

ETF3231/5231

Business forecasting

Week 6: Exponential smoothing

<https://bf.numbat.space/>



Outline

- 1 ETS models
- 2 Forecasting with ETS models

Outline

- 1 ETS models
- 2 Forecasting with ETS models

ETS models

General notation

ETS : ExponenTial Smoothing

Error Trend Season

ETS(y ~ **error**() + **trend**() + **season**())

Error: Additive ("A") or multiplicative ("M")

ETS models

General notation E T S : ExponenTial Smoothing

 ↗ ↑ ↖
 Error Trend Season

ETS(y ~ **error**() + **trend**() + **season**())

Error: Additive ("A") or multiplicative ("M")

Trend: None ("N"), additive ("A"), multiplicative ("M"), or damped ("Ad" or "Md").

ETS models

General notation

ETS : ExponenTial Smoothing

Error Trend Season

ETS(y ~ **error**() + **trend**() + **season**())

Error: Additive ("A") or multiplicative ("M")

Trend: None ("N"), additive ("A"), multiplicative ("M"), or damped ("Ad" or "Md").

Seasonality: None ("N"), additive ("A") or multiplicative ("M")

ETS(A,N,N): SES with additive errors

Measurement equation

$$y_t = l_{t-1} + \varepsilon_t$$

State equation

$$l_t = l_{t-1} + \alpha \varepsilon_t$$

where $\varepsilon_t \sim \text{NID}(0, \sigma^2)$.

- **innovations** or **single source of error** because equations have the same error process, ε_t .
- Measurement equation: relationship between observations and states.
- State equation(s): evolution of the state(s) through time.

Holt's methods method with additive errors.

Forecast equation

$$\hat{y}_{t+h|t} = \ell_t + hb_t$$

Observation equation

$$y_t = \ell_{t-1} + b_{t-1} + \varepsilon_t$$

State equations

$$\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$$

$$b_t = b_{t-1} + \beta\varepsilon_t$$

- Innovation residuals: $\hat{\varepsilon}_t = y_t - \hat{y}_{t|t-1}$

Holt-Winters additive method with additive errors.

Forecast equation

$$\hat{y}_{t+h|t} = \ell_t + hb_t + s_{t+h-m(k+1)}$$

Observation equation

$$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$$

State equations

$$\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$$

$$b_t = b_{t-1} + \beta\varepsilon_t$$

$$s_t = s_{t-m} + \gamma\varepsilon_t$$

- Innovation residuals: $\hat{\varepsilon}_t = y_t - \hat{y}_{t|t-1}$
- k is integer part of $(h - 1)/m$.

Holt-Winters multiplicative method with multiplicative errors.

Forecast equation	$\hat{y}_{t+h t} = (\ell_t + hb_t)s_{t+h-m(k+1)}$
Observation equation	$y_t = (\ell_{t-1} + b_{t-1})s_{t-m}(1 + \varepsilon_t)$
State equations	$\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t$ $s_t = s_{t-m}(1 + \gamma\varepsilon_t)$

- Innovation residuals: $\hat{\varepsilon}_t = (y_t - \hat{y}_{t|t-1})/\hat{y}_{t|t-1}$
- k is integer part of $(h - 1)/m$.

ETS model specification

```
ETS(y ~ error("A") + trend("N") + season("N"))
```

By default, optimal values for α , β , γ , and the states at time 0 are used.

The values for α , β and γ can be specified:

```
trend("A", alpha = 0.5, beta = 0.2)  
trend("A", alpha_range = c(0.2, 0.8), beta_range = c(0.1, 0.4))  
season("M", gamma = 0.04)  
season("M", gamma_range = c(0, 0.3))
```

Exponential smoothing methods

		Seasonal Component		
		N	A	M
Trend Component		(None)	(Additive)	(Multiplicative)
N	(None)	(N,N)	(N,A)	(N,M)
A	(Additive)	(A,N)	(A,A)	(A,M)
A _d	(Additive damped)	(A _d ,N)	(A _d ,A)	(A _d ,M)

(N,N): Simple exponential smoothing

(A,N): Holt's linear method

(A_d,N): Additive damped trend method

(A,A): Additive Holt-Winters' method

(A,M): Multiplicative Holt-Winters' method

(A_d,M): Damped multiplicative Holt-Winters' method

Exponential smoothing methods

		Seasonal Component		
		N	A	M
Trend Component		(None)	(Additive)	(Multiplicative)
N	(None)	(N,N)	(N,A)	(N,M)
A	(Additive)	(A,N)	(A,A)	(A,M)
A _d	(Additive damped)	(A _d ,N)	(A _d ,A)	(A _d ,M)

(N,N): Simple exponential smoothing

(A,N): Holt's linear method

(A_d,N): Additive damped trend method

(A,A): Additive Holt-Winters' method

(A,M): Multiplicative Holt-Winters' method

(A_d,M): Damped multiplicative Holt-Winters' method

There are also multiplicative trend methods (not recommended).

ETS models

Additive Error

		Seasonal Component		
Trend Component		N	A	M
		(None)	(Additive)	(Multiplicative)
N	(None)	A,N,N	A,N,A	A,N,M
A	(Additive)	A,A,N	A,A,A	A,A,M
A _d	(Additive damped)	A,A _d ,N	A,A _d ,A	A,A _d ,M

Multiplicative Error

		Seasonal Component		
Trend Component		N	A	M
		(None)	(Additive)	(Multiplicative)
N	(None)	M,N,N	M,N,A	M,N,M
A	(Additive)	M,A,N	M,A,A	M,A,M
A _d	(Additive damped)	M,A _d ,N	M,A _d ,A	M,A _d ,M

Exponential smoothing models

Additive Error

		Seasonal Component		
Trend Component		N	A	M
		(None)	(Additive)	(Multiplicative)
N	(None)	A,N,N	A,N,A	A,N,M
A	(Additive)	A,A,N	A,A,A	A,A,M
A _d	(Additive damped)	A,A _d ,N	A,A _d ,A	A,A_d,M

Multiplicative Error

		Seasonal Component		
Trend Component		N	A	M
		(None)	(Additive)	(Multiplicative)
N	(None)	M,N,N	M,N,A	M,N,M
A	(Additive)	M,A,N	M,A,A	M,A,M
A _d	(Additive damped)	M,A _d ,N	M,A _d ,A	M,A _d ,M

Additive error models

Trend	Seasonal		
	N	A	M
N	$y_t = \ell_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha \varepsilon_t$	$y_t = \ell_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha \varepsilon_t$ $s_t = s_{t-m} + \gamma \varepsilon_t$	$y_t = \ell_{t-1} s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha \varepsilon_t / s_{t-m}$ $s_t = s_{t-m} + \gamma \varepsilon_t / \ell_{t-1}$
A	$y_t = \ell_{t-1} + b_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha \varepsilon_t$ $b_t = b_{t-1} + \beta \varepsilon_t$	$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha \varepsilon_t$ $b_t = b_{t-1} + \beta \varepsilon_t$ $s_t = s_{t-m} + \gamma \varepsilon_t$	$y_t = (\ell_{t-1} + b_{t-1}) s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha \varepsilon_t / s_{t-m}$ $b_t = b_{t-1} + \beta \varepsilon_t / s_{t-m}$ $s_t = s_{t-m} + \gamma \varepsilon_t / (\ell_{t-1} + b_{t-1})$
A _d	$y_t = \ell_{t-1} + \phi b_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha \varepsilon_t$ $b_t = \phi b_{t-1} + \beta \varepsilon_t$	$y_t = \ell_{t-1} + \phi b_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha \varepsilon_t$ $b_t = \phi b_{t-1} + \beta \varepsilon_t$ $s_t = s_{t-m} + \gamma \varepsilon_t$	$y_t = (\ell_{t-1} + \phi b_{t-1}) s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha \varepsilon_t / s_{t-m}$ $b_t = \phi b_{t-1} + \beta \varepsilon_t / s_{t-m}$ $s_t = s_{t-m} + \gamma \varepsilon_t / (\ell_{t-1} + \phi b_{t-1})$

Multiplicative error models

Trend	Seasonal		
	N	A	M
N	$y_t = \ell_{t-1}(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$	$y_t = (\ell_{t-1} + s_{t-m})(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1} + \alpha(\ell_{t-1} + s_{t-m})\varepsilon_t$ $s_t = s_{t-m} + \gamma(\ell_{t-1} + s_{t-m})\varepsilon_t$	$y_t = \ell_{t-1}s_{t-m}(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$ $s_t = s_{t-m}(1 + \gamma\varepsilon_t)$
A	$y_t = (\ell_{t-1} + b_{t-1})(1 + \varepsilon_t)$ $\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t$	$y_t = (\ell_{t-1} + b_{t-1} + s_{t-m})(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha(\ell_{t-1} + b_{t-1} + s_{t-m})\varepsilon_t$ $b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1} + s_{t-m})\varepsilon_t$ $s_t = s_{t-m} + \gamma(\ell_{t-1} + b_{t-1} + s_{t-m})\varepsilon_t$	$y_t = (\ell_{t-1} + b_{t-1})s_{t-m}(1 + \varepsilon_t)$ $\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t$ $s_t = s_{t-m}(1 + \gamma\varepsilon_t)$
A _d	$y_t = (\ell_{t-1} + \phi b_{t-1})(1 + \varepsilon_t)$ $\ell_t = (\ell_{t-1} + \phi b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = \phi b_{t-1} + \beta(\ell_{t-1} + \phi b_{t-1})\varepsilon_t$	$y_t = (\ell_{t-1} + \phi b_{t-1} + s_{t-m})(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha(\ell_{t-1} + \phi b_{t-1} + s_{t-m})\varepsilon_t$ $b_t = \phi b_{t-1} + \beta(\ell_{t-1} + \phi b_{t-1} + s_{t-m})\varepsilon_t$ $s_t = s_{t-m} + \gamma(\ell_{t-1} + \phi b_{t-1} + s_{t-m})\varepsilon_t$	$y_t = (\ell_{t-1} + \phi b_{t-1})s_{t-m}(1 + \varepsilon_t)$ $\ell_t = (\ell_{t-1} + \phi b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = \phi b_{t-1} + \beta(\ell_{t-1} + \phi b_{t-1})\varepsilon_t$ $s_t = s_{t-m}(1 + \gamma\varepsilon_t)$

Model selection

Akaike's Information Criterion

$$AIC = -2 \log(L) + 2k$$

where L is the likelihood and k is the number of parameters initial states estimated in the model.

Corrected AIC

$$AIC_c = AIC + \frac{2k(k+1)}{T-k-1}$$

which is the AIC corrected (for small sample bias).

Bayesian (Schwartz) Information Criterion

$$BIC = AIC + k[\log(T) - 2] = -2 \log(L) + \ln(T)k$$

Minimizing the AIC assuming Gaussian residuals is asymptotically equivalent to minimizing one-step time series cross validation MSE.

Automatic forecasting

From Hyndman et al. (IJF, 2002):

- Apply each model that is appropriate to the data. Optimize parameters and initial values using MLE (or some other criterion).
- Select best model using AICc:
- Produce forecasts using best model.
- Obtain forecast intervals using underlying state space model.

Method performed very well in M3 competition.

Outline

- 1 ETS models
- 2 Forecasting with ETS models

Forecasting with ETS models

Traditional point forecasts: iterate the equations for
 $t = T + 1, T + 2, \dots, T + h$.

Forecasting with ETS models

Traditional point forecasts: iterate the equations for $t = T + 1, T + 2, \dots, T + h$.

- Not the same as $E(y_{t+h} | \mathbf{x}_t)$ unless seasonality is additive.
- fable uses $E(y_{t+h} | \mathbf{x}_t)$.
- Point forecasts for $ETS(A, *, *)$ are identical to $ETS(M, *, *)$ if the parameters are the same.

Example: ETS(A,A,N)

$$y_{T+1} = \ell_T + b_T + \varepsilon_{T+1}$$

$$\hat{y}_{T+1|T} = \ell_T + b_T$$

$$y_{T+2} = \ell_{T+1} + b_{T+1} + \varepsilon_{T+2}$$

$$= (\ell_T + b_T + \alpha\varepsilon_{T+1}) + (b_T + \beta\varepsilon_{T+1}) + \varepsilon_{T+2}$$

$$\hat{y}_{T+2|T} = \ell_T + 2b_T$$

etc.

Example: ETS(M,A,N)

$$y_{T+1} = (\ell_T + b_T)(1 + \varepsilon_{T+1})$$

$$\hat{y}_{T+1|T} = \ell_T + b_T.$$

$$y_{T+2} = (\ell_{T+1} + b_{T+1})(1 + \varepsilon_{T+2})$$

$$= \{(\ell_T + b_T)(1 + \alpha\varepsilon_{T+1}) + [b_T + \beta(\ell_T + b_T)\varepsilon_{T+1}]\} (1 + \varepsilon_{T+2})$$

$$\hat{y}_{T+2|T} = \ell_T + 2b_T$$

etc.

Forecasting with ETS models

Prediction intervals: can only be generated using the models.

- The prediction intervals will differ between models with additive and multiplicative errors.
- Exact formulae for some models.
- More general to simulate future sample paths, conditional on the last estimate of the states, and to obtain prediction intervals from the percentiles of these simulated future paths.

Prediction intervals

PI for most ETS models: $\hat{y}_{T+h|T} \pm c\sigma_h$, where c depends on coverage probability and σ_h is forecast standard deviation.

$$(A,N,N) \quad \sigma_h = \sigma^2 \left[1 + \alpha^2(h-1) \right]$$

$$(A,A,N) \quad \sigma_h = \sigma^2 \left[1 + (h-1) \left\{ \alpha^2 + \alpha\beta h + \frac{1}{6}\beta^2 h(2h-1) \right\} \right]$$

$$(A,A_d,N) \quad \sigma_h = \sigma^2 \left[1 + \alpha^2(h-1) + \frac{\beta\phi h}{(1-\phi)^2} \{2\alpha(1-\phi) + \beta\phi\} - \frac{\beta\phi(1-\phi^h)}{(1-\phi)^2(1-\phi^2)} \{2\alpha(1-\phi^2) + \beta\phi(1+2\phi-\phi^h)\} \right]$$

$$(A,N,A) \quad \sigma_h = \sigma^2 \left[1 + \alpha^2(h-1) + \gamma k(2\alpha + \gamma) \right]$$

$$(A,A,A) \quad \sigma_h = \sigma^2 \left[1 + (h-1) \left\{ \alpha^2 + \alpha\beta h + \frac{1}{6}\beta^2 h(2h-1) \right\} + \gamma k \{2\alpha + \gamma + \beta m(k+1)\} \right]$$

$$(A,A_d,A) \quad \sigma_h = \sigma^2 \left[1 + \alpha^2(h-1) + \frac{\beta\phi h}{(1-\phi)^2} \{2\alpha(1-\phi) + \beta\phi\} - \frac{\beta\phi(1-\phi^h)}{(1-\phi)^2(1-\phi^2)} \{2\alpha(1-\phi^2) + \beta\phi(1+2\phi-\phi^h)\} \right. \\ \left. + \gamma k(2\alpha + \gamma) + \frac{2\beta\gamma\phi}{(1-\phi)(1-\phi^m)} \{k(1-\phi^m) - \phi^m(1-\phi^{mk})\} \right]$$