$$-\frac{Vl}{m\pi}\cos\frac{m\pi x}{l}\Big|_{-\delta+\frac{1}{2}l}^{\delta+\frac{1}{2}l} = \frac{A_m l m\pi}{2}$$

$$-\frac{Vl}{m\pi}\cos\left(\frac{m\pi x}{l}\right)\Big|_{-\delta+\frac{1}{2}l}^{\delta+\frac{1}{2}l} = -\frac{Vl}{m\pi}\left[\cos\frac{m\pi(\delta+\frac{1}{2}l)}{l} - \cos\frac{m\pi(-\delta+\frac{1}{2}l)}{l}\right]$$

$$= \frac{2Vl}{m\pi}\sin\left(\frac{m\pi\delta}{l}\right)\sin\left(\frac{m\pi}{2}\right)$$

$$u(x,t) = \sum_{n=1}^{\infty}\frac{4Vl}{c(2n-1)^2\pi^2}\sin\left(\frac{(2n-1)\pi\delta}{l}\right)\sin\left(\frac{(2n-1)\pi}{2}\right)\sin\left(\frac{(2n-1)\pi x}{l}\right)\sin\left(\frac{c(2n-1)\pi t}{l}\right)$$

$$\boxed{u(x,t) = \frac{4Vl}{c\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\left(\frac{(2n-1)\pi\delta}{l}\right)\sin\left(\frac{(2n-1)\pi x}{l}\right)\sin\left(\frac{c(2n-1)\pi t}{l}\right)}$$

Use the nth harmonic equation

$$E_n = \frac{1}{2}\int_0^l\left[\rho\left[\frac{dh}{dt}\right]^2 + T\left[\frac{dh}{dt}\right]^2\right]dx$$

$$= \frac{8\rho V^2}{n^2\pi^2}\sin^2\left(\frac{n\pi\delta}{l}\right)\sin^2\left(\frac{n\pi}{2}\right)\int_0^l\left[\sin^2\left(\frac{n\pi x}{l}\right)\cos^2\left(\frac{cn\pi t}{l}\right) + \cos^2\left(\frac{n\pi x}{l}\right)\sin^2\left(\frac{cn\pi t}{l}\right)\right]dx$$

$$= \frac{8\rho V^2}{n^2\pi^2}\sin^2\left(\frac{n\pi\delta}{l}\right)\left[\int_0^l\sin^2\left(\frac{n\pi x}{l}\right)dx\cos^2\left(\frac{cn\pi t}{l}\right) + \int_0^l\cos^2\left(\frac{n\pi x}{l}\right)dx\sin^2\left(\frac{cn\pi t}{l}\right)\right]$$

(the integral of sin squared & cos squared is $l/2$)

$$= \frac{8\rho V^2}{n^2\pi^2}\sin^2\left(\frac{n\pi\delta}{l}\right)\sin^2\left(\frac{n\pi}{2}\right)\left(\frac{l}{2}\right)\left[\cos^2\left(\frac{cn\pi t}{l}\right) + \sin^2\left(\frac{cn\pi t}{l}\right)\right]$$

$$= \begin{cases}\frac{4l\rho V^2}{n^2\pi^2}\sin^2\left(\frac{n\pi\delta}{l}\right), & n \text{ is odd}\\ 0, & n \text{ is even}\end{cases}$$

$$\boxed{E_n \approx \frac{4\rho V^2\delta^2}{l} \text{ for fixed } n \text{ \& odd } n \text{ \& small } \delta}$$

1. **For each of the following functions, state whether it is even or odd or periodic. If periodic, what is its smallest period?**

(a) $\sin ax \quad (a > 0)$

(b) $e^{ax} \quad (a > 0)$

(c) $x^m \quad (m = \text{integer})$

(d) $\tan x^2$

(e) $|\sin(x/b)| \quad (b > 0)$

(f) $x \cos ax \quad (a > 0)$

(a)

it's periodic let's find the period

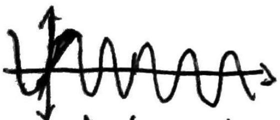
$\sin ax = \sin a(x+p)$

$= \sin ax \cos ap + \cos ax \sin ap$

$\sin ax(1-\cos ap) = \cos ax \sin ap$

true for x only if $1-\cos p = \sin ap = 0$

$ap = 2\pi \Rightarrow p = \frac{2\pi}{a}$



$\sin(-ax) = \sin(-ax+2\pi)$

$= -\sin ax \cos 2\pi + \cos ax \sin 2\pi$

function is odd $= -\sin ax$

[redacted] (f)

$-x\cos(-ax) = -x(\cos ax) = -(x\cos(ax))$

[The function is odd

& the function is non-periodic]

no smallest interval for which it increases

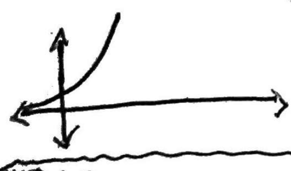
(b)

assume $x > 0$, $-ax < 0$, since the function increases

$e^{-ax} > e^0 = 1$

but $0 < e^{ax} < 1 \Rightarrow -1 < -e^{ax} < 0$

so it's not odd nor even



it is increasing so it's not periodic

[redacted] (c) [split into cases]

1.) $m = 0 \Rightarrow f(x) = 1$

function is constant & hence even

the function is not periodic because no smallest interval

2.) $m = \text{odd}$, $f(-x) = (-x)^m = (-1)^m x^m = -f(m)$ repeating

function is odd & not periodic

no smallest interval in which it repeats

3.) $m = \text{even}$, $f(-x) = (-x)^m = (-1)^m x^m = f(m)$

function is even & not periodic

no smallest interval in which it repeats

[redacted] (d)

the figure is this function

$\tan(-x)^2 = \tan^2 x)$

function is even but not periodic because ther is no smallest interval for which it repeats

[redacted] (e)

$|\sin(\frac{-x}{b})| = |-\sin(\frac{x}{b})| = |\sin(\frac{x}{b})|$

so the function is even

find the period p,

$= |\sin(\frac{x}{b})| = |\sin(\frac{x+p}{b})|$ true for $x = b$

$= |\sin(\frac{x}{b})\cos(\frac{p}{b}) + \cos(\frac{x}{b})\sin(\frac{p}{b})|$ periodic period $b\pi$

$\frac{b}{\pi}$

$= \frac{1}{9}$

5. **Show that the Fourier sine series on $(0, l)$ can be derived from the full Fourier series on $(-l, l)$ as follows. Let $\phi(x)$ be any (continuous) function on $(0, l)$. Let $\tilde{\phi}(x)$ be its odd extension. Write the full series for $\tilde{\phi}(x)$ on $(-l, l)$. [Assume that its sum is $\tilde{\phi}(x)$.] By Exercise 4, this series has only sine terms. Simply restrict your attention to $0 < x < l$ to get the sine series for $\phi(x)$.**

Let a function defined on $(-l, l)$ be

$$\psi(x) \begin{cases} \phi(x) & x \in (0,l) \\ -\phi(-x) & x \in (-l,0) \end{cases}$$

ψ is an odd extension of ϕ, then the full fourier series over $(-l,l)$ is

$$\psi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi x}{l}\right) + B_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

the formula for the coefficient are

$$A_n = \frac{1}{l}\int_{-l}^{l} \psi(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

for $n = 0, 1, 2, 3, \ldots$

because ψ is odd & cos is an even function

$$\psi(-x)\cos\left(\frac{-n\pi x}{l}\right)dx = -(\underbrace{\psi(x)\cos\left(\frac{n\pi x}{l}\right)dx}_{\text{odd}})$$

$\boxed{\text{So, } A_n = 0,\ n = 0, 1, 2, \ldots}$

The formula for the remaining coefficients are

$$b_n = \frac{1}{l}\int_{-l}^{l}\psi(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$n = 1, 2, 3, \ldots$ because ψ & sin are odd functions

$$\psi(-x)\sin\left(\frac{-n\pi x}{l}\right)dx = (\underbrace{\psi(x)\sin\left(\frac{n\pi x}{l}\right)dx}_{\text{is even}})$$

$$b_n = \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{n\pi x}{l}\right)dx,\quad n = 1, 2, 3, \ldots$$

The full series expansion of ψ on $(-l,l)$ is

$$\psi(x) = \sum_{n=1}^{\infty}\frac{2}{l}\sin\left(\frac{n\pi x}{l}\right)\int_0^l \psi(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

restricted on $(0,l)$

$$\boxed{\phi(x) = \sum_{n=1}^{\infty}\frac{2}{l}\sin\left(\frac{n\pi x}{l}\right)\int_0^l \phi(y)\sin\left(\frac{n\pi y}{l}\right)dy}$$

given the restriction

5. Show that the Fourier sine series on $(0, l)$ can be derived from the full Fourier series on $(-l, l)$ as follows. Let $\phi(x)$ be any (continuous) function on $(0, l)$. Let $\tilde{\phi}(x)$ be its odd extension. Write the full series for $\tilde{\phi}(x)$ on $(-l, l)$. [Assume that its sum is $\tilde{\phi}(x)$.] By Exercise 4, this series has only sine terms. Simply restrict your attention to $0 < x < l$ to get the sine series for $\phi(x)$.

let a function defined on $(-l,l)$ be

$$\psi(x) \begin{cases} \phi(x) & x \in (0,l) \\ -\phi(-x) & x \in (-l,0) \end{cases}$$

ψ is an odd extension of ϕ, then the full fourier series over $(-l,l)$ is

$$\psi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi x}{l}\right) + B_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

the formula for the coefficient are

$$A_n = \frac{1}{l}\int_{-l}^{l} \psi(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

for $n = 0,1,2,3,\ldots$

because ψ is odd & cos is an even function

$$\psi(-x)\cos\left(\frac{-n\pi x}{l}\right)dx = -\underbrace{\left(\psi(x)\cos\left(\frac{n\pi x}{l}\right)dx\right)}_{\text{odd}}$$

$$\boxed{\text{So, } A_n = 0,\ n = 0,1,2,\ldots}$$

The formula for the remaining coefficients are

$$B_n = \frac{1}{l}\int_{-l}^{l} \psi(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$n = 1,2,3,\ldots$ because ψ & sin are odd functions

$$\psi(-x)\sin\left(\frac{-n\pi x}{l}\right)dx = \underbrace{\left(\psi(x)\sin\left(\frac{n\pi x}{l}\right)dx\right)}_{\text{is even}}$$

$$B_n = \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{n\pi x}{l}\right)dx,\quad n = 1,2,3,\ldots$$

the full series expansion of ψ on $(-l,l)$ is

$$\psi(x) = \sum_{n=1}^{\infty}\frac{2}{l}\sin\left(\frac{n\pi x}{l}\right)\int_0^l \psi(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

restricted on $(0,l)$,

given the restriction

$$\boxed{\phi(x) = \sum_{n=1}^{\infty}\frac{2}{l}\sin\left(\frac{n\pi x}{l}\right)\int_0^l \phi(y)\sin\left(\frac{n\pi y}{l}\right)dy}$$

7. **Show how the full Fourier series on $(-l, l)$ can be derived from the full series on $(-\pi, \pi)$ by changing variables $w = (\pi/l)x$. (This is called a *change of scale;* it means that one unit along the x axis becomes π/l units along the w axis.)**

Let ϕ be a function, then the full fourier series over $(-\pi,\pi)$ is

$$\phi(w) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos(nw) + B_n \sin(nw)\right]$$

formula for coefficients

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(w)\cos(nw)\,dw$$

for $n = 0,1,2,3\ldots$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(w)\sin(nw)\,dw$$

for $n = 1,2,3\ldots$

make the substitution $\left\{\begin{matrix} w = \frac{\pi}{l}x \\ dw = \frac{\pi}{l}dx \end{matrix}\right\}$

$$\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi x}{l}\right) + B_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(w)\cos(nw)\,dw + \sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{l}\right)\left(\frac{1}{\pi}\right)\int_{-\pi}^{\pi}\phi(w)\cos(nw)\,dw$$
$$+ \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{l}\right)\left(\frac{1}{\pi}\right)\int_{-\pi}^{\pi}\phi(w)\sin(nw)\,dw$$

make same substitution under integrals

$$= \frac{1}{2\pi}\int_{-l}^{l}\phi(y)\cos\left(\frac{n\pi y}{l}\right)\left(\frac{\pi}{l}\right)dy + \sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{l}\right)\left(\frac{1}{\pi}\right)\int_{-l}^{l}\phi(y)\cos\left(\frac{n\pi y}{l}\right)\frac{\pi}{l}dy$$
$$+ \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{l}\right)\left(\frac{1}{\pi}\right)\int_{-l}^{l}\phi(y)\sin\left(\frac{n\pi y}{l}\right)\left(\frac{\pi}{l}\right)dy$$

$$\boxed{= \frac{1}{2l}\int_{-l}^{l}\phi(y)\cos\left(\frac{n\pi y}{l}\right)dy + \sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{l}\right)\left(\frac{1}{l}\right)\int_{-l}^{l}\phi(y)\cos\left(\frac{n\pi y}{l}\right)dy + \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{l}\right)\left(\frac{1}{l}\right)\int_{-l}^{l}\phi(y)\sin\left(\frac{n\pi y}{l}\right)dy}$$

4. Consider the problem $u_t = ku_{xx}$ for $0 < x < l$, with the boundary conditions $u(0,t) = U$, $u_x(l,t) = 0$, and the initial condition $u(x,0) = 0$, where U is a constant.
   (a) Find the solution in series form. (*Hint:* Consider $u(x,t) - U$.)
   (b) Using a direct argument, show that the series converges for $t > 0$.
   (c) If ϵ is a given margin of error, estimate how long a time is required for the value $u(l,t)$ at the endpoint to be approximated by the constant U within the error ϵ. (*Hint:* It is an alternating series with first term U, so that the error is less than the next term.)

(a) $v(x,t) = u(x,t) - U$

It satisfies $\begin{cases} v_t = kv_{xx} & ,0<x<l \\ v(0,t)=0 & ,v_{xx}(l,t)=0 \\ v(x,0) = -U \end{cases}$

$v(x,t) = X(x)T(t)$

$v_t = k v_{xx} \Leftrightarrow \frac{T'}{kT} = \frac{X''}{X} = -\lambda^2$

X has to satisfy $X(0) = X'(l) = 0$

find the eigen values & corresponding eigen functions

$X(x) = A\sin\beta x + B\cos\beta x$

Boundary conditions gives us

$0 = X(0) = B \Rightarrow 0 = X'(l) = A\beta\cos\beta l \overset{A\neq 0}{\Rightarrow} \beta = \frac{(2n-1)\pi}{2l}$

$T(t) = Ae^{-k\lambda}$

the eigenfunction corresponding to $-\beta^2$ is

$$\boxed{u_n(x,t) = A_n e^{-\left(\frac{(2n-1)^2\pi^2 kt}{4l^2}\right)} \sin\left(\frac{(2n-1)\pi x}{2l}\right)}$$

$X(x) = Ax + B$

$0 = X(0) = B \Rightarrow 0 = X'(l) = A$

so the eigenfunction is trivial & $\lambda = 0$ is not an eigenfunction

assume $\lambda<0$ there exists such that $\lambda = -\beta^2$

then $X(x) = A\sinh\beta x + B\cosh\beta x$

the boundary conditions gives us

$0 = X(0) = B \Rightarrow 0 = X'(l) = A\beta\sinh\beta l \Rightarrow A = 0$

hence, $\lambda<0$ is not an eigenfunction.

$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{(2n-1)^2\pi^2 kt}{4l^2}\right)} \sin\left(\frac{(2n-1)\pi x}{2l}\right)$

$-U = v(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{(2n-1)\pi x}{2l}\right)$.

$\int_0^l -U\sin\left(\frac{(2m-1)\pi x}{2l}\right)dx = \sum_{n=1}^{\infty} A_n \int_0^l \sin\left(\frac{(2n-1)\pi x}{2l}\right)\sin\left(\frac{(2m-1)\pi x}{2l}\right)dx$

(4)

$-\frac{2lU}{(2m-1)\pi} = \frac{l}{2}A_m$, $A_n = -\frac{4U}{(2n-1)\pi}$

Final solution

$$\boxed{\begin{aligned} u(x,t) &= v(x,t) + U \\ &= \sum_{n=1}^{\infty} -\frac{4U}{(2n-1)\pi} e^{-\left(\frac{(2n-1)^2\pi^2 kt}{4l^2}\right)} \sin\left(\frac{(2n-1)\pi x}{2l}\right) \\ &\quad + U \end{aligned}}$$

(c)

~~Note that it converges absolutely~~ (c)

$$|u(x,t)| = U - \sum_{n=1}^{\infty} \frac{4U}{(2n-1)\pi} e^{-\left(\frac{(2n-1)^2\pi^2 kt}{4l^2}\right)} \sin\left(\frac{(2n-1)\pi x}{2l}\right)$$

(b) $\leq \underbrace{|U|}_{>0} + \sum_{n=1}^{\infty} \left| \underbrace{\frac{4U}{(2n-1)\pi}}_{>0} \underbrace{e^{-\left(\frac{(2n-1)^2\pi^2 kt}{4l^2}\right)}}_{>0} \sin\left(\frac{(2n-1)\pi x}{2l}\right) \right|$

$u(x,t) \leq U + \sum_{n=1}^{\infty} \frac{4U}{(2n-1)\pi} e^{-\left(\frac{(2n-1)^2\pi^2 kt}{4l^2}\right)} \left| \sin\frac{(2n-1)\pi x}{2l} \right|$

$\leq U + \sum_{n=1}^{\infty} \frac{4U}{(2n-1)\pi} e^{-\left(\frac{(2n-1)^2\pi^2 kt}{4l^2}\right)} \underbrace{(-1)^n}_{\leq 1}$

$= U + \sum_{n=1}^{\infty} \frac{4U}{(2n-1)\pi} e^{-\left(\frac{(2n-1)\ldots}{4l^2}\right)}$

the lower bound $U-\epsilon$, let it be the magnitude of the first term in the series

$\epsilon = \frac{4U}{\pi} e^{-k\frac{\pi^2}{4l^2}t}$

Solve for t, $t = -\frac{4l^2}{k\pi^2}\ln\left(\frac{\pi\epsilon}{4U}\right)$

the time required to reach $U-\epsilon$ at the endpoint is

$$\boxed{t = \frac{4l^2}{k\pi^2}\ln\left(\frac{4U}{\pi\epsilon}\right)}$$

$$\lim_{n\to\infty} \frac{\frac{4U}{(2n+1)\pi} e^{-\left(\frac{(2n+1)^2\pi^2 kt}{4l^2}\right)}}{\frac{4U}{(2n-1)\pi} e^{-\left(\frac{(2n-1)^2\pi^2 kt}{4l^2}\right)}}$$

$= \cancel{\lim_{x\to\infty}\left(\frac{(2n-1)}{(2n+1)}\right) e^{\ldots}}$

$= \lim_{x\to\infty} \left(\frac{(2n-1)}{(2n+1)}\right) e^{-\left(\frac{(2n+1+2n-1)(2n+1-2n+1)\pi^2 kt}{4l^2}\right)}$

$= \lim_{x\to\infty} \left(\frac{(2n-1)}{(2n+1)}\right) e^{-\left(\frac{8n\pi^2 kt}{4l^2}\right)}$

$= 0$

, so it converges absolutely

Problem 13: Diffusion Equation Scheme

Consider the following scheme for the diffusion equation:

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = \frac{u_{j+1}^n + u_{j-1}^n - u_j^{n+1} - u_j^{n-1}}{(\Delta x)^2}$$

It uses a centered difference for u_t and a modified form of the centered difference for u_{xx}.

(a) Solve it for u_j^{n+1} in terms of s and the previous time steps.

(b) Show that it is stable for all s.

(a) we have to solve the equation

$$\frac{U_j^{n+1} - U_j^{n-1}}{2\Delta t} = \frac{U_{j+1}^n + U_{j-1}^n - U_j^{n+1} - U_j^{n-1}}{(\Delta x)^2}$$

for u_j^{n+1}, let $s = \frac{\Delta t}{(\Delta x)^2}$,

$$u_j^{n+1} - u_j^{n-1} = 2s(u_{j+1}^n + u_{j-1}^n - u_j^{n+1} - u_j^{n-1})$$

$$(1+2s)u_j^{n+1} = 2s(u_{j+1}^n + u_{j-1}^n) + (1-2s)u_j^{n-1}$$

$$u_j^{n+1} = \boxed{\frac{2s(u_{j+1}^n + u_{j-1}^n) + (1-2s)U_j^{n-1}}{1+2s}}$$

(b) $U_j^n = (e^{ik\Delta x})^j(\xi(k))^n$

$$u_j^{n+1} = \frac{2s(u_{j+1}^n + u_{j-1}^n) + (1-2s)u_j^{n-1}}{1+2s}$$

$$(e^{ik\Delta x})^j(\xi(k))^{n+1} = (e^{ik\Delta x})^j(\xi(k))^{n-1}\left[\frac{2s\xi(k)(e^{ik\Delta x}+e^{-ik\Delta x}) + (1-2s)}{(1+2s)}\right]$$

$$(\xi(k))^2 = \frac{4s\xi(k)\cos k\Delta x + (1-2s)}{1+2s}$$

then

$$\xi(k) = \frac{2s\cos k\Delta x \pm \sqrt{1-4s^2\sin^2 k\Delta x}}{1+2s}$$

the scheme is stable if $|\xi(k)| \leq 1$

$$1 - 4s^2\sin^2 k\Delta x \geq 0 \Rightarrow s \leq \frac{1}{2|\sin k\Delta x|}$$

thus, $\frac{2s\cos k\Delta x + \sqrt{1-4s^2\sin^2 k\Delta x}}{1+2s} \leq 1$

has to be true, $0 \leq (1-\cos k\Delta x)(4s+8s^2)$

so, $\xi \geq -1$ for all s

meaning that it's stable

Problem 8: Crank–Nicolson Scheme

(a) Write down the Crank–Nicolson scheme ($\theta = \frac{1}{2}$) for $u_t = u_{xx}$.

(b) Consider the solution in the interval $0 \le x \le 1$ with $u = 0$ at both ends. Assume $u(x, 0) = \phi(x)$ and $\phi(1-x) = \phi(x)$. Show, using uniqueness, that the exact solution must be even around the midpoint $x = \frac{1}{2}$. [That is, $u(x,t) = u(1-x,t)$.]

(c) Let $\Delta x = \Delta t = \frac{1}{6}$. Let the initial data be 0 0 1 0 0 0. Compute the solution by the Crank–Nicolson scheme for one time step ($t = \frac{1}{6}$). (Hint: Use part (b) to halve the computation.)

(a) $$\frac{u_j^{n+1} - u_j^n}{\Delta t} = (1-\theta)\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} + \theta\left[\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\Delta x)^2}\right]$$

multiply Δt, plug in $\theta = \frac{1}{2}$ $\quad s = \frac{\Delta t}{(\Delta x)^2}$

$$u^{n+1} - u_j^n = \frac{s}{2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) + \frac{s}{2}\left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right)$$

re-arrangement gives

$$\boxed{\frac{s}{2}u_{j-1}^{n+1} - (1+s)u_j^{n+1} + \frac{s}{2}u_{j+1}^{n+1} = -\frac{s}{2}u_{j-1}^n + (s-1)u_j^n - \frac{s}{2}u_{j+1}^n}$$

(b) $v(x,t) = u(1-x,t)$

$v(0,t) = u(1,t) = 0$ & $v(1,t) = u(0,t) = 0$

$v(x,0) = u(1-x,0) = \phi(1-x) = \phi(x)$

$v_t(x,t) = u_t(1-x,t) = u_{xx}(1-x,t) = (-1)^2 v_{xx}(x,t) = v_{xx}(x,t)$

$u = v$

$u(1-x,t) = v(x,t) = u(x,t)$

u is even around the midpoint $x = \frac{1}{2}$

(c) $u_0^1, u_1^1, u_2^1, u_3^1$

$s = 6$

$$3u_{j-1}^1 - 7u_j^1 + 3u_{j+1}^1 = -3u_{j-1}^0 + 5u_j^0 - 3u_{j+1}^0$$

$u_0^1 = u_6^1 = 0$

$$\begin{cases} 3u_0^1 - 7u_1^1 + 3u_2^1 = -3u_0^0 + 5u_1^0 - 3u_2^0 \\ 3u_1^1 - 7u_2^1 + 3u_3^1 = -3u_1^0 + 5u_2^0 - 3u_3^0 \\ 3u_2^1 - 7u_3^1 + 3u_4^1 = -3u_2^0 + 5u_3^0 - 3u_4^0 \end{cases}$$

Plug in initial conditions $u_4^1 = u_2^1$

$$\begin{cases} -7u_1^1 + 3u_2^1 = 0. \\ 3u_1^1 - 7u_2^1 + 3u_3^1 = -3. \\ 6u_2^1 - 7u_3^1 = 5. \end{cases}$$

Solve the system to get $\boxed{0, \frac{9}{77}, \frac{3}{11}, -\frac{37}{77}, \frac{3}{11}, \frac{9}{77}, 0}$

Problem 7: Harmonic Functions

(a) If $u(x,y) = f(x/y)$ is a harmonic function, solve the ODE satisfied by f.

(b) Show that $\partial u/\partial r \equiv 0$, where $r = \sqrt{x^2+y^2}$ as usual.

(c) Suppose that $v(x,y)$ is any function in $\{y > 0\}$ such that $\partial v/\partial r \equiv 0$. Show that $v(x,y)$ is a function of the quotient x/y.

(d) Find the boundary values $\lim_{y\to 0} u(x,y) = h(x)$.

(e) Show that your answer to parts (c) and (d) agrees with the general formula from Exercise 6.

(a) $u(x,y) = f\left(\frac{x}{y}\right)$

$0 = \Delta u$

$= \partial_{xx}u + \partial_{yy}u$

$= \partial_{xx} f\left(\frac{x}{y}\right) + \partial_{yy} f\left(\frac{x}{y}\right)$

$= \partial_x\left(f'\left(\frac{x}{y}\right)\frac{1}{y}\right) + \partial_y\left(f'\left(\frac{x}{y}\right)\frac{-x}{y^2}\right)$

$= f''\left(\frac{x}{y}\right)\frac{1}{y^2} + f''\left(\frac{x}{y}\right)\frac{x^2}{y^4} + f'\left(\frac{x}{y}\right)\frac{2x}{y^3}$

$f''(t) + t^2 f''(t) + 2t f'(t) = 0$

$[f'(t) + t^2 f'(t)]' = 0$

$f'(t)(1+t^2) = C$

$f'(t) = \frac{C}{1+t^2}$

$\boxed{f(t) = C\arctan t + D}$

(b) In polar form, $u(r,\theta) = f\left(\frac{\cos\theta}{\sin\theta}\right) = f(\cot\theta) = C\arctan(\cot\theta) + D$, so u only of θ, $\partial_r u(r) = 0$

(c) Let $v(x,y)$ be a harmonic function, $\partial_r v = 0$. (Then in polar coordinates, v is a function of θ.) $v(x,y) = v(\theta)$. Sub in $\theta = \arctan\frac{y}{x}$, $v(x,y) = v\left(\arctan\frac{y}{x}\right)$ ... $v(x,y) = v\left(\arctan\left(\frac{y}{x}\right)\right) = g\left(\frac{x}{y}\right)$.

(d) $\lim_{y\to 0} u(x,y) = \lim_{y\to 0} f\left(\frac{x}{y}\right) = \lim_{y\to 0} C\arctan\left(\frac{x}{y}\right) + D$

(e)

Problem 7: Harmonic Functions

(a) If $u(x, y) = f(x/y)$ is a harmonic function, solve the ODE satisfied by f.

(b) Show that $\partial u/\partial r \equiv 0$, where $r = \sqrt{x^2+y^2}$ as usual.

(c) Suppose that $v(x, y)$ is any function in $\{y > 0\}$ such that $\partial v/\partial r \equiv 0$. Show that $v(x, y)$ is a function of the quotient x/y.

(d) Find the boundary values $\lim_{y\to 0} u(x, y) = h(x)$.

(e) Show that your answer to parts (c) and (d) agrees with the general formula from Exercise 6.

(a) $u(x,y) = f\left(\frac{x}{y}\right)$

$0 = \Delta u$

$= \partial_{xx} u + \partial_{yy} u$

$= \partial_{xx} f\left(\frac{x}{y}\right) + \partial_{yy} f\left(\frac{x}{y}\right)$

$= \partial_x \left(f'\left(\frac{x}{y}\right)\frac{1}{y}\right) + \partial_y \left(f'\left(\frac{x}{y}\right)\frac{-x}{y^2}\right)$

$= f''\left(\frac{x}{y}\right)\frac{1}{y^2} + f''\left(\frac{x}{y}\right)\frac{x^2}{y^4} + f'\left(\frac{x}{y}\right)\frac{2x}{y^3}$

funct f mus satisfy

$f''(t) + t^2 f''(t) + 2t f'(t) = 0$

$[f'(t) + t^2 f'(t)]' = 0$

$f'(t)(1+t^2) = C$

Moreover $f'(t) = \frac{C}{1+t^2}$

giving solution $\boxed{f(t) = C\,\text{atan}\,t + D}$

(b) in polar terms

$v(r,\theta) = f\left(\frac{r\cos\theta}{r\sin\theta}\right) = f(\cot\theta) = C\,\text{atan}\cot\theta + D$

So v only of θ, $\partial_r v(\theta) = 0$

(c) let $v(x,y)$ be a function such,

$\partial_r v = 0$

(then in polar coordinates v is a function of θ)

$v(x,y) = v(\theta)$

Sub in $\to \theta = \text{atan}\,\frac{y}{x}$

$v(x,y) = v\left(\text{atan}\,\frac{y}{x}\right)$

there exist

$g(t) = v\left(\text{atan}\,\frac{1}{t}\right)$

$v(x,y) = v\left(\text{atan}\,\frac{1}{\left(\frac{x}{y}\right)}\right) = g\left(\frac{x}{y}\right)$

(d) $\lim_{y\to 0} u(x,y) = \lim_{y\to 0} f\left(\frac{x}{y}\right)$

$= \lim_{y\to 0} C\,\text{atan}\left(\frac{x}{y}\right) + D$

$= \begin{cases} \pi/2\, C + D & x>0 \\ -\pi/2\, C + D & x<0 \end{cases}$ $\to \pm\infty$

(e)

$\frac{y}{\pi}\int_{-\infty}^{\infty} \frac{h(s)}{(s-x)^2+y^2}\,ds$

$= \frac{y}{\pi}\int_{-\infty}^{0} \frac{-\pi/2\, C + D}{(s-x)^2+y^2}\,ds + \frac{y}{\pi}\int_{0}^{\infty} \frac{\pi/2\, C + D}{(s-x)^2+y^2}\,ds = \frac{-\frac{\pi}{2}C + D}{\pi}\left[\text{atan}\left(\frac{s-x}{y}\right)\right]_{-\infty}^{0} + \frac{\frac{\pi}{2}C + D}{\pi}\left[\text{atan}\left(\frac{s-x}{y}\right)\right]_{0}^{\infty}$

$\Uparrow\ \vdots$

$= C\,\text{atan}\left(\frac{x}{y}\right) + D$

$= f\left(\frac{x}{y}\right)$

$= u(x,y)$

Problem 1: Forward Difference Scheme

(a) Solve the problem $u_t = u_{xx}$ in the interval $[0, 4]$ with $u = 0$ at both ends and $u(x, 0) = x(4 - x)$, using the forward difference scheme with $\Delta x = 1$ and $\Delta t = 0.25$. Calculate four time steps (up to $t = 1$).

(b) Do the same with $\Delta x = 0.50$ and $\Delta t = 0.0625 = \frac{1}{16}$. Calculate four time steps (up to $t = 0.25$).

(c) Compare your answers with each other. How close are they at $x = 2.0, t = 0.25$?

(a) $\Delta x = 1$ $\quad 0 \leq j \leq 4$

$t = 1$, when $\Delta t = .25$

$0 \leq n \leq 4$

$j = 0$ or $j = 4$

$u_j^n = 0$

$n = 0$ $\quad u_j^0 = j\Delta x(4 - j\Delta x)$

For $1 \leq j \leq 3$, $u_j^{n+1} = s(u_{j+1}^n + u_{j-1}^n) + (1 - 2s)u_j^n$

where $s = \dfrac{\Delta t}{(\Delta x)^2} = \dfrac{.25}{1} = .25$

Calculate u_j^n for $\forall_{0 \leq j}$, $\forall_{n \leq 4}$

$n = 0$: 0 3 . . 0

$n = 1$: 0 5/2 . . 0

$n = 2$: 0 17/8 . . 0

$n = 3$: 0 29/16 . . 0

$n = 4$: 0 99/64 . . 0

(b) $\Delta x = .5$ $\quad 0 \leq j \leq 8$

$t = .25$, when $\Delta t = .0625$

$0 \leq n \leq 4$

when $j = 0$ or $j = 8$

when $n = 0$ $\quad u_j^0 = j\Delta x(4 - j\Delta x)$

For, $1 \leq j \leq 3$

$u_j^{n+1} = s(u_{j+1}^n + u_{j-1}^n) + (1 - 2s)u_j^n$, where $s = \dfrac{\Delta t}{(\Delta x)^2} = \dfrac{.0625}{.25} = .25$

Calculate u_j^n for $\forall_{0 \leq j \leq 8}$ & $\forall_{0 \leq n \leq 4}$

$n = 0$: 0 7/4 0

$n = 1$: 0 13/8 . 0

$n = 2$: 0 49/32 . 0

$n = 3$: 0 93/64 . 0

$n = 4$: 0 709/512 . 0

(c) The approximation of $u(2.0, .25)$

$u_1^2 = \frac{7}{2}$

$u_4^4 = \frac{7}{2}$

at $(x, t) = (2.0, .25)$ the approximations are equal.

Problem 1: Taylor Expansion Error

The Taylor expansion written in Section 8.1 is valid if u is a C^4 function. If $u(x)$ is merely a C^3 function, the best we can say is that the Taylor expansion is valid only with a $o(\Delta x)^3$ error. [This notation means that the error is $(\Delta x)^3$ times a factor that tends to zero as $\Delta x \to 0$.] If merely a C^2 function, it is only valid with a $o(\Delta x)^2$ error, and so on.

(a) If $u(x)$ is merely a C^3 function, what is the error in the first derivative due to its approximation by the centered difference?

(b) What if $u(x)$ is merely a C^2 function?

(a) taylor expansion is only valid with $O(\Delta x)^3$ error:

$$u(x+\Delta x) = u(x) + u'(x)\Delta x + \frac{1}{2}u''(x)(\Delta x)^2 + \frac{1}{6}u'''(x)(\Delta x)^3 + O(\Delta x)^3$$

&

$$u(x-\Delta x) = u(x) - u'(x)\Delta x + \frac{1}{2}u''(x)(\Delta x)^2 - \frac{1}{6}u'''(x)(\Delta x)^3 + O(\Delta x)^3.$$

the approx of the first derivative by the centered difference is

$$u'(x) = \frac{u(x+\Delta x) - u(x-\Delta x)}{\Delta x} + \underbrace{\frac{1}{3}u'''(x)(\Delta x)^2}_{O(\Delta x)^2} + \underbrace{O(\Delta x)^3}_{O(\Delta x)^2}$$

thus

$$u'(x) = \frac{u(x+\Delta x) - u(x-\Delta x)}{\Delta x} + O(\Delta x)^2$$

(b) taylor expansion is only valid with $O(\Delta x)^3$ error;

$$u(x+\Delta x) = u(x) + u'(x)\Delta x + \frac{1}{2}u''(x)(\Delta x)^2 + O(\Delta x)^2$$

&

$$u(x-\Delta x) = u(x) - u'(x)\Delta x + \frac{1}{2}u''(x)(\Delta x)^2 + O(\Delta x)^2.$$

approx of the first derivative by the centered difference

$$u'(x) = \frac{u(x+\Delta x) - u(x-\Delta x)}{\Delta x} + \underbrace{\frac{O(\Delta x)^2}{\Delta x}}_{O(\Delta x)^2}$$

thus,

$$\boxed{u'(x) = \frac{u(x+\Delta x) - u(x-\Delta x)}{\Delta x} + O(\Delta x)}$$

Problem 11: First-Order Equation

Consider the first-order equation $u_t + au_x = 0$.

(a) Solve it exactly with the initial condition $u(x,0) = \phi(x)$.

(b) Write down the finite difference scheme which uses the forward difference for u_t and the centered difference for u_x.

(c) For which values of Δx and Δt is the scheme stable?

(a)

$$u(x,t) = X(x)\,T(t)$$

$$u_t + a u_x = 0$$

$$u(x,t) = f(x-at)$$

$$\phi(x) = u(x,0) = f(x)$$

$$\boxed{u(x,t) = \phi(x-at)}$$

(b)

(b) Approx u_t (forward diff) & u_x (centered diff)

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -a\,\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

let $s = \dfrac{\Delta t}{2\Delta x}$

$$u_j^{n+1} = -as\left(u_{j+1}^n - u_{j-1}^n\right) + u_j^n$$

(c)

(c)

$$2as \leq 1 \leftrightarrow \boxed{\Delta t \leq \tfrac{1}{a}\Delta x}$$

Problem 7: Jacobi and Gauss–Seidel

Solve $u_{xx} + u_{yy} = 0$ in the unit square $\{0 \le x \le 1, 0 \le y \le 1\}$ with the boundary conditions: $u(x,0) = u(0,y) = 0$, $u(x,1) = x$, $u(1,y) = y$. Use $\Delta x = \Delta y = \frac{1}{4}$, so that there are nine interior points for the scheme (2).

(a) Use two steps of Jacobi iteration, with the initial guess that the value at each of the nine points equals 1.

(b) Use two steps of Gauss–Seidel iteration, with the same initial guess.

(c) Compare parts (a) and (b) and the exact solution.

(a)

we start with intitral values $u^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1/4 \\ 0 & 1 & 1 & 1 & 1/2 \\ 0 & 1 & 1 & 1 & 3/4 \\ 0 & 1/4 & 1/2 & 3/4 & 1 \end{pmatrix}$

formula of Gaus-seidel iteration

$$U^{n+1}_{j,k} = \frac{1}{4}\left(U^{n}_{j+1,k} + U^{n+1}_{j-1,k} + U^{n}_{j,k+1} + U^{n+1}_{j,k-1}\right), \quad j,k = 1,2,3.$$

(b)

Computing iterations

(c)

finding exact solution

Problem 4: Finite Elements (Wave Equation)

(Finite elements for the wave equation) Consider the problem $u_{tt} = u_{xx}$ in $[0, l]$, with $u = 0$ at both ends, and some initial conditions. For simplicity, suppose that l is an integer and partition the interval into l equal sub-intervals. Each of the $l - 1 = N$ interior vertices has the trial function defined in Exercise 3. The approximate solution is defined by the formula $u_N(x) = U_1(t)v_1(x) + \cdots + U_N(t)v_N(x)$, where the coefficients are unknown functions of t.

(a) Show that a reasonable requirement is that

$$\sum_{i=1}^{N} U_i''(t) \int_0^l v_i(x)v_j(x)\,dx + \sum_{i=1}^{N} U_i(t) \int_0^l \frac{\partial v_i}{\partial x}\frac{\partial v_j}{\partial x}\,dx = 0$$

for $j = 1, \ldots, N$.

(b) Show that the finite element method reduces in this case to a system of ODEs: $K d^2U/dt^2 + MU = 0$ with an initial condition $U(0) = \Phi$. Here K and M are $N \times N$ matrices, $U(t)$ is an N-vector function, and Φ is an N-vector.

(a) The approx must satisfy

$$(U_N)_{tt} = (U_N)_{xx}$$

for an arbitrary $j = 1, 2, \ldots, N$, multiply by $v_j(x)$ & integrate respect to x over $(0, l)$

$$\int_0^l (u_N)_{tt}(x,t)\,v_j(x)\,dx = \int_0^l (u_N)_{xx}(x,t)\,v_j(x)\,dx$$

$$\int_0^l \sum_{i=1}^{N} U_i''(t)v_i(x)\,v_j(x)\,dx = \int_0^l \sum_{i=1}^{N} U_i(t)\,v_i''(x)\,v_j(x)\,dx$$

$$\sum_{i=1}^{N} U_i''(t)\int_0^l v_i(x)\,v_j(x)\,dx = \sum_{i=1}^{N} U_i(t)\int_0^l v_i''(x)\,v_j(x)\,dx$$

$$= \sum_{i=1}^{N} U_i(t)\left(v_j(x)v_i'(x)\Big|_0^l - \int_0^l v_i'(x)\,v_j'(x)\,dx\right)$$

$$\sum_{i=1}^{N} U_i''(t)\int_0^l v_i(x)\,v_j(x)\,dx + \sum_{i=1}^{N} U_i(t)\int_0^l v_i'(x)\,v_j'(x)\,dx = 0$$

(b) Define K as a matrix $[k_{ij}]_{i,j=1,\ldots,N}$

$$k_{ij} = \int_0^l v_i(x)v_j(x)\,dx \Rightarrow m_{ij} = \int_0^l v_i'(x)\,v_j'(x)\,dx \Rightarrow K\left(\frac{d^2U}{dt^2}\right) + MU = 0 \Rightarrow U(0) = \phi$$

$M = [m_{ij}]_{i,j=1,\ldots,N}$

unique solution if we add initial conditions

Problem 6: Green's Function

(a) Find the Green's function for the half plane $\{(x,y) : y > 0\}$

(b) Use it to solve the Dirichlet problem in the half plane with boundary values $h(x)$

(c) Calculate the solution with $u(x,0) = 1$

(a) Let D be the given half-space in two dimensions $(y > 0)$.
Each point $x = (x,y)$ in D has a reflected point $x^* = (x,-y)$ that is not in D.

$$-\frac{1}{2\pi}\log|x - x_0|$$

satisfy 2 of 3 conditions
(it doesn't satisfy the condition to vanish on the boundary)
So define $G(x, x_0) = \frac{1}{2\pi}\left(\log|x - x_0| - \log|x - x_0^*|\right)$

when $y = 0$, $|x - x_0| = |x - x_0^*| \Rightarrow G(x, x_0) = 0$

Therefore G is Green's function for half-space: $G(x,x_0) = \frac{1}{4\pi}\left[\log((x-x_0)^2 + (y-y_0)^2) - \log((x-x_0)^2 + (y+y_0)^2)\right]$

(b)

$$u(x_0) = \int_{bdy\, D}\left(h(x)\,\frac{\partial G(x,0;x_0,y_0)}{\partial n}\right)dx$$

$n = (0,-1)$

$$\frac{\partial G}{\partial n}(x,y;x_0,y_0) = -\frac{\partial G}{\partial y}(x,y) = \frac{1}{4\pi}\frac{\partial}{\partial y}\left[-\log((x-x_0)^2 + (y-y_0)^2) + \log((x-x_0)^2 + (y+y_0)^2)\right]$$

$$= \frac{1}{2\pi}\left[-\frac{y - y_0}{(x-x_0)^2 + (y-y_0)^2} + \frac{y + y_0}{(x-x_0)^2 + (y+y_0)^2}\right]$$

$$\frac{\partial G}{\partial n}(x,0;x_0,y_0) = \frac{y_0}{\pi}\frac{1}{(x-x_0)^2 + y_0^2},$$

$$u(x_0) = \frac{y_0}{\pi}\int_{-\infty}^{\infty}\frac{h(x)}{(x-x_0)^2 + y_0^2}\,dx$$

(c) $u(x,0) = h(x) = 1$

$$u(x_0) = \frac{y_0}{\pi}\int_{-\infty}^{\infty}\frac{1}{(x-x_0)^2 + y_0^2}\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{\left(\frac{x-x_0}{y_0}\right)^2 + 1}\,\frac{dx}{y_0} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{t^2+1}\,dt$$

$$= \frac{1}{\pi}\arctan t\Big|_{-\infty}^{\infty} = \frac{1}{\pi}\cdot\pi = \boxed{1}$$

Problem 6: Green's Function

(a) Find the Green's function for the half-plane $\{(x, y) : y > 0\}$.

(b) Use it to solve the Dirichlet problem in the half-plane with boundary values $h(x)$.

(c) Calculate the solution with $u(x, 0) = 1$.

(a) Let D be the given half-space in two dimensions ($y>0$). Each point $x=(x,y)$ in D has a reflected point $x^*=(x,-y)$ that is not in D.

$$-\frac{1}{2\pi}\log|x-x_0|$$

satisfy 2 of 3 conditions

(it doesn't satisfy the condition to vanish on the boundary)

So define $G(x,x_0)=\frac{1}{2\pi}\left(\log|x-x_0|-\log|x-x_0^*|\right)$

when $y=0$, $|x-x_0|=|x-x_0^*| \Rightarrow G(x,x_0)=0$

therefore G is Green's function for half-space: $G(x,x_0)=\frac{1}{4\pi}\left[\log((x-x_0)^2+(y-y_0)^2)-\log((x-x_0)^2+(y+y_0)^2)\right]$

(b)

$$u(x_0)=\int_{bdy\,D}\left(h(x)\,\frac{\partial G(x,0;x_0,y_0)}{\partial n}\right)dx$$

$n=(0,-1)$

$$\frac{\partial G}{\partial n}(x,y;x_0,y_0)=-\frac{\partial G}{\partial y}(x,y)=\frac{1}{4\pi}\frac{\partial}{\partial y}\left[-\log((x-x_0)^2+(y-y_0)^2)+\log((x-x_0)^2+(y+y_0)^2)\right]$$

$$=\frac{1}{2\pi}\left[-\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}+\frac{y+y_0}{(x-x_0)^2+(y+y_0)^2}\right]$$

$$\frac{\partial G}{\partial n}(x,0;x_0,y_0)=\frac{y_0}{\pi}\frac{1}{(x-x_0)^2+y_0^2},$$

$$u(x_0)=\frac{y_0}{\pi}\int_{-\infty}^{\infty}\frac{h(x)}{(x-x_0)^2+y_0^2}dx$$

(c) $u(x,0)=h(x)=1$

$$u(x_0)=\frac{y_0}{\pi}\int_{-\infty}^{\infty}\frac{1}{(x-x_0)^2+y_0^2}dx=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{\left(\frac{x-x_0}{y_0}\right)^2+1}\frac{dx}{y_0}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{t^2+1}dt$$

$$=\frac{1}{\pi}\arctan t\Big|_{-\infty}^{\infty}=\frac{1}{\pi}\cdot\pi=\boxed{1}$$

Problem 21: Neumann Function

The Neumann function $N(x, y)$ for a domain D is defined exactly like the Green's function in Section 7.3 except that (ii) is replaced by the Neumann boundary condition

$$\text{(ii)*} \qquad \frac{\partial N}{\partial n} = c \quad \text{for } x \in \text{bdy } D.$$

for a suitable constant c.

(a) Show that $c = 1/A$, where A is the area of bdy D. ($c = 0$ if $A = \infty$)

(b) State and prove the analog of Theorem 7.3.1, expressing the solution of the Neumann problem in terms of the Neumann function.

(a)

$$u(X_0) = \iint_{bdy\,D} v\frac{dG}{dn} - G\left(\frac{du}{dn}\right) ds$$

where, $G(X, X_0) = \dfrac{-1}{4\pi|X-X_0|}$

let the harmonic function be such

$$N(X, X_0) = G(X, X_0) + H(X, X_0)$$

$$\frac{dN}{dn}(X, X_0) = c$$

plug $v = 1$, $1 = \iint_{bdy\,D} c$

$$= c\iint_{bdy\,D}$$

$$= c \cdot A$$

$$\boxed{c = \frac{1}{A}}$$

(b)

$$\begin{cases} \Delta u = 0, & \text{in } D \\ \frac{du}{dn} = h, & \text{on bdy } D \end{cases}$$

assume that

$$\iint_{bdy\,D} u\,ds = 0$$

Solution: $u(X_0) = -\iint_{bdy\,D} N(x, X_0)h(x)\,ds + c$

condi-

Problem 5

In the three-dimensional half-space $\{(x, y, z) : z > 0\}$, solve the Laplace equation with $u(x, y, 0) = \delta(x, y)$, where δ denotes the delta function, as follows.

(a) Show that

$$u(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ikx+ily} e^{-z\sqrt{k^2+l^2}} \frac{dkdl}{4\pi^2}.$$

(b) Letting $\rho = \sqrt{k^2+l^2}$, $r = \sqrt{x^2+y^2}$, and θ be the angle between (x, y) and (k, l), so that $xk + yl = \rho r \cos\theta$, show that

$$u(x,y,z) = \int_0^{2\pi}\int_0^{\infty} e^{i\rho r\cos\theta} e^{-z\rho}\,\rho\,d\rho \frac{d\theta}{4\pi^2}.$$

(c) Carry out the integral with respect to ρ and then use an extensive table of integrals to evaluate the θ integral.

(a) $u(x,y,z) = \frac{1}{4\pi^2}\iint_{\mathbb{R}^2} e^{ixk+iyl}\hat{U}(k,l,z)\,dk\,dl$

$(ik)^2\hat{u} + (il)^2\hat{u} + \hat{u}_{zz} = 0$

$\hat{U}(k,l,z) = Ce^{\sqrt{k^2+l^2}z} + De^{-\sqrt{k^2+l^2}z}$

$= De^{-\sqrt{k^2+l^2}z}$

$= \frac{1}{4\pi^2}\iint_{\mathbb{R}^2} e^{-ixk+iyl} e^{-\sqrt{k^2+l^2}z}\,dk\,dl$

$\hat{U}(x,y,0) = \delta(x,y) \Rightarrow \hat{u}(k,l,0) = 1$

$D = 1,$

(b)

$k = \rho\cos\theta$

$l = \rho\sin\theta$

$r = \sqrt{x^2+y^2}$

$u(x,y,z) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{\infty} e^{ir\rho\cos\theta - \rho z}\,d\rho\,d\theta$

(c) $u(x,y,z) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{\infty} e^{\rho ir\cos\theta - \rho z}\rho\,d\rho\,d\theta \overset{\text{integration by parts}}{=} \frac{1}{4\pi^2}\int_0^{2\pi} \left.\frac{\rho e^{\rho ir\cos\theta - \rho z}}{ir\cos\theta - z}\right|_0^{\infty} d\theta - \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{\infty} \frac{e^{\rho ir\cos\theta - \rho z}}{ir\cos\theta - z}\,d\theta$

$u(x,y,z) = -\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{\infty} \frac{e^{\rho ir\cos\theta - \rho z}}{ir\cos\theta - z}\,d\rho\,d\theta = \frac{1}{4\pi^2}\int_0^{2\pi} \frac{1}{(ir\cos\theta - z)^2}\,d\theta$

$$\boxed{= \frac{2\pi z}{\sqrt{(r^2+z^2)^3}}}$$

For a solution $u(x,t)$ of the wave equation with $\rho = T = c = 1$, the energy density is defined as $e = \frac{1}{2}(u_t^2 + u_x^2)$ and the momentum density as $p = u_t u_x$.

(a) Show that $\partial e/\partial t = \partial p/\partial x$ and $\partial p/\partial t = \partial e/\partial x$.

(b) Show that both $e(x,t)$ and $p(x,t)$ also satisfy the wave equation.

$u_{tt} = c^2 u_{xx}$, $u_{tt} = (1)^2 u_{xx}$, $u_{tt} + u_{xx}$

(a)

$e = \frac{1}{2}(u_t^2 + u_x^2)$

$p = u_t u_x$

$\frac{de}{dt} = \frac{d}{dt}\left(\frac{1}{2}(u_t^2 + u_x^2)\right)$

$= \frac{1}{2}\frac{d}{dt}(u_t^2) + \frac{d}{dt}(u_x^2)$

$= \frac{1}{2}\left(\frac{d}{dt}(u_t^2) + \frac{d}{dt}(u_x^2)\right)$

$= \frac{1}{2}\left(2u_t\frac{d}{dt}(u_t) + 2u_x\frac{d}{dt}(u_x)\right)$

$= \frac{1}{2}(2u_t u_{tt} + 2u_x u_{xx})$

$\frac{de}{dt} = u_t u_{tt} + u_x u_{xt}$

$\frac{dp}{dx} = \frac{d}{dx}(u_t u_x)$

$= u_t \frac{d}{dx}(u_x) + (u_x)\frac{d}{dx}(u_t)$

$= u_t u_{xx} + u_x u_{tx}$

So $\frac{de}{dt} = \frac{dp}{dx}$

(b) $e(x,t) = \frac{1}{2}(u_t^2 + u_x^2)$

$e_t = \frac{de}{dt}$

$= \frac{d}{dt}\left(\frac{1}{2}(u_t^2 + u_x^2)\right)$

$= \frac{1}{2}\frac{d}{dt}(u_t^2 + u_x^2)$

$= \frac{1}{2}\left(\frac{d}{dt}u_t^2 + \frac{d}{dt}u_x^2\right)$

$= \frac{1}{2}(2u_t u_{tt} + 2u_x u_{xt})$

$u_t u_{tt} + u_x u_{xt}$

$e_{tt} = \frac{d e_t}{dt}$

$= \frac{d}{dt}(u_t u_{tt} + u_x u_{xt})$

$= \frac{d}{dt}(u_t u_{tt}) + \frac{d}{dt}(u_x u_{xt})$

$= \left(u_t \frac{d}{dt}(u_{tt}) + u_{tt}\frac{d}{dt}(u_t)\right)$

$+ \left(u_x \frac{d}{dt}(u_{xt}) + u_{xt}\frac{d}{dt}(u_x)\right)$

$= (u_t u_{ttt} + u_{tt}u_{tt}) + (u_x u_{xtt} + u_{xt}u_{xt})$

$\frac{de_t}{dt} = (u_t u_{ttt} + u_{tt}^2) + (u_x u_{xtt} + u_{xt}^2)$

$e_x = \frac{de}{dx}$

$= \frac{d}{dx}\left(\frac{1}{2}(u_t^2 + u_x^2)\right)$

$= \frac{1}{2}\frac{d}{dx}(u_t^2 + u_x^2)$

$= \frac{1}{2}\left(\frac{d}{dx}u_t^2 + \frac{d}{dx}u_x^2\right)$

$= \frac{1}{2}(2u_t u_{tx} + 2u_x u_{xx})$, $\frac{de}{dx} = u_t u_{tx} + u_x u_{xx}$

$e_{xx} = \frac{de_x}{dx}$

$= \frac{d}{dx}(u_t u_{tx} + u_x u_{xx})$

$= \frac{d}{dx}(u_t u_{tx}) + \frac{d}{dx}(u_x u_{xx})$

$= u_t \frac{d}{dx}u_{tx} + u_{tx}\frac{d}{dx}u_t + u_x\frac{d}{dx}u_{xx} + u_{xx}\frac{d}{dx}u_x$

$= u_t(u_{txx}) + u_{tx}u_{tx} + u_x u_{xxx} + u_{xx}u_{xx}$

$e_{xx} - e_{tt} = 0$

with the Robin boundary condi-
If $a_0 > 0$ and

1. Consider the solution $1 - x^2 - 2kt$ of the diffusion equation. Find the locations of its maximum and its minimum in the closed rectangle $\{0 \le x \le 1, 0 \le t \le T\}$.

using the maximum principle

For $t=0$, $u(x,0) = 1 - x^2$

For $x=0$, $u(0,t) = 1 - 2kt$

For $x=1$, $u(1,t) = -2kt$

Now combine the 3 cases together & look at the maximum value over the entire interval.

the max of u is 1 & occurs at $(x,t) = (0,0)$.

the minimum point is $-2kT$ which is located on point $(x,t) = (1,T)$

$$\boxed{\begin{array}{l}\text{Max}: 1 \text{ at } (0,0) \\ \text{Min}: -2kT \text{ at } (1,T)\end{array}}$$

... on $(0, l)$ with the Robin boundary conditions ... and ... $u(x, t) = 0$ ... solution ...

4. Consider the diffusion equation $u_t = u_{xx}$ in $\{0 < x < 1, 0 < t < \infty\}$ with $u(0, t) = u(1, t) = 0$ and $u(x, 0) = 4x(1 - x)$.
(a) Show that $0 < u(x, t) < 1$ for all $t > 0$ and $0 < x < 1$.
(b) Show that $u(x, t) = u(1 - x, t)$ for all $t \geq 0$ and $0 \leq x \leq 1$.
(c) Use the energy method to show that $\int_0^1 u^2\,dx$ is a strictly decreasing function of t.

(a)

given the maximum principle the maximum occurs initially

$$u(\tfrac{1}{2},0) = 1$$

& hence $0 < u(x,t) < 1$

when $x \in (0,1)$ & $t>0$

(b) since $u(x,t)$ is a unique solution of the given equation let's show $u(1-x,t)$ is als a solution

$\frac{d}{dt}u(1-x,t) = u_t(1-x,t)$

$\frac{d}{dx}u(1-x,t) = -u_x(1-x,t)$

$\frac{d}{dx} - u_x(1-x,t) = -(-u_{xx}(1-x,t)) = u_{xx}(1-x,t)$

So, $\frac{d}{dt}u(1-x,t) = \frac{d^2}{dx^2}u(1-x,t)$

Also the initial conditions are satisfied

$u(1-1,t) = u(0,t) = 0$

$u(1-0) = u(1,t) = 0$

$u(1-x,0) = 4(1-x)(1-1+x)$

$= 4x(1-x)$

So $u(1-x,t) = u(x,t)$

ok

(c) multiple the diffusion eq by u & integrate over rod length

$$\int_0^l uu_t\,dx = \int_0^l (uu_{xx})\,dx$$

$$\int_0^l \frac{1}{2}(u)^2_t\,dx = \int_0^l (uu_x)_x - u_x^2\,dx$$

$$\int_0^l \frac{1}{2}\frac{d}{dt}(u)^2\,dx = \underbrace{(u(l)u_x(l) - u(0)u_x(0))}_{[0-0=0]} - \int_0^l u_x^2\,dx$$

$$\frac{1}{2}\frac{d}{dt}\int_0^l (u)^2\,dx = -\int_0^l u_x^2\,dx$$

$u_x^2 > 0 \Rightarrow \int_0^l u_x^2\,dx > 0$ & thus

$$\frac{d}{dt}\int_0^l (u)^2\,dx = -2\int_0^1 u_x^2\,dx < 0$$

Thus $\int_0^l (u)^2\,dx$ is a decreasing function in respect to t

... on $(0, l)$ with the Robin boundary condi-
... $u(l, t) = 0$. If $a_0 > 0$ and
... contribute to

4. Consider the diffusion equation $u_t = u_{xx}$ in $\{0 < x < 1,\ 0 < t < \infty\}$ with $u(0, t) = u(1, t) = 0$ and $u(x, 0) = 4x(1 - x)$.
   (a) Show that $0 < u(x, t) < 1$ for all $t > 0$ and $0 < x < 1$.
   (b) Show that $u(x, t) = u(1 - x, t)$ for all $t \geq 0$ and $0 \leq x \leq 1$.
   (c) Use the energy method to show that $\int_0^1 u^2\,dx$ is a strictly decreasing function of t.

(a)

given the maximum principle the maximum occurs initially

$$u(\tfrac{1}{2}, 0) = 1$$

& hence $0 < u(x,t) < 1$

when $x \in (0,1)$ & $t > 0$

(b) since $u(x,t)$ is a unique solution of the given equation lets show $u(1-x,t)$ is als a solution

$\frac{d}{dt} u(1-x, t) = u_t(1-x, t)$

$\frac{d}{dx} u(1-x,t) = -u_x(1-x,t)$

$\frac{d}{dx} -u_x(1-x,t) = -(-u_{xx}(1-x,t)) = u_{xx}(1-x,t)$

So, $\frac{d}{dt} u(1-x,t) = \frac{d^2}{dx^2} u(1-x,t)$

Also the initial conditions are satisfied

$u(1-1,t) = u(0,t) = 0$

$u(1-0) = u(1,t) = 0$

$u(1-x,0) = 4(1-x)(1-1+x)$
$= 4x(1-x)$

So $u(1-x,t) = u(x,t)$

(c) multiple the diffusion eq by u & integrate over rod length

$$\int_0^l u u_t\,dx = \int_0^l (u u_{xx})\,dx$$

$$\int_0^l \frac{1}{2}(u)^2_t\,dx = \int_0^l (u u_x)_x - u_x^2\,dx$$

$$\int_0^l \frac{1}{2}\frac{1}{dt}(u)^2\,dx = \underbrace{(u(l)u_x(l) - u(0)u_x(0))}_{[0-0=0]} - \int_0^l u_x^2\,dx$$

$$\frac{1}{2}\frac{d}{dt}\int_0^l (u)^2\,dx = -\int_0^l u_x^2\,dx$$

$u_x^2 > 0 \Rightarrow \int_0^l u_x^2\,dx > 0$ & thus

$$\frac{d}{dt}\int_0^l (u)^2\,dx = -2\int_0^l u_x^2\,dx < 0$$

Thus $\int_0^l (u)^2\,dx$ is a decreasing function in respect to t

8. Consider the diffusion equation on $(0, l)$ with the Robin boundary conditions $u_x(0, t) - a_0 u(0, t) = 0$ and $u_x(l, t) + a_l u(l, t) = 0$. If $a_0 > 0$ and $a_l > 0$, use the energy method to show that the endpoints contribute to the decrease of $\int_0^l u^2(x, t)\,dx$. (This is interpreted to mean that part of the "energy" is lost at the boundary, so we call the boundary conditions "*radiating*" or "*dissipative*.")

$$\int_0^l u u_t \, dx = k \int_0^l (u u_{xx})\, dx$$

$$\int_0^l \frac{1}{2} (u^2)_t \, dx = k \int_0^l (u u_x)_x - u_x^2 \, dx$$

$$\int_0^l \frac{1}{2} \frac{d}{dt} (u)^2 \, dx = k\,(u(l,t)\,u_x(l,t) - u(0,t)\,u_x(0,t)) - k \int_0^l u_x^2 \, dx$$

$(u_x(0,t) = a_0 u(0,t))$

$(u_x(l,t) = -a_l u(l,t))$

$$\frac{d}{dt} \int_0^l (u)^2 \, dx = -2k\,(a_l u(l,t)^2 + a_0 u(0,t)^2 + \int_0^l u_x^2 \, dx$$

$$2k(\underbrace{a_l u(l,t)^2}_{>0} + \underbrace{a_0 u(0,t)^2}_{>0} + \underbrace{\int_0^l u_x^2 \, dx}_{>0}) > 0$$

thus

$$\frac{d}{dt} \int_0^l (u)^2 \, dx < 0$$

So, $\int_0^l (u)^2 \, dx$ is a decreasing function in respect to t.

8. **Consider the diffusion equation on $(0, l)$ with the Robin boundary conditions $u_x(0,t) - a_0 u(0,t) = 0$ and $u_x(l,t) + a_l u(l,t) = 0$. If $a_0 > 0$ and $a_l > 0$, use the energy method to show that the endpoints contribute to the decrease of $\int_0^l u^2(x,t)\,dx$. (This is interpreted to mean that part of the "energy" is lost at the boundary, so we call the boundary conditions "*radiating*" or "*dissipative*.")**

$$\int_0^l u u_t \, dx = k \int_0^l (u u_{xx})\, dx$$

$$\int_0^l \frac{1}{2} (u)^2_t \, dx = k \int_0^l (u u_x)_x - u_x^2 \, dx$$

$$\int_0^l \frac{1}{2} \frac{d}{dt} (u)^2 dx = k\,(u(l,t)\,u_x(l,t) - u(0,t)\,u_x(0,t)) - k \int_0^l u_x^2\, dx$$

$$(u_x(0,t) = a_0 u(0,t))$$
$$(u_x(l,t) = -a_l u(l,t))$$

$$\frac{d}{dt}\int_0^l (u)^2 dx = -2k\,(a_l u(l,t)^2 + a_0 u(0,t)^2 + \int_0^l u_x^2\, dx$$

$$2k(\underbrace{a_l u(l,t)^2}_{>0} + \underbrace{a_0 u(0,t)^2}_{>0} + \underbrace{\int_0^l u_x^2\, dx)}_{>0} > 0$$

thus

$$\frac{d}{dt}\int_0^l (u)^2 dx < 0$$

So, $\int_0^l (u)^2 dx$ is a decreasing function in respect to t.

... the initial ...

1. Solve the diffusion equation with the initial condition

$$\phi(x) = 1 \quad \text{for } |x| < l \qquad \text{and} \qquad \phi(x) = 0 \quad \text{for } |x| > l.$$

Write your answer in terms of $\mathscr{E}\mathrm{rf}(x)$.

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\left(\frac{(x-y)^2}{4kt}\right)} \phi(y)\, dy$$

if $l \leq 0$ then $\phi = 0$ & $u = 0$

Assume $l > 0$ then

$$\phi(x) = \begin{cases} 1, & -l < x < l \\ 0, & \text{otherwise} \end{cases}$$

The solution reduced to

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-l}^{l} e^{-\left(\frac{(x-y)^2}{4kt}\right)} dy$$

Substitute y by $p\sqrt{4kt} + x$

$$= \frac{1}{\sqrt{\pi}} \int_{\left(\frac{-l-x}{\sqrt{4kt}}\right)}^{\left(\frac{l-x}{\sqrt{4kt}}\right)} e^{-p^2} dy$$

$$= \frac{1}{2}\left(\frac{2}{\sqrt{\pi}} \int_{\left(\frac{-l-x}{\sqrt{4kt}}\right)}^{0} e^{-p^2} dy + \frac{2}{\sqrt{\pi}} \int_{0}^{\left(\frac{l-x}{\sqrt{4kt}}\right)} e^{-p^2} dy\right)$$

$$= \frac{1}{2}\left(-\frac{2}{\sqrt{\pi}} \int_{0}^{\left(\frac{-l-x}{\sqrt{4kt}}\right)} e^{-p^2} dy + \frac{2}{\sqrt{\pi}} \int_{0}^{\left(\frac{l-x}{\sqrt{4kt}}\right)} e^{-p^2} dy\right)$$

$$\boxed{u(x,t) = \frac{1}{2}\left(\mathscr{E}rf\left(\frac{l-x}{\sqrt{4kt}}\right) - \mathscr{E}rf\left(\frac{-l-x}{\sqrt{4kt}}\right)\right)}$$

9. Solve the diffusion equation $u_t = ku_{xx}$ with the initial condition $u(x, 0) = x^2$ by the following special method. First show that u_{xxx} satisfies the diffusion equation with *zero* initial condition. Therefore, by uniqueness, $u_{xxx} \equiv 0$. Integrating this result thrice, obtain $u(x, t) = A(t)x^2 + B(t)x + C(t)$. Finally, it's easy to solve for A, B, and C by plugging into the original problem.

$$V(x,t) = u_{xxx}(x,t)$$

$$V_t = \frac{d}{dt} u_{xxx} = u_{xxxt} = u_{txxx}$$

$$V_{xx} = \frac{d^2}{dx^2} u_{xxx} = u_{xxxxx}$$

v is a solution, hence

$$V_t - kV_{xx} = u_{txxx} - ku_{xxxxx}$$
$$= (u_t - ku_{xx})_{xxx}$$
$$= (0)_{xxx}$$
$$= 0$$

$$V(x,0) = u(x,0)_{xxx} = (x^2)_{xxx} = 0$$

V is a solution of the equation

$$V_t = kV_{xx}, \quad V(x,0) = 0$$

One solution to V is $V=0$, the solution is unique hence

$$u_{xxx} = 0$$
$$u_{xx} = A(t)$$
$$u_x = A(t)x + B(t)$$
$$u(x,t) = A(t)\frac{x^2}{2} + B(t)x + C(t)$$

u has to now satisfy these conditions $\quad u_t = ku_{xx}$

$$A'(t)\frac{x^2}{2} + B'(t)x + C'(t) = kA(t)$$

$$B'(t) = A'(t) = 0$$
$$C'(t) = kA(t)$$
$$\Rightarrow B(t) = B \;\&\; A(t) = A$$
$$C(t) = kAt + C$$
$$u(x,t) = \frac{A}{2}x^2 + Bx + kAt + C$$
$$x^2 = u(x,0) = \frac{A}{2}x^2 + Bx + C$$

if $A = 2$ & $B = C = 0$, our solution is: $\boxed{u(x,t) = x^2 + 2kt}$

....wtkt)"

we can ignore all the other terms in the series.

12. **The purpose of this exercise is to calculate $Q(x, t)$ approximately for large t. Recall that $Q(x, t)$ is the temperature of an infinite rod that is initially at temperature 1 for $x > 0$, and 0 for $x < 0$.**

(a) **Express $Q(x, t)$ in terms of $\mathscr{E}$rf.**

(b) **Find the Taylor series of $\mathscr{E}\text{rf}(x)$ around $x = 0$. (*Hint:* Expand e^z, substitute $z = -y^2$, and integrate term by term.)**

(c) **Use the first two nonzero terms in this Taylor expansion to find an approximate formula for $Q(x, t)$.**

(d) ***Why* is this formula a good approximation for x fixed and t large?**

(a)
$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-p^2}\,dp$$
$$= \frac{1}{2} + \frac{1}{2} * \frac{2}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-p^2}\,dp$$
$$= \boxed{\frac{1}{2} + \frac{1}{2} \mathscr{E}\text{rf}(x/\sqrt{4kt})}$$

(b) The taylor series of e^z
$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
Substitute z by $-y^2$
$$e^{-y^2} = 1 - y^2 + \frac{y^4}{2!} - \frac{y^6}{3!} + \dots = \sum_{n=0}^{\infty} \left(\frac{(-y^2)^n}{n!}\right)$$
$$\mathscr{E}\text{rf}(x) = \int_0^x e^{-y^2}\,dy = \int_0^x \sum_{n=0}^{\infty} \left(\frac{(-y^2)^n}{n!}\right) dy = \sum_{n=0}^{\infty} \int_0^x \left(\frac{(-y^2)^n}{n!}\right) dy = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)n!}\Big|_0^x$$

taylor series around $x = 0$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}$$

(c) x & $-\frac{x^3}{3}$ are the first 2 nonzero terms in the expansion.
$$\boxed{Q(x,t) \approx \frac{1}{2}\left(1 + \frac{x}{\sqrt{4kt}} - \frac{x^3}{3(\sqrt{4kt})^3}\right)}$$

$Q(x,t) = \uparrow$

(d)

For arbitrary $n \geq 2$, $\lim_{t \to \infty} \frac{x^n}{(2n+1)n!(\sqrt{4kt})^n} = 0$

to get a good approx of large T we can ignore all the other terms in the series.

Problem 5

(a) Write Schrödinger's equation in *two dimensions* in polar coordinates $iu_t = -\frac{1}{2}(u_{rr} + u_r/r + u_{\theta\theta}/r^2) + V(r)u$, with a radial potential $V(r)$. Find the separated eigenfunctions $u = T(t)R(r)\Theta(\theta)$, leaving R in the form of a solution of an ODE.

(b) Assume that $V(r) = \frac{1}{2}r^2$. Substitute $\rho = r^2$ and $R(r) = e^{-\rho/2}\rho^{-n/2}L(\rho)$ to show that L satisfies the Laguerre ODE

$$L_{\rho\rho} + \left[-1 + \frac{\nu+1}{\rho}\right] L_\rho + \frac{\mu}{\rho} L = 0$$

for some constants ν and μ.

(a)

$$iu_t = -\frac{1}{2}\left(u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2}\right) + V(r)u$$

$$u = T(t)\,R(r)\,\Theta(\theta)$$

Plug in

$$iT'R\Theta = -\frac{1}{2}\left(TR''\Theta + \frac{TR'\Theta}{r} + \frac{TR\Theta''}{r^2}\right) + V(r)TR\Theta$$

$$\frac{2iT'}{T} = -\left(\frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta}\right) + 2V(r)$$

$$T' = -\frac{\lambda i}{2}T$$

λ being a constant

moreover

$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\lambda - 2r^2V(r) = -\frac{\Theta''}{\Theta}$$

$$\gamma = n^2,\ n \in \mathbb{N}$$

$$\boxed{R'' + \frac{1}{r}R' + \left(\lambda - \frac{n^2}{r^2} - 2V(r)\right)R = 0}$$

(b) $V(r) = \frac{1}{2}r^2$

$$R'' + \frac{1}{r}R' + \left(\lambda - \frac{n^2}{r^2} - r^2\right)R = 0$$

substitute $R(r) = r^{-n}w(r)$

$$0 = (n(n+1)r^{-n-2}w - 2nr^{-n-1}w' + r^{-n}w'') + \frac{1}{r}(-nr^{-n-1}w + r^{-n}w') + \left(\lambda - \frac{n^2}{r^2} - r^2\right)r^{-n}w$$

$$= n(n+1)r^{-n-2}w - 2nr^{-n-1}w' + r^{-n}w'' - nr^{-n-2}w + r^{-n-1}w' + \lambda r^{-n}w - n^2r^{-n-2}w - r^{-n+2}w$$

$$0 = w'' + \frac{1-2n}{r}w' + (\lambda - r^2)w$$

$\rho = r^2$

$$0 = (4\rho w_{\rho\rho} + 2w_\rho) + \frac{1-2n}{\sqrt{\rho}}2\sqrt{\rho}w_\rho + (\lambda - \rho)w$$

$$= 4\rho w_{\rho\rho} + (4-4n)w_\rho + (\lambda - \rho)w$$

$$0 = w_{\rho\rho} + \frac{1-n}{\rho}w_\rho + \left(\frac{\lambda}{4\rho} - \frac{1}{4}\right)w$$

$$w(r) = e^{-\rho/2}L(\rho)$$

$$0 = \left(L_{\rho\rho} - L_\rho + \frac{1}{4}L\right)e^{-\rho/2} + \left(\frac{1-n}{\rho}\right)\left(L_\rho - \frac{1}{2}L\right)e^{-\rho/2} + \left(\frac{\lambda}{4\rho} - \frac{1}{4}\right)e^{-\rho/2}L$$

$$0 = L_{\rho\rho} - L_\rho + \frac{1}{4}L + \left(\frac{1-n}{\rho}\right)\left(L_\rho - \frac{1}{2}L\right) + \left(\frac{\lambda}{4\rho} - \frac{1}{4}\right)L$$

$$0 = L_{\rho\rho} + \left(-1 + \frac{1-\overset{=-\nu}{n}}{\rho}\right)L_\rho + \underbrace{\left(\frac{\lambda}{4} - \frac{1-n}{2}\right)}_{=\mu}\frac{1}{\rho}L$$

L satisfies the Laguerre ODE

$$\boxed{0 = L_{\rho\rho} + \left(-1 + \frac{1+\nu}{\rho}\right)L_\rho + \frac{\mu}{\rho}L}$$

Problem 1

Assuming in (5) that $\mathbf{F} = \mathbf{0}$, and that $\mathbf{v}$ is a gradient ($\mathbf{v} = \nabla\phi$), which means that the flow is irrotational and unforced, show that $\int dp/\rho + \partial\phi/\partial t + \frac{1}{2}|\nabla\phi|^2 = \text{constant}$. (*Hint*: Into (5) substitute $\mathbf{v} = \nabla\phi$ and $p = f(\rho)$.) This is called Bernoulli's Law.

$$\nabla(|v|^2) = 2(v\cdot\nabla)v + 2v \times (\nabla \times v)$$

irrotational
unforced flow
(meaning the force is generated by pressure)

$$\Rightarrow \begin{cases} \frac{dv}{dt} + (v\cdot\nabla)v = -\frac{1}{\rho}\nabla p \\ v = \nabla\phi \\ p = f(\rho) \end{cases}$$

$$\frac{1}{2}\nabla(|v|^2) = (v\cdot\nabla)v$$

$$\nabla\left(\frac{\partial\phi}{\partial t} + \frac{|v|^2}{2}\right) = -\frac{1}{\rho}\nabla p = -\frac{1}{\rho}f'(\rho)\nabla\rho$$

$$h'(\rho) = \frac{f'(\rho)}{\rho}$$

$$h'(\rho)\nabla\rho = \nabla(h(\rho))$$

$$\nabla\left(\frac{\partial\phi}{\partial t} + \frac{|v|^2}{2} + h(\rho)\right) = 0$$

$$\frac{\partial\phi}{\partial t} + \frac{|v|^2}{2} + h(\rho) = C$$

$$h(\rho) + C = \int h'(\rho)\,d\rho = \int \frac{f'(\rho)}{\rho}\,d\rho = \int \frac{dp}{\rho}$$

Problem 1

Let S be a characteristic surface for which $S \cap \{(x, y, z) : t = 0\}$ is the sphere $\{x^2+y^2+z^2 = a^2\}$. **Describe S geometrically.**

Imagine the sphere $\{x^2+y^2+z^2=a^2\}$ in space-time as a circle in the xyz-plane.

The light rays of the slope c for which every point on the circle are the characteristic surface of S

$$S=\{(x,y,z,t);\ x^2+y^2+z^2=(a\pm ct)^2\}$$

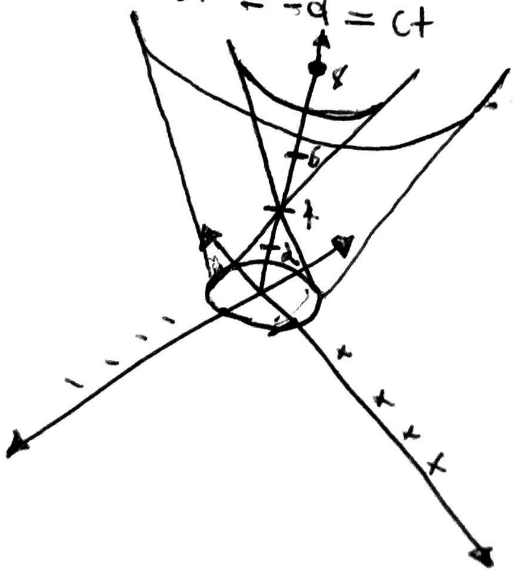
rays intersect when

$$a+ct=-a-ct$$

or

$$a-ct=-a=ct$$

at $\boxed{t=\pm\frac{a}{c}}$



Problem 9

(a) Show that if A is a real symmetric matrix and λ_n^* is its largest eigenvalue, then

$$\lambda_n^* = \max_{\mathbf{c}\neq 0} \frac{A\mathbf{c}\cdot\mathbf{c}}{|\mathbf{c}|^2}.$$

(b) If B is another real symmetric matrix, B is positive definite, and λ_n^* is the largest root of the polynomial equation $\det(A-\lambda B)=0$, show that

$$\lambda_n^* = \max_{\mathbf{c}\neq 0} \frac{A\mathbf{c}\cdot\mathbf{c}}{B\mathbf{c}\cdot\mathbf{c}}.$$

[Hint for (b): Use the fact that B has a unique square root that is positive definite symmetric.]

(3) Consider $-\Delta$ in the square $(0,\pi)^2$ with Dirichlet BCs. Compute the Rayleigh quotient with the trial function $xy(\pi-x)(\pi-y)$. Compare with the first eigenvalue.

$$\nabla f(x,y) = (y(\pi-y)(\pi-2x),\ x(\pi-x)(\pi-2y))$$

$$\|\nabla f\|^2 = \int_0^\pi\int_0^\pi y^2(\pi-y)^2(\pi-2x)^2 + x^2(\pi-x)^2(\pi-2y)^2\,dy\,dx$$

$$= 2\int_0^\pi y^2(\pi-y)^2\,dy\int_0^\pi(\pi-2x)^2\,dx = \frac{\pi^8}{45}$$

$$\|w\|^2 = \int_0^\pi\int_0^\pi x^2y^2(\pi-x)^2(\pi-y)^2\,dy\,dx$$

$$= \int_0^\pi\int_0^\pi x^2(\pi-x)^2\,dx \int_0^\pi y^2(\pi-y)^2\,dy$$

$$= \frac{\pi^{10}}{900}$$

The Rayleigh quotient for f is

$$\frac{\|\nabla f\|^2}{\|f\|^2} = \frac{20}{\pi^2} \approx 2.026$$

$$\lambda_1 = 1^2+1^2 = 2$$

(first eigenvalue)

Problem 7

Let D be the semidisk $\{x^2 + y^2 < b^2, y > 0\}$. Consider the diffusion equation in D with the conditions: $u = 0$ on bdy D and $u = \phi(r, \theta)$ when $t = 0$. Write the complete expansion for the solution $u(r, \theta, t)$, including the formulas for the coefficients.

Wave eq in polar coordinates

$$\begin{cases} u_t = k u_{rr} + \frac{k}{r} u_r + \frac{k}{r^2} u_{\theta\theta} \\ u(r,0,t) = u(r,\pi,t) = r(b,\theta,t) = 0 \\ u(r,\theta,0) = \phi(r,\theta) \end{cases}$$

$$v(r,\theta,t) = R(r)\Theta(\theta)T(t)$$

$$\frac{T'}{kT} = \frac{R'' + \frac{1}{r}R'}{R} + \frac{\Theta''}{r^2\Theta} = \lambda$$

$$\frac{r^2 R'' + rR'}{R} - r^2\lambda = -\frac{\Theta''}{\Theta} = \mu$$

$$\begin{cases} \Theta'' + \mu\Theta = 0 \\ \Theta(0) = \Theta(\pi) = 0 \end{cases}$$

$$\Theta_n(\theta) = \sin(n\theta)$$

$\lambda \geq 0$ $\quad R(b) = 0, \quad R(0) < \infty$

let $\lambda = -\beta^2 < 0$

$$r^2 R'' + rR' + (\beta^2 r^2 - n^2)R = 0$$

$$R(r) = A J_n(\beta r) + B Y_n(\beta r),$$

$Y_n \to \infty$ as $r \to 0$

$$R(a) = A J_n(\beta b) = 0$$

If $A = 0$ the solution is trivial & hence β such that

$$J_n(\beta b) = 0 \Rightarrow \beta_n = \frac{z_{mn}}{b}$$

$$T' = -\beta_{mn}^2 kT$$

$$T_n(t) = \exp\left(-\frac{z_{mn}^2 kt}{b^2}\right)$$

$$v(r,t) = \sum_{m,n=1}^{\infty} A_{mn} \exp\left(-\frac{z_{mn}^2 kt}{b^2}\right) J_n\left(\frac{z_{mn} r}{b}\right)\sin n\theta$$

$$\phi(r,\theta) = \sum_{n=1}^{\infty} A_{mn} J_n\left(\frac{z_{mn} r}{b}\right)\sin n\theta$$

$$\int_0^{\pi} \phi(r,\theta)\sin k\theta \, d\theta$$

$$= \sum_{n=1}^{\infty} A_{mn} J_n\left(\frac{z_{mn} r}{b}\right)\int_0^{\pi} \sin(n\theta)\sin(k\theta)\,d\theta$$

$$= \frac{\pi}{2}\sum_{m=1}^{\infty} A_{mk} J_k\left(\frac{z_{mk} r}{b}\right)$$

multiply & integrate $(0,b)$

$$\int_0^b\int_0^{\pi} \phi(r,\theta)\sin k\theta \, J_k\left(\frac{z_{1k} r}{a}\right) r\,d\theta\,dr = \frac{\pi}{2}\sum_{m=1}^{\infty} A_{mk}\int_0^b J_k\left(\frac{z_{mk} r}{b}\right) J_k\left(\frac{z_{1k} r}{a}\right) r\,dr$$

$$= A_{1k}\int_0^b J_k\left(\frac{z_{1k} r}{b}\right)^2 r\,dr$$

$$= \frac{\pi}{2} A_{1k}\left(\frac{b^2}{2}\right) J_{k+1}^2(z_{1k})$$

$$\boxed{A_{1k} = \frac{4}{\pi b^2 J_{k+1}^2(z_{1k})}\int_0^b\int_0^{\pi} \phi(r,\theta)\sin k\theta \, J_k\left(\frac{z_{1k} r}{a}\right) r\,d\theta\,dr}$$

Problem 7

Let D be the semidisk $\{x^2 + y^2 < b^2, y > 0\}$. Consider the diffusion equation in D with the conditions: $u = 0$ on bdy D and $u = \phi(r, \theta)$ when $t = 0$. Write the complete expansion for the solution $u(r, \theta, t)$, including the formulas for the coefficients.

Wave eq in polar coordinates

$$\begin{cases} u_t = k u_{rr} + \frac{k}{r} u_r + \frac{k}{r^2} u_{\theta\theta} \\ u(r,0,t) = u(r,\pi,t) = v(b,\theta,t) = 0 \\ u(r,\theta,0) = \phi(r,\theta) \end{cases}$$

$$V(r,\theta,t) = R(r)\Theta(\theta)T(t)$$

$$\frac{T'}{kT} = \frac{R'' + \frac{1}{r}R'}{R} + \frac{\Theta''}{r^2\Theta} = \lambda$$

$$\frac{r^2 R'' + rR'}{R} - r^2\lambda = -\frac{\Theta''}{\Theta} = \mu$$

$$\begin{cases} \Theta'' + \mu\Theta = 0 \\ \Theta(0) = \Theta(\pi) = 0 \end{cases}$$

$$\Theta_n(\theta) = \sin(n\theta)$$

$\lambda \geq 0$

$R(b) = 0, \quad R(0) < \infty$

let $\lambda = -\beta^2 < 0$

$$r^2 R'' + rR' + (\beta^2 r^2 - n^2)R = 0$$

$$R(r) = A J_n(\beta r) + B Y_n(\beta r),$$

$Y_n \to \infty$ as $r \to 0$

$$R(a) = A J_n(\beta b) = 0$$

if $A = 0$ the solution is trivial & hence β such that

$$J_n(\beta b) = 0 \Rightarrow \beta_n = \frac{z_{mn}}{b}$$

$$T' = -\beta_{mn}^2 kT$$

$$T_n(t) = \exp\left(-\frac{z_{mn}^2 kt}{b^2}\right)$$

$$v(r,t) = \sum_{m,n=1}^{\infty} A_{mn} \exp\left(-\frac{z_{mn}^2 kt}{b^2}\right) J_n\left(\frac{z_{mn} r}{b}\right) \sin n\theta$$

$$\phi(r,\theta) = \sum_{n=1}^{\infty} A_{mn} J_n\left(\frac{z_{mn} r}{b}\right) \sin n\theta$$

$$\int_0^{\pi} \phi(r,\theta) \sin k\theta \, d\theta$$

$$= \sum_{n=1}^{\infty} A_{mn} J_n\left(\frac{z_{mn} r}{b}\right) \int_0^{\pi} \sin(n\theta)\sin(k\theta)\,d\theta$$

$$= \frac{\pi}{2} \sum_{m=1}^{\infty} A_{mk} J_k\left(\frac{z_{mk} r}{b}\right)$$

multiply & integrate (a,b)

$$\int_0^b \int_0^{\pi} \phi(r,\theta)\sin k\theta J_k\left(\frac{z_{1k} r}{a}\right) r\,d\theta\,dr = \frac{\pi}{2} \sum_{m=1}^{\infty} A_{mk} \int_0^b J_k\left(\frac{z_{mk} r}{b}\right) J_k\left(\frac{z_{1k} r}{a}\right) r\,dr$$

$$= A_{1k} \int_0^b J_k\left(\frac{z_{1k} r}{b}\right)^2 r\,dr$$

$$= \frac{\pi}{2} A_{1k}\left(\frac{b^2}{2}\right) J_{k+1}^2(z_{1k})$$

$$\boxed{A_{1k} = \frac{4}{\pi b^2 J_{k+1}^2(z_{1k})} \int_0^b \int_0^{\pi} \phi(r,\theta)\sin k\theta J_k\left(\frac{z_{1k} r}{a}\right) r\,d\theta\,dr}$$

Problem 8

Solve the diffusion equation in the solid cone $\{x^2+y^2+z^2<a^2, \theta<\alpha\}$ with $u=0$ on the **whole boundary** and with general initial conditions. [*Hint*: Separate variables and write **the solution** as a series with terms of the separated form $T(t)R(r)q(\phi)p(\cos\theta)$. Show **that $p(s)$** satisfies the associated Legendre equation. Expand $p(s)$ in powers of $(s-1)$. **In terms** of such a function, write the equations that determine the eigenvalues.]

$$u_t = k\Delta u$$

$$\{r<a,\ 0\le\theta<\alpha\}$$

$$u(t,r,\theta,\phi) = T(t)R(r)P(\theta)q(\phi)$$

$$T(t) = e^{-\lambda k t}$$

$$R_{rr} + \frac{2}{r}R_r + \left(\lambda - \frac{\gamma}{r^2}\right)R = 0$$

$$\frac{(d/d\theta)[\sin\theta(dP/d\theta)]}{\sin\theta} + \left(\gamma - \frac{m^2}{\sin^2\theta}\right)P = 0$$

$$q(\phi) = A\cos m\phi + B\sin m\phi$$

$R(0)$ is finite, $R(r) = \dfrac{J_{\sqrt{\gamma+1/4}}(\sqrt{\lambda}\, r)}{\sqrt{r}}$

$t=\cos(\theta)$,

$$[(1-t)^2 P_t]_t + \left(\gamma - \frac{m^2}{1-s^2}\right)P = 0$$

$P(1)$ is finite $\qquad P(\cos\alpha)=0$

$$P(t) = C(s-1)^{\sigma_1}\sum_{n=0}^{\infty} P_n(s-1)^n + D(s-1)^{\sigma_2}\sum_{n=0}^{\infty} Q_n(s-1)^n.$$

$$P_{tt} - \frac{2t}{1-t^2}P_t + \left(\frac{\gamma}{1-t^2} - \frac{m^2}{(1-s^2)^2}\right)P = 0$$

$\sigma(\sigma-1)+B\sigma+\delta=0$ or $\sigma^2 = \dfrac{m^2}{4}$

$$\sigma_{1,2} = \pm\frac{m}{2}$$

$$P(t) = \underbrace{C(s-1)^{-m/2}\sum_{n=0}^{\infty} P_n(s-1)^n}_{C=0 \text{ because } P(1) \text{ is infinite}} + D(s-1)^{m/2}\sum_{n=0}^{\infty} Q_n(s-1)^n$$

$$P(t) = D(s-1)^{m/2}\sum_{n=0}^{\infty} Q_n(s-1)^n$$

let $D=1$

$$\theta = P_{\gamma m}(\cos\alpha)$$

$$\boxed{u = \sum_{m l j} A_{mlj}\, e^{-\lambda_{mlj} k t} e^{im\phi}\frac{1}{\sqrt{r}} J_{\sqrt{\gamma_{ml}+1/4}}(\sqrt{\lambda_{mlj}}\, r)\, P_{\gamma_{ml}}^{m}(\cos\theta)}$$

where

$$P_{\gamma_{ml}}^{m}(\cos\alpha) = 0$$

& $J_{\sqrt{\gamma_{ml}+1/4}}(\sqrt{\lambda_{mlj}}\, a) = 0$

Problem 8

(a) Let B be the ball $\{x^2+y^2+z^2<a^2\}$. Find all the *radial* eigenfunctions of $-\Delta$ in B with the Neumann BCs. By "radial" we mean "depending only on the distance r to the origin." [*Hint:* A simple method is to let $v(r)=ru(r)$.]

(b) Find a simple explicit formula for the eigenvalues.

(c) Write the solution of $u_t=k\Delta u$ in B, $u_r=0$ on bdy B, $u(\mathbf{x},0)=\phi(r)$ as an infinite series, including the formulas for the coefficients.

(d) In part (c), why does $u(\mathbf{x},t)$ depend only on r and t?

(a) $-\Delta u=\lambda u$

$\Delta u = 0$

$v_{rr}+\frac{2}{r}v_r=-\lambda u$

$v(r)=r(u(r))$

$\frac{1}{r}v_{rr}=u_{rr}+\frac{2}{r}u_r=-\lambda u$

$=-\lambda\frac{1}{r}v \Rightarrow \boxed{v_{rr}=-\lambda v}$

observe the different cases

1.) $\lambda=0$

$v(r)=Ar+B\Rightarrow u(r)=A+\frac{B}{r}\Rightarrow B=0\Rightarrow \boxed{u(r)=A}$

2.) $\lambda=-\beta^2$

$V(r)=A\sinh\beta r+B\cosh\beta r\Rightarrow u(r)=\frac{A}{r}\sinh\beta r+\frac{B}{r}\cosh\beta r$

$A\sinh\beta-\beta aA\cosh\beta a=0\Rightarrow A=0$ or $\beta=0$

3.) $\lambda=\beta^2$

$v(r)=A\sin\beta r+B\cos\beta r\Rightarrow u(r)=\frac{A}{r}\sin\beta r+\frac{B}{r}\cos\beta r$

$A\sin\beta a-\beta aA\cos\beta a=0\Rightarrow A=0$ or $\tan\beta a=\beta a$

If $A=0$ u is trivial

$\boxed{U_n(r)=\frac{1}{r}\sin\beta_n r}$ where β_n is the nth positive root of $\boxed{\tan\beta a=\beta}$

(b) Plug in $\lambda=\beta^2$, $\tan\sqrt{\lambda}a=\sqrt{\lambda}a$

(c) $v_t=k\Delta v$

$v(t,x)=T(t)u(x)$

$\frac{T'}{kT}=\frac{\Delta v}{v}=-\lambda$

$v(t,r)=A_0+\sum_{n=1}^{\infty}\frac{A_n}{r}\sin\beta_n r\,e^{-k\beta_n^2 t}$,

$\tan\beta_n a=\beta_n a$

$\phi(r)=A_0+\sum_{n=1}^{\infty}A_n\frac{1}{r}\sin\beta_n r$.

$\int_0^a\phi(r)r^2dr=\int_0^a A_0 r^2dr+\sum_{n=1}^{\infty}\frac{A_n}{\beta_n}\int_0^a r\sin\beta_n r\,dr$

$\to A_0\frac{a^3}{3}dr+\sum_{n=1}^{\infty}\frac{A_n}{\beta_n^3}\underbrace{(\sin\beta_n a-\beta_n a\cos\beta_n a)}_{=0}$

thus, $A_0=\frac{3}{a^3}\int_0^a\phi(r)r^2dr$

$\int_0^a\phi(r)r\sin\beta_n r\,dr=\int_0^a A_0 r\sin\beta_n r\,dr+\sum_{m=1}A_m\int_0^a\sin\beta_m r\sin\beta_n r\,dr$

$=A_n\int_0^a\sin^2\beta_n r\,dr$

$=\frac{1}{2}A_n\int_0^a 1-\cos 2\beta_n r\,dr$

$=\frac{a}{2}A_n(1-\beta_n\cos^2\beta_n a)$

$\boxed{A_n=\frac{2}{a(1-\beta_n\cos^2\beta_n a)}\int_0^a\phi(r)r\sin\beta_n r\,dr}$

(d) the solution depends on t & r because the initial condition depends only on r & BC is homongenous.

Problem 12

A rectangular plate $(0 < x < a, 0 < y < b)$ initially has a hot spot at its center so that its initial temperature distribution is $u(x, y, 0) = M\delta(x - \frac{a}{2}, y - \frac{b}{2})$. Its edges are maintained at zero temperature. Let k be the diffusion constant. Find the temperature at any later time in the form of a series.

$$u(x,y,t) = \sum_{m,n} A_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \exp -\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\pi^2 k t$$

$$\sum_{m,n} A_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) = M\delta\left(x - \frac{a}{2}, y - \frac{b}{2}\right)$$

$$\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{k\pi y}{b}\right)$$

$$\sum_{m,n} A_{mn} \int_0^b\int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{k\pi y}{b}\right) dx\,dy$$

$$= \sum_{m,n} A_{mn} \int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{l\pi x}{a}\right)dx \int_0^b \sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{k\pi y}{b}\right)dy$$

$$M\int_0^b\int_0^a \delta\left(x - \frac{a}{2}, y - \frac{b}{2}\right)\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{k\pi y}{b}\right)dx\,dy \quad = A_{lk}\int_0^a \sin^2\left(\frac{l\pi x}{a}\right)dx\int_0^b \sin^2\left(\frac{k\pi y}{b}\right)dy$$

$$= A_{lk}\left(\frac{ab}{4}\right)$$

$$= M\int_0^a \delta\left(x - \frac{a}{2}\right)\sin\left(\frac{l\pi x}{a}\right)dx \int_0^b \delta\left(y - \frac{b}{2}\right)\sin\left(\frac{k\pi y}{b}\right)dy$$

$$= M\int_0^a \delta\left(x - \frac{a}{2}\right)\sin\left(\frac{l\pi x}{a}\right)dx \int_0^b \delta\left(y - \frac{b}{2}\right)\sin\left(\frac{k\pi y}{b}\right)dy$$

$$= M\sin\left(\frac{l\pi}{2}\right)\sin\left(\frac{k\pi}{2}\right)$$

$$= M\begin{cases}\frac{1}{2}\left(\cos\left(\frac{(l-k)\pi}{2}\right) - \cos\left(\frac{(l+k)\pi}{2}\right)\right) & l, k \text{ odd} \\ 0 & \text{otherwise}\end{cases}$$

$$= M\begin{cases}\frac{(-1)^{\left(\frac{l-k}{2}\right)} - (-1)^{\left(\frac{l+k}{2}\right)}}{2} & l, k \text{ odd} \\ 0\end{cases}$$

$$A_{mn} = \begin{cases}[illegible] & m, n \text{ odd} \\ 0 & \text{otherwise}\end{cases}$$

$$\boxed{u(x,y,t) = [illegible]}$$

Problem 12

A rectangular plate $[0 \le x \le a, 0 \le y \le b]$ initially has a hot spot at its center so that its initial temperature distribution is $u(x, y, 0) = M\delta(x - \frac{a}{2}, y - \frac{b}{2})$. Its edges are maintained at zero temperature. Let k be the diffusion constant. Find the temperature at any later time in the form of a series.

$$u(x,y,t) = \sum_{m,n} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \exp\left(-\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\pi^2 k t\right)$$

$$\sum_{m,n} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) = M\delta\left(x - \frac{a}{2}, y - \frac{b}{2}\right)$$

$$\sin\left(\frac{k\pi x}{a}\right) \sin\left(\frac{l\pi y}{b}\right)$$

$$\sum_{m,n} A_{mn} \int_0^b \int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{k\pi x}{a}\right) \sin\left(\frac{l\pi y}{b}\right) dx\, dy$$

$$= \sum_{m,n} A_{mn} \int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{k\pi x}{a}\right) dx \int_0^b \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{l\pi y}{b}\right) dy$$

$$= A_{kl} \int_0^a \sin^2\left(\frac{k\pi x}{a}\right) dx \int_0^b \sin^2\left(\frac{l\pi y}{b}\right) dy$$

$$= A_{kl} \left(\frac{ab}{4}\right)$$

$$M \int_0^b \int_0^a \delta\left(x - \frac{a}{2}, y - \frac{b}{2}\right) \sin\left(\frac{k\pi x}{a}\right) \sin\left(\frac{l\pi y}{b}\right) dx\, dy$$

$$= M \int_0^a \delta\left(x - \frac{a}{2}\right) \sin\left(\frac{k\pi x}{a}\right) dx \int_0^b \delta\left(y - \frac{b}{2}\right) \sin\left(\frac{l\pi y}{b}\right) dy$$

$$= M \int_0^a \delta\left(x - \frac{a}{2}\right) \sin\left(\frac{k\pi x}{a}\right) dx \int_0^b \delta\left(y - \frac{b}{2}\right) \sin\left(\frac{l\pi y}{b}\right) dy$$

$$= M \sin\left(\frac{k\pi}{2}\right) \sin\left(\frac{l\pi}{2}\right)$$

$$= M \begin{cases} \frac{1}{2}\left(\cos\left(\frac{(k-l)\pi}{2}\right) - \cos\left(\frac{(k+l)\pi}{2}\right)\right) & k \text{ \& } l \text{ are odd} \\ 0 & \text{otherwise} \end{cases}$$

$$= M \begin{cases} \dfrac{(-1)^{\left(\frac{k-l}{2}\right)} - (-1)^{\left(\frac{k+l}{2}\right)}}{2}, & k \text{ \& } l \text{ are odd} \\ 0 & \text{otherwise} \end{cases}$$

$$A_{mn} = \begin{cases} 2M\left[\dfrac{(-1)^{(m-n)/2} - (-1)^{(m+n)/2}}{ab}\right] & m \text{ \& } n \text{ are odd} \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{u(x,y,t) = 2M \sum_{m,n} \frac{(-1)^{m-n} - (-1)^{m+n}}{ab} \sin\left(\frac{(2m-1)\pi x}{a}\right) \sin\left(\frac{(2n-1)\pi y}{b}\right) \exp\left(-\left(\frac{(2m-1)^2}{a^2} + \frac{(2n-1)^2}{b^2}\right)\pi^2 k t\right)}$$

Problem 12

A rectangular plate [illegible] $\le x \le a,\ 0 \le y \le b$ initially has a hot spot at the center [illegible] [illegible] $= M\delta$ [illegible]. The edges are [illegible]. Let k be the diffusion constant [illegible] and the temperature at [illegible] in the form of a series.

$$u(x,y,t) = \sum_{m,n} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \exp\left(-\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\pi^2 k t\right)$$

$$\sum_{m,n} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) = M\delta\left(x-\frac{a}{2},\, y-\frac{b}{2}\right)$$

$$\sin\left(\frac{p\pi x}{a}\right) \sin\left(\frac{q\pi y}{b}\right)$$

$$\sum_{m,n} A_{mn} \int_0^b\!\!\int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{p\pi x}{a}\right) \sin\left(\frac{q\pi y}{b}\right) dx\, dy$$

$$= \sum_{m,n} A_{mn} \int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{p\pi x}{a}\right) dx \int_0^b \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{q\pi y}{b}\right) dy$$

$$M\int_0^b\!\!\int_0^a \delta\left(x-\frac{a}{2},\, y-\frac{b}{2}\right) \sin\left(\frac{p\pi x}{a}\right) \sin\left(\frac{q\pi y}{b}\right) dx\, dy = A_{pq} \int_0^a \sin^2\left(\frac{p\pi x}{a}\right) dx \int_0^b \sin^2\left(\frac{q\pi y}{b}\right) dy$$

$$= A_{pq}\left(\frac{ab}{4}\right)$$

$$M\int_0^a \delta\left(x-\frac{a}{2}\right) \sin\left(\frac{p\pi x}{a}\right) dx \int_0^b \delta\left(y-\frac{b}{2}\right) \sin\left(\frac{q\pi y}{b}\right) dy$$

$$= M\int_0^a \delta\left(x-\frac{a}{2}\right) \sin\left(\frac{p\pi x}{a}\right) dx \int_0^b \delta\left(y-\frac{b}{2}\right) \sin\left(\frac{q\pi y}{b}\right) dy$$

$$= M \sin\left(\frac{p\pi}{2}\right) \sin\left(\frac{q\pi}{2}\right)$$

$$= M \begin{cases} \frac{1}{2}\left(\cos\left(\frac{(p-q)\pi}{2}\right) - \cos\left(\frac{(p+q)\pi}{2}\right)\right) & p \,\&\, q \text{ are odd} \\ 0 & \text{otherwise} \end{cases}$$

$$= M \begin{cases} \dfrac{(-1)^{\left(\frac{p-q}{2}\right)} - (-1)^{\left(\frac{p+q}{2}\right)}}{2} & p \,\&\, q \text{ are odd} \\ 0 & \text{otherwise} \end{cases}$$

$$A_{mn} = \begin{cases} 2M\left[\dfrac{(-1)^{\frac{m-n}{2}} - (-1)^{\frac{m+n}{2}}}{ab}\right] & m \,\&\, n \text{ are odd} \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{u(x,y,t) = 2M \sum_{m,n} \frac{(-1)^{m-n} - (-1)^{m+n}}{ab} \sin\left(\frac{(2m-1)\pi x}{a}\right) \sin\left(\frac{(2n-1)\pi y}{b}\right) \exp\left(-\left(\frac{(2m-1)^2}{a^2} + \frac{(2n-1)^2}{b^2}\right)\pi^2 k t\right)}$$

Problem 12

A rectangular plate $\{0 \le x \le a, 0 \le y \le b\}$ initially has a hot spot at its center so that its **initial temperature** distribution is $u(x,y,0) = M\delta(x - \frac{a}{2}, y - \frac{b}{2})$. Its edges are maintained **at zero temperature**. Let k be the diffusion constant. Find the temperature at any later **time in the form of a series.**

$$u(x,y,t) = \sum_{m,n} A_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \exp-\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\pi^2 k t$$

$$\sum_{m,n} A_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) = M\delta\left(x-\frac{a}{2}, y-\frac{b}{2}\right)$$

$$\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{k\pi y}{b}\right)$$

$$\sum_{m,n} A_{mn} \int_0^b\int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{k\pi y}{b}\right) dx\,dy$$

$$= \sum_{m,n} A_{mn} \int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{l\pi x}{a}\right)dx \int_0^b \sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{k\pi y}{b}\right)dy$$

$$M\int_0^b\int_0^a \delta\left(x-\frac{a}{2}, y-\frac{b}{2}\right)\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{k\pi y}{b}\right)dx\,dy = A_{lk}\int_0^a \sin^2\left(\frac{l\pi x}{a}\right)dx \int_0^b \sin^2\left(\frac{k\pi y}{b}\right)dy$$

$$= A_{lk}\left(\frac{ab}{4}\right)$$

$$= M\int_0^a \delta\left(x-\frac{a}{2}\right)\sin\left(\frac{l\pi x}{a}\right)dx \int_0^b \delta\left(y-\frac{b}{2}\right)\sin\left(\frac{k\pi y}{b}\right)dy$$

$$= M\int_0^a \delta\left(x-\frac{a}{2}\right)\sin\left(\frac{l\pi x}{a}\right)dx \int_0^b \delta\left(y-\frac{b}{2}\right)\sin\left(\frac{k\pi y}{b}\right)dy$$

$$= M\sin\left(\frac{l\pi}{2}\right)\sin\left(\frac{k\pi}{2}\right)$$

$$= M\begin{cases}\frac{1}{2}\left(\cos\left(\frac{(k-l)\pi}{2}\right)-\cos\left(\frac{(k+l)\pi}{2}\right)\right) & k \,\&\, l \text{ are odd}\\ 0 & \text{otherwise}\end{cases}$$

$$= M\begin{cases}\frac{(-1)^{\left(\frac{(k-l)}{2}\right)}-(-1)^{\left(\frac{(k+l)}{2}\right)}}{2}, & k \,\&\, l \text{ are odd}\\ 0 & \text{otherwise}\end{cases}$$

$$A_{mn} = \begin{cases}2M\left[\frac{(-1)^{(m-n)/2}-(-1)^{(m+n)/2}}{ab}\right] & m \,\&\, n \text{ are odd}\\ 0 & \text{otherwise}\end{cases}$$

$$\boxed{u(x,y,t) = 2M\sum_{m,n}\frac{(-1)^{m-n}-(-1)^{m+n}}{ab}\sin\left(\frac{(2m-1)\pi x}{a}\right)\sin\left(\frac{(2n-1)\pi y}{b}\right)\exp-\left(\frac{(2m-1)^2}{a^2}+\frac{(2n-1)^2}{b^2}\right)\pi^2 k t}$$

Problem 3

Show that

$$\frac{1}{2\pi^2 cr}\int_0^\infty \sin kct \sin kr \, dk = \frac{1}{8\pi^2 cr}\int_{-\infty}^{\infty} [e^{ik(ct-r)} - e^{ik(ct+r)}]\, dk$$

$$= \frac{1}{4\pi cr}[\delta(ct-r) - \delta(ct+r)].$$

$$\frac{1}{2\pi^2 cr}\int_0^\infty \sin kct \sin kr \, dk = \frac{1}{8\pi^2 cr}\int_0^\infty \left[e^{ik(ct-r)} - e^{ik(ct+r)}\right] dk$$

$$\sin kct \sin kr = \sinh(i(kct))\sinh(ikr)$$

$$= -\frac{1}{4}\left[e^{ikct} - e^{-ikct}\right]\left[e^{ikr} - e^{-ikr}\right]$$

$$= -\frac{1}{4}\left[e^{ik(ct+r)} - e^{-ik(ct-r)} - e^{ik(ct-r)} + e^{-ik(ct+r)}\right]$$

$$= -\frac{1}{4}\left[e^{ik(ct+r)} - e^{ik(ct-r)} + e^{-ik(ct+r)} - e^{-ik(ct-r)}\right]$$

$$\frac{1}{2\pi^2 cr}\int_0^\infty \sin kct \sin kr \, dk = -\frac{1}{8\pi^2 cr}\int_0^\infty \left[e^{ik(ct+r)}\right.$$

$$= -\frac{1}{8\pi^2 cr} + \int_0^\infty \left[e^{-ik(ct+r)} - e^{-ik(ct-r)}\right] dk$$

$$= -\frac{1}{8\pi^2 cr}\int_0^\infty \left[e^{ik(ct+r)} - e^{-ik(ct-r)}\right] dk$$

$$= -\frac{1}{8\pi^2 cr}\int_{-\infty}^0 \left[e^{ik(ct+r)} - e^{ik(ct-r)}\right] dk$$

$$= \boxed{\frac{1}{8\pi^2 cr}\int_{-\infty}^\infty \left[e^{ik((ct+r)} - e^{ik(ct+r)}\right] dk \ (2)}$$

$$\boxed{\frac{1}{8\pi^2 cr}\int_0^\infty \left[e^{ik(ct-r)} - e^{(ik(ct+r))}\right] dk}$$

$$= \frac{1}{8\pi^2 cr}\left(2\pi\delta(ct-r) - 2\pi\delta(ct+r)\right)$$

$$\boxed{= \frac{1}{4\pi cr}\left(\delta(ct-r) - \delta(ct+r)\right)}$$

$\boxed{\text{Hint: } \sinh x = -i\sin ix}$

Problem 1

Verify directly from the definition that $\phi \mapsto \int_{-\infty}^{\infty} f(x)\phi(x)\,dx$ is a distribution if $f(x)$ is **any function that** is integrable on each bounded set.

$$(F,\phi) = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx$$

Prove linearity:

$$\boxed{(F, a\phi + b\psi)} = \int_{-\infty}^{\infty} f(x)[a\phi(x) + b\psi(x)]\,dx$$

$$= a\int_{-\infty}^{\infty} f(x)\phi(x)\,dx + b\int_{-\infty}^{\infty} f(x)\psi(x)\,dx$$

$$= \boxed{a(F,\phi) + b(F,\psi)}$$

Prove continuity:

$$|(F,\phi_n) - (F,\phi)| \overset{\text{linearity}}{=} |(F, \phi_n - \phi)|$$

$$= \left|\int_I f(x)[\phi_n - \phi]\,dx\right|$$

$$= \underbrace{\max_I |\phi_n - \phi|}_{\text{finite}} \underbrace{\int_I |f(x)|\,dx}_{\text{finite}}$$

there exists $N \in \mathbb{N}$ such that for all $n > N$

$$\boxed{\max_I |\phi_n - \phi| \int_I |f(x)|\,dx < \epsilon}$$

Problem 2

For the hydrogen atom if $\lambda > 0$, why would you expect equation (4) not to have a solution that satisfies the condition at infinity?

The equation is

$$-R_{rr} - \frac{2}{r} R_r - \frac{2}{r} R = \lambda R$$

At ∞ R has to satisfy

$$\int_0^\infty |R(r)|^2 r^2 dr < \infty$$

as $r \to \infty$

$$\frac{2}{r}(R_r + R) \to 0$$

$\therefore$ thus as $r \to \infty$

$$\boxed{-R_{rr} = \lambda R}$$

If $\lambda > 0$ the solution in the limit is of the form

$$\boxed{\lim_{r \to \infty} R(r) = A\cos(\sqrt{\lambda} r) + B\sin(\sqrt{\lambda} r)}$$

2. Consider a traveling wave $u(x, t) = f(x - at)$ where f is a given function **of one variable.**

(a) **If it is a** solution of the wave equation, show that the speed must be $a = \pm c$ (unless f is a linear function).

(b) **If it is a** solution of the diffusion equation, find f and show that the **speed** a is arbitrary.

(a)

$$u_{tt} = c^2 u_{xx}$$

$$a^2 f''(x-at) = u_{tt} = c^2 u_{xx} = c^2 f''(x-at)$$

$$(a-c)(a+c) f''(x-at) = 0$$

1. $a - c = 0 \Rightarrow \boxed{a = c}$
2. $a + c = 0 \Rightarrow \boxed{a = -c}$
3. $f''(x-at) = 0$ for all $(x,t) \in [0,L] \times [0,T]$

$$f''(y) = 0 \Rightarrow f(y) = cy + a$$

f is a linear function

in order for f to be a solution either f is linear or $a = \pm c$.

(b)

$$u(x,t) = f(x-at)$$

$$u_t = -k u_{xx}$$

$$-af'(x-at) = -kf''(x-at)$$

$$kf''(x-at) - af'(x-at) = 0$$

For all $(x,t) \in [0,L] \times [0,T]$

Solve the ODE:

$$kf''(y) - af'(y) = 0$$

$$f(y) = Ce^{-(a/k)y} + B$$

u is a solution of the diffusion iff

$$\boxed{f(x) = Ce^{-(a/k)x} + B}$$

where C & B are constants

19. (a) Show that $S_2(x, y, t) = S(x, t)S(y, t)$ satisfies the diffusion equation $S_t = k(S_{xx} + S_{yy})$.
(b) Deduce that $S_2(x, y, t)$ is the source function for two-dimensional diffusions.

(a)

$$S(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\left(\frac{x^2+y^2}{4kt}\right)}$$

$$S_2(x,y,t) = \frac{1}{4\pi k t} e^{-\left(\frac{x^2+y^2}{4kt}\right)}$$

$$\frac{dS_2}{dt} = -\frac{1}{4\pi k t^2} e^{-\left(\frac{x^2+y^2}{4kt}\right)} + \frac{1}{4\pi k t} e^{-\left(\frac{x^2+y^2}{4kt}\right)}\left(\frac{x^2+y^2}{4kt^2}\right)$$

$$= \frac{1}{4\pi k t^2} e^{-\left(\frac{x^2+y^2}{4kt}\right)}\left(-1 + \frac{x^2+y^2}{4kt}\right)$$

$$= \left(\frac{x^2+y^2-4kt}{16\pi k^2 t^3}\left(e^{-\left(\frac{x^2+y^2}{4kt}\right)}\right)\right)$$

$$\frac{dS_2}{dx} = \frac{1}{4\pi k t} e^{-\left(\frac{x^2+y^2}{4kt}\right)}\left(\frac{-2x}{4kt}\right)$$

$$= \frac{-x}{8\pi k^2 t^2} e^{-\left(\frac{x^2+y^2}{4kt}\right)}$$

$$\frac{d^2S_2}{dx^2} = \frac{x^2-2kt}{16\pi k^2 t^3} e^{-\left(\frac{x^2+y^2}{4kt}\right)}$$

$$\frac{d^2S_2}{dy^2} = \frac{y^2-2kt}{16\pi k^2 t^3} e^{-\left(\frac{x^2+y^2}{4kt}\right)}$$

$$\frac{d^2S_2}{dx^2} + \frac{d^2S_2}{dy^2} = \frac{x^2-2kt}{16\pi k^2 t^3} e^{-\left(\frac{x^2+y^2}{4kt}\right)} + \frac{y^2-2kt}{16\pi k^2 t^3} e^{-\left(\frac{x^2+y^2}{4kt}\right)}$$

$$= \left(\frac{x^2+y^2-4kt}{16\pi k^2 t^3}\right) e^{-\left(\frac{x^2+y^2}{4kt}\right)}$$

$$= \frac{1}{k} \cdot \frac{dS_2}{dt}$$

S_2 satisfies the two-dimensional diffusion equation

(b)

$$u_{xx} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d^2}{dx^2} S_2(x-r, y-s, t)\phi(r,s)\,dr\,ds$$

$$u_{yy} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d^2}{dy^2} S_2(x-r, y-s, t)\phi(r,s)\,dr\,ds$$

$$u_t = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d}{dt} S_2(x-r, y-s, t)\phi(r,s)\,dr\,ds$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right) S_2(x-r, y-s, t)\phi(r,s)\,dr\,ds$$

$$k\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d^2}{dx^2} S_2(x-r, y-s, t)\phi(r,s)\,dr\,ds + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d^2}{dy^2} S_2(x-r, y-s, t)\phi(r,s)\,dr\,ds\right)$$

$$= k(u_{xx} + u_{yy})$$

$$u(x,y,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_2(x-r, y-s, 0)\phi(r,s)\,dr\,ds$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-r)\,\delta(y-s)\,\phi(r,s)\,dr\,ds$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(r)\,\delta(s)\,\phi(x-r, y-s)\,dr\,ds$$

$$= \boxed{\phi(x,y)}$$

[Thus u satisfies the 2 dimensional diffusion equation]

19. **(a)** Show that $S_2(x, y, t) = S(x, t)S(y, t)$ satisfies the diffusion equation $S_t = k(S_{xx} + S_{yy})$.

(b) Deduce that $S_2(x, y, t)$ is the source function for two-dimensional diffusions.

(a)

$$S(x,t) = \frac{1}{\sqrt{4\pi k t}} * e^{-\left(\frac{x^2+y^2}{4kt}\right)}$$

$$S_2(x,y,t) = \frac{1}{4\pi k t} * e^{-\left(\frac{x^2+y^2}{4kt}\right)}$$

$$\frac{dS_2}{dx} = \frac{-1}{4\pi k t^2} e^{-\left(\frac{x^2+y^2}{4kt}\right)} + \frac{1}{4\pi k t} e^{-\left(\frac{x^2+y^2}{4kt}\right)}\left(\frac{x^2+y^2}{4kt^2}\right)$$

$$= \frac{1}{4\pi k t^2} e^{-\left(\frac{x^2+y^2}{4kt}\right)}\left(-1 + \frac{x^2+y^2}{4kt}\right)$$

$$= \left(\frac{x^2+y^2-4kt}{16\pi k^2 t^3}\left(e^{-\left(\frac{x^2+y^2}{4kt}\right)}\right)\right)$$

$$\frac{dS_2}{dx} = \frac{1}{4\pi k t} e^{-\left(\frac{x^2+y^2}{4kt}\right)}\left(\frac{-2x}{4kt}\right)$$

$$= \frac{-x}{8\pi k^2 t^2} e^{-\left(\frac{x^2+y^2}{4kt}\right)}$$

$$\frac{d^2S_2}{dx^2} = \frac{x^2-2kt}{16\pi k^3 t^3} e^{-\left(\frac{x^2+y^2}{4kt}\right)}$$

$$\frac{d^2S_2}{dy^2} = \frac{y^2-2kt}{16\pi k^3 t^3} e^{-\left(\frac{x^2+y^2}{4kt}\right)}$$

$$\frac{d^2S_2}{dx^2} + \frac{d^2S_2}{dy^2} = \frac{x^2-2kt}{16\pi k^3 t^3} e^{-\left(\frac{x^2+y^2}{4kt}\right)} + \frac{y^2-2kt}{16\pi k^3 t^3} e^{\left(\frac{x^2+y^2}{4kt}\right)}$$

$$= \left(\frac{x^2+y^2-4kt}{16\pi k^3 t^3}\right) e^{-\left(\frac{x^2+y^2}{4kt}\right)}$$

$$= \frac{1}{k} * \frac{dS_2}{dt}$$

S_2 satisfies the two-dimensional diffusion equation

(b)

$$U_{xx} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d^2}{dx^2} S_2(x-r, y-s, t)\phi(r,s)\,dr\,ds$$

$$U_{yy} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d^2}{dy^2} S_2(x-r, y-s, t)\phi(r,s)\,dr\,ds$$

$$U_t = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d}{dt} S_2(x-r, y-s, t)\phi(r,s)\,dr\,ds$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right) S_2(x-r, y-s, t)\phi(r,s)\,dr\,ds$$

$$k\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d^2}{dx^2} S_2(x-r, y-s, t)\phi(r,s)\,dr\,ds + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d^2}{dy^2} S_2(x-r, y-s, t)\phi(r,s)\,da\,ds\right)$$

$$= k(U_{xx} + U_{yy})$$

$$U(x,y,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_2(x-r, y-s, 0)\phi(r,s)\,dr\,ds$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-r)\,\delta(y-s)\,\phi(r,s)\,dr\,ds$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(r)\,\delta(s)\,\phi(x-r, y-s)\,dr\,ds$$

$$= \boxed{\phi(x,y)}$$

[Thus U satisfies the 2 dimensional diffusion equation]

[illegible]

[illegible]

[illegible]

$$= \bar{\lambda} \int_0^1 u \bar{u}$$

$$(\lambda - \bar{\lambda}) \int_0^1 u \bar{u} = 0 = (\lambda - \bar{\lambda}) \int_0^1 |u|^2 \, dx$$

hence, $\lambda - \bar{\lambda} = 0$

Problem 7

Consider the operator $v \mapsto (pv^{(m)})^{(m)}$, where the superscript denotes the mth derivative, in an interval with the boundary conditions $v = v' = v'' = \cdots = v^{(m-1)} = 0$ at both ends. Show that its eigenvalues are real.

$$(p v^{(m)})^{(m)} = \lambda v$$

$$\lambda \int_a^b v \bar{v} \, dx = \int_a^b (p v^m)^m \bar{v} \, dx$$

$$= \int_a^b (p v^m)^{m-1} \bar{v}' \, dx + (p v^m)^{m-1} \bar{v} \Big|_a^b$$

recall $v = v' = \ldots = v^m = 0$

$$\lambda \int_a^b v \bar{v} \, dx = (-1)^m \int_a^b p (v)^{(m)} \bar{v}^{(m)} \, dx$$

$$= (-1)^m \int_a^b v^m (p \bar{v}^{(m)}) \, dx$$

$$= \int_a^b v (p \bar{v}^{(m)})^m \, dx + 0$$

$$\lambda \int_a^b v \bar{v} \, dx = \int_a^b v (p \bar{v}^{(m)})^{(m)} \, dx$$

$$= \int_a^b v \bar{\lambda} \bar{v} \, dx$$

$$= \bar{\lambda} \int_a^b v \bar{v}$$

$$(\lambda - \bar{\lambda}) \int_a^b v \bar{v} x = 0 = (\lambda - \bar{\lambda}) \int_a^b |v|^2 \, dx$$

hence, $\lambda - \bar{\lambda} = 0$

Problem 4: Lorentz Invariance

(Lorentz invariance of the wave equation) Thinking of the coordinates of space-time as 4-vectors (x, y, z, t), let Γ be the diagonal matrix with the diagonal entries $1, 1, 1, -1$. Another matrix L is called a *Lorentz transformation* if L has an inverse and $L^{-1} = \Gamma\, {}^tL\, \Gamma$, where ${}^tL$ is the transpose.

(a) If L and M are Lorentz, show that LM and L^{-1} also are.

(b) Show that L is Lorentz if and only if $m(L\mathbf{v}) = m(\mathbf{v})$ for all 4-vectors $\mathbf{v} = (x, y, z, t)$, where $m(\mathbf{v}) = x^2 + y^2 + z^2 - t^2$ is called the *Lorentz metric*.

(c) If $u(x, y, z, t)$ is any function and L is Lorentz, let $U(x, y, z, t) = u(L(x, y, z, t))$. Show that

$$U_{xx} + U_{yy} + U_{zz} - U_{tt} = u_{xx} + u_{yy} + u_{zz} - u_{tt}.$$

(d) Explain the meaning of a Lorentz transformation in more geometrical terms. (*Hint*: Consider the level sets of $m(\mathbf{v})$.)

is a linear transformation

(a) let L & M be Lorentz.

$L^{-1} = \Gamma\, {}^tL\, \Gamma$

$M^{-1} = \Gamma\, {}^tM\, \Gamma$

$\Gamma = {}^t\Gamma = \Gamma^{-1}$

the inverse of LM is

$(LM)^{-1} = M^{-1}L^{-1}$

$= \Gamma\, {}^tM\, \underbrace{\Gamma\Gamma^{-1}}_{I}\, {}^tL\, \Gamma$

$= \Gamma\, {}^tM\, {}^tL\, \Gamma = \Gamma\, {}^t(LM)\, \Gamma$

LM is Lorentz

the inverse of L^{-1} is

$(L^{-1})^{-1} = L = {}^t(\Gamma^{-1}\, {}^tL^{-1}\, \Gamma^{-1})$

$= {}^t(\Gamma\, {}^tL^{-1}\, \Gamma) = {}^t\Gamma\, {}^{tt}L^{-1}\, {}^t\Gamma$

$= \Gamma\, {}^tL^{-1}\, \Gamma$

L^{-1} is Lorentz

(b) Assume L is Lorentz

$m(v) = x^2 + y^2 + z^2 - t^2 = {}^tv\Gamma v$.

we can derive $\Gamma = {}^tL\Gamma L$

$m(v) = {}^tv\, {}^tL\Gamma L v = {}^t(Lv)\Gamma(Lv) = m(Lv)$

now assume that there exists a matrix L such for every vector v $m(v) = m(Lv)$

${}^tv\Gamma v = {}^tv\, {}^tL\Gamma L v$

for all vectors v

$\Gamma = {}^tL\Gamma L$ which is = (1) part ⇒ is Lorentz.

(c) let $u(x,y,z,t)$ be an arbitrary function & L Lorentz, let U be

$U(x,y,z,t) = u(L(x,y,z,t))$

denote $v = (v_1, v_2, v_3, v_4)$ then

$$U_{xx} + U_{yy} + U_{zz} - U_{tt} = \sum_{i=1}^{4}\sum_{j=1}^{4} \Gamma_{ij} \frac{\partial}{\partial i}\frac{\partial}{\partial j} u(Lv) = \sum_{i=1}^{4}\sum_{j=1}^{4} \Gamma_{ij} \sum_{k=1}^{4}\sum_{l=1}^{4} \frac{\partial^2 u(Lv)}{\partial v_k \partial v_l} \frac{\partial}{\partial v_i}(Lv)_k \frac{\partial}{\partial v_j}(Lv)_l =$$

$$\sum_{i=1}^{4}\sum_{j=1}^{4} \Gamma_{ij} \sum_{k=1}^{4}\sum_{l=1}^{4} \frac{\partial^2 u(Lv)}{\partial v_k \partial v_l}\left(\sum_{n=1}^{4} L_{kn} \underbrace{\frac{\partial v_n}{\partial v_i}}_{\delta_{ni}}\right)\left(\sum_{n=1}^{4} L_{ln} \underbrace{\frac{\partial v_n}{\partial v_j}}_{\delta_{nj}}\right) = \sum_{i=1}^{4}\sum_{j=1}^{4} \Gamma_{ij} \sum_{k=1}^{4}\sum_{l=1}^{4} \frac{\partial^2 u(Lv)}{\partial v_k \partial v_l} L_{ki} L_{lj} =$$

$$\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4}\sum_{l=1}^{4} L_{ki}\Gamma_{ij}L_{lj}\left(\frac{\partial^2 u(Lv)}{\partial v_k \partial v_l}\right) \overset{\Gamma = L\Gamma\, {}^tL}{=} \sum_{k=1}^{4}\sum_{l=1}^{4} \Gamma_{kl}{}^{t}\left(\frac{\partial^2 U(v)}{\partial v_k \partial v_l}\right) = \boxed{U_{xx} + U_{yy} + U_{zz} - U_{tt}}$$

∴ so the equation is invariant under lorentz transformations

(d) the lorentz metric of a vector $v = (x, y, z, t)$ is $M(v) = |(x, y, z, it)|^2$, represents magnitude in space. this transformation is an

(linear transformation, move at constant velocity, rotation in space)