Background

"Time is money", as mentioned in the lecture, bond is one of the ways for government to generate money. This assignment aims to give us the basic understanding on bonds. We are asked to interpret the yield rates and forward rates of five bonds issued by the government of Canada in 1, 2, 3, 4 and 5-year terms. To do so, we first pick five bonds with maturity in five consecutive years and track the prices of each bond in a two-week period.

Bond Prices

Coupon	Maturity	2018-01-15	2018-01-16	2018-01-17	2018-01-18	2018-01-19	2018-01-22	2018-01-23	2018-01-24	2018-01-25	2018-01-26
1.25	2018-03-01	100.03	100.03	100.02	100.02	100.02	100.02	100.02	100.02	100.02	100.02
1.75	2019-03-01	100.19	100.17	100.17	100.13	100.14	100.14	100.14	100.13	100.14	100.13
1.5	2020-03-01	99.49	99.45	99.44	99.38	99.4	99.4	99.4	99.37	99.4	99.38
0.75	2021-03-01	96.7	96.61	96.58	96.48	96.53	96.53	96.53	96.48	96.51	96.48
0.5	2022-03-01	94.43	94.31	94.26	94.18	94.21	94.21	94.21	94.13	94.17	94.11

We round the day counting to the nearest month for calculation convenience. Thus, the day counting would be Feb 1st, 2018, and assume today is Feb 1st, 2018. We notice that all the Maturity of the five bonds are on March 1st, so the last payment dates are all Sep 1st, 2017. Firstly, Bond 1 is zero coupon, and the last payment is on its maturity. $\left(ie. t_1 = \frac{1}{12}\right)$. Secondly, Bond 2 has three more future cash flows, which are on Mar 1st & Sep 1st, 2018 and Mar 1st, 2019. $\left(ie. t_1 = \frac{1}{12}, t_2 = \frac{7}{12}, t_3 = \frac{13}{12}\right)$. Thirdly, Bond 3 has five more future cash flows, which are on Mar 1st & Sep 1st, 2018, Mar 1st & Sep 1st, 2019 and Mar 1st, 2020. $(ie. t_1 = \frac{1}{12}, t_2 = \frac{7}{12}, t_3 = \frac{13}{12}, t_4 = \frac{19}{12}, t_5 = \frac{25}{12})$. Then, Bond 4 has seven more future cash flows, which are on Mar 1st & Sep 1st, 2018, Mar 1st & Sep 1st, 2019, Mar 1st & Sep 1st, 2020 and Mar 1st, 2021 $(ie. t_1 = \frac{1}{12}, t_2 = \frac{7}{12}, t_3 = \frac{13}{12}, t_4 = \frac{19}{12}, t_5 = \frac{25}{12}, t_6 = \frac{31}{12}, t_7 = \frac{37}{12})$. Last but not least, Bond 5 has nine more future cash flows, which are on Mar 1st & Sep 1st, 2018, Mar 1st & Sep 1st, 2019, Mar 1st & Sep 1st, 2020, Mar 1st & Sep 1st, 2021 and Mar 1st, 2022. $(ie. t_1 = \frac{1}{12}, t_2 = \frac{7}{12}, t_3 = \frac{13}{12}, t_4 = \frac{19}{12}, t_5 = \frac{25}{12}, t_6 = \frac{31}{12}, t_7 = \frac{37}{12}, t_8 = \frac{43}{12}, t_9 = \frac{49}{12})$

In this assignment, Face Value is defined as \$ 100. The notional, which is the payment at maturity is defined as 100 + C/2, call it N. Thus, for bond 1, $N = 100.625$; for bond 2, $N = 100.875$; for bond 3, $N = 100.75$; for bond 4, $N = 100.375$; for bond 5, $N = 100.25$. In the above chart, bond prices are referred to clean price. We also need to be careful that the coupons are paid semi-annually in this case.

With the above information on mind, we can calculate the yield rate of the selected bonds issued by the government of Canada.

Calculation of Yield Rates

Dirty Price:

$$Dirty\ Price = Accrued\ Interest + Clean\ Price$$

Accrued Interest

$$Accrued\ Interest = \frac{\#days\ since\ last\ payment}{365} * Annual\ coupon\ rate$$

Thus, we obtain the following chart with Dirty Price:

Coupon	Maturity	2018-01-15	2018-01-16	2018-01-17	2018-01-18	2018-01-19	2018-01-22	2018-01-23	2018-01-24	2018-01-25	2018-01-26
1.25	2018-03-01	100.550833	100.550833	100.540833	100.540833	100.540833	100.540833	100.540833	100.540833	100.540833	100.540833
1.75	2019-03-01	100.919167	100.899167	100.899167	100.859167	100.869167	100.869167	100.869167	100.859167	100.869167	100.859167
1.5	2020-03-01	100.115	100.075	100.065	100.005	100.025	100.025	100.025	99.995	100.025	100.005
0.75	2021-03-01	97.0125	96.9225	96.8925	96.7925	96.8425	96.8425	96.8425	96.7925	96.8225	96.7925
0.5	2022-03-01	94.6383333	94.5183333	94.4683333	94.3883333	94.4183333	94.4183333	94.4183333	94.3383333	94.3783333	94.3183333

First, we obtain accrued interest by multiplying the annual coupon rate with the ratio of number of days since last payment with the number of a year. By adding the accrued interests to the clean price, which is what we obtain from the website, we then get the dirty prices of the five bonds on each of the ten days. The dirty price is corresponding to the summation of future cash flows with individual's yield rates. Then, we get:

$$Dirty\ Price = \sum_i p_i e^{-r(t_i)t_i}$$

We first calculate the yield rates of the first bond. Notice that the first bond is zero-coupon, so we can use the formula $r_T = \frac{-\ln\left(\frac{P}{N}\right)}{T}$. P is the dirty price, which we have obtained in the previous step, and N is the notional, which I have included in the "bond prices" section. $Plug\ into\ r_{\frac{1}{12}},\ get\ r_{\frac{1}{12}} = r_{\frac{7}{12}} = r_1$ (Assume the same yield rate less than one year). Then, we are able to get the following chart:

Date	r1
15-Jan	0.015672032
16-Jan	0.017442741
17-Jan	0.016645306
18-Jan	0.0172179
19-Jan	0.016931589
22-Jan	0.016931589
23-Jan	0.0172179
24-Jan	0.017504241
25-Jan	0.0172179
26-Jan	0.017504241

Then we use the second bond to calculate the 2-year yield rate. By looking at the second bond, we notice that there are three coupon payments remain. One is on Mar 1st, 2018, one is Sep 1st, 2018 and the last one is on maturity, which is Mar 1st, 2019. Using the formula:

$$Dirty\ price = \sum_i p_i e^{-rt_i}$$

$$\frac{C}{2}e^{-r(t_1)t_1} + \frac{C}{2}e^{-r(t_2)t_2} + \left(100 + \frac{C}{2}\right)e^{-r(t_3)t_3} = Dirty\ Price$$

Here $t_1 = \frac{1}{12}, t_2 = \frac{7}{12}, t_3 = \frac{13}{12}. And\ r_{t_1} = r_{t_2} = \mathrm{r}_1, \mathrm{C} = 1.75.$ Solve this equation and $get\ r_{\frac{13}{12}}$. By linearity, assume the slope between year 1 yield rate to year 2 yield rate is m, so $m = \frac{r_{\frac{13}{12}} - r_1}{\frac{13}{12} - 1} = \frac{r_2 - r_1}{2-1} \rightarrow r_2 - r_1 = m \rightarrow r_2 = m + r_1$. Then, we get r2 table.

Date	r2
15-Jan	0.017239754
16-Jan	0.017442228
17-Jan	0.017483636
18-Jan	0.017789592
19-Jan	0.017691239
22-Jan	0.017691239
23-Jan	0.017721309
24-Jan	0.017852056
25-Jan	0.017852056
26-Jan	0.017842993

Follow the same procedures, for bond 3 we have:

$$\frac{C}{2}e^{-r_1 t_1}+\frac{C}{2}e^{-r_1 t_2}+\frac{c}{2}e^{-r_{\frac{13}{12}}*\left(\frac{13}{12}\right)}+\frac{c}{2}e^{-r_{\frac{19}{12}}*\left(\frac{19}{12}\right)}+\left(100+\frac{C}{2}\right)e^{-r_{\frac{25}{12}}*\frac{25}{12}}=\ Dirty\ Price\ 3$$

By linearity, we can obtain $r_{\frac{19}{12}}$ from the slope we calculate in the last step, or from the graph we will get afterward. And we can get $r_{\frac{25}{12}}$ by solving the equation on the top of the page. Then calculate r_3 using slope by assuming linearity between r_2 & r_3. Then for all 10 days:

Date	r3
15-Jan	0.01805352
16-Jan	0.018364069
17-Jan	0.018481723
18-Jan	0.018796188
19-Jan	0.018660881
22-Jan	0.01872259
23-Jan	0.018699968
24-Jan	0.018878889
25-Jan	0.018803765
26-Jan	0.018913728

Date	r4
15-Jan	0.018739685
16-Jan	0.018739685
17-Jan	0.01922247
18-Jan	0.019436316
19-Jan	0.019373029
22-Jan	0.019400622
23-Jan	0.019410648
24-Jan	0.019633949
25-Jan	0.019543954
26-Jan	0.019708968

To calculate yield rate on year 4, use $\frac{C}{2}e^{-r_1 t_1}+\frac{C}{2}e^{-r_1 t_2}+\frac{c}{2}e^{-r_{\frac{13}{12}}*\left(\frac{13}{12}\right)}+\frac{c}{2}e^{-r_{\frac{19}{12}}*\left(\frac{19}{12}\right)}+\frac{c}{2}e^{-\frac{r_{25}}{12}*\frac{25}{12}}+\frac{c}{2}e^{-\frac{r_{31}}{12}*\frac{31}{12}}+\left(100+\frac{c}{2}\right)e^{-\frac{r_{37}}{12}*\frac{37}{12}}=\ Dirty\ Price\ 4$. Similarly, we get the r4 chart.

Lastly, we calculate yield rate on year 5, $\frac{C}{2}e^{-r_1 t_1}+\frac{C}{2}e^{-r_1 t_2}+\frac{c}{2}e^{-r_{\frac{13}{12}}*\left(\frac{13}{12}\right)}+\frac{c}{2}e^{-r_{\frac{19}{12}}*\left(\frac{19}{12}\right)}+\frac{c}{2}e^{-\frac{r_{25}}{12}*\frac{25}{12}}+\frac{c}{2}e^{-\frac{r_{31}}{12}*\frac{31}{12}}+\frac{c}{2}e^{-r_{\frac{37}{12}}*\frac{37}{12}}+\frac{c}{2}e^{-\frac{r_{43}}{12}*\frac{43}{12}}+\left(100+\frac{c}{2}\right)e^{-\frac{r_{49}}{12}*\frac{49}{12}}=$ $Dirty\ Price\ 5$

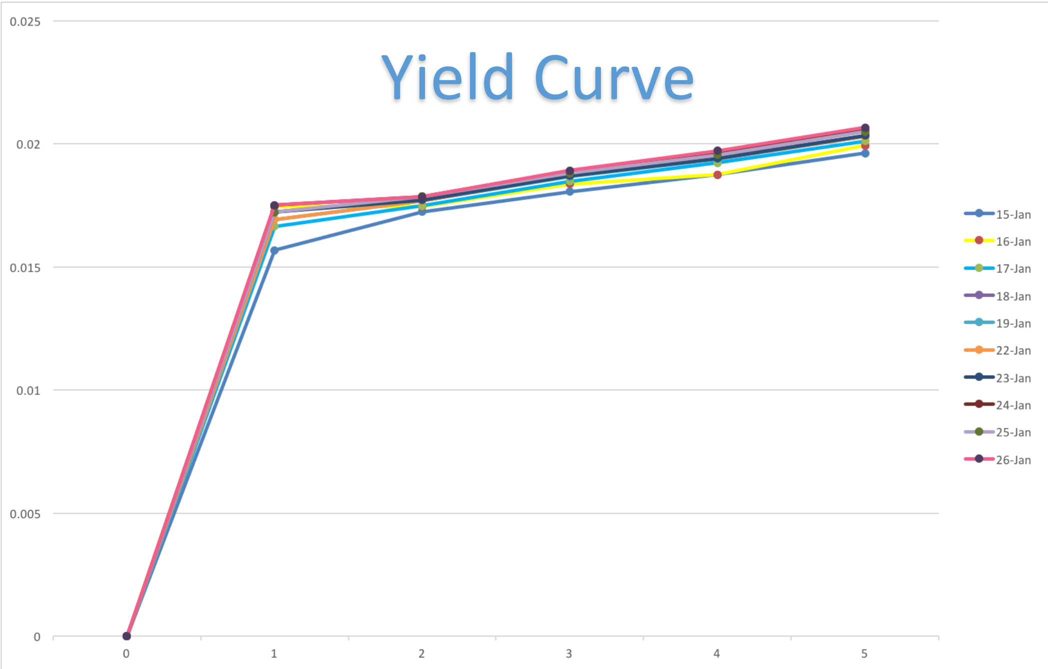
Obtain,

Date	r5
15-Jan	0.019622999
16-Jan	0.019931851
17-Jan	0.020104123
18-Jan	0.020338074
19-Jan	0.020289887
22-Jan	0.020293227
23-Jan	0.020341364
24-Jan	0.020541245
25-Jan	0.020484722
26-Jan	0.020655947

Now, we can attain the yield curve for each day.

Yield Curve

Use excel to draw the Yield Curves, we show the 10 yield curves for 10 different days in a single graph, they are similar due to the similar price on each day.



As the graph shown above, we observed that the yield rate is increasing as the number of years become larger. The upward sloping of the yield curves is due to the higher yields of long-term bond. By research, "Long term yields are higher because of the liquidity premium and the risk premium added by the risk of default from holding a security over the long term." [1]

[1] *Yield curve." Wikipedia. Wikipedia.org, 4 February 2018, last edited on 4 February 2018, https://en.wikipedia.org/wiki/Yield_curve.*

Forward Rate

After obtaining the yield rates for 1 to 5-year term, we can now calculate the forward rate. As explained in the slide,

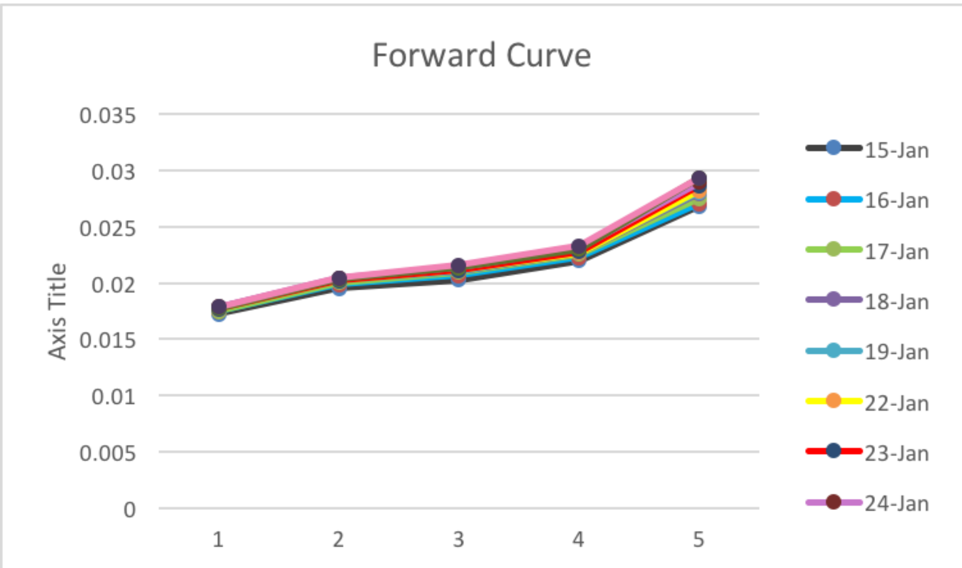
$$P(t,T) = e^{-r(T-t)}\ (formula\ 1)$$

$$r = -\frac{logP(t,T)}{(T-t)}$$

We use 1, 2, 3, 4 and 5-year yield rates to generate the prices of zero-coupon bonds, which pays $1 at time T and has no other intermediate payments, yet have maturity in 1, 2, 3, 4 & 5-year. That is, we discount the face value by the i-year yield rate to calculate the i-year zero coupon bond's price. Then, we obtain formula 1 defined above, taking log on both sides, formula 2 is then acquired. The forward rate is defined to be $f(t,T) = r(t,T,T) = -\frac{\partial}{\partial T} logP(t,T)$; however, the price here is discrete since we only use 5 bonds to generate the rates. Thus, the function is not differentiable. To solve this problem, we can assume the linear interpolation in order to attain forward rates. That is, $f(t,T) = r(t,T,T) = \frac{\log(p_{t+1}) - \log(p_{t-1})}{2}$.

Forward Curve

We use the similar way as we draw yield curves to draw forward curves.



The above chart demonstrates the upward sloping curves, which imply the higher yield on longer maturity bonds. Thus, we conclude that as time getting larger, the forward rate increases since the payoff is expected to be higher for further maturity date.

Two charts with yield curves & forward curves

We use Excel to generate the forward rates for each days and arrange them into a table with yield rates we obtained previously.

Date	r1	r2	r3	r4	r5	f1	f2	f3	f4	f5
15-Jan	0.01567203	0.01723975	0.01805352	0.01873969	0.019623	0.017239754	0.01950231	0.02023962	0.02197722	0.02682204
16-Jan	0.01744274	0.01744223	0.01836407	0.01873969	0.01993185	0.017442228	0.01988702	0.02067727	0.02228352	0.0270512
17-Jan	0.01664531	0.01748364	0.01848172	0.01922247	0.02010412	0.017483636	0.0200363	0.0209613	0.02253772	0.02748292
18-Jan	0.0172179	0.01778959	0.01879619	0.01943632	0.02033807	0.017789592	0.02035821	0.02108304	0.0226509	0.02797053
19-Jan	0.01693159	0.01769124	0.01866088	0.01937303	0.02028989	0.017691239	0.02017975	0.02105482	0.0227334	0.02806333
22-Jan	0.01693159	0.01769124	0.01872259	0.01940062	0.02029323	0.01770773	0.02027535	0.02109351	0.02264918	0.02818433
23-Jan	0.0172179	0.01772131	0.01869997	0.01941065	0.02034136	0.017721309	0.02019689	0.02109999	0.02280346	0.02863233
24-Jan	0.01750424	0.01785206	0.01887889	0.01963395	0.02054125	0.017852056	0.02043717	0.02141584	0.02303478	0.02893569
25-Jan	0.0172179	0.01785206	0.01880376	0.01954395	0.02048472	0.01772542	0.02034933	0.02136249	0.02300616	0.02932387
26-Jan	0.01750424	0.01784299	0.01891373	0.01970897	0.02065595	0.017842993	0.02043406	0.02157494	0.02326928	0.02933398

Covariance Matrix

To calculate the covariance matrix of the time series of daily log-returns of yield and forward rates. Use the formula given: $X_{i,j} = \log\left(\frac{r_{i,j+1}}{r_{i,j}}\right), i = 1,..5,\ j = 1, \dots, 9$, Where i refers to i-th year yield rate, and j refers to the j-th day. To get the covariance matrix, we use the following:

$$cov(X) = \frac{1}{N-1}(X - \bar{X})^T (X - \bar{X}), \qquad N\ is\ the\ degree\ of\ freedom.$$

Then we use R to attain the covariance matrix of X_i (Time series related to yield rates)

```
             [,1]         [,2]         [,3]         [,4]         [,5]
[,1]  0.010646291  0.011676159  0.017055314    0.0169353  0.015616702
[,2]  0.003545541  0.002371193  0.006386294  0.008501132  0.008605926
[,3]  0.019299377  0.017348215   0.01687179  0.011063348  0.011569787
[,4] -0.003130522  -0.00554405  -0.00722469 -0.003261413 -0.002372154
[,5] -0.000389393  0.000931749  0.003301396  0.001423268  0.000164628
[,6]  0.005686002  0.000766547 -0.001208987  0.000516691  0.002369268
[,7]  0.003571926  0.007350855  0.009522494   0.01143838  0.009778374
[,8] -0.003157283 -0.007118879 -0.003987184 -0.004594226 -0.002755511
[,9]  0.010158957  0.006611079  0.005830872  0.008407786  0.008323964
```

Then generate cov(X)

```
> cov(X)
             V1           V2           V3           V4           V5
V1 5.363962e-05 5.351285e-05 5.137828e-05 3.978459e-05 3.819564e-05
V2 5.351285e-05 6.151225e-05 6.352955e-05 5.116081e-05 4.613354e-05
V3 5.137828e-05 6.352955e-05 7.258980e-05 5.910826e-05 5.233005e-05
V4 3.978459e-05 5.116081e-05 5.910826e-05 5.430700e-05 4.773320e-05
V5 3.819564e-05 4.613354e-05 5.233005e-05 4.773320e-05 4.289345e-05
```

We then find the eigenvalues and eigenvectors.

```
> x <- cov(X)
> eigen(x)
$values
[1] 2.608443e-04 1.822685e-05 4.563910e-06 1.220596e-06 8.641106e-08

$vectors
           [,1]       [,2]       [,3]       [,4]        [,5]
[1,] -0.4064476  0.7346997 -0.3573811  0.3146624  0.26131010
[2,] -0.4757493  0.2841861  0.2717340 -0.7339082 -0.28361988
[3,] -0.5165699 -0.1855876  0.6711395  0.4968364  0.03792213
[4,] -0.4343598 -0.4988837 -0.3088079 -0.2687798  0.62836437
[5,] -0.3910882 -0.3100434 -0.5026418  0.2080321 -0.67453460
```

Similarly, use R to attain the covariance matrix of Y_i (Time series related to forward rates)

```
> Y
            V1           V2           V3           V4          V5
1   0.011676159  0.019534617  0.021393234  0.013841197 0.008507127
2   0.002371193  0.007478203  0.013642958  0.011342965 0.015833374
3   0.017348215  0.015938958  0.005790813  0.005009251 0.017586905
4  -0.005544050 -0.008805053 -0.001339399  0.003635236 0.003312266
5   0.000931749  0.004726451  0.001836079 -0.003711198 0.004302318
6   0.000766547 -0.003877149  0.000306892  0.006788422 0.015770360
7   0.007350855  0.011826698  0.014858521  0.010093037 0.010539243
8  -0.007118879 -0.004307402 -0.002494558 -0.001243400 0.013326357
9   0.006611079  0.004154892  0.009896149  0.011372049 0.000344597
```

Then take cov(Y)

```
> cov(Y)
             V1           V2            V3           V4            V5
V1 5.467755e-05 5.992257e-05  3.780273e-05 2.187982e-05  9.165306e-06
V2 5.992257e-05 8.223903e-05  5.864380e-05 2.610156e-05  1.104217e-05
V3 3.780273e-05 5.864380e-05  6.155201e-05 3.670330e-05 -6.653900e-08
V4 2.187982e-05 2.610156e-05  3.670330e-05 3.205846e-05  1.561136e-06
V5 9.165306e-06 1.104217e-05 -6.653900e-08 1.561136e-06  3.431793e-05
```

Finally, find the eigenvalues and eigenvectors.

```
> y <- cov(Y)
> eigen(y)
$values
[1] 1.891549e-04 3.968141e-05 2.490887e-05 1.100530e-05 9.450739e-08

$vectors
            [,1]       [,2]       [,3]       [,4]        [,5]
[1,] -0.48257063  0.2676526 -0.3467431  0.6588607  0.37571207
[2,] -0.63556010  0.2006590 -0.3129049 -0.4007880 -0.54521450
[3,] -0.51987613 -0.4081761  0.2929429 -0.3871048  0.57223725
[4,] -0.29503204 -0.3516550  0.5657318  0.4946964 -0.47383321
[5,] -0.07664102  0.7731941  0.6131817 -0.1034262  0.09802143
```

Reference

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