Solving directional dynamic games for all Markov perfect equilibria

Dynamic Programming and Structural Econometrics #19

Fedor Iskhakov, Australian National University
John Rust, Georgetown University
Bertel Schjerning, University of Copenhagen

ROAD MAP for the next two lectures

1. Collusion of Australian corrugated fibre packaging (CFP) producers
   - Collusion between Amcor and Visy
   - Bertrand pricing and investment game
   - Solution concept: Markov perfect equilibrium (MPE)
2. Experiment with the model
3. State recursion algorithm
   - Theory of directional dynamic games (DDGs)
4. Recursive lexicographical search (RLS) algorithm
5. Full solution for the leapfrogging game
6. Structural estimation of directional dynamic games with Nested RLS method

Overview of results

- We extend the standard static textbook model of Bertrand price competition by allowing duopolists to undertake cost-reducing investments in discrete time
- Technological progress is exogenous and stochastic
- Each firm has a binary decision to acquire the state of the art production technology
- Even though this is a small extension of the classic static model of Bertrand price competition, surprisingly little is known about Bertrand competition in the presence of production cost uncetainty, especially in dynamic settings
- We show how to compute all equilibria of this game and show that this dynamic model of Bertrand price competition has surprisingly rich, complex, and counter-intuitive equilibrium outcomes.

How do you find *all* Markov Perfect Equilibria?

The Markov Perfect Equilibrium (MPE) concept of Maskin and Tirole (1988) is now a widely used in *empirical IO*. However computing MPE remains a daunting computation problem

Quote (Hörner *et. al. Econometrica* 2011)

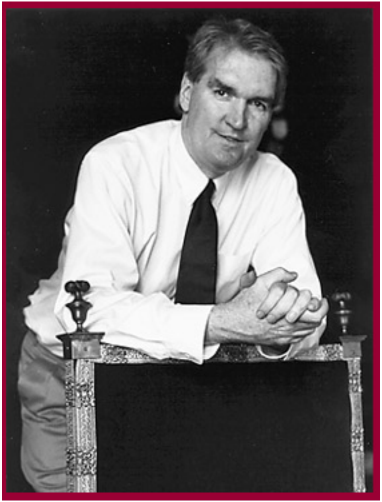
"Dynamic games are difficult to solve. In repeated games, finding some equilibrium is easy, as any repetition of a stage-game Nash equilibrium will do. This is not the case in stochastic games. The characterization of even the most elementary equilibria for such games, namely (stationary) Markov equilibria, in which continuation strategies depend on the current state only, turns out to be often challenging."

Finding even a single MPE is challenging!

- How do people "find" MPEs?
- Theorists: Guess and Verify
- Applied people: Iterate on the player Bellman equations
- Pakes and McGuire (1994): some of the earliest work on computing MPE. Proposed a deterministic, iterative algorithm to compute MPE. Found a curse of dimensionality in trying to solve MPE model of firm dynamics with even moderate numbers of firms
- Pakes and McGuire (2001): Proposed a *stochastic algorithm* to approximate an MPE, in an attempt to break this curse of dimensionality

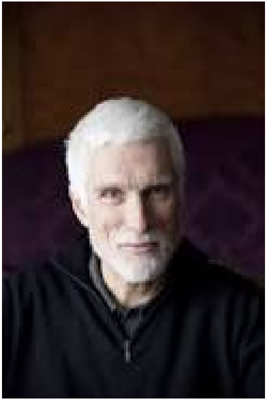
Eric Maskin: taught game theory class at MIT

One of John's classmates: Tim Kehoe



Another classmate: David Levine

Another classmate: Drew Fudenberg



Another classmate: Jean Tirole

Motivation: Collusion on the beach

Peter Brown: Amcor Managing Director

Harry Debney: Visy CEO

Russell Jones: Chairman of Amcor

Richard Pratt: Owner of Visy

The Australian cardboard market

- The Australian market for *cardboard* (CFP) is essentially a duopoly
- Between 2000 and 2005 the two firms, *Visy* and *Amcor* colluded to raise the price of CFP
- I was hired to estimate the damage caused by the collusion, which requires predicting what CFP prices would have been in the absence of collusion
- My opinion was that the "but-for" CFP prices are those predicted by Bertrand price competition in the short run, with *leapfrogging investments* by the two firms over the longer run as they vie for low-cost leadership

Amcor's New B9 Paper Mill

Main Mill Site, Botany Bay Road, Botany Bay NSW

Source: Amcor

B9 is an example of leapfrogging

- Amcor's existing paper plant was over 50 years old
- "The B9 paper machine, so named as it is the ninth paper machine to operate at the company's Botany site, will produce more than 400,000 tonnes of paper annually when operating at full capacity and will deliver significant environmental benefits."
- Cost: $500 million, the largest single investment in Amcor's 144 year history. "Largest and most innovative recycled paper machine of its kind in Australasia"
- "The machine is 330 metres long, and 22 metres high, and produces 1.6 km of paper per minute and reduces water consumption by 26%, energy usage by 34% and the amount of waste sent to landfill by 75%" (Nigel Garrard, Amcor CEO)

But collusion caused B9 to be abandoned

- Amcor had planned B9 back in 1999, and at that time internal studies estimated huge rate of return for this investment because it would enable it to leapfrog Visy to become the low-cost producer of CFP in Australia.
- Amcor and Visy were locked in a price war that started in 1999, around the time the Amcor Board authorized the B9 investment.
- However when Visy and Amcor started to collude in 2000, the B9 project was curiously scrapped. B9 was not actually started until 2011, well after the end of the collusion in 2005. B9 only came online in February 2013.

Justification for Bertrand pricing

- cardboard is a highly standardized product
- the consumers of cardboard are firms that are highly rational and interested in buying inputs at least possible cost
- further, firms acquire these inputs via *tenders* that create strong incentives for Bertrand-like price cutting
- In the case, we lacked good data on *aggregate demand* for cardboard facing Amcor and Visy before and after collusion
- but there was good data on their *costs of production*
- cardboard is made on production lines with machinery that is well-approximated as constant returns to scale with constant marginal costs

A cardboard corrugator

Technological progress via cost-reducing investments

- in this industry, Amcor and Visy do minimal amounts of R&D since there is limited scope for new product innovations to replace cardboard
- however the firms do spend considerable amounts on *cost reducing investments*
- these investments consist of building new plants or upgrading existing plants with the latest technology and machinery for producing cardboard
- rather than developing these machines themselves, Amcor and Visy purchase these machines from other companies that specialize in doing the R&D and product development to develop the machines that produce cardboard at the least possible cost

Leapfrogging by Amcor lead to a price war

- the proximate cause of the collusion between Amcor and Visy was a price war in cardboard
- a key input to cardboard is *paper* and Amcor had a severe cost disadvantage relative to Visy due to its outdated paper production plant, with machines that had not been replaced/upgraded in decades
- Visy on the other hand, has aggressively invested in the latest and most cost-efficient technology and maintained a persistent edge as the low cost leader
- however Amcor planned to invest in a new paper mill, B9, enabling it to produce CFP at substantially lower costs, thereby leapfrogging Visy to become the low cost leader in Australia

Are price wars evidence of tacit collusion?

- The economic experts defending Amcor and Visy dismissed theory of Bertrand competition and leap frogging investments as naive and out of touch with reality
- They claim that there is a huge body of research and empirical work in IO that supports a theory *tacit collusion* for repeatedly interacting duopolists
- In particular, they claimed that duopolists could achieve via tacit collusion the same discounted profits as they could via *explicit collusion.*
- This implies that the damage is *zero.*
- But if this is the case, and if tacit collusion is *legal,* why would Amcor and Visy have had any incentive to do illegal explicit collusion?

Paucity of empirical support for tacit collusion

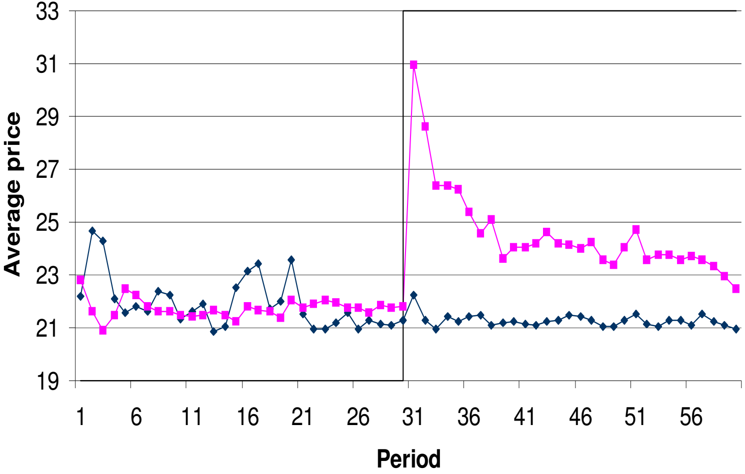
- Tacit collusion is hard to "observe" by the very fact that it is tacit
- We need good data on costs and demands to calculate what the cartel price would be
- Most of the empirical work on tacit collusion comes from laboratory experiments
- Hundreds of experiments done on tacit collusion have found that it is extremely difficult to "grow" tacit collusion in laboratory settings
- There are very few "field studies" that find evidence of tacit collusion outside of Breshnahan's (1987) JIE paper, "Competition and Collusion in the American Automobile Industry: the 1955 Price War"

Conclusions of meta-study of over 500 experiments

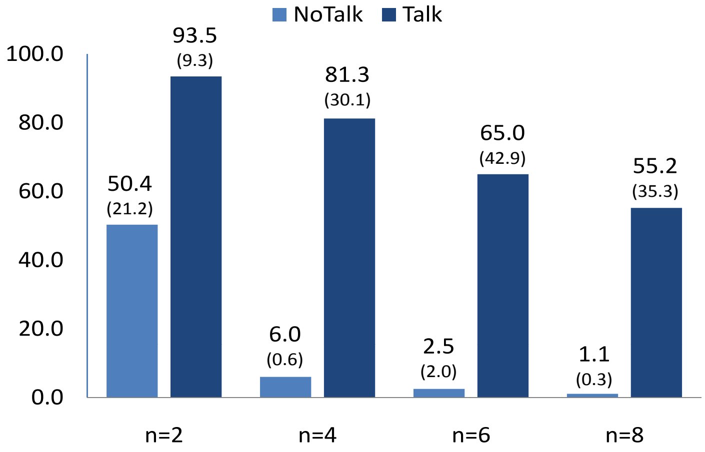
- Christoph Engel (2007) "Tacit Collusion The Neglected Experimental Evidence"
- Econometric meta-analysis of 510 laboratory experiments finds no systematic evidence supporting tacit collusion
- D. Engelmann and W. Müller (2008) "Collusion through price ceilings? A search for a focal point effect"
- "Note that the Folk Theorem (see for example Tirole, 1988) predicts that infinitely many prices can occur as outcomes of collusive equilibria in infinitely repeated games if the discount factor is sufficiently high. This suggests a coordination problem when firms attempt to collude." (p. 2)

Results of a laboratory duopoly

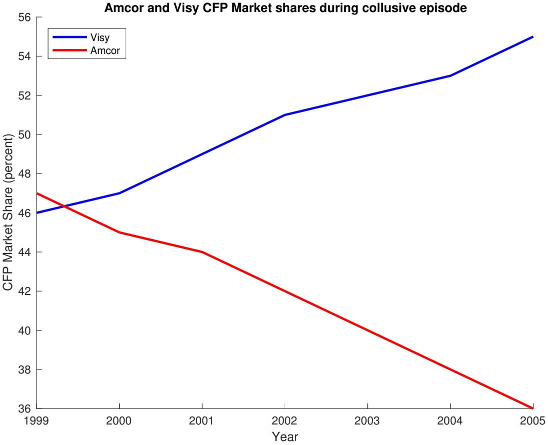
(Note: the Bertrand price is 21, the maximum cartel price is 48 and 28 is the price ceiling)



Explicit communication is necessary for collusion



Amcor and Visy collusive market shares



David Rapson's reanalysis of Breshnahan 1987

- David Rapson (2009) "Tacit Collusion in the 1950s Automobile Industry? Revisiting Bresnahan (1987)"
- "This paper reexamines the competitive landscape in the 1950s U.S. automobile industry, and tests the robustness of the famous result from Bresnahan (1987) that firms were engaged in tacit collusion."
- Rapson uses a random coefficients logit model allows for more realistic demand behavior, including a broad set of possible substitution patterns in characteristic space.
- This enables firms to engage in a more realistic set of potential actions, including intrabrand or intrafirm, multiproduct strategic pricing.

David Rapson's reanalysis of Breshnahan 1987

- "Relative to Bresnahan's framework, these improvements increase the likelihood that high price-cost markups will be attributed properly to either strategic oligopoly behavior or collusion." (p. 21).
- "*For no year can either of the forms of Bertrand competition be rejected in favor of tacit collusion.* This stands in contrast to Bresnahan's finding that firms were colluding in 1954 and 1956, with a price war in 1955."
- "These results accentuate the paucity of empirical evidence in favor of tacit collusion.
- Bresnahan's (1987) famous paper is one of the only studies that claim evidence of its occurrence."

Nicolas de Roos' analysis of the US lysine cartel

- Theorists model collusion as an incentive-compatible, self-enforcing mechanism where price wars either do not exist in equilibrium, or if they exist, they are on the equilibrium path of a dynamic game of asymmetric information with cost or demand shocks.
- Papers such as "Cramton and Palfrey (1990) show that efficient collusion is attainable in which the lowest cost firm is allocated full production and the monopoly price is set."
- However de Roos argues that these theories may be out of touch with reality "Both price wars contain elements of a bargaining or negotiation problem. Disagreements persisted about the appropriate market shares for the participants as well as the fundamental issue of exactly what form the cartel should take."

de Roos' analysis of the US lysine cartel, cont.

- "A second such issue relates to the existence of cheating in the lysine market. It appears that cheating occurred or was at least heavily suspected by cartel participants." ... "where cheating is a problem for a cartel, this suggests the lack of an incentive compatible enforcement mechanism."
- A price war, prior to the cartel, was a result of ADM's leap-frogging: "In 1988, ADM acquired a fermentation technique for lysine and, observed by its incumbent rivals, began production of the world's largest lysine factory in 1989. ADM's plant began production in February 1991, precipitating a severe price war. "

The Bertrand Investment Paradox

Why should Bertrand competitors undertake cost-reducing investments?

- Suppose a pair of duopolists simultaneously invest in the state of the art low cost production technology with marginal cost c
- Bertrand price competition following these investments will lead to a price of $p = c$ and *zero profits for each firm*
- If each firm earns zero profits *ex post*, why would either have incentive to invest *ex ante*?

The investment stage game is an anti-coordination game. Can the firms dynamically coordinate their investments in equilibrium, in order to avoid "bad" simultaneous investment outcomes?

The Riordan and Salant Conjecture

- In their 1994 *Journal of Industrial Economics* article, Riordan and Salant proved that in continuous time, if duopolists move alternately and technological progress is deterministic, then investment preemption is the only possible equilibrium outcome
- Further, they show this equilibrium is *completely inefficient* due to the excessively frequent investments of the preempting firm, a result they call rent dissipation
- They conjectured that their result does not depend on the alternating move assumption and that preemption (as opposed to leapfrogging) will be the generic equilibrium outcome in models of Bertrand price competition with cost-reducing investements.

Solution to the Bertrand Investment Paradox

We show:

- Endogenous coordination is possible in equilibrium
  - leapfrogging (alternating investments) is possible
  - We show that the Riordan and Salant conjecture is wrong: leapfrogging, not preemption, is the generic outcome
- Price paths are piecewise flat and non-increasing
  - *Price wars* occur when the high cost firm leapfrogs its rival to become the new low-cost leader
  - These price wars lead to *permanent* price declines, unlike the conventional interpretation of price wars as punishments for periodic breakdowns in tacit collusion
- Equilibria are generally inefficient due to overinvestment
  - duplicative investments
  - excessively frequent investments

Computing all equilibria

Our findings are based on the computation of all Markov perfect equilibria of this dynamic game

- New solution approach consisting of:
  1. State recursion algorithm for finding stage equilibria
  2. Recursive Lexicographic equilibrium Search (RLS) algorithm for finding all MPE paths
- Traditional solution approach (value function iterations, i.e. time recursion) fails in this model due to multiplicity of equilibria
  - Implementation of the Bellman operator induces an equilibrium selection rule
  - Not a contraction mapping, convergence is not guaranteed
- Danger of imposing symmetry
  - Most of MPE equilibria we find are asymmetric

ROAD MAP

1. Collusion of Australian corrugated fibre packaging (CFP) producers
   - Collusion between Amcor and Visy
   - Bertrand pricing and investment game
   - Solution concept: Markov perfect equilibrium (MPE)
2. Experiment with the model
3. State recursion algorithm
4. Recursive lexicographical search (RLS) algorithm
5. Full solution for the leapfrogging game
6. Structural estimation of directional dynamic games with Nested RLS method

Dynamic Bertrand price competition

Stochastic dynamic game

- Two Bertrand competitors, $n = 2$, no entry or exit
- Discrete time, infinite horizon ($t = 1, 2, \dots, \infty$)
- Each firm maximizes expected discounted profits, common discount factor $\beta \in (0, 1)$
- Each firm has two choices in each period:
  1. Price for the product
  2. Whether or not to buy the state of the art technology

Static Bertrand price competition in each period

- Continuum of consumers make static purchase decision
- No switching costs: buy from the lower price supplier

Cost-reducing investments

State-of-the-art production cost c process

- Initial value c_0, lowest value 0: $0 \leq c \leq c_0$
- Discretized with n points
- Follows exogenous Markov process and only improves
- Markov transition probability $\pi(c_{t+1}|c_t)$
  $\pi(c_{t+1}|c_t) = 0$ if $c_{t+1} > c_t$

Investment choice: dichotomous

- Investment cost of $K(c)$ to obtain marginal cost c
- One period construction time: production with technology obtained at t starts at $t+1$

State space and information structure

Common knowledge

- State of the game: cost structure (c_1, c_2, c)
- State space is $S = (c_1, c_2, c) \subset R^3$: $c_1 \geq c,\ c_2 \geq c$
- Actions are observable

Private information

- In each period each firm incurs additive costs (benefits) from not investing and investing $\eta\epsilon_{i,I}$ and $\eta\epsilon_{i,N}$
- $\epsilon_{i,I}$ and $\epsilon_{i,N}$ are extreme value distributed, independent across choice, time and firms
- $\eta \geq 0$ is a scaling parameter
- Investment choice probabilities have logit form for $\eta > 0$

Timing of moves

Pricing decisions are made simultaneously

Expected one period profit of firm i from Bertrand game ($j \neq i$)

$$r_i(c_1, c_2) = \begin{cases} 0 & \text{if } c_i \geq c_j \\ c_j - c_i & \text{if } c_i < c_j \end{cases}$$

Two versions regarding investment decisions

1. Simultaneous moves:
   - Investment decisions are made simultaneously
2. Alternating moves:
   - The "right to move" state variable $m \in \{1, 2\}$,
   - When $m = i$, only firm i can make a cost reducing investment
   - m follows an own Markov process
     (deterministic alternation as a special case).

Actions and behavior strategies

Two choices in each period

- $p_i(c_1, c_2, c) = \max(c_1, c_2)$ – Bertrand pricing decision
- $P_i^I(c_1, c_2, c)$ – probability of firm i to invest in state-of-the-art production technology

  $P_i^N(c_1, c_2, c) = 1 - P_i^I(c_1, c_2, c)$ – probability not to invest

Strategy profile

- $\sigma = (\sigma_1, \sigma_2)$ – pair of Markovian *behavior* strategies

  $\sigma_i = \Big(p_i(c_1, c_2, c), P_i^I(c_1, c_2, c)\Big) \in \mathbb{R}_+ \times [0, 1]$
- Pure strategies included as special case

Definition of Markov Perfect Equilibium

Definition (Markov perfect equilibrium (MPE))

MPE of Bertrand investment stochastic game is a pair of

- strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*)$, and
- pair of *value functions* $V(s) = \Big(V_1(s), V_2(s)\Big)$, $V_i : S \to R$,

such that

1. Bellman equations (below) are satisfied for each firm, and
2. strategies σ_1^* and σ_2^* constitute mutual best responses, and assign positive probabilities only to the actions in the set of maximizers of the Bellman equations.

Bellman equations, firm $i = 1$, simultaneous moves

$$V_i(c_1, c_2, c) = \max \left[v_i^I(c_1, c_2, c) + \eta\epsilon_{i,I}, v_i^N(c_1, c_2, c) + \eta\epsilon_{i,N} \right]$$

$$\begin{aligned} v_i^N(c_1, c_2, c) &= r^i(c_1, c_2) + \beta EV_i(c_1, c_2, c|N) \\ v_i^I(c_1, c_2, c) &= r^i(c_1, c_2) - K(c) + \beta EV_i(c_1, c_2, c|I) \end{aligned}$$

With extreme value shocks, the investment probability is

$$P_i^I(c_1, c_2, c) = \frac{\exp\{v_I^I(c_1, c_2, c)/\eta\}}{\exp\{v_i^I(c_1, c_2, c)/\eta\} + \exp\{v_i^N(c_1, c_2, c)/\eta\}}$$

Bellman equations, firm $i = 1$, simultaneous moves

The expected values are given by

$$
\begin{aligned}
EV_i(c_1, c_2, c|N) &= \int_0^c \left[P_j^I(c_1, c_2, c)H_i(c_1, c, c') + \right. \\
&\qquad \left. P_j^N(c_1, c_2, c)H_i(c_1, c_2, c')\right] \pi(dc'|c) \\
EV_i(c_1, c_2, c|I) &= \int_0^c \left[P_j^I(c_1, c_2, c)H_i(c, c, c') + \right. \\
&\qquad \left. P_j^N(c_1, c_2, c)H_i(c, c_2, c')\right] \pi(dc'|c)
\end{aligned}
$$

where

$$H_i(c_1, c_2, c) = \eta \log \left[\exp\left(v_i^N(c_1, c_2, c)/\eta\right) + \exp\left(v_i^I(c_1, c_2, c)/\eta\right)\right]$$

is the "smoothed max" or logsum function

ROAD MAP

1. Collusion of Australian corrugated fibre packaging (CFP) producers
    - Collusion between Amcor and Visy
    - Bertrand pricing and investment game
    - Solution concept: Markov perfect equilibrium (MPE)
2. **Experiment with the model**
3. State recursion algorithm
    - Theory of directional dynamic games (DDGs)
4. Recursive lexicographical search (RLS) algorithm
5. Full solution for the leapfrogging game
6. Structural estimation of directional dynamic games with Nested RLS method

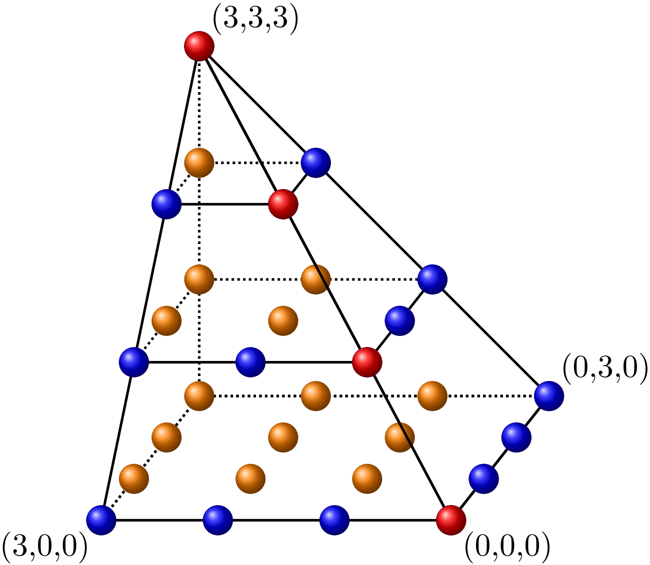
Leapfrogging Game

ROAD MAP

1. Collusion of Australian corrugated fibre packaging (CFP) producers
   - Collusion between Amcor and Visy
   - Bertrand pricing and investment game
   - Solution concept: Markov perfect equilibrium (MPE)
2. Experiment with the model
3. State recursion algorithm
   - Theory of directional dynamic games (DDGs)
4. Recursive lexicographical search (RLS) algorithm
5. Full solution for the leapfrogging game
6. Structural estimation of directional dynamic games with Nested RLS method

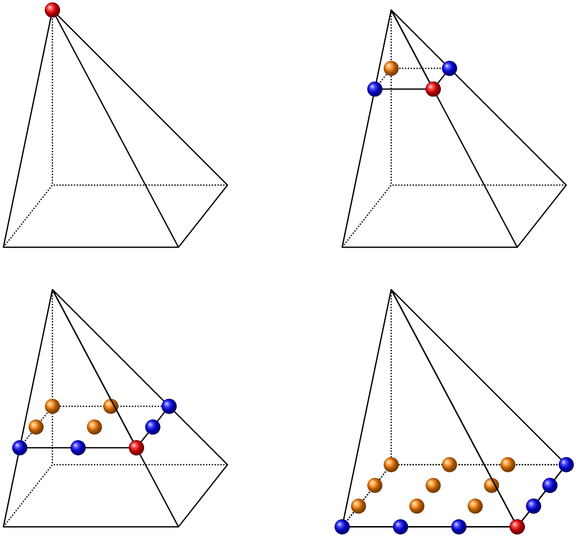
Discretized state space = a "quarter pyramid"

$S = \{(c_1, c_2, c) | c_1 \geq c,\ c_2 \geq c, c \in [0,3]\},\ n = 4$



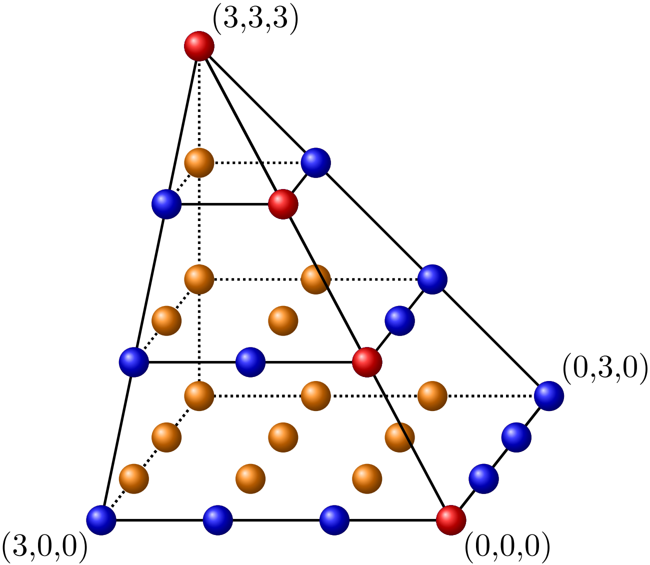
Transitions due to technological progress

As c decreases, the game falls through the layers of the pyramid



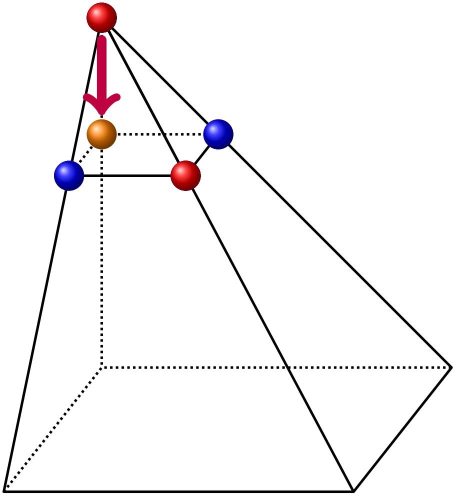
Discretized state space = a "quarter pyramid"

$S = \{(c_1, c_2, c) | c_1 \geq c,\ c_2 \geq c, c \in [0, \bar{c}]\},\ N = 4$



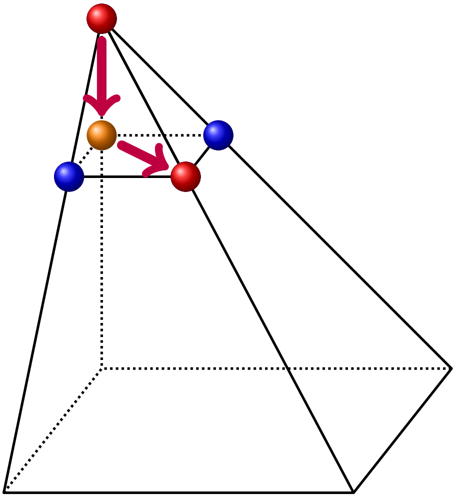
Game dynamics: example

The game starts at the apex, as some point technology improves



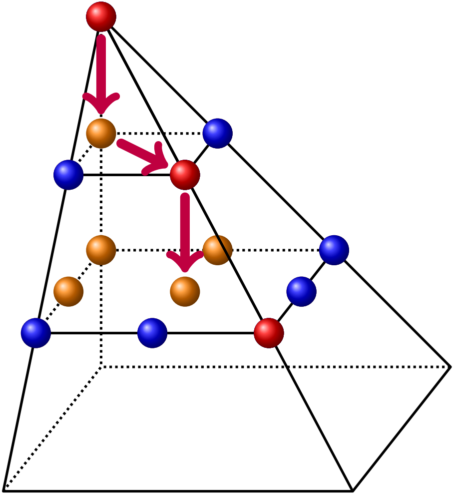
Game dynamics: example

Both firms buy new technology $c = 2 \rightsquigarrow (c_1, c_2, c) = (2, 2, 2)$



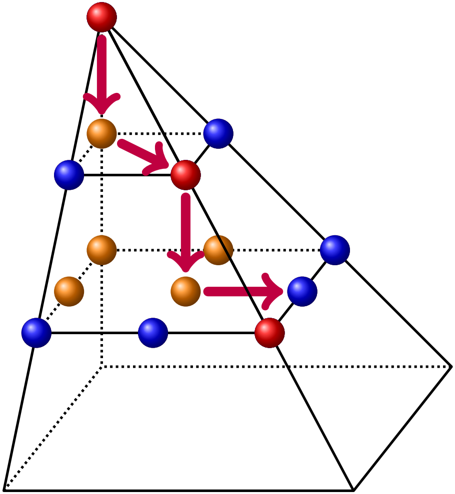
Game dynamics: example

State-of-the-art technology becomes $c = 1 \rightsquigarrow (c_1, c_2, c) = (2, 2, 1)$



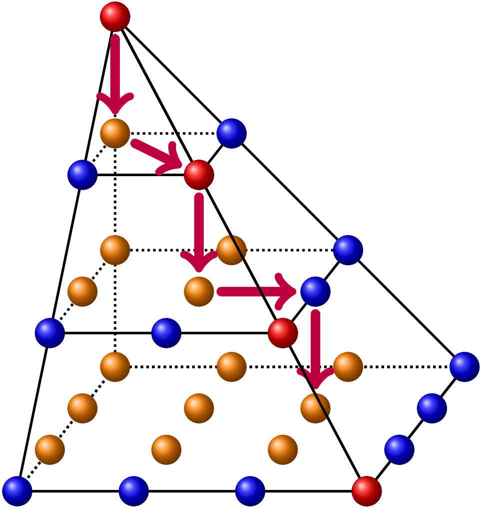
Game dynamics: example

Firm 1 invests and becomes cost leader $\rightsquigarrow (c_1, c_2, c) = (1, 2, 1)$



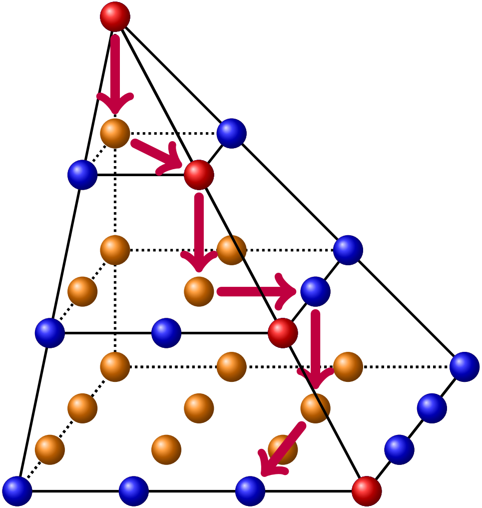
Game dynamics: example

State-of-the-art technology becomes $c = 0 \rightsquigarrow (c_1, c_2, c) = (1, 2, 0)$



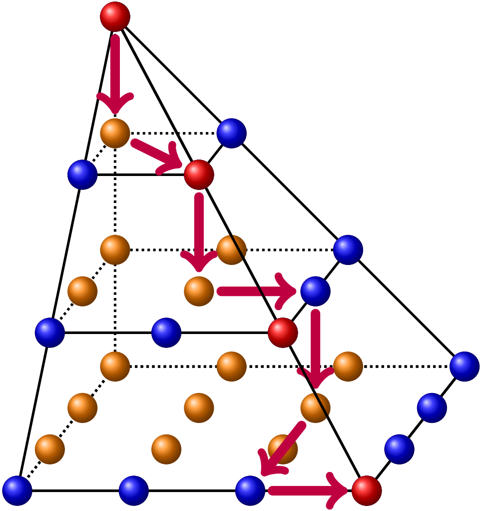
Game dynamics: example

Firm 2 leapfrogs firm 1 to become new cost leader $\rightsquigarrow (c_1, c_2, c) = (1, 0, 0)$



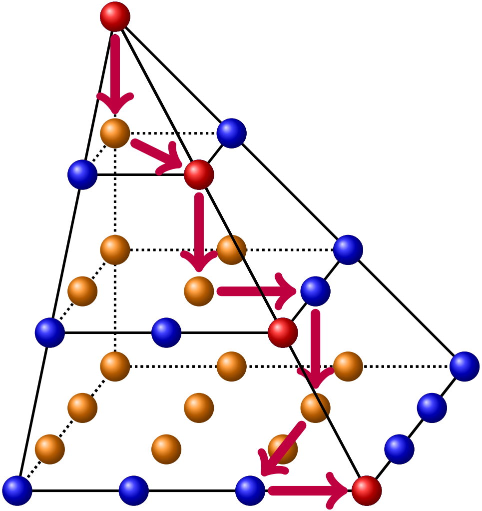
Game dynamics: example

Firm 1 invests, and the game reaches terminal state $\rightsquigarrow (c_1, c_2, c) = (0, 0, 0)$



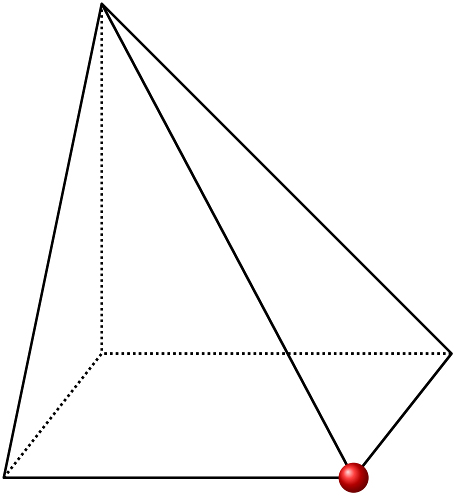
Game dynamics: example

Firm 1 invests, and the game reaches terminal state $\rightsquigarrow (c_1, c_2, c) = (0, 0, 0)$



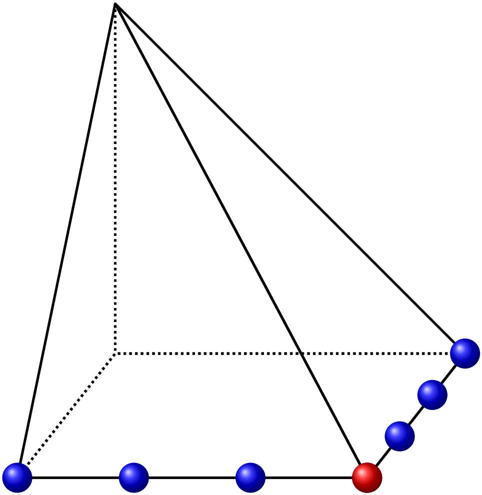
State recursion

Solve terminal state $(c_1, c_2, c) = (0, 0, 0)$



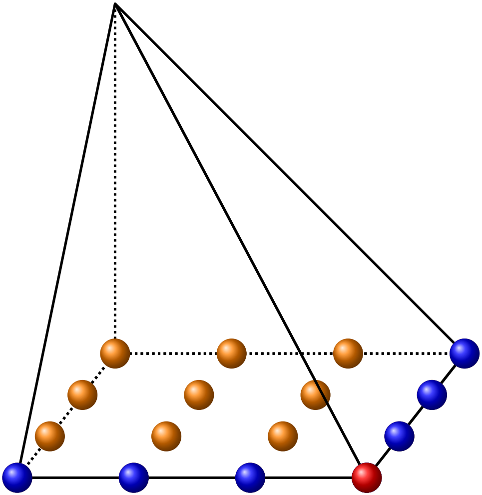
State recursion

Solve bottom layer edges $(c_1, c_2, c) = (0, c_2, 0)$ and $(c_1, 0, 0)$



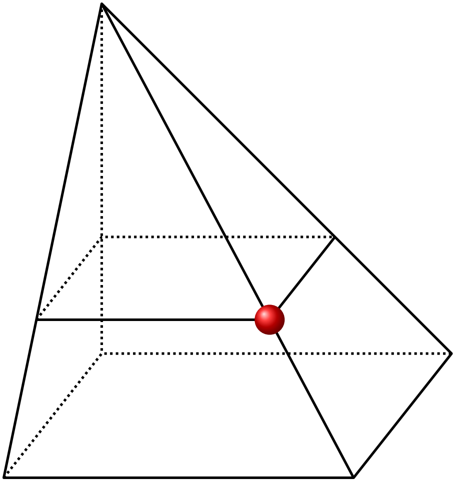
State recursion

Solve bottom layer interior $(c_1, c_2, c) = (c_1, c_2, 0)$



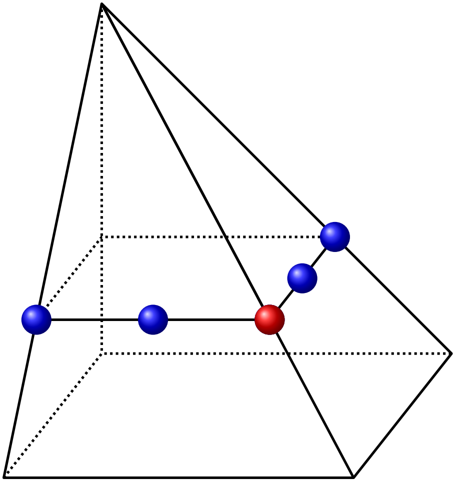
State recursion

Solve the corner $(c_1, c_2, c) = (1, 1, 1)$



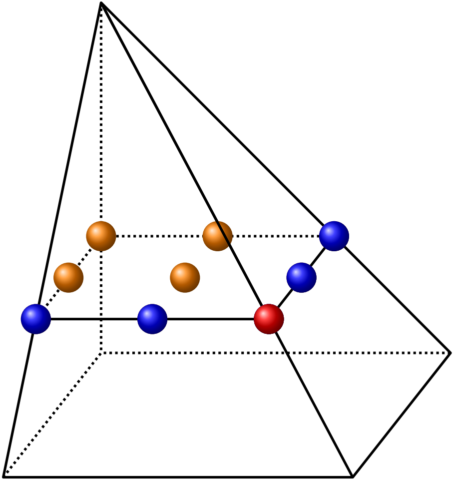
State recursion

Solve edges $(c_1, c_2, c) = (1, c_2, 1)$ and $(c_1, 1, 1)$



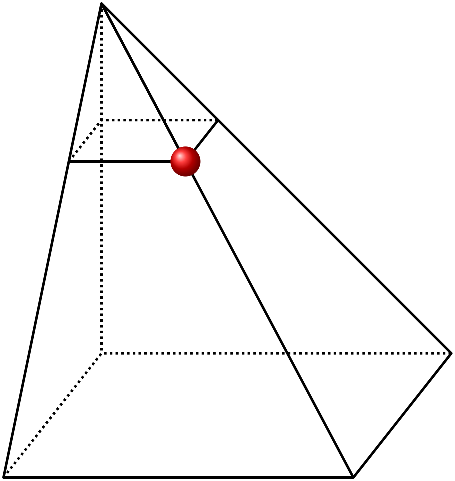
State recursion

Solve interior $(c_1, c_2, c) = (c_1, c_2, 1)$



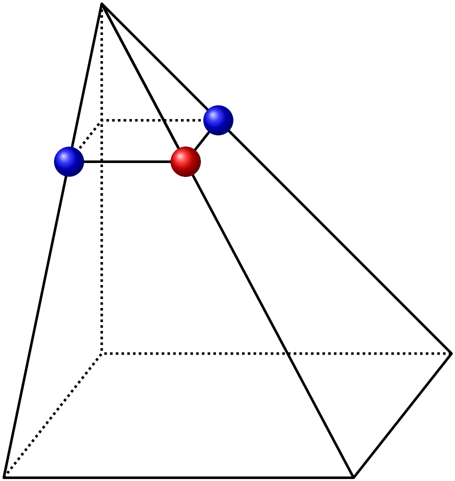
State recursion

Solve the corner $(c_1, c_2, c) = (2, 2, 2)$



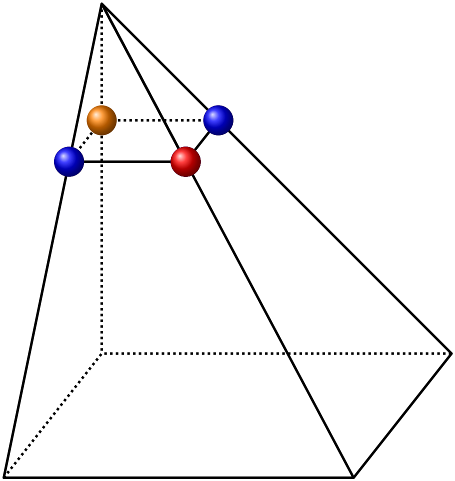
State recursion

Solve edges $(c_1, c_2, c) = (2, c_2, 2)$ and $(c_1, 2, 2)$



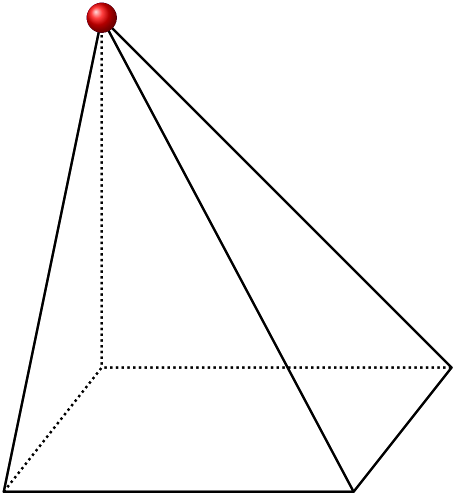
State recursion

Solve interior $(c_1, c_2, c) = (c_1, c_2, 2)$



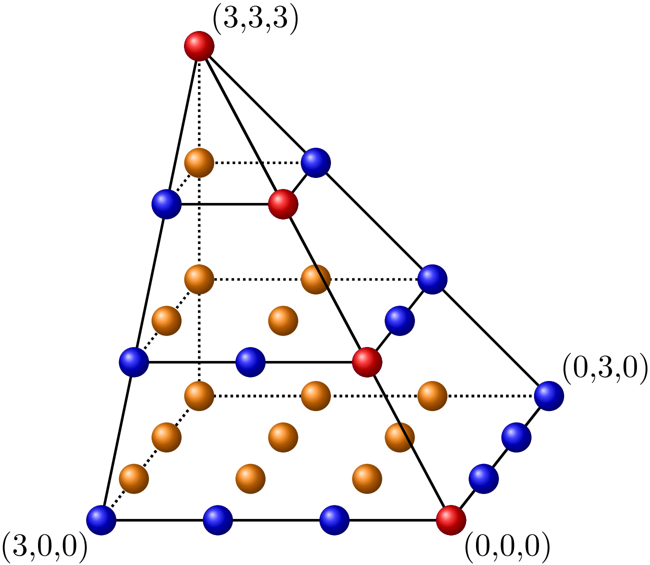
State recursion

Finally solve the apex $(c_1, c_2, c) = (3, 3, 3)$



State recursion

The whole game is solved!

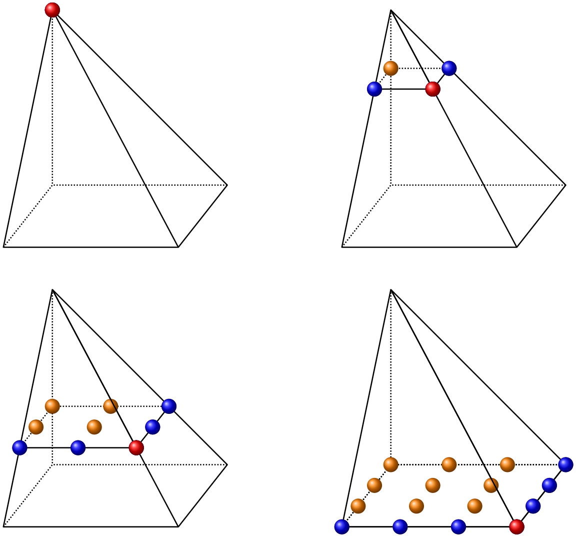


ROAD MAP

1. Collusion of Australian corrugated fibre packaging (CFP) producers
   - Collusion between Amcor and Visy
   - Bertrand pricing and investment game
   - Solution concept: Markov perfect equilibrium (MPE)
2. Experiment with the model
3. State recursion algorithm
   - Theory of directional dynamic games (DDGs)
4. Recursive lexicographical search (RLS) algorithm
5. Full solution for the leapfrogging game
6. Structural estimation of directional dynamic games with Nested RLS method

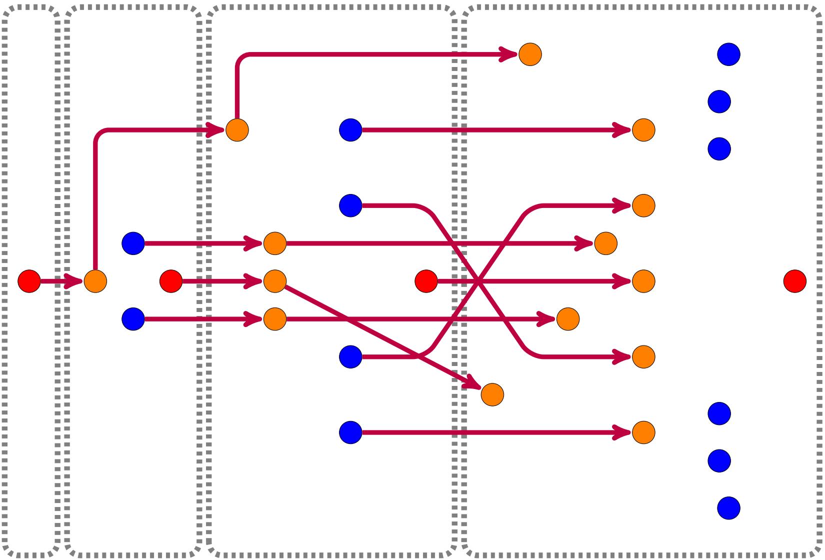
Transitions due to technological progress

As c decreases, the game falls through the layers of the pyramid



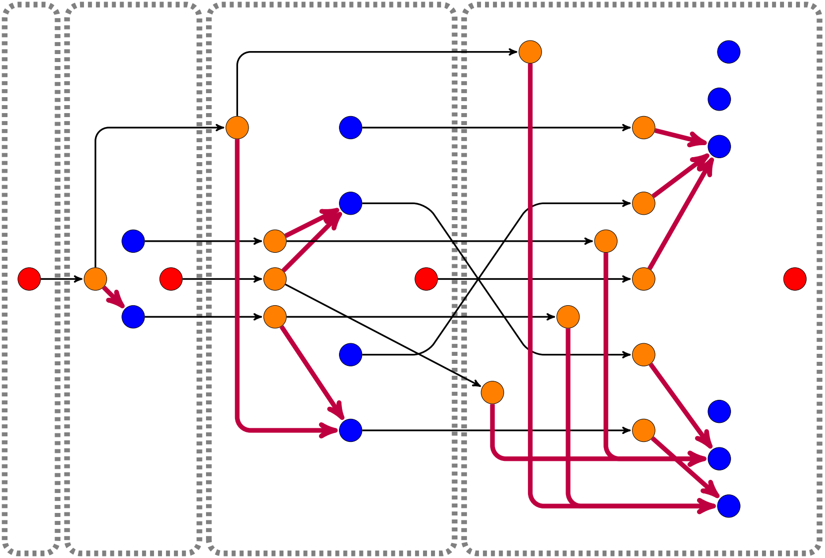
Transitions due to technological progress

As c decreases, the game falls through the layers of the pyramid



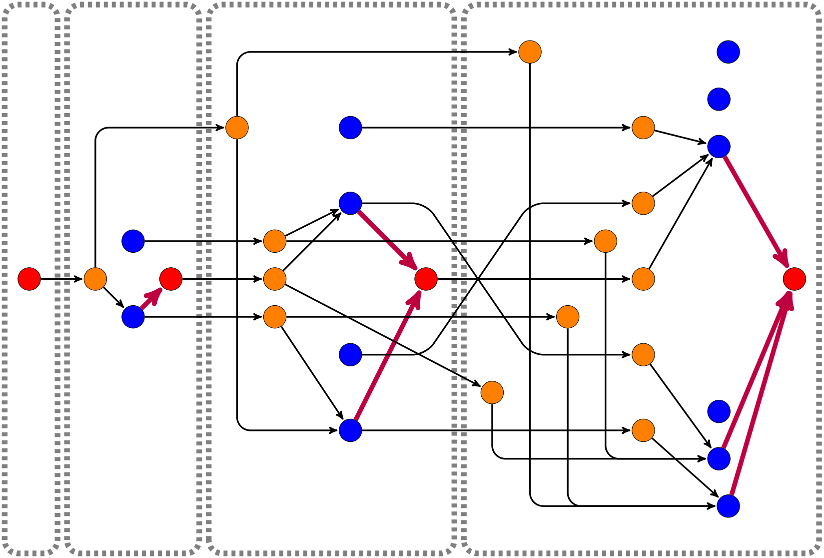
Strategy-specific partial order on S

Strategy σ_1 of firm 1: invest at all interior points



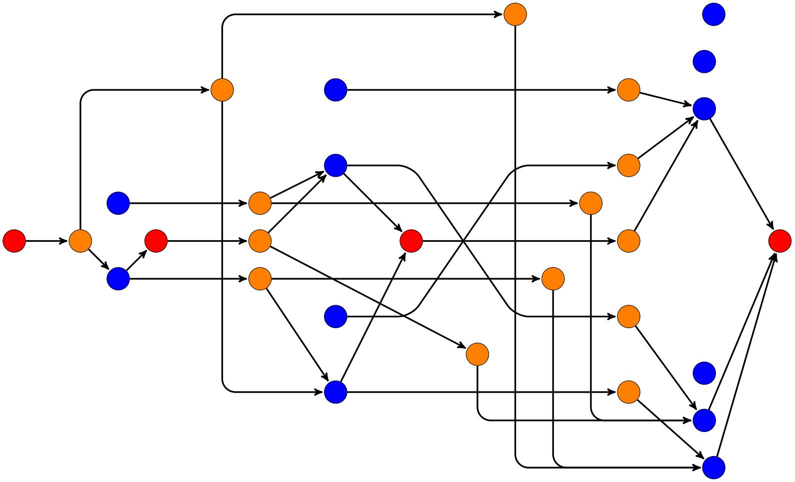
Strategy-specific partial order on S

Strategy σ_2 of firm 2: invest at all edge points



Strategy-specific partial order on S

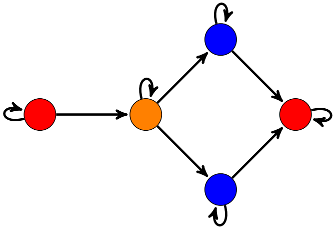
Strategy $\sigma = (\sigma_1, \sigma_2)$ of both firms

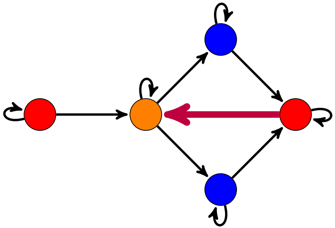


No loop (anti-cycling) condition

Hypothetical strategy profile inducing cycles

Self-loops appear when the game remains in the same state for two or more consecutive periods of time



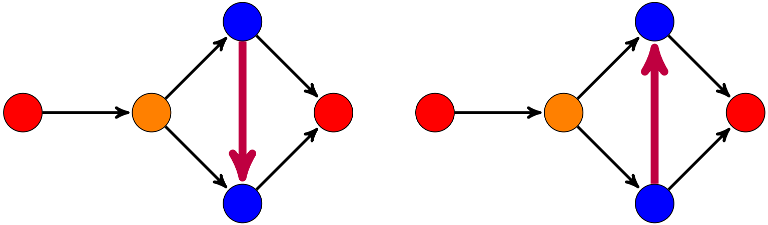


But loops between different states are not allowed

Consistency of strategy specific partial orders

Two hypothetical inconsistent strategies

Two strategies that induce opposite transitions are inconsistent



Note that in both cases the no-loop condition is satisfied

Definition of the Dynamic Directional Games

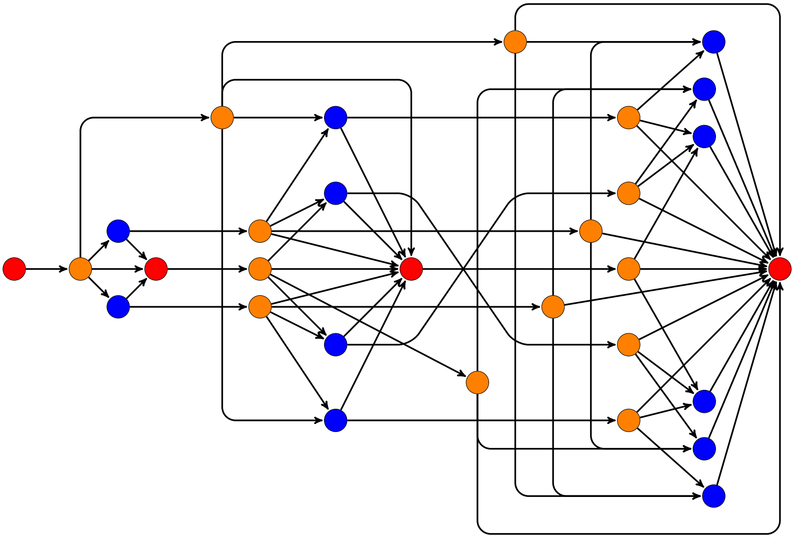
Definition (Dynamic Directional Games, DDG)

Finite state Markovian stochastic game is a DDG if it holds:

1. Every feasible Markovian strategy σ satisfies the no loop condition.
2. Every pair of feasible Markovian strategies σ and σ' induce consistent partial orders on the state space.

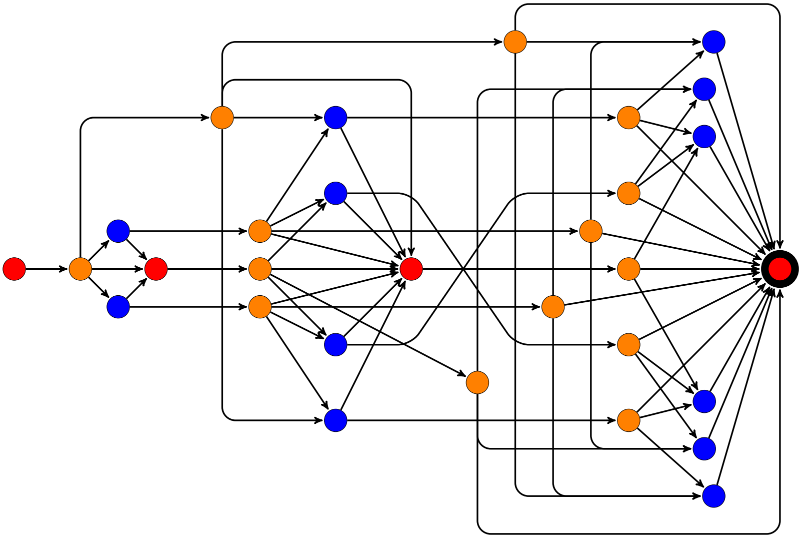
Strategy independent partial order on S

Coarsest common refinement of partial orders induced by all strategies



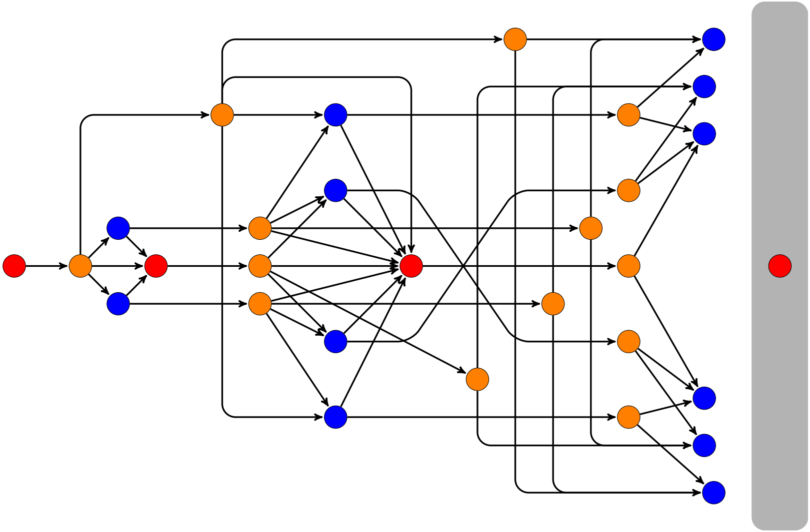
DAG recursion to partition S into stages

Identify terminal states



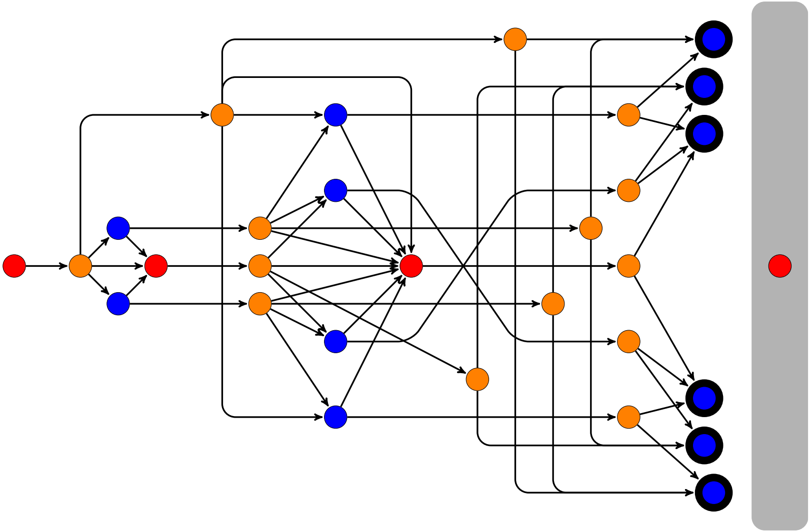
DAG recursion to partition S into stages

Remove terminal states



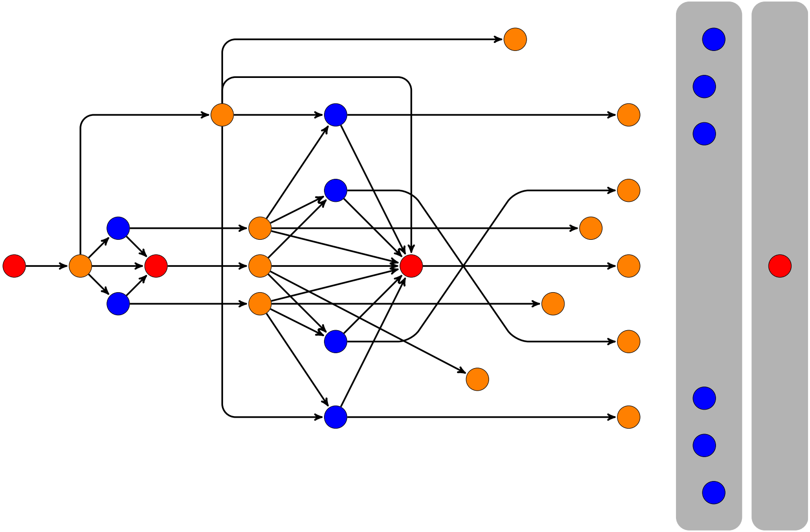
DAG recursion to partition S into stages

Identify terminal states



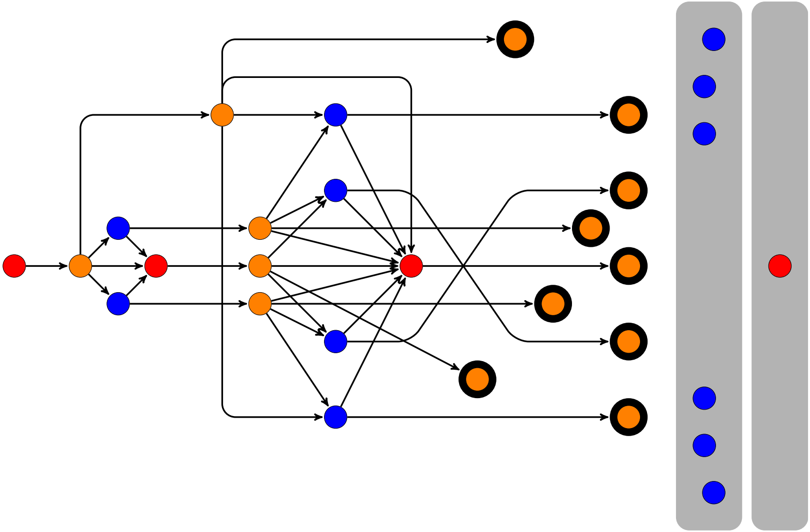
DAG recursion to partition S into stages

Remove terminal states



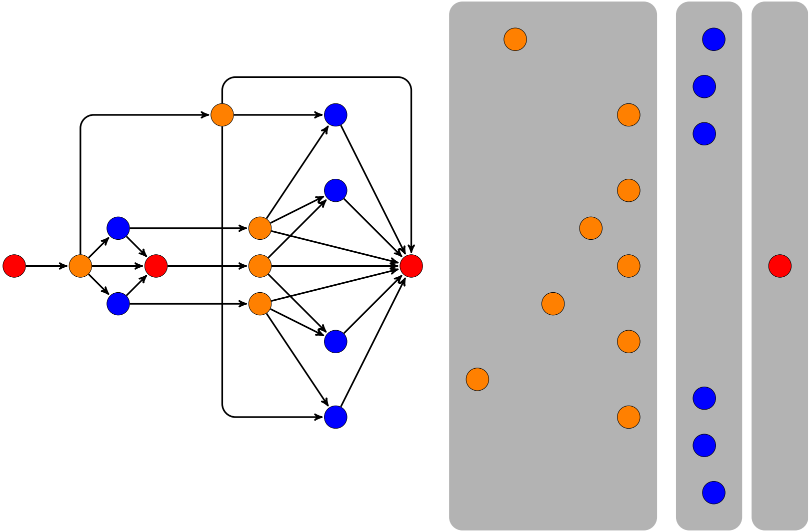
DAG recursion to partition S into stages

Identify terminal states



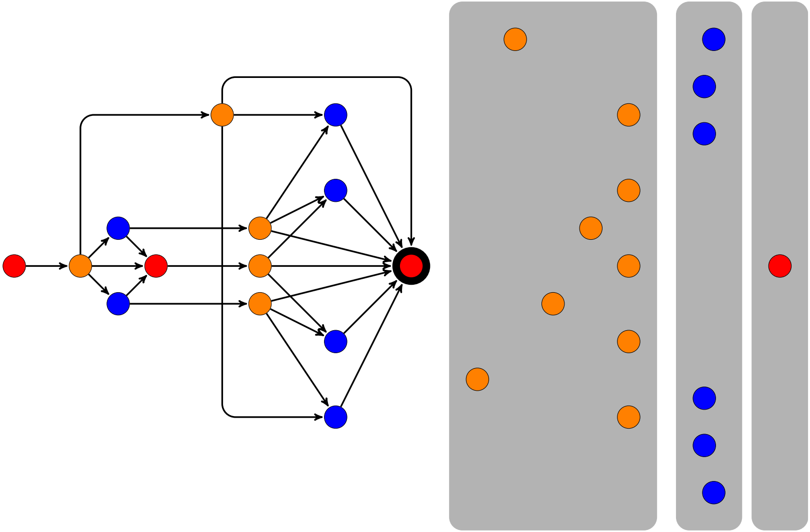
DAG recursion to partition S into stages

Remove terminal states



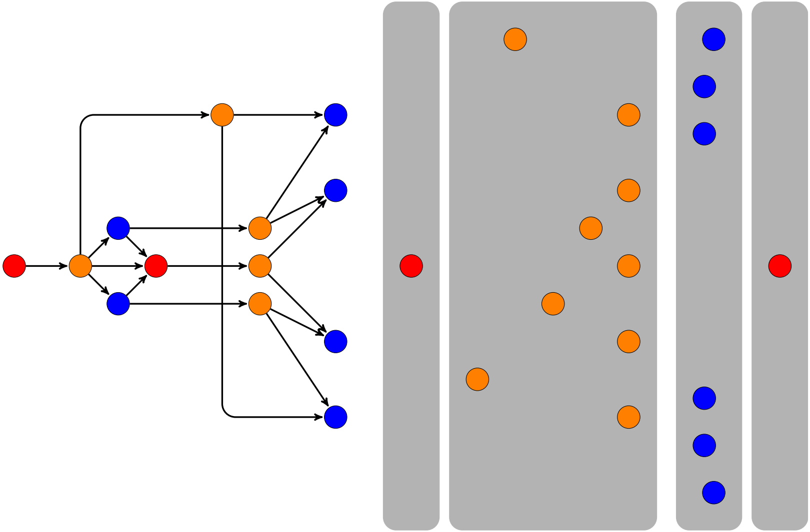
DAG recursion to partition S into stages

Identify terminal states



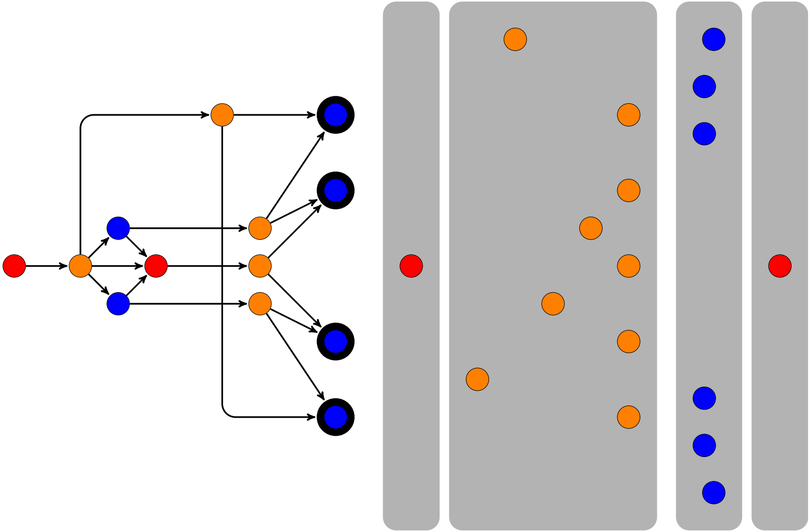
DAG recursion to partition S into stages

Remove terminal states



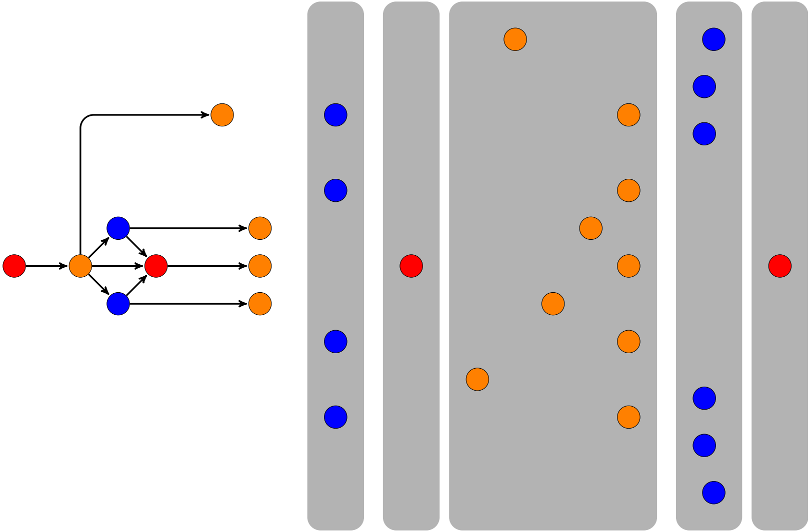
DAG recursion to partition S into stages

Identify terminal states



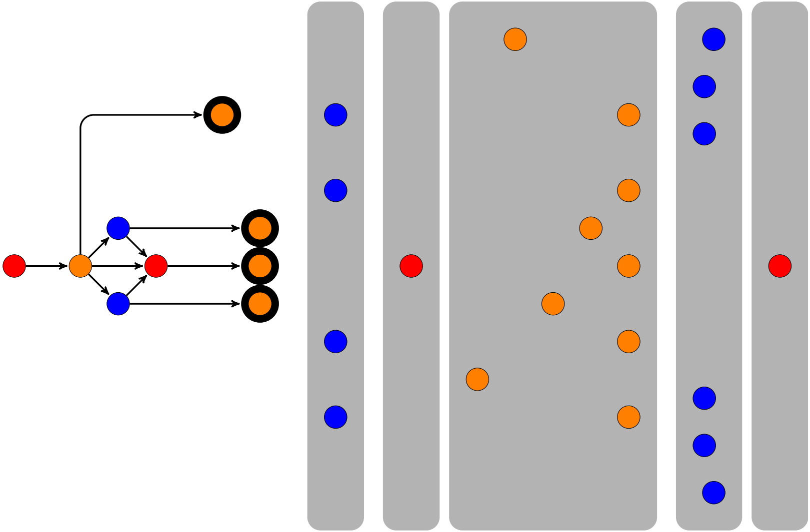
DAG recursion to partition S into stages

Remove terminal states



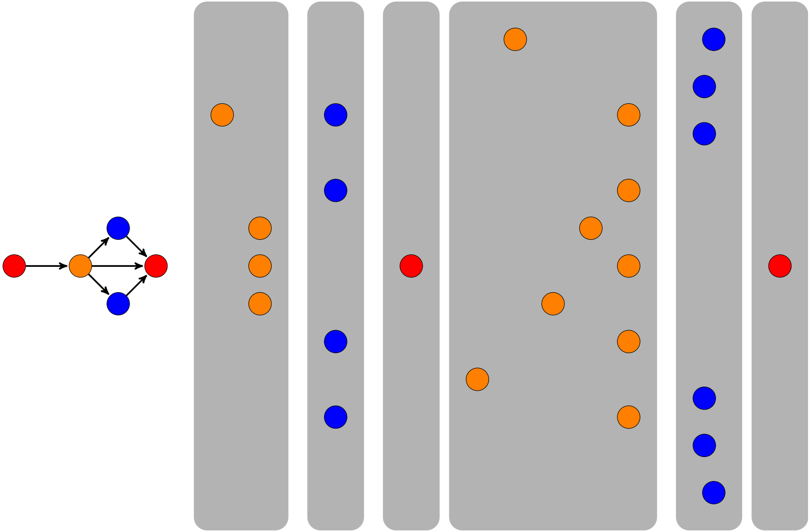
DAG recursion to partition S into stages

Identify terminal states



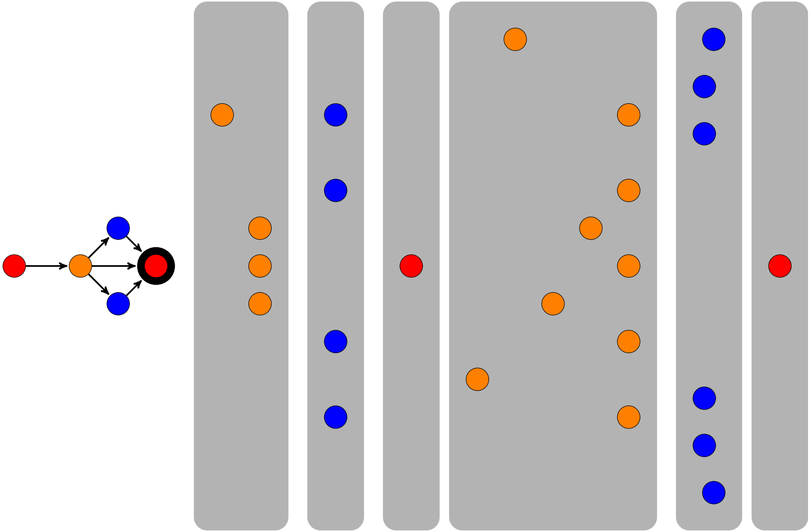
DAG recursion to partition S into stages

Remove terminal states



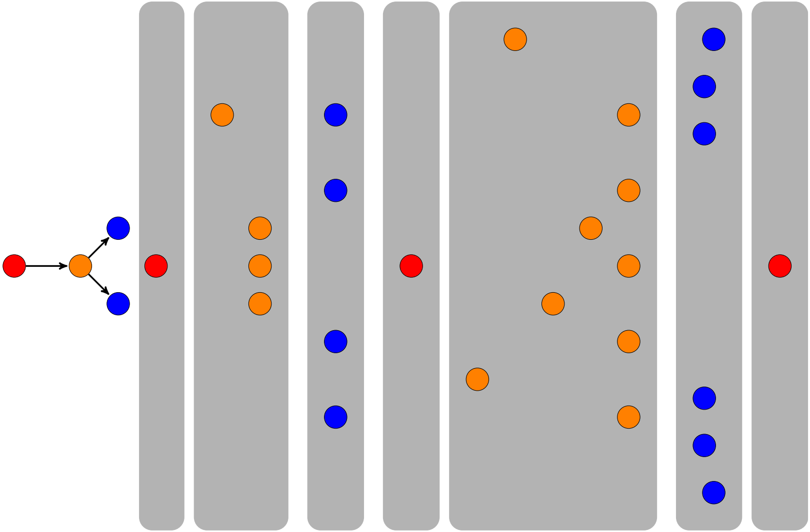
DAG recursion to partition S into stages

Identify terminal states



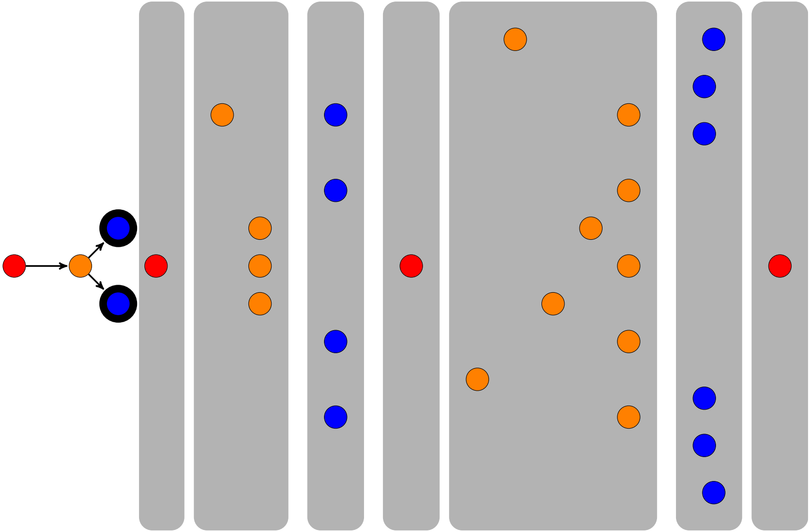
DAG recursion to partition S into stages

Remove terminal states



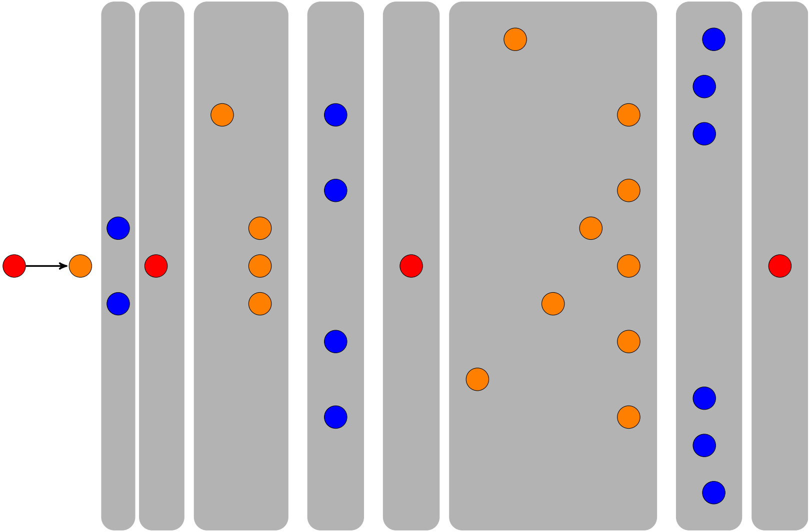
DAG recursion to partition S into stages

Identify terminal states



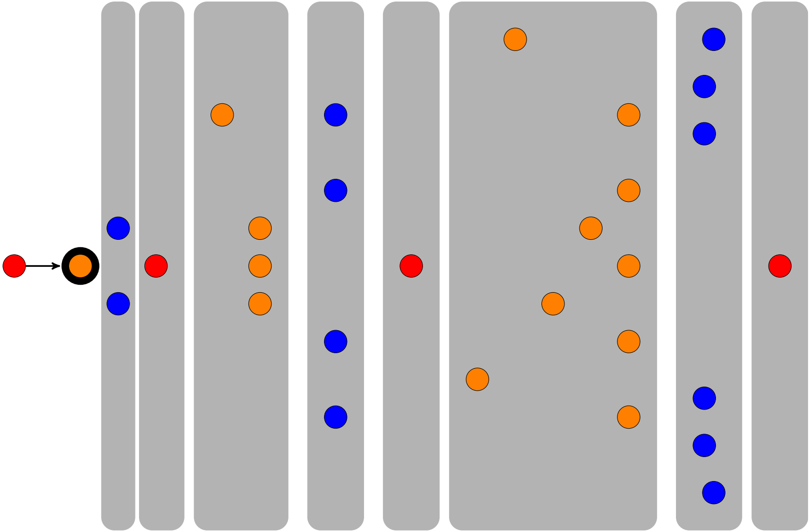
DAG recursion to partition S into stages

Remove terminal states



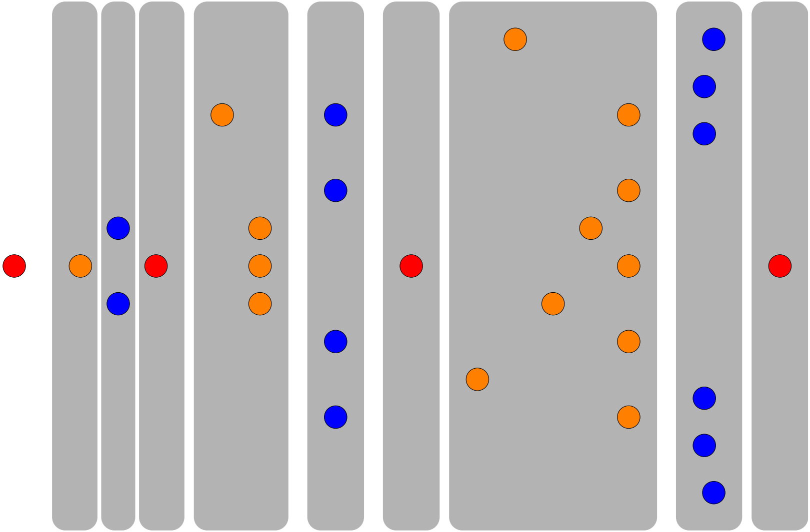
Resulting partition of S into stages

Identify terminal states



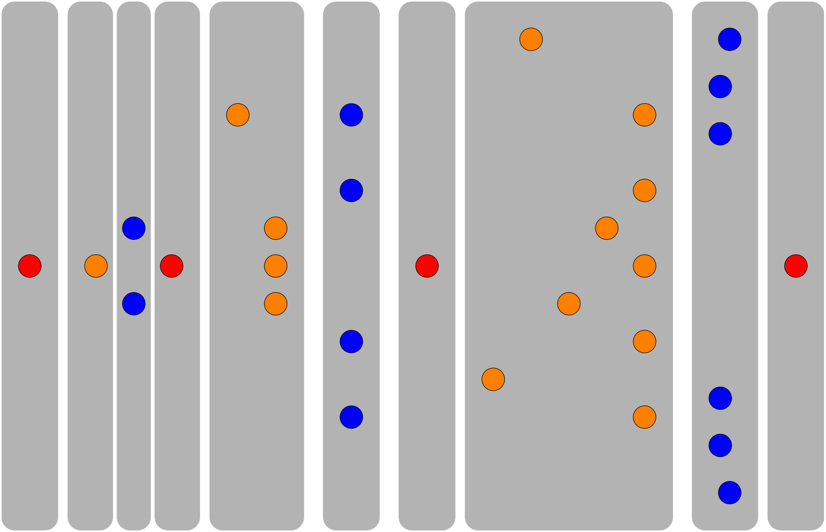
Resulting partition of S into stages

Remove terminal states



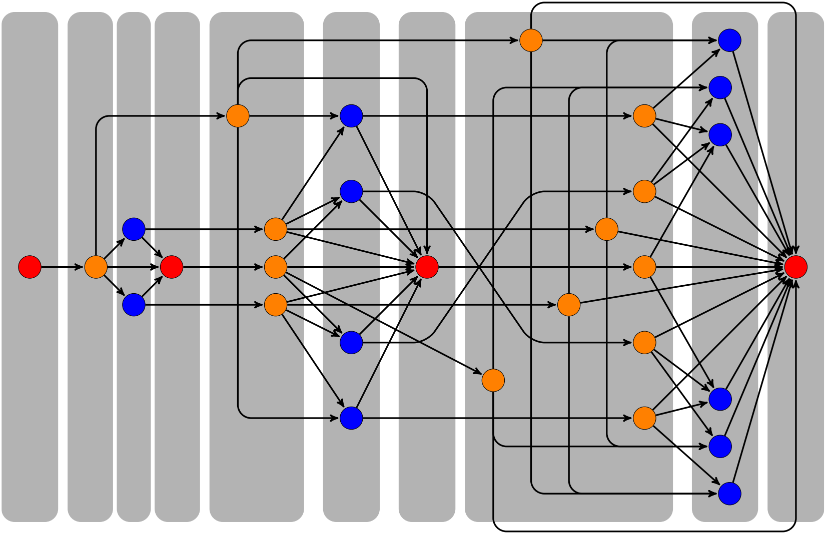
Resulting partition of S into stages

The stages of the game are totally ordered



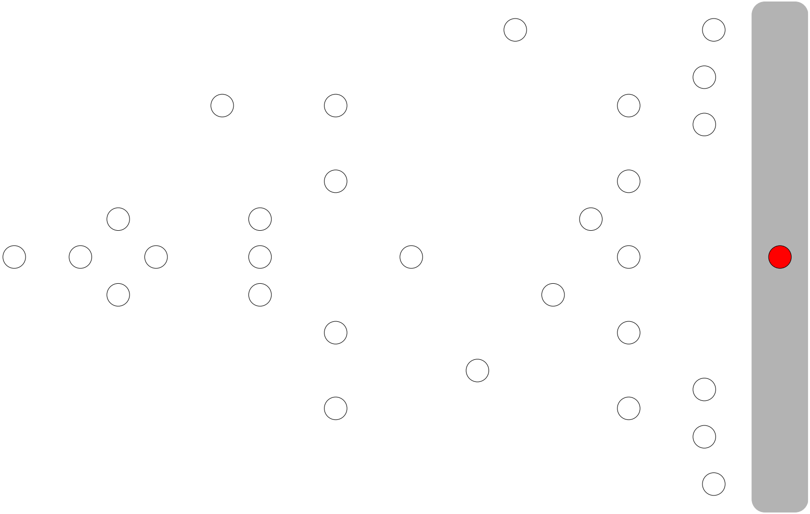
Total order on the set of stages

Subgames of DDG follow the order of stages



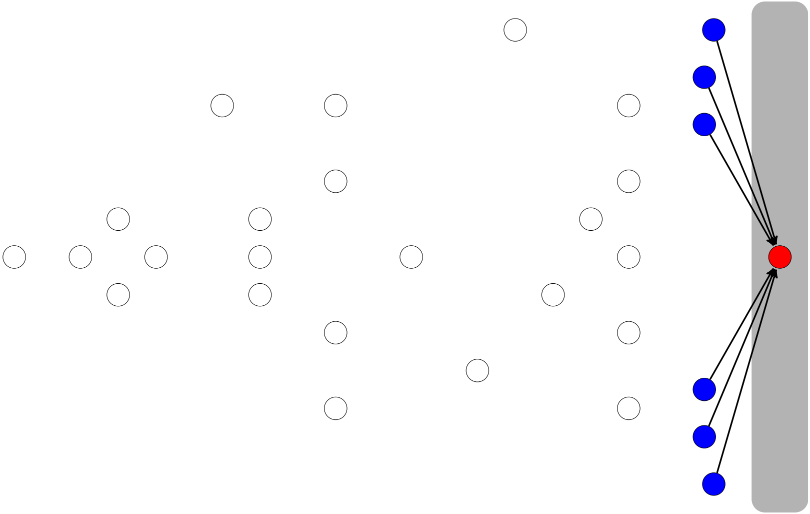
State recursion algorithm

Backward induction on stages of DDG



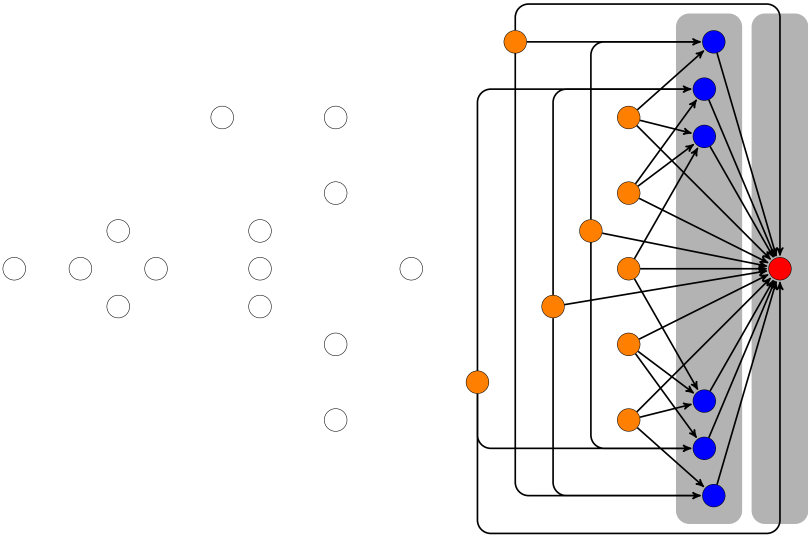
State recursion algorithm

Backward induction on stages of DDG



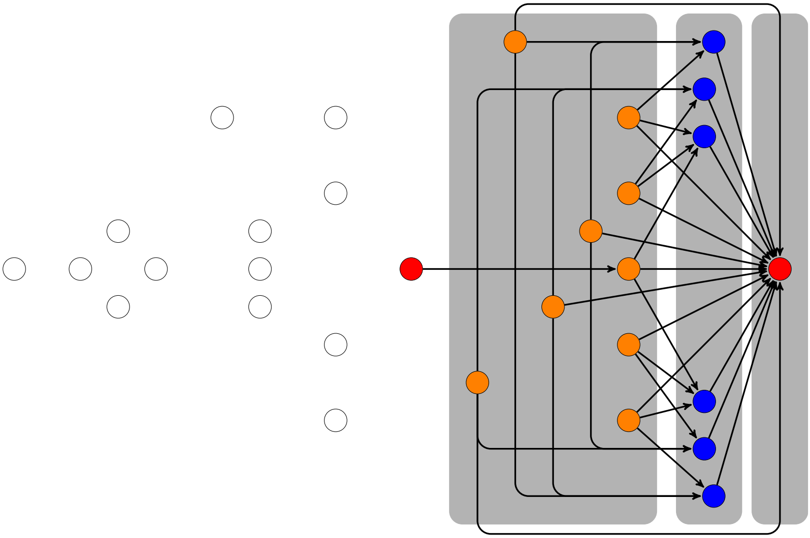
State recursion algorithm

Backward induction on stages of DDG



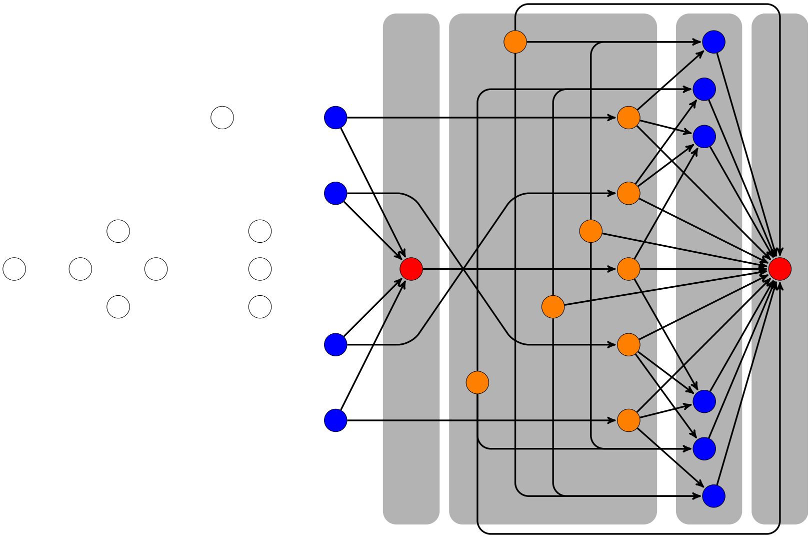
State recursion algorithm

Backward induction on stages of DDG



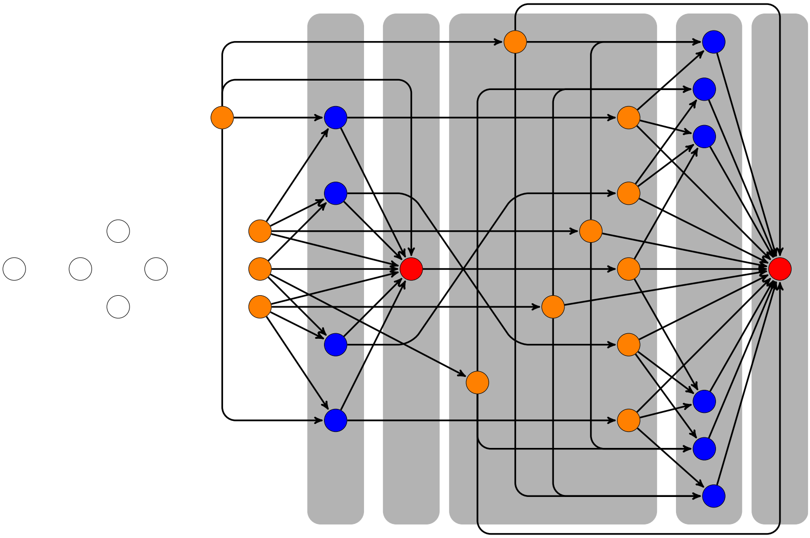
State recursion algorithm

Backward induction on stages of DDG



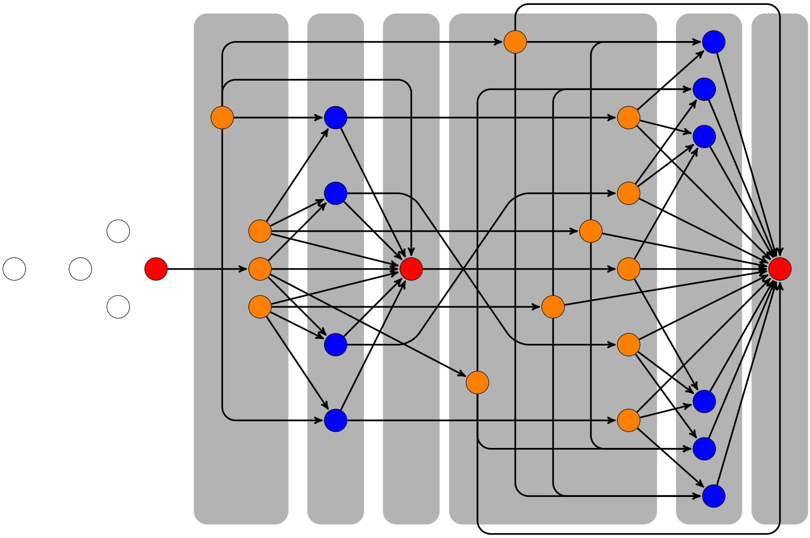
State recursion algorithm

Backward induction on stages of DDG



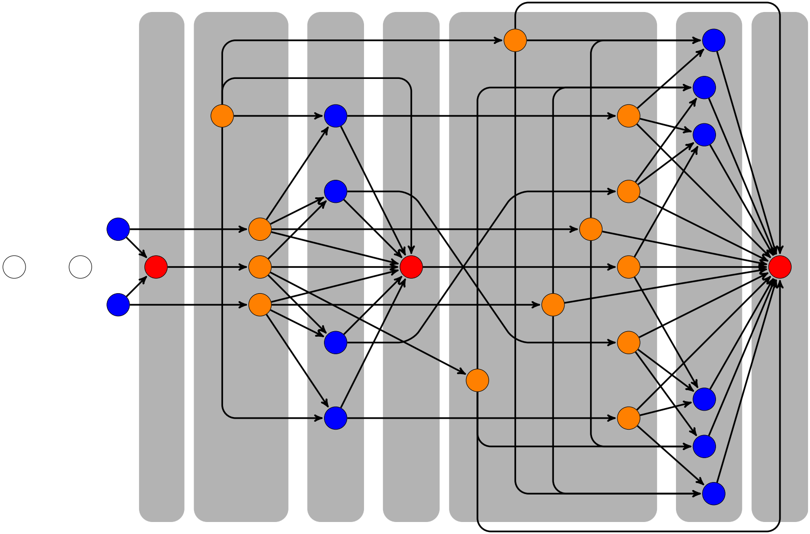
State recursion algorithm

Backward induction on stages of DDG



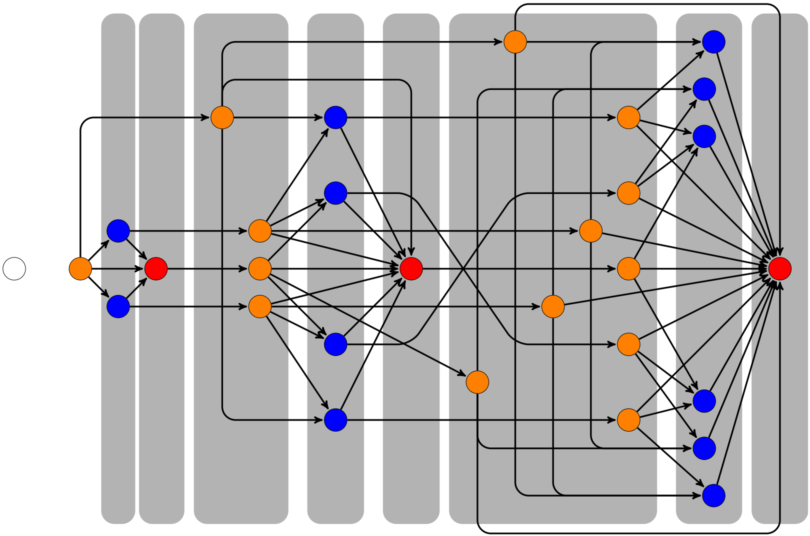
State recursion algorithm

Backward induction on stages of DDG



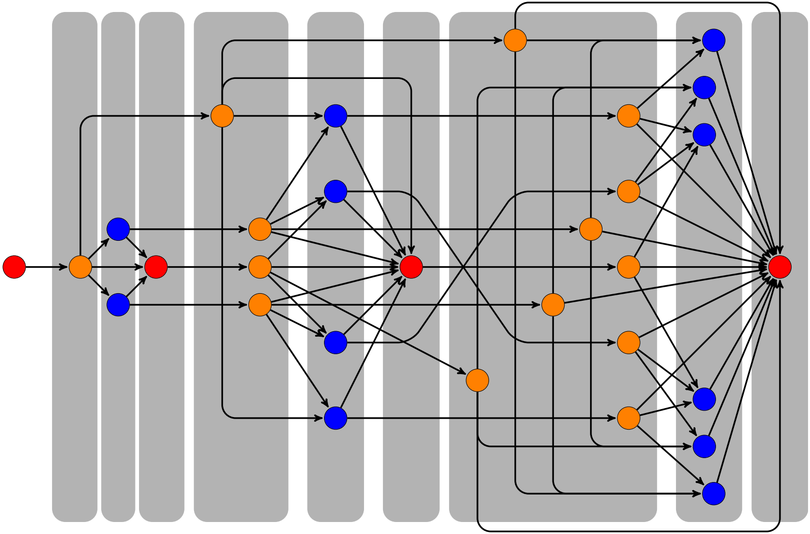
State recursion algorithm

Backward induction on stages of DDG



State recursion algorithm

Backward induction on stages of DDG

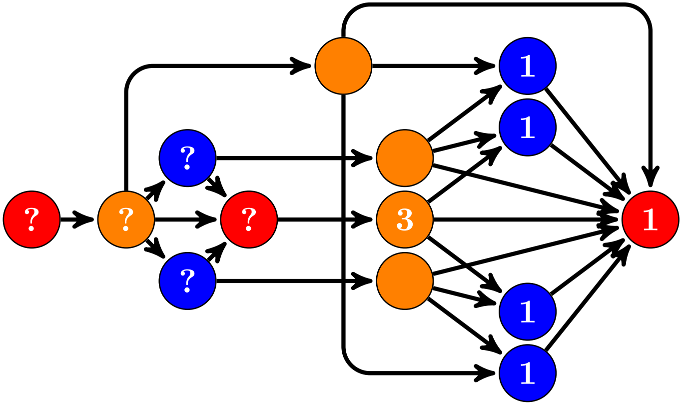


State Recursion versus Backward Induction

- State recursion – generalization of backward induction
- Runs on state space instead of time periods
- Time (t) evolves as $t \to t+1$ with probability 1
- For stages of state space (τ) transitions are stochastic and not necessarily sequential
- Yet, probability of going $\tau \to \tau'$ is zero when $\tau' < \tau$
- With multiplicity, state recursion is performed conditional of a particular equilibrium selection rule (ESR)

Multiplicity of equibiria

Number of equilibria in the higher stages depends on the selected equilibria

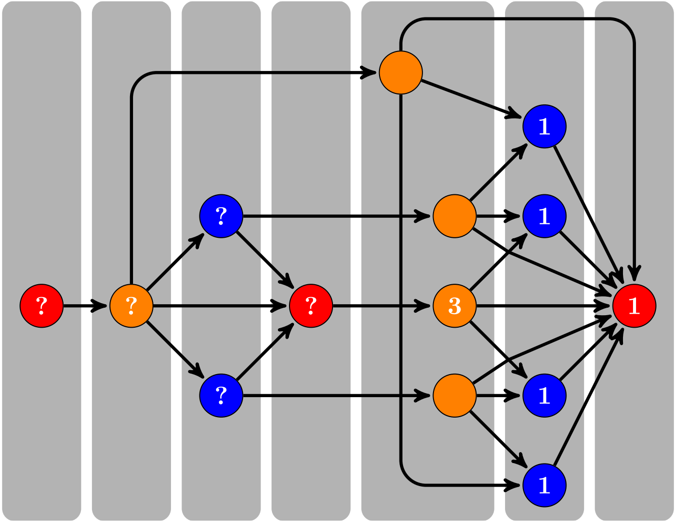


ROAD MAP

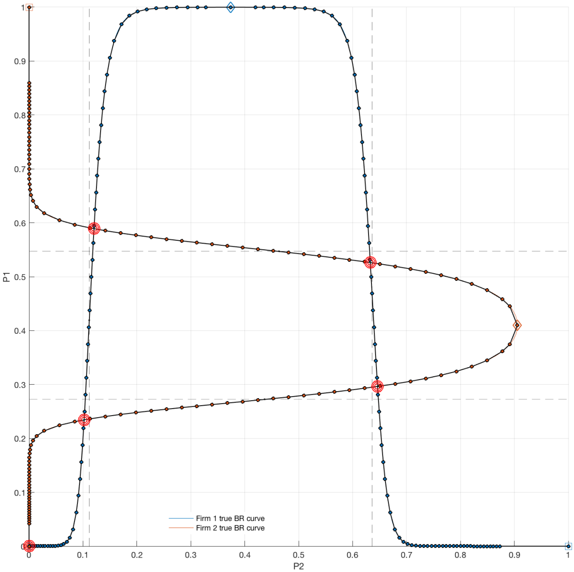
1. Collusion of Australian corrugated fibre packaging (CFP) producers
   - Collusion between Amcor and Visy
   - Bertrand pricing and investment game
   - Solution concept: Markov perfect equilibrium (MPE)
2. Experiment with the model
3. State recursion algorithm
   - Theory of directional dynamic games (DDGs)
4. Recursive lexicographical search (RLS) algorithm
5. Full solution for the leapfrogging game
6. Structural estimation of directional dynamic games with Nested RLS method

Multiplicity of stage equibiria

Number of equilibria in the higher stages depends on the selected equilibria



Best response correspondences of the two firms



Recursive Lexicographic Search Algorithm

Building blocks of RLS algorithm:

1. State recursion algorithm solves the game conditional on equilibrium selection rule (ESR)
2. RLS algorithm efficiently cycles through all feasible ESRs

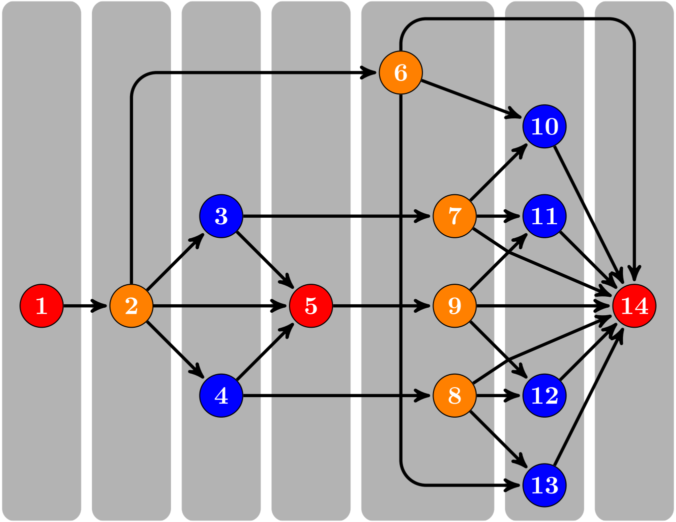
Challenge:

- Choice of a particular MPE for any stage game at any stage
- may alter the set and even the number of stage equilibria at earlier stages

Need to find feasible ESRs

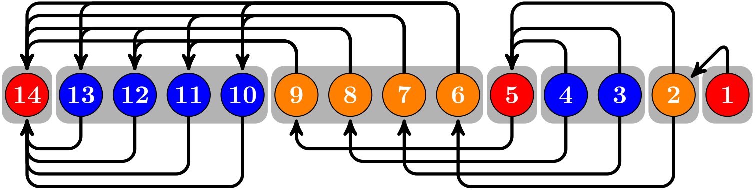
Index of points in the state space

Lower index for dependent points, highest for terminal stage

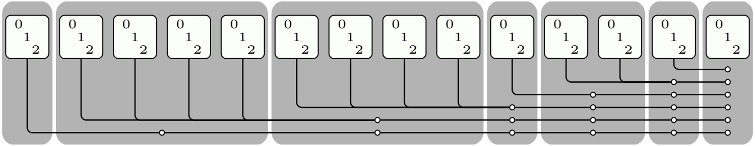


ESR strings

Formalization of the ESR as stings of digits



- Digits arranged to preserve the dependence structure



All possible ESR in lexicographic order, $K = 3$

	c	e	e	e	e	i	i	i	i	c	e	e	i	c
ESR string	14	13	12	11	10	9	8	7	6	5	4	3	2	1
Lexicograph	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	2
	0	0	0	0	0	0	0	0	0	0	0	0	1	0
	0	0	0	0	0	0	0	0	0	0	0	0	1	1
	0	0	0	0	0	0	0	0	0	0	0	0	1	2
	0	0	0	0	0	0	0	0	0	0	0	0	2	0
	0	0	0	0	0	0	0	0	0	0	0	0	2	1
	0	0	0	0	0	0	0	0	0	0	0	0	2	2
	0	0	0	0	0	0	0	0	0	0	0	1	0	0
	0	0	0	0	0	0	0	0	0	0	0	1	0	1
														...
	2	2	2	2	2	2	2	2	2	2	2	2	2	0
	2	2	2	2	2	2	2	2	2	2	2	2	2	1
	2	2	2	2	2	2	2	2	2	2	2	2	2	2

4,782,969

Recursive Lexicographic Search (RLS) Algorithm

- Start at ESR $= (0, \ldots, 0)$
- Cycle through all possible ESR strings
  Upper bound $= \big(\text{max number of stage equlibria}\big)^{(\text{nr points in } S)}$
- Efficiently skip infeasible ESR strings using information on number of stage equilibria found on previous iteration
- Stopping rule: run out of digits

Recalculation of feasibility condition for new ESR

Avoid recalculation of subgames

	c	e	e	e	e	i	i	i	i	c	e	e	i	c	
ESR string	14	13	12	11	10	9	8	7	6	5	4	3	2	1	
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	always admissible
Nr of eqb	1	1	1	1	1	3	3	3	3	1	1	1	**3**	1	
	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**	
	0	0	0	0	0	0	0	0	0	0	0	0	0	**2**	
	0	0	0	0	0	0	0	0	0	0	0	0	**1**	0	admissible, solve
	1	1	1	1	1	3	3	3	3	1	1	1	3	*	

No changes in the solution of the game including the number of stage equilibria

Might have change

Jumping over blocks of infeasibles ESRs

	c	e	e	e	e	i	i	i	i	c	e	e	i	c	
ESR string	14	13	12	11	10	9	8	7	6	5	4	3	2	1	Iteration:
	0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	1
	1	1	1	1	1	3	3	3	3	1	1	1	**3**	**1**	1a
	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**	
	0	0	0	0	0	0	0	0	0	0	0	0	0	2	
	0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	2
	1	1	1	1	1	3	3	3	3	1	1	1	**3**	**1**	2a
	0	0	0	0	0	0	0	0	0	0	0	0	1	**1**	
	0	0	0	0	0	0	0	0	0	0	0	0	1	2	
	0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**2**	**0**	3
	1	1	1	1	1	3	3	3	**3**	**1**	**1**	**1**	**3**	**1**	
	0	0	0	0	0	0	0	0	0	0	0	0	2	**1**	3a
	0	0	0	0	0	0	0	0	0	0	0	0	2	2	
	0	0	0	0	0	0	0	0	0	0	0	**1**	0	0	3b
														…	
	0	0	0	0	0	0	0	0	0	0	**1**	0	0	0	3c
														…	
	0	0	0	0	0	0	0	0	0	**1**	0	0	0	0	3d
														…	
	0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	4
														…	

Variable base arithmetics

- Replace the base-K ($mod(K)$) arithmetics with variable base arithmetics
- Let (3 1 2) be bases $\rightarrow$
- Allowed digits in the numbers are $\{0, 1, 2\}$, $\{0\}$ and $\{0, 1\}$
- The 3-digit numbers in this system are:

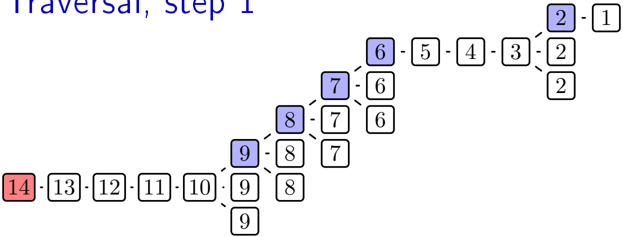
$$
\begin{array}{lll}
0\ 0\ 0 & +1 & \rightarrow \\
0\ 0\ 1 & +1 & \rightarrow \\
1\ 0\ 0 & +1 & \rightarrow \\
1\ 0\ 1 & +1 & \rightarrow \\
2\ 0\ 0 & +1 & \rightarrow \\
2\ 0\ 1 & &
\end{array}
$$

Recursive Lexicographic Search (RLS) Algorithm

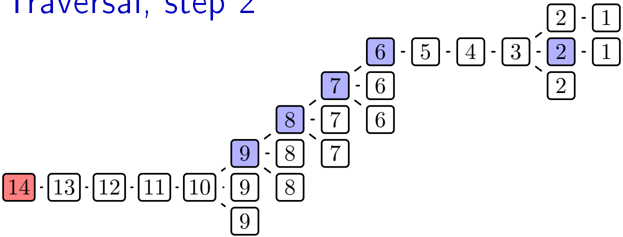
1. Set ESR $= (0, \ldots, 0)$
2. Run State Recursion using the current ESR
3. Save the number of equilibria in every stage game as *ne*(ESR)
4. Add 1 to the ESR in bases *ne*(ESR) to obtain new feasible ESR
5. Stopping rule: successor function exceeds the maximum number with given number of digits
6. Return to step 2

RLS = tree traversal!

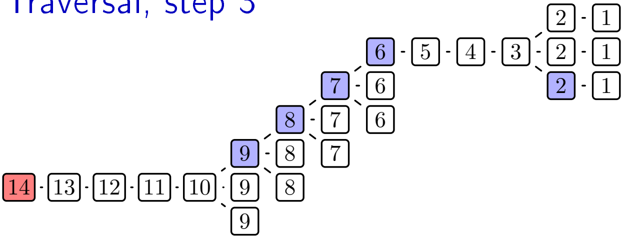
RLS Tree Traversal, step 1



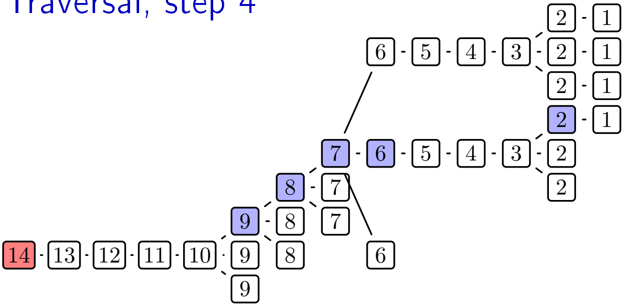
RLS Tree Traversal, step 2



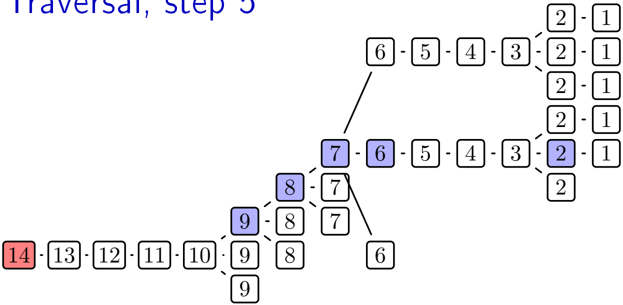
RLS Tree Traversal, step 3



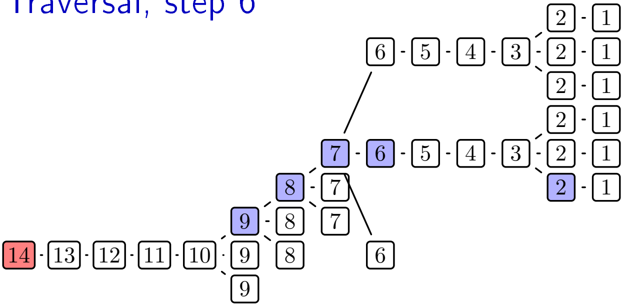
RLS Tree Traversal, step 4



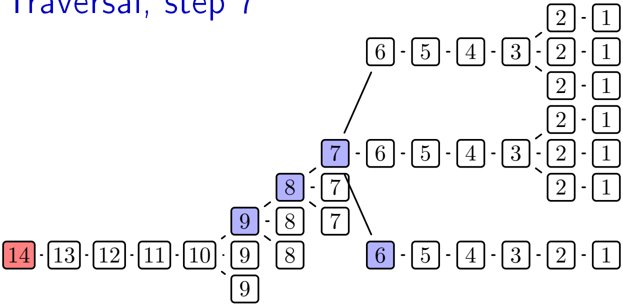
RLS Tree Traversal, step 5



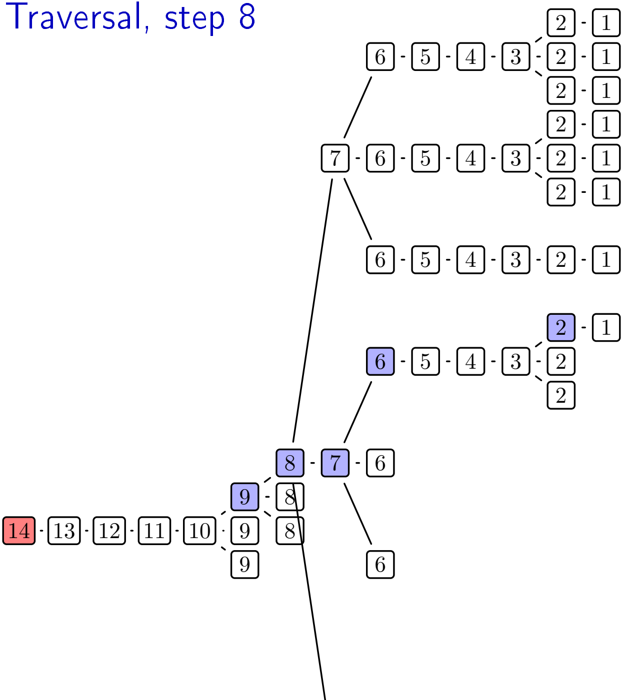
RLS Tree Traversal, step 6



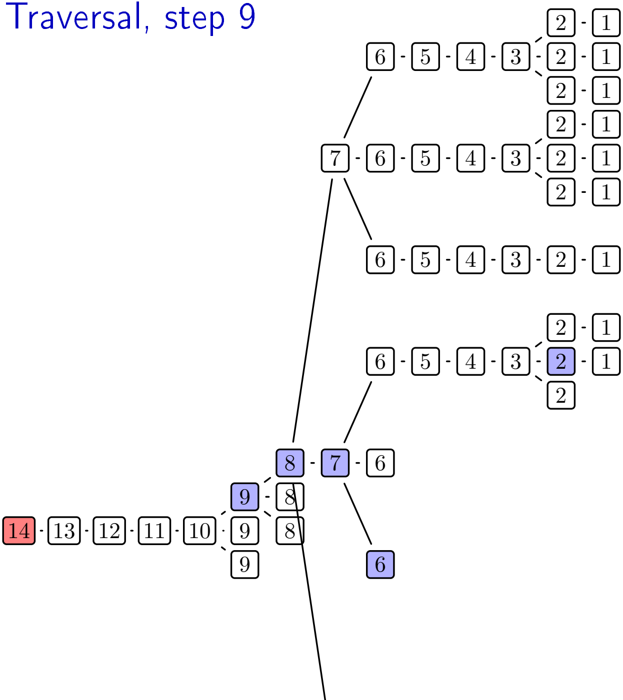
RLS Tree Traversal, step 7



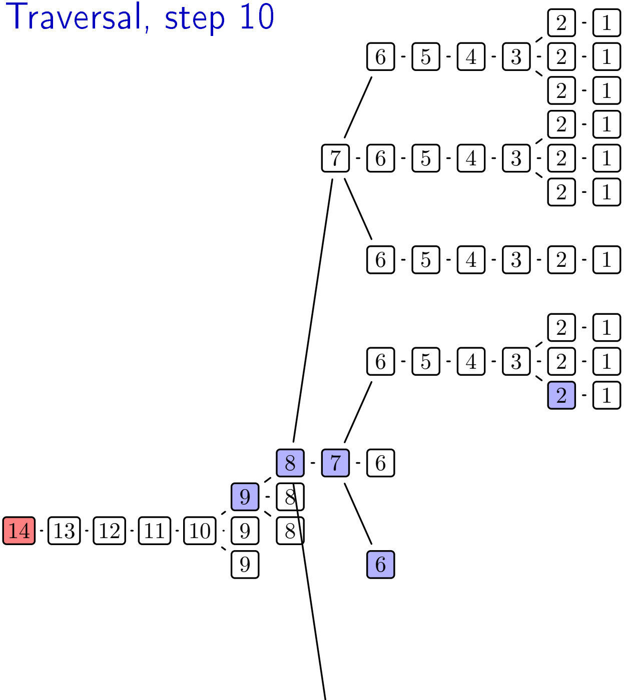
RLS Tree Traversal, step 8



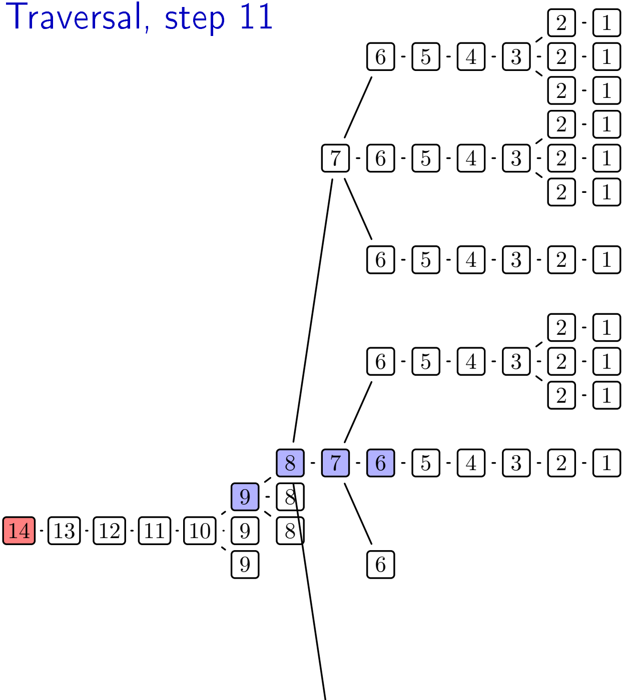
RLS Tree Traversal, step 9



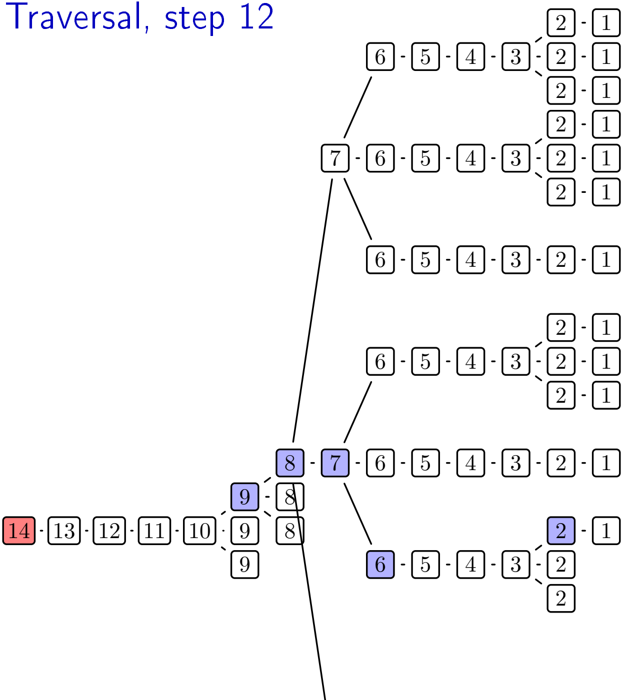
RLS Tree Traversal, step 10



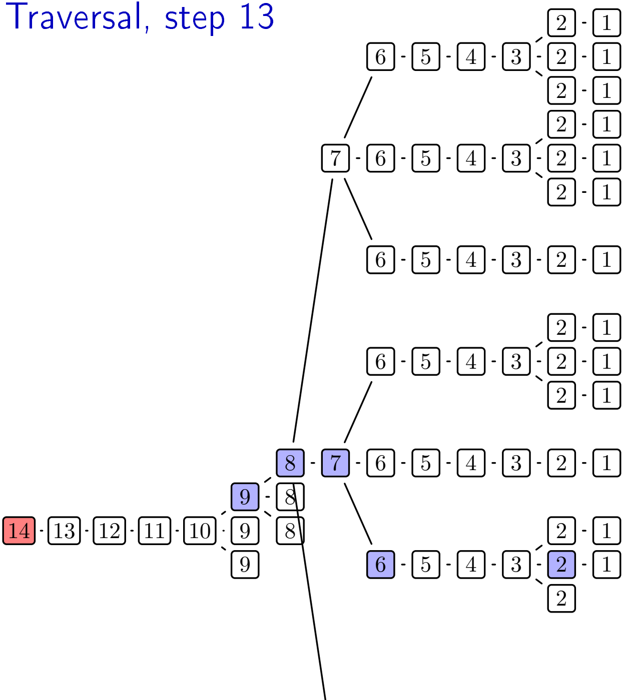
RLS Tree Traversal, step 11



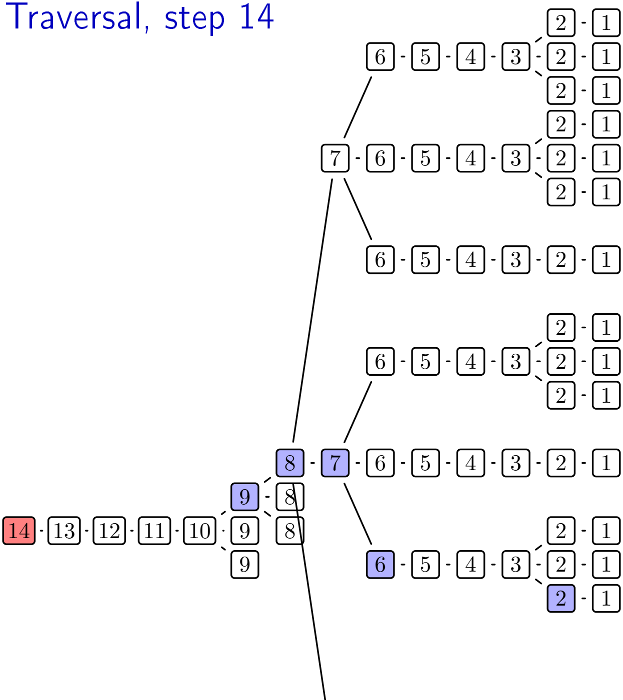
RLS Tree Traversal, step 12



RLS Tree Traversal, step 13



RLS Tree Traversal, step 14



Main result of the RLS Algorithm

Theorem (Decomposition theorem, strong)

Assume there exists an algorithm that can find all MPE of every stage game of the DDG, and that the number of these equilibria is finite in every stage game.
Then the RLS algorithm finds all MPE of the DDG in a finite number of steps, which equals the total number of MPE.

Iskhakov, Rust and Schjerning, 2016

Main result of the RLS Algorithm

Theorem (Decomposition theorem, weak)

Assume there exists an algorithm that can at least one MPE of every stage game of the DDG, and that the number of these equilibria is finite in every stage game.
Then the RLS algorithm finds some (at least one) MPE of the DDG in a finite number of steps, which does not exceed the total number of MPE.

RLS algorithm: running times

$K = 3$

Simultaneous moves	$n = 3$	$n = 4$
Upper bound on number of MPE	4,782,969	3,948,865,611
Actual number of equilibria	127	46,707
Time used	0.008 sec.	0.334 sec.

Simultaneous moves	$n = 5$
Upper bound on number of MPE	174,449,211,009,120,166,087,753,728
Actual number of equilibria	192,736,405
Time used	45 min.

Alternating moves	$n = 5$
Upper bound on number of MPE	174,449,211,009,120,166,087,753,728
Actual number of equilibria	1
Time used	0.006 sec.

ROAD MAP

1. Collusion of Australian corrugated fibre packaging (CFP) producers
   - Collusion between Amcor and Visy
   - Bertrand pricing and investment game
   - Solution concept: Markov perfect equilibrium (MPE)
2. Experiment with the model
3. State recursion algorithm
   - Theory of directional dynamic games (DDGs)
4. Recursive lexicographical search (RLS) algorithm
5. Full solution for the leapfrogging game
6. Structural estimation of directional dynamic games with Nested RLS method

Solving for stage game MPE

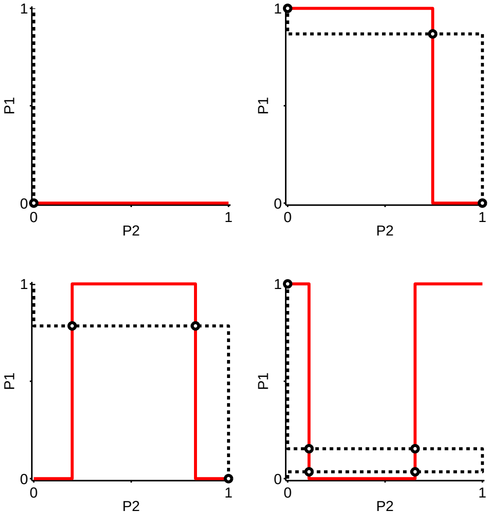
Theorem (1, 3 or 5 MPE in simultaneous move stage games)

When $\eta = 0$ *there exists an exact, non-iterative algorithm that finds all MPE of every d-stage game in the Bertrand pricing and investment game with simultaneous moves.*
The number of MPE in any d-stage game is either 1, 3 or 5.

Theorem (1 or 3 MPE in alternating move stage games)

When $\eta = 0$ *there exists an exact, non-iterative algorithm that finds all MPE of every d-stage game in the Bertrand pricing and investment game with alternating moves.*
The number of MPE in any d-stage game is either 1 or 3.

Best response correspondences of the two firms



Resolutions to Bertrand Investment Paradox

Earlier work:

- Fudenberg et al. (1983 RIE), Reinganum (1985 QJE), Fudenberg and Tirole (1985 ReStud), ...
- Riordan and Salant (1994 JIE):
  Preemption and rent dissipation (unique equilibrium)

We show:

1. Many types of endog. coordination is possible in equilibrium
   - Leapfrogging (alternating investments)
   - Preemption (investment by cost leader)
   - Duplicative (simultaneous investments)
2. The equilibria are generally inefficient due to over-investment
   - Duplicative or excessively frequent investments

Resolution to the Bertrand investment paradox

Theorem (Solution to Bertrand investment paradox)

If investment is socially optimal at a state point $(c_1, c_2, c) \in S$, then

- *no investment by both firms cannot be an MPE outcome in the subgame starting from (c_1, c_2, c) in either the simultaneous or alternating move versions of the dynamic game.*

Multiplicity of equilibria

Theorem (Sufficient conditions for uniqueness)

In the dynamic Bertrand investment and pricing game a sufficient condition for the MPE to be unique is that

1. *firms move in alternating fashion (i.e. $m \neq 0$), and,*
2. *for each $c > 0$ in the support of π we have $\pi(c|c) = 0$.*

1. If firms move simultaneously,
   equilibrium is generally not unique.
2. If technological change is stochastic,
   equilibrium is generally not unique.

Multiplicity of equilibria

Theorem (Number of equilibria in simultaneous move game)

If investment is socially optimal, and the support of the Markov process $\{c_t\}$ for the state of the art marginal costs is the full interval $[0, c_0]$ (i.e. continuous state version),

- *the simultaneous move Bertrand investment and pricing game has a continuum of MPE.*

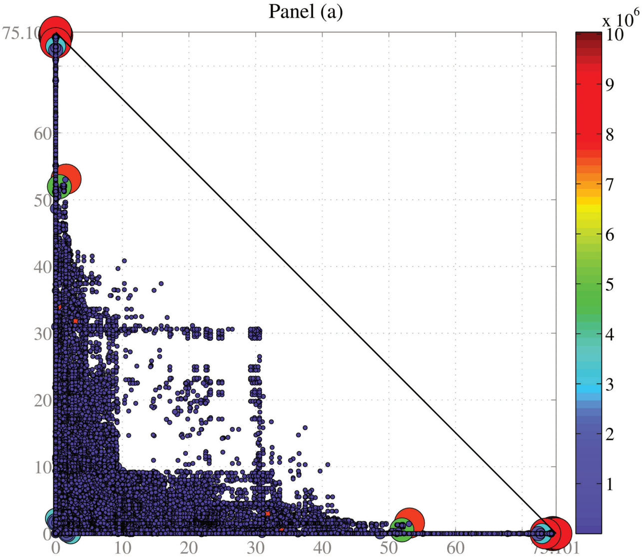
Pay-offs in the simultaneous move game

Theorem (Triangular payoffs in the simultaneous move game)

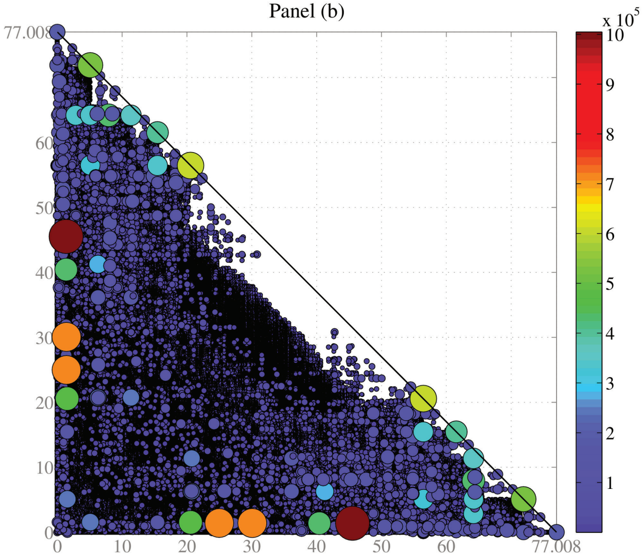
Suppose that the $\{c_t\}$ process has finite support, that there are no idiosyncratic shocks to investment (i.e. $\eta = 0$) and that firms move simultaneously

- *The (convex hull of the) set of the expected discounted equilibrium payoffs at the apex state $(c_0, c_0, c_0) \in S$ is a triangle*
- *The vertices of this triangle are at the points $(0, 0)$, $(0, V_M)$ and $(V_M, 0)$ where $V_M = v_{N,i}(c_0, c_0, c_0)$ is the expected discounted payoff to firm i in the monopoly equilibrium where firm i is the monopolist investor.*

Pay-offs (deterministic tech progress)



Pay-offs (stochastic tech progress)

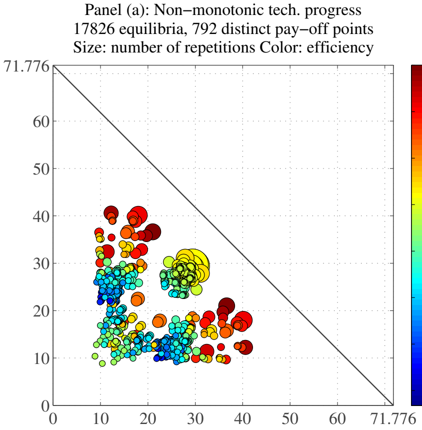


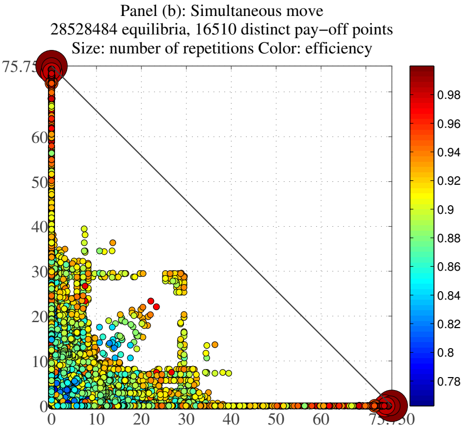
Pay-offs in the alternating move game

Theorem (Equilibrium payoffs in the alternating move game)

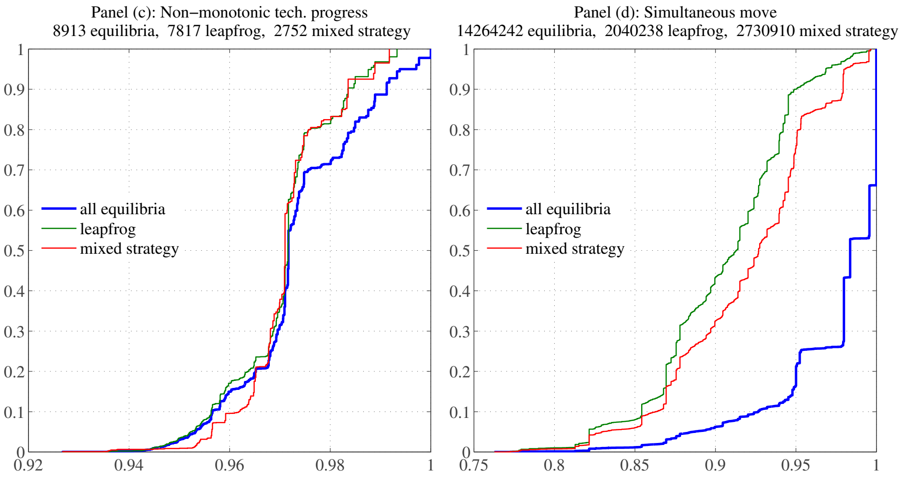
The (convex hull of the) set of expected discounted equilibrium payoffs at the apex state $(c_0, c_0, c_0) \in S$ *of the alternating game* *is a strict subset of the triangle with the vertices* $(0, 0)$, $(0, V_M)$ *and* $(V_M, 0)$

Pay-offs: alternating vs simultaneous move games





Efficiency: alternating vs simultaneous move games



Efficiency of equilibria

Simultaneous move game

Theorem (Inefficiency of mixed strategy equilibria)

A necessary condition for efficiency in the dynamic Bertrand investment and pricing game is that along MPE path only pure strategy stage equilibria are played.

Riordan and Salant: Full Preemption

Theorem (Riordan and Salant, 1994)

The continuous time investment game where

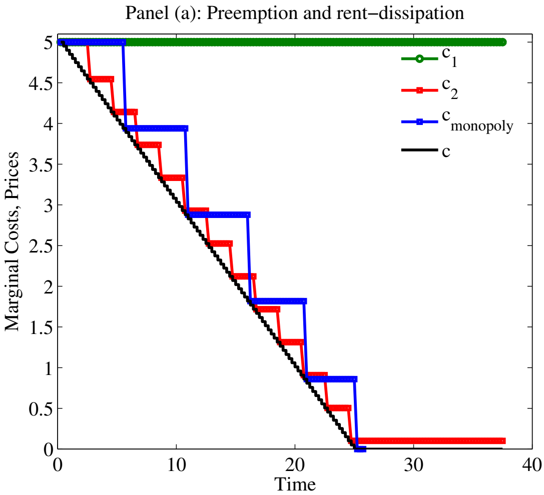
1. *right to move alternates deterministically.*
2. $K(c) = K$ *and is not prohibitively high.*
3. *technological progress is deterministic:* $c(t)$ *is a continuous, decreasing function*

has a unique MPE with

- *preemptive investments: by only one firm and no investment in equilibrium by its opponent.*
- *rent dissipation: discounted payoffs of both firms in equilibrium is* 0*, so the entire surplus is wasted on excessively frequent investments by the preempting firm.*

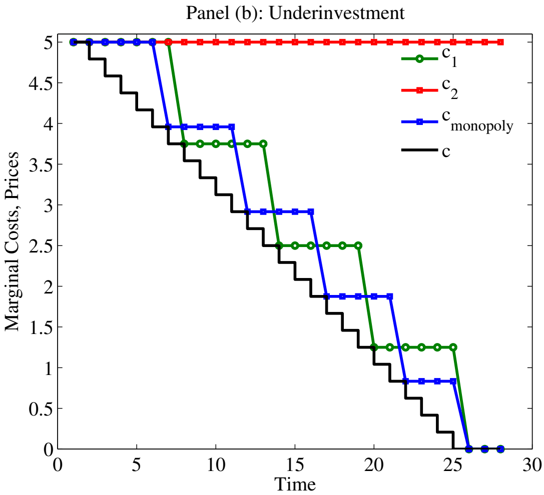
Full preemption and rent dissipation

Confirm R&S the result with high K and small dt



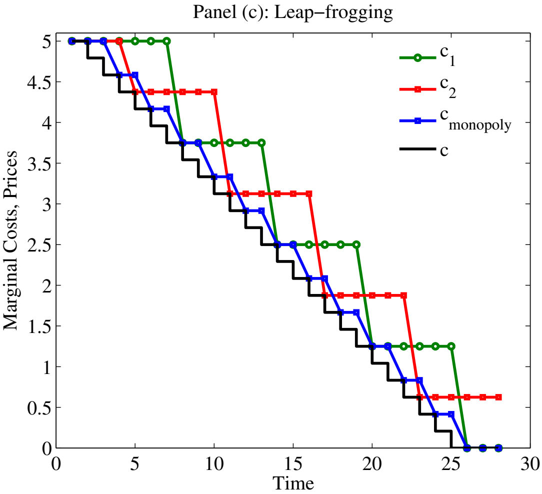
Underinvestment

Rent-dissapation is not a general outcome - disappears when K is low relative dt



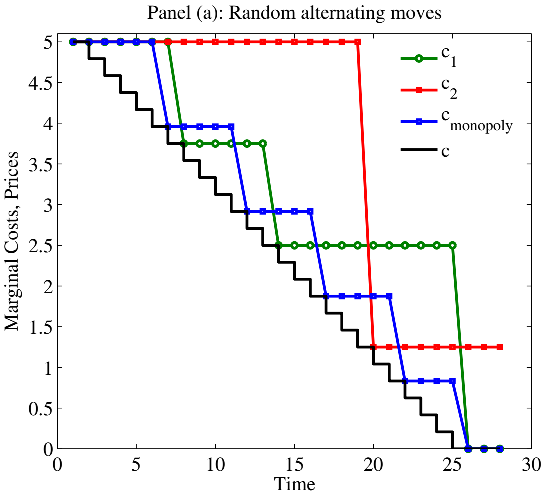
Leap-frogging

Preemption is not the general outcome - disappears when K is even lower



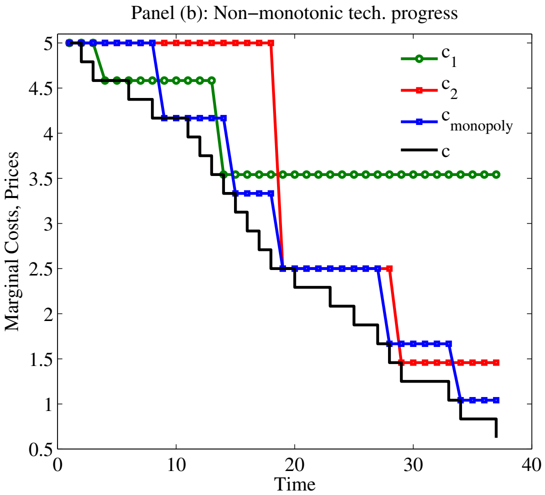
Random alternation → Leapfrogging

Riordan and Salant's result is not robust



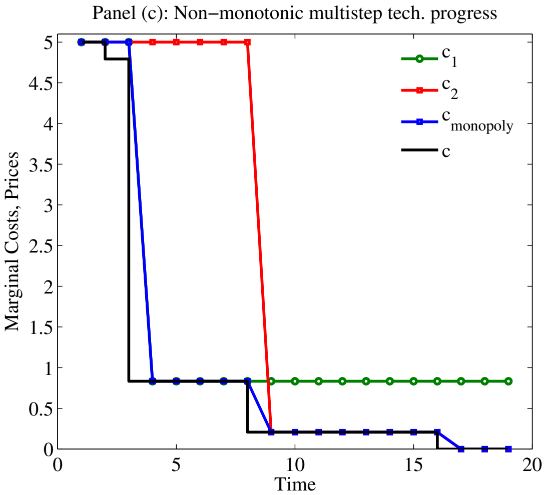
Random onestep technology → Leapfrogging

Riordan and Salant's result is not robust



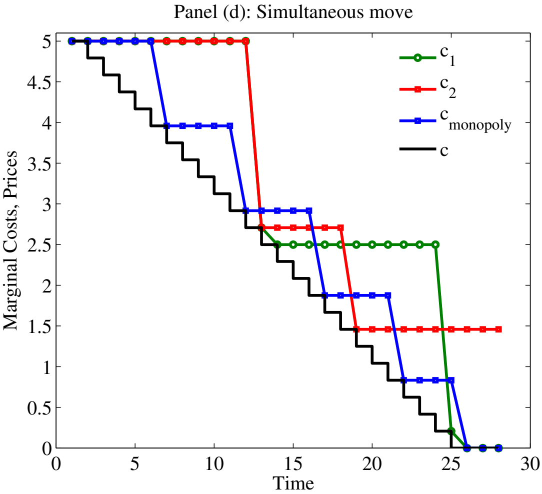
Random multistep technology → Leapfrogging

Riordan and Salant's result is not robust

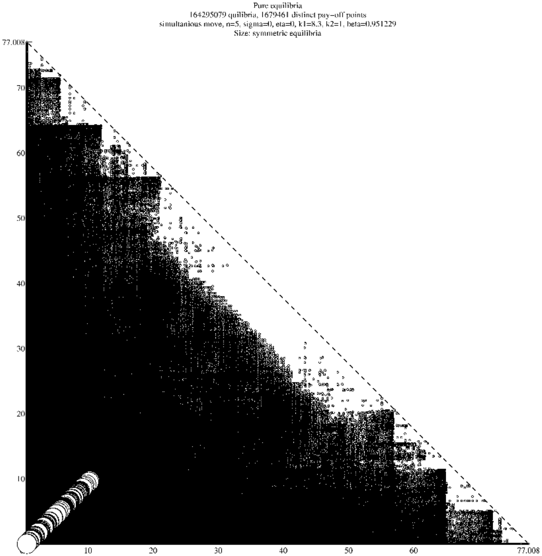


Simultaneous moves: Leapfrogging

Riordan and Salant's conjecture is wrong



Symmetric equilibria: $V_1(c_1, c_2, c) = V_2(c_2, c_1, c)$



Failure of homotopy approach

Homotopy parameter: η

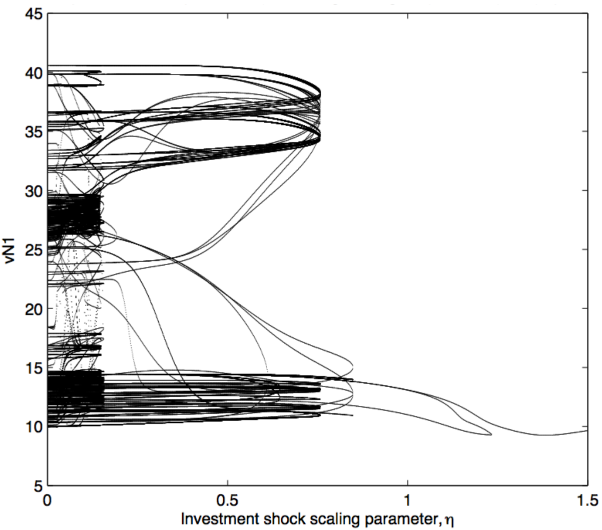
- In each period each firm incurs additive random costs/benefit from not investing and investing
- η is a scaling parameter that index variance of idiosyncratic shocks to investment
- High $\eta \rightarrow$ unique equilibrium $\eta \rightarrow 0 \rightarrow$ multiple equilibria

Problems:

- Multiplicity of equilibria $\rightarrow$ too many bifurcations along the path
- Equilibrium correspondence is not lower hemi-continuous

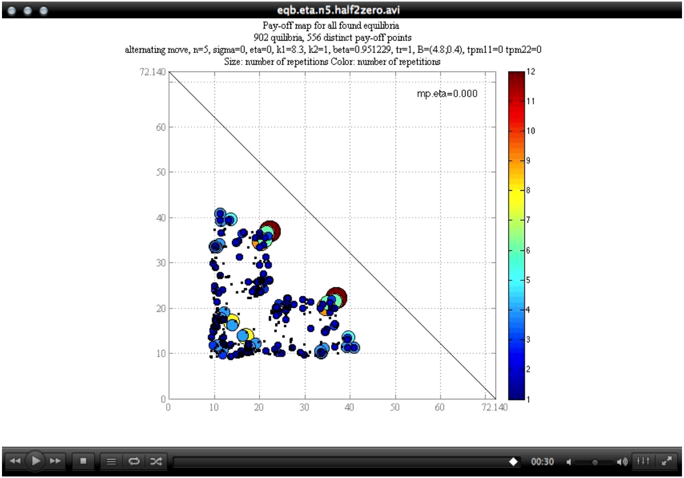
Failure of homotopy approach

Equilibrium correspondance, alternating move game: $V_{N,1}(c_0, c_0, c_0)$ vs. η



Failure of homotopy approach

Video: Set of equilibrium outcomes as variance of shocks decreases to zero



Conclusions: Bertrand investments model

- Many types of endogenous coordination is possible in equilibrium
  - Leapfrogging (alternating investments)
  - Preemption (investment by cost leader)
  - Duplicative (simultaneous investments)
- Full rent dissipation and monopoly outcomes are supported as MPE.
- Numerous MPE equilibria and “Folk theorem”-like result
- The equilibria are generally inefficient due to over-investment
  - Duplicative or excessively frequent investments

Conclusions: Solution of dynamic games

- When equilibrium is not unique the computation algorithm inadvertently acts as an equilibrium selection mechanism
- When directionality in the state space is present, state recursion algorithm is preferred to time iterations
- Plethora of Markov perfect equilibria poses new challenges:
  - How firms manage to coordinate on a particular equilibrium?
  - Increased difficulties for empirical applications.
  - Daunting perspectives for identification of equilibrium selection rule from the data.
- Estimation of dynamic games with multiple equilibria
  Nested Recursive Lexicographical Search (NRLS)