

# ETF3231/5231

## Business forecasting

Week 6: Exponential smoothing

<https://bf.numbat.space/>



# Outline

- 1 ETS models
- 2 Forecasting with ETS models

# Outline

- 1 ETS models
- 2 Forecasting with ETS models

# ETS models

**General notation**      E T S : ExponenTial Smoothing

                            ↑    ↑    ↑

                            Error Trend Season

**ETS**(y ~ **error**( ) + **trend**( ) + **season**( ))

**Error:** Additive ("A") or multiplicative ("M")

# ETS models

**General notation**      E T S : ExponenTial Smoothing

                            ↗    ↑    ↖  
                            Error Trend Season

**ETS**(y ~ **error**( ) + **trend**( ) + **season**( ))

**Error:** Additive ("A") or multiplicative ("M")

**Trend:** None ("N"), additive ("A"), multiplicative ("M"), or damped ("Ad" or "Md").

# ETS models

## General notation

ETS : Exponential Smoothing

Error Trend Season

**ETS**(y ~ **error**( ) + **trend**( ) + **season**( ))

**Error:** Additive ("A") or multiplicative ("M")

**Trend:** None ("N"), additive ("A"), multiplicative ("M"), or damped ("Ad" or "Md").

**Seasonality:** None ("N"), additive ("A") or multiplicative ("M")

## ETS(A,N,N): SES with additive errors

Observation equation

$$y_t = l_{t-1} + \varepsilon_t$$

State equation

$$l_t = l_{t-1} + \alpha \varepsilon_t$$

where  $\varepsilon_t \sim \text{NID}(0, \sigma^2)$ .

- **innovations** or **single source of error** because equations have the same error process,  $\varepsilon_t$ .
- Measurement equation: relationship between observations and states.
- State equation(s): evolution of the state(s) through time.

# ETS(A,A,N)

Holt's methods method with additive errors.

Forecast equation

$$\hat{y}_{t+h|t} = \ell_t + hb_t$$

Observation equation

$$y_t = \ell_{t-1} + b_{t-1} + \varepsilon_t$$

State equations

$$\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$$

$$b_t = b_{t-1} + \beta\varepsilon_t$$

- Forecast errors:  $\varepsilon_t = y_t - \hat{y}_{t|t-1}$



# ETS(A,A,A)

Holt-Winters additive method with additive errors.

Forecast equation

$$\hat{y}_{t+h|t} = \ell_t + hb_t + s_{t+h-m(k+1)}$$

Observation equation

$$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$$

State equations

$$\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$$

$$b_t = b_{t-1} + \beta\varepsilon_t$$

$$s_t = s_{t-m} + \gamma\varepsilon_t$$

- Forecast errors:  $\varepsilon_t = y_t - \hat{y}_{t|t-1}$
- $k$  is integer part of  $(h - 1)/m$ .

# ETS(M,A,M)

Holt-Winters multiplicative method with multiplicative errors.

Forecast equation	$\hat{y}_{t+h t} = (\ell_t + hb_t)s_{t+h-m(k+1)}$
Observation equation	$y_t = (\ell_{t-1} + b_{t-1})s_{t-m}(1 + \varepsilon_t)$
State equations	$\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t$ $s_t = s_{t-m}(1 + \gamma\varepsilon_t)$

- Forecast errors:  $\varepsilon_t = (y_t - \hat{y}_{t|t-1})/\hat{y}_{t|t-1}$
- $k$  is integer part of  $(h - 1)/m$ .

# ETS model specification

```
ETS(y ~ error("A") + trend("N") + season("N"))
```

By default, optimal values for  $\alpha$ ,  $\beta$ ,  $\gamma$ , and the states at time 0 are used.

The values for  $\alpha$ ,  $\beta$  and  $\gamma$  can be specified:

```
trend("A", alpha = 0.5, beta = 0.2)  
trend("A", alpha_range = c(0.2, 0.8), beta_range = c(0.1, 0.4))  
season("M", gamma = 0.04)  
season("M", gamma_range = c(0, 0.3))
```

# Exponential smoothing methods

		Seasonal Component		
		N	A	M
Trend Component		(None)	(Additive)	(Multiplicative)
N	(None)	(N,N)	(N,A)	(N,M)
A	(Additive)	(A,N)	(A,A)	(A,M)
A <sub>d</sub>	(Additive damped)	(A <sub>d</sub> ,N)	(A <sub>d</sub> ,A)	(A <sub>d</sub> ,M)

(N,N): Simple exponential smoothing

(A,N): Holt's linear method

(A<sub>d</sub>,N): Additive damped trend method

(A,A): Additive Holt-Winters' method

(A,M): Multiplicative Holt-Winters' method

(A<sub>d</sub>,M): Damped multiplicative Holt-Winters' method

# Exponential smoothing methods

		Seasonal Component		
		N	A	M
Trend Component		(None)	(Additive)	(Multiplicative)
N	(None)	(N,N)	(N,A)	(N,M)
A	(Additive)	(A,N)	(A,A)	(A,M)
A <sub>d</sub>	(Additive damped)	(A <sub>d</sub> ,N)	(A <sub>d</sub> ,A)	(A <sub>d</sub> ,M)

(N,N): Simple exponential smoothing

(A,N): Holt's linear method

(A<sub>d</sub>,N): Additive damped trend method

(A,A): Additive Holt-Winters' method

(A,M): Multiplicative Holt-Winters' method

(A<sub>d</sub>,M): Damped multiplicative Holt-Winters' method

There are also multiplicative trend methods (not recommended).

# ETS models

## Additive Error

Trend Component		Seasonal Component		
		N (None)	A (Additive)	M (Multiplicative)
N	(None)	A,N,N	A,N,A	A,N,M
A	(Additive)	A,A,N	A,A,A	A,A,M
A <sub>d</sub>	(Additive damped)	A,A <sub>d</sub> ,N	A,A <sub>d</sub> ,A	A,A <sub>d</sub> ,M

## Multiplicative Error

Trend Component		Seasonal Component		
		N (None)	A (Additive)	M (Multiplicative)
N	(None)	M,N,N	M,N,A	M,N,M
A	(Additive)	M,A,N	M,A,A	M,A,M
A <sub>d</sub>	(Additive damped)	M,A <sub>d</sub> ,N	M,A <sub>d</sub> ,A	M,A <sub>d</sub> ,M

# Exponential smoothing models

## Additive Error

		Seasonal Component		
Trend Component		N	A	M
		(None)	(Additive)	(Multiplicative)
N	(None)	A,N,N	A,N,A	<del>A,N,M</del>
A	(Additive)	A,A,N	A,A,A	<del>A,A,M</del>
A <sub>d</sub>	(Additive damped)	A,A <sub>d</sub> ,N	A,A <sub>d</sub> ,A	<del>A,A<sub>d</sub>,M</del>

## Multiplicative Error

		Seasonal Component		
Trend Component		N	A	M
		(None)	(Additive)	(Multiplicative)
N	(None)	M,N,N	M,N,A	M,N,M
A	(Additive)	M,A,N	M,A,A	M,A,M
A <sub>d</sub>	(Additive damped)	M,A <sub>d</sub> ,N	M,A <sub>d</sub> ,A	M,A <sub>d</sub> ,M

# Additive error models

Trend	Seasonal		
	N	A	M
N	$y_t = \ell_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha \varepsilon_t$	$y_t = \ell_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha \varepsilon_t$ $s_t = s_{t-m} + \gamma \varepsilon_t$	$y_t = \ell_{t-1} s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha \varepsilon_t / s_{t-m}$ $s_t = s_{t-m} + \gamma \varepsilon_t / \ell_{t-1}$
A	$y_t = \ell_{t-1} + b_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha \varepsilon_t$ $b_t = b_{t-1} + \beta \varepsilon_t$	$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha \varepsilon_t$ $b_t = b_{t-1} + \beta \varepsilon_t$ $s_t = s_{t-m} + \gamma \varepsilon_t$	$y_t = (\ell_{t-1} + b_{t-1}) s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha \varepsilon_t / s_{t-m}$ $b_t = b_{t-1} + \beta \varepsilon_t / s_{t-m}$ $s_t = s_{t-m} + \gamma \varepsilon_t / (\ell_{t-1} + b_{t-1})$
A <sub>d</sub>	$y_t = \ell_{t-1} + \phi b_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha \varepsilon_t$ $b_t = \phi b_{t-1} + \beta \varepsilon_t$	$y_t = \ell_{t-1} + \phi b_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha \varepsilon_t$ $b_t = \phi b_{t-1} + \beta \varepsilon_t$ $s_t = s_{t-m} + \gamma \varepsilon_t$	$y_t = (\ell_{t-1} + \phi b_{t-1}) s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha \varepsilon_t / s_{t-m}$ $b_t = \phi b_{t-1} + \beta \varepsilon_t / s_{t-m}$ $s_t = s_{t-m} + \gamma \varepsilon_t / (\ell_{t-1} + \phi b_{t-1})$



# Multiplicative error models

Trend	Seasonal		
	N	A	M
N	$y_t = \ell_{t-1}(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$	$y_t = (\ell_{t-1} + s_{t-m})(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1} + \alpha(\ell_{t-1} + s_{t-m})\varepsilon_t$ $s_t = s_{t-m} + \gamma(\ell_{t-1} + s_{t-m})\varepsilon_t$	$y_t = \ell_{t-1}s_{t-m}(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$ $s_t = s_{t-m}(1 + \gamma\varepsilon_t)$
A	$y_t = (\ell_{t-1} + b_{t-1})(1 + \varepsilon_t)$ $\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t$	$y_t = (\ell_{t-1} + b_{t-1} + s_{t-m})(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha(\ell_{t-1} + b_{t-1} + s_{t-m})\varepsilon_t$ $b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1} + s_{t-m})\varepsilon_t$ $s_t = s_{t-m} + \gamma(\ell_{t-1} + b_{t-1} + s_{t-m})\varepsilon_t$	$y_t = (\ell_{t-1} + b_{t-1})s_{t-m}(1 + \varepsilon_t)$ $\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t$ $s_t = s_{t-m}(1 + \gamma\varepsilon_t)$
A <sub>d</sub>	$y_t = (\ell_{t-1} + \phi b_{t-1})(1 + \varepsilon_t)$ $\ell_t = (\ell_{t-1} + \phi b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = \phi b_{t-1} + \beta(\ell_{t-1} + \phi b_{t-1})\varepsilon_t$	$y_t = (\ell_{t-1} + \phi b_{t-1} + s_{t-m})(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha(\ell_{t-1} + \phi b_{t-1} + s_{t-m})\varepsilon_t$ $b_t = \phi b_{t-1} + \beta(\ell_{t-1} + \phi b_{t-1} + s_{t-m})\varepsilon_t$ $s_t = s_{t-m} + \gamma(\ell_{t-1} + \phi b_{t-1} + s_{t-m})\varepsilon_t$	$y_t = (\ell_{t-1} + \phi b_{t-1})s_{t-m}(1 + \varepsilon_t)$ $\ell_t = (\ell_{t-1} + \phi b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = \phi b_{t-1} + \beta(\ell_{t-1} + \phi b_{t-1})\varepsilon_t$ $s_t = s_{t-m}(1 + \gamma\varepsilon_t)$

# Model selection

## Akaike's Information Criterion

$$AIC = -2 \log(L) + 2k$$

where  $L$  is the likelihood and  $k$  is the number of parameters initial states estimated in the model.

## Corrected AIC

$$AIC_c = AIC + \frac{2k(k+1)}{T-k-1}$$

which is the AIC corrected (for small sample bias).

## Bayesian (Schwartz) Information Criterion

$$BIC = AIC + k[\log(T) - 2] = -2 \log(L) + \ln(T)k$$

Minimizing the AIC assuming Gaussian residuals is asymptotically equivalent to minimizing one-step time series cross validation MSE.

# Automatic forecasting

**From Hyndman et al. (IJF, 2002):**

- Apply each model that is appropriate to the data. Optimize parameters and initial values using MLE (or some other criterion).
- Select best model using AICc:
- Produce forecasts using best model.
- Obtain forecast intervals using underlying state space model.

Method performed very well in M3 competition.

# Residuals

## Residuals (response)

$$e_t = y_t - \hat{y}_{t|t-1}$$

## Innovation residuals

Additive error model:

$$\hat{\varepsilon}_t = y_t - \hat{y}_{t|t-1}$$

Multiplicative error model:

$$\hat{\varepsilon}_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}}$$

# Outline

- 1 ETS models
- 2 Forecasting with ETS models

# Forecasting with ETS models

**Traditional point forecasts:** iterate the equations for  
 $t = T + 1, T + 2, \dots, T + h$ .

# Forecasting with ETS models

**Traditional point forecasts:** iterate the equations for  $t = T + 1, T + 2, \dots, T + h$ .

- Not the same as  $E(y_{t+h} | \mathbf{x}_t)$  unless seasonality is additive.
- fable uses  $E(y_{t+h} | \mathbf{x}_t)$ .
- Point forecasts for  $ETS(A, *, *)$  are identical to  $ETS(M, *, *)$  if the parameters are the same.



## Example: ETS(A,A,N)

$$y_{T+1} = \ell_T + b_T + \varepsilon_{T+1}$$

$$\hat{y}_{T+1|T} = \ell_T + b_T$$

$$y_{T+2} = \ell_{T+1} + b_{T+1} + \varepsilon_{T+2}$$

$$= (\ell_T + b_T + \alpha\varepsilon_{T+1}) + (b_T + \beta\varepsilon_{T+1}) + \varepsilon_{T+2}$$

$$\hat{y}_{T+2|T} = \ell_T + 2b_T$$

etc.

## Example: ETS(M,A,N)

$$y_{T+1} = (\ell_T + b_T)(1 + \varepsilon_{T+1})$$

$$\hat{y}_{T+1|T} = \ell_T + b_T.$$

$$y_{T+2} = (\ell_{T+1} + b_{T+1})(1 + \varepsilon_{T+2})$$

$$= \{(\ell_T + b_T)(1 + \alpha\varepsilon_{T+1}) + [b_T + \beta(\ell_T + b_T)\varepsilon_{T+1}]\} (1 + \varepsilon_{T+2})$$

$$\hat{y}_{T+2|T} = \ell_T + 2b_T$$

etc.

# Forecasting with ETS models

**Prediction intervals:** can only be generated using the models.

- The prediction intervals will differ between models with additive and multiplicative errors.
- Exact formulae for some models.
- More general to simulate future sample paths, conditional on the last estimate of the states, and to obtain prediction intervals from the percentiles of these simulated future paths.

# Prediction intervals

PI for most ETS models:  $\hat{y}_{T+h|T} \pm c\sigma_h$ , where  $c$  depends on coverage probability and  $\sigma_h$  is forecast standard deviation.

$$(A,N,N) \quad \sigma_h = \sigma^2 \left[ 1 + \alpha^2(h-1) \right]$$

$$(A,A,N) \quad \sigma_h = \sigma^2 \left[ 1 + (h-1) \left\{ \alpha^2 + \alpha\beta h + \frac{1}{6}\beta^2 h(2h-1) \right\} \right]$$

$$(A,A_d,N) \quad \sigma_h = \sigma^2 \left[ 1 + \alpha^2(h-1) + \frac{\beta\phi h}{(1-\phi)^2} \{2\alpha(1-\phi) + \beta\phi\} - \frac{\beta\phi(1-\phi^h)}{(1-\phi)^2(1-\phi^2)} \{2\alpha(1-\phi^2) + \beta\phi(1+2\phi-\phi^h)\} \right]$$

$$(A,N,A) \quad \sigma_h = \sigma^2 \left[ 1 + \alpha^2(h-1) + \gamma k(2\alpha + \gamma) \right]$$

$$(A,A,A) \quad \sigma_h = \sigma^2 \left[ 1 + (h-1) \left\{ \alpha^2 + \alpha\beta h + \frac{1}{6}\beta^2 h(2h-1) \right\} + \gamma k \{2\alpha + \gamma + \beta m(k+1)\} \right]$$

$$(A,A_d,A) \quad \sigma_h = \sigma^2 \left[ 1 + \alpha^2(h-1) + \frac{\beta\phi h}{(1-\phi)^2} \{2\alpha(1-\phi) + \beta\phi\} - \frac{\beta\phi(1-\phi^h)}{(1-\phi)^2(1-\phi^2)} \{2\alpha(1-\phi^2) + \beta\phi(1+2\phi-\phi^h)\} \right. \\ \left. + \gamma k(2\alpha + \gamma) + \frac{2\beta\gamma\phi}{(1-\phi)(1-\phi^m)} \{k(1-\phi^m) - \phi^m(1-\phi^{mk})\} \right]$$