y)^2}{4k(t-s)}\right)} g(y,s)\,dy\,ds + \int_0^t \int_0^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)} g(y,s$$

$$= \int_0^t \int_0^{\infty} \left(e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)} - e^{-\left(\frac{(x+y)^2}{4k(t-s)}\right)}\right) \frac{1}{\sqrt{4\pi k(t-s)}}\, g(y,s)\,dy\,ds$$

now that we have our v as the sum of w & I the solution for v is restricte

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left(e^{-\left(\frac{(x-y)^2}{4kt}\right)} - e^{-\left(\frac{(x+y)^2}{4kt}\right)}\right) (\phi(y) - h(0))\,dy$$

$$+ \int_0^t \int_0^{\infty} \left(e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)} - e^{-\left(\frac{(x+y)^2}{4k(t-s)}\right)}\right) \frac{1}{\sqrt{4\pi k(t-s)}} (f(y,s) - h'(s))\,dy\,ds = \underline{\text{Partial In}}$$

Final Answer: $V(x,t) = H(x) + \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left(e^{-\left(\frac{(x-y)^2}{4kt}\right)} - e^{-\left(\frac{(x+y)^2}{4kt}\right)}\right) \phi(y)\,dy$

$$\cancel{\frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left(e^{-\left(\frac{(x-y)^2}{4kt}\right)} - e^{-\left(\frac{(x+y)^2}{4kt}\right)}\right) h(0)\,dy} + \int_0^t \int_0^{\infty} \left(e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)} - e^{-\left(\frac{(x+y)^2}{4k(t-s)}\right)}\right) \frac{1}{\sqrt{4\pi k(t-s)}} f(y,s)$$

$$\cancel{\frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left(e^{-\left(\frac{(x-y)^2}{4kt}\right)} - e^{-\left(\frac{(x+y)^2}{4kt}\right)}\right)} h(0)\,dy + \int_0^t \int_0^{\infty} \frac{d}{ds}\left(\left(e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)} - e^{-\left(\frac{(x+y)^2}{4k(t-s)}\right)}\right) \frac{1}{\sqrt{4\pi k(t-s)}}\right) h(s)$$