

ETF3231/5231

Business forecasting

Week 5: Exponential smoothing

<https://bf.numbat.space/>



Outline

- 1 Exponential smoothing
- 2 Simple exponential smoothing

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Historical perspective

- Proposed in the late 1950s (Brown 1959, Holt 1957 and Winters 1960 are key pioneering works) as methods (algorithms) to produce point forecasts.
- Forecasts are **weighted averages** of past observations, with the **weights decaying exponentially** as the observations get older.
- Framework generates reliable forecasts quickly and for a wide spectrum of time series. A great advantage and of major importance to applications in industry.

Combine components

- Combine components: level ℓ_t , trend (slope) b_t and seasonal s_t to describe a time series

$$y_t = f(\ell_t, b_t, s_t)$$

- The rate of change of the components are controlled by “smoothing parameters”: α , β and γ respectively.
- Need to choose best values for the smoothing parameters (and initial states).
- Add error ε_t to get equivalent ETS state space models developed in the 1990s and 2000s.

Big idea: control the rate of change (smoothing)

α controls the flexibility of the level ℓ_t

- If $\alpha = 0$, the level never updates (mean)
- If $\alpha = 1$, the level updates completely (naive)

β controls the flexibility of the trend b_t

- If $\beta = 0$, the trend is linear (regression trend)
- If $\beta = 1$, the trend updates every observation

γ controls the flexibility of the seasonality s_t

- If $\gamma = 0$, the seasonality is fixed (seasonal means)
- If $\gamma = 1$, the seasonality updates completely (seasonal naive)

A model for levels, trends, and seasonalities

We want a model that captures the level (ℓ_t), trend (b_t) and seasonality (s_t).

How do we combine these elements?

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Additively?

$$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$$

Multiplicatively?

$$y_t = \ell_{t-1} b_{t-1} s_{t-m} (1 + \varepsilon_t)$$

Perhaps a mix of both?

$$y_t = (\ell_{t-1} + b_{t-1}) s_{t-m} + \varepsilon_t$$

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How do the level, trend and seasonal components evolve over time?

ETS models

General notation

ETS : ExponenTial Smoothing



Error Trend Season

The diagram shows three arrows pointing from the words 'Error', 'Trend', and 'Season' below to the letters 'E', 'T', and 'S' respectively in the 'ETS' acronym above.

ETS(y ~ **error**() + **trend**() + **season**())

Error: Additive ("A") or multiplicative ("M")

ETS models

General notation E T S : ExponenTial Smoothing

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 Error Trend Season

ETS(y ~ **error**() + **trend**() + **season**())

Error: Additive ("A") or multiplicative ("M")

Trend: None ("N"), additive ("A"), multiplicative ("M"), or damped ("Ad" or "Md").

ETS models

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ETS(y ~ **error**() + **trend**() + **season**())

Error: Additive ("A") or multiplicative ("M")

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Seasonality: None ("N"), additive ("A") or multiplicative ("M")

Models and methods

Methods

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Models

- Generate same point forecasts but can also generate forecast distributions.
- A stochastic (or random) data generating process that can generate an entire forecast distribution.
- Allow for “proper” model selection.

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Simple Exponential Smoothing - SES

Iterative form

$$\hat{y}_{t+1|t} = \alpha y_t + (1 - \alpha) \hat{y}_{t|t-1}$$

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$$\hat{y}_{t+1|t} = \alpha y_t + (1 - \alpha) \hat{y}_{t|t-1}$$

Weighted average form

$$\hat{y}_{T+1|T} = \sum_{j=0}^{T-1} \alpha (1 - \alpha)^j y_{T-j} + (1 - \alpha)^T \ell_0$$

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Component form

Forecast equation

$$\hat{y}_{t+1|t} = \ell_t$$

Smoothing equation

$$\ell_t = \alpha y_t + (1 - \alpha) \ell_{t-1}$$

ETS(A,N,N): SES with additive errors

Component form

Forecast equation

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Component form

Forecast equation

$$\hat{y}_{t+1|t} = \ell_t$$

Smoothing equation

$$\ell_t = \alpha y_t + (1 - \alpha)\ell_{t-1}$$

Residual: $e_t = y_t - \hat{y}_{t|t-1} = y_t - \ell_{t-1}$.

ETS(A,N,N): SES with additive errors

Component form

Forecast equation

$$\hat{y}_{t+1|t} = \ell_t$$

Smoothing equation

$$\ell_t = \alpha y_t + (1 - \alpha)\ell_{t-1}$$

Residual: $e_t = y_t - \hat{y}_{t|t-1} = y_t - \ell_{t-1}$.

Error correction form

$$y_t = \ell_{t-1} + e_t$$

$$\ell_t = \ell_{t-1} + \alpha(y_t - \ell_{t-1})$$

$$= \ell_{t-1} + \alpha e_t$$

ETS(A,N,N): SES with additive errors

Component form

Forecast equation

$$\hat{y}_{t+1|t} = l_t$$

Smoothing equation

$$l_t = \alpha y_t + (1 - \alpha)l_{t-1}$$

Residual: $e_t = y_t - \hat{y}_{t|t-1} = y_t - l_{t-1}$.

Error correction form

$$y_t = l_{t-1} + e_t$$

$$l_t = l_{t-1} + \alpha(y_t - l_{t-1})$$

$$= l_{t-1} + \alpha e_t$$

Specify probability distribution for e_t , we assume $e_t = \varepsilon_t \sim \text{NID}(0, \sigma^2)$. 12

ETS(A,N,N): SES with additive errors

Measurement equation

$$y_t = l_{t-1} + \varepsilon_t$$

State equation

$$l_t = l_{t-1} + \alpha \varepsilon_t$$

where $\varepsilon_t \sim \text{NID}(0, \sigma^2)$.

- **innovations** or **single source of error** because equations have the same error process, ε_t .
- Measurement equation: relationship between observations and states.
- State equation(s): evolution of the state(s) through time.

ETS(M,N,N): SES with multiplicative errors.

- Specify relative errors $\varepsilon_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}} \sim \text{NID}(0, \sigma^2)$
- Substituting $\hat{y}_{t|t-1} = \ell_{t-1}$ gives:
 - ▶ $y_t = \ell_{t-1} + \ell_{t-1}\varepsilon_t$
 - ▶ $e_t = y_t - \hat{y}_{t|t-1} = \ell_{t-1}\varepsilon_t$

ETS(M,N,N): SES with multiplicative errors.

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Measurement equation

$$y_t = \ell_{t-1}(1 + \varepsilon_t)$$

State equation

$$\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$$

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State equation

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- Models with additive and multiplicative errors with the same parameters generate the same point forecasts but different prediction intervals.