MONASH University

MONASH BUSINESS SCHOOL

ETF3231/5231
Business forecasting

* We are interested in using these for forecasting.

Ch7. Regression models

https://bf.numbat.space/

* pure time series models v models with predictors

AACSB ACCREDITED EQUIS ACCREDITED ASSOCIATION OF MBAS ACCREDITED

Outline

Outline

REVISION

Multiple regression and forecasting

- response
- dependent
- regressand
- outcome

$$y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \cdots + \beta_k x_{k,t} + \varepsilon_t.$$

- predictors
- Indep. variables
- regressors
- explanatory

- y_t is the variable we want to predict: the "response'' variable
- Each $x_{j,t}$ is numerical and is called a "predictor''. They are usually assumed to be known for all past and future times.
  → not always the case.
- The coefficients $\beta_1, \ldots, \beta_k$ measure the effect of each predictor after taking account of the effect of all other predictors in the model.

That is, the coefficients measure the marginal effects.

- ε_t is a white noise error term

Example: US consumption expenditure

```
fit_cons <- us_change %>%
  model(lm = TSLM(Consumption ~ Income))
report(fit_cons)
```

$E(y_t/x_t) = \beta_0 + \beta_1 x_t$ Economic model

$y_t = \beta_0 + \beta_1 x_t + \varepsilon_t$ Statistical model

$\hat{y}_t = \hat{\beta}_0 + \hat{\beta}_1 x_t$

$= b_0 + b_1 x_t$ Estimated model

$= 0.54 + 0.27 x_t$

```
## Series: Consumption
## Model: TSLM
##
## Residuals:
##     Min      1Q Median      3Q     Max
## -2.582 -0.278  0.019  0.323  1.422
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   0.5445     0.0540   10.08  < 2e-16 ***
## Income        0.2718     0.0467    5.82   2.4e-08 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.591 on 196 degrees of freedom
## Multiple R-squared: 0.147,   Adjusted R-squared: 0.143
## F-statistic: 33.8 on 1 and 196 DF, p-value: 2e-08
```

Always included

Example: US consumption expenditure

```
fit_consMR <- us_change %>%
  model(lm = TSLM(Consumption ~ Income + Production + Savings + Unemployment))
report(fit_consMR)
```

y_t X_{1t} X_{2t} X_{3t} X_{4t}

* Intercept always included unless

$y \sim 0 + \ldots$

```
## Series: Consumption
## Model: TSLM
##
## Residuals:
##     Min      1Q Median      3Q     Max
## -0.906 -0.158 -0.036  0.136  1.155
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)   0.25311    0.03447    7.34  5.7e-12 ***
## Income        0.74058    0.04012   18.46  < 2e-16 ***
## Production    0.04717    0.02314    2.04    0.043 *
## Savings      -0.05289    0.00292  -18.09  < 2e-16 ***
## Unemployment -0.17469    0.09551   -1.83    0.069 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.31 on 193 degrees of freedom
## Multiple R-squared: 0.768,   Adjusted R-squared: 0.763
## F-statistic:  160 on 4 and 193 DF, p-value: <2e-16
```

b_0 b_1 b_2 b_3 b_4

Compare to 2-way correlations (we don't care about p-values)

∘ Forecasting ✓ Inference

Outline

Trend

Linear trend

$$x_t = t \qquad y_t = \beta_0 + \beta_1 t + \varepsilon_t$$

- $t = 1, 2, \ldots, T$
- Strong assumption that trend will continue. → very strong assumptio

Nonlinear trend

Piecewise linear trend with bend "knot" at τ

$$x_{1,t} = t$$

$$x_{2,t} = (t - \tau)_+ = \begin{cases} 0 & t < \tau \\ (t - \tau) & t \geq \tau \end{cases}$$

- β_1 trend slope before time τ
- $\beta_1 + \beta_2$ trend slope after time τ
- More knots can be added forming more $(t - \tau)_+$

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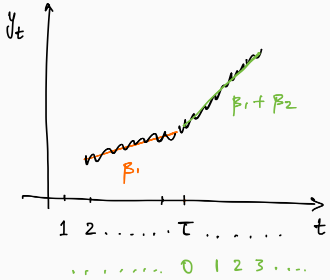

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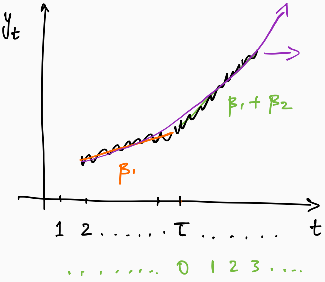

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- $\beta_1 + \beta_2$ trend slope after time τ
- More knots can be added forming more $(t-\tau)_+$

Quadratic or higher order trend

$$x_{1,t} = t, \quad x_{2,t} = t^2, \quad \dots$$

NOT RECOMMENDED!

Uses of dummy variables

Seasonal dummies

- For quarterly data: use 3 dummies why?
- For monthly data: use 11 dummies
- For daily data: use 6 dummies
- What to do with weekly data?
  - not exactly 52 weeks ($365/7 = 52.14$)
  - 51 dummy variables?

Outliers

- If there is an outlier, you can use a dummy variable to remove its effect.
  - Sydney Olympics (spike in bus travel the quarter after)

DUMMY VARIABLE TRAP

C	Yes	No	Y + N = C
1	1	0	1
1	1	0	1
1	0	1	1
⋮	⋮		⋮
1	1	0	1

Holidays

For monthly data

- Christmas: always in December so part of monthly seasonal effect
- Easter: use a dummy variable $v_t = 1$ if any part of Easter is in that month, $v_t = 0$ otherwise.
- Ramadan and Chinese new year similar.

For daily data

- If it is a public holiday, dummy=1, otherwise dummy=0.

Weekend v working day

Fourier series

Periodic seasonality can be handled using pairs of Fourier terms:

Set up these: $s_k(t) = \sin\left(\frac{2\pi kt}{m}\right) \qquad c_k(t) = \cos\left(\frac{2\pi kt}{m}\right)$

seasonal period

$$y_t = \underbrace{a}_{\text{int}} + \underbrace{bt}_{\text{trend}} + \sum_{k=1}^{K} \underbrace{[\alpha_k s_k(t) + \beta_k c_k(t)]}_{\text{these are included as pairs}} + \varepsilon_t$$

- Every periodic function can be approximated by sums of sin and cos terms for large enough K.
- Choose K by minimizing AICc.
- Called "harmonic regression" → as K increases we get harmonics of the first two Fourier terms.

```
TSLM(y ~ trend() + fourier(K))
```

specify K

for $k = 1$ $\quad s_1(t) = \sin\left(\frac{2\pi t}{m}\right)$ $\quad c_1(t) = \cos\left(\frac{2\pi t}{m}\right)$

$k = 2$ $\quad s_2(t) = \sin\left(\frac{2\pi 2t}{m}\right)$ $\quad c_2(t) = \cos\left(\frac{2\pi 2t}{m}\right)$

for $k=1$ $\quad s_1(t) = \sin\left(\frac{2\pi t}{m}\right)$ $\quad c_1(t) = \cos\left(\frac{2\pi t}{m}\right)$

$m=4$ $\quad = \sin\left(\frac{\pi}{2} t\right)$ $\quad = \cos\left(\frac{\pi}{2} t\right)$

$k=2$ $\quad s_2(t) = \sin\left(\frac{2\pi 2t}{m}\right)$ $\quad c_2(t) = \cos\left(\frac{2\pi 2t}{m}\right)$

$= \underline{\sin(\pi t) = 0}$ $\quad = \cos(\pi t)$

true for $k = \frac{m}{2}$ in fact $\max(k) = \frac{m}{2}$ or $k \leq \frac{m}{2}$

Distributed lags

Lagged values of a predictor.

Example: x is advertising which has a delayed effect

$$
\begin{aligned}
x_1 &= \text{advertising for previous month;} \\
x_2 &= \text{advertising for two months previously;} \\
&\vdots \\
x_m &= \text{advertising for } m \text{ months previously.}
\end{aligned}
$$

Outline

Multiple regression and forecasting

For forecasting purposes, we require the following assumptions:

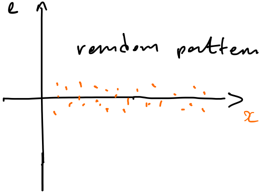
$\sum_{t=1}^{T} e_t = 0$ $\sum_{t=1}^{T} x_{k,t} e_t = 0$

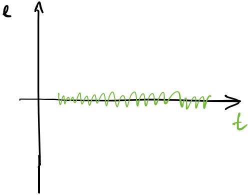
- ε_t are uncorrelated and zero mean
- Extra ε_t are uncorrelated with each $x_{j,t}$.

• normal equations as long as we have intercept. ε_t may not: endogeneity

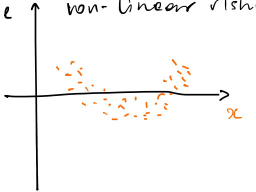
It is useful to also have $\varepsilon_t \sim \mathrm{N}(0, \sigma^2)$ when producing prediction intervals or doing statistical tests.

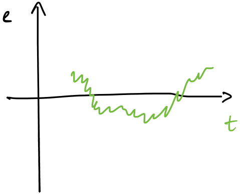
random pattern



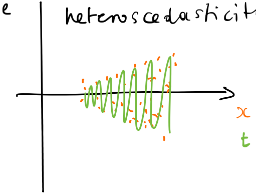


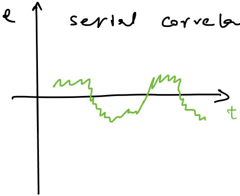
non-linear rlship





heteroscedasticity





serial correlation (exclusive to ts)
(ACF)

Outline

Comparing regression models

- R^2 does not allow for "degrees of freedom' '.
- Adding *any* variable tends to increase the value of R^2, even if that variable is irrelevant.

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To overcome this problem, we can use *adjusted* R^2:

$$\bar{R}^2 = 1 - (1 - R^2)\frac{T-1}{T-k-1}$$

where k = no. predictors and T = no. observations.

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where k = no. predictors and T = no. observations.

Maximizing $\bar{R}^2$ is equivalent to minimizing $\hat{\sigma}^2$.

estimated residual variance

$$\hat{\sigma}^2 = \frac{1}{T-k-1}\sum_{t=1}^{T}\varepsilon_t^2$$

WE CAN DO BETTER

Akaike's Information Criterion

predictors

intercept + variance

$$\text{AIC} = -2\log(L) + 2(\underset{\text{predictors}}{k} + 2)$$

- L = likelihood
- k = # predictors in model.
- AIC penalizes terms more heavily than $\bar{R}^2$. — smaller models

Akaike's Information Criterion

predictors

intercept + variance

$$\text{AIC} = -2\log(L) + 2(k+2)$$

- L = likelihood
- k = # predictors in model.
- AIC penalizes terms more heavily than $\bar{R}^2$. – smaller models

$$\text{AIC}_\text{C} = \text{AIC} + \frac{2(k+2)(k+3)}{T-k-3}$$

- Minimizing the AIC or AICc is asymptotically equivalent to minimizing MSE via **leave-one-out cross-validation** (for any linear regression).

will show soon, useful for prediction

Bayesian Information Criterion

$$\text{BIC} = -2\log(L) + (k+2)\log(T)$$

where L is the likelihood and k is the number of predictors in the model.

- BIC penalizes terms more heavily than AIC ← even smaller models if needed
- Also called SBIC and SC.
- Minimizing BIC is asymptotically equivalent to leave-v-out cross-validation when $v = T[1 - 1/(log(T) - 1)]$.

Leave-one-out cross-validation

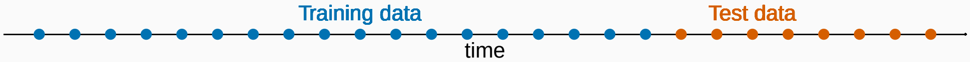
For regression, leave-one-out cross-validation is faster and more efficient than time-series cross-validation.

only with one regression (will show this).

- Select one observation for test set, and use *remaining* observations in training set. Compute error on test observation.
- Repeat using each possible observation as the test set.
- Compute accuracy measure over all errors.

Cross-validation

Traditional evaluation



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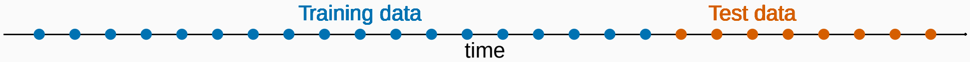
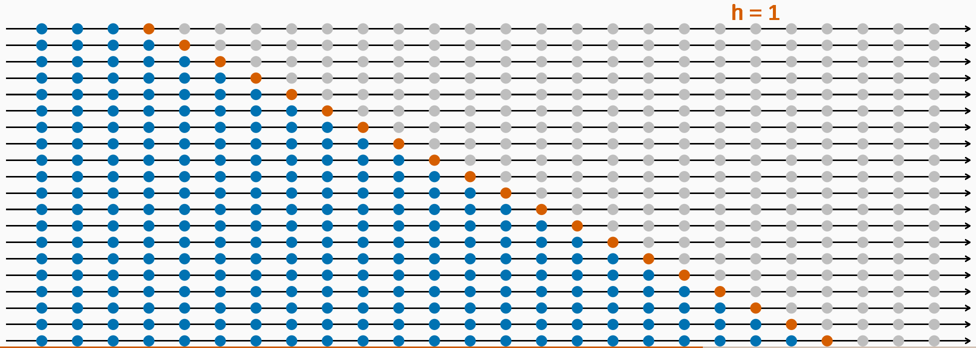


Time series cross-validation



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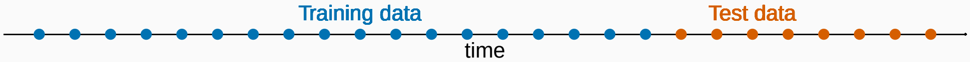
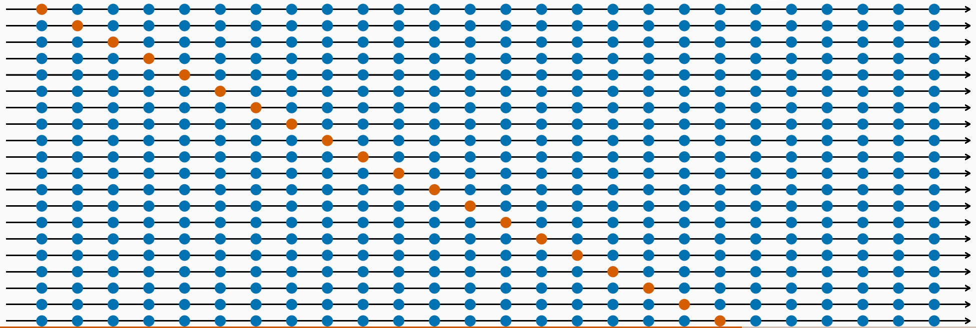


Leave-one-out cross-validation



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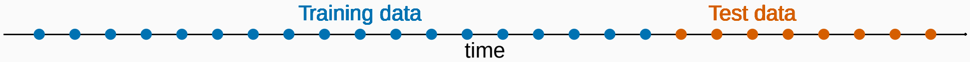
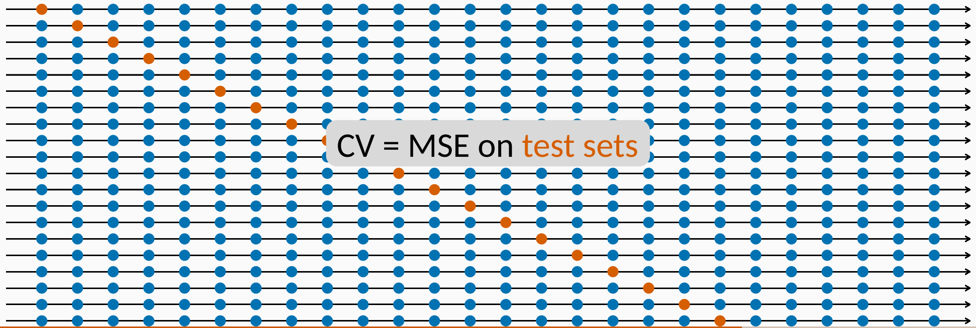


Leave-one-out cross-validation

* This is very fast - there is a trick



Choosing regression variables

Best subsets regression

- Fit all possible regression models using one or more of the predictors.
- Choose the best model based on one of the measures of predictive ability (CV, AIC, AICc).

Choosing regression variables

Best subsets regression

- Fit all possible regression models using one or more of the predictors.
- Choose the best model based on one of the measures of predictive ability (CV, AIC, AICc).

Warning!

- If there are a large number of predictors, this is not possible.
- For example, 44 predictors leads to 18 trillion possible models!

Choosing regression variables

Backwards stepwise regression

- Start with a model containing all variables.
- Try subtracting one variable at a time. Keep the model if it has lower CV or AICc.
- Iterate until no further improvement.

Forwards stepwise regression

* useful when you cannot fit all variables $K > T$

- Start with a model containing only a constant.
- Add one variable at a time. Keep the model if it has lower CV or AICc.
- Iterate until no further improvement.

Hybrid backwards and forwards also possible.

- Stepwise regression is not guaranteed to lead to the best possible model.

What should you use?

Notes

- Stepwise regression is not guaranteed to lead to the best possible model.
- Inference on coefficients of final model will be wrong. * Common mistake

Choice: CV, AIC, AICc, BIC, $\bar{R}^2$

- BIC tends to choose models too small for prediction (however can be useful for large k).
- $\bar{R}^2$ tends to select models too large.
- AIC also slightly biased towards larger models (especially when T is small).
- Empirical studies in forecasting show AIC is better than BIC for forecast accuracy.

Choice between AICc and CV (double check AIC and BIC where possible).

Outline

Ex-ante versus ex-post forecasts

- Ex ante forecasts are made using only information available in advance.
  - require forecasts of predictors * or future values are given to us.
- Ex post forecasts are made using later information on the predictors. * scenario based forecasting.
  - useful for studying behaviour of forecasting models.
- trend, seasonal and calendar variables are all known in advance, so these don't need to be forecast.

x In all cases prediction intervals are underestimated

Outline

Multiple regression forecasts

$X = \begin{bmatrix} 1 & X_{1,1} & \cdots & X_{k,1} \\ 1 & \vdots & & \\ \vdots & \vdots & & \\ 1 & X_{1,T} & \cdots & X_{k,T} \end{bmatrix}$ $\hat{\beta} = \begin{bmatrix} \hat{\beta}_1 \\ \vdots \\ \beta_k \end{bmatrix}$

Fitted values

$$\underset{(T\times 1)}{\hat{\boldsymbol{y}}} = \underset{(T\times k)(k\times 1)}{\boldsymbol{X}\hat{\boldsymbol{\beta}}} = \underbrace{\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'}_{(T\times T)}\boldsymbol{y} = \boldsymbol{H}\boldsymbol{y}$$

where $\boldsymbol{H} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$ is the "hat matrix' ".

Projection

Residual $e = y - \hat{y} = Ry$
where $R = I - H$

Leave-one-out residuals

Let $h_1, \dots, h_T$ be the diagonal values of $\boldsymbol{H}$, then the cross-validation statistic is

$$\text{CV} = \frac{1}{T}\sum_{t=1}^{T}[e_t/(1-h_t)]^2,$$

where e_t is the residual obtained from fitting the model to all T observations.

- So you can calculate the CV from only fitting one model

Outline

Correlation is not causation

correlation $\not\Rightarrow$ causation ; causation $\Rightarrow$ correlation BUT

- When x is useful for predicting y, it is not necessarily causing y.
- e.g., predict number of drownings y using number of ice-creams sold x. Cyclists on St Kilda Rd do not cause rain BUT . . . .
- Correlations are useful for forecasting, even when there is no causality.
- Better models usually involve causal relationships (e.g., temperature x and people z to predict drownings y). → but you can still forecast without causation.