J. Appl. Prob. **14**, 89–97 (1977)
Printed in Israel

TOWARDS CONSENSUS: SOME CONVERGENCE THEOREMS ON REPEATED AVERAGING

S. CHATTERJEE, *New York University*
E. SENETA, *Australian National University*

Abstract

The problem of tendency to consensus in an information-exchanging operation is connected with the ergodicity problem for backwards products of stochastic matrices. For such products, weak and strong ergodicity, defined analogously to these concepts for forward products of inhomogeneous Markov chain theory, are shown (in contrast to that theory) to be equivalent. Conditions for ergodicity are derived and their relation to the consensus problem is considered.

CONSENSUS; REPEATED AVERAGING; STOCHASTIC MATRICES; BACKWARDS PRODUCTS; ERGODICITY; INHOMOGENEOUS MARKOV CHAINS

1. Introduction

A group of individuals, each of whom has an estimate of an unknown quantity, engage in an information-exchanging operation. This unknown quantity may be the value of an unknown parameter, or a probability. When the individuals are made aware of each others' estimates, they may modify their own estimate by taking into account the opinion of others.

De Groot [5] has proposed a model for this estimate-modification process. The essence of this is that an individual weighs the several estimates according to his opinion of their reliabilities. Thus, the weights reflect his 'trust' in the diverse opinions. De Groot [5] and Chatterjee [2] have then studied conditions under which a consensus is reached; the present paper generalizes the results of [5], and extends and makes rigorous the results of [2].

The quantitative formulation of the model is as follows. Let there be n individuals, and let their initial estimates be $\boldsymbol{F}_0' = (F_1^0, F_2^0, \cdots, F_n^0)$. Let $p_{ij}(1)$ be the initial weight which the ith individual attaches to the opinion of the jth individual. After the first interchange of information, then, the ith individual's estimate becomes

$$F_i^1 = \sum_j p_{ij}(1) F_j^0,$$

Received in revised form 20 July 1976.

where the $p_{ij}(1)$'s can be normalized so that

$$\sum_j p_{ij}(1) = 1, \quad \text{all} \quad i.$$

From this it is clear that the F_i's may be elements of any convex set in an appropriate linear space, rather than just real numbers. In particular they may be probability distributions, in which case the problem is one of attaining agreement about subjective probability distributions. This is the motivation of de Groot [5], who gives a range of references; for one survey, see Winkler [13]. Another situation of applicability arises in forecasting, where several individuals interact with each other while engaged in making the forecast (Delphi method; see [4]).

Now write $P_m = \{p_{ij}(m)\}$, $i, j = 1, \cdots, n$, where $p_{ij}(m)$ is the weight attached by the ith individual to the estimate of the jth individual after m interchanges, properly normalized. If $\boldsymbol{F}_m$ is the estimate vector after m interchanges, it follows that

$$\boldsymbol{F}_m = P_m P_{m-1} \cdots P_1 \boldsymbol{F}_0 \tag{1.1}$$

where P_k, $k = 1, 2, \cdots$, are each $(n \times n)$ stochastic matrices. There is clearly some resemblance here to an inhomogeneous Markov chain—see [11], §4.3—except that in (1.1) the matrix product goes 'backwards'. Nevertheless, we shall see that we may use a number of results from the inhomogeneous Markov chain theory, since they are essentially 'direction-free'. In actual fact, the scheme (1.1) represents a procedure described by Feller [6] as 'repeated averaging', though Feller, like de Groot, is concerned only with the homogeneous case (where all P_k are the same).

Our chief concern, then, is the behaviour of (1.1) as $m \to \infty$. We shall proceed to this by first formulating the problem in a mathematically more general setting. In the sequel all matrices are $n \times n$.

Let $\{P_k\}$, $k \geqq 1$, be a fixed sequence of stochastic matrices, and let $U_{r,k}$ be the stochastic matrix defined by

$$U_{r,k} = P_{r+k} \cdots P_{r+2} P_{r+1}, \tag{1.2}$$

where $P_k = \{p_{ij}(k)\}$, $U_{r,k} = \{u_{i,j}^{(r,k)}\}$. We study the behaviour of $U_{r,k}$ as $k \to \infty$, for each $r \geqq 0$. The sequence $\{P_k\}$ is said to be *weakly ergodic* (in a backwards direction), if for all $i, j, s = 1, \cdots, n$ and $r \geqq 0$,

$$(u_{i,s}^{(r,k)} - u_{j,s}^{(r,k)}) \to 0$$

as $k \to \infty$; that is to say, all rows of $U_{r,k}$ tend to become the same as $k \to \infty$, even if they do not tend to a limit vector. The sequence is said to be *strongly ergodic* (in a backwards direction), if

$$(1.3) \qquad \lim_{k\to\infty} U_{r,k} = \mathbf{1}\boldsymbol{D}'_r, \quad r \geqq 0,$$

elementwise, where $\boldsymbol{D}_r$ is necessarily a probability vector (i.e. $\boldsymbol{D}_r \geqq \mathbf{0}$, $\boldsymbol{D}'_r\mathbf{1} = 1$) but will in general depend on r. Both definitions are analogous to those of Markov chain theory, where the products are taken in a forward direction; and as in that theory, strong ergodicity clearly implies weak ergodicity.

It is evident, considering (1.1), that weak and strong ergodicity both describe a tendency to consensus — a situation where opinions tend to become the same. In the strong ergodicity situation, opinions not only tend to become the same, but this common opinion tends to stabilize, in time, to a fixed opinion. From a practical point of view, this is the kind of consensus of interest. Likewise, it might seem adequate, from such a viewpoint, to study the situation in the case $r = 0$ only. The reason (as for Markov chains) for the more general definition is to exclude trivial cases such as when for some k, $k = k_0$ say, P_k has the form $\mathbf{1}\,\boldsymbol{v}'$ where $\boldsymbol{v}$ is a probability vector; for then

$$(1.4) \qquad U_{0,k} = \mathbf{1}\,\boldsymbol{v}'U_{0,k_0-1}, \quad \text{all} \quad k \geqq k_0,$$

which already represents, for $k \geqq k_0$, a state of uninteresting stable consensus according to the less restrictive definition with $r = 0$ only, irrespective of P_k for $k > k_0$. In any case, our study gives substantial information on the consensus problem, while in reality being a mathematical study of (1.2).

While the definitions of weak and strong ergodicity are analogous to those of Markov chain theory, in the present setting, where each *row* of $U_{r,k+1}$ is a weighted average of the *rows* of $U_{r,k}$, we have a fundamental difference from that theory, in the following result.

Theorem 1. *For the matrix product* (1.2), *weak and strong ergodicity are equivalent.*

Thus, following this, we shall need only speak of *ergodicity* in the succeeding theorems, even though we shall generally prove the 'weak ergodicity' variant. This is an added advantage in defining 'consensus' as coinciding with ergodicity in the sense of the above discussion.

Note that, for convenience, proofs are deferred to a final section.

2. Convergence theorems

For further progress we need a number of concepts from Markov chain theory [11]. Let G_1 be the class of *regular* stochastic matrices (i.e. stochastic matrices having a single eigenvalue of modulus unity); and M the class of $(n \times n)$ *Markov* matrices (i.e. stochastic matrices with at least one column entirely positive). Thus a stochastic $P = \{p_{ij}\}$ is Markov, if and only if $\lambda(P) > 0$, where

$$\lambda(P) = \max_j \left(\min_i p_{ij} \right). \tag{2.1}$$

It is well known (e.g. [11]) that $P \in G_1$, if and only if for some r, $P^r \in M$.

Additionally ([7], [12]) any scalar function $\mu(\cdot)$ continuous on the set of $(n \times n)$ stochastic matrices P and satisfying $0 \leqq \mu(P) \leqq 1$ is called a *coefficient of ergodicity*. It is said to be *proper* if

$$\mu(P) = 1 \quad \text{if and only if} \quad P = \mathbf{1}\,\boldsymbol{v}'$$

where $\boldsymbol{v}'$ is any probability vector (i.e. whenever P has all rows the same). We shall be interested in situations where for any set of stochastic matrices $P^{(i)}$ $(i = 1, \cdots, k)$, and each k,

$$m(P^{(1)}P^{(2)} \cdots P^{(k)}) \leqq \prod_{i=1}^{k} (1 - \mu(P^{(i)})) \tag{2.2}$$

where $1 - m(\cdot)$ is a proper coefficient of ergodicity, and μ a coefficient of ergodicity. For example, this is true with $m(P) \equiv a(P)$, and $\mu(P) \equiv \lambda(P)$, where $\lambda(\cdot)$ is as in (2.1) and

$$a(P) = \tfrac{1}{2} \max_{i,j} \sum_s | p_{i,s} - p_{j,s} | ;$$

or indeed with $m(P) \equiv a(P)$ and $\mu(P) \equiv 1 - a(P)$. Notice that weak ergodicity is then equivalent to

$$\mu(U_{r,k}) \to 1, \quad k \to \infty, \quad r \geqq 0$$

where μ is a proper coefficient of ergodicity.

We now state necessary and sufficient conditions (Theorems 2 and 3) for ergodicity, in respect to $\{P_k\}$. These are followed by Theorems 4 and 5 which each give relatively simple sufficient conditions for ergodicity. These need at most outline proofs.

Theorem 2. *Suppose* $1 - m(\cdot)$ *and* $\mu(\cdot)$ *in* (2.2) *are both proper coefficients of ergodicity. Then* $\{P_k\}$, $k \geqq 1$, *is ergodic if and only if there exists a strictly increasing subsequence* $\{i_j\}$, $j = 1, 2, \cdots$ *of the positive integers such that*

$$\sum_{j=1}^{\infty} \mu(U_{i_j, i_{j+1} - i_j}) = \infty. \tag{2.3}$$

This result is analogous in both statement and proof to the Doeblin–Hajnal condition ([7], [12]) for weak ergodicity for Markov chains.

A set of absolute probability vectors for the sequence $\{P_k\}$, $k \geqq 1$, is a sequence $\{\boldsymbol{v}_k\}$, $k \geqq 0$, of probability vectors such that

$$\boldsymbol{v}'_{r+k} U_{r,k} = \boldsymbol{v}'_r, \quad r \geqq 0, \quad k \geqq 1.$$

Theorem 3. *The sequence* $\{P_k\}$, $k \geqq 1$, *is ergodic if and only if there is only one set of absolute probability vectors*, $\{v_k\}$. *In this case* (1.3) *holds with* $\boldsymbol{D}_r = \boldsymbol{v}_r$.

This result is due to Kolmogorov [9]; for a succinct reworking, see Blackwell [1]. (A set of absolute probability vectors always exists.)

The set $\{P_k\}$, $k \geqq 1$, corresponds to an inhomogeneous Markov chain starting in the infinitely remote past and ending at time 0, P_k being the transition matrix at time $-k$, and $U_{r,k}$, as defined by (1.2), being the k-step transition matrix between time $-(r+k)$ and time $-r$. Then v_k is a vector of absolute probabilities for the chain at time $-k$. Kolmogorov's proof was formulated for Markov chains infinite in both time directions but applies for chains infinite only in the backward direction.

Theorem 4. *For the sequence* $\{P_k\}$, $k \geqq 1$, *to be ergodic, it suffices that*

$$\sum_{k=1}^{\infty} \lambda(P_k) = \infty.$$

Corollary. A condition sufficient for ergodicity is $\Sigma \varepsilon_k = \infty$, *where* $\varepsilon_k = \min_{i,j} p_{ij}(k)$.

Theorem 5. *If for each* $r \geqq 0$, $U_{r,k} \in G_1$ *for all* $k \geqq 1$, *and*

$$\min_{i,j}{}^{+} p_{ij}(k) \geqq \delta > 0$$

uniformly for all $k \geqq 1$ (*where* $\min^+$ *refers to the minimum non-zero entry*), *then ergodicity obtains.*

Theorem 5 is analogous in both statement and proof to the Sarymsakov–Mustafin Theorem for weak ergodicity of Markov chains ([11], Theorem 4.9), and it will be seen from its proof that, likewise, ergodicity is attained at a rate independent of r in (1.3) and geometric. If all $P_k \equiv P \in G_1$, then the conditions of the theorem are satisfied, so ergodicity obtains; in view of the remark concerning the relation between G_1 and M preceding (2.1), this result is due to de Groot [5]. There is in fact a simple converse: if $P_k \equiv P$ and ergodicity obtains, then $P \in G_1$. The converse is easily deducible by considering the behaviour of the powers P^r for a P of general structure ([11], Chapter 1 and §4.2). We shall also deduce the following.

Corollary. If $P_k \to P \in G_1$ *as* $k \to \infty$, *then ergodicity obtains.*

This corollary resembles a result on strong ergodicity of Markov chains of Mott ([10]; [11], Exercise 4.26) and thus generalizes the essence of de Groot's result.

To conclude this section we direct the interested reader to Theorem 3 of Blackwell [1] for the rather complex behaviour of $U_{r,k}$ in general as $k \to \infty$; and to Cohn [3] for an explanation in terms of the tail σ-field of a reverse Markov chain.

3. Examples

We shall discuss the implications of the ergodicity results for consensus by first considering two weighting schemes which may be regarded as extreme situations.

(a) *Open-minded participants.* A group of individuals enter a problem with specialized information. The individuals are open-minded; with exchange of information the knowledge tends to become group knowledge and participants tend to give equal weight to all opinions. Thus as $k \to \infty$

$$p_{ij}(k) \to n^{-1}.$$

Thus by the Corollary to Theorem 5, consensus is approached.

(b) *Hardening-position scheme.* Suppose the information exchange process causes people to put more weight on their own opinions and less on those of others, with a tendency in the limit to put all the weight on their own opinion. Thus as $k \to \infty$

$$p_{ij}(k) \to \delta_{ij}, \begin{cases} = 1 & \text{if} \quad i = j \\ = 0 & \text{if} \quad i \neq j\,. \end{cases} \tag{3.1}$$

Even here consensus may be reached. For example, suppose as $k \to \infty$

$$p_{ij}(k) = \delta_{ij} + a_{ij}k^{-1} + o(k^{-1}), \; i, j = 1, \cdots, n$$

where

$$\sum_j a_{ij} = 0; \; -1 < a_{ii} < 0; \; a_{ij} > 0, \; i \neq j.$$

Then for k sufficiently large,

$$\varepsilon_k = \min_{i,j} p_{ij}(k) \geqq \gamma k^{-1}$$

for some $\gamma > 0$ independent of k; and the result follows from the Corollary of Theorem 4.

Another application of Theorem 4 is to a situation where there is an infinite number of occasions when there is at least one participant to whose opinion everyone attaches a weight of at least $\delta > 0$. Then consensus will be reached.

4. Proof outlines

Theorem 1. We need only prove weak ergodicity implies strong ergodicity. Fix $r \geqq 0$ and $\varepsilon > 0$; then by weak ergodicity

$$-\varepsilon \leqq u_{i,s}^{(r,k)} - u_{j,s}^{(r,k)} \leqq \varepsilon$$

for $k \geqq k_0$, uniformly for all $i, j, s = 1, \cdots, n$. Since $U_{r,k+1} = P_{r+k+1} U_{r,k}$

$$\sum_{j=1}^{n} p_{hj}(r+k+1)(u_{i,s}^{(r,k)} - \varepsilon)$$

$$\leqq \sum_{j=1}^{n} p_{hj}(r+k+1) u_{j,s}^{(r,k)} \leqq \sum_{j=1}^{n} p_{hj}(r+k+1)(u_{i,s}^{(r,k)} + \varepsilon)$$

i.e.

$$u_{i,s}^{(r,k)} - \varepsilon \leqq u_{h,s}^{(r,k+1)} \leqq u_{i,s}^{(r,k)} + \varepsilon$$

for $k \geqq k_0$. Thus we can prove (by induction) that for any $q \geqq 1$, and any $i, j, s = 1, \cdots, n$, for $k \geqq k_0$

$$|u_{j,s}^{(r,k+q)} - u_{i,s}^{(r,k)}| \leqq \varepsilon$$

and in particular we can put $j = i$; in which case the Cauchy convergence criterion for sequences tells us that as $k \to \infty$, $u_{i,s}^{(r,k)}$ approaches a limit.

Theorem 2. Analogous to the proof of Hajnal's [7] Theorem 3; ([12], Theorem 1, Lemma 1). We give only the proof of sufficiency. Take $r \geqq 0$ fixed but arbitrary and consider k large in $U_{r,k}$. Let β be such that i_β is the minimal member of the sequence $\{i_j\}$ to satisfy $i_j \geqq r+1$; and let $j(k)$ be such that $i_{j(k)}$ is the maximal member to satisfy $i_j < k + r$. Then since

$$\begin{aligned} U_{r,k} &= U_{i_{j(k)}, k+r-i_{j(k)}} U_{i_\beta, i_{j(k)}-i_\beta} U_{r, i_\beta - r} \\ &= U_{i_{j(k)}, k+r-i_{j(k)}} \left(\prod_{j=\beta}^{j=j(k)-1} U_{i_j, i_{j+1}-i_j} \right) U_{r, i_\beta - r} \end{aligned}$$

it follows from (2.2) that

$$\begin{aligned} m(U_{r,k}) &\leqq (1 - \mu(U_{i_{j(k)}, k+r-i_{j(k)}})) \left(\prod_{j=\beta}^{j=j(k)-1} (1 - \mu(U_{i_j, i_{j+1}-i_j})) \right) (1 - \mu(U_{r, i_\beta - r})) \\ &\leqq \prod_{j=\beta}^{j=j(k)-1} (1 - \mu(U_{i_j, i_{j+1}-i_j})) \end{aligned}$$

and the right-hand side diverges to zero as $k \to \infty$ in view of (2.3).

Theorem 4. It follows from Theorem 2 that $\{P_k\}$ is ergodic if $\Sigma_k\{1 - a(P_k)\} = \infty$. On the other hand ([7], [12])

$$1 - a(P) \geqq \lambda(P) \geqq \min_{i,j} p_{ij}$$

so the result and its corollary follow. The corollary itself is a consequence of a well-known result ([8], Theorem 4.1.3; [11], Lemma 3.4).

Theorem 5. This needs a set of preliminary results which are analogous, and proved in an analogous way to, Lemmas 4.2, 4.3 and 4.5 of [11]; these we now state. (Two $(n \times n)$ stochastic matrices are said to have the same pattern if they have zero elements and positive elements in the same positions.)

Lemma 1. *If P and Q are stochastic, and Q or $P \in M$, then PQ and $QP \in M$.*

Lemma 2. *If $Q \in Q_1$, P is stochastic, and QP has the same pattern as P, then $P \in M$.*

Lemma 3. *If for $r \geqq 0$, $U_{r,k} \in G_1$ for all $k \geqq 1$, then $U_{r,k} \in M$ for $k \geqq t+1$, where t is the number of distinct patterns of matrices in G_1.*

Notice that the assumption in Theorem 5 and Lemma 3 that $U_{r,k} \in G_1$ for all r, k may be replaced by the (stronger but more simply verifiable) assumption that P_i is a *scrambling matrix* for each i (see [11]).

To prove the theorem itself we first note that

$$U_{0,k(t+1)} = U_{(k-1)(t+1),t+1} \cdots U_{2(t+1),t+1} U_{t+1,t+1} U_{0,t+1}$$

where $U_{i(t+1),t+1}$, $i \geqq 0$, $\in M$ in virtue of Lemma 3. The assumption of the theorem on the P_i's implies that $\lambda(U_{i(t+1),t+1}) \geqq \delta^{t+1}$; thus the sequence $\{U_{i(t+1),t+1}\}$, $i \geqq 0$ of stochastic matrices is weakly ergodic by application of (2.2). A totally analogous argument applies for any sequence $\{U_{r+i(t+1),t+1}\}$ $i \geqq 0$, for any fixed $r \geqq 0$. Now consider fixed r, and k large; then

$$U_{r,k} = \bar{U}_{(k)} U_{r+h(t+1),t+1} \cdots U_{r+(t+1),t+1} U_{r,t+1}$$

where $h \equiv h(k)$ is the largest positive integer so that $(h+1)(t+1) \leqq k$ and $\bar{U}_{(k)}$ is some stochastic matrix. Again by (2.2) we have

$$a(U_{r,k}) \leqq \prod_{i=0}^{h(k)} (1 - \lambda(U_{r+i(t+1),t+1}))$$

since $0 \leqq 1 - \lambda(\bar{U}_{(k)}) \leqq 1$;

$$\leqq (1 - \delta^{t+1})^{h(k)} \to 0, \quad k \to \infty.$$

Corollary. Since $P_i \to P \in G_1$, for sufficiently large i, P_i will have positive entries in at least the same positions as P. Indeed by the remark relating G_1 and M, we can find a k_0 such that $\lambda(U_{r,k_0}) > \alpha > 0$, for some $\alpha > 0$ uniformly for all $r \geqq r_0$. Hence considering any fixed $r \geqq 0$ and 'large' k, we have

$$U_{r,k} = \bar{U}_{(k)} \left(\prod_{j=0}^{h(k)} U_{r_0+jk_0,k_0} \right) U_{r,r_0-r}$$

if $r < r_0$ (and a similar expression for any r), $\bar{U}_{(k)}$ being some stochastic matrix and $h \equiv h(k)$ being the largest positive integer such that $r_0 + (h+1)k_0 \leqq r + k$. Proceeding now as in the proof of Theorem 5 yields that $a(U_{r,k}) \to 0$ as $k \to \infty$.

References

[1] Blackwell, D. (1945) Finite non-homogeneous chains. *Ann. Math.* **46**, 594–599.

[2] Chatterjee, S. (1975) Reaching a consensus; some limit theorems. *Proc. Int. Statist. Inst.* 159–164.

[3] Cohn, H. (1974) A ratio limit theorem for the finite non-homogeneous Markov chains. *Israel J. Math.* **19**, 329–334.

[4] Dalkey, N. C. (1969) The Delphi method: an experimental study of group opinion. The Rand Corporation, RM–5888.

[5] De Groot, M. H. (1974) Reaching a consensus. *J. Amer. Statist. Assoc.* **69**, 118–121.

[6] Feller, W. (1968) *An Introduction to Probability Theory and its Applications.* Vol. 1, 3rd edn. Wiley, New York.

[7] Hajnal, J. (1958) Weak ergodicity in non-homogeneous Markov chains. *Proc. Camb. Phil. Soc.* **54**, 233–246.

[8] Kemeny, J. G. and Snell, L. J. (1960) *Finite Markov Chains.* Van Nostrand, Princeton, N.J.

[9] Kolmogorov, A. N. (1936) Zur theorie der Markoffschen Ketten. *Math. Ann.* **112**, 155–160.

[10] Mott, J. L. (1957) Conditions for the ergodicity of non-homogeneous finite Markov chains. *Proc. Roy. Soc. Edinburgh* A**64**, 369–380.

[11] Seneta, E. (1973) *Non-Negative Matrices.* Allen and Unwin, London.

[12] Seneta, E. (1973) On the historical development of the theory of finite inhomogeneous Markov chains. *Proc. Camb. Phil. Soc.* **74**, 507–513.

[13] Winkler, R. L. (1968) The consensus of subjective probability distributions. *Management Sci.* **15**, B61–B75.