1 Mathematical Induction

Mathematical Induction is a powerful and elegant technique for proving certain types of mathematical statements: general propositions which assert that something is true for all positive integers or for all positive integers from some point on.

Let us look at some examples of the type of result that can be proved by induction.

Proposition 1. The sum of the first n positive integers $(1, 2, 3, \ldots)$ is $\frac{1}{2}n(n+1)$.

Proposition 2. In a convex polygon with n vertices, the greatest number of diagonal that can be drawn is $\frac{1}{2}n(n-3)$.

Note, we give an example of a convex polygon together with one that is not convex in Figure 1.

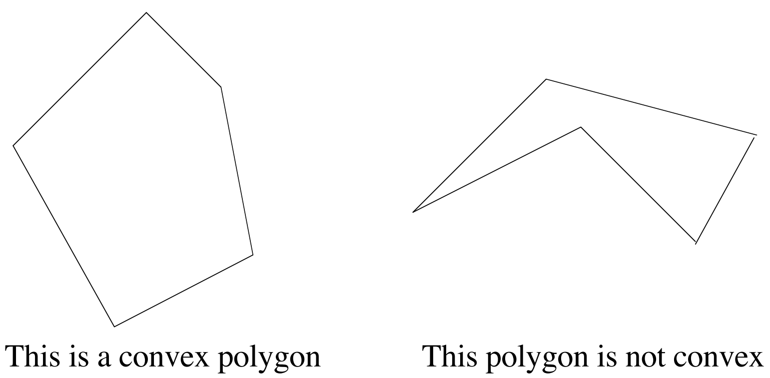

Figure 1: Examples of polygons

A polygon is said to be convex if any line joining two vertices lies within the polygon or on its boundary.

By a diagonal, we mean a line joining any two non-adjacent vertices.

As you see, the subject matter of the statements can vary widely. It can include algebra, geometry and many other topics. What is common to all the examples is the number n that appears in the statement. In all cases it is either stated, or implicitly assumed, that n can be any positive integer.

1.1 Why do we need proof by induction?

A natural starting point for proving many mathematical results is to look at a few simple cases. This helps us understand what is being claimed and may even give us some pointers for finding a proof.

Let's do this with Proposition 1. The results are recorded in the following table.

n	1	2	3	4
sum of first n numbers	1	$1+2=3$	$1+2+3=6$	$1+2+3+4=10$
$\frac{1}{2}n(n+1)$	$\frac{1}{2}\times 1\times 2=1$	$\frac{1}{2}\times 2\times 3=3$	$\frac{1}{2}\times 3\times 4=6$	$\frac{1}{2}\times 4\times 5=10$

You may find this quite convincing and wonder whether any further proof is needed. If a statement is true for all numbers we have tested, can we conclude that it is true for all values of n?

To answer this, let's look at another example.

Proposition 3. If p is any prime number, $2^p - 1$ is also a prime. Let us try some special cases here too.

p	2	3	5	7
$2^p - 1$	3	7	31	127

Since 3, 7, 31, 127 are all primes, we may be satisfied the result is always true. But if we try the next prime, 11, we find that

$$2^{11} - 1 = 2047 = 23 \times 89.$$

So it is not a prime, and our general assertion is therefore FALSE.

Let us turn again to Proposition 1, and ask how many cases we would need to check, before we could say for certain that it is true. Imagine getting a computer on the job and setting it the task. No matter how many values of n we found the proposition to be true for, we could never be sure that there was not an ever bigger value for which it was false. This explains the need for a general proof which covers all values of n. Mathematical induction is one way of doing this.

1.2 What is proof by induction?

One way of thinking about mathematical induction is to regard the statement we are trying to prove as not *one* proposition, but a whole sequence of propositions, one for each n. The trick used in mathematical induction is to prove the first statement in the sequence, and then prove that if any particular statement is true, then the one after it is also true. This enables us to conclude that *all* the statements are true.

Let's state these two steps in more formal language.

The initial step

Prove the proposition is true for $n = 1$.

(Or, if the assertion is that the proposition is true for $n \geq a$, prove it for $n = a$.)

Inductive step

Prove that *if* the proposition is true for $n = k$, then it *must* also be true for $n = k + 1$.

This step is the difficult part, and it may help you if we break it up into several stages.

Stage 1 Write down what the proposition asserts for the case $n = k$. This is what you are going to assume. It is often called the *inductive hypothesis*.

Stage 2 Write down what the proposition asserts for the case $n = k + 1$. This is what you have to prove. Keep this clearly in mind as you go.

Stage 3 Prove the statement in Stage 2, using the assumption in Stage 1. We can't give you any recipe for how to do this. It varies from problem to problem, depending on the mathematical content. You have to use your ingenuity, common sense and knowledge of mathematics here. The question to ask is "how can I get from Stage 1 to Stage 2?"

Once the initial and the inductive step have been carried out, we can conclude immediately that the proposition is true for all $n \geq 1$ (or for all $n \geq a$ if we started at $n = a$.)

To explain this, it may help to think of mathematical induction as an authomatic "statement proving" machine.

We have proved the proposition for $n = 1$.

By the inductive step, since it is true for $n = 1$, it is also true for $n = 2$. Again, by the inductive step, since it is true for $n = 2$, it is also true for $n = 3$. And since it is true for $n = 3$, it is also true for $n = 4$, and so on.

Because we have proved the inductive step, this process will never come to an end. We could set the "machine" running, and it would keep going forever, eventually reaching any n, no matter how big it may be.

Suppose there was a number N for which the statement was false. Then when we get to the number $N - 1$, we would have the following situation:

The statement is true for $n = N - 1$, but false for $n = N$.

This contradicts the inductive step, so it cannot possible happen. Hence the statement must be true for *all* positive integers n.

If you are familiar with computer programming, it may be helpful for you to compare this argument with a looping process, in which a computation is carried out, an indexing variable is advanced by one, and the computation is repeated.

The two processes have much in common. In a computer program you must begin by setting the initial value of your variables (this is analogous to our initial step). Then you must set up the loop, calling on the previous values of your variables to calculate new values (this is analogous to our inductive step).

There is one other thing necessary in a computer program: you must set up a "stop" condition otherwise your program will run forever. That has no analogy in our process – our theoretical machine *will* run forever! That is why we can be certain our result is true for all positive integers.

Now let's see how this works in practice, by proving Proposition 1.

Proposition 1. The sum of the first n positive integers is $\frac{1}{2}n(n+1)$.

Initial step: If $n = 1$, the sum is simply 1.

Now, for $n = 1$, $\frac{1}{2}n(n+1) = \frac{1}{2} \times 1 \times 2 = 1$. So the result is true for $n = 1$.

Inductive step:

Stage 1: Our assumption (the inductive hypothesis) asserts that

$$1 + 2 + 3 + \cdots + k = \frac{1}{2}k(k+1).$$

Stage 2: We want to prove that

$$1 + 2 + 3 + \cdots + (k+1) = \frac{1}{2}(k+1)[(k+1)+1] = \frac{1}{2}(k+1)(k+2).$$

Stage 3: How can we get to stage 2 from stage 1?

The answer here is that we get the left hand side of stage 2 from the left hand side of stage 1 by adding $(k+1)$.

So,

$$\begin{aligned}
1 + 2 + 3 + \cdots + (k+1) &= 1 + 2 + 3 + \cdots + k + (k+1) \\
&= \frac{1}{2}k(k+1) + (k+1) && \text{using the inductive hypothesis} \\
&= (k+1)(\frac{1}{2}k+1) && \text{factorising} \\
&= \frac{1}{2}(k+1)(k+2) && \text{which is what we wanted.}
\end{aligned}$$

This completes the inductive step.

Hence, the result is true for all $n \geq 1$.

As a further example, let's try proving Proposition 2.

Proposition 2. The number of diagonals of a convex polygon with n vertices is $\frac{1}{2}n(n-3)$, for $n \geq 4$.

Initial step: If $n = 4$, the polygon is a quadrilateral, which has two diagonals as shown in Figure 2.

Also, for $n = 4$, $\frac{1}{2}n(n-3) = \frac{1}{2}(4)(1) = 2$.

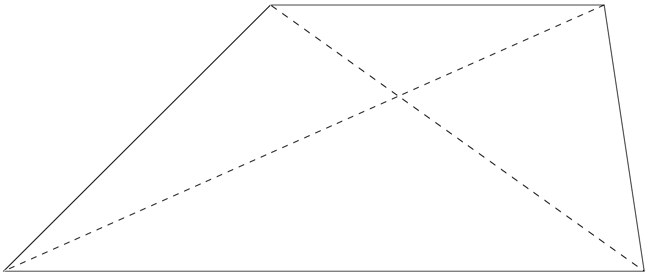

Figure 2: A quadrilateral showing 2 diagonals.

Inductive step:

Stage 1: The inductive hypothesis asserts that the number of diagonals of a polygon with k vertices is $\frac{1}{2}k(k-3)$.

Stage 2: We want to prove that the number of diagonals of a polygon with $(k+1)$ vertices is $\frac{1}{2}(k+1)[(k+1)-3] = \frac{1}{2}(k+1))(k+2)$.

Stage 3: How can we get to stage 2 from stage 1?

The answer here is to "add another vertex". Let's do this and see if we can count how many additional diagonals can be drawn as a result. Figure 3 will help here.

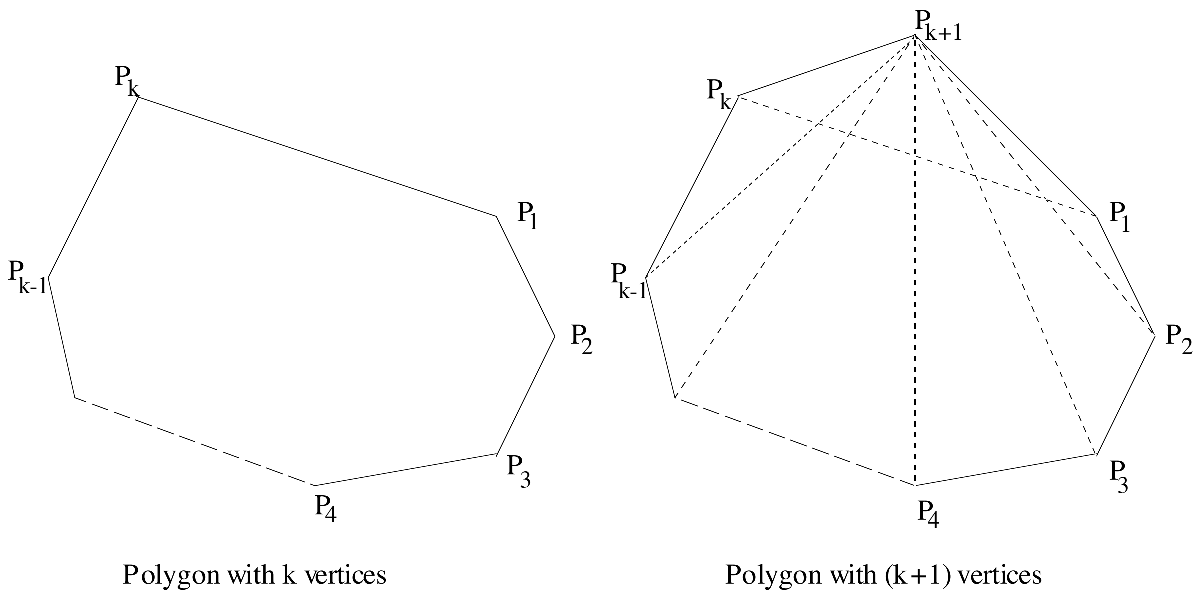

Figure 3: Adding another vertex to a polygon with k vertices.

When we add an extra vertex to a polygon with n vertices, all the lines which were diagonals of the original polygon will still be diagonals of the new one. The inductive hypothesis says that there are $\frac{1}{2}k(k-3)$ existing diagonals. In addition, new diagonals can be drawn from the extra vertex P_{k+1} to all other vertices except the two adjoining it (P_1 and P_k) giving us $k-2$ extra diagonals. Finally, the line joining P_1 and P_k, which used to be a side of the polygon, is now a diagonal.

This gives us a total of

$$\frac{1}{2}k(k-3)+(k-2)+1 \quad \text{diagonals.}$$

But,

$$\begin{aligned}\frac{1}{2}k(k-3)+(k-2)+1 &= \frac{1}{2}[k(k-3)+2k-2]\\ &= \frac{1}{2}(k^2-k-2)\\ &= \frac{1}{2}(k+1)(k+2), \quad \text{as required.}\end{aligned}$$

This completes the inductive step. Hence the result is true for all $n \geq 4$.

At this point, it would be a good idea to go back and read over the explanation of the process of mathematical induction thinking about the general explanation in the light of the two examples we have just completed.

Next, we illustrate this process again, by using mathematical induction to give a proof of an important result, which is frequently used in algebra, calculus, probability and other topics.

1.3 The Binomial Theorem

The Binomial Theorem states that if n is an integer greater than 0,

$$(x+a)^n = x^n+nx^{n-1}a+\frac{n(n-1)}{2!}x^{n-2}a^2+\frac{n(n-1)(n-2)}{3!}x^{n-3}a^3+\cdots+nxa^{n-1}+a^n. \qquad (1)$$

For example,

$$(x+a)^3 = x^3+3x^2a+3xa^2+a^3.$$

This useful theorem can be more compactly expressed as

$$(x+a)^n = \sum_{r=0}^{n} nCrx^ra^{n-r}$$

but if you are not familiar with this notation, use equation (1).

Let's prove this theorem by mathematical induction.

Initial step: Let $n=1$. Then the left hand side of (1) is $(x+a)^1$ and the right hand side of (1) is x^1+a^1 which both equal $x+a$.

Inductive step:

Step 1: Assume the theorem is true for $n = k$, ie that

$$(x+a)^k = x^k + kx^{k-1}a + \frac{k(k-1)}{2!}x^{k-2}a^2 + \cdots + kxa^{k-1} + a^k.$$

Stage 2: We want to prove that the theorem is true for $n = k+1$, ie that

$$(x+a)^{k+1} = x^{k+1} + (k+1)x^k a + \frac{(k+1)(k)}{2!}x^{k-1}a^2 + \cdots + (k+1)xa^k + a^{k+1}. \quad (2)$$

Stage 3: How do we get to stage 2 from stage 1?

Look at the left hand side of (2),

$$(x+a)^{k+1} = (x+a)(x+a)^k.$$

but by the inductive hypothesis,

$$\begin{aligned}
(x+a)(x+a)^k &= (x+a)[x^k + kx^{k-1}a + \frac{k(k-1)}{2!}x^{k-2}a^2 + \cdots + kxa^{k-1} + a^k] \\
&= x^{k+1} + kx^k a + \frac{k(k-1)}{2!}x^{k-1}a^2 + \cdots + kx^2a^{k-1} + xa^k \\
&\qquad + x^k a + \qquad kx^{k-1}a^2 + \qquad \cdots \qquad + kxa^k + a^{k+1} \\
\text{(adding)} \qquad &= x^{k+1} + (k+1)x^k a + \frac{(k+1)(k)}{2!}x^{k-1}a^2 + \cdots + (k+1)xa^k + a^{k+1}.
\end{aligned}$$

This is the right hand side as required. Hence the result is true for all $n \geq 1$.

Now try some of the exercises for yourself.

1.4 Exercises

Prove the following statements by mathematical induction.

Note that many of these statements can also be proved by other methods, and sometimes these other proofs will be neater and simpler. However, the purpose of these exercises is to practice proof by induction, so please try to use this method, even if if you can see an easier way.

1. The n^{th} odd number is $2n - 1$.

2. The sum of the first n odd numbers is n^2.

3. **a.** For every positive integer n, $n(n+1)$ is even.

b. For every positve integer $n \geq 2$, $n^3 - n$ is a multiple of 6.