Problem 12

(a) Solve by hand the nonlinear PDE $u_t = u_{xx} + (u)^3$ for all x using the standard forward difference scheme with $(u)^3$ treated as $(u_j^n)^3$. Use $s = \frac{1}{4}$, $\Delta t = 1$, and initial data $u_j^0 = 1$ for $j = 0$ and $u_j^0 = 0$ for $j \neq 0$. Solve for u_0^3.

(b) Compare your answer to the same problem without the nonlinear term.

(c) Exactly solve the ODE $dv/dt = (v)^3$ with the condition $v(0) = 1$. Use it to explain why u_0^3 is so large in part (a).

(d) Repeat part (a) with the same initial data but for the PDE $u_t = u_{xx} - (u)^3$. Compare with the answer in part (a) and explain.

(a)

$$\begin{cases} u_t = \dfrac{u_j^{n+1} - u_j^n}{\Delta t} \\ u_{xx} = \dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} \end{cases}$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} + (u_j^n)^3$$

$$u_j^{n+1} - u_j^n = s(u_{j+1}^n - 2u_j^n + u_{j-1}^n) + \Delta t\,(u_j^n)^3$$

$$u_j^{n+1} = s u_{j+1}^n + (1-2s)u_j^n + s u_{j-1}^n + \Delta t\,(u_j^n)^3$$

$$u_j^{n+1} = \frac{1}{4} u_{j+1}^n + \frac{1}{2} u_j^n + \frac{1}{4} u_{j-1}^n + (u_j^n)^3$$

$n=0$:	0	0	0	1	0	0	0
$n=1$:	0	0	1/4	3/2	1/4	0	0
$n=2$:	0	1/16	.	.	.	.	0
$n=3$:	1/64	.	.	.	.	.	.

$\boxed{u_0^3 = 79.1484}$

(b)

$$u_j^{n+1} = s u_{j+1}^n + (1-2s) u_j^n + s u_{j-1}^n$$

$$u_j^{n+1} = \frac{1}{4} u_{j+1}^n + \frac{1}{2} u_j^n + \frac{1}{4} u_{j-1}^n$$

$n=0$:	0	0	0	1	0	0	0
$n=1$:	0	0	1/4	1/2	1/4	0	0
$n=2$:	0	1/16	.	.	.	1/16	0
$n=3$:	1/64	.	.	.	.	.	.

$\boxed{u_0^3 = \frac{5}{16} = .3125}$

(c)

$$\int \frac{dv}{v^3} = \int dt$$

$$-\frac{1}{v^2} = t + C$$

$$v = \frac{1}{\sqrt{C - 2t}} \rightarrow 1 = v(0) = \frac{1}{\sqrt{C}} \Rightarrow C = 1$$

$\boxed{v = \frac{1}{\sqrt{1-2t}}}$ ← Solution

as $t \to \frac{1}{2}$, $v(t) \to \infty$, so u_0^n grows & explodes rapidly

(d)

u_t approx with forward diff u_{xx} approx with center diff

Plug in $s = \frac{1}{4}$, $\Delta t = 1$

$$u_j^{n+1} = s u_{j+1}^n + (1-2s)u_j^n + s u_{j-1}^n - \Delta t (u_j^n)^3$$

$$u_j^{n+1} = \frac{1}{4}(u_{j+1}^n) + \frac{1}{2}(u_j^n) + \frac{1}{4}(u_{j-1}^n) - (u_j^n)^3$$

$n=0$:	0	0	0	1	0	0	0
$n=1$:	0	0	1/4	-1/2	1/4	0	0
$n=2$:	0	1/16	-1/64	.	-1/64	1/16	0
$n=3$:	.	.	.	.	.	.	.

$\boxed{u_0^3 = -0.0078}$, the solution decays at $x=0$ & explodes (a) in $x=0$

(b) The Max temp is reached such that x is

$$\frac{du}{dx}(x)=0$$

u being function from (a)

So $x \in (\frac{l}{2}, l)$ because the rod is subjected to heat on Interval

$$0 = -\frac{H}{8}(-4(l-x)-(l-4x)) \Rightarrow \frac{5l}{8}-x=0$$

hottest temp reached: $x=\frac{5l}{8}$

& the temp is $u(\frac{5l}{8}) = \frac{9Hl^2}{128}$

4. A rod occupying the interval $0 \leq x \leq l$ is subject to the heat source $f(x) = 0$ for $0 < x < \frac{l}{2}$, and $f(x) = H$ for $\frac{l}{2} < x < l$ where $H > 0$. The rod has physical constants $c = \rho = \kappa = 1$, and its ends are kept at zero temperature.
(a) Find the steady-state temperature of the rod.
(b) Which point is the hottest, and what is the temperature there?

(a) At steady state temperature is a constant in respect to t

$$u_t = 0$$

the temp satisfies $u_t = \Delta u + f(x,t)$

the rod in 1 dimension : $\frac{d^2}{dx^2} + f(x) = 0$

the solution for $0<x<\frac{l}{2}$ is: $u(x) = Ax+B$

& for $\frac{l}{2}<x<l$ is $u(x) = -\frac{1}{2}Hx^2 + Cx + D$

let's find A, B, C, D, mind the end kept at 0

$0 = u(0,t) = B$, $0 = u(l,t) = -\frac{1}{2}Hl^2 + Cl + D$

temp & Heat flux is continuous

$\frac{Al}{2} = \lim_{x\to\frac{l}{2}} u(x) = \lim_{x\to\frac{l}{2}^+} u(x) = \frac{l}{8}(3Hl-4C)$

so, $u(x) = \begin{cases} \frac{Hl}{8}x, & 0<x<\frac{l}{2} \\ -\frac{H}{8}(l-4x)(l-x), & \frac{l}{2}<x<l \end{cases}$

The function given the conditions is

$$u(x) = \begin{cases} \dfrac{10 \cdot 1cp}{3 \cdot 1cp + 2 \cdot 2cp} x, & 0 < x < 3 \\ 10 - \dfrac{10 \cdot 2cp}{3 \cdot 1cp + 2 \cdot 2cp}(5-x), & 3 < x < 5 \end{cases}$$

the final form is :
$$u(x) = \begin{cases} \frac{10}{7} x, & 0 < x < 3 \\ 10 + \frac{10}{7}(2x-3), & 3 < x < 5 \end{cases}$$

$$u(x) = \begin{cases} \dfrac{k_1 T}{k_2 L_1 + k_1 L_2} x, & 0 < x < L_1 \\ T - \dfrac{k_1 T}{k_2 L_1 + k_1 L_2}(L_1 + L_2 - x), & L_1 < x < L_1 + L_2 \end{cases}$$

6. Two homogeneous rods have the same cross section, specific heat c, and density ρ but different heat conductivities κ_1 and κ_2 and lengths L_1 and L_2. Let $k_j = \kappa_j / c\rho$ be their diffusion constants. They are welded together so that the temperature u and the heat flux κu_x at the weld are continuous. The left-hand rod has its left end maintained at temperature zero. The right-hand rod has its right end maintained at temperature T degrees.
   (a) Find the *equilibrium* temperature distribution in the composite rod.
   (b) Sketch it as a function of x in case $k_1 = 2$, $k_2 = 1$, $L_1 = 3$, $L_2 = 2$, and $T = 10$. (This exercise requires a lot of elementary algebra, but it's worth it.)

(a) At the equilibrium temp is a constant in respect to t,

so $U_t = 0$

The Temp satisfies since the bar has different physical properties

$$c\rho u_t = \begin{cases} \nabla \cdot (k_1 \nabla u), & 0 < x < L_1 \\ \nabla \cdot (k_2 \nabla u), & L_1 < x < L_1 + L_2 \end{cases}$$

Moreover, the rod is 1 dimensional & $u_t = 0$ at equilibrium

$$0 = \begin{cases} \frac{d^2 u}{dx^2}, & 0 < x < L_1 \\ \frac{d^2 u}{dx^2}, & L_1 < x < L_1 + L_2 \end{cases}$$

$$-k_1 A = \lim_{x \to L_1^-} \frac{du}{dx} = \lim_{x \to L_1^+} \frac{du}{dx} = -k_2 C$$

The temp & heat flux must be continuous

$$AL_1 + B = \lim_{x \to L_1^-} u(x) = \lim_{x \to L_1^+} u(x) = CL_1 + D$$

lets find A, B, C, D

Solve the two ode

$$0 = u(0,t) = B, \; T = u(L_1 + L_2) = C(L_1 + L_2) + D$$

$$u(x) = \begin{cases} Ax + B, & 0 < x < L_1 \\ Cx + D, & L_1 < x < L_1 + L_2 \end{cases}$$

(b) integrate ODE from 0 to l

$$\int_0^l u''(x) + u'(x)\,dx = \int_0^l f(x)\,dx$$

$$u'(l) + u(l) - u'(0) - u(0) = \int_0^l f(x)\,dx$$

using the boundary conditions

$$0 = u'(l) + u(l) - \tfrac{1}{2}(u'(l) + u(l)) - \tfrac{1}{2}(u'(l) + u(l)) = \int_0^l f(x)\,dx$$

So $\int_0^l f(x)\,dx = 0$

so the avg is zero

2. Consider the problem

$$u''(x) + u'(x) = f(x)$$
$$u'(0) = u(0) = \tfrac{1}{2}[u'(l) + u(l)],$$

with $f(x)$ a given function.

(a) Is the solution *unique*? Explain.

(b) Does a solution necessarily *exist*, or is there a condition that $f(x)$ must satisfy for existence? Explain.

(a)

Assume the solution exists

let u be a solution to the given ode

define v as $v(x) = u(x) + e^{-x} - 2$

So $v \neq u$

let's show v is also a solution to the ode

1. $v''(x) + v'(x) = u''(x) + e^{-x} + u'(x) - e^{-x}$
$= u''(x) + u'(x)$
$= f(x)$

2. $v'(0) = u'(0) - 1$
$v(0) = u(0) + 1 - 2 = u(0) - 1$
$\frac{1}{2}(v'(l) + v(l)) = \frac{1}{2}(u'(l) - e^{-l} + u(l) + e^{-l} - 2) = \frac{1}{2}(u'(l) + u(l)) - 1$

So, the solution is not unique

5. Consider the equation

$$u_x + yu_y = 0$$

with the boundary condition $u(x, 0) = \phi(x)$.
(a) For $\phi(x) \equiv x$, show that no solution exists.
(b) For $\phi(x) \equiv 1$, show that there are many solutions.

(a) $u_x + yu_y = 0$

$\frac{dy}{dx} = y \Rightarrow ye^{-x} = C$

$u(x,y) = f(ye^{-x})$ where $x \in \mathbb{R}$, this does not exist because $x = f(0)$ for all $x \in \mathbb{R}$

(b) $u(x,0) = x$

so this exists

$1 = u(x,0) = f(0)$

for all $x \in \mathbb{R}$

This obviously does exist a trivial function is $f(x) = 1$ for all $x \in \mathbb{R}$

6. Solve the equation $u_x + 2xy^2u_y = 0$.

$$u_x + 2xy^2 u_y = 0$$

using characteristics.

$$\frac{dy}{dx} = \frac{2xy^2}{1}$$

$$\frac{dy}{y^2} = 2x\,dx$$

$$-\frac{1}{y} = x^2 + c$$

$$u(x,y) = f\left(\frac{1}{y} + x^2\right)$$

1. What is the type of each of the following equations?

(a) $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0.$

(b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0.$

(a) $a_{12}^2 - a_{11}a_{22} = 9/4 - (1)(1)$

$= 5/4 > 0$

Then the equation (a) is hyperbolic

$u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0$

$\Leftarrow$ Compare this equation with

$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0$

(b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$

Compare this equation with

$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0$

$\Downarrow$

Since $a_{12}^2 - a_{11}a_{22} = 9 - (9)(1)$

$= 0$

thus the equation is parabolic.

4. What is the *type* of the equation

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0?$$

Show by direct substitution that $u(x, y) = f(y + 2x) + xg(y + 2x)$ is a solution for arbitrary functions f and g.

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0$$

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0$$

here $a_{11} = 1, a_{12} = -2, a_{22} = 4$

$$a_{12}^2 - a_{11}a_{22} = 4-4$$
$$= 0$$

Thus equation 1 is parabolic.

$$u(x,y) = f(y+2x) + xg(y+2x)$$

$$u_x = 2f_x(y+2x) + 2xg_x(y+2x) + g(y+2x)$$

$$u_{xx} = 4f_{xx}(y+2x) + 4xg_{xx}(y+2x) + 2g_x(y+2x) + g_x(y+2x)$$

$$u_y = f_y(y+2x) + xg_y(y+2x)$$

$$u_{yy} = f_{yy}(y+2x) + xg_{yy}(y+2x)$$

$$u_{xy} = 2f_{xy}(y+2x) + 2\cancel{x}g_{xy}(y+2x) + g_y(y\pm 2x)$$

thus

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0$$

1. Solve $u_{tt} = c^2 u_{xx}$, $u(x,0) = e^x$, $u_t(x,0) = \sin x$.

The problem gives
$$\left\{\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u(x,0) &= e^x = \phi(x) \\ u_t(x,0) &= \sin x = \psi(x) \end{aligned}\right\}$$

let's use D'Alembert's formula

$$u(x,t) = \frac{1}{2}\left[e^{x+ct} + e^{x-ct}\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(s)\,ds$$

$$= \frac{e^x}{2}\left(e^{ct} + e^{-ct}\right) - \frac{1}{2c}\left(\cos(x+ct) - \cos(x-ct)\right)$$

$$u(x,t) = e^x \cosh(ct) + \frac{1}{c}\sin x \sin ct$$

↑

we used trigonometric identity to simplify the expression.

3. The midpoint of a piano string of tension T, density ρ, and length l is hit by a hammer whose head diameter is $2a$. A flea is sitting at a distance $l/4$ from one end. (Assume that $a < l/4$; otherwise, poor flea!) How long does it take for the disturbance to reach the flea?

The speed of the wave is given by

$$c^2 = \frac{T}{\rho}$$

~~B~~ it is constant hence the distance is given by

$$x = vt$$

The wave that is going to reach the flea is the one starting in $l/2 + a$ traveling to the right thus the distance is

$$x = \frac{3l}{4} - \left(\frac{l}{4} + a\right) = \frac{l}{4} - a$$

Moreover, the wave speed is

$$c = \sqrt{\frac{T}{\rho}}$$

So we have everything to know when the disturbance will reach the flea

$$t = \frac{x}{v} = \frac{\frac{l}{4} - a}{\sqrt{\frac{T}{\rho}}}, \text{ so } t = \left(\frac{l}{4} - a\right)\sqrt{\frac{\rho}{T}}$$

(c) find solution for v if the initial conditions are

$$v(r,0) = ru(r,0) = r\phi(r)$$
$$v_t(r,0) = ru_t(r,0) = r\psi(r)$$
$$v(r,t) = \frac{1}{2}\left((r+ct)\phi(r+ct) + (r-ct)\phi(r-ct)\right) + \frac{1}{2}\int_{r-ct}^{r-ct} s\psi(s)\,ds$$

expressing u using v

$$u(r,t) = \frac{1}{2r}\left((r+ct)\phi(r+ct) + (r-ct)\phi(r-ct)\right) + \frac{1}{2rc}\int_{r-ct}^{r-ct} s\psi(s)\,ds$$

8. A *spherical wave* is a solution of the three-dimensional wave equation of the form $u(r, t)$, where r is the distance to the origin (the spherical coordinate). The wave equation takes the form

$$u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right) \quad \text{("spherical wave equation").}$$

(a) Change variables $v = ru$ to get the equation for v: $v_{tt} = c^2 v_{rr}$.
(b) Solve for v using (3) and thereby solve the spherical wave equation.
(c) Use (8) to solve it with initial conditions $u(r, 0) = \phi(r)$, $u_t(r, 0) = \psi(r)$, taking both $\phi(r)$ and $\psi(r)$ to be even functions of r.

(a)

Let function v be defined as

$$v = ru$$

It's easy to find the expression

$$v_{tt} = r u_{tt}$$

$$v_{rr} = (v_r)_r = (u + ru_r)_r = 2(u_r) + r u_{rr}$$

combine the results to find the solution

$$v_{tt} = r u_{tt} = r c^2\left(u_{rr} + \frac{2}{r}u_r\right) = c^2(r u_{rr} + 2u_r)$$

$\boxed{v_{tt} = c^2 v_{rr}}$ final solution

(b)

From (3) solution

is $v(r,t) = f(r+ct) + g(r-ct)$

for arbitrary functions f & g.

. Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0$, $u(x,0) = x^2$, $u_t(x,0) = e^x$. (*Hint*: Factor the operator as we did for the wave equation.)

We are given equation $u(x,y) = \begin{cases} u_{xx} - 3u_{xt} - 4u_{tt} = 0 \\ u(x,0) = x^2 \\ u_t(x,0) = e^x \end{cases}$

The factored operator is:

$$(\partial_x + \partial_t)(\partial_x - 4\partial_t)u = 0$$

$$v = u_x - 4u_t$$

(once we factor the operator we proceed in this fashion)

after applying the first factored operator we get new equation:

$$v_x + v_t = 0$$

we know the equation general solution is:

$$v = h(x-t)$$

from here we have

$v = u_x - 4u_t = h(x-t)$

this is a transport equation

it's a non-homongeneous equation with 1 partial derivative, so it can be solved as for each variable

$u(x,t) = f(x-t) + g(4x-t)$

Next we take care of the initial conditions, so

$f(x) + g(4x) = x^2$

$-f'(x) + g'(4x) = e^x$

from above it follows

$f'(x) = \frac{2x}{5} - \frac{4e^x}{5}$ → $f(x) = \frac{x^2}{5} - \frac{4e^x}{5}$ → $g'(s) = \frac{s}{10} + \frac{e^{\frac{s}{4}}}{5}$

$g(s) = \frac{s^2}{20} + \frac{4e^{\frac{s}{4}}}{5}$

$u(x,t) = \frac{(x-t)^2}{5} - \frac{4e^{x-t}}{5} + \frac{(4x+t)^2}{20} + \frac{4e^{\frac{4x+t}{4}}}{5}$

(simplified)

$\frac{4}{5}\left[e^{\frac{x+t}{4}} - e^{x-t}\right] + x^2 + \frac{t^2}{4}$

6. Prove that, among all possible dimensions, only in three dimensions can one have distortionless spherical wave propagation with attenuation. This means the following. A spherical wave in n-dimensional space satisfies the PDE

$$u_{tt} = c^2 \left(u_{rr} + \frac{n-1}{r} u_r \right),$$

where r is the spherical coordinate. Consider such a wave that has the special form $u(r, t) = \alpha(r) f(t - \beta(r))$, where $\alpha(r)$ is called the

attenuation and $\beta(r)$ the delay. The question is whether such solutions exist for "arbitrary" functions f.

(a) Plug the special form into the PDE to get an ODE for f.
(b) Set the coefficients of f'', f', and f equal to zero.
(c) Solve the ODEs to see that $n = 1$ or $n = 3$ (unless $u \equiv 0$).
(d) If $n = 1$, show that $\alpha(r)$ is a constant (so that "there is no attenuation").

(T. Morley, *American Mathematical Monthly*, Vol. 27, pp. 69–71, 1985)

(a) find the following expressions

$u_t = \alpha f'$

$u_{tt} = \alpha f''$

$u_r = \alpha' f - \alpha \beta' f'$

$u_{rr} = (u_r)_r = \alpha'' f - 2\alpha'\beta' f - \alpha\beta'' f' + \alpha(\beta')^2 f''$

substitute expressions into this given EQ: $\alpha f'' = u_{tt}$

$= c^2(\alpha'' f - 2\alpha'\beta' f - \alpha\beta'' f' + \alpha(\beta')^2 f'' + \frac{n-1}{r}(\alpha' f - \alpha\beta' f'))$

outcome

(5) $0 = c^2(\alpha'' + \frac{n-1}{r}\alpha') f - c^2(2\alpha'\beta' + \alpha\beta'' + \frac{n-1}{r}\alpha\beta') f' + \alpha(c^2\beta'^2 - 1) f''$

(b) set coefficient from (a) $= 0$

$c^2(\alpha'' + \frac{n-1}{r}\alpha') = 0$

$-c^2(2\alpha'\beta' + \alpha\beta'' + \frac{n-1}{r}\alpha\beta') = 0$

$\alpha(c^2\beta'^2 - 1) = 0$

(c) first solve (1), it is reduced to

$a'(r) = a(r)$

$\alpha'' + \frac{n-1}{r}\alpha' = 0$

$\frac{a'}{a} = \frac{1-n}{r}$

$\alpha' = a = c_1 r^{1-n}$

$\alpha = \frac{c_1}{2-n} r^{2-n} + c_2$

$\beta' = \pm\frac{1}{c} \Rightarrow \beta = \pm\frac{1}{c} r + c_3$

$0 = (2\alpha' + \frac{n-1}{r}\alpha)\beta' + \alpha\beta''$

$= (2c_1 r^{1-n} + \frac{c_1(n-1)}{2-n} r^{1-n} + \frac{c_2(n-1)}{r}) \frac{1}{c} \neq 0$

$2c_1 r^{1-n} + \frac{c_1(n-1)}{2-n} r^{1-n} + \frac{c_2(n-1)}{r}$

$\frac{3-n}{2-n} c_1 r^{1-n} + \frac{c_2(n-1)}{r}$

(d) If $n=1$ then in (5)

$\frac{3-n}{2-n} c_1 = 0 \Rightarrow c_1 = 0$

thus $\alpha(r) = c_2$ is a contant

4. Here is a direct relationship between the wave and diffusion equations. Let $u(x, t)$ solve the wave equation on the whole line with bounded second derivatives. Let

$$v(x,t) = \frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-s^2c^2/4kt} u(x,s)\, ds.$$

(a) Show that $v(x, t)$ solves the diffusion equation!
(b) Show that $\lim_{t\to 0} v(x,t) = u(x, 0)$.

(*Hint:* (a) Write the formula as $v(x,t) = \int_{-\infty}^{\infty} H(s,t)u(x,s)\, ds$, where $H(x, t)$ solves the diffusion equation with constant k/c^2 for $t > 0$. Then differentiate $v(x, t)$ using Section A.3. (b) Use the fact that $H(s, t)$ is essentially the source function of the diffusion equation with the spatial variable s.)

(a) $H(x,t) = \frac{c}{\sqrt{4\pi kt}} e^{-\left(\frac{x^2c^2}{4kt}\right)}$ (step 1)

$H_t = \frac{c}{\sqrt{4\pi k}}\left(-\frac{1}{2t\sqrt{t}} e^{-\left(\frac{x^2c^2}{4kt}\right)} + \frac{x^2c^2}{4kt^2\sqrt{t}} e^{-\left(\frac{x^2c^2}{4kt}\right)}\right)$

$= \frac{c}{4t\sqrt{\pi t k}} e^{-\left(\frac{x^2c^2}{4kt}\right)}\left(-1 + \frac{x^2c^2}{2kt}\right)$

$H_x = -\frac{c^3 x}{4kt\sqrt{\pi kt}} e^{-\left(\frac{x^2c^2}{4kt}\right)}$

$H_{xx} = \frac{c^3}{4kt\sqrt{\pi kt}} e^{-\left(\frac{x^2c^2}{4kt}\right)}\left(-1 + \frac{x^2c^2}{2kt}\right)$

So, $H_t = \frac{k}{c^2} H_{xx}$

H solves the diffusion equation with constant $\frac{k}{c^2}$

(step 2) Now prove $v(x,t) = \int_{-\infty}^{\infty} H(s,t)u(x,s)ds$

$v_t = \int_{-\infty}^{\infty} H_t(s,t)u(x,s)ds$

$= \frac{k}{c^2}\int_{-\infty}^{\infty} H_{xx}(s,t)u(x,s)ds$

$= -\frac{k}{c^2}\left(H_x(s,t)u(s,t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H_x(s,t)u(x,s)ds\right)$ Partial integration

$= \frac{k}{c^2}\int_{-\infty}^{\infty} H_x(s,t)u_x(x,s)ds$

$= -\frac{k}{c^2}\left(H(s,t)u_x(x,s)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H(s,t)u_{xx}(x,s)ds\right)$

$= \frac{k}{c^2}\int_{-\infty}^{\infty} H(s,t)u_{xx}(x,s)ds$

$v_{xx} = \int_{-\infty}^{\infty} H(s,t)u_{xx}(x,s)ds$

thus, $v_t = \frac{k}{c^2}v_{xx}$

(step 3) $H(x,0) = \delta(x)$,

where δ is the Dirac function.

$\lim_{t\to 0} v(x,t) = \lim_{t\to 0}\int_{-\infty}^{\infty} H(s,t)u(x,s)ds$

$= \int_{-\infty}^{\infty} \lim_{t\to 0} H(s,t)u(x,s)ds$

$= \int_{-\infty}^{\infty}\left[\lim_{t\to 0} H(s,t)\right]u(x,s)ds$

$= \int_{-\infty}^{\infty} \delta(s)u(x,s)ds$

$= u(x,0)$

1. Solve $u_t = k u_{xx}$; $u(x,0) = e^{-x}$; $u(0,t) = 0$ on the half-line $0 < x < \infty$.

from the diffusion equation we see that
$u(x,0) = \phi(x)$ for $t=0$
we see that

$$\phi(x) = e^{-x}$$

now we find the solution to the diffusion equation $u(x,t)$

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_0^\infty \left(e^{-\left(\frac{(x-y)^2}{4kt}\right)} - e^{-\left(\frac{(x+y)^2}{4kt}\right)} \right) \phi(y)\, dy$$

$$= \frac{1}{\sqrt{4\pi k t}} \int_0^\infty \left(e^{-\left(\frac{(x-y)^2}{4kt}\right)} - e^{-\left(\frac{(x+y)^2}{4kt}\right)} \right) e^{-y}\, dy$$

$$u(x,y) = \frac{1}{\sqrt{4\pi k t}} \int_0^\infty e^{-\left(\frac{(x-y)^2}{4kt}\right) - y} - \frac{1}{\sqrt{4\pi k t}} \int_0^\infty e^{-\left(\frac{(x+y)^2}{4kt}\right) - y}\, dy$$

$$= \frac{1}{\sqrt{4\pi k t}} \int_0^\infty e^{\left(\frac{4kt(kt-x)}{4kt}\right)} e^{-\left(\frac{y + (2kt - x)}{\sqrt{4kt}}\right)^2} dy$$

$$= e^{kt-x} \cdot \frac{1}{\sqrt{4\pi k t}} \int_0^\infty e^{-\left(\frac{y + (2kt - x)}{\sqrt{4kt}}\right)^2} dy$$

now we need the following substitution

(Now put the integral in terms of y)

$$z = \frac{y + (2kt - x)}{\sqrt{4kt}} \quad \text{then} \quad dz = \frac{1}{\sqrt{4kt}}\, dy$$

$$dy = \sqrt{4kt}\, dz$$

find the lower limit of z (make $y=0$)

$$z = \frac{2kt - x}{\sqrt{4kt}}$$

now find upper limit of y

$$\frac{1}{\sqrt{4\pi k t}} \int_0^\infty e^{-\frac{(x-y)^2}{4kt} - y}\, dy = e^{kt-x} \frac{1}{\sqrt{4\pi k t}} \int_z^\infty e^{-z^2} \sqrt{4kt}\, dz$$

$$= e^{kt-x} \frac{1}{\sqrt{\pi}} \int_z^\infty e^{-z^2}\, dz$$

$$\frac{1}{\sqrt{4\pi k t}} \int_0^\infty e^{-\left(\frac{(x-y)^2}{4kt}\right) - y}\, dy$$

$$= \frac{e^{kt-x}}{\sqrt{\pi}} \int_z^\infty e^{-y^2}\, dy$$

$$= \frac{e^{kt-x}}{\sqrt{\pi}} \left(\int_0^\infty e^{-y^2} dy - \int_0^z e^{-y^2} dy \right)$$

$$= \frac{e^{kt-x}}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \operatorname{Erf}\left(\frac{2kt - x}{\sqrt{4kt}} \right) \right)$$

$$= \frac{e^{kt-x}}{\sqrt{\pi}} \left(1 - \operatorname{Erf}\left(\frac{2kt - x}{\sqrt{4kt}} \right) \right)$$

thus,
$$u(x,t) = \frac{e^{kt-x}}{2} \left(\left(1 - \operatorname{Erf}\left(\frac{2kt - x}{\sqrt{4kt}} \right) \right) - \left(1 - \operatorname{Erf}\left(\frac{2kt + x}{\sqrt{4kt}} \right) \right) \right)$$

4. Consider the following problem with a Robin boundary condition:

$$\begin{array}{lll} \text{DE:} & u_t = ku_{xx} & \text{on the half-line } 0 < x < \infty \\ & & (\text{and } 0 < t < \infty) \\ \text{IC:} & u(x,0) = x & \text{for } t = 0 \text{ and } 0 < x < \infty \\ \text{BC:} & u_x(0,t) - 2u(0,t) = 0 & \text{for } x = 0. \end{array} \qquad (*)$$

The purpose of this exercise is to verify the solution formula for (*). Let $f(x) = x$ for $x > 0$, let $f(x) = x + 1 - e^{2x}$ for $x < 0$, and let

$$v(x,\ t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\,dy.$$

(a) What PDE and initial condition does $v(x,t)$ satisfy for $-\infty < x < \infty$?

(b) Let $w = v_x - 2v$. What PDE and initial condition does $w(x,t)$ satisfy for $-\infty < x < \infty$?

(c) Show that $f'(x) - 2f(x)$ is an odd function (for $x \neq 0$).

(d) Use Exercise 2.4.11 to show that w is an odd function of x.

(e) Deduce that $v(x,\ t)$ satisfies (*) for $x > 0$. Assuming uniqueness, deduce that the solution of (*) is given by

$$u(x,\ t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\,dy.$$

(a) $u_t = k u_{xx}$ for $0<x<\infty,\ 0<t<\infty$

$u(x,0) = x$ for $0<x<\infty$

$u_x(0,t) - 2u(0,t) = 0$

$f(x) = \begin{cases} x & \text{for } x> \\ x+1-e^{2x} & \text{for } x<0 \end{cases}$

$V(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} f(y)\,dy$

(b)

$w = V_x - 2V$

$u_t = k u_{xx}$

$u_x(0,t) - 2u(0,t) = 0$

$w(x,t)$ for $-\infty<x<\infty$

$w_t = k w_{xx}$

$w(x,0) = f'(x) - 2f(x)$

$$u(x,t) = u(-x,t)$$

e) $$= -u(x,t)$$

since this is true

$$u(0,t)=0$$
$$u(0,t)=0$$

thus u, v satisfy exactly the same pde for $\forall x$ & initial conditions at $x>0$. So we can determine $u(x,t)$ solution

c) $f'(x) - 2f(x) = h(x)$

sub in: $x = -x$

$$h(-x) = f'(-x) - 2f(-x)$$

note for an odd function

$$f(-x) = -f(x)$$
$$h(-x) = f'(-x) - 2f(-x)$$
$$-h(x) = -f'(x) + 2f(x)$$
$$= -[f'(x) - 2f(x)]$$
$$h(x) = f'(x) - 2f(x)$$

so the function is odd

d) now we're trying to prove $w = v_x - 2v$

substitute $x = -x$

$$w(x,t) = v_x - 2v$$
$$w(-x) = -(v_x - 2v)$$
$$w(-x) = -w(x)$$

which is an odd function of x

2. The longitudinal vibrations of a semi-infinite flexible rod satisfy the wave equation $u_{tt} = c^2 u_{xx}$ for $x > 0$. Assume that the end $x = 0$ is free ($u_x = 0$); it is initially at rest but has a constant initial velocity V for $a < x < 2a$ and has zero initial velocity elsewhere. Plot u versus x at the times $t = 0, a/c, 3a/2c, 2a/c$, and $3a/c$.

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{external}(s)\,ds$$

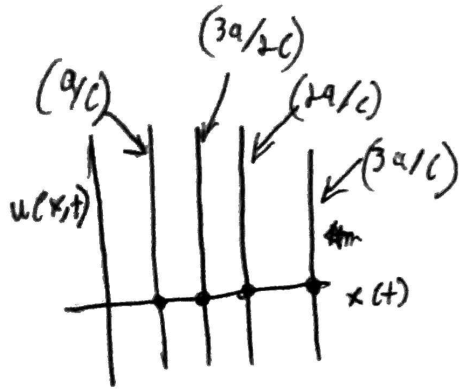
$$\psi_{external}(s) = V\begin{Bmatrix} a<s<2a \\ -2a<s<-a \end{Bmatrix}$$

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{external}(s)\,ds$$

$$= \frac{1}{2c}\int_{x-ct}^{x+ct} V\,ds$$

$$= \frac{V}{2c}\left[x + ct - x + ct\right]$$

$$u(x,t) = Vt$$



when $t=0$ we use $u(x,t) = x(t)\,y(t)$ & $V = a = c = 1 \Rightarrow \frac{Va}{c} = 1$

thus $u(x,t) = Vt$

where $x(t) = 1$ & $y(t) = t$

now when $t = 0$ $\quad u(x,t) = V(0) = 0$

$u(x,t) = 0$

when $t = \frac{3a}{c}$

$= V\frac{3a}{c}$

$u(t,x) = \frac{3Va}{c}$

when $t = \frac{a}{c}$

$u(x,t) = \frac{a}{c}t$ where $x(t) = 1$ & $y(t) = t$

$u(x,t) = Vt$

3 seperate lines

$= V(\frac{a}{c})$

$u(x,t) = \frac{Va}{c}$

when $t = \frac{2a}{c}$

$= V(\frac{2a}{c})$

$u(x,t) = \frac{2Va}{c}$

when $t = \frac{3a}{2c}$

$= V\frac{3a}{2c}$

$u(x,t) = \frac{3Va}{2c}$

~~Solve $u_{tt} = 9u_{xx}$ in $0 < x < \pi/2$, $u(x,0) = \cos x$, $u_t(x,0) = 0$, $u_x(0,t) = 0$, $u(\pi/2,t) = 0$.~~

If we're given
$$\begin{cases} u_{tt} = v_{xx} & -\infty < x < \infty,\ 0 < t \\ v(x,0) = \phi_{ext}(x) \\ u_t(x,0) = \psi_{ext}(x) \end{cases}$$

$$\phi_{ext}(x) = \cos(x),\ \psi_{ext}(x) = 0$$

v is given by d'Alembert's formula

$$v(x,t) = \frac{1}{2}\left(\phi_{ext}(x+3t) + \phi_{ext}(x-3t)\right) + \frac{1}{2\cdot 3}\int_{x-3t}^{x+3t} \psi_{ext}(s)\,ds$$

$$= \frac{1}{2}\left(\cos(x+3t) + \cos(x-3t)\right) + 0$$

$$= \frac{1}{2}\left(\cos x \cos 3t - \sin x \sin 3t + \cos x \cos 3t + \sin x \sin 3t\right)$$

$$= \frac{1}{2}\left(\cos x \cos 3t + \cos x \cos 3t\right)$$

$$= \cos x \cos 3t$$

$$u(x,t) = \cos x \cos 3t,\quad 0 < x < \frac{\pi}{2}$$

2. **Solve the completely inhomogeneous diffusion problem on the half-line**

$$v_t - kv_{xx} = f(x,t) \quad \text{for } 0 < x < \infty, \quad 0 < t < \infty$$
$$v(0,t) = h(t) \quad v(x,0) = \phi(x),$$

15 step problem

$$w(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\left(\frac{(x-y)^2}{4kt}\right)} \psi_{odd}(y)\,dy$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{0} e^{-\left(\frac{(x-y)^2}{4kt}\right)} \psi_{odd}(y)\,dy + \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty}$$

$$\vdots$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left(e^{-\left(\frac{(x-y)^2}{4kt}\right)} - e^{-\left(\frac{(x+y)^2}{4kt}\right)}\right) \psi(y)\,dy$$

use Duhamel's principle

$$P(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)} g_{odd}(y,s)\,dy\,ds$$

$$\int_0^t \int_{-\infty}^{0} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\left(\frac{(x+y)^2}{4k(t-s)}\right)} g_{odd}(y,s)\,dy\,ds + \int_0^t \int_0^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)} g_{odd}(y,s)\,dy\,ds$$

$$\boxed{\int_0^t \int_0^{\infty} \left(e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)} - e^{-\left(\frac{(x+y)^2}{4k(t-s)}\right)}\right) \frac{1}{\sqrt{4\pi k(t-s)}} g(y,s)\,dy\,ds}$$

$v(x,t)$

$$\boxed{= h(t) + \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left(e^{-} - e^{-}\right) \phi(y)\,dy}$$

$$\boxed{+ \int_0^t \int_0^{\infty} \left(e^{-} - e^{-}\right) \ldots (y,s)\,dy\,ds}$$

$$\boxed{+ \int_0^t \int_0^{\infty} \frac{1}{ds} \left(e^{-} - e^{-}\right) \ldots\,dy\,ds}$$

us v as the sum of w & p

$$V(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left(e^{-\left(\frac{(x-y)^2}{4kt}\right)} - e^{-\left(\frac{(x+y)^2}{4kt}\right)}\right) (\phi(y) - h(0))$$
$$+ \int_0^t \int_0^{\infty} e^{-\left(\frac{(x+y)^2}{4k(t-s)}\right)} \frac{1}{\sqrt{4\pi k(t-s)}} (f(y,s) - h'(s))\,dy\,ds$$

partial integration ....

$$w(x,t)=\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\psi_{odd}(y)dy$$

$$=\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{0} e^{-\left(\frac{(x-y)^2}{4kt}\right)}\psi_{odd}(y)dy+\frac{1}{\sqrt{4\pi kt}}\int_{0}^{\infty} e^{-\left(\frac{(x-y)^2}{4kt}\right)}\psi_{odd}(y)dy$$

$$=\frac{1}{\sqrt{4\pi kt}}\int_{0}^{\infty} e^{-\left(\frac{(x+y)^2}{4kt}\right)}\psi_{odd}(-y)+\frac{1}{\sqrt{4\pi kt}}\int_{0}^{\infty} e^{-\left(\frac{(x-y)^2}{4kt}\right)}\psi_{odd}(y)dy$$

$$=-\frac{1}{\sqrt{4\pi kt}}\int_{0}^{\infty} e^{-\left(\frac{(x+y)^2}{4kt}\right)}\psi(y)dy+\frac{1}{\sqrt{4\pi kt}}\int_{0}^{\infty} e^{-\left(\frac{(x-y)^2}{4kt}\right)}\psi(y)dy$$

$$=\frac{1}{\sqrt{4\pi kt}}\int_{0}^{\infty}\left(e^{-\left(\frac{(x-y)^2}{4kt}\right)}-e^{-\left(\frac{(x+y)^2}{4kt}\right)}\right)\psi(y)dy$$

Now find ℓ solution using Duhamel's principle

$$\ell(x,t)=\int_0^t\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi k(t-s)}}e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)}g_{odd}(y,s)dyds$$

$$=\int_0^t\int_{-\infty}^{0}\frac{1}{\sqrt{4\pi k(t-s)}}e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)}g_{odd}(y,s)dyds+\int_0^t\int_0^{\infty}\frac{1}{\sqrt{4\pi k(t-s)}}e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)}g_{odd}(y,s)dyds$$

$$=\int_0^t\int_0^{\infty}\frac{1}{\sqrt{4\pi k(t-s)}}e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)}g_{odd}(-y,s)dyds+\int_0^t\int_0^{\infty}\frac{1}{\sqrt{4\pi k(t-s)}}e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)}g_{odd}(y,s)dyds$$

$$=-\int_0^t\int_0^{\infty}\frac{1}{\sqrt{4\pi k(t-s)}}e^{-\left(\frac{(x+y)^2}{4k(t-s)}\right)}g(y,s)dyds+\int_0^t\int_0^{\infty}\frac{1}{\sqrt{4\pi k(t-s)}}e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)}g(y,s)dyds$$

$$=\int_0^t\int_0^{\infty}\left(e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)}-e^{-\left(\frac{(x+y)^2}{4k(t-s)}\right)}\right)\frac{1}{\sqrt{4\pi k(t-s)}}g(y,s)dyds$$

Now that we have our v as the sum of w & ℓ the solution for V is restricted

$$V(x,t)=\frac{1}{\sqrt{4\pi kt}}\int_0^{\infty}\left(e^{-\left(\frac{(x-y)^2}{4kt}\right)}-e^{-\left(\frac{(x+y)^2}{4kt}\right)}\right)(\phi(y)-h(0))dy$$
$$+\int_0^t\int_0^{\infty}\left(e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)}-e^{-\left(\frac{(x+y)^2}{4k(t-s)}\right)}\right)\frac{1}{\sqrt{4\pi k(t-s)}}(f(y,s)-h'(s))dyds=\underline{\text{Partial Integration}}$$

Final Answer: $V(x,t)=H(x)+\frac{1}{\sqrt{4\pi kt}}\int_0^{\infty}\left(e^{-\left(\frac{(x-y)^2}{4kt}\right)}-e^{-\left(\frac{(x+y)^2}{4kt}\right)}\right)\phi(y)dy$

$$\cancel{-\frac{1}{\sqrt{4\pi kt}}\int_0^{\infty}\left(e^{-\left(\frac{(x-y)^2}{4kt}\right)}-e^{-\left(\frac{(x+y)^2}{4kt}\right)}\right)}h(0)dy+\int_0^t\int_0^{\infty}\left(e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)}-e^{-\left(\frac{(x+y)^2}{4k(t-s)}\right)}\right)\frac{1}{\sqrt{4\pi k(t-s)}}f(y,s)dyds$$

$$\cancel{\frac{1}{\sqrt{4\pi kt}}\int_0^{\infty}\left(e^{-\left(\frac{(x-y)^2}{4kt}\right)}-e^{-\left(\frac{(x+y)^2}{4kt}\right)}\right)}h(0)dy+\int_0^t\int_0^{\infty}\frac{d}{ds}\left(\left(e^{-\left(\frac{(x-y)^2}{4k(t-s)}\right)}-e^{-\left(\frac{(x+y)^2}{4k(t-s)}\right)}\right)\frac{1}{\sqrt{4\pi k(t-s)}}\right)h(s)dyds$$

Done!!!

3. Solve the inhomogeneous Neumann diffusion problem on the half-line

$$\begin{aligned} &w_t - kw_{xx} = 0 && \text{for } 0 < x < \infty, \quad 0 < t < \infty \\ &w_x(0,t) = h(t) && w(x,0) = \phi(x), \end{aligned}$$

by the subtraction method indicated in the text.

Same idea as last

Solve $u_{tt} = c^2 u_{xx} + \cos x, \quad u(x,0) = \sin x, \quad u_t(x,0) = 1 + x.$

Since $u(x,0) = \sin x$ then $\phi(x) = \sin(x)$

$$\frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] = \frac{1}{2}\left[\sin(x+ct) + \sin(x-ct)\right] = \underline{\sin(x)\cos(ct)}$$

since $u_t(x,0) = 1 + x$ then $\psi(x) = 1 + x$

$$\frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x) = \frac{1}{2c}\int_{x-ct}^{x+ct} (1+z)\,dz$$

$$= \frac{1}{2c}\left[z + \frac{z^2}{2}\right]_{x-ct}^{x+ct} = \frac{1}{2c}(x+ct-x+ct) + \frac{1}{4c}\left[(x+ct)^2 - (x-ct)^2\right]$$

$$= \frac{1}{2c}(2ct) + \frac{1}{4c}(4xct)$$

$$= \underline{\underline{t + xt}} = (1+x)t$$

the solution is over a characteristic triangle so,

$$\frac{1}{2c}\iint_{\Delta} f = \frac{1}{2c}\int_0^{t_0}\int_{x-c(t_0-t)}^{x+c(t_0-t)} \cos(x)\,dx\,dt$$

$$= \frac{1}{2c}\int_0^{t_0}\left[-\sin(x)\right]_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} dt$$

$$= \frac{1}{2c}\int_0^{t_0} \sin(x_0\, c(t_c-t)) - \sin(x_0 + c(t_0-t))\,dt$$

$$= \frac{1}{2c}\int_0^{t_0} 2\cos(x_0)\sin(c(t_0-t))\,dt$$

$$= \frac{\cos(x_0)}{c}\int_0^{t_0} \sin(c(t_0-t))\,dt = \frac{\cos(x_0)}{c^2}\left[\cos(c(t_0-t))\right]_0^{t_0}$$

$$= \underline{\underline{\frac{1}{c^2}(1-\cos(ct_0))\cos(x_0)}}$$

$$u(x,t) = \sin(x)\cos(ct) + (1+x)t + \frac{1}{c^2}(1-\cos(ct_0))\cos(x_0)$$

Done!!!

1. Prove that if ϕ is any piecewise continuous function, then

$$\frac{1}{\sqrt{4\pi}}\int_0^{\pm\infty} e^{-p^2/4}\phi(x+\sqrt{kt}\,p)\,dp \to \pm\frac{1}{2}\phi(x\pm) \quad \text{as } t \searrow 0.$$

If $\phi(x)$ is any piecewise function

By use $U_{tt} = kU_{xx}$ $\quad -\infty < x < \infty,\ 0 < t < \infty$

we see that

$$\lim_{t \to 0} u(x,t) = \phi(x)$$

Since the integral of the solution is a solution as well

if we let $S(x,t)$ a solution then $S(x-y,t)$ is a solution as well.

this means $\quad u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\,\phi(y)\,dy\ ,\ t>0$

The solution of equation 1 is the convolution of ϕ with S means

$$x - y = z$$

Putting it into equation 2 means

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\,\phi(y)\,dy = \int_{-\infty}^{\infty} S(z,t)\,\phi(x-z)\,dz$$

now since

$$S = \frac{dQ}{dx} = \frac{1}{\sqrt{4\pi kt}}\, e^{\frac{-x^2}{4kt}}\ ,\ t>0$$

this means that

$$S(z,t) = \frac{1}{\sqrt{4\pi kt}}\, e^{-\left(\frac{x^2}{4kt}\right)}$$

now we substitute $p = \frac{z}{\sqrt{kt}}$

$$u(x,t) = \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} e^{\frac{-x^2}{4}}\,\phi(x - p\sqrt{kt})\,dp$$

since $\phi(x)$ is a piecewise continuous function

$$\lim_{t \to 0} u(x,t) = \pm\frac{1}{2}\phi(x\pm)$$

1. (a) Use the Fourier expansion to explain why the note produced by a violin string rises sharply by one octave when the string is clamped exactly at its midpoint.

(b) Explain why the note rises when the string is tightened.

(a) The Fourier expansion of the solution to the wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx} & 0 < x < l \\ u(0,t) = u(l,t) = 0 \end{cases}$$

given the formula: $u(x,t) = \sum_n \left(A_n \cos \frac{n\pi c t}{l} + B_n \sin \frac{n\pi c t}{l} \right) \sin \frac{n\pi x}{l}$

(the coefficient of t give the frequency of the violin note)

$$\frac{n\pi c}{l} = \boxed{\frac{n\pi}{l}\sqrt{\frac{T}{\rho}}}, \quad n = 1, 2, \ldots$$

In this case

$$c^2 = \frac{T}{\rho}$$

if the string gets clamped at midpoint $\Rightarrow$ the length gets cut in half, then the frequency is given by

$$2\frac{n\pi}{l}\sqrt{\frac{T}{\rho}}, \quad n = 1, 2, \ldots$$

(b) As the string is tightened the pressure rises. Furthermore, the frequency is proportional to the square root of tension, therefore the not rises.

3. A quantum-mechanical particle on the line with an infinite potential outside the interval $(0, l)$ ("particle in a box") is given by Schrödinger's equation $u_t = iu_{xx}$ on $(0, l)$ with Dirichlet conditions at the ends. Separate the variables and use (8) to find its representation as a series.

The given PDE is schrodinger equatron with
Dirichlet conditions at the end

$$\begin{cases} u_t = iu_{xx}, & 0<x<l \\ u(0,t)=u(l,t)=0 \end{cases}$$

Solve it by separating varıables
$u(x,t)=T(t)X(x)$ & get

$$\frac{T'}{iT}=\frac{X''}{X}=-\lambda$$

first solve

$$\begin{cases} X''+\lambda X=0 \\ X(0)=X(l)=0 \end{cases}$$

&

$$T'=-\lambda i T$$

for all natural numbers we have

$$\lambda_n=\left(\frac{n\pi}{l}\right)^2 \Rightarrow \begin{cases} X_n(x)=\sin\frac{\pi n x}{l} \\ T_n(t)=e^{\left(\frac{n\pi}{l}\right)^2 it} \end{cases}$$

The Solution is

$$u(x,t)=\sum_{n=1}^{\infty} A_n e^{\left(\frac{n\pi}{l}\right)^2} \sin\left(\frac{n\pi x}{l}\right)$$

1. **Solve the diffusion problem** $u_t = ku_{xx}$ in $0 < x < l$, **with the mixed boundary conditions** $u(0,t) = u_x(l,t) = 0$.

Solve

$$\begin{cases} U_t = kU_{xx} & 0<x<l \\ U(0,t) = U_x(l,t) = 0 \end{cases}$$

Solve ~~?~~ separate variables

$$u(x,t) = T(t)X(x)$$

~~?~~ get

$$\frac{T'(t)}{kT} = \frac{X''}{X} = -\lambda^2$$

First solve

$$\begin{cases} X'' + \lambda^2 X = 0 \\ X(0) = X'(l) = 0 \end{cases}$$

$$\frac{T'}{kT} = -\lambda^2$$

$X(x) = A\sin\lambda x + B\cos\lambda x \Rightarrow B=0 \Rightarrow \cos l\lambda = 0$

thus

~~[illegible]~~

$$X_n(x) = B_n \sin\lambda_n x, \quad n=1,2,\dots$$

$$\lambda_n = \frac{(2n-1)\pi}{2l}$$

$$T_n(t) = C_n e^{-\lambda_n^2 kt}, \quad n=1,2,\dots$$

So the final solution is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} D_n e^{-\frac{(2n-1)^2\pi^2 kt}{4l^2}} \sin\left(\frac{(2n-1)\pi x}{2l}\right)}$$

Consider the equation $u_{tt} = c^2 u_{xx}$ for $0 < x < l$, with the boundary conditions $u_x(0, t) = 0$, $u(l, t) = 0$ (Neumann at the left, Dirichlet at the right).

(a) Show that the eigenfunctions are $\cos[(n + \frac{1}{2})\pi x/l]$.

(b) Write the series expansion for a solution $u(x, t)$.

(a)

Solve

$$\begin{cases} U_{tt} = c^2 U_{xx} & 0 < x < l \\ U_x(0,t) = U(l,t) = 0 \end{cases}$$

$$U(x,t) = T(t)\,X(x)$$

$$\frac{T''(t)}{c^2 T} = \frac{X''}{X} = -\lambda^2$$

First solve $\begin{cases} X'' + \lambda^2 X = 0 \\ X(l) = X'(0) = 0 \end{cases}$

$$\frac{T''}{T} = -(\lambda c)^2$$

$$X(x) = A \sin\lambda x + B\cos\lambda x \Rightarrow A = 0 \Rightarrow \cos l\lambda = 0$$

$$X_n(x) = B_n \cos\lambda_n x,\ n = 1, 2, \ldots$$

$$\lambda_n = \frac{(2n-1)\pi}{2l}$$

So for every natural (n) the eigen functions are

$$\boxed{\cos\left(n - \frac{1}{2}\right)\frac{\pi x}{l}}$$

(b)

For every natural ~~mode~~ the solution is

$$T_n(t) = C_n \sin\frac{(2n-1)c\pi t}{2l} + D_n \cos\frac{(2n-1)c\pi t}{2l},\ n = 1, 2, \ldots$$

The series expansion for this solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left(C_n \sin\left(\frac{(2n-1)c\pi t}{2l}\right) + D_n \cos\left(\frac{(2n-1)c\pi t}{2l}\right)\right)\cos\frac{(2n-1)\pi x}{2l}$$

$\{X(0) = 0 \quad \& \quad X''(l) + \lambda c^2 X(l) = 0\}$

3. Solve the Schrödinger equation $u_t = iku_{xx}$ for real k in the interval $0 < x < l$ with the boundary conditions $u_x(0,t) = 0$, $u(l,t) = 0$.

$$u(x,t) = T(t)\,X(x)$$

$$\frac{T'}{ikT} = \frac{X''}{X} = -\lambda^2$$

$$\begin{cases} X'' + \lambda^2 X = 0 \\ X'(0) = X(l) = 0 \end{cases}$$

$$\frac{T'}{ikT} = -\lambda^2$$

$$X(x) = A\sin\lambda x + B\cos\lambda x \Rightarrow A = 0 \Rightarrow \cos l\lambda = 0$$

$$X_n(x) = B_n \sin\lambda_n x, \quad n = 1, 2, \dots$$

$$\lambda_n = \frac{(2n-1)\pi}{2l}$$

$$T_n(t) = C_n e^{-\lambda_n^2 kit}, \quad n = 1, 2, \dots$$

$$u(x,t) = \sum_{n=1}^{\infty} D_n e^{-\left(\frac{(2n-1)^2\pi^2 kit}{4l^2}\right)} \sin\left(\frac{(2n-1)\pi x}{2l}\right)$$

4. **Consider diffusion inside an enclosed circular tube. Let its length (circumference) be $2l$. Let x denote the arc length parameter where $-l \le x \le l$. Then the concentration of the diffusing substance satisfies**

$$u_t = k u_{xx} \quad \text{for } -l \le x \le l$$

$$u(-l,t) = u(l,t) \quad \text{and} \quad u_x(-l,t) = u_x(l,t).$$

(a) Show that the eigen values are $\lambda = (n\pi/l)^2$ for $n = 0,1,2,3,\dots$

(b) Show that the concentration is

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}\right)e^{-n^2\pi^2 kt/l^2}.$$

(a) $u(x,t) = T(t)X(x)$

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda^2$$

first solve $\begin{cases} X'' + \lambda^2 X = 0 \\ X(l) - X(-l) = X'(l) - X'(-l) = 0 \end{cases}$

$$\frac{T'}{KT} = -\lambda^2$$

$$X(x) = A\sin\lambda x + b\cos\lambda x$$

$$0 = X(l) - X(-l)$$
$$= (A\sin\lambda l + b\cos\lambda l) - (A\sin -\lambda l + b\cos -\lambda l)$$
$$= 2A\sin\lambda l$$

$$0 = X'(l) - X'(-l)$$
$$= (A\lambda\cos\lambda l - B\lambda\cos\lambda l) - (A\lambda\cos -\lambda l - B\lambda\cos -\lambda l)$$
$$= 2B\sin\lambda l$$

thus $\sin\lambda l \Rightarrow \lambda_n = \frac{n\pi}{l}$, $n = 0,1,2,3,\dots$

the eigen values are $\lambda_n^2 = \frac{n^2\pi^2}{l^2}$, $n = 0,1,2,3,\dots$

(b) $T_n(t) = C_n e^{-(\lambda_n^2 kt)}$, $n = 1,2,\dots$

the series expansion for the concentration

$$u(x,t) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} e^{-\left(\frac{n^2\pi^2 kt}{l^2}\right)}\left(A_n \sin\frac{n\pi x}{l} + B_n\cos\frac{n\pi x}{l}\right)$$

4. **Consider the Robin eigenvalue problem. If**

$$a_0 < 0, \quad a_l < 0 \quad \text{and} \quad -a_0 - a_l < a_0 a_l l,$$

show that there are *two* negative eigenvalues. This case may be called "substantial absorption at both ends." (*Hint:* Show that the rational curve $y = -(a_0 + a_l)\gamma/(\gamma^2 + a_0 a_l)$ has a single maximum and crosses the line $y = 1$ in two places. Deduce that it crosses the tanh curve in two places.)

The negative eigenvalues are $\lambda = -\gamma^2$

we have to find coefficients

$$X(x) = C\cosh\gamma x + D\sinh\gamma x$$

$$\begin{cases} X'(0) - a_0 X(0) = 0 \\ X'(l) + a_l X(l) = 0 \end{cases}$$

where γ, satisfies

$$\tanh(\gamma l) = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l}$$

which is an eigenvalue equation for the negative eigenvalues $\lambda = -\gamma^2$

we know $\tanh(\gamma l) \leq 1$

$$y(\gamma) = -\frac{(a_0 + a_1)\gamma}{\gamma^2 + a_0 a_1}$$

(let's show that there is 1 maximum that crosses $y = 1$ at 2 points.)

$$g(\gamma) = 1$$

$$-(a_0 + a_l)\gamma = \gamma^2 + a_0 a_l$$

$$\gamma^2 + (a_0 + a_l)\gamma + a_0 a_l = 0$$

$$\gamma_{1,2} = \frac{-(a_0 + a_1) \pm \sqrt{(a_0 + a_1)^2 - 4a_0 a_1}}{2} = \frac{-(a_0 + a_l) \pm \sqrt{(a_0 - a_1)^2}}{2} = \frac{-(a_0 + a_1) \pm (a_0 - a_1)}{2}$$

$$g''(\sqrt{a_0 a_1}) = \frac{2(a_0 + a_l)\sqrt{a_0 a_l}(3a_0 a_l - a_0 a_l)}{(a_0 a_l + a_0 a_l)^3} = \frac{(a_0 + a_l)}{2\sqrt{a_0 a_l}^3} < 0$$

$$g''(-\sqrt{a_0 a_1}) = \frac{-2(a_0 + a_l)\sqrt{a_0 a_l}(3a_0 a_l - a_0 a_l)}{(a_0 a_l + a_0 a_l)^3} = -\frac{(a_0 + a_l)}{2\sqrt{a_0 a_l}^3} > 0$$

$$g''(\gamma) = \frac{2(a_0 + a_l)\gamma(3a_0 a_l - \gamma^2)}{(\gamma^2 + a_0 a_l)^3}$$

find the max

$$g'(\gamma) = 0$$

$$g'(\gamma) = a_0 a_1 - \gamma^2 = 0$$

So funct g Max point $= \gamma = -\sqrt{a_0 a_l}$ intersecting $y = 1$

$(X(0) = 0 \quad k X'(l) + \lambda c^2 X(l) = 0)$

8. Consider again Robin BCs at both ends for arbitrary a_0 and a_l.

(a) In the a_0a_l plane sketch the hyperbola $a_0 + a_l = -a_0a_ll$. Indicate the asymptotes. For (a_0, a_l) on this hyperbola, zero is an eigenvalue, according to Exercise 2(a).

(b) Show that the hyperbola separates the whole plane into three regions, depending on whether there are two, one, or no negative eigenvalues.

(c) Label the directions of increasing absorption and radiation on each axis. Label the point corresponding to Neumann BCs.

(d) Where in the plane do the Dirichlet BCs belong?

(a) rewrite the equation:

$$\frac{1}{-la_0} + \frac{1}{-la_1} = 1$$

because

$$\lim_{a_l \to \infty} a_0 = \lim_{a_l \to \infty} \frac{1}{-l + \frac{1}{a_l}} = \frac{-1}{l},$$

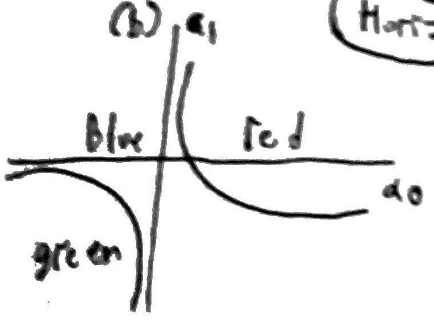
$\boxed{a_0 = \frac{-1}{l}}$ & $\boxed{a_l = \frac{-1}{l}}$

asymtotes

Horizontal

Vertical

(b)



red : $a_0 + a_l > -la_0a_l$

Blue : $a_0 + a_l < -la_0a_l$

Green : $a_0 < \frac{-1}{l}$ & $a_l < \frac{-1}{l}$

(c)

as $a_0 \to +\infty$ or $a_l \to +\infty$ radiation increases

as $a_0 \to -\infty$ or $a_l \to -\infty$ absorption increases

(d) In order to get $X(0) = X(l) = 0$ in

$$\begin{Bmatrix} X(0) = \frac{X'(0)}{a_0} \\ X(l) = \frac{X'(l)}{a_l} \end{Bmatrix}$$

we require $a_0, a_1 \to \pm\infty$ Therefore the Dirichlet boundary conditions are at four corner of a_0-a_l plane.

$\{X(0) = 0 \quad k X'(l) + \lambda c^2 X(l) = 0\}$

$\sqrt{\lambda}\, c^2$

13. Consider a string that is fixed at the end $x = 0$ and is free at the end $x = l$ except that a load (weight) of given mass is attached to the right end.

(a) Show that it satisfies the problem

$$u_{tt} = c^2 u_{xx} \qquad \text{for } 0 < x < l$$

$$u(0,t) = 0 \qquad u_{tt}(l,t) = -k u_x(l,t)$$

for some constant k.

(b) What is the eigenvalue problem in this case?

(c) Find the equation for the positive eigenvalues and find the eigenfunctions.

(a)

Displacement of the string

$U_{tt} = c^2 U_{xx}$ for $0 < x < l$.

at the left end, $x=0$, is fixed at 0 meaning the string doesnt move

$u(0,t) = 0$

~~xxxxxx~~

at the right end $x=l$, it is free, a load of a given mass is attached.

$u_{tt}(l,t) = -k u_x(l,t)$, k is constant

(b) $u(x,t) = X(x)T(t)$

$$\frac{T''}{c^2 T} = -\frac{X''}{X}$$

$X'' = -\lambda X$

$X(0)T(t) = 0 \Rightarrow X(0) = 0$

for all $t > 0$

$T''(t)X(l) = -kT(t)X'(l)$

$$\underbrace{\frac{T''(t)}{T(t)}}_{\lambda} = -k\frac{X'(l)}{X(l)}$$

$$\lambda c^2 = -k\frac{X'(l)}{X(l)}$$

The eigenvalue problem is

$$\begin{cases} X'' = -\lambda X & 0 < x < l \\ X(0) = 0 \quad kX'(l) + \lambda c^2 X(l) = 0 \end{cases}$$

(c) Assume $\lambda > 0$

$\lambda = \beta^2$

$X(x) = A\sin\beta x + B\cos\beta x$

+ has to satisfy the boundary conditions (the first being)

$0 = X(0) = B$

$0 = kX' + \lambda c^2 X = Ak\beta\cos\beta x + A\lambda c^2\sin\beta x$

thus λ being an eigen value must satisfy

$$\tan\beta l = -\frac{k}{\sqrt{\lambda}\, c^2}$$

13. **Consider a string that is fixed at the end $x = 0$ and is free at the end $x = l$ except that a load (weight) of given mass is attached to the right end.**

(a) **Show that it satisfies the problem**

$$u_{tt} = c^2 u_{xx} \qquad \text{for } 0 < x < l$$

$$u(0,t) = 0 \qquad u_{tt}(l,t) = -k u_x(l,t)$$

for some constant k.

(b) **What is the eigenvalue problem in this case?**

(c) **Find the equation for the positive eigenvalues and find the eigenfunctions.**

(a)

Displacement of the string

$U_{tt} = c^2 U_{xx}$ for $0 < x < l$.

at the left end, $x=0$, is fixed at 0 meaning the string doesnt move

$u(0,t) = 0$

~~x=0 ext~~

at the right end $x=l$, it is free, a load of a given mass is attatched.

$u_{tt}(l,t) = -k u_x(l,t)$, k is constant

(b) $u(x,t) = X(x)T(t)$

$$\frac{T''}{c^2 T} = -\frac{X''}{X}$$

$X'' = -\lambda X$

$X(0)T(t) = 0 \Rightarrow X(0) = 0$

for all $t > 0$

$T''(t)X(l) = -kT(t)X'(l)$

$$\underbrace{\frac{T''(t)}{T(t)}}_{s} = -k\frac{X'(l)}{X(l)}$$

$$\lambda c^2 = -k\frac{X'(l)}{X(l)}$$

The eigen value problem is

$$\left\{\begin{array}{ll} X'' = -\lambda X & 0 < x < l \\ X(0) = 0 & kX'(l) + \lambda c^2 X(l) = 0 \end{array}\right\}$$

(c) Assume $\lambda > 0$

$\lambda = \beta^2$

$X(x) = A\sin\beta x + B\cos\beta x$

+ has to satisfy the boundary conditions (the first being)

$0 = X(0) = B$

$0 = kX' + \lambda c^2 X = Ak\beta\cos\beta x + A\lambda c^2 \sin\beta x$

thus λ being an eigen value must satisfy

$$\tan\beta l = -\frac{k}{\sqrt{\lambda}\, c^2}$$

4. Solve the eigenvalue problem $x^2u'' + 3xu' + \lambda u = 0$ for $1 < x < e$, with $u(1) = u(e) = 0$. Assume that $\lambda > 1$. (*Hint:* Look for solutions of the form $u = x^m$.)

assume the solution is form

$$U = x^m$$

plug into equation

$$x^2 \cdot m \cdot (m-1) x^{m-2} + 3x m \cdot x^{m-1} + \lambda x^m = 0$$

this is satisfied for all $x \in (1, e)$

$$m(m-1) + 3m + \lambda = m^2 + 2m + \lambda = 0$$

compute m

$$m = -1 \pm \sqrt{1-\lambda} \overset{\lambda > 1}{=} -1 \pm i\sqrt{\lambda - 1}$$

we can express the solution: $u(x) = A x^{-1} \sin(\sqrt{\lambda-1}\ln x) + B x^{-1} \cos(\sqrt{\lambda - 1}\ln x)$

$$u(1) = 0 \rightarrow u(1) = B = 0$$

$$u(e) = 0 \rightarrow u(e) = A e^{-1} \sin(\sqrt{\lambda - 1}) = 0$$

for u not to be trivial $(A \neq 0)$

$$\sin(\sqrt{\lambda - 1}) = 0 \Leftrightarrow \lambda = k^2\pi^2 + 1$$

for every $k \in \mathbb{N}$

$$\lambda_k = k^2\pi^2 + 1$$

the eigen function is

$$\boxed{u_k(x) = A_k x^{-1} \sin(k\pi \ln x)}$$

18. A tuning fork may be regarded as a pair of vibrating flexible bars with a certain degree of stiffness. Each such bar is clamped at one end and is approximately modeled by the fourth-order PDE $u_{tt} + c^2 u_{xxxx} = 0$. It has initial conditions as for the wave equation. Let's say that on the end $x = 0$ it is clamped (fixed), meaning that it satisfies

$u(0, t) = u_x(0, t) = 0$. On the other end $x = l$ it is free, meaning that it satisfies $u_{xx}(l, t) = u_{xxx}(l, t) = 0$. Thus there are a total of four boundary conditions, two at each end.

(a) Separate the time and space variables to get the eigenvalue problem $X'''' = \lambda X$.

(b) Show that zero is not an eigenvalue.

(c) Assuming that all the eigenvalues are positive, write them as $\lambda = \beta^4$ and find the equation for β.

(d) Find the frequencies of vibration.

(e) Compare your answer in part (d) with the overtones of the vibrating string by looking at the ratio β_2^2/β_1^2. Explain why you hear an almost pure tone when you listen to a tuning fork.

(a) $u(x,t) = X(x)T(t)$

$$XT'' = c^2X^4T \to \frac{T''}{c^2T} = \frac{X^{(4)}}{X} = \lambda$$

plug in BCs to get the eigenvalue problem

$$\begin{cases} X^{(4)} = \lambda X \\ X(0) = X'(0) = X''(l) = X'''(l) = 0 \end{cases}$$

(b) Assume $\lambda = 0$ is an eigenvalue then

$$X^{(4)} = 0$$
$$\Downarrow$$
$$X(x) = Ax^3 + Bx^2 + Cx + D$$
$$\Downarrow$$
$$X(0) = D = 0$$
$$X'(0) = C = 0$$
$$X''(l) = 6Al + 2B = 0$$
$$X'''(l) = 6A = 0$$

$X(x) = 0$, since X is a trivial, λ is Not an eigenvalue

(c) $\lambda = \beta^4$

$X(x) = e^{\alpha x}$

$\beta^4 e^{\alpha x} = \lambda X = X^4 = \alpha^4 e^{\alpha x}$

$\alpha = \pm 1, \pm i$

thus the general solution is $\boxed{X(x) = A\cosh\beta x + B\sinh\beta x + C\cos\beta x + D\sin\beta x}$

Plug in boundaries

$A + C = 0$

$B + D = 0$

$A\cosh\beta l + B\sinh\beta l - C\cos\beta l - D\sin\beta l = 0$

$A\sinh\beta l + B\cosh\beta l - C\sin\beta l - D\cos\beta l = 0$

so $\boxed{\cos\beta l \cosh\beta l = -1}$

(d) It has infinite solutions via (c)

The frequency of vibrations is this function

$$u(x,t) = \begin{cases} \sum_{\lambda=\beta^4} (A_n \cos c\beta^2 t + B_n \sin c\beta^2 t) \times \\ [(\sin\beta l + \sinh\beta l)(\cosh\beta x - \cos\beta x) + (\cos\beta l + \cosh\beta l)(\sin\beta x - \sinh\beta x)] \end{cases}$$

3. Consider the function $\phi(x) \equiv x$ on $(0, l)$. On the same graph, *sketch* the following functions.
   (a) The sum of the first three (nonzero) terms of its Fourier sine series.
   (b) The sum of the first three (nonzero) terms of its Fourier cosine series.

(a)

$$x = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$$

$$x \sin\left(\frac{m\pi x}{l}\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right)$$

$$\int_0^l \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right) dx = \int_0^l x \sin\left(\frac{m\pi x}{l}\right) dx$$

$$\sum_{n=1}^{\infty} A_n \int_0^l \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right) dx =$$

since $m = n$

$$\frac{l}{2} A_n = \int_0^l x \sin\left(\frac{n\pi x}{l}\right) dx$$

$$A_n = \frac{2}{l} \int_0^l x \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l}\left[-\frac{l}{n\pi} x \cos\left(\frac{n\pi x}{l}\right)\Big|_0^l - \int_0^l \left(-\frac{l}{n\pi}\right) \cos\left(\frac{n\pi x}{l}\right) dx\right]$$

$$\frac{2}{l}\left[-\frac{l}{n\pi}(l \cos(n\pi - 0) + \underbrace{\frac{l^2}{n^2\pi^2} \sin\left(\frac{n\pi x}{l}\right)\Big|_0^l}_{0}\right]$$

$$= \frac{2}{l}\left[-\frac{l^2}{n\pi} \cos(n\pi)\right]$$

$$= \frac{-2l}{n\pi}(-1)^n, \text{ thus } A_n = \frac{2l}{n\pi}(-1)^{n+1}$$

our serie is

$$\sum_{n=1}^{\infty} \frac{2l}{n\pi}(-1)^{n+1} \sin\left(\frac{n\pi x}{l}\right) = x$$

$$\frac{2l}{\pi} \sin\left(\frac{\pi x}{l}\right) - \frac{l}{\pi} \sin\left(\frac{2\pi x}{l}\right) + \frac{2l}{3\pi} \sin\left(\frac{3\pi x}{l}\right) \approx$$

(b)

$$A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) = x$$

$$\int_0^l \left[A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right)\right] dx$$

$$\int_0^l x\, dx$$

$$A_0 \int_0^l dx + \sum_{n=1}^{\infty} A_n \int_0^l \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{x^2}{2}\Big|_0^l$$

$$A_0 l + \sum_{n=1}^{\infty} A_n \frac{l}{n\pi} \sin\left(\frac{n\pi x}{l}\right)\Big|_0^l = \frac{l^2}{2}$$

$$A_0 l = \frac{l^2}{2} \Rightarrow \boxed{A_0 = \frac{l}{2}}$$

solve for A_n multiplying both sides by $\cos\left(\frac{m\pi x}{l}\right)$

$$A_0 \cos\left(\frac{m\pi}{l}\right) + \sum_{n=1}^{\infty} A_n \cos\left(\frac{\pi n x}{l}\right) \cos\left(\frac{m\pi x}{l}\right)$$

2. now integrate 0 to l

$$A_0 \int_0^l \cos\left(\frac{\pi m x}{l}\right) dx + \sum_{n=1}^{\infty} A_n \int_0^l \cos\left(\frac{n\pi x}{l}\right) \cos\left(\frac{m\pi x}{l}\right) dx$$

$$= \int_0^l x \cos\left(\frac{m\pi x}{l}\right) dx$$

3. $$A_0 \cdot \frac{l}{n\pi} \sin\left(\frac{n\pi x}{l}\right)\Big|_0^l + A_n \cdot \frac{l}{2} = \frac{l}{n\pi} x \sin\left(\frac{n\pi x}{l}\right)\Big|_0^l - \int_0^l \frac{l}{n\pi} \sin\left(\frac{n\pi x}{l}\right) dx$$

$$A_n \cdot \frac{l}{2} = \frac{l^2}{n^2\pi^2}(\cos(n\pi - 1))$$

$$= \frac{l^2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$A_n = \frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right]$$

Now we can state the series in reference to the problem

$$\frac{l}{2} + \sum_{n=1}^{\infty} \frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right] \cos\left(\frac{n\pi x}{l}\right) = x$$

$$\frac{l}{2} - \frac{4l}{\pi^2} \cos\left(\frac{\pi x}{l}\right) - \frac{4l}{9\pi^2} \cos\left(\frac{3\pi x}{l}\right) \approx x$$

$\cos n \neq m$

1. In the expansion $1 = \sum_{n \text{ odd}} (4/n\pi) \sin n\pi$, valid for $0 < x < \pi$, put $x = \pi/4$ to calculate the sum

$$\left(1 - \tfrac{1}{5} + \tfrac{1}{9} - \tfrac{1}{13} + \cdots\right) + \left(\tfrac{1}{3} - \tfrac{1}{7} + \tfrac{1}{11} - \tfrac{1}{15} + \cdots\right)$$
$$= 1 + \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} + \cdots$$

(*Hint:* Since each of the series converges, they can be combined as indicated. However, they cannot be arbitrarily rearranged because they are only conditionally, not absolutely, convergent.)

sin(nπ) should actually be sin(nx)

$$1 = \sum_{n=\text{odd}} \frac{4}{n\pi} \sin(nx)$$

$x = \frac{\pi}{4}$ to get $1 = \sum_{n=\text{odd}} \frac{4}{n\pi} \sin \frac{\pi n}{4}$

slighty expand the summation

$$1 = \frac{4}{\pi} \sin\frac{\pi}{4} + \frac{4}{3\pi} \cdots$$

$$= \frac{2\sqrt{2}}{\pi} + \frac{2\sqrt{2}}{3\pi} - \frac{2\sqrt{2}}{5\pi} - \frac{2\sqrt{2}}{7\pi} + \frac{2\sqrt{2}}{9\pi}$$

$$= \frac{2\sqrt{2}}{\pi}\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9}\right)$$

$$\boxed{\frac{\pi}{2\sqrt{2}} \approx 1.111}$$

↑

solution

10. A string (of tension T and density ρ) with fixed ends at $x = 0$ and $x = l$ is hit by a hammer so that $u(x, 0) = 0$, and $\partial u/\partial t(x, 0) = V$ in $[-\delta + \frac{1}{2}l, \delta + \frac{1}{2}l]$ and $\partial u/\partial t(x, 0) = 0$ elsewhere. Find the solution explicitly in series form. Find the energy

$$E_n(h) = \frac{1}{2}\int_0^l \left[\rho\left(\frac{\partial h}{\partial t}\right)^2 + T\left(\frac{\partial h}{\partial x}\right)^2\right] dx$$

of the nth harmonic $h = h_n$. Conclude that if δ is small (a concentrated blow), each of the first few overtones has almost as much energy as the fundamental. We could say that the tone is saturated with overtones.

The Pde we must solve

$$\left\{\begin{array}{ll} u_{tt} = c^2 u_{xx} & 0 < x < \pi \\ u(0,t) = u(l,t) = 0 & \\ u(x,0) = 0 & \\ u_t(x,0) = \begin{cases} V & x \in [-\delta + \frac{1}{2}l, \delta + \frac{1}{2}l] \\ 0 & \text{otherwise} \end{cases} & \end{array}\right\}$$

$u(x,t) = X(x)T(t)$

then $T''X = u_{tt} = u_{xx} = c^2 TX''$

$$\Downarrow$$

$$\frac{T''}{c^2 T} = \frac{X''}{X} = \lambda$$

$X(0) = X(l) = 0$

Split into cases.

1.) $\lambda > 0 \Rightarrow \lambda = \beta^2$

$X(x) = A\sinh\beta x + B\cosh\beta x$: Solution

$0 = X(0) = B \Rightarrow 0 = X(l) = A\sinh\beta l \Rightarrow A = 0$ or $l = 0$: boundary conditions

thus u is trivial $\therefore$ $\lambda > 0$ is not an eigenvalue

2.) $\lambda = 0$

$X(x) = Ax + B$: Solution

$0 = X(0) = B \Rightarrow 0 = X(l) = Al \Rightarrow A = 0$ or $l = 0$: Boundary Conditions

thus u is trivial $\therefore$ $\lambda = 0$ is not an eigenvalue.

3.) $\lambda < 0 \Rightarrow \lambda = -\beta^2$

$X(x) = A\sin\beta x + B\cos\beta x$: Solution

$0 = X(0) = B \Rightarrow 0 = X(l) = A\sin\beta l \overset{A \neq 0}{\Longrightarrow} \beta = \frac{n\pi}{l}$ $n \in \mathbb{N}$: Boundary Conditions

So for eigenvalue $\lambda = -n^2$, the eigenfunction is

$u_n(x,t) = X_n T_n = \sin\left(\frac{n\pi x}{l}\right)\cdot\left(A_n\sin\left(\frac{cn\pi t}{l}\right) + B_n\cos\left(\frac{cn\pi t}{l}\right)\right)$

the Solution is a linear combination

$u(x,t) = \sum_{n=1}^{\infty}\left[\sin\left(\frac{n\pi x}{l}\right)\cdot\left(A_n\sin\left(\frac{cn\pi t}{l}\right) + B_n\cos\left(\frac{cn\pi t}{l}\right)\right)\right]$

$0 = u(x,0) = \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{l}\right)$; initial Condition

thus $\boxed{B_n = 0}$, for all $n \in \mathbb{N}$

Second init cond

$u_t(x,0) = \sum_{n=1}^{\infty}\left(\frac{A_n cn\pi}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$

$\int_0^l [u_t(x,0)]\sin\left(\frac{m\pi x}{l}\right)dx = \sum_{n=1}^{\infty}\frac{A_n cn\pi}{l}\int_0^l \sin\frac{n\pi x}{l}\ldots$

$V\int_{-\delta+\frac{1}{2}l}^{\delta+\frac{1}{2}l}\sin\frac{m\pi x}{l}\,dx = \sum_{n=1}^{\infty}\frac{A_n cn\pi}{l}\int_0^l \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)dx$

the only non trivial interval is the m-th

$\int_0^l \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)dx = \int_0^l \frac{\cos\frac{(n-m)\pi x}{l} + \cos\frac{(n+m)\pi x}{l}}{2}dx = \begin{cases} l/2 & n = m \\ 0 & n \neq m \end{cases}$