MONASH University

MONASH BUSINESS SCHOOL

ETF3231/5231 Business forecasting

Week 6: Exponential smoothing

https://bf.numbat.space/

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Outline

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Historical perspective

- Proposed in the late 1950s (Brown 1959, Holt 1957 and Winters 1960 are key pioneering works) as methods (algorithms) to produce point forecasts.
- Forecasts are weighted averages of past observations, with the weights decaying exponentially as the observations get older.
- Framework generates reliable forecasts quickly and for a wide spectrum of time series. A great advantage and of major importance to applications in industry.

Combine components

- Combine components: level ℓ_t , trend (slope) b_t and seasonal s_t to describe a time series
$$y_t = f(\ell_t, b_t, s_t)$$
- The rate of change of the components are controlled by "smoothing parameters": α, β and γ respectively.
- Need to choose best values for the smoothing parameters (and initial states).
- Add error ε_t to get equivalent ETS state space models developed in the 1990s and 2000s.

Big idea: control the rate of change (smoothing)

α controls the flexibility of the level ℓ_t

- If $\alpha = 0$, the level never updates (mean)
- If $\alpha = 1$, the level updates completely (naive)

β controls the flexibility of the trend b_t

- If $\beta = 0$, the trend is linear (regression trend)
- If $\beta = 1$, the trend updates every observation

γ controls the flexibility of the seasonality s_t

- If $\gamma = 0$, the seasonality is fixed (seasonal means)
- If $\gamma = 1$, the seasonality updates completely (seasonal naive)

A model for levels, trends, and seasonalities

We want a model that captures the level (ℓ_t), trend (b_t) and seasonality (s_t).

How do we combine these elements?

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Additively?

$$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$$

Multiplicatively?

$$y_t = \ell_{t-1} b_{t-1} s_{t-m}(1 + \varepsilon_t)$$

Perhaps a mix of both?

$$y_t = (\ell_{t-1} + b_{t-1}) s_{t-m} + \varepsilon_t$$

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How do the level, trend and seasonal components evolve over time?

ETS models

General notation E T S : **E**xponen**T**ial **S**moothing

Error **T**rend **S**eason

```
ETS(y ~ error( ) + trend( ) + season( ))
```

Error: Additive (`"A"`) or multiplicative (`"M"`)

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Trend: None (`"N"`), additive (`"A"`), multiplicative (`"M"`), or damped (`"Ad"` or `"Md"`).

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Seasonality: None (`"N"`), additive (`"A"`) or multiplicative (`"M"`)

Models and methods

Methods

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Models

- Generate same point forecasts but can also generate forecast distributions.
- A stochastic (or random) data generating process that can generate an entire forecast distribution.
- Allow for "proper" model selection.

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Simple Exponential Smoothing - SES

Iterative form

$$\hat{y}_{t+1|t} = \alpha y_t + (1 - \alpha)\hat{y}_{t|t-1}$$

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$$\hat{y}_{T+1|T} = \sum_{j=0}^{T-1} \alpha(1 - \alpha)^j y_{T-j} + (1 - \alpha)^T \ell_0$$

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Component form

Forecast equation	$\hat{y}_{t+1|t} = \ell_t$
Smoothing equation	$\ell_t = \alpha y_t + (1 - \alpha)\ell_{t-1}$

ETS(A,N,N): SES with additive errors

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Error correction form

$$\begin{aligned} y_t &= \ell_{t-1} + e_t \\ \ell_t &= \ell_{t-1} + \alpha(y_t - \ell_{t-1}) \\ &= \ell_{t-1} + \alpha e_t \end{aligned}$$

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$$

Specify probability distribution for e_t, we assume $e_t = \varepsilon_t \sim \text{NID}(0, \sigma^2)$.

ETS(A,N,N): SES with additive errors

Measurement equation	$y_t = \ell_{t-1} + \varepsilon_t$
State equation	$\ell_t = \ell_{t-1} + \alpha\varepsilon_t$

where $\varepsilon_t \sim \text{NID}(0, \sigma^2)$.

- innovations or single source of error because equations have the same error process, ε_t.
- Measurement equation: relationship between observations and states.
- State equation(s): evolution of the state(s) through time.

ETS(M,N,N): SES with multiplicative errors.

- Specify relative errors $\varepsilon_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}} \sim \text{NID}(0, \sigma^2)$
- Substituting $\hat{y}_{t|t-1} = \ell_{t-1}$ gives:
  - $y_t = \ell_{t-1} + \ell_{t-1}\varepsilon_t$
  - $e_t = y_t - \hat{y}_{t|t-1} = \ell_{t-1}\varepsilon_t$

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Measurement equation	$y_t = \ell_{t-1}(1 + \varepsilon_t)$
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- Models with additive and multiplicative errors with the same parameters generate the same point forecasts but different prediction intervals.