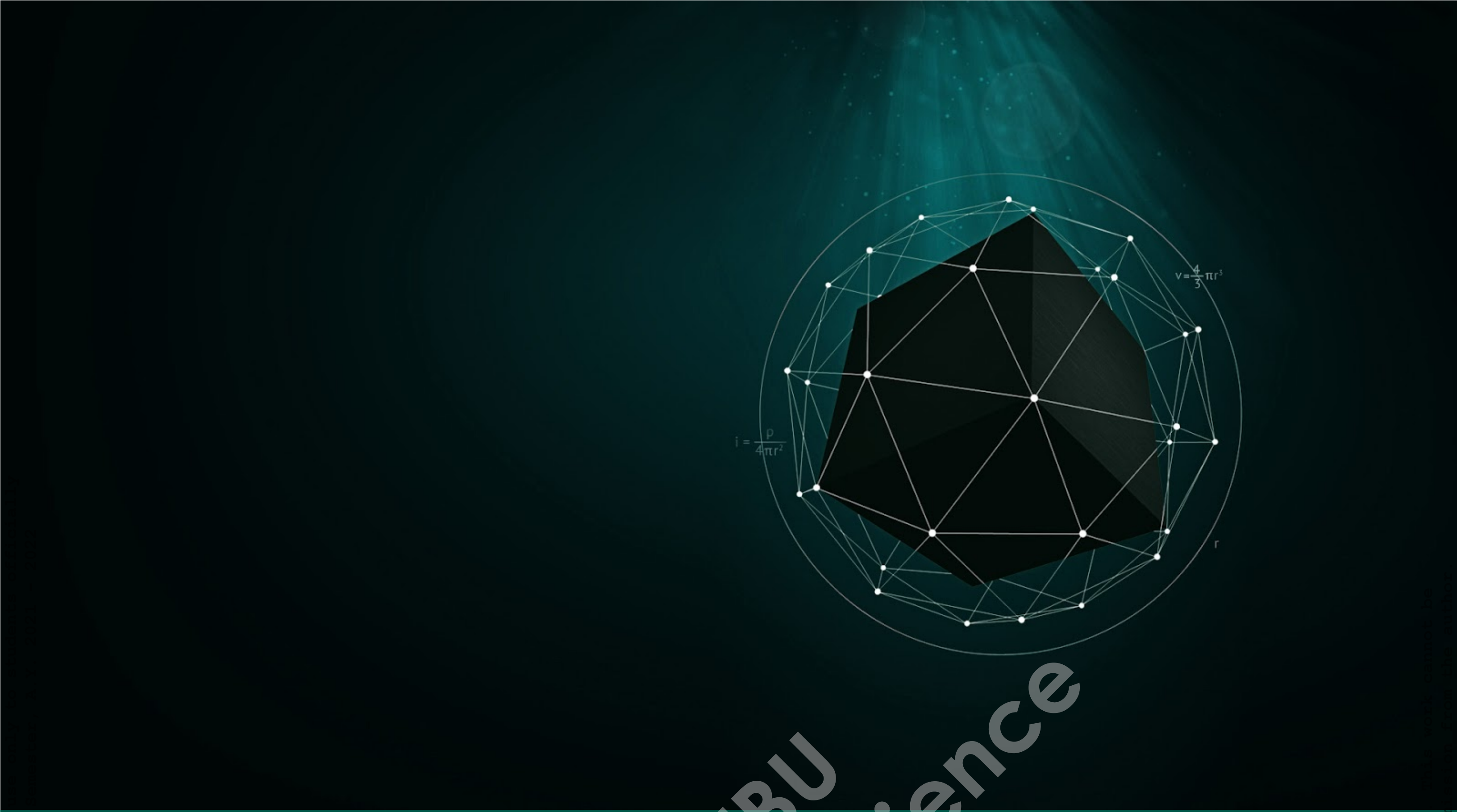

ELEMENTARY ANALYSIS II

M A T H 5 4

Learner's Module

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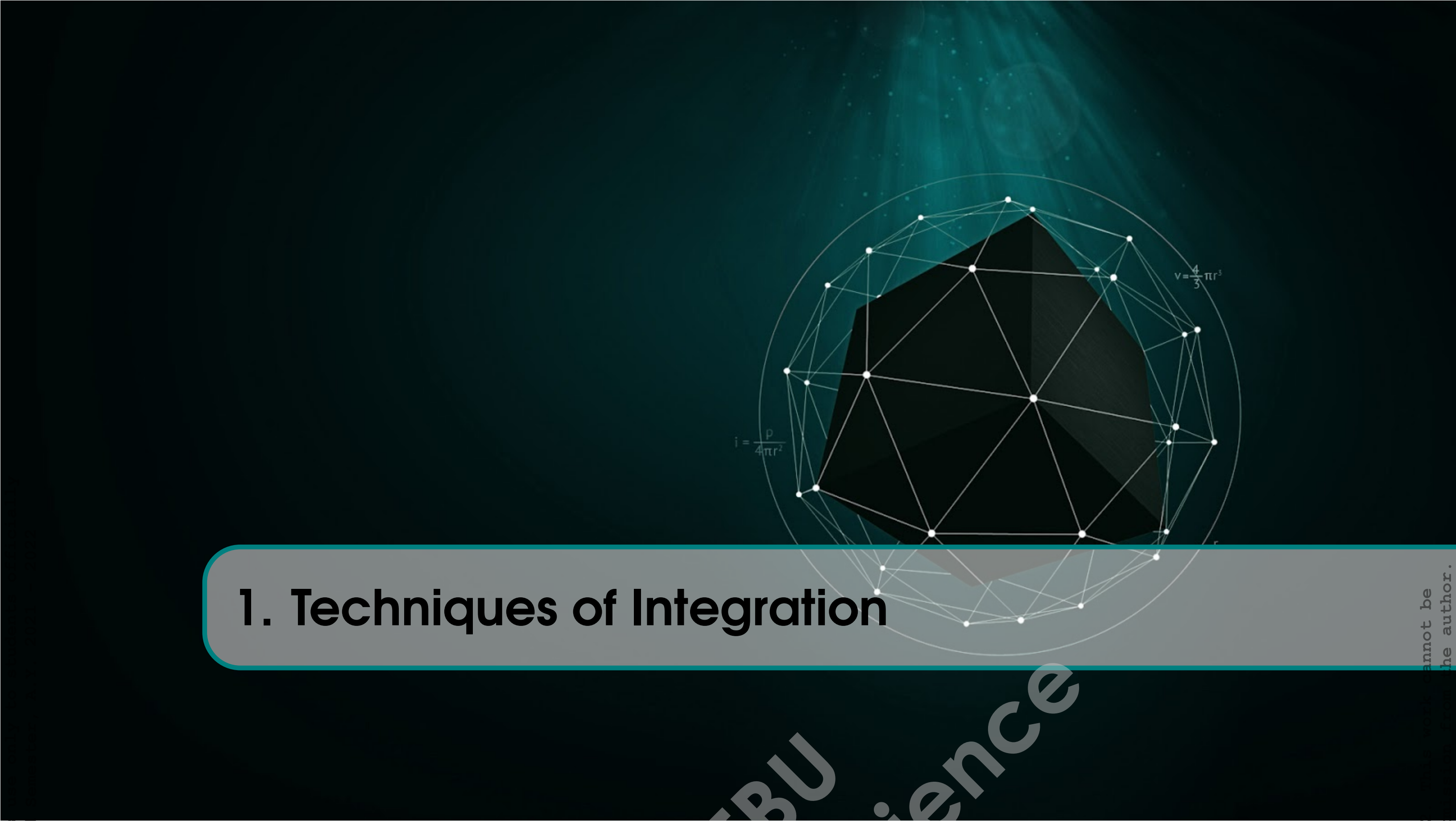
I

Week One



1. Techniques of Integration

At the end of this topic you are expected to:

- find integrals using integration by parts
- solve trigonometric integrals

Below is a list of some important integrals that we have studied so far in Math 53.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad\qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad\qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad\qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad\qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad\qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C,\ a>0$$

1.1 Integration by Parts

Integration by parts is used to a wide variety of functions and is particularly useful for integrands involving products of algebraic and transcendental functions. Here are some integrals that uses integration by parts:

$$\int x\ln x\,dx \qquad \int x^2 e^x\,dx \qquad \int e^x \cos x\,dx$$

Integration by parts is based on the formula for the derivative of a product

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx} = uv' + vu'$$

where u and v are differentiable functions of x. After interchanging both sides of the equation we obtain

$$uv = \int uv'\,dx + \int vu'\,dx = \int u\,dv + \int v\,du$$

or

$$uv - \int v\,du = \int u\,dv$$

and so we have the following theorem.

Theorem 1.1.1 (Integration by Parts)

If u and v are functions of x and have continuous derivatives, then

$$\int u\,dv = uv - \int v\,du.$$

■ **Example 1.1** Find $\int x e^x\,dx$.

Solution: We need to form the integral as in the above theorem. There are only two things to pick, those are, u and dv. Usually, we select u to be the function with a much simpler derivative and the remaining parts of the integral will be our dv. In our case,

$$\text{Let } u = x \text{ and } dv = e^x dx$$

Then we differentiate u and integrate dv giving us

$$du = dx \text{ and } v = e^x$$

We will just remove the constant of integration C in v for it will just cancel out in the integration process later. Thus, we have

$$\begin{aligned}\int u\,dv &= uv - \int v\,du \\ \int xe^x\,dx &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + C.\end{aligned}$$

Have you ever wondered on what is the integral of the function $f(x) = \ln x$? Integration by parts will give us the answer.

■ **Example 1.2** Find the $\int \ln x\,dx$.

Solution: Let us form the formula by choosing u and dv.

$$\text{Let } u = \ln x \text{ and } dv = dx.$$

Then we differentiate u and integrate dv giving us

$$du = \tfrac{1}{x}dx \text{ and } v = x.$$

Thus, we can form our integral as

$$\begin{aligned}\int u\,dv &= uv - \int v\,du \\ \int \ln x\,dx &= x\ln x - \int x \cdot \frac{1}{x}\,dx \\ &= x\ln x - \int dx \\ &= x\ln x - x + C.\end{aligned}$$

There are also times when integration by parts are to be performed more than once as in our next example.

■ **Example 1.3** Find $\int x^2 \sin x \, dx$.

Solution: Let us form the formula by choosing u and dv.

$$\text{Let } u = x^2 \text{ and } dv = \sin x \, dx.$$

Then we differentiate u and integrate dv giving us

$$du = 2x \, dx \text{ and } v = -\cos x.$$

Thus, we can form our integral as

$$\begin{aligned} \int u \, dv &= uv - \int v \, du \\ \int x^2 \sin x \, dx &= -x^2 \cos x - \int (-2x \cos x) \, dx \\ &= -x^2 \cos x + \int 2x \cos x \, dx. \end{aligned}$$

Note that the integral on the right $\int 2x \cos x \, dx$ calls for another integration by parts. Let us focus on this integral and perform the integration by parts for the second time.

$$\text{Let } u = 2x \text{ and } dv = \cos x \, dx.$$

Then we differentiate u and integrate dv giving us

$$du = 2 \, dx \text{ and } v = \sin x.$$

So,

$$\begin{aligned} \int u \, dv &= uv - \int v \, du \\ \int 2x \cos x \, dx &= 2x \sin x - \int 2 \sin x \, dx \\ &= 2x \sin x + 2 \cos x + C. \end{aligned}$$

Therefore,

$$\begin{aligned} \int x^2 \sin x \, dx &= -x^2 \cos x + \int 2x \cos x \, dx \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + C. \end{aligned}$$

There are also cases when the given integral reappears on the right-hand side of the equation upon applying integration by parts repeatedly like in our next example.

Example 1.4 Find $\int e^x \sin x\, dx$.

Solution: Choosing $u = e^x$ and $dv = \sin x\, dx$ and finding du and v, we have

$$du = e^x\, dx \text{ and } v = -\cos x.$$

So,

$$\int e^x \sin x\, dx = -e^x \cos x + \int e^x \cos x\, dx$$

The integral $\int e^x \cos x\, dx$ is as pretty the same as our given integral. Hence, we employ integration by parts for the second time. This time we use $u = e^x$ and $dv = \cos x\, dx$. Thus, we have

$$du = e^x\, dx \text{ and } v = \sin x$$

Then

$$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx$$

Upon inspection, we have seen an integral on the right exactly the same as what we have started earlier. Moreover,

$$\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx.$$

This means that

$$\begin{aligned} \int e^x \sin x\, dx + \int e^x \sin x\, dx &= -e^x \cos x + e^x \sin x \\ 2\int e^x \sin x\, dx &= -e^x \cos x + e^x \sin x \end{aligned}$$

Therefore,

$$\int e^x \sin x\, dx = \frac{1}{2} e^x (\sin x - \cos x) + C.$$

If we combine the formula for integration by parts and the Fundamental Theorem of Calculus, we obtain the formula

$$\boxed{\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx}$$

■ **Example 1.5** Evaluate $\int_0^1 \tan^{-1} x\,dx$.

Solution: Let

$$u = \tan^{-1} x \text{ and } dv = dx.$$

Then,

$$du = \frac{1}{1+x^2}\,dx \text{ and } v = x.$$

Hence,

$$\begin{aligned}\int_0^1 \tan^{-1} x\,dx &= x\tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx \\ &= 1\cdot\tan^{-1}(1) - 0\cdot\tan^{-1}(0) - \int_0^1 \frac{x}{1+x^2}\,dx \\ &= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx\end{aligned}$$

Let us evaluate $\int_0^1 \frac{x}{1+x^2}\,dx$ by substitution. Let $t = 1+x^2$. Then $dt = 2x\,dx$. Moreover, when $x = 0$, $t = 1$ and when $x = 1$, $t = 2$. Hence,

$$\begin{aligned}\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_1^2 \frac{dt}{t} &= \frac{1}{2}\ln|t|\Big|_1^2 \\ &= \frac{1}{2}(\ln 2 - \ln 1) \\ &= \frac{1}{2}\ln 2\end{aligned}$$

Therefore,

$$\int_0^1 \tan^{-1} x\,dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx = \frac{\pi}{4} - \frac{1}{2}\ln 2.$$

1.2 Trigonometric Integrals

In this section, we use trigonometric identities to integrate certain combinations of trigonometic functions. We begin with powers of sine and cosine.

■ **Example 1.6** Find $\int \cos^3 x\, dx$.

Solution: Here, we can separate one cosine factor and convert the remaining $\cos^2 x$ factor to an expression involving sine using the identity $\sin^2 x + \cos^2 x = 1$:

$$\cos^3 x = \cos^2 x \cdot \cos x = (1 - \sin^2 x)\cos x.$$

If $u = \sin x$, then $du = \cos x\, dx$ and

$$\begin{aligned}\int \cos^3 x\, dx &= \int \cos^2 x \cdot \cos x\, dx = \int (1-\sin^2 x)\cos x\, dx\\ &= \int (1-u^2)du = u - \frac{1}{3}u^3 + C\\ &= \sin x - \frac{1}{3}\sin^3 x + C.\end{aligned}$$

■ **Example 1.7** Find $\int \sin^5 x \cos^2 x\, dx$.

Solution: Here, we separate a single sine factor and rewrite the remaining $\sin^4 x$ factor in terms of $\cos x$:

$$\sin^5 x \cos^2 x = (\sin^2 x)^2 \cos^2 x \sin x = (1-\cos^2 x)^2 \cos^2 x \sin x$$

Substituting $u = \cos x$, we have $du = -\sin x\, dx$ and so

$$\begin{aligned}\int \sin^5 x \cos^2 x\, dx &= \int (\sin^2 x)^2 \cos^2 x \sin x\, dx\\ &= \int (1-\cos^2 x)^2 \cos^2 x \sin x\, dx\\ &= \int (1-u^2)^2 u^2(-du) = -\int (u^2 - 2u^4 + u^6)\, du\\ &= -\left(\frac{u^3}{3} - \frac{2}{5}u^5 + \frac{1}{7}u^7\right) + C\\ &= -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C.\end{aligned}$$

An odd power of sine or cosine enabled us to separate a single factor and convert the remaining even power. However, if the integrand contains an even power of both sine and cosine, this strategy fails. In cases like these, we use the half-angle identities.

$$\sin^2 x = \frac{1-\cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

■ **Example 1.8** Evaluate $\int_0^\pi \sin^2 x\, dx$.

Solution: Using the half-angle formula for $\sin^2 x$, we have

$$\begin{aligned}\int_0^\pi \sin^2 x\, dx &= \frac{1}{2}\int_0^\pi (1-\cos 2x)\, dx \\ &= \left[\frac{1}{2}\left(x-\frac{1}{2}\sin 2x\right)\right]_0^\pi \\ &= \frac{1}{2}\left(\pi-\frac{1}{2}\sin 2\pi\right)-\frac{1}{2}\left(0-\frac{1}{2}\sin 0\right)=\frac{\pi}{2}.\end{aligned}$$

To summarize, we list the guidelines in evaluating integrals of the form $\int \sin^m x \cos^n x\, dx$, where $m \geq 0$ and $n \geq 0$ are integers.

Strategy for Evaluating $\int \sin^m x \cos^n x\, dx$

(a) If the power of cosine is odd $(n = 2k+1)$, save one cosine factor and use $\cos^2 x = 1-\sin^2 x$ to express the remaining factors in terms of sine:

$$\begin{aligned}\int \sin^m x \cos^{2k+1} x\, dx &= \int \sin^m x\, (\cos^2 x)^k \cos x\, dx \\ &= \int \sin^m x\, (1-\sin^2 x)^k \cos x\, dx\end{aligned}$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd $(m = 2k+1)$, save one sine factor and use $\sin^2 x = 1-\cos^2 x$ to express the remaining factors in terms of cosine:

$$\begin{aligned}\int \sin^{2k+1} x \cos^n x\, dx &= \int (\sin^2 x)^k \cos^n x \sin x\, dx \\ &= \int (1-\cos^2 x)^k \cos^n x \sin x\, dx\end{aligned}$$

Then substitute $u = \cos x$.

If the powers of both sine and cosine are odd, then either (a) or (b) can be used.

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \text{and} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

A similar strategy can be used to evaluate integrals of the form $\int \tan^m x \sec^n x\,dx$ which we will summarize af follows.

Strategy for Evaluating $\displaystyle\int \tan^m x \sec^n x\,dx$

(a) If the power of secant is even $(n = 2k)$, save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\begin{aligned}\int \tan^m x \sec^{2k} x\,dx &= \int \tan^m x\,(\sec^2 x)^{k-1} \sec^2 x\,dx\\ &= \int \tan^m x\,(1+\tan^2 x)^{k-1} \sec^2 x\,dx\end{aligned}$$

Then substitute $u = \tan x$.

(b) If the power of tangent is odd $(m = 2k+1)$, save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\begin{aligned}\int \tan^{2k+1} x \sec^n x\,dx &= \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x\,dx\\ &= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x\,dx\end{aligned}$$

Then substitute $u = \sec x$.

■ **Example 1.9** Find $\displaystyle\int \tan^6 x \sec^4 x\,dx$.

Solution: This is case (a). We will evaluate the integral by substituting $u = \tan x$ so that $du = \sec^2 x\,dx$.

$$\begin{aligned}\int \tan^6 x \sec^4 x\,dx &= \int \tan^6 x \sec^2 x \sec^2 x\,dx\\ &= \int \tan^6 x\,(1+\tan^2 x) \sec^2 x\,dx\\ &= \int u^6(1+u^2)du = \int (u^6+u^8)\,du\\ &= \frac{u^7}{7} + \frac{u^9}{9} + C\\ &= \frac{1}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C.\end{aligned}$$

■ **Example 1.10** Find $\int \tan^5\theta \sec^7\theta \, d\theta$

Solution: This is case (b). We can evaluate the integral by substituting $u = \sec\theta$, so that $du = \sec\theta \tan\theta \, d\theta$:

$$\begin{aligned}\int \tan^5\theta \sec^7\theta \, d\theta &= \int \tan^4\theta \sec^6\theta \sec\theta \tan\theta \, d\theta \\ &= \int (\sec\theta - 1)^2 \sec^6\theta \sec\theta \tan\theta \, d\theta \\ &= \int (u^2-1)^2 u^6 \, du \\ &= \int (u^{10} - 2u^8 + u^6) du \\ &= \frac{1}{11}u^{11} - \frac{2}{9}u^9 + \frac{1}{7}u^7 + C \\ &= \frac{1}{11}\sec^{11}\theta - \frac{2}{9}\sec^9\theta + \frac{1}{7}\sec^7\theta + C.\end{aligned}$$

■ **Example 1.11** Find $\int \tan^3 x \, dx$.

Solution: Here only $\tan x$ occurs, so we use $\tan^2 x = \sec^2 x - 1$:

$$\begin{aligned}\int \tan^3 x \, dx = \int \tan x \tan^2 x \, dx &= \int \tan x (\sec^2 x - 1) \, dx \\ &= \int \tan x \sec^2 x \, dx - \int \tan x \, dx \\ &= \frac{\tan^2 x}{2} - \ln|\sec x| + C.\end{aligned}$$

Some powers of $\sec x$ may require integrations by parts, as shown in the next example.

■ **Example 1.12** Find $\int \sec^3 x \, dx$.

Solution: Here, we integrate by parts with

$$\begin{aligned} u &= \sec x & dv &= \sec^2 x \, dx \\ du &= \sec x \tan x \, dx & v &= \tan x \end{aligned}$$

Then

$$\begin{aligned}
\int \sec^3 x\, dx &= \sec x \tan x - \int \sec x \tan^2 x\, dx \\
&= \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right)\, dx \\
&= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx \\
&= \sec x \tan x - \int \sec^3 x\, dx + \ln|\sec x + \tan x| + C_1.
\end{aligned}$$

This means that

$$\begin{aligned}
2\int \sec^3 x &= \sec x \tan x + \ln|\sec x + \tan x| + C_1 \\
\int \sec^3 x &= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x| + C_1)
\end{aligned}$$

Therefore, $\displaystyle\int \sec^3 x = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C.$

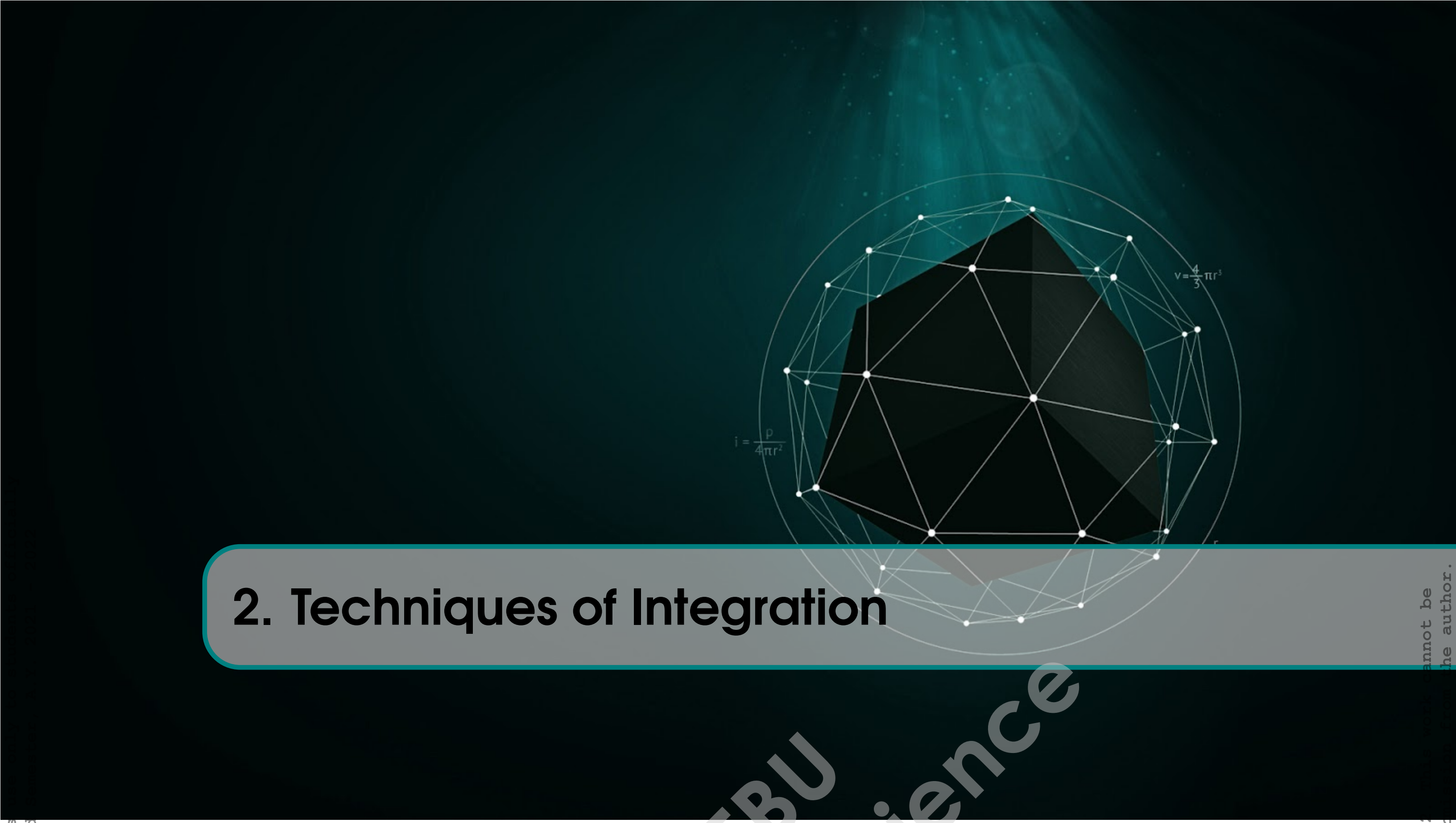
1.3 Exercises

Exercise 1.1 Evaluate the following integrals:

1. $\displaystyle\int x^2 \sin 4x\, dx$
2. $\displaystyle\int x \sec^{-1} x\, dx$
3. $\displaystyle\int \cos^4 x\, dx$
4. $\displaystyle\int_0^{\pi/4} \tan^4 x\, dx$
5. $\displaystyle\int \tan^5 x \sec^4 x\, dx$

II

Week Two



2. Techniques of Integration

At the end of this topic you are expected to:

- find integrals using trigonometric substitution

2.1 Trigonometric Substitution

In the first week, we have learned to find integrals involving powers of trigonometric functions. This time, we will use the technique called **trigonometric substitution** to find the integrals involving radicals of the form

$$\sqrt{a^2-u^2}, \quad \sqrt{a^2+u^2}, \quad \text{and} \quad \sqrt{u^2-a^2}.$$

The objective with trigonometric substitution is to eliminate the radical in the integrand. The Pythagorean identities will aid us in our transformation.

$$\cos^2\theta = 1 - \sin^2\theta$$
$$\sec^2\theta = 1 + \tan^2\theta$$
$$\tan^2\theta = \sec^2\theta - 1$$

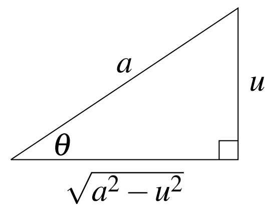
For example, for $a > 0$, if $u = a\sin\theta$, where $\pi/2 \leq \theta \leq \pi/2$. Then

$$\begin{aligned}\sqrt{a^2 - u^2} &= \sqrt{a^2 - a^2\sin^2\theta}\\ &= \sqrt{a^2(1-\sin^2\theta)}\\ &= \sqrt{a^2\cos^2\theta}\\ &= a\cos\theta.\end{aligned}$$

Trigonometric Substitution ($a > 0$)

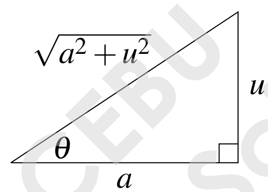
1. For integrals involving $\sqrt{a^2 - u^2}$, let $u = a \sin\theta$.

   Then $\sqrt{a^2 - u^2} = a\cos\theta$, where $-\pi/2 \le \theta \le \pi/2$.



2. For integrals involving $\sqrt{a^2 + u^2}$, let $u = a \tan\theta$.

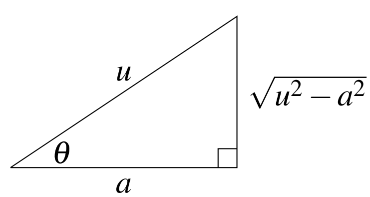
   Then $\sqrt{a^2 + u^2} = a\sec\theta$, where $-\pi/2 < \theta < \pi/2$.



3. For integrals involving $\sqrt{u^2 - a^2}$, let $u = a \sec\theta$.

   Then

$$\sqrt{u^2 - a^2} = \begin{cases} a\tan\theta & \text{for } u > a, \text{ where } 0 \le \theta < \pi/2 \\ -a\tan\theta & \text{for } u < -a, \text{ where } \pi/2 < \theta \le \pi. \end{cases}$$



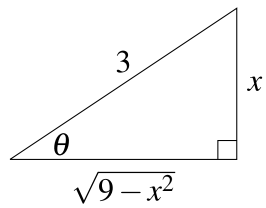
Remark 2.1.1 The restrictions on θ ensure the function that defines the substitution is one-to-one. In fact, these are the same intervals over which the arcsine, arctangent, and arcsecant are defined.

■ **Example 2.1** Find $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}}$.

Solution: We cannot use the basic integration rules here. This calls for a trigonometric substitution. Note that our integrand contains $\sqrt{9-x^2}$ which is of the form $\sqrt{a^2-u^2}$. So we let

$$x = 3\sin\theta$$

Thus, our reference triangle would be



Moreover, $dx = 3\cos\theta\, d\theta$, $\sqrt{9-x^2} = 3\cos\theta$, and $x^2 = 9\sin^2\theta$.
Hence,

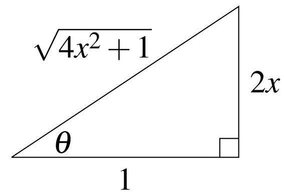
$$\begin{aligned}\int \frac{dx}{x^2\sqrt{9-x^2}} &= \int \frac{3\cos\theta\, d\theta}{(9\sin^2\theta)(3\cos\theta)}\\ &= \frac{1}{9}\int \frac{d\theta}{\sin^2\theta}\\ &= \frac{1}{9}\int \csc^2\theta\, d\theta\\ &= -\frac{1}{9}\cot\theta + C\\ &= -\frac{1}{9}\left(\frac{\sqrt{9-x^2}}{x}\right) + C\\ &= -\frac{\sqrt{9-x^2}}{9x} + C.\end{aligned}$$

■ **Example 2.2** Find $\displaystyle\int \frac{dx}{4x^2+1}$.

Solution: Our integrand contains $\sqrt{4x^2+1} = \sqrt{(2x)^2+1^2}$ which is of the form $\sqrt{u^2+a^2}$. So we let

$$2x = 1\tan\theta$$

Thus, our reference triangle would be



Moreover, $dx = \frac{1}{2}\sec^2\theta\,d\theta$, and $\sqrt{4x^2+1} = \sec\theta$.

Following the appropriate substitution gives us,

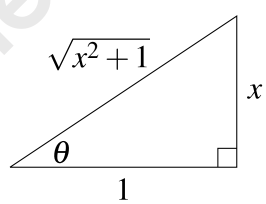
$$\begin{aligned}\int \frac{dx}{\sqrt{4x^2+1}} &= \frac{1}{2}\int \frac{\sec^2\theta\,d\theta}{\sec\theta} \\ &= \frac{1}{2}\int \sec\theta\,d\theta \\ &= \frac{1}{2}\ln|\sec\theta+\tan\theta| + C \\ &= \frac{1}{2}\ln|\sqrt{4x^2+1}+2x| + C.\end{aligned}$$

■ **Example 2.3** Find $\displaystyle\int \frac{dx}{(x^2+1)^{3/2}}$.

Solution: Our integrand contains $(x^2+1)^{3/2}$ which we can write as $(\sqrt{x^2+1})^3$. So we let

$$x = 1\tan\theta$$

Thus, our reference triangle would be



Moreover, $dx = \sec^2\theta\,d\theta$, and $\sqrt{x^2+1} = \sec\theta$.

Following the appropriate substitution gives us,

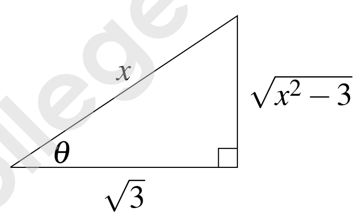
$$\begin{aligned}\int \frac{dx}{(x^2+1)^{3/2}} &= \int \frac{dx}{(\sqrt{x^2+1})^3}\\ &= \int \frac{\sec^2\theta\, d\theta}{\sec^3\theta}\\ &= \int \frac{d\theta}{\sec\theta}\\ &= \int \cos\theta\, d\theta\\ &= \sin\theta + C\\ &= \frac{x}{\sqrt{x^2+1}} + C.\end{aligned}$$

■ **Example 2.4** Evaluate $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx$.

Solution: Note that $\sqrt{x^2-3}$ is of the form $\sqrt{u^2-a^2}$. So we let

$$x = \sqrt{3}\sec\theta$$

Thus, our reference triangle would be



Moreover, $dx = \sqrt{3}\sec\theta\tan\theta\, d\theta$, and $\sqrt{x^2-3} = \sqrt{3}\tan\theta$.

Let us determine the upper and lower limits of integration. When $x = \sqrt{3}$, $\sec\theta = 1$ and so, $\theta = 0$. When $x = 2$, $\sec\theta = \frac{2}{\sqrt{3}}$ ans so, $\theta = \frac{\pi}{6}$.

Hence,

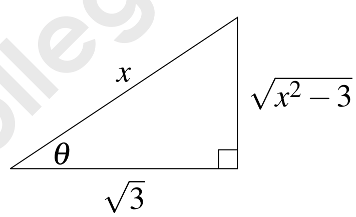
$$\begin{aligned}
\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx &= \int_0^{\pi/6} \frac{(\sqrt{3}\tan\theta)(\sqrt{3}\sec\theta\tan\theta)}{\sqrt{3}\sec\theta}\,d\theta \\
&= \int_0^{\pi/6} \sqrt{3}\tan^2\theta\,d\theta \\
&= \sqrt{3}\int_0^{\pi/6} (\sec^2\theta - 1)\,d\theta \\
&= \sqrt{3}\Big[\tan\theta - \theta\Big]_0^{\pi/6} \\
&= \sqrt{3}\left(\frac{1}{\sqrt{3}} - \frac{\pi}{6}\right) \\
&= 1 - \frac{\sqrt{3}}{6}\pi.
\end{aligned}$$

■ **Example 2.5** Evaluate $\displaystyle\int_{-2}^{-\sqrt{3}} \frac{\sqrt{x^2-3}}{x}\,dx$.

Solution: Note that $\sqrt{x^2-3}$ is of the form $\sqrt{u^2-a^2}$. So we let

$$x = \sqrt{3}\sec\theta$$

Thus, our reference triangle would be



Moreover, $dx = \sqrt{3}\sec\theta\tan\theta\,d\theta$, and $\sqrt{x^2-3} = -\sqrt{3}\tan\theta$, since $a = \sqrt{3}$ and $u = x$ in $[-2, -\sqrt{3}]$ would imply $u < -a$.

Let us determine the upper and lower limits of integration. When $x = -2$, $\sec\theta = \dfrac{-2}{\sqrt{3}}$ and so, $\theta = \dfrac{5\pi}{6}$. When $x = -\sqrt{3}$, $\sec\theta = -1$ ans so, $\theta = \pi$. (θ is chosen within $\pi/2 < \theta \le \pi$)

Hence,

$$\begin{aligned}
\int_{-2}^{-\sqrt{3}} \frac{\sqrt{x^2-3}}{x}\,dx &= \int_{5\pi/6}^{\pi} \frac{(-\sqrt{3}\tan\theta)(\sqrt{3}\sec\theta\tan\theta)}{\sqrt{3}\sec\theta}\,d\theta \\
&= \int_{5\pi/6}^{\pi} -\sqrt{3}\tan^2\theta\,d\theta \\
&= -\sqrt{3}\int_{5\pi/6}^{\pi} (\sec^2\theta - 1)\,d\theta \\
&= -\sqrt{3}\Big[\tan\theta - \theta\Big]_{5\pi/6}^{\pi} \\
&= -\sqrt{3}\left[(0-\pi) - \left(-\frac{1}{\sqrt{3}} - \frac{5\pi}{6}\right)\right] \\
&= -1 + \frac{\sqrt{3}}{6}\pi.
\end{aligned}$$

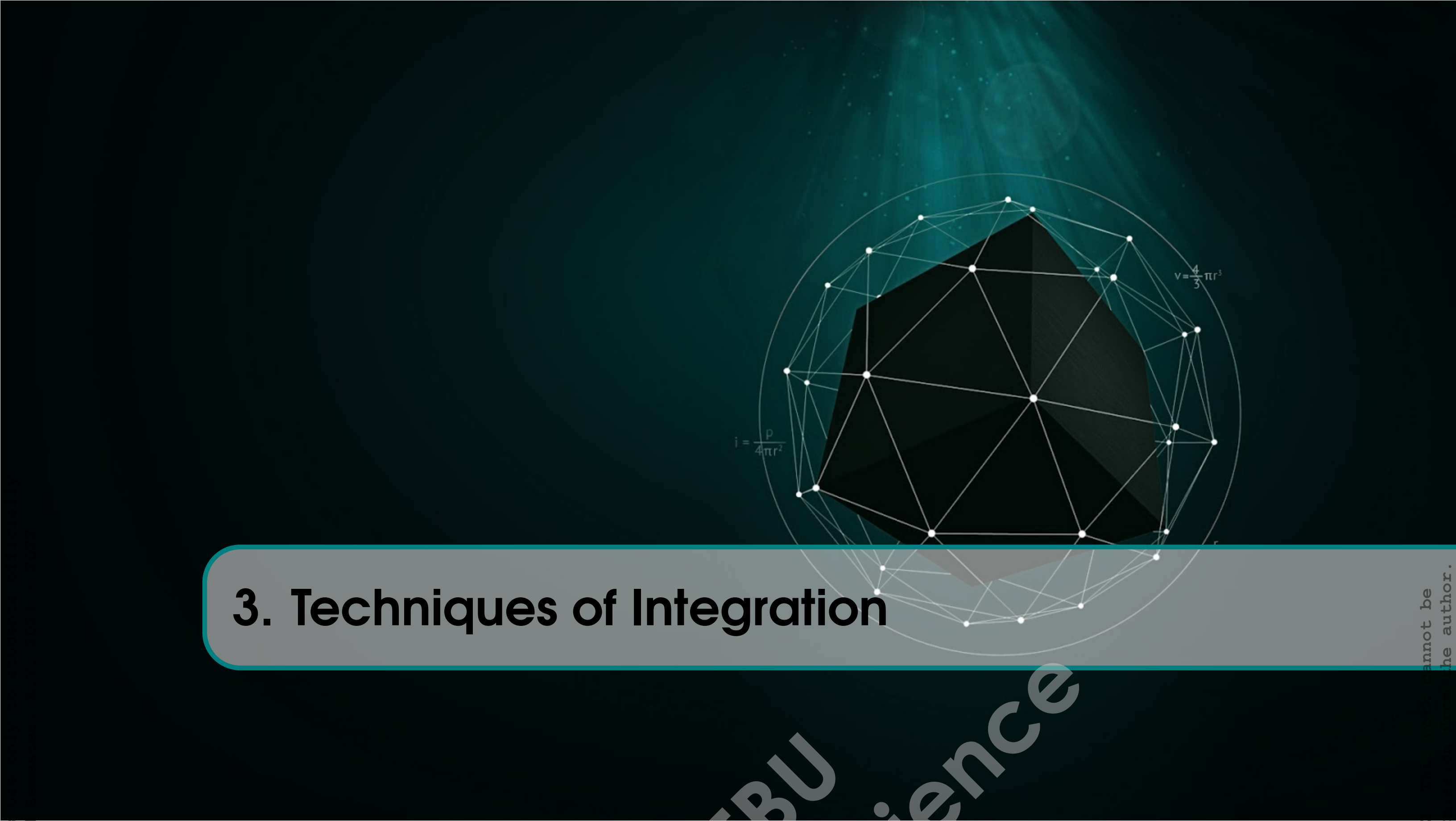
2.2 Exercises

Exercise 2.1 Evaluate the following integrals:

1. $\displaystyle\int \frac{1}{(16-x^2)^{3/2}}\,dx$
2. $\displaystyle\int \frac{4}{x^2\sqrt{16-x^2}}\,dx$
3. $\displaystyle\int \frac{x^3}{x^2-25}\,dx$
4. $\displaystyle\int \frac{\sqrt{16-x^2}}{x}\,dx$
5. $\displaystyle\int_4^6 \frac{x^2}{\sqrt{x^2-9}}\,dx$

III

Week Three



3. Techniques of Integration

At the end of this topic you are expected to:

- decompose rational functions as sums of partial fractions
- integrate rational functions using partial fractions

3.1 Integration of Rational Functions by Partial Fractions

In this section we show how to integrate any rational function (a ratio of polynomials) by expressing it as a sum of simpler fractions, called *partial fractions*, that we already know how to integrate. To illustrate the method, observe that by taking the fractions $2/(x-1)$ and $1/(x+2)$ to a common denominator we obtain

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2}$$

If we now reverse the procedure, we see how to integrate the function on the right side of this equation:

$$\int \frac{x+5}{x^2+x-2}dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right) dx$$
$$= 2\ln|x-1| - \ln|x+2| + C$$

To see how the method of partial fractions works in general, let's consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. It's possible to express f as a sum of simpler fractions provided that the degree of P is less than the degree of Q. Such a rational function is called *proper*. Recall that if

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$, then the degree of P is n and we write $\deg(P) = n$.

If f is *improper*, that is, $\deg(P) \geq \deg(Q)$, then we must take the preliminary step of dividing Q into P (by long division) until a remainder $R(x)$ is obtained such that $\deg(R) < \deg(Q)$. The division statement is

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where S and R are also polynomials.

Example 3.1 Find $\int \frac{x^3+x}{x-1}dx$.

Solution: Since the degree of the numerator is greater than the degree of the denominator, we first perform the long division. This enables us to write

$$\int \frac{x^3+x}{x-1}dx = \int \left(x^2+x+2+\frac{2}{x-1}\right) dx$$
$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C.$$

In the case of a rational equations whose denominator is more complicated, the next step is to factor the denominator $Q(x)$ as far as possible. Then express the proper rational function $R(x)/Q(x)$ as a sum of **partial fractions** of the form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^j}$$

A theorem in algebra guarantees that it is always possible to do this. We explain the details for the four cases that occur.

3.1.1 Case I: The denominator $Q(x)$ is a product of distinct linear factors.

This means that we can write

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$$

where no factor is repeated (and no factor is a constant multiple of another). In this case the partial fraction theorem states that there exist constants $A_1, A_2, \ldots, A_k$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}$$

■ **Example 3.2** Evaluate $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx$.

Solution: Since the degree of the numerator is less than the degree of the denominator, we don't need to divide. We factor the denominator as

$$2x^3+3x^2-2x = x(2x^2+3x-2) = x(2x-1)(x+2)$$

Since the denominator has three distinct linear factors, the partial fraction decomposition of the integrand has the form

$$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

To determine the values of A, B, and C, we multiply both sides of this equation by the product of the denominators, $x(2x-1)(x+2)$, obtaining

$$x^2+2x-1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

Expanding the right side and writing it in the standard form for polynomials, we get

$$x^2+2x-1 = (2A+B+2C)x^2 + (3A+2B-C)x - 2A$$

The polynomials in the previous equation are identical, so their coefficients must be equal. The coefficient of x^2 on the right side, $2A+B+2C$, must equal the coefficient of x^2 on the left side-namely, 1. Likewise, the coefficients of x are equal and the constant terms are equal. This gives the following system of equations for A, B, and C:

$$2A+B+2C=1$$
$$3A+2B-C=2$$
$$-2A=-1$$

Solving, we get $A=\frac{1}{2}$, $B=\frac{1}{5}$, and $C=-\frac{1}{10}$, and so

$$\begin{aligned}
\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx &= \int \left(\frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}\right)dx \\
&= \int \left(\frac{1}{2}\frac{1}{x}+\frac{1}{5}\frac{1}{2x-1}-\frac{1}{10}\frac{1}{x+2}\right)dx \\
&= \frac{1}{2}\ln|x|+\frac{1}{10}\ln|2x-1|-\frac{1}{10}\ln|x+2|+K
\end{aligned}$$

In integrating the middle term we have made the mental substitution $u=2x-1$, which gives $du=2dx$ and $dx=\dfrac{1}{2}du$.

Remark 3.1.1 We can use an alternative method to find the coefficients A, B, and C in the previous example. Let's choose values of x that simplify the equation

$$x^2+2x-1=A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1).$$

If we put $x=0$ in the equation, then the second and third terms on the right side vanish and the equation then becomes $-2A=-1$, or $A=\frac{1}{2}$. Likewise, $x=\dfrac{1}{2}$ gives $5B/4=\dfrac{1}{4}$ and $x=-2$ gives $10C=-1$, so $B=\dfrac{1}{5}$ and $C=-\dfrac{1}{10}$.

■ **Example 3.3** Find $\displaystyle\int \frac{dx}{x^2-a^2}$, where $a\neq 0$.

Solution: The method of partial fractions gives

$$\frac{1}{x^2-a^2}=\frac{1}{(x-a)(x+a)}=\frac{A}{x-a}+\frac{B}{x+a}$$

and therefore

$$A(x+a)+B(x-a)=1$$

Using the method of the preceding note, we put $x=a$ in this equation and get $A(2a)=1$, so $A=1/(2a)$. If we put $x=-a$, we get $B(-2a)=1$, so $B=-1/(2a)$. Thus,

$$\begin{aligned}
\int \frac{dx}{x^2-a^2} &= \frac{1}{2a}\int\left(\frac{1}{x-a}-\frac{1}{x+a}\right)dx \\
&= \frac{1}{2a}(\ln|x-a|-\ln|x+a|)+C
\end{aligned}$$

Since $\ln x - \ln y = \ln(x/y)$, we can write the integral as

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

3.1.2 Case II: $Q(x)$ is a product of linear factors, some of which are repeated.

Suppose the first linear factor (a_1x+b_1) is repeated r times; that is, $(a_1x+b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_i/(a_1x+b_1)$, we would use

$$\frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \cdots + \frac{A_r}{(a_1x+b_1)^r}$$

By way of illustration, we could write

$$\frac{x^3-x+1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

■ **Example 3.4** Find $\displaystyle\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx.$

Solution: The first step is to divide. The result of long division is

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}$$

The second step is to factor the denominator $Q(x) = x^3-x^2-x+1$. Since $Q(1) = 0$, we know that $x-1$ is a factor and we obtain

$$x^3-x^2-x+1 = (x-1)(x^2-1) = (x-1)(x-1)(x+1) = (x-1)^2(x+1)$$

Since the linear factor $x-1$ occurs twice, the partial fraction decomposition is

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

Multiplying by the least common denominator, $(x-1)^2(x+1)$, we get

$$\begin{aligned} 4x &= A(x-1)(x+1) + B(x+1) + C(x-1)^2 \\ &= (A+C)x^2 + (B-2C)x + (-A+B+C) \end{aligned}$$

Now we equate coefficients:

$$\begin{aligned} A+C &= 0 \\ B-2C &= 4 \\ -A+B+C &= 0 \end{aligned}$$

Solving, we obtain $A = 1$, $B = 2$, and $C = -1$, so

$$\begin{aligned}\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx &= \int \left[x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}\right] dx \\ &= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + K \\ &= \frac{x^2}{2} + x - \frac{2}{x-1} + \ln\left|\frac{x-1}{x+1}\right| + K\end{aligned}$$

3.1.3 Case III: $Q(x)$ contains irreducible quadratic factors, none of which is repeated

If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, the expression for $R(x)/Q(x)$ will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

where A and B are constants to be determined. For instance, the function given by $f(x) = x/[(x-2)(x^2+1)(x^2+4)]$ has a partial fraction decomposition of the form

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{x^2+4}$$

Example 3.5 Evaluate $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx.$

Solution: Since $x^3 + 4x = x(x^2+4)$ can't be factored further, we write

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2+4}$$

Multiplying by $x(x^2+4)$, we have

$$2x^2 - x + 4 = A(x^2+4) + (Bx + C)x = (A+B)x^2 + Cx + 4A$$

Equating coefficients, we obtain $A + B = 2$, $C = -1$, and $4A = 4$. Therefore, $A = 1$, $B = 1$, and $C = -1$ and so,

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx = \int \left(\frac{1}{x} + \frac{x-1}{x^2+4}\right) dx$$

In order to integrate the second term we split it into two parts:

$$\int \frac{x-1}{x^2+4}\,dx = \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx.$$

We make the substitution $u = x^2 + 4$ in the first of these integrals so that $du = 2xdx$. We evaluate the second integral by means of the previous result, $\int \frac{dx}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$. Therefore,

$$\begin{aligned}\int \frac{2x^2-x+4}{x^3+4x}dx &= \int \frac{1}{x}dx + \int \frac{x}{x^2+4}dx - \int \frac{1}{x^2+4}dx \\ &= \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}(x/2) + K\end{aligned}$$

3.1.4 Case IV: $Q(x)$ contains a repeated irreducible quadratic factor

If $Q(x)$ has the factor $(ax^2+bx+c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction, the sum

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$. Each of the terms can be integrated by using a substitution or by first completing the square if necessary.

■ **Example 3.6** Write out the form of the partial fraction decomposition of the function

$$\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3}.$$

Solution:

$$\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}$$

■ **Example 3.7** Evaluate $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}dx$.

Solution: The form of the partial fraction decomposition is

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiplying by $x(x^2+1)^2$, we have

$$\begin{aligned}-x^3+2x^2-x+1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\ &= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex \\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A\end{aligned}$$

If we equate coefficients, we get the system

$$A+B=0, \quad C=-1, \quad 2A+B+D=2, \quad C+E=-1, \quad A=1$$

which has the solution $A=1$, $B=-1$, $C=-1$, $D=1$, and $E=0$. Thus,

$$\begin{aligned}
\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}dx &= \int \left(\frac{1}{x}+\frac{x+1}{x^2+1}+\frac{x}{(x^2+1)^2}\right)dx \\
&= \int \frac{dx}{x} - \int \frac{x}{x^2+1}dx - \int \frac{dx}{x^2+1} + \int \frac{xdx}{(x^2+1)^2}dx \\
&= \ln|x| - \frac{1}{2}\ln(x^2+1) - \tan^{-1}x - \frac{1}{2(x^2+1)} + K.
\end{aligned}$$

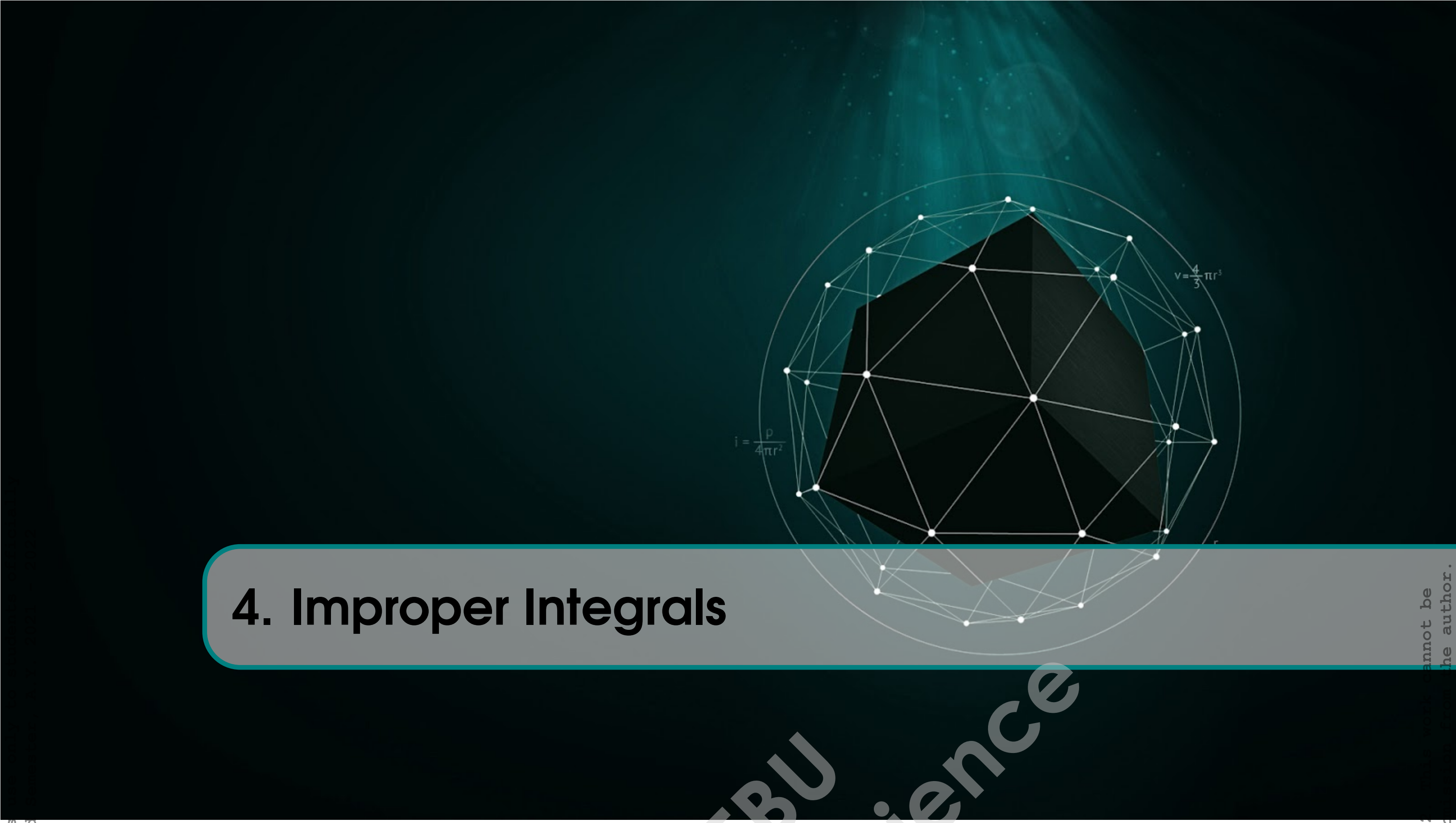
3.2 Exercises

Exercise 3.1 Find the following integrals.

1. $\displaystyle\int \frac{x^4}{x-1}dx$
2. $\displaystyle\int \frac{2}{2x^2+3x+1}dx$
3. $\displaystyle\int \frac{x^2+x+1}{(x+1)^2(x+2)}dx$
4. $\displaystyle\int \frac{y}{(y+4)(2y-1)}dy$
5. $\displaystyle\int \frac{1}{x^3-1}dx$
6. $\displaystyle\int \frac{x^2-3x+7}{(x^2-4x+6)^2}dx$
7. $\displaystyle\int \frac{x^4+9x^2+x+2}{x^2+9}dx$
8. $\displaystyle\int \frac{x^3-2x^2+2x-5}{x^4+4x^2+3}dx$

IV

Week Four



4. Improper Integrals

At the end of this topic you are expected to:

- evaluate an improper integral that has an infinite limit of integration
- evaluate an improper integral that has an infinite discontinuity

In defining the definite integral

$$\int_a^b f(x)\,dx$$

it was assumed that the limits of integration were finite and that the integrand was continuous for every x in the closed interval $[a,b]$. If either of these conditions are not satisfied, then the integral is called an **improper integral**.

Recall also that a function f is said to have an **infinite discontinuity** if

$$\lim_{x\to c} f(x) = +\infty \quad \text{or} \quad \lim_{x\to c} f(x) = -\infty$$

4.1 Improper Integrals with Infinite Limits of Integration

Definition 4.1.1 (Improper Integrals with Infinite Integration Limits)

1. If f is continuous on the interval $[a,+\infty)$, then
$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to+\infty}\int_a^b f(x)\,dx.$$
2. If f is continuous on the interval $(-\infty,b]$, then
$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$
3. if f is continuous on the interval $(-\infty,+\infty)$, then
$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{+\infty} f(x)\,dx$$
where c is any real number.

In the first two cases, the improper integral **converges** when the limit exists–otherwise, the improper integral **diverges**. In the third case, the improper integral on the left diverges when either of the improper integrals on the right diverges.

■ **Example 4.1** Evaluate $\displaystyle\int_1^{+\infty}\frac{dx}{x}$.

Solution:

$$\begin{aligned}\int_1^{+\infty}\frac{dx}{x} &= \lim_{b\to+\infty}\int_1^b \frac{dx}{x}\\ &= \lim_{b\to+\infty}\Big[\ln x\Big]_1^b\\ &= \lim_{b\to+\infty}(\ln b - 0)\\ &= +\infty.\end{aligned}$$

Therefore, $\displaystyle\int_1^{+\infty}\frac{dx}{x}$ diverges.

■ **Example 4.2** Evaluate the foloowing integrals (a) $\displaystyle\int_0^{\infty} e^{-x}\,dx$ (b) $\displaystyle\int_0^{+\infty}\frac{1}{x^2+1}\,dx$

Solution: (a)

$$\begin{aligned}\int_0^{+\infty} e^{-x}\,dx &= \lim_{b\to+\infty}\int_0^b e^{-x}\,dx\\ &= \lim_{b\to+\infty}\Big[-e^{-x}\Big]_0^b\\ &= \lim_{b\to+\infty}(-e^{-b}+1)\\ &= 1.\end{aligned}$$

Therefore, $\int_0^{\infty} e^{-x}\,dx$ is convergent.

(b)

$$\begin{aligned}\int_0^{+\infty} \frac{1}{x^2+1}\,dx &= \lim_{b\to+\infty}\int_0^b \frac{1}{x^2+1}\,dx\\ &= \lim_{b\to+\infty}\Big[\tan^{-1}x\Big]_0^b\\ &= \lim_{b\to+\infty}\tan^{-1}b\\ &= \frac{\pi}{2}.\end{aligned}$$

Therefore, $\int_0^{\infty} \frac{1}{x^2+1}\,dx$ is convergent.

■ **Example 4.3** Evaluate $\int_1^{+\infty}(1-x)e^{-x}\,dx$

Solution: Here, we apply first integration by partsof the indefinite integral $\int(1-x)e^{-x}\,dx$. Let $u=(1-x)$ and $dv=e^{-x}dx$. Then $du=-dx$ and $v=-e^{-x}$. Thus,

$$\begin{aligned}\int(1-x)e^{-x}\,dx &= -e^{-x}(1-x)-\int e^{-x}\,dx\\ &= -e^{-x}+xe^{-x}+e^{-x}+C\\ &= xe^{-x}+C\end{aligned}$$

Now, applying the definition of an improper integral, we have

$$\begin{aligned}\int_1^{+\infty}(1-x)e^{-x}\,dx &= \lim_{b\to+\infty}\left[xe^{-x}\right]_1^b\\ &= \lim_{b\to+\infty}\left(\frac{b}{e^b}-\frac{1}{e}\right)\\ &= \lim_{b\to+\infty}\frac{b}{e^b}-\frac{1}{e}.\end{aligned}$$

Since $\frac{b}{e^b}$ is of the form $\frac{\infty}{\infty}$, we apply L'Hôpital's Rule.

$$\lim_{b\to+\infty}\frac{b}{e^b} = \lim_{b\to+\infty}\frac{1}{e^b} = 0$$

Hence,

$$\begin{aligned}\int_1^{+\infty}(1-x)e^{-x}\,dx &= \lim_{b\to+\infty}\frac{b}{e^b}-\frac{1}{e}\\ &= 0-\frac{1}{e}\\ &= -\frac{1}{e}\end{aligned}$$

Therefore, $\int_1^{+\infty}(1-x)e^{-x}\,dx$ is convergent.

■ **Example 4.4** Evaluate $\displaystyle\int_{-\infty}^{+\infty}\frac{e^x}{1+e^{2x}}\,dx.$

Solution: Note that the given integrand is continuous for all real number x. To evaluate the integral , we can break it into two parts. We chose $c=0$ since it is a convenient value.

We have to solve first for the idefinite integral $\displaystyle\int\frac{e^x}{1+e^{2x}}\,dx$. Let $u=e^x$. Then $du=e^x\,dx$. So,

$$\begin{aligned}\int\frac{e^x}{1+(e^x)^2}\,dx &= \int\frac{du}{1+u^2}\\ &= \tan^{-1}u+C\\ &= \tan^{-1}e^x+C.\end{aligned}$$

Now, we will evaluate the improper integral according to the definition.

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{e^x}{1+e^{2x}}\,dx &= \int_{-\infty}^{0} \frac{e^x}{1+e^{2x}}\,dx + \int_{0}^{+\infty} \frac{e^x}{1+e^{2x}}\,dx \\
&= \lim_{a\to-\infty} \int_{a}^{0} \frac{e^x}{1+e^{2x}}\,dx + \lim_{b\to\infty} \int_{0}^{b} \frac{e^x}{1+e^{2x}}\,dx \\
&= \lim_{a\to-\infty} \left[\tan^{-1} e^x\right]_a^0 + \lim_{b\to+\infty} \left[\tan^{-1} e^x\right]_0^b \\
&= \lim_{a\to-\infty} \left(\frac{\pi}{4} - \tan^{-1} e^a\right) + \lim_{b\to+\infty} \left(\tan^{-1} e^b - \frac{\pi}{4}\right) \\
&= \frac{\pi}{4} - 0 + \frac{\pi}{2} - \frac{\pi}{4} \\
&= \frac{\pi}{2}.
\end{aligned}$$

Therefore, $\int_{-\infty}^{+\infty} \frac{e^x}{1+e^{2x}}\,dx$ is convergent.

4.2 Improper Integrals with Infinite Discontinuities

Another basic type of improper intergal is one that has infinite discontinuity at or between the limits of integration.

Definition 4.2.1 (Improper Integrals with Infinite Discontinuities)

1. If f is continuous on the interval $[a,b)$ and has an infinite discontinuity at b, then
$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$
2. If f is continuous on the interval $(a,b]$ and has an infinite discontinuity at a, then
$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$
3. if f is continuous on the interval $[a,b]$, except for some c in (a,b), then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In the first two cases, the improper integral **converges** when the limit exists–otherwise, the improper integral **diverges**. In the third case, the improper integral on the left diverges when either of the improper integrals on the right diverges.

■ **Example 4.5** Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}}$.

Solution: The integrand has an infinite discontinuity at $x = 0$. Thus,

$$\begin{aligned}
\int_0^1 \frac{dx}{\sqrt[3]{x}} &= \lim_{b\to 0^+} \int_b^1 \frac{dx}{\sqrt[3]{x}} \\
&= \lim_{b\to 0^+} \left[\frac{3}{2}x^{2/3}\right]_b^1 \\
&= \lim_{b\to 0^+} \frac{3}{2}\left(1 - b^{2/3}\right) \\
&= \frac{3}{2}.
\end{aligned}$$

Therefore, $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}}$ is convergent.

■ **Example 4.6** Evaluate $\displaystyle\int_0^2 \frac{dx}{x^3}$.

Solution: The integrand has an infinite discontinuity at $x = 0$. Thus,

$$\begin{aligned}
\int_0^2 \frac{dx}{x^3} &= \lim_{b\to 0^+} \int_b^2 \frac{dx}{x^3} \\
&= \lim_{b\to 0^+} \left[-\frac{1}{2x^2}\right]_b^2 \\
&= \lim_{b\to 0^+} \left(-\frac{1}{8} + \frac{1}{2b^2}\right) \\
&= +\infty.
\end{aligned}$$

Therefore, $\displaystyle\int_0^2 \frac{dx}{x^3}$ is divergent.

■ **Example 4.7** Evaluate $\displaystyle\int_{-1}^2 \frac{dx}{x^3}$.

Solution: The integral is improper because the integrand has an infinite discontinuity at the interior point $x = 0$. Now, we can write the given integral as

$$\int_{-1}^2 \frac{dx}{x^3} = \int_{-1}^0 \frac{dx}{x^3} + \int_0^2 \frac{dx}{x^3}.$$

Notice that the second integral on the right is divergent (from Example 4.6). Therefore, $\displaystyle\int_{-1}^2 \frac{dx}{x^3}$ is also divergent.

There might be instances when you fail recognize that a particular integral is improper just like in Example 4.7. If you are not careful, you will have obtained a result as illusrated below

$$\int_{-1}^{2} \frac{dx}{x^3} = \left[-\frac{1}{2x^2}\right]_{-1}^{2} = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8} \quad \text{which is not correct.}$$

There are also integrals which are doubly improper. The next example has this property – one limit is infinite and the integrand has an infinite discontinuity at the other limit of integration.

■ **Example 4.8** Evaluate $\displaystyle\int_0^{+\infty} \frac{dx}{\sqrt{x}(x+1)}$.

Solution: To evaluate this integral, we select a point where we split the given integral. Suppose we split the integral at $x = 1$. Then

$$\begin{aligned}
\int_0^{+\infty} \frac{dx}{\sqrt{x}(x+1)} &= \int_0^{1} \frac{dx}{\sqrt{x}(x+1)} + \int_1^{+\infty} \frac{dx}{\sqrt{x}(x+1)} \\
&= \lim_{b\to 0^+} \int_b^{1} \frac{dx}{\sqrt{x}(x+1)} + \lim_{c\to +\infty} \int_1^{c} \frac{dx}{\sqrt{x}(x+1)} \\
&= \lim_{b\to 0^+} \left[2\tan^{-1}\sqrt{x}\right]_b^1 + \lim_{c\to +\infty} \left[2\tan^{-1}\sqrt{x}\right]_1^c \\
&= \lim_{b\to 0^+} \left(2\tan^{-1}(1) - 2\tan^{-1}\sqrt{b}\right) + \lim_{c\to +\infty} \left(2\tan^{-1}\sqrt{c} - 2\tan^{-1}(1)\right) \\
&= 2\left(\frac{\pi}{4}\right) - 0 + 2\left(\frac{\pi}{2}\right) - 2\left(\frac{\pi}{4}\right) \\
&= \pi.
\end{aligned}$$

Therefore, $\displaystyle\int_0^{+\infty} \frac{dx}{\sqrt{x}(x+1)}$ is convergent.

4.3 Exercises

Exercise 4.1 Determine whether the given improper integral is convergent or divergent.

1. $\displaystyle\int_{-\infty}^{1} e^z \, dz$
2. $\displaystyle\int_5^{+\infty} \frac{dy}{\sqrt{y-1}}$
3. $\displaystyle\int_0^{1} \frac{dx}{\sqrt{1-x}}$
4. $\displaystyle\int_2^{+\infty} \frac{dx}{x\sqrt{x^2-4}}$

5. $\displaystyle\int_0^4 \frac{dx}{x^2 - 2x - 3}$
6. $\displaystyle\int_1^{13/5} \frac{dy}{\sqrt{y^2 - 1}}$

V

Week Five

5. Infinite Sequence and Series

At the end of this topic you are expected to:

- evaluate the limit of a sequence
- evaluate the sum of some infinite series

5.1 Sequences

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

The number a_1 is called the *first term*, a_2 is the *second term*, and in general a_n is the *nth term*. We will deal exclusively with infinite sequences and so each term a_n will have a successor a_{n+1}.

Notice that for every positive integer n there is a corresponding number a_n and so a sequence can be defined as afuction whose domain is the set of positive integers. But we usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n.

NOTATION The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by

$$a_n \qquad \text{or} \qquad \{a_n\}_{n=1}^{\infty}$$

Example 5.1 Some sequences can be defined by giving a formula for the nth term. In the following eamples we give three descriptions of the sequence: one by using the preceeding notation, another by using the defining formula, and a third by writing out the terms of sequences. Notice that n doesn't have to start at 1.

(a) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$ $\quad a_n = \dfrac{n}{n+1}$ $\quad \left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots, \dfrac{n}{n+1}, \ldots\right\}$

(b) $\left\{\dfrac{(-1)^n(n+1)}{3^n}\right\}_{n=1}^{\infty}$ $\quad a_n = \dfrac{(-1)^n(n+1)}{3^n}$ $\quad \left\{-\dfrac{2}{3}, \dfrac{3}{9}, \dfrac{-4}{27}, \dfrac{5}{81}, \ldots, \dfrac{(-1)^n(n+1)}{3^n}, \ldots\right\}$

(c) $\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$ $\quad a_n = \sqrt{n-3},\ n \geq 3$ $\quad \left\{0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots\right\}$

(d) $\left\{\cos\dfrac{n\pi}{6}\right\}_{n=0}^{\infty}$ $\quad a_n = \cos\dfrac{n\pi}{6},\ n \geq 0$ $\quad \left\{1, \dfrac{\sqrt{3}}{2}, \dfrac{1}{2}, 0, \ldots, \cos\dfrac{n\pi}{6}, \ldots\right\}$

Example 5.2 Find a formula for the general term a_n of the sequence

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \ldots\right\}$$

assuming that the pattern of the first few terms continues.

Solution: We are given that

$$a_1 = \frac{3}{5} \qquad a_2 = -\frac{4}{25} \qquad a_3 = \frac{5}{125} \qquad a_4 = -\frac{6}{625} \qquad a_5 = \frac{7}{3125}$$

Notice that the numerators of these fractions start with 3 and increase by 1 whenever we go to the next term. The second term has numerator 4, the third term has numerator 5; in general, the nth term will have numerator $n+2$. The denominators are the powers of 5, so a_n has denominator 5^n. The signs of the terms are alternately positive and negative, so we need to multiply by a power of -1. Here we want to start with a positive term and so we use $(-1)^{n-1}$ or $(-1)^{n+1}$. Therefore,

$$a_n = (-1)^{n-1}\frac{n+2}{5^n}.$$

Example 5.3 Here are some sequences that don't have a simple defining equation.

(a) The sequence $\{p_n\}$ where p_n is the population of the world as of January 1 in the year n.

(b) If we let a_n be the digit in the nth decimal plate of the number e, then $\{a_n\}$ is a well-defined sequence whose first few terms are

$$\{7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, \dots\}$$

(c) **The Fibonacci Sequence** $\{f_n\}$ is defined recursively by the conditions

$$f_1 = 1 \qquad f_2 = 1 \qquad f_n = f_{n-1} + f_{n-2} \qquad n \geq 3$$

Each term is the sum of the two preceding terms. The first few terms are

$$\{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$$

This sequence arose when the 13th-century Italian mathematician known as Fibonacci solved a problem concerning the breeding of rabbits.

A sequence such as the one in Example 5.1(a), $a_n = \dfrac{n}{n+1}$, can be pictured either by plotting its terms on a number line, or by plotting its graph. Note that, since a sequence is a function whose domain is the set of positive integers, its graph consists of isolated points with coordinates

$$\boxed{(1, a_1) \qquad (2, a_2) \qquad (3, a_3) \qquad \dots \qquad (n, a_n) \qquad \dots}$$

It appears that the terms of the sequence $a_n = \dfrac{n}{n+1}$ are approaching 1 as n becomes large. In fact, the difference

$$\boxed{1 - \frac{n}{n+1} = \frac{1}{n+1}}$$

can be made small as we like by taking n sufficiently large. We indicate this by writing

$$\boxed{\lim_{n \to \infty} \frac{n}{n+1} = 1}$$

In general, the notation

$$\lim_{n \to \infty} a_n = L$$

means that means that the terms of the sequence $\{a_n\}$ approach L as n becomes large. Notice that the following definition of the limit of a sequence is very similar to the definition of a limit of a function at infinity on previous discussion.

Definition 5.1.1 A sequence $\{a_n\}$ has the **limit** L and we write

$$\boxed{\lim_{n\to\infty} a_n = L} \qquad \text{or} \qquad a_n \to L \quad \text{as} \quad n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

A more precise version of the above definition is as follows.

Definition 5.1.2 A sequence $\{a_n\}$ has the **limit** L and we write

$$\boxed{\lim_{n\to\infty} a_n = L} \qquad \text{or} \qquad a_n \to L \quad \text{as} \quad n \to \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad |a_n - L| < \varepsilon$$

Note that the only difference between $\lim_{n\to\infty} a_n = L$ and $\lim_{x\to\infty} f(x) = L$ is that n is required to be an integer. Thus we have the following theorem.

Theorem 5.1.1 If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

In particular, since we know that $\lim_{x\to\infty}\left(\frac{1}{x^r}\right) = 0$ when $r > 0$ we have

$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \qquad \text{if } r > 0.$$

Remark 5.1.1 If a_n becomes large as n becomes large, we use the notation $\lim_{n\to\infty} a_n = \infty$. The following precise definition is similar to the definition of limits at infinity.

Definition 5.1.3 $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad a_n > M$$

If $\lim_{n\to\infty} a_n = \infty$, then the sequence $\{a_n\}$ is divergent but in a special way. We say that $\{a_n\}$ diverges to ∞.

The Limit Laws discussed in Math 53 also hold for the limits of sequences and theirproofs are similar.

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is constant, then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$
$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$
$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

The squeeze theorem can also be adapted for sequences as follows.

Theorem 5.1.2 If $a_n \le b_n \le c_n$ for $n \ge 0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Another useful fact about limits of sequences is given by the following theorem, whose proof is left as exercise.

Theorem 5.1.3 If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

■ **Example 5.4** Find $\lim_{n\to\infty} \dfrac{n}{n+1}$.

Solution: The method is similar to the one we used previously: Divide numerator and denominator by the highest power of n that occurs in the denominator and then use the limit laws.

$$\lim_{n\to\infty}\frac{n}{n+1}=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}=\frac{\lim\limits_{n\to\infty}1}{\lim\limits_{n\to\infty}1+\lim\limits_{n\to\infty}\frac{1}{n}}$$
$$=\frac{1}{1+0}=1.$$

■ **Example 5.5** Is the sequence $a_n=\dfrac{n}{\sqrt{10+n}}$ convergent or divergent?

Solution: We divide numerator and denominator by n:

$$\lim_{n\to\infty}\frac{n}{\sqrt{10+n}}=\lim_{n\to\infty}\frac{1}{\sqrt{\frac{10}{n^2}+\frac{1}{n}}}=\infty$$

because the numerator is constant and the denominator approaches 0. So $\{a_n\}$ is divergent.

■ **Example 5.6** Calculate $\lim\limits_{n\to\infty}\dfrac{\ln n}{n}$.

Solution: Notice that both numerator and denominator approach infinity as $n\to\infty$. We can't apply L'Hopital's Rule directly because it applies not to sequences but to functions of a real variable. However, we can apply L'Hopital's Rule to the related function $f(x)=\dfrac{\ln x}{x}$ and obtain

$$\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=0$$

Therefore, by Theorem 5.1.1, we have

$$\lim_{n\to\infty}\frac{\ln n}{n}=0.$$

■ **Example 5.7** Determine whether the sequence $a_n=(-1)^n$ is convergent or divergent.

Solution: If we write out the terms of the sequence, we obtain

$$\{-1,1,-1,1,-1,1,-1,\ldots\}$$

The graph of this sequence is shown in Figure 8. Since the terms oscillate between 1 and -1 infinitely often, an does not approach any number. Thus $\lim\limits_{n\to\infty}(-1)^n$ does not exist; that is, the sequence $\{(-1)^n\}$ is divergent.

■ **Example 5.8** Evaluate $\lim\limits_{n\to\infty}\dfrac{(-1)^n}{n}$ if it exists.

Solution: We first calculate the limit of the absolute value:

$$\lim_{n\to\infty}\left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty}\frac{1}{n} = 0.$$

Therefore, by Theorem 5.1.3,

$$\lim_{n\to\infty}\frac{(-1)^n}{n} = 0.$$

The following theorem says that if we apply a continuous function to the terms of a convergent sequence, the result is also convergent.

Theorem 5.1.4 If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L).$$

■ **Example 5.9** Find $\lim_{n\to\infty}\sin\frac{\pi}{n}$.

Solution: Because the sine function is continuous at 0, Theorem 5.1.4 enables us to write

$$\lim_{n\to\infty}\sin\frac{\pi}{n} = \sin\left(\lim_{n\to\infty}\frac{\pi}{n}\right) = \sin 0 = 0.$$

■ **Example 5.10** Discuss the convergence of the sequence $a_n = \frac{n!}{n^n}$.

Solution: Both numerator and denominator approach infinity as $n\to\infty$ but here we have no corresponding function for use with L'Hopital's Rule ($x!$ is not defined when x is not an integer). Let's write out a few terms to get a feeling for what happens to an as n gets large:

$$a_1 = 1 \qquad a_2 = \frac{1\cdot 2}{2\cdot 2} \qquad a_3 = \frac{1\cdot 2\cdot 3}{3\cdot 3\cdot 3}$$

$$a_n = \frac{1\cdot 2\cdot 3\cdot\cdots\cdot n}{n\cdot n\cdot n\cdot\cdots\cdot n} \tag{*}$$

It appears from these expressions that the terms are decreasing and perhaps approach 0. To confirm this, observe from Equation * that

$$a_n = \frac{1}{n}\left(\frac{2\cdot 3\cdot\cdots\cdot n}{n\cdot n\cdot\cdots\cdot n}\right)$$

Notice that the expression in parentheses is at most 1 because the numerator is less than (or equal to) the denominator. So

$$\boxed{0 < a_n \le \frac{1}{n}}$$

We know that $\frac{1}{n} \to 0$ as $n \to \infty$. Therefore $a_n \to 0$ as $n \to \infty$ by the Squeeze Theorem.

The next theorem will give us an answer as to what values of r will the sequence $\{r^n\}$ be convergent and its proof will be left as an exercise.

Theorem 5.1.5 The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r. Moreover,

$$\boxed{\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}}$$

Definition 5.1.4 A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \ldots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is **monotonic** if it is either increasing or decreasing.

Example 5.11 The sequence $\left\{\frac{3}{n+5}\right\}$ is decreasing because

$$\frac{3}{n+5} > \frac{3}{(n+1)+5} = \frac{3}{n+6}$$

and so $a_n > a_{n+1}$ for all $n \geq 1$.

Example 5.12 Show that the sequence $a_n = \frac{n}{n^2+1}$ is decreasing.

Solution: We must show that $a_n > a_{n+1}$, that is

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$$

This inequality is equivalent to the one we get by cross-multiplication:

$$\begin{aligned} \frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} &\Leftrightarrow (n+1)(n^2+1) < n[(n+1)^2+1] \\ &\Leftrightarrow n^3+n^2+n+1 < n^3+2n^2+2n \\ &\Leftrightarrow 1 < n^2+n \end{aligned}$$

Since $n \geq 1$, we know that $1 < n^2+n$ is true. Therefore $a_n > a_{n+1}$ and so $\{a_n\}$ is decreasing.

Solution: (Alternative Solution) Consider the function $f(x) = \dfrac{x}{x^2+1}$:

$$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0 \qquad \text{whenever } x^2 > 1$$

Thus, f is decreasing on $(1,\infty)$ and so $f(n) > f(n+1)$. Therefore, $\{a_n\}$ is decreasing.

Definition 5.1.5 A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M \qquad \text{for all } n \geq 1$$

A sequence $\{a_n\}$ is **bounded below** if there is a number m such that

$$m \leq a_n \qquad \text{for all } n \geq 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

For instance, the sequence $a_n = n$ is bounded below $(a_n > 0)$ but not above. The sequence $a_n = \dfrac{n}{n+1}$ is bounded because $0 < a_n < 1$ for all n.

Theorem 5.1.6 **Monotonic Sequence Theorem**

Every bounded, monotonic sequence is convergent.

We will not show the proof of this theorem but the its proof of shows that a sequence that is increasing and bounded above is convergent. (Likewise, a decreasing sequence that is bounded below is convergent.) This fact is used many times in dealing with infinite series.

5.2 Series

What do we mean when we express a number as an infinite decimal? For instance, what does it mean to write

$$\pi = 3.14159265358979323846264338327950288\ldots$$

The convention behind our decimal notation is that any number can be written as an infinite sum. Here it means that

$$\pi = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \ldots$$

where the three dots $(\ldots)$ indicate that the sum continues forever, and the more terms we add, the closer we get to the actual value of π.

In general, if we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots \tag{5.1}$$

which is called an **infinite series** (or just a series) and is denoted, for short, by the symbol

$$\sum_{n=1}^{\infty} a_n \qquad \text{or} \qquad \sum a_n$$

Does it make sense to talk about the sum of infinitely many terms?

We consider the **partial sums**

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$s_4 = a_1 + a_2 + a_3 + a_4$$

and, in general,

$$s_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i=1}^{n} a_i$$

These partial sums form a new sequence $\{s_n\}$, which may or may not have a limit. If $\lim_{n\to\infty} s_n = s$ exists (as a finite number), then, as in the preceeding example, we call it the sum of the infinite series $\sum a_n$.

Definition 5.2.1 Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots$, let s_n denote its nth partial sum:

$$\boxed{s_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i=1}^{n} a_i}$$

If the sequence $\{s_n\}$ is convergent and $\sum_{n=1}^{\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

Thus the sum of a series is the limit of the sequence of partial sums. So when we write $\sum_{n=1}^{\infty} a_n = s$, we mean that by adding sufficiently many terms of the series we can get as close as we like to the number s. Notice that

$$\boxed{\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i.}$$

▪ **Example 5.13** Suppose that we know that the sum of the first n terms of the series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = a_1 + a_2 + \cdots + a_n = \frac{2n}{3n+5}$$

Then the sum of the series is the limit of the sequence $\{s_n\}$:

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2n}{3n+5} = \lim_{n\to\infty} \frac{2}{3+\frac{5}{n}} = \frac{2}{3}.$$

▪ **Example 5.14** An important example of an infinite series is the **geometric series**

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

Each term is obtained from the preceeding one by multiplying it by the **commmon ratio** r.

Remark 5.2.1 The geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

is convergent if $|r| < 1$ and its sum is

$$\boxed{\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad , \qquad |r| < 1}$$

If $|r| \geq 1$, the geometric series is divergent.

▪ **Example 5.15** Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots.$$

Solution: The first term is $a = 5$ and thecommon ratio is $r = -\frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series is convergent and its sum is

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = \frac{5}{1-\left(-\frac{2}{3}\right)} = 3.$$

■ **Example 5.16** Is the series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent?

Solution: Let's rewrite the nth term of the series in the form ar^{n-1}:

$$\sum_{n=1}^{\infty} 2^{2n}3^{1-n} = \sum_{n=1}^{\infty} (2^2)^n 3^{-(n-1)} = \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}$$

We recognize this series as a geometric series with $a = 4$ and $r = \frac{4}{3}$. Since $r > 1$, the series diverges.

■ **Example 5.17**

$$1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad |x| < 1.$$

■ **Example 5.18** Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

Solution: This is not a geometric series, so we go back to the definition of a convergent series and compute the partial sums.

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)}$$

We can simplify this expression if we use the partial fraction decomposition

$$\boxed{\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}}$$

Thus we have

$$\begin{aligned} s_n &= \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

and so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1.$$

■ **Example 5.19** The **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.

Theorem 5.2.1 If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

The converse of Theorem 5.2.1 is not true in general. If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that $\sum a_n$ is convergent. Observe that for the harmonic series $\sum 1/n$ we have $a_n = 1/n \to 0$ as $n \to \infty$, but the harmonic series is divergent.

Theorem 5.2.2 **Test for Divergence**
If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

■ **Example 5.20** Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

Solution:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4} = \frac{1}{5} \neq 0$$

So the series diverges by the test for divergence.

Theorem 5.2.3 If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum(a_n+b_n)$, $\sum(a_n-b_n)$, and

(i) $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\sum_{n=1}^{\infty} (a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\sum_{n=1}^{\infty} (a_n-b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

■ **Example 5.21** Find the sum of the series $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$.

Solution: The series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$, so

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1.$$

On the other hand (by previous example),

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

So by Theorem 5.2.3, the given series is convergent and

$$\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right) = 3 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n} = 3 \cdot 1 + 1 = 4.$$

Remark 5.2.2 A finite number of terms doesn't affect the convergence or divergence of a series.

For instance, suppose that we were able to show that the series

$$\sum_{n=4}^{\infty} \frac{n}{n^3+1}$$

is convergent. Since

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1} = \frac{1}{2} + \frac{2}{9} + \frac{3}{28} + \sum_{n=4}^{\infty} \frac{n}{n^3+1}$$

it follows that the entire series $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ is convergent. Similarly, if it is known that the series $\sum_{n=N+1}^{\infty} a_n$ converges, then the full series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

is also convergent.

5.3 Exercises

Exercise 5.1 Do as directed.

1. Determine whether the given sequence is increasing, decreasing or not monotonic. Is the sequence bounded? Explain your answer.
   (a) $a_n = \dfrac{1}{2n+3}$
   (b) $a_n = \dfrac{1-n}{2+n}$
   (c) $a_n = 2 + \dfrac{(-1)^n}{n}$.
2. Explain the difference between
   (a) $\sum_{i=1}^{n} a_i$ and $\sum_{j=1}^{n} a_j$

(b) $\sum_{i=1}^{n} a_i$ and $\sum_{i=1}^{n} a_j$.

3. Determine whether the given series is convergent or divergent. If it is convergent, find its sum.

   (a) $\sum_{n=1}^{\infty} [(-0.2)^n + (0.6)^{n-1}]$

   (b) $\sum_{n=1}^{\infty} \frac{2^n + 4^n}{e^n}$

   (c) $\sum_{k=0}^{\infty} (\sqrt{2})^{-k}$.

4. Use geometric series to express the following as a ratio of two integers

   (a) $0.\overline{8}$

   (b) $10.1\overline{35}$.

VI

Week Six

6. Infinite Sequence and Series

At the end of this topic you are expected to:

- determine the convergence or divergence of a series using integral test
- determine the convergence or divergence of a series using comparison tests
- identify whether an alternating series is convergent or divergent

6.1 The Integral Test

In general, it is difficult to find the exact sum of a series. We were able to accomplish this for geometric series and the series $\sum 1/[n(n+1)]$ because in each of those cases we could find a simple formula for the nth partial sum s_n. But usually it isn't easy to discover such a formula. Therefore, in the next few sections, we develop several tests that enable us to determine whether a series is convergent or divergent without explicitly finding its sum. (In some cases, however, our methods will enable us to find good estimates of the sum.) Our first test involves improper integrals.

Theorem 6.1.1 The Integral Test

Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

1. If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
2. If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Remark 6.1.1 When we use the Integral Test, it is not necessary to start the series or the integral at $n = 1$. For instance, in testing the series

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2},$$

we use

$$\int_4^{\infty} \frac{1}{(x-3)^2}\,dx.$$

Also, it is not necessary that f be always decreasing. What is important is that f be ultimately decreasing, that is, decreasing for x larger than some number N. Then $\sum_{n=N}^{\infty} a_n$ is convergent, so $\sum_{n=1}^{\infty} a_n$ is convergent.

Example 6.1 Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

Solution: The function $f(x) = 1/(x^2+1)$ is continuous, positive, and decreasing on $[1,\infty)$ so we use the Integral Test:

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2+1}\,dx &= \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1}\,dx \\
&= \lim_{t\to\infty} \tan^{-1} x \Big|_1^t \\
&= \lim_{t\to\infty} \left(\tan^{-1} t - \frac{\pi}{4}\right) \\
&= \frac{\pi}{2} - \frac{\pi}{4} \\
&= \frac{\pi}{4}
\end{aligned}$$

Thus, $\int_1^{\infty} \frac{1}{x^2+1} dx$ is a convergent integral and so, by the Integral Test, the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent.

■ **Example 6.2** For what values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

Solution: If $p < 0$, then $\lim_{n\to\infty}(1/n^p) = \infty$. If $p = 0$, then $\lim_{n\to\infty}(1/n^p) = 1$. In either case $\lim_{n\to\infty}(1/n^p) \neq 0$, so the given series diverges by the Test for Divergence.

If $p > 0$, then the function $f(x) = 1/x^p$ is clearly continuous, positive, and decreasing on $[1,\infty)$. We found that

$$\int_1^{\infty} \frac{1}{x^p} dx \text{ converges if } p > 1 \text{ and diverges if } p \leq 1.$$

It follows from the Integral Test that the series $\sum 1/n^p$ converges if $p > 1$ and diverges if $0 < p \leq 1$. (For $p = 1$, this series is the harmonic series).

The series in the above example is called the **p-series**. It is important in the rest of this chapter.

Theorem 6.1.2 The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

■ **Example 6.3** 1. The series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$$

is convergent because it is a p-series with $p = 3 > 1$.

2. The series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$$

is divergent because it is a p-series with $p = \frac{1}{3} < 1$.

Remark 6.1.2 We should not infer from the Integral Test that the sum of the series is equal to the value of the integral. In fact,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{whereas} \quad \int_1^{\infty} \frac{1}{x^2} dx = 1$$

Therefore, in general,

$$\sum_{n=1}^{\infty} a_n \neq \int_1^{\infty} f(x)dx.$$

■ **Example 6.4** Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

Solution: The function $f(x) = (\ln n)/x$ is positive and continuous for $x > 1$ because the logarithm function is continuous. But it is not obvious whether or not f is decreasing, so we compute its derivative:

$$f'(x) = \frac{(1/x)x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

Thus $f'(x) < 0$ when $\ln x > 1$, that is, when $x > e$. It follows that f is decreasing when $x > e$ and so we can apply the Integral Test:

$$\begin{aligned}
\int_1^{\infty} \frac{\ln x}{x}\,dx &= \lim_{t\to\infty} \int_1^t \frac{\ln x}{x}\,dx \\
&= \lim_{t\to\infty} \left.\frac{(\ln x)^2}{2}\right|_1^t \\
&= \lim_{t\to\infty} \frac{(\ln t)^2}{2} = \infty
\end{aligned}$$

Since this improper integral is divergent, the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is also divergent by the Integral Test.

6.2 The Comparison Tests

The idea in the comparison test is to compare a given series from a series that is known to converge or diverge. For instance,

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$$

would make us remember the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

which is a geometric series $a = 1/2$ and $r = 1/2$ and is therefore convergent. Thus, we can consider the second series as our *comparison series*. Comparing their respective nth terms we have,

$$\frac{1}{2^n + 1} < \frac{1}{2^n},$$

for all $n \geq 1$. This means also that

$$\sum_{n=1}^{\infty} \frac{1}{2^n+1} < \sum_{n=1}^{\infty} \frac{1}{2^n}.$$

We all know that the series on the right is equal to 1. Hence,

$$\sum_{n=1}^{\infty} \frac{1}{2^n+1} < 1.$$

This implies that the given series is convergent since its sum is less than 1.

On the other hand, if a given series is greater than a series which is known to diverge, then the given series must also diverge. The following theorem will give the general rule of the above statements.

Theorem 6.2.1 The Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

Example 6.5 Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges or diverges.

Solution: We can compare the given series with $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a convergent, since it is $5/2$ times a convergent p-series ($p = 2$. Observe that

$$\frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$$

for all n. This means that

$$\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3} < \sum_{n=1}^{\infty} \frac{5}{2n^2}.$$

Since the series on the right is convergent and by part (i) of the Comparison Test, the given series is also convergent.

Example 6.6 Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence or divergence.

Solution: The most useful series that we can think of at the moment is the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ which is known to diverge. Observe that $\ln n > 1$ for $n \geq 3$. Thus,

$$\frac{\ln n}{n} > \frac{1}{n}, \qquad n \geq 3.$$

This means that

$$\sum_{n=3}^{\infty} \frac{\ln n}{n} > \sum_{n=3}^{\infty} \frac{1}{n}, \qquad n \geq 3.$$

The series on the right is divergent since it is just the harmonic series less than the sum of the first two terms. Thus, the series on the left is also divergent by part (ii) of the Comparison test. Therefore, the given series is divergent.

Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$. You might to compare it with the convergent geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ and notice that

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1} > \sum_{n=1}^{\infty} \frac{1}{2^n}.$$

Here, the given series is greater than our (convergent) comparison series. Hence, we cannot use the Comaparison Test for this. In such cases the following test can be used.

Theorem 6.2.2 The Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

■ **Example 6.7** We use the Limit Comparison Test with

$$a_n = \frac{1}{2^n - 1} \text{ and } b_n = \frac{1}{2^n}.$$

Note that b_n is the nth term of a convergent geometric series. Now,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(2^n - 1)}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - 1/2^n} = 1 > 0.$$

Therefore, by the Limit Comparison Test, the given series is convergent.

6.3 Alternating Series

An **alternating series** is a series whose terms are alternately positive and negative. Here are two examples:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$$
$$-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\frac{4}{5}-\frac{5}{6}+\frac{6}{7}+\cdots=\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n+1}$$

We see from these examples that the nth term of an alternating series is of the form

$$a_n=(-1)^{n-1}b_n \qquad \text{or} \qquad a_n=(-1)^n b_n$$

where b_n is a positive number. (In fact $b_n=|a_n|$.)

The following test says that if the terms of an alternating series decrease toward 0 in absolute value, then the series converges.

Theorem 6.3.1 The Alternating Series Test
If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n=b_1-b_2+b_3-b_4+b_5-b_6+\cdots \qquad , \qquad b_n>0$$

satisfies

(i) $b_{n+1}\leq b_n$

(ii) $\lim_{n\to\infty} b_n=0$

then the series is convergent.

■ **Example 6.8** The alternating harmonic series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$$

satisfies

(i) $b_{n+1}<b_n$ because $\dfrac{1}{n+1}<\dfrac{1}{n}$

(ii) $\lim_{n\to\infty} b_n=\lim_{n\to\infty}\dfrac{1}{n}=0$

so the series is convergent by the Alternating Series Test.

■ **Example 6.9** The series $\sum_{n=1}^{\infty}(-1)^n \frac{3n}{4n-1}$ is alternating,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \frac{3}{4-\frac{1}{n}} = \frac{3}{4}$$

so condition (ii) is not satisfied. Instead, we look at the limit of the nth term of the series:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n 3n}{4n-1}$$

This limit does not exist, so the series diverges by the Test for Divergence.

6.4 Exercises

Exercise 6.1 Determine whether the following series is convergent or divergent.

1. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
2. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$
3. $\sum_{n=1}^{\infty} \frac{1}{\ln n}$
4. $\sum_{n=1}^{\infty} \frac{1}{n2^n}$
5. $\sum_{n=3}^{\infty} (-1)^{n+1} \frac{1}{\ln n}$

VII

Week Seven

7. Infinite Sequence and Series

At the end of this topic you are expected to:

- determine the convergence or divergence of series using ratio and root tests
- apply the different strategies for testing the convergence or divergence of series

7.1 Absolute Convergence

Given any series $\sum a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

whose terms are the absolute values of the terms of the original series.

Definition 7.1.1 A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

Notice that if $\sum a_n$ is a series with positive terms, then $|a_n| = a_n$ and so absolute convergence is the same as convergence in this case.

■ **Example 7.1** The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is a convergent p-series ($p = 2$).

■ **Example 7.2** We know that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

is convergent, but it is not absolutely convergent because the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

which is the harmonic series (p-series with $p = 1$) and is therefore divergent.

Definition 7.1.2 A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

■ **Example 7.3** The alternating harmonic series is conditionally convergent since this series is convergent but the corresponding series of absolute values is divergent as shown in Example 7.2.

Thus it is possible for a series to be convergent but not absolutely convergent. However, the next theorem shows that absolute convergence implies convergence.

Theorem 7.1.1 If a series $\sum a_n$ is absolutely convergent, then it is convergent.

■ **Example 7.4** Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$

is convergent or divergent.

Solution: This series has both positive and negative terms, but it is not alternating. (The first term is positive, the next three are negative, and the following three are positive: the signs change irregularly.) We can apply the Comparison Test to the series of absolute values

$$\sum_{n=1}^{\infty}\left|\frac{\cos n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{|\cos n|}{n^2}$$

Since $|\cos n| \leq 1$ for all n, we have

$$\frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$$

We know that $\sum 1/n^2$ is convergent (p-series with $p = 2$) and therefore $\sum |\cos n|/n^2$ is convergent by the Comparison Test. Thus the given series $\sum(\cos n)/n^2$ is absolutely convergent and therefore convergent by Theorem 7.1.1.

The following test is very useful in determining whether a given series is absolutely convergent.

7.1.1 The Ratio Test

Theorem 7.1.2 The Ratio Test

(i) If $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$ or then the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive, that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

■ **Example 7.5** Test the series $\sum\limits_{n=1}^{\infty}\frac{(-1)^n n^3}{3^n}$ for absolute convergence.

Solution: We use the Ratio Test with $a_n = \frac{(-1)^n n^3}{3^n}$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\dfrac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\dfrac{(-1)^n n^3}{3^n}}\right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$
$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1+\frac{1}{n}\right) \to \frac{1}{3} < 1$$

Thus, by the Ratio Test, the given series is absolutely convergent.

■ **Example 7.6** Test the convergence of the series $\sum\limits_{n=1}^{\infty} \dfrac{n^n}{n!}$.

Solution: Since the terms $a_n = \dfrac{n^n}{n!}$ are positive, we don't need the absolute value signs.

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(n+1)^{n+1}}{(n+1)!}}{\dfrac{n^n}{n!}} = \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n}$$
$$= \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n \to e \quad \text{as} \quad n \to \infty$$

Since $e > 1$, the given series is divergent by the Ratio Test.

The following test is convenient to apply when nth powers occur.

7.1.2 The Root Test

Theorem 7.1.3 The Root Test

(i) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim\limits_{n\to\infty} \sqrt{|a_n|} = \infty$ or then the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive, that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then part (iii) of the Root test says the tests gives no information. The series $\sum a_n$ could converge or diverge. (If $L = 1$ in the Ratio Test, don't try the Root Test because

L will again be 1. And if $L = 1$ in the Root Test, don't try the Ratio Test because it will fail too.)

■ **Example 7.7** Test the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n$.

Solution:

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2+\frac{3}{n}}{3+\frac{2}{n}} \to \frac{2}{3} < 1$$

Thus, the given series is absolutely convergent (and therefore convergent) by the Root Test.

7.2 Rearrangements

If we rearrange the order of the terms in a finite sum, then of course the value of the sum remains unchanged. But this is not always the case for an infinite series. By **rearrangement** of an infinite series $\sum a_n$, we mean a series obtained by simply changing the order of the terms. For instance, a rearragement of $\sum a_n$ could start as follows:

$$a_1 + a_2 + a_5 + a_3 + a_4 + a_{15} + a_6 + a_7 + a_{20} + \cdots$$

It turns out that

if $\sum a_n$ is an absolutely convergent series with sum s,
then any rearrangement of $\sum a_n$ has the same sum s.

However, any conditionally convergent series can be rearranged to give a different sum. To illustrate this fact, let's consider the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \ln 2 \tag{7.1}$$

Multiplying both sides by $\frac{1}{2}$, we get

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots = \frac{1}{2}\ln 2 \tag{7.2}$$

Inserting zeros between the terms of this series, we have

$$0+\frac{1}{2}+0-\frac{1}{4}+0+\frac{1}{6}+0-\frac{1}{8}+\cdots=\tfrac{1}{2}\ln 2 \tag{7.3}$$

Finally, adding the first and last equations, we have

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}\cdots=\tfrac{3}{2}\ln 2 \tag{7.4}$$

Notice that the resulting series contains the same terms as in the very first equation but rearranged. The sums of these series, however, are different. In fact, Riemann proved that

if $\sum a_n$ is a conditionally convergent series and r is any real number whatsoever, then there is a rearrangement of $\sum a_n$ that has a sum equal to r.

7.3 Strategy for Testing Series

We now have several ways of testing a series for convergence or divergence; the problem is to decide which test to use on which series. In this respect, testing series is similar to integrating functions. Again there are no hard and fast rules about which test to apply to a given series, but you may find the following advice of some use. It is not wise to apply a list of the tests in a specific order until one finally works. That would be a waste of time and effort. Instead, as with integration, the main strategy is to classify the series according to its form.

1. If the series is of the form $\sum 1/n^p$, it is a p-series, which we know to be convergent if $p>1$ and divergent if $p<1$.
2. If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, it is a geometric series, which converges if $|r|<1$ and diverges if $|r|\geq 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.
3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if an is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series. The comparison tests apply only to series with positive terms, but if $\sum a_n$ has some negative terms, then we can apply the Comparison Test to $\sum a_n$ and test for absolute convergence.

4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.
5. If the series is of the form $\sum(-1)^{n-1}b_n$ or $\sum(-1)^n b_n$, then the Alternating Series Test is an obvious possibility.
6. Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should not be used for such series.
7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.
8. If $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

In the following examples we don't work out all the details but simply indicate which tests should be used.

Example 7.8 $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

Since $a_n \to \frac{1}{2} \neq 0$ as $n \to \infty$, we should use the test for divergence.

Example 7.9 $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

Since a_n is an algebraic function of n, we compare the given series with a p-series. The comparison series for the Limit Comparison Test is $\sum b_n$, where

$$b_n = \frac{\sqrt{n^3}}{3n^3} = \frac{n^{\frac{3}{2}}}{3n^3} = \frac{1}{3n^{\frac{3}{2}}}$$

Example 7.10 $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

Since the integral $\int_1^\infty xe^{-x^2}\,dx$ is easily evaluated, we use the integral test. The Ratio Test also works.

Example 7.11 $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$

Since the series is alternating, we use the Alternating Series Test.

■ **Example 7.12** $\sum_{k=1}^{\infty} \frac{2^k}{k!}$
Since the series involves $k!$, we use the Ratio Test.

■ **Example 7.13** $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$
Since the series is closely related to the geometric series $\sum 1/3^n$, we use the Comparison Test.

7.4 Exercises

Exercise 7.1 Test the following series for convergence or divergence.

1. $\sum_{n=1}^{\infty} \frac{n^2-1}{n^3+1}$
2. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^3+1}$
3. $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$
4. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$
5. $\sum_{k=1}^{\infty} \frac{5^k}{3^k+4^k}$

VIII

Week Eight

8. Infinite Sequences and Series

At the end of this topic you are expected to:

- determine the convergence of a power series
- solve for the radius and interval of convergence of a power series

8.1 Power Series

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

where x is a variable and the c_n's are constants called the coefficients of the series. For each fixed x, the series $\sum_{n=0}^{\infty} c_n x^n$ is a series of constants that we can test for convergence or divergence. A power series may converge for some values of x and diverge for other values of x. The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

whose domain is the set of all x for which the series converges. Notice that f resembles a polynomial. The only difference is that f has infinitely many terms.

For instance, if we take $c_n = 1$ for all n, the power series becomes the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

which converges when $-1 < x < 1$ and diverges when $|x| \geq 1$.

More generally, a series of the form

$$\boxed{\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots}$$

is called a **power series in $(x-a)$ ad or a power series centered at a or a power series about a.**

■ **Example 8.1** For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

Solution: We use the Ratio Test. If we let a_n, as usual, denote the nth term of the series, then $a_n = n!x^n$. If $x \neq 0$, we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| = \lim_{n\to\infty} (n+1)|x| = \infty$$

By the Ratio Test, the series diverges when $x \neq 0$. Thus the given series converges only when $x = 0$.

■ **Example 8.2** For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

Solution: Let $a_n = (x-3)^n/n$. Then

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} \right| \\ &= \frac{1}{1 + \frac{1}{n}} |x-3| \to |x-3| \quad n \to \infty \end{aligned}$$

By the Ratio Test, the given series is absolutely convergent, and therefore convergent, when $|x-3| < 1$ and divergent when $|x-3| > 1$. Now

$$|x-3| < 1 \Leftrightarrow -1 < x-3 < 1 \Leftrightarrow 2 < x < 4$$

so the series converges when $2 < x < 4$ and diverges when $x < 2$ or $x > 4$.

The Ratio Test gives no information when $|x-3|=1$ so we must consider $x=2$ and $x=4$ separately. If we put $x=4$ in the series, it becomes $\sum 1/n$, the harmonic series, which is divergent. If $x=2$, the series is $\sum(-1)^n/n$, which converges by the Alternating Series Test. Thus, the given power series converges for $2 \leq x < 4$.

■ **Example 8.3** Find the domain of the **Bessel function** of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

Solution: Let $a_n = (-1)^n x^{2n}/[2^{2n}(n!)^2]$. Then

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right| \\
&= \frac{x^{2n+2}}{2^{2n+2}(n+1)^2(n!)^2} \cdot \frac{2^{2n}(n!)^2}{x^{2n}} \\
&= \frac{x^2}{4(n+1)^2} \to 0 < 1 \qquad \text{for all } x
\end{aligned}$$

Thus, by the Ratio Test, the given series converges for all values of x. In other words, the domain of the Bessel function J_0 is $(-\infty,\infty) = \mathbb{R}$.

Theorem 8.1.1 For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:

(i) The series converges only when $x=a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The number R in case (iii) is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R=0$ in case (i) and $R=\infty$ in case (ii). The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges.

■ **Example 8.4** Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}.$$

Solution: Let $a_n = (-3)^n x^n / \sqrt{n+1}$. Then

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n}\right| \\
&= \left|-3x\sqrt{\frac{n+1}{n+2}}\right| \\
&= 3\sqrt{\frac{1+(1/n)}{1+(2/n)}}|x| \to 3|x| \quad \text{as } n \to \infty
\end{aligned}$$

By the Ratio Test, the given series converges if $3|x| < 1$ and diverges if $3|x| > 1$. Thus it converges if $|x| < \dfrac{1}{3}$ and diverges if $|x| > \dfrac{1}{3}$. This means that the radius of convergence is $R = \dfrac{1}{3}$.

We know the series converges in the interval $(-\frac{1}{3}, \frac{1}{3})$, but we must now test for convergence at the endpoints of this interval. If $x = -\frac{1}{3}$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-3)^n(-\frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

which diverges. (Use the Integral Test or simply observe that it is a p-series with $p = \frac{1}{2} < 1$.) If $x = \frac{1}{3}$, the series is

$$\sum_{n=0}^{\infty} \frac{(-3)^n(\frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

which converges by the Alternating Series Test. Therefore, the given power series converges when $-\frac{1}{3} < x \le \frac{1}{3}$, so the interval of convergence is $(-\frac{1}{3}, \frac{1}{3}]$.

■ **Example 8.5** Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}.$$

Solution: Let $a_n = n(x+2)^n / 3^{n+1}$. Then

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n}\right| \\
&= \left(1+\frac{1}{n}\right)\frac{|x+2|}{3} \to \frac{|x+2|}{3} \quad \text{as } n \to \infty
\end{aligned}$$

Using the Ratio Test, we see that the series converges if $|x+2|/3 < 1$ and it diverges if $|x+2|/3 > 1$. So it converges if $|x+2| < 3$ and diverges if $|x+2| > 3$. Thus the radius of convergence is $R = 3$.

The inequality $|x+2| < 3$ can be written as $-5 < x < 1$, so we test the series at the endpoints -5 and 1. When $x = -5$, the series is

$$\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n n$$

which diverges by the Test for Divergence [$(-1)^n n$ doesn't converge to 0]. When $x = 1$, the series is

$$\sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty} n$$

which also diverges by the Test for Divergence. Thus, the series converges only when $-5 < x < 1$, so the interval of convergence is $(-5, 1)$.

8.2 Exercises

Exercise 8.1 Find the radius of convergence and interval of convergence of the following series:

1. $\sum_{n=1}^{\infty} (-1)^n n x^n$
2. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$
3. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$
4. $\sum_{n=1}^{\infty} n^n x^n$
5. $\sum_{n=1}^{\infty} 2^n n^2 x^n$.

IX

Week Nine

9. Infinite Sequences and Series

At the end of this topic you are expected to:

- write representations of functions as power series
- find the Maclaurin and Taylor series of a function

9.1 Representations of Functions as Power Series

In this section we will learn how to represent certain types of fucntions as sums of power series by manipulating geometric series or by differentiating or integrating such series. You might wonder why we would ever want to express a known function as a sum of infinitely many terms.

We begin with an equation that we have encountered before:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

We have encountered this equation before where we obtain it by observing that the series is a geometric series with $a = 1$ and $r = x$. But our point of view is different. We now regard the above equation as expressing the function $f(x) = 1/(1-x)$ as a sum of a power series.

Example 9.1 Express $1/(1+x^2)$ as the sum of a power series and find the interval of convergence.

Solution: Replacing x by $-x^2$ as in the given series above, w have

$$\begin{aligned}\frac{1}{1+x^2} &= \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n \\ &= \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots\end{aligned}$$

Beacause this is a geometric series, it converges when $|-x^2| < 1$, that is $x^2 < 1$, or $|x| < 1$. Therefore, the interval of convergence is $(-1,1)$.

Example 9.2 Find a power series representation for $\dfrac{1}{x+2}$

Solution: We factor first a 2 from the denominator:

$$\begin{aligned}\frac{1}{2+x} &= \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2\left[1-\left(-\frac{x}{2}\right)\right]} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n\end{aligned}$$

This series converges when $|-x/2| < 1$, that is, $|x| < 2$. So the interval of convergence is $(-2,2)$.

9.2 Differentiation and Integration of Power Series

The sum of a power series is a function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ whose domain is the interval of convergence of the series. We would like to be able to differentiate and integrate such functions, and the following theorem says that we can do so by differentiating or integrating each individual term in the series. This process is known as **term-by-term differentiation or integration**.

Theorem 9.2.1 If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$

The radii of convergence of the power series in equations (i) and (ii) are both R.

Example 9.3 Express $1/(1-x)^2$ as a power series by differentiating the power series for $1/(1-x)$. What is the radius of convergence?

Solution: Differentiating each side of the equation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

we get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} nx^{n-1}.$$

If we wish, we can replace n by $n+1$ and write the answer as

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n.$$

According to the above theorem, the radius of convergence of the differentiated series is the same as the radius of covergence of the original series which is $R = 1$.

Example 9.4 Find a power series representation for $\ln(1+x)$ and its radius of convergence.

Solution: We notice that the derivative of this function is $1/(1+x)$. Thus, after integrating each side of the power series

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \cdots \qquad |x| < 1$$

we have

$$\begin{aligned}
\ln(1+x) &= \int \frac{1}{1+x}\,dx = \int (1-x+x^2-x^3+\cdots)\,dx \\
&= x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+C \\
&= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}+C \qquad |x|<1.
\end{aligned}$$

To determine the value of C we put $x=0$ in this equation and obtain $\ln(1+0)=C$ which implies that $C=0$. Thus,

$$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} \qquad |x|<1.$$

The radius of convergence is the same as for the orginal series: $R=1$.

■ **Example 9.5** Find the power series representation for $f(x)=\tan^{-1}x$.

Solution: We can find the required series by integrating the power series for $1/(1+x^2)$.

$$\begin{aligned}
\tan^{-1}x &= \int \frac{1}{1+x^2}\,dx = \int (1-x^2+x^4-x^6+\cdots)\,dx \\
&= C+x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots
\end{aligned}$$

To find C, we put $x=0$ and obtain $C=0$. Therefore,

$$\begin{aligned}
\tan^{-1}x &= x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}.
\end{aligned}$$

Since the radius of convergence of the series for $1/(1+x^2)$ is 1, the radius of convergence of this series for $\tan^{-1}x$ is also 1.

9.3 Taylor and Maclaurin Series

In the precceding section, we were able to find the power series representaions for a certain restricted class of functions. Here we investigate more general problems: Which functions have power series representations? How can we find such representations?

We start by supposing that f is any function that can be represented by a power series

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots \qquad |x-a| < R$$

If we put $x = a$ in the above equation, then all the terms after the first are 0 and we get

$$f(a) = c_0.$$

Also,

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \qquad |x-a| < R$$

and substitution of $x = a$ in this equation gives

$$f'(a) = c_1.$$

Following the same process we obtain the following results:

$$f''(a) = 2c_2 \qquad \text{and} \qquad f'''(a) = 2 \cdot 3c_3 = 3!c_3.$$

If we continue to differntiate and substitute $x = a$, we obtaib

$$f^n(a) = 2 \cdot 3 \cdot 4 \cdot nc_n = n!c_n.$$

Solving this equation for the nth coefficient, we get

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

This formula remains valid even when n is 0 if we adopt the conventions that $0! = 1$ and $f^{(0)} = f$. Thus, we have the following theorem.

Theorem 9.3.1 If f has the power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Substituting this formula for c_n back into the series, we have

$$\boxed{\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}}$$

The series in the boxed equation is called the **Taylor series of the function** f **at** a or (**about** a). For the special case where $a = 0$, the Taylor series becomes

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots}$$

This case arises frequently enough tht it is given the special name **Maclaurin series**.

▪ **Example 9.6** Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

Solution: If $f(x) = e^x$ then $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = e^0 = 1$ for all n. Therefore, the Taylor series for f at 0 (that is, the Maclaurin series) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

To find the radius of convergence we let $a_n = x^n/n!$. Then

$$\left|\frac{a_n+1}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0 < 1$$

so by the Ratio Test, the series converges for all x and the radius of convergence is $R = \infty$.

It can be shown also that e^x is equal to the sum of its Maclaurin series, that is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{for all } x$$

.

In particular, if we put $x = 1$, we obtain

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

.

Example 9.7 Find the Taylor series for $f(x) = e^x$ at $a = 2$.

Solution: We have $f^{(n)}(2) = e^2$ and so, putting $a = 2$ in the definition of a Taylor series, we get

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n.$$

Again it can be verified that the radius of convergence is $R = \infty$. Therefore,

$$e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n \qquad \text{for all } x.$$

Example 9.8 Find the Maclaurin series for $\sin x$.

Solution: We arrange our computation as follows

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

Since the derivatives repeat in a cycle of four, we can write the Maclaurin series as follows:

$$\begin{aligned} f(0) &+ \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \\ &= x - \frac{x^3}{3} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}. \end{aligned}$$

It can be shown also that $\sin x$ is equal to the sum of its Maclaurin series, that is,

$$\begin{aligned} \sin x &= x - \frac{x^3}{3} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{for all } x. \end{aligned}$$

Example 9.9 Find the Maclaurin series for $\cos x$.

Solution: It would be easier to obtain the Maclaurin series of the given function by differentiating the Macalurin series for $\sin x$ knowing that the derivative of $\sin x$ is $\cos x$.

$$\begin{aligned} \cos x &= \frac{d}{dx}(\sin x) = \frac{d}{dx}\left(x - \frac{x^3}{3} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\ &= 1 - \frac{3x^2}{3!} + = \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{for all } x. \end{aligned}$$

Example 9.10 Find the Maclaurin series for the function $f(x) = x\cos x$.

Solution: Instead of computing derivatives and substituting to the formula of a Taylor series, it is easier to multiply the series for $\cos x$ by x:

$$x\cos x = x\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!}.$$

9.4 Exercises

Exercise 9.1 Do as directed.

1. Find the power series of the first function by differentiating the power series of the second function:

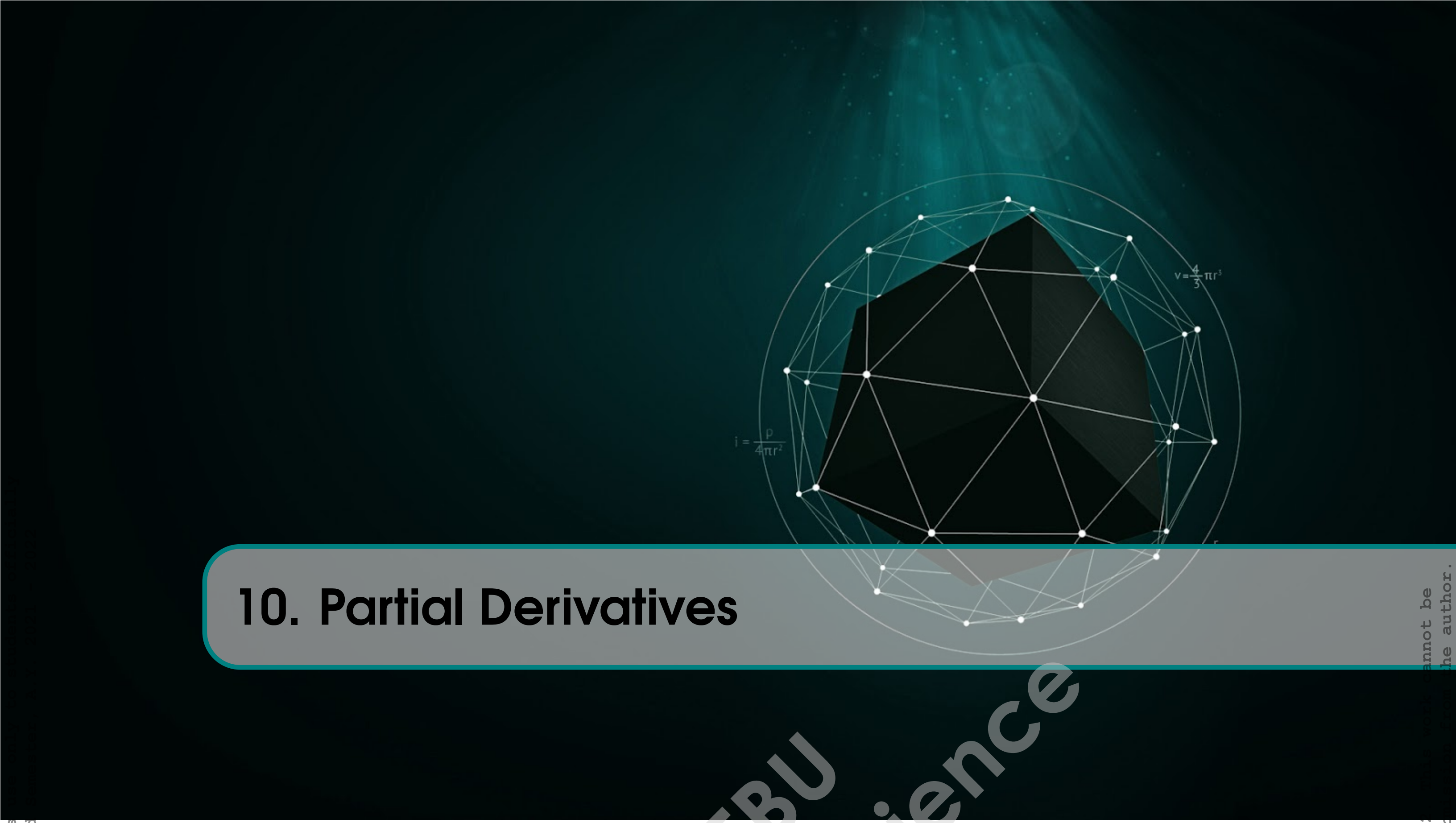
   (a) $\sec^2 x$; $\tan x$

   (b) $\sin x + x\cos x$; $x\sin x$.

2. Find the power series of the first function by differentiating the power series of the second function:
   (a) $\ln \cos x$; $\tan x$
   (b) $\sin^{-1} x$; $\dfrac{1}{\sqrt{1-x^2}}$.

X

Week Ten



10. Partial Derivatives

At the end of this topic you are expected to:

- identify the domain of functions in two variables
- find the limit of some functions in two variables

10.1 Three-Dimensional Coordinate Systems

10.1.1 3D Space

To locate a point in a plane, we need two numbers. We know that any point in the plane can be represented as an ordered pair (a,b) of real numbers, where a is the x-coordinate and b is the y-coordinate. For this reason, a plane is called two-dimensional. To locate a point in space, three numbers are required. We represent any point in space by an ordered triple (a,b,c) of real numbers.

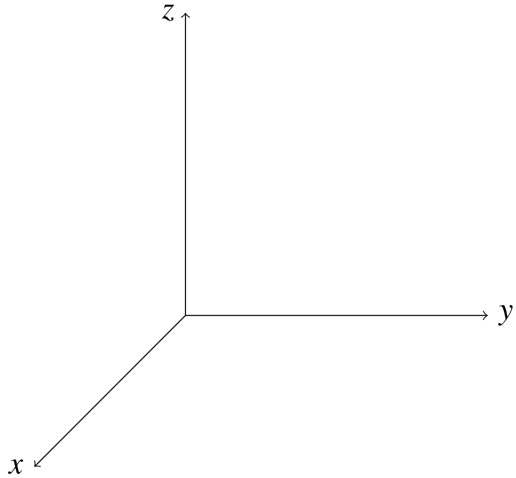


Figure 1.1: A 3-Dimensional Coordinate System.

Because many people have some difficulty visualizing diagrams of three-dimensional figures, you may find it helpful to do the following [Figure 1.2]. Look at any bottom corner of a room and call the corner the origin. The wall on your left is in the xz-plane, the wall on your right is in the yz-plane, and the floor is in the xy-plane. The x-axis runs along the intersection of the floor and the left wall. The y-axis runs along the intersection of the floor and the right wall. The z-axis runs up from the floor toward the ceiling along the intersection of the two walls. You are situated in the first octant, and you can now imagine seven other rooms situated in the other seven octants [Figure 1.3] (three on the same floor and four on the floor below), all connected by the common corner point O.

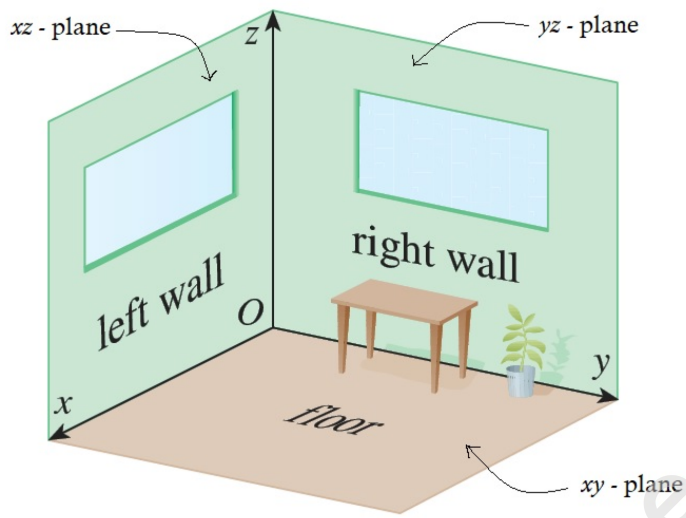


Figure 1.2: A model of xy-, xz- and yz- plane.

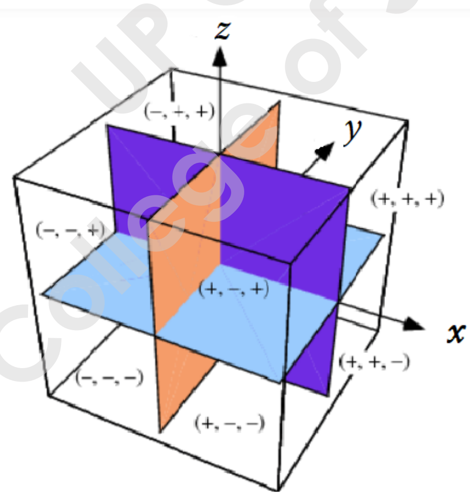


Figure 1.3: A 3-D model the octants.

Usually we think of the $x-$ and $y-$axes as being horizontal and the $z-$axis as being vertical. The direction of the $z-$axis is determined by the **right-hand rule** as illustrated below [Figure 1.4]: If you curl the fingers of your right hand around the z-axis in the direction of a 90°

counterclockwise rotation from the positive x-axis to the positive y-axis, then your thumb points in the positive direction of the z-axis.

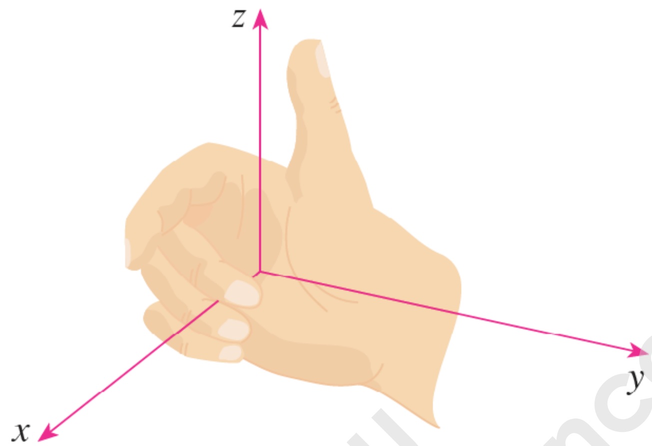


Figure 1.4: The Right-Hand Rule.

The point $P(a,b,c)$ determines a rectangular box as in Figure 1.5 . If we drop a perpendicular from P to the xy-plane, we get a point Q with coordinates $(a,b,0)$ called the projection of P onto the xy-plane. Similarly, $R(0,b,c)$ and $S(a,0,c)$ are the projections of P onto the yz-plane and xz-plane, respectively.

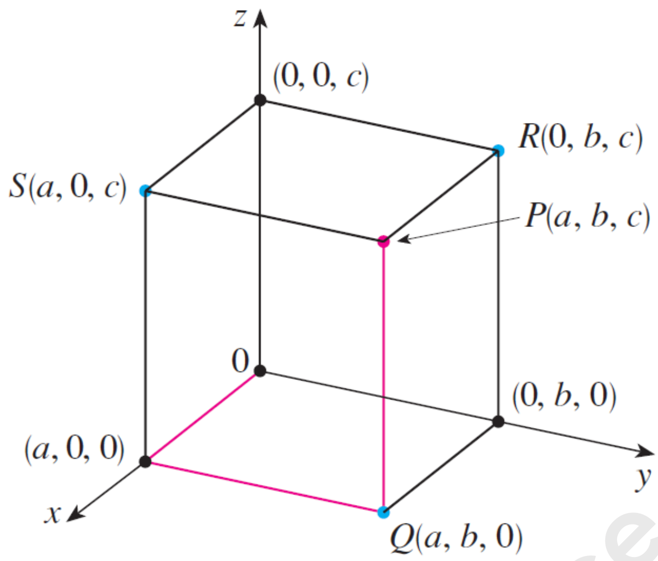
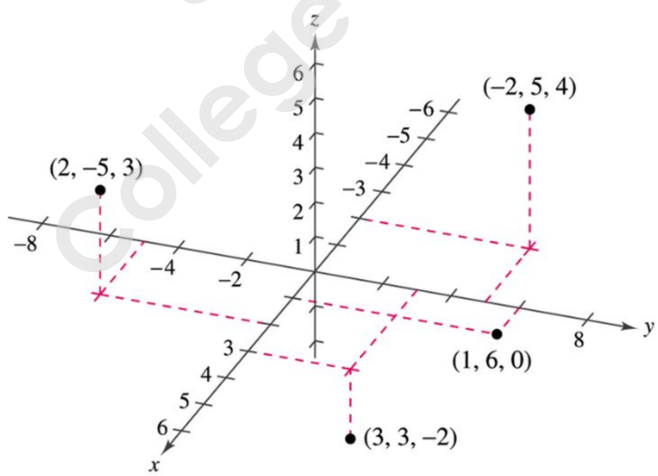


Figure 1.5

Example. Plot the points $(1,6,0)$, $(3,3,-2)$, $(-2,5,4)$ and $(2,-5,3)$.

Solution:



The Cartesian product $\mathbb{R}\times\mathbb{R}\times\mathbb{R} = \{(x,y,x)\,|\,x,y,z\in\mathbb{R}\}$ is the set of all ordered triples of real numbers and is denoted by $\mathbb{R}^3$. We have given a one-to-one correspondence between points

P in space and ordered triples (a,b,c) in $\mathbb{R}^3$. It is called a **three-dimensional rectangular coordinate system**. Notice that, in terms of coordinates, the first octant can be described as the set of points whose coordinates are all positive.

10.2 Functions of Several Variables

In our previous calculus courses, we dealt with the functions of a single variable. This is usually denoted by $f(x)$. This time, we will extend this to functions of several variables.

The following are examples of functions of several variables.

1. The volume of a right circular cylinder, $V = \pi r^2 h$, is a function of two variables.
2. The area of a rectangle, $A = lw$, is a function of two variables.
3. The volume of a rectangular solid, $V = lwh$, is a function of three variables.

We use the notation $f(x,y)$ to denote a function in two variables x and y. Similarly, for three vaiables, we have $f(x,y,z)$.

Example 10.1 The following are functioons of more than one variabe:

1. $f(x,y) = x^2 + 2xy + y^2$
2. $f(x,y,z) = x^2 + yx + 4z$

Definition 10.2.1 (Function of Two Variables)
Let D be the set of ordered pairs of real numbers. If to each ordered pair (x,y) in D there corresponds a unique real number $f(x,y)$, then f is a **function** of x and y. The set D is the **domain** of f, and the corresponding set of values for $f(x,y)$ is the range of f. For thefunction

$$z = f(x,y),$$

x and y are called the **independent variables** and z called the **dependent variable**.

Example 10.2 Find the domain of the function $f(x,y) = \ln(y^2 - x)$.

Solution: Since $\ln(y^2 - x)$ is only defined when $y^2 - x > 0$, the domain of f is therefore $D = \{(x,y) \,|\, x < y^2\}$.

Example 10.3 Find the domain and range of $f(x,y) = x^2 + y^2$.

Solution: Since f is defined for any ordered pair (x,y), the domain is the entire xy-plane or $\{(x,y) : x,y \in \mathbb{R}\}$. Notice also that f nonnegative $(x^2+y^2 \geq 0)$. Thus, the range is the set $[0,+\infty)$ or the set of nonnegative real numbers.

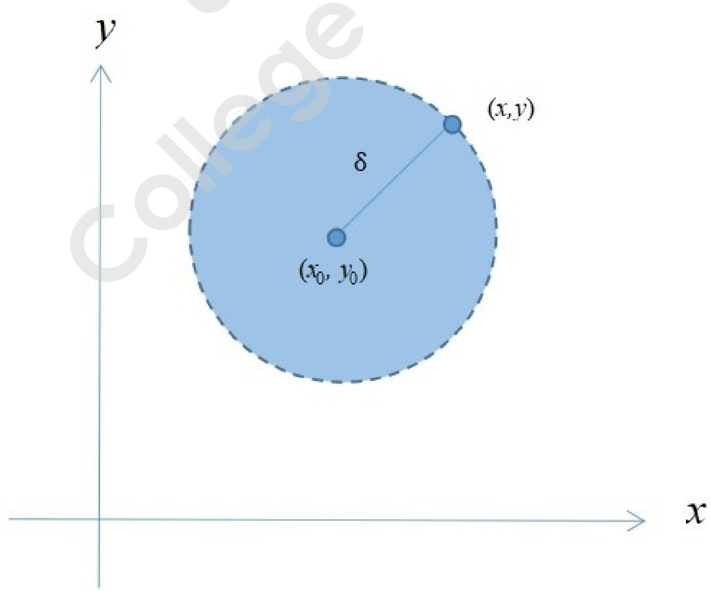
10.3 Limits and Continuity

The study of the limit of a function of two variables begins by defining a two-dimensional analog to an interval on the real number line.

Using the formula for the distance between two points (x,y) and (x_0,y_0) in the plane, we can define the **δ-neighborhood** about (x_0,y_0) to be the **disk** centered at (x_0,y_0) with radius $\delta > 0$

$$\{(x,y) : \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta\}.$$

Because the inequality sign is strictly less than δ, we have an open disk centered at (x_0,y_0) as in the figure below.



Definition 10.3.1 (Limit of a Function of Two Variables)

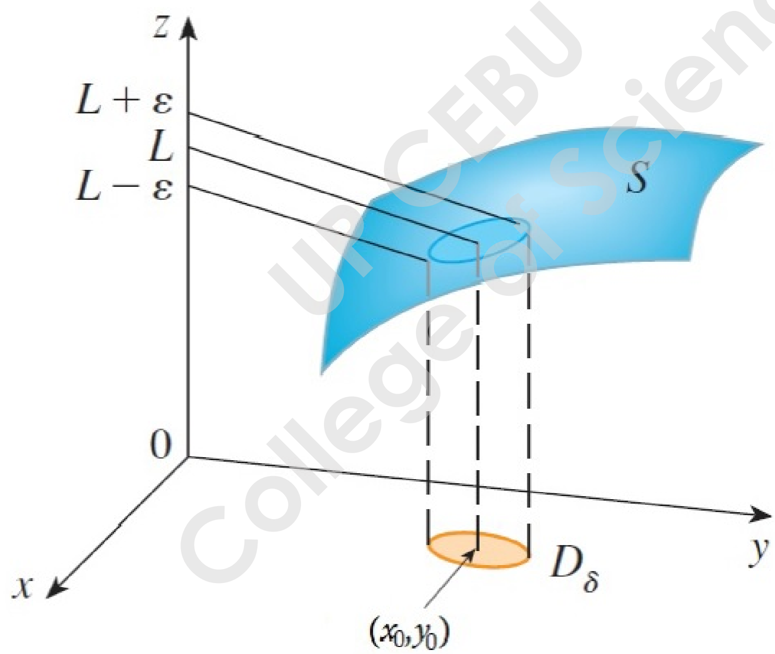
Let f be a function of two variables defined, except possibly at (x_0, y_0) on an open disk centered at (x_0, y_0), and let L be a real number. Then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if for every $\varepsilon > 0$, there corresponds $\delta > 0$ such that

$$|f(x,y) - L| < \varepsilon \text{ whenever } 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta.$$

An illustration of the above definition is shown below.



Example 10.4 Find $\displaystyle\lim_{(x,y)\to(1,2)} \frac{4x^2y}{x^2+y^2}$.

Solution: Note that the limit of the quotient is just the quotient of the limits. In our case,

$$\lim_{(x,y)\to(1,2)} 4x^2y = 4(1)^2(2) = 8 \text{ and } \lim_{(x,y)\to(1,2)} x^2+y^2 = 1^2+2^2 = 5$$

Therefore,

$$\lim_{(x,y)\to(1,2)} \frac{4x^2y}{x^2+y^2} = \frac{8}{5}.$$

Remember that in two dimensions when we approach a real number x on the x-axis, we have only two directions, either from the right or from the left. In three dimensions, a point (x,y) in the plane may be approached in an infinitely many directions.

■ **Example 10.5** Does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ exist?

Solution: By just looking at the function, you will immediately notice that direct substitution would not give you the limit for both the numerator and denominator will be zero.

Let us consider approaching $(0,0)$ about the x-axis $(y=0)$. Then

$$\lim_{(x,0)\to(0,0)} \frac{x(0)}{x^2+0^2} = \lim_{(x,0)\to(0,0)} 0 = 0.$$

This time let us consider approaching $(0,0)$ about the y-axis $(x=0)$. Then

$$\lim_{(0,y)\to(0,0)} \frac{(0)y}{0^2+y^2} = \lim_{(0,y)\to(0,0)} 0 = 0.$$

However, these are not the only directions of approach and will not immediatelytell us the existence of the limit.

Let us consider approaching $(0,0)$ about the the line $y=x$. Then

$$\lim_{(x,x)\to(0,0)} \frac{xx}{x^2+x^2} = \lim_{(0,y)\to(0,0)} \frac{x^2}{2x^2} = \lim_{(x,x)\to(0,0)} \frac{1}{2} = \frac{1}{2}.$$

Now, we have found another value for the limit from a different direction.

Therefore, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

In your previous calculus course, showing that a function of single variable is continuous at a real number a is simply done by showing that $\lim\limits_{x\to a} f(x) = f(a)$. Similarly, continuity of functions of two variables are also defined by direct substitution.

■ **Example 10.6** Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2} = 0.$

Solution: Let $\varepsilon > 0$ be given. First, note that

$$|y| \le \sqrt{x^2+y^2} \quad \text{and} \quad \frac{x^2}{x^2+y^2} \le 1$$

Then in a δ-neighborhood of $(0,0)$,

$$0 < \sqrt{x^2+y^2} < \delta,$$

$$\begin{aligned}|f(x,y)-0| &= \left|\frac{2x^2y}{x^2+y^2}\right| \\ &= 2|y|\left|\frac{x^2}{x^2+y^2}\right| \\ &\leq 2|y| \\ &\leq 2\sqrt{x^2+y^2} \\ &< 2\delta.\end{aligned}$$

If we take $\delta = \frac{\varepsilon}{2}$, then

$$|f(x,y)-0| < 2\delta = 2(\varepsilon/2) = \varepsilon.$$

Thus, we have shown that whenever

$$0 < \sqrt{x^2+y^2} < \delta, \text{ we have } |f(x,y)-0| < \varepsilon.$$

Therefore,

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2} = 0.$$

Definition 10.3.2 (Continuity of a Function of Two Variables)

A function f of two variables is said to be continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

We say that f is continuous on D if f is continuous at every point (a,b) in D.

Here are some continuous functions.

1. Polynomial functions are continuous on $\mathbb{R}$.
2. Rational functions are continuous on their respective domains.

■ **Example 10.7** Find $\lim\limits_{(x,y)\to(1,3)} (2xy^2 - x^2y + 2x - 3y)$.

Solution: Since $f(x,y) = 2xy^2 - x^2y + 2x - 3y$ is a polynomial function, it is continuous everywhere and so, we can apply direct substitution. Thus,

$$\lim_{(x,y)\to(1,3)} (2xy^2 - x^2y + 2x - 3y) = 2(1)(3)^2 - (1)^2(3) + 2(1) - 3(3) = 8.$$

■ **Example 10.8** Where is the function $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ continuous?

Solution: Note that f is a rational function and thus, it is continuous on its domain which is $D = \{(x,y) : (x,y) \neq (0,0)\}$.

■ **Example 10.9** Determine whether the function below is continuous at $(0,0)$ or not.

$$f(x) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Solution: Note that $f(0,0) = 0$, thus $f(0,0)$ is defined.

Let us take a look at $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$. Now,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

does not exists (shown previously).

Therefore, f is not continuous at $(0,0)$.

■ **Example 10.10** Dtermine whether the given function below is continuous at $(0,0)$ ot not

$$f(x) = \begin{cases} \dfrac{2x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Solution: Note that $f(0,0) = 0$, thus $f(0,0)$ is defined.

Let us take a look at $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$. Now,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2} = 0.$$

The solution for this limit is already shown before.

Finally, we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$$

Therefore, f continuous at $(0,0)$.

10.4 Exercises

Exercise 10.1 Do as directed.

1. Suppose that $\lim_{(x,y)\to(3,1)} f(x,y) = 6$. What can you say about the value of $f(3,1)$? What is f is continuous?

2. Find the limit, if it exists, or show that the limit does not exist.

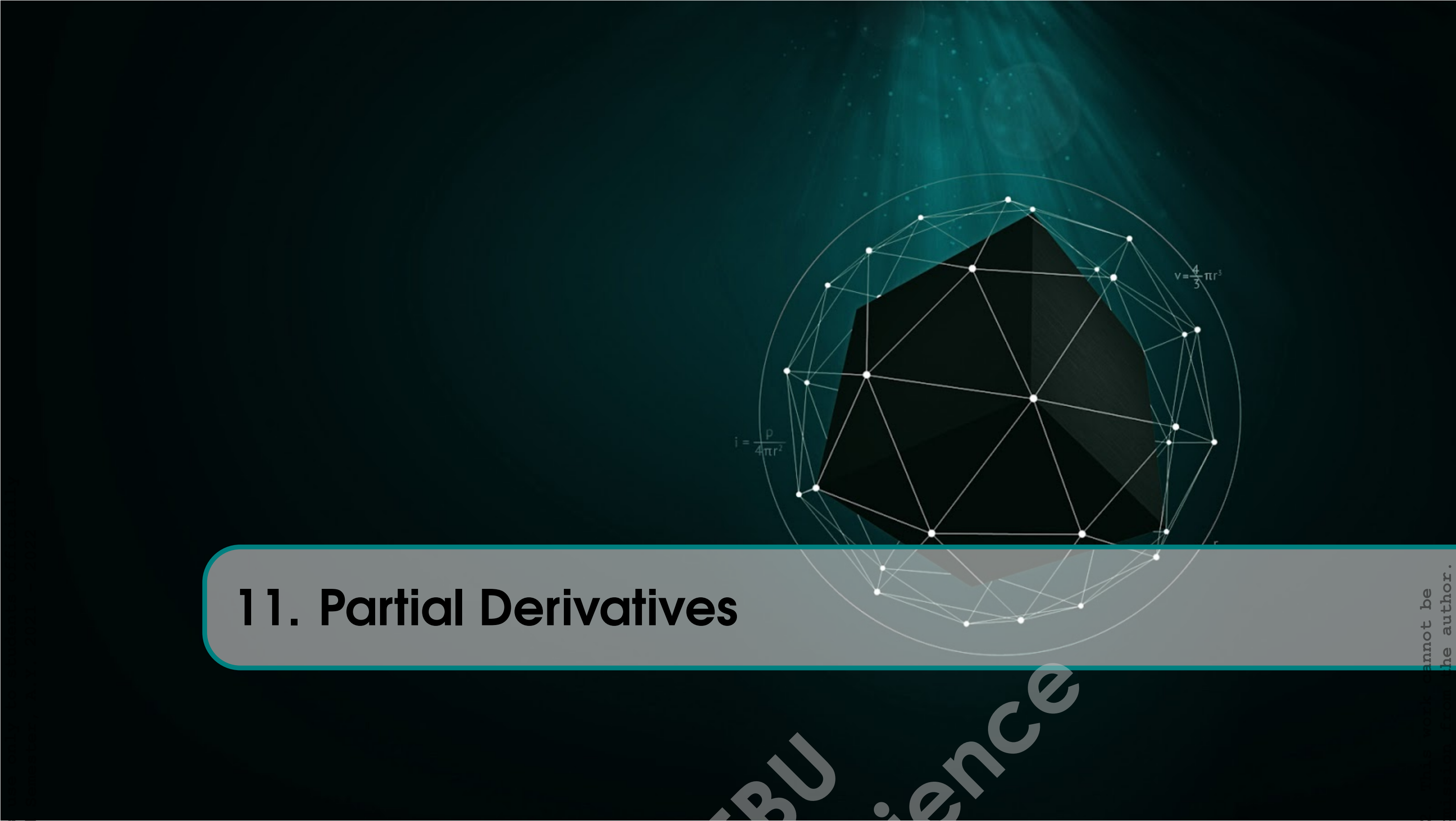
   (a) $\lim_{(x,y)\to(2,-1)} \dfrac{x^2y+xy^2}{x^2-y^2}$

   (b) $\lim_{(x,y)\to(\pi,\pi/2)} y\sin(x-y)$

   (c) $\lim_{(x,y)\to(0,0)} \dfrac{xy^4}{x^2+y^8}$

XI

Week Eleven



11. Partial Derivatives

At the end of this topic you are expected to:

- evaluate partial derivatives
- use chain rule in obtaining derivatives of composite functions

11.1 Partial Derivatives

You might ask "What will happen to the value of a function when there is a change in one of the two independent variables?". This commonly happens in a science experiments when a researcher wants to know the effect if one of the the variables in an experiment is changing while keeping other variables fixed. For instance, an experiment fixing the temperature while pressure is changing. Similarly, we can determine the rate of change of a function f with respect to one of its several variables. This procedure is known as **partial differentiation** and the resulting function is called the **partial derivative**.

Definition 11.1.1 (Partial Derivative of a Function of Two Variables)

If $z = f(x,y)$, then the first partial derivatives of f with respect to x and y are the functions defined by

$$\boxed{f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}}$$

and

$$\boxed{f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}}$$

provided that the limits exist.

The above definition tells that if you want to find f_x, treat y as constant and differentiate with respect to x. Similarly, if you want to find f_y, treat x as constant and differentiate with respect to y.

Example 11.1 Below are examples of partial derivatives.

1. To find f_x for $f(x,y) = 5x^2 - x^2y^2 + 3xy^3$, we treat y as constant and differentiate with respect to x. Thus,
$$f_x(x,y) = 10x - 2xy^2 + 3y^3.$$
To find f_y for $f(x,y) = 5x^2 - x^2y^2 + 3xy^3$, we treat x as constant and differentiate with respect to y. Thus,
$$f_y(x,y) = -2x^2y + 9xy^2.$$
2. To find f_x for $f(x,y) = (\ln x)(\tan x^2y)$, we treat y as constant and differentiate with respect to x. Thus,
$$f_x(x,y) = (\ln x)(\sec^2 x^2y)(2xy) + (\tan x^2y)\left(\frac{1}{x}\right)$$
To find f_y for $f(x,y) = (\ln x)(\tan x^2y)$, we treat x as constant and differentiate with respect to y. Thus,
$$f_y(x,y) = (\ln x)(\sec^2 x^2y)(x^2)$$

Notations for First Partial Derivatives

For $z = f(x,y)$, the partial derivatives f_x and f_y are denoted by

$$\frac{\partial}{\partial x} f(x,y) = f_x(x,y) = z_x = \frac{\partial z}{\partial x}$$

and

$$\frac{\partial}{\partial y} f(x,y) = f_y(x,y) = z_y = \frac{\partial z}{\partial y}$$

From evaluation of partial derivatives at point (a,b), we have

$$\left.\frac{\partial z}{\partial x}\right|_{(a,b)} = f_x(a,b)$$

and

$$\left.\frac{\partial z}{\partial y}\right|_{(a,b)} = f_y(a,b).$$

■ **Example 11.2** For $f(x,y) = xe^{x^2y}$, find f_x and f_y and evaluate eacch at $(1, \ln 2)$.

Solution: For f_x, we have

$$f_x(x,y) = xe^{x^2y}(2xy) + e^{x^2y}$$

Thus,

$$f_x(1, \ln 2) = e^{\ln 2}(2 \ln 2) + e^{\ln 2} = 4 \ln 2 + 2.$$

For f_y, we have

$$f_y(x,y) = xe^{x^2y}(x^2) = x^3 e^{x^2y}.$$

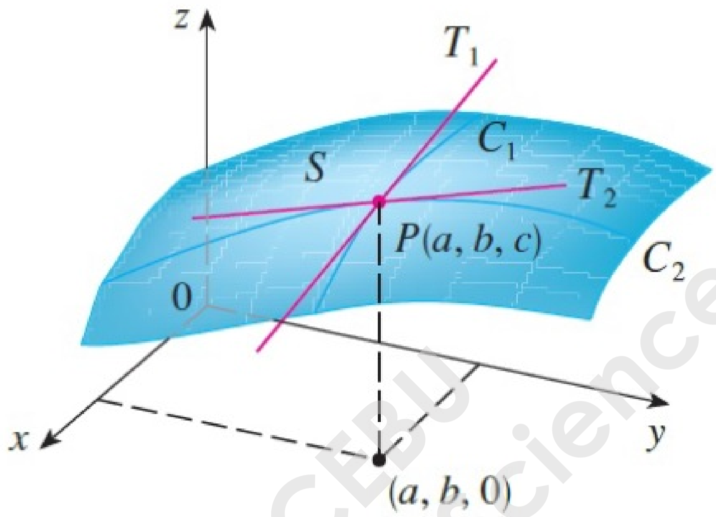
Thus,

$$f_y(1, \ln 2) = e^{\ln 2} = 2.$$

11.1.1 Geometric Interpretations of Partial Derivatives

In the two-dimensional coordinate system, we have learned the geometric interpretation of the the derivative of a funcion at the point (a,b) which is just the slope of the tangent line at that point.

Informally, the values of f_x and f_y at the point (a,b,c) on the surface denote the **slopes of the surface in the** $x-$ **and** $y-$ **directions**, respectively as shown in the figure below.



■ **Example 11.3** Find the slopes in the x-direction and y-direction of the surface

$$f(x,y) = \frac{x^2}{y^3}$$

at the point $(4,2,2)$.

Solution: Solving for the partial derivatives, we get

$$f_x(x,y) = \frac{2x}{y^3} \text{ and } f_y(x,y) = -\frac{3x^2}{y^4}.$$

Evaluating each of these partial derivatives at $(4,2)$, we have

$$f_x(4,2) = \frac{2(4)}{2^3} = 1 \text{ and } f_y(4,2) = -\frac{3(4)^2}{2^4} = -3.$$

Therefore, the slopes of the surface in the x- and y-direction at the point $(4,2,2)$ are 1 and -3 respectively.

We can also apply implicit differentiation to equation in three variables in which one variable, say z, is implicitly defined as a function of the other variables, say x and y.

■ **Example 11.4** Find $\partial z/\partial x$ and $\partial z/\partial y$ if z is defined implicitly as a function of x and y by the equation

$$x^3+y^3+z^3+3xyz=0.$$

Solution: To find $\partial z/\partial x$, we differentiate implicitly with respect to x and carefully treating y as constant.

$$\begin{aligned}
3x^2+0+3z^2\frac{\partial z}{\partial x}+3yz+3xy\frac{\partial z}{\partial x} &= 0\\
x^2+z^2\frac{\partial z}{\partial x}+yz+xy\frac{\partial z}{\partial x} &= 0\\
(z^2+xy)\frac{\partial z}{\partial x} &= -x^2-yz\\
\frac{\partial z}{\partial x} &= -\frac{x^2+yz}{z^2+xy}.
\end{aligned}$$

Similarly, implicit differentiation with respect to y gives

$$\frac{\partial z}{\partial y}=-\frac{y^2+xz}{z^2+xy}.$$

11.1.2 Higher-Order Partial Derivatives

Higher-order derivatives are denoted by the order in which the differentiation occurs. For instance, the function $z=f(x,y)$ has the following partial derivatives.

1. Differentiate twice with respect to x:
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}=f_{xx}$$
2. Differentiate twice with respect to y:
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}=f_{yy}$$
3. Differentiate first with respect to x, then with respect to y:
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\,\partial x}=f_{xy}$$

4. Differentiate first with respect to y, then with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$$

The third and the fourth cases are called **mixed partial derivatives**.

■ **Example 11.5** Find all second partial derivatives of the function

$$f(x,y) = 2x^2y - 3y + 3x^2y^2.$$

and determine the value of $f_{xy}(1,2)$.

Solution: Solving for the first partial derivatives, we get

$$f_x(x,y) = 4xy + 6xy^2 \quad \text{and} \quad f_y(x,y) = 2x^2 - 3 + 6x^2y.$$

Now, differentiate each of these with respect to x and y, we obtain

$$f_{xx}(x,y) = 4y - 6y^2 \quad \text{and} \quad f_{xy}(x,y) = 4x + 12xy$$

$$f_{yx}(x,y) = 4x + 12xy \quad \text{and} \quad f_{yy}(x,y) = 6x^2$$

Finally, $f_{xy}(1,2) = 4(1) + 12(1)(2) = 28$.

11.2 The Chain Rule

In your previous calculus course, you have learned Chain Rule. That is: If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then

$$\frac{dy}{dt} = \frac{dy}{dx} + \frac{dx}{dt}.$$

For functions of more than one variable, the Chain Rule has several versions, each of them giving a rule for differentiating a composite function.

The Chain Rule (Case 1)

Suppose $z = f(x,y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Usually, we write $\partial z/\partial x$ in place of $\partial f/\partial x$ and wer can rewrite the chain rule as

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Example 11.6 Let $z = x^2y - y^2$, where $x = \sin t$ and $y = e^t$. Find dz/dt when $t = 0$.

Solution: By the chain rule,

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\\ &= 2xy(\cos t) + (x^2 - 2y)e^t\\ &= 2(\sin t)(e^t)(\cos t) + (\sin^2 t - 2e^t)e^t\\ &= 2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}.\end{aligned}$$

When $t = 0$,

$$\left.\frac{dz}{dt}\right|_{t=0} = \left(2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}\right)\Big|_{t=0} = -2.$$

The Chain Rule (Case 2)

Suppose that $z = f(x,y)$ is a differentiable function of x and y, where $x = g(s,t)$ and $y = h(s,t)$ are both differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Example 11.7 If $z = e^x \sin y$, where $x = st^2$ and $y = s^2t$, find $\partial z/\partial s$ and $\partial z/\partial t$.

Solution: Using the chain rule (case 2), we have

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y)(2st)\\ &= t^2 e^{st^2}\sin(s^2t) + 2ste^{st^2}\cos(s^2t)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2)\\ &= 2ste^{st^2} + s^2 e^{st^2}\cos(s^2 t).\end{aligned}$$

11.2.1 Implicit Differentiation

The implicit differentiation that we used to know may be completely described by the chain rule.

Suppose that y is a differntiable function of x, then applying implicit differentiation to the equation $x^3 + y^3 = 6xy$ to find dy/dx, we have

$$\begin{aligned} x^3 + y^3 &= 6xy\\ 3x^2 + 3y^2\frac{dy}{dx} &= 6y + 6x\frac{dy}{dx}\\ (3y^2 - 6x)\frac{dy}{dx} &= 6y - 3x^2\\ \frac{dy}{dx} &= \frac{6y - 3x^2}{3y^2 - 6x}\\ \frac{dy}{dx} &= \frac{2y - x^2}{y^2 - 2x}.\end{aligned}$$

The next theorem will give us an alternative formula in finding derivatives of implicitly defined functions as well as their partial derivatives.

Theorem 11.2.1 (Chain Rule: Implicit Differentiation)
If the equation $F(x,y) = 0$ defines y implicitly as a differentiable function of x, then

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)}, \quad F_y(x,y) \neq 0$$

If the equation $F(x,y,z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)} \quad F_z(x,y,z) \neq 0.$$

■ **Example 11.8** Find dy/dx if $x^3 + y^3 = 6xy$.

Solution: This is already solved above. Let us confirm this using the theorem. First, we rewrite the equation as $x^3+y^3-6xy=0$. So,

$$F(x,y)=x^3+y^3-6xy.$$

Then

$$F_x(x,y)=3x^2-6y \text{ and } F_y(x,y)=3y^2-6x.$$

Thus,

$$\frac{dy}{dx}=-\frac{F_x(x,y)}{F_y(x,y)}=-\frac{3x^2-6y}{3y^2-6x}=-\frac{x^2-2y}{y^2-2x}.$$

■ **Example 11.9** Find $\partial z/\partial x$ and $\partial z/\partial y$ for

$$3x^2z-x^2y^2+2z^3+3yz-5=0.$$

Solution: Let

$$f(x,y,z)=3x^2z-x^2y^2+2z^3+3yz-5.$$

Then

$$F_x(x,y,z)=6xz-2xy^2, \qquad F_y(x,y,z)=-2x^2y$$

and

$$F_z(x,y,z)=3x^2+6z^2+3y.$$

Using the theorem, we have

$$\frac{\partial z}{\partial x}=-\frac{F_x(x,y,z)}{F_z(x,y,z)}=\frac{2x^2y-6xz}{3x^2+6z^2+3y}$$

and

$$\frac{\partial z}{\partial y}=-\frac{F_y(x,y,z)}{F_z(x,y,z)}=\frac{2x^2y-3z}{3x^2+6z^2+3y}.$$

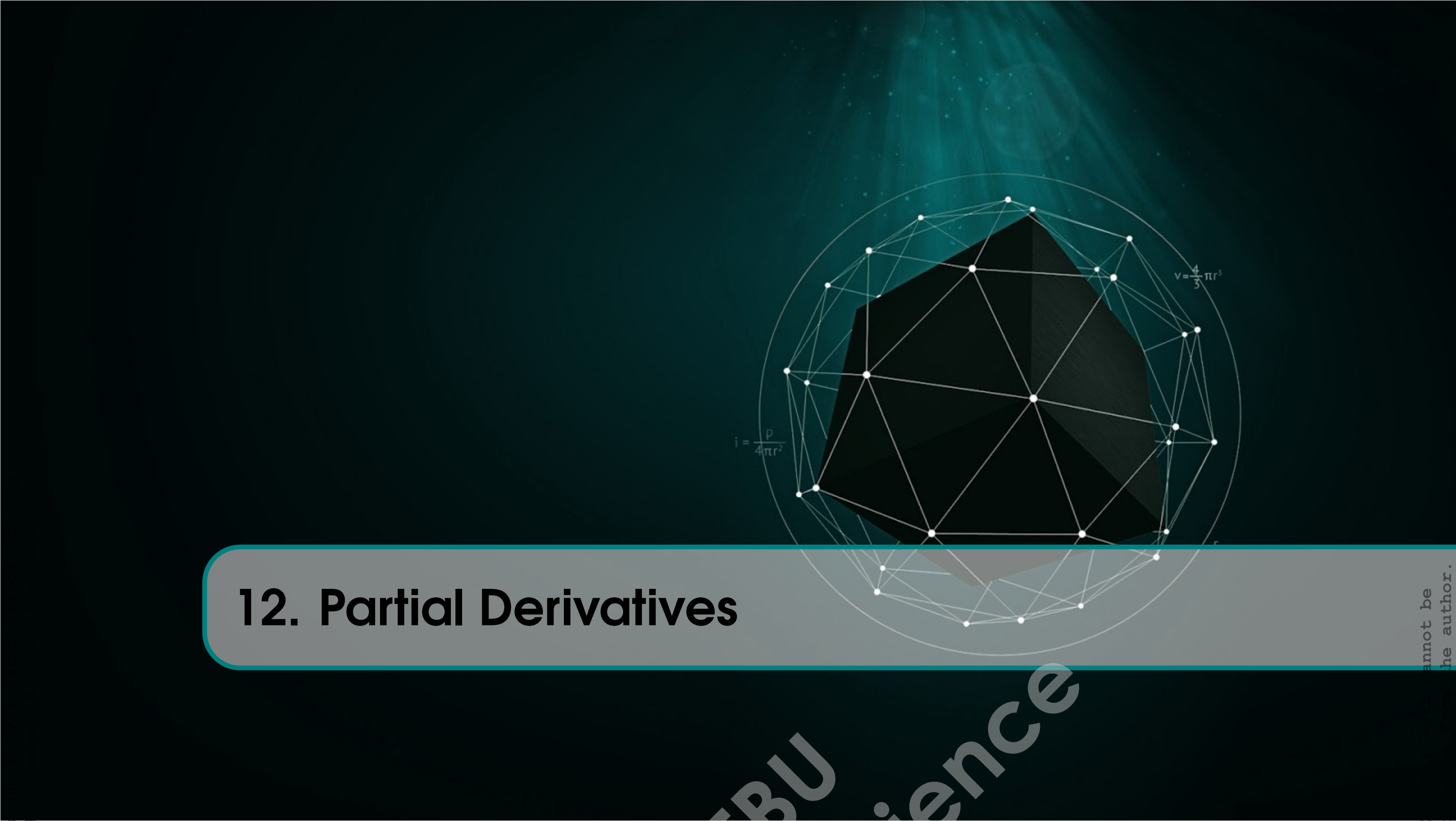
11.3 Exercises

Exercise 11.1 Do as directed.

1. Find the first partial derivatives of the following functions
   (a) $f(x,y) = x^4 + 5xy^3$
   (b) $z = \ln(x + t^2)$
   (c) $f(x,y) = \dfrac{x}{y}$
2. Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$ of the following.
   (a) $x^2 + 2y^2 + 3z^2 = 1$
   (b) $yz + x\ln y = z^2$

XII

Week Twelve



12. Partial Derivatives

At the end of this topic you are expected to:

- find the critical point of functions of several variables
- use the partial derivative tests in finding extrema of functions

12.1 Maximum and Minimum Values of Functions

We will extend our techniques in finding extrema of functions in two variables to functions in three variables. An analogous approach will be dealt.

Theorem 12.1.1 (Extreme Value Theorem)
Let f be a continuous function of two variables x and y defined on a closed and bounded region R in the xy-plane

1. There is at least one point in R at which f takes on a minimum value.
2. There is at least one point in R at which f takes on a maximum value.

Definition 12.1.1 (Relative Extrema)

Let f be a function defined on a region R containing (x_0, y_0).

1. The function f has a **relative mimimum** at (x_0, y_0) if $f(x,y) \geq f(x_0, y_0)$ for all (x,y) in the open disk containing (x_0, y_0).
2. The function f has a **relative maximum** at (x_0, y_0) if $f(x,y) \leq f(x_0, y_0)$ for all (x,y) in the open disk containing (x_0, y_0).

Definition 12.1.2 (Critical Point)

Let f be a function defined on a region R containing (x_0, y_0). The point (x_0, y_0) is a **critical point** of f if one of the following is true.

1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$
2. $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Theorem 12.1.2 (Relative Extrema Occur Only at Critical Points)

If f has a relative extremum at (x_0, y_0) on an open region R, then (x_0, y_0) is a critical point of f.

■ **Example 12.1** Find all relative extrema of the function

$$f(x,y) = x^2 + y^2 - 2x - 6y + 14.$$

Solution: Let us find first f_x and f_y. In our case,

$$f_x(x,y) = 2x - 2 \text{ and } f_y(x,y) = 2x - 6.$$

Setting $f_x(x,y) = 0$ and $f_y(x,y) = 0$, and solving for x and y, we obtain $x = 1$ and $y = 3$. Hence, $(1,3)$ is the only critical point. Moreover, $f(1,3) = 4$.

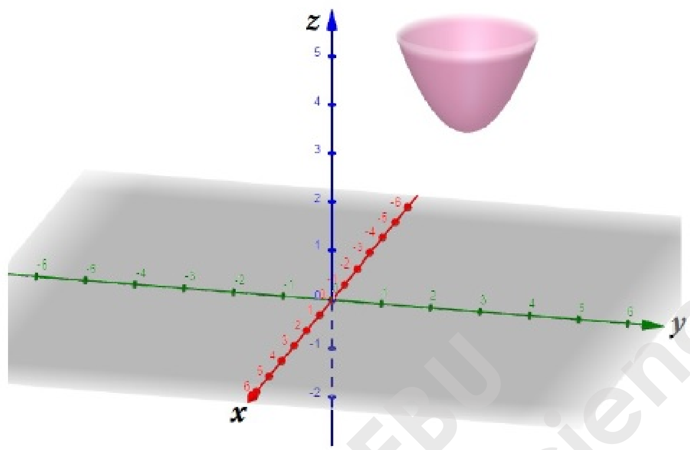
This time we have no tools yet to confirm whether this point corresponds to a relative maximum or a relative minimum. But we can still manage to confirm it by completing the square.

We can rewrite the given function as

$$f(x,y) = 4 + (x-1)^2 + (y-3)^2.$$

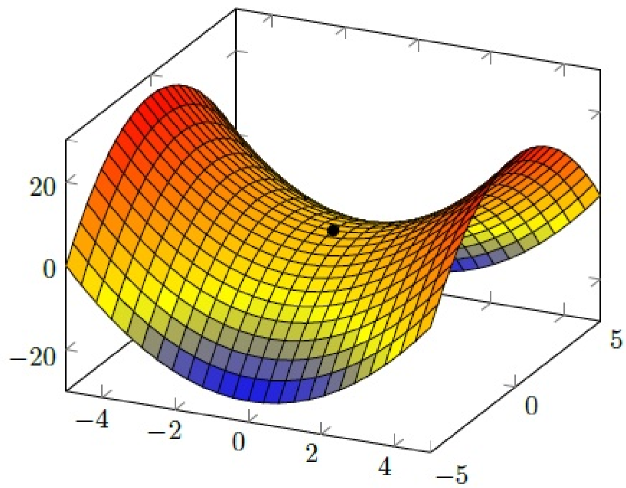
Note that $(x-1)^2 \geq 0$ and $(y-3)^2 \geq 0$. Hence, $f(x,y) \geq 4$ for all x and y.

Therefore, $f(1,3) = 4$ is a relative minimum. In fact it is the absolute minimum as confirmed in the graph below.



An elliptic paraboloid, minimum at the vertex $(1,3,4)$

Just like for functions in one variable, a critical point of a function of two variables do not always yield relative maxima or minima. There are times when a critical point yield a **saddle point**, which is neither relative maximum or relative minimum. The graph of the surface below is the hyperbolic paraboloid $z = x^2 - y^2$ with saddle point indicated by a black dot. A shape similar to this is the "pringles" potato chip.



A hyperbolic paraboloid with saddle point $(0,0,0)$-black dot

The next theorem will give us the answer on how do we identify critical points in relation to extrema.

Theorem 12.1.3 (Second Partials Tests)

Suppose that f have continuous second partial derivatives on an open region containing a point (a,b) for which

$$f_x(x,y) = 0 \quad \text{and} \quad f_y(a,b) = 0.$$

To test for relative extrema of f, consider the quantity

$$d = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2.$$

1. If $d > 0$ and $f_{xx}(a,b) > 0$, then f has a relative minimum at (a,b).
2. If $d > 0$ and $f_{xx}(a,b) < 0$, then then f has a relative maximum at (a,b).
3. If $d < 0$, then $(a,b,f(a,b))$ is a saddle point.
4. The test is inconclusive if $d = 0$.

Example 12.2 Find the relative extrema of $f(x,y) = -x^3 + 4xy - 2y^2 + 1$.

Solution: We will find first the critical points. Taking the first partials,

$$f_x(x,y) = -3x^2 + 4y \text{ and } f_y(x,y) = 4x - 4y$$

and setting them to 0 we have,

$$-3x^2 + 4y = 0 \text{ and } 4x - 4y = 0$$

From the second equation, we have $x = y$. Substituting $x = y$ to the first equation, we have $-3x^2 + 4x = 0$ or $-x(3x - 4) = 0$ which means that $x = 0$ or $x = 4/3$.

Since $x = y$, the points $(0,0)$ and $(4/3, 4/3)$ are critical points. Taking the second partials, we have

$$f_{xx}(x,y) = -6x, \quad f_{yy}(x,y) = -4, \quad \text{and} \quad f_{xy}(x,y) = 4$$

For the critical point $(0,0)$,

$$d = f_{xx}(0,0) f_{yy}(0,0) - [f_{xy}(0,0)]^2 = 0 - 16 < 0.$$

Thus, by the Second Partials Test, we can conclude that $(0,0,1)$ is a saddle point of f.

For the critical point $(4/3, 4/3)$, we have

$$d = f_{xx}(4/3,4/3) f_{yy}(4/3,4/3) - [f_{xy}(4/3,4/3)]^2 = -8(-4) - 16 = 16 > 0.$$

and since $f_{xx}(\frac{4}{3}, \frac{4}{3}) = -8$ by the Second Partials Test, we can conclude that $(\frac{4}{3}, \frac{4}{3}, \frac{59}{27})$ is a relative maximum of f.

Example 12.3 A rectangular box with no lid is to be made from $12\,\mathrm{m}^2$ cardboard. Find the maximum volume of such box.

Solution: Let the length, width and height of the box be represented by the variables x, y, and z respectively. Then the volume of the box is

$$V = xyz.$$

The total surface area of the box is $2xz + 2yz + xy$. Hence,

$$2xz + 2yz + xy = 12$$

Solving for z in $2xz+2yz+xy=12$, we have

$$z=\frac{12-xy}{2(x+y)}$$

Thus, we can write V as

$$V=xy\frac{12-xy}{2(x+y)}=\frac{12xy-x^2y^2}{2(x+y)}$$

Taking the partial derivatives of V, we obtain

$$V_x=\frac{y^2(12-2xy-x^2)}{2(x+y)^2} \quad \text{and} \quad V_y=\frac{x^2(12-2xy-y^2)}{2(x+y)^2}$$

One possible critical point is $(0,0)$ beacause $x=0$ and $y=0$ makes both partial derivatives 0. However, this would also mean that $V=0$ which we do not want.

The other point that would make V_x and V_y equal to zero is requiring that

$$12-2xy-x^2=0 \quad \text{and} \quad 12-2xy-y^2=0$$

From the second equation we have $2xy=12-y^2$ and substituting to the first equation we obtain $y^2-x^2=0$ which means that $(y-x)(y+x)=0$. Hence, $x=y$ or $x=-y$. We reject $x=-y$ since x and y are all positive (lengths).

If $x=y$, then from $12-2xy-x^2=0$, we have $12-2x^2-x^2=0$ or $12-3x^2=0$ for which $x=2$ and it follows that $y=2$.

Solving for z, we have $z=\dfrac{12-2(2)}{2(2+2)}=1$.

We can use the second partial derivatives test to verify that the point $(2,2)$ will give us the maximum.

Therefore, the maximum volume that can be obtained is $V=2\cdot 2\cdot 1=4\,\text{m}^3$.

12.2 Exercises

Exercise 12.1 Do as directed.

1. Identify any extrema of the function by recognizing its given form or its form after completing the square.
   (a) $f(x,y)=(x-1)^2+(y-3)^2$
   (b) $f(x,y)=x^2+y^2+2x-6y+6$
2. Find all relative extrema and saddle points of the function. Use the Second Partials Test where applicable.

(a) $f(x,y) = x^2 + y^2 + 8x - 12y - 3$

(b) $f(x,y) = x^2 - y^2 - x - y$

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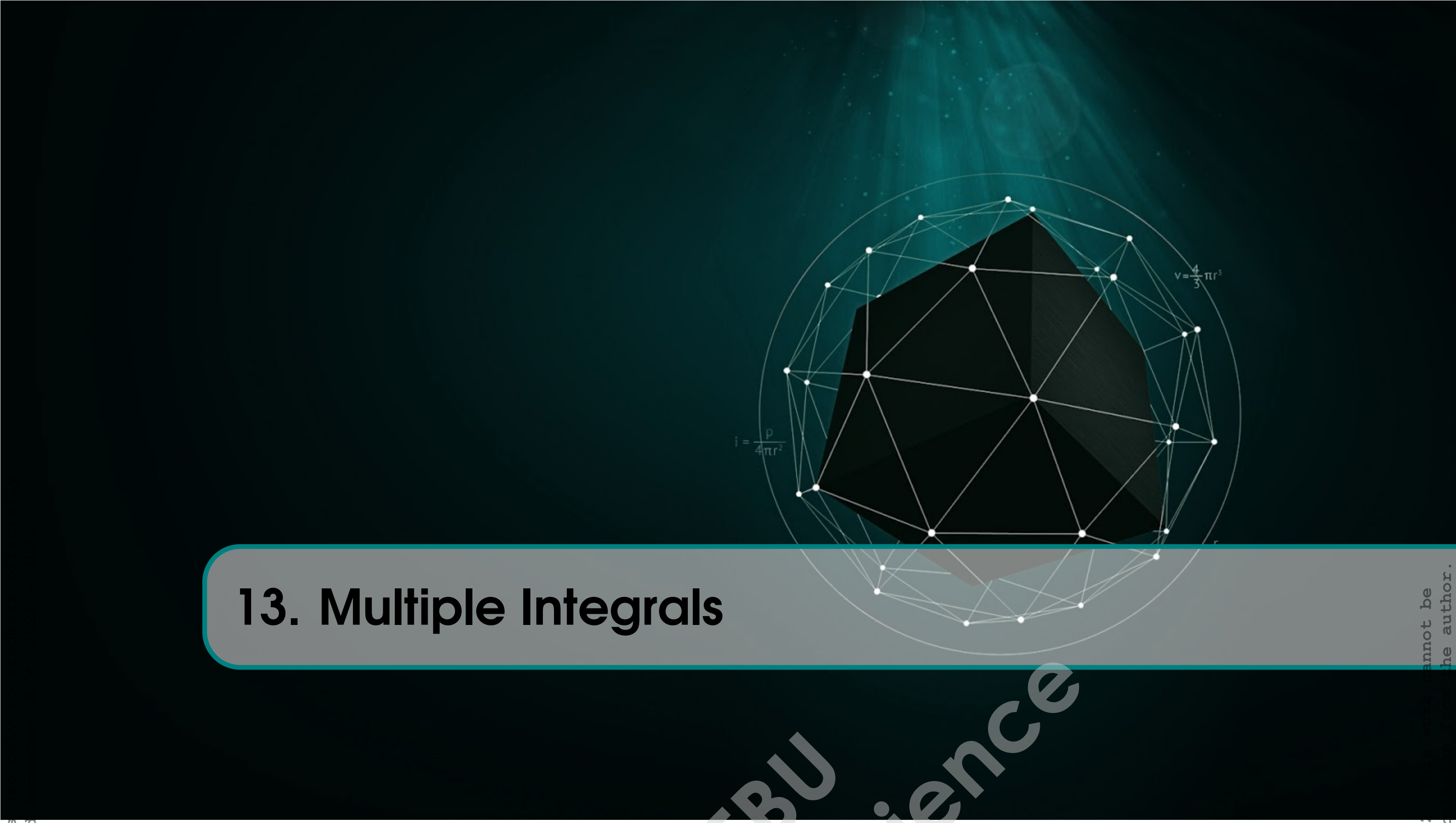
Week Thirteen



13. Multiple Integrals

At the end of this topic you are expected to:

- evaluate multiple integrals
- evaluate double integrals over rectangles

13.1 Double Integrals Over Rectangles

For integrals of functions of a single variable, we begin by a continuous function f defined on a closed interval $[a,b]$ and then dividing this interval into n subintervals, each with length $\Delta x = (b-a)/n$. Then we form the Riemann sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

which is just the sum of the areas of n rectangles. Next, we take the limit of the above sum as $n \to \infty$ and we obtain

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx$$

which is called the definite integral of f evaluated from a to b and is interpreted as the area of the region under the curve f, above the x-axis and between the lines $x = a$ and $x = b$.

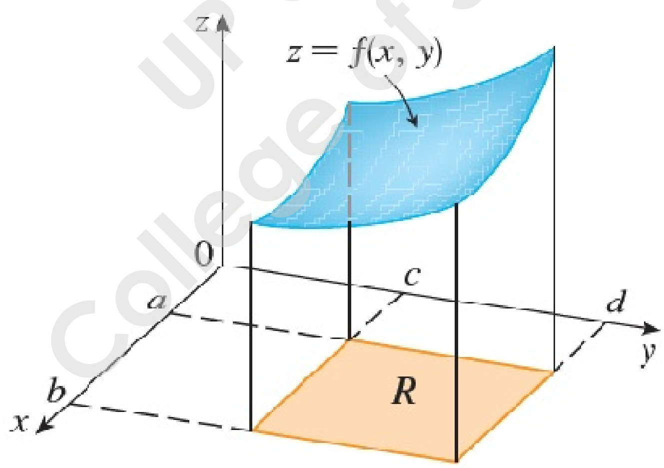
13.1.1 Volumes and Double Integrals

A similar approach for functions in two variables will be considered in obtainingthe volume of a solid.

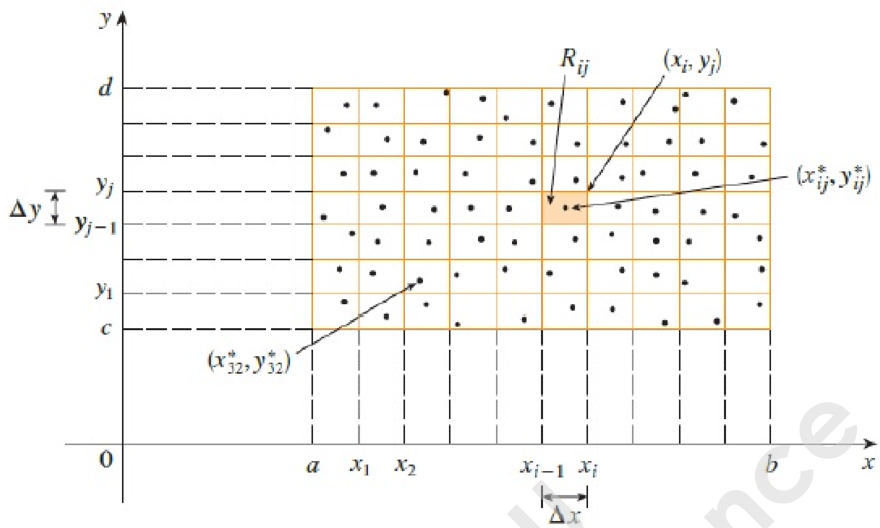
Let f be a function defined on a a closed rectangle

$$R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b, c \leq y \leq d\}$$

with $f(x,y) \geq 0$. Let S be the solid above the region R and under the surface $z = f(x,y)$ as shown in the figure below

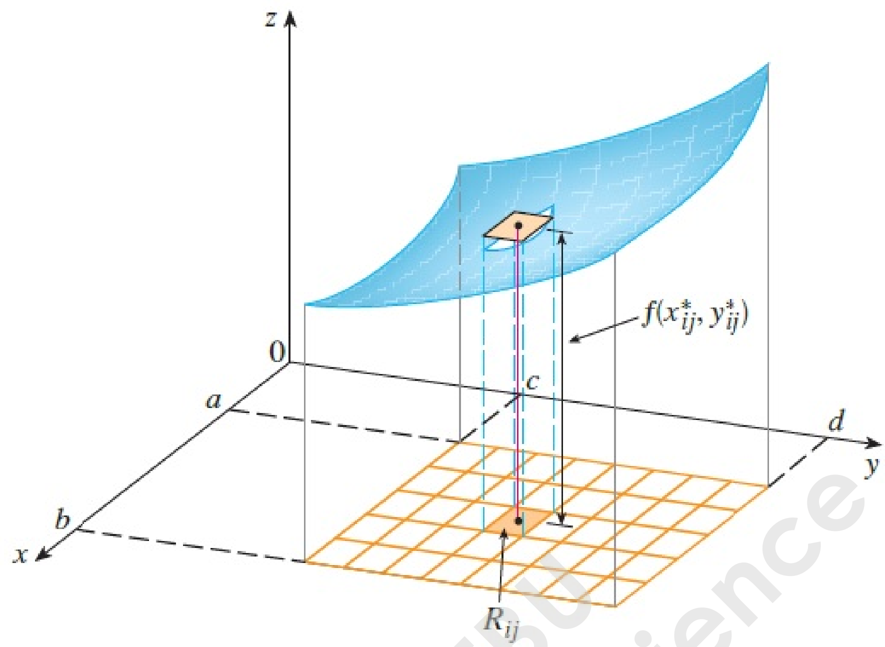


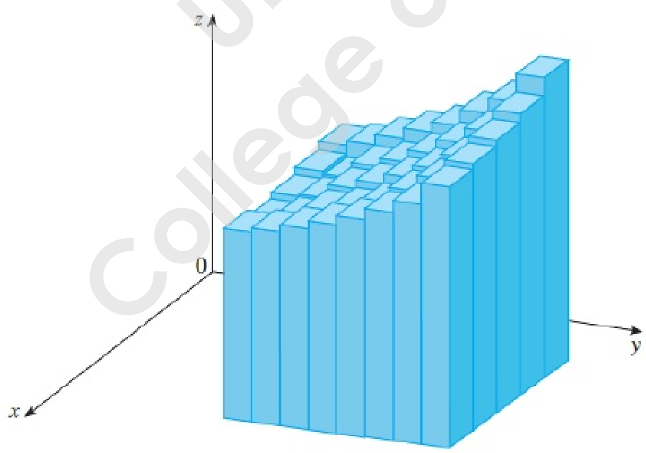
Thus, $S = \{(x,y,z) \in \mathbb{R}^3 : 0 \leq z \leq f(x,y), (x,y) \in R\}$. Then we subdivide R into subrectangles with width (along x-axis) $\Delta x = (b-a)/m$ and width (along y-axis) $\Delta y = (c-d)/n$ as seen in the figue below.



The shaded rectangle in figure above is called R_{ij} or the rectangle in the ith row and jth column. The area of this rectangle is $\Delta A = \Delta x \Delta y$. Suppose we pick a **sample point** (x_{ij}^*, y_{ij}^*) on R_{ij} . Then we can construct a small box whose base area is ΔA and whose height is $f(x_{ij}^*, y_{ij}^*)$ as shown in the figure below.

Similarly, for any other reactngles there corresponds a rectangular box after picking a sample point. This $m \times n$ rectangles will approximate the volume of the solid as illustared in the blue blocks below.





If we add the volumes of these rectangular solid, we have an approximate volume for S, that is

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \Delta A.$$

As m and n get larger, we have

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

Definition 13.1.1 (Double Integral)

The double integral of f over the reactangle R is

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

if this limit exists.

Theorem 13.1.1 (Properties of Double Integrals)

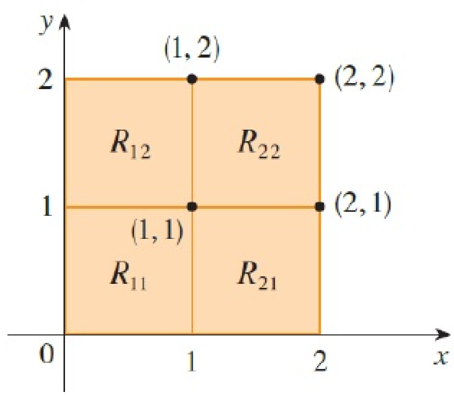
Let f and g be continuous over a closed, bounded plane region R, and let c be a constant. Then

1. $\displaystyle\iint_R c\,f(x,y)\,dA = c\iint_R f(x,y)\,dA$
2. $\displaystyle\iint_R [f(x,y) \pm g(x,y)]\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$
3. $\displaystyle\iint_R f(x,y)\,dA \geq 0, \quad \text{if} \quad f(x,y) \geq 0$
4. $\displaystyle\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA, \quad \text{if } f(x,y) \geq g(x,y)$
5. $\displaystyle\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$

   where R is the union of two nonoverlapping subregions R_1 and R_2.

■ **Example 13.1** Estimate the volume of the solid that lies above the square $R = [0,2]\times[0,2]$ and below $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square.

Solution: We will subdivide the square into four equal regions



The area of each square is $\Delta A = 1$. Approximating the volume, we have

$$\begin{aligned} V &\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A \\ &= f(1,1)\,\Delta A + f(1,2)\,\Delta A + f(2,1)\,\Delta A + f(2,2)\,\Delta A \\ &= 13(1)+7(1)+10(1)+4(1) = 34. \end{aligned}$$

Therfore, $V \approx 34$ cubic units.

13.1.2 The Midpoint Rule

The methods that we used for approximating single integrals (the Midpoint Rule, the Trapezoidal Rule, Simpson's Rule) all have counterparts for double integrals. Here we consider only the Midpoint Rule for double integrals. This means that we use a double Riemann sum to approximate the double integral, where the sample point (x_{ij}^*, y_{ij}^*) in R_{ij} is chosen to be the center $(\bar{x}_i, \bar{y}_j)$. In other words, $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_i$ is the midpoint of $[y_{i-1}, y_i]$.

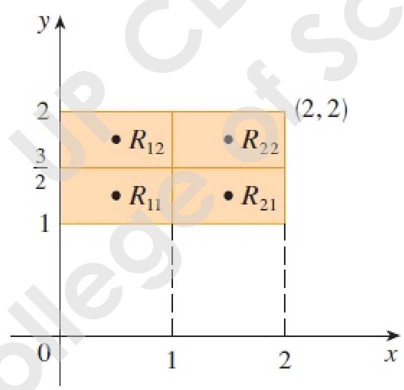
Midpoint Rule for Double Integrals

$$\iint_R f(x,y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\overline{x}_i, \overline{y}_j)\Delta A$$

where $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_i$ is the midpoint of $[y_{i-1}, y_i]$.

■ **Example 13.2** Use midpoint rule for double integrals with $n = m = 2$ to estimate the value of the integral $\iint_R (x - 3y^2)\, dA$, where $R = \{(x,y)\,|\, 0 \le x \le 2,\ 1 \le y \le 2\}$.

Solution: In using the midpoint rule with $m = n = 2$, we evaluate $f(x,y) = x - 3y^2$ at the centers of the four subrectangles as shown below.



So, $\overline{x}_1 = \frac{1}{2}, \overline{x}_2 = \frac{3}{2}, \overline{y}_1 = \frac{5}{4}$, and $\overline{y}_2 = \frac{7}{4}$. Thus, the desired midpoints are $(\frac{1}{2}, \frac{5}{4}), (\frac{1}{2}, \frac{7}{4}), (\frac{3}{2}, \frac{5}{4})$

and $(\frac{3}{2}, \frac{7}{4})$. The area of each subrectangle is $\Delta A = \frac{1}{2}$. Hence,

$$\begin{aligned}
\iint_R (x - 3y^2)\, dA &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\bar{x}_i, \bar{y}_j)\Delta A \\
&= f(\bar{x}_1, \bar{y}_1)\,\Delta A + f(\bar{x}_1, \bar{y}_2)\,\Delta A + f(\bar{x}_2, \bar{y}_1)\,\Delta A + f(\bar{x}_2, \bar{y}_2)\,\Delta A \\
&= f\left(\frac{1}{2}, \frac{5}{4}\right)\Delta A + f\left(\frac{1}{2}, \frac{7}{4}\right)\Delta A + f\left(\frac{3}{2}, \frac{5}{4}\right)\Delta A + f\left(\frac{3}{2}, \frac{7}{4}\right)\Delta A \\
&= \left(-\frac{67}{16}\right)\frac{1}{2} + \left(-\frac{139}{16}\right)\frac{1}{2} + \left(-\frac{51}{16}\right)\frac{1}{2} + \left(-\frac{123}{16}\right)\frac{1}{2} \\
&= -\frac{95}{8}
\end{aligned}$$

Therefore, $\iint_R (x - 3y^2)\, dA \approx -11.875.$

13.2 Exercises

Exercise 13.1 Do as directed.

1. Estimate the volume of the solid that lies below the surface $z = xy$ and above the rectangle $R = \{(x, y) \,|\, 0 \leq x \leq 6,\ 0 \leq y \leq 4\}$
   (a) using Riemann sum with $m = 3$, $n = 2$, and take the sample point to be the upper right corner of each square
   (b) using midpoint rule with same values of m and n in problem (a).

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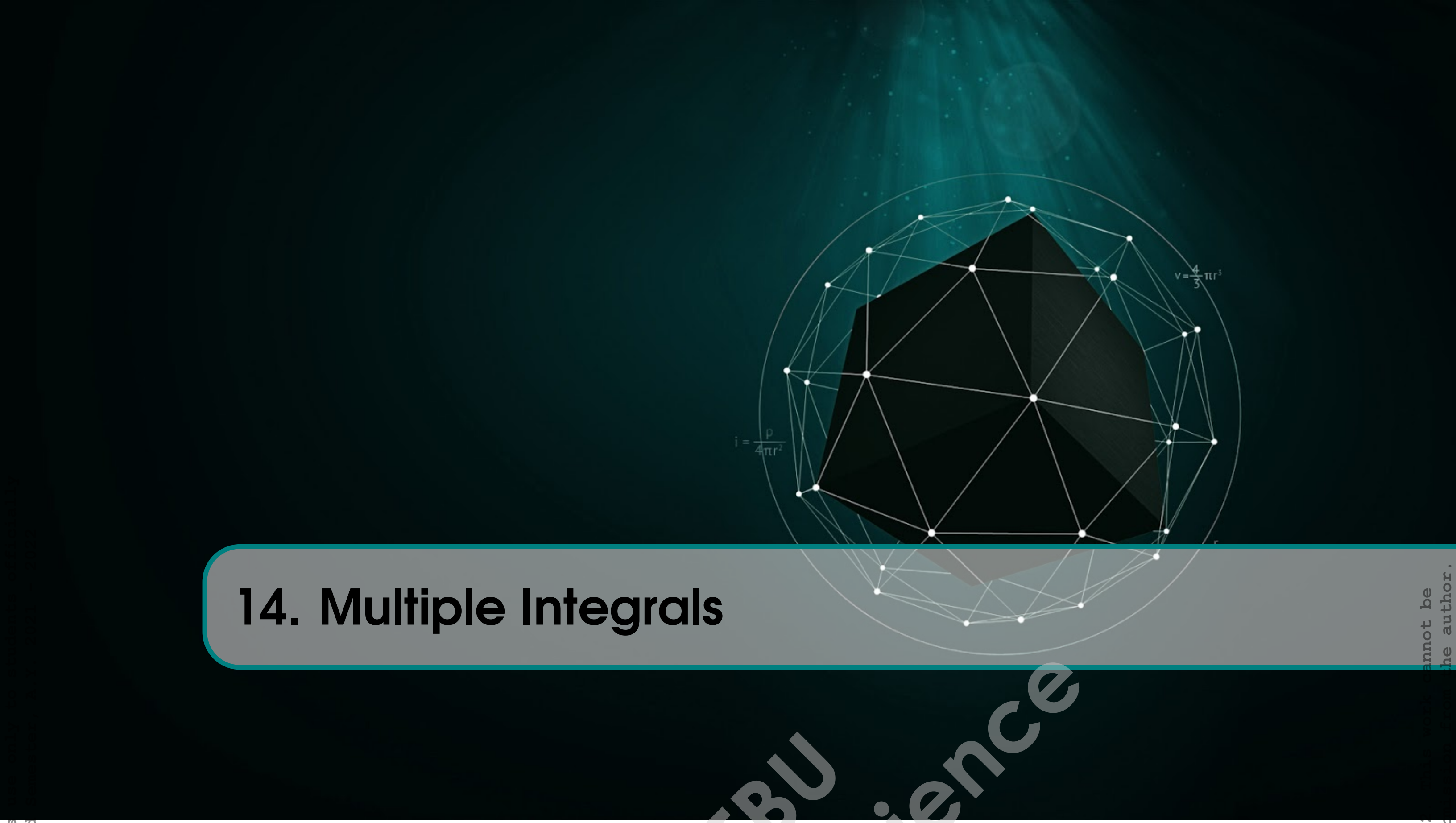
Week Fourteen



14. Multiple Integrals

At the end of this topic you are expected to:

- evaluate iterated integrals
- find the area of plane regions using iterated integrals

14.1 Iterated Integrals

Suppose that f is a function of two variables that is integrable on the rectangle $R = [a,b] \times [c,d]$. We use the notation

$$\int f(x,y)\,dx$$

to mean that $f(x,y)$ is integrated with respect to x and the variable y is held fixed. This is also called partial integration with respect to x.

■ **Example 14.1** Find $\displaystyle\int 2xy\,dx$

Solution:

$$\begin{aligned}\int 2xy\,dx &= 2y\int x\,dx\\ &= 2y\left(\frac{x^2}{2}\right)+C(y)\\ &= x^2y+C(y)\end{aligned}$$

where $C(y)$ is a function of y.

Now if $\int_c^d f(x,y)\,dy$, a number that depends on the value of x, so it defines a function of x, say,

$$A(x)=\int_c^d f(x,y)\,dy.$$

If we integrate A with respect to x from $x=a$ to $x=b$, we get

$$\int_a^b A(x)\,dx=\int_a^b\left[\int_c^d f(x,y)\,dy\right]\,dx.$$

The integral on the right side of the equation is called an **iterated integral**. Most of the times the brackets are omitted.

Example 14.2 Evaluate the following integrals

1. $\displaystyle\int_0^3\int_1^2 x^2y\,dy\,dx$ 2. $\displaystyle\int_1^2\int_0^3 x^2y\,dx\,dy$

Solution:

1. $$\begin{aligned}\int_0^3\int_1^2 x^2y\,dy\,dx &= \int_0^3\left[\int_1^2 x^2y\,dy\right]\,dx=\int_0^3\left[x^2\frac{y^2}{2}\right]_{y=1}^{y=2}\,dx\\ &= \int_0^3\left[x^2\left(\frac{2^2}{2}\right)-x^2\left(\frac{1^2}{2}\right)\right]\,dx\\ &= \int_0^3\frac{3}{2}x^2\,dx\\ &= \left[\frac{x^3}{2}\right]_0^3=\frac{27}{2}.\end{aligned}$$

2. $$\int_1^2 \int_0^3 x^2 y\,dx\,dy = \int_1^2 \left[\int_1^2 x^2 y\,dx\right]\,dy = \int_0^3 \left[\frac{x^3}{3}y\right]_{x=0}^{x=3}\,dy$$
$$= \int_1^2 \left[\left(\frac{3^3}{3}\right)y - \left(\frac{0^3}{3}\right)y\right]\,dy$$
$$= \int_1^2 9y\,dy$$
$$= \left[\frac{9}{2}y^2\right]_1^2 = \frac{27}{2}.$$

Notice that in the example above, we obtained the same result. In general, two iterated integrals are equal, that is, the order of integration doesn't matter whether you are integrating with respect to x or y first.

The following theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral (in either order).

Theorem 14.1.1 (Fubini's Theorem)
If f is continuous on the rectangle $R = \{(x,y)\,|\,a \leq x \leq b,\ c \leq y \leq d\}$, then
$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

■ **Example 14.3** Evaluate the double integral $\iint_R (x-3y^2)\,dA$, where R is the region given by $R = \{(x,y)\,|\,0 \leq x \leq 2,\ 1 \leq y \leq 2\}$.

Solution: Fubini's Theorem gives
$$\iint_R (x-3y^2)\,dA = \int_0^2 \int_1^2 (x-3y^2)\,dy\,dx$$
$$= \int_0^2 \left[xy - y^3\right]_{y=1}^{y=2}\,dx$$
$$= \int_0^2 (x-7)\,dx$$
$$= \left(\frac{x^2}{2} - 7x\right)\Big|_0^2 = -12.$$

Note that you can also use the other way and arrived at the same value, that is

$$\iint_R (x-3y^2)\, dA = \int_1^2 \int_0^2 (x-3y^2)\, dx\, dy = -12.$$

Notice the negative answer in the example above; nothing is wrong with that. The function f is not a positive function, so its integral doesn't represent a volume. Here, f is always negative on R, so the value of the integral is the negative of the volume that lies above the graph of f and below R.

■ **Example 14.4** Evaluate $\displaystyle\iint_R y\sin(xy)\, dA$, where $R = [1,2]\times[0,\pi]$

Solution: If we first integrate with respect to x, we get

$$\begin{aligned}
\iint_R y\sin(xy)\, dA &= \int_0^{\pi}\int_1^2 y\sin(xy)\, dx\, dy \\
&= \int_0^{\pi} \big[-\cos(xy)\big]_{x=1}^{x=2}\, dy \\
&= \int_0^{\pi} (-\cos 2y + \cos y)\, dy \\
&= \left(-\frac{1}{2}\sin 2y + \sin\, y\right)\Bigg|_0^{\pi} \\
&= 0.
\end{aligned}$$

For a function f that takes on both positive and negative values, $\iint_R y\sin(xy)\, dA$ is a difference of volumes: V_1 and V_2, where V_1 is the volume above R and below the graph of f, and V_2 is the volume below R and above the graph. The fact that the integral in Example 14.4 is 0 means that these two volumes V_1 and V_2 are equal.

If we reverse the order of integration and first integrate with respect to y in Example 14.4, we get

$$\iint_R y\sin(xy)\, dA = \int_1^2 \int_0^{\pi} y\sin(xy)\, dy\, dx$$

but this order of integration is much more difficult than the method given in the example because it involves integration by parts twice. Therefore, when we evaluate double integrals it is wise to choose the order of integration that gives simpler integrals.

In the special case where $f(x,y)$ can be factored as the product of a function of x only and a function of y only, the double integral of f can be written in a particularly simple form. To be specific, suppose that $f(x,y) = g(x)h(y)$ and $R = [a,b] \times [c,d]$. Then Fubini's Theorem gives

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b g(x)\,h(y)\,dx\,dy = \int_c^d \left[\int_a^b g(x)\,h(y)\,dx\right] dy$$

In the inner integral, y is a constant, so $h(y)$ is a constant and we can write

$$\int_c^d \left[\int_a^b g(x)\,h(y)\,dx\right] dy = \int_c^d \left[h(y)\left(\int_a^b g(x)\,dx\right)\right] dy = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

Therefore, in this case the double integral of f can be written as the product of two single integrals:

$$\iint_R g(x)\,h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy, \quad \text{where } R = [a,b] \times [c,d]$$

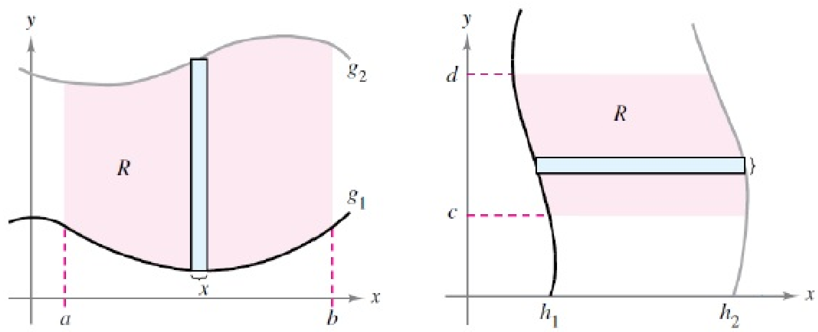
Example 14.5

$$\begin{aligned}
\int_0^3 \int_1^2 x^2 y\,dy\,dx &= \int_0^3 x^2\,dx \int_1^2 y\,dy \\
&= \left(\frac{1}{3}x^3\right)\Bigg|_0^3 \cdot \left(\frac{1}{2}y^2\right)\Bigg|_1^2 \\
&= (9-0)\cdot\left(2-\frac{1}{2}\right) \\
&= 9\cdot\frac{3}{2} \\
&= \frac{27}{2}.
\end{aligned}$$

Iterated integrals can also be used in finding the area of plane regions. In your previous calculus course, you have learned that the area between two curves can be obtained by integration. Particularly, if R is the region between continuous functions $f(x)$ and $g(x)$ such that $g(x) \le R \le f(x)$ from $x = a$ to $x = b$, then the area A of region R is given by

$$A = \int_a^b [f(x) - g(x)]\,dx$$

The region $g(x) \le R \le f(x)$ is called vertically simple while $g(y) \le R \le f(y)$ is called horizontally simple. The figures below are simple regions.



14.1.1 Area of a Plane Region

Now, we will deal with area of plane regions but this time we will make use of iterated integrals.

Area of a Plane Region

1. If R is defined by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where g_1 and g_2 are continuous on $[a,b]$, then the area of R is

$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx.$$

2. If R is defined by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, where h_1 and h_2 are continuous on $[c,d]$, then the area of R is

$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy.$$

■ **Example 14.6** Use iterated integrals to find the area of the simple region which is bounded by the graphs of $f(x) = \sin x$ and $g(x) = \cos x$ between $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

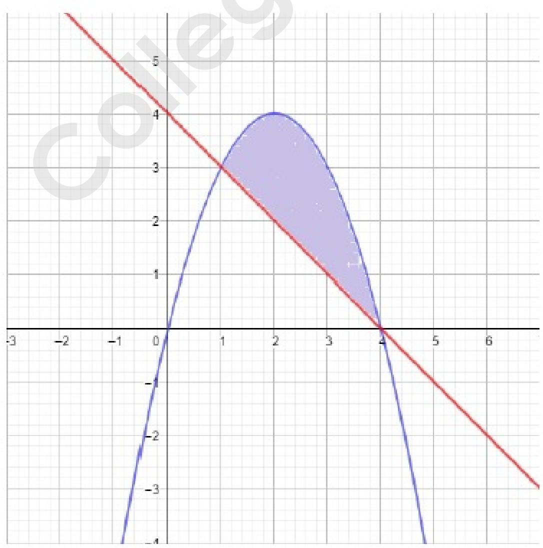
Solution: We can easily verify by sketching that this is indeed a simple region with $f(x) = \sin x$ as the upper curve and $g(x) = \cos x$ as the lower curve. Thus,

$$
\begin{aligned}
A &= \int_{\pi/4}^{5\pi/4} \int_{\cos x}^{\sin x} dy\, dx \\
&= \int_{\pi/4}^{5\pi/4} y \Big|_{\cos x}^{\sin x} dx \\
&= \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx \\
&= (-\cos x - \sin x) \Big|_{\pi/4}^{5\pi/4} \\
&= 2\sqrt{2}.
\end{aligned}
$$

Therefore, the area of the region is $2\sqrt{2}$ square units.

■ **Example 14.7** Use iterated integrals to find the area of the simple region which is bounded by the graphs of $f(x) = 4x - x^2$ and $g(x) = 4 - x$.

Solution: First, we sketch the region



We can see that the region is vertically simple. Thus

$$\begin{aligned}
A &= \int_1^4 \int_{4-x}^{4x-x^2} dy\, dx \\
&= \int_1^4 y\Big|_{4-x}^{4x-x^2} dx \\
&= \int_1^4 [(4-x)-(4x-x^2))]\, dx \\
&= \int_1^4 (-x^2+5x-4)\, dx \\
&= \left(-\frac{1}{3}x^3+\frac{5}{2}x^2-4x\right)\Big|_1^4 \\
&= \frac{9}{2}.
\end{aligned}$$

Therefore, the area of the region is $\frac{9}{2}$ square units.

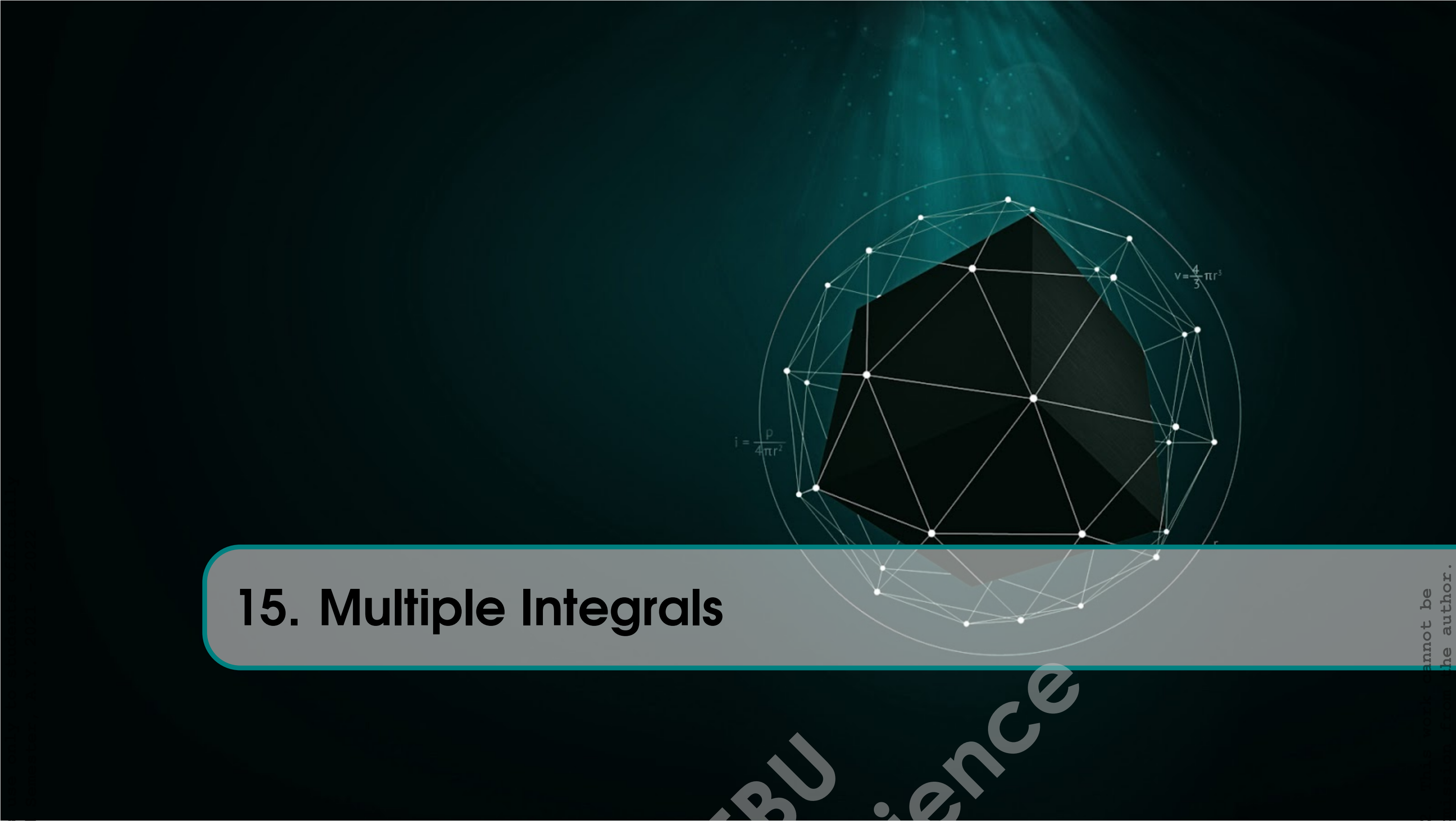
14.2 Exercises

Exercise 14.1 Do as directed.

1. Evaluate the following double integrals.
   (a) $\iint\limits_R (2x+1)\, dA, \quad R = \{(x,y)\,|\,2 \le x \le 6,\ 0 \le y \le 4\}$
   (b) $\iint\limits_R (4-2y)\, dA, \quad R = [0,1]\times[0,1]$
2. Calculate the following iterated integrals.
   (a) $\int_0^1 \int_0^1 (x+y)^2\, dx\, dy$
   (b) $\int_0^{\pi/6} \int_0^{\pi/2} (\sin x + \sin y)\, dy\, dx$
   (c) $\int_1^4 \int_1^2 \left(\frac{x}{y}+\frac{y}{x}\right) dy\, dx$

XV

Week Fifteen



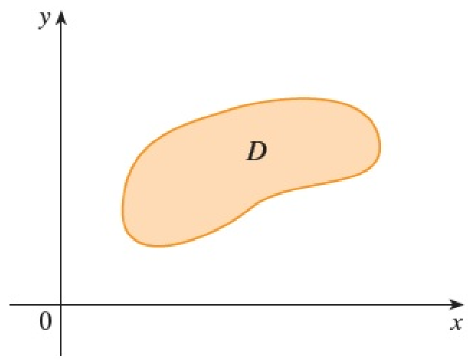
15. Multiple Integrals

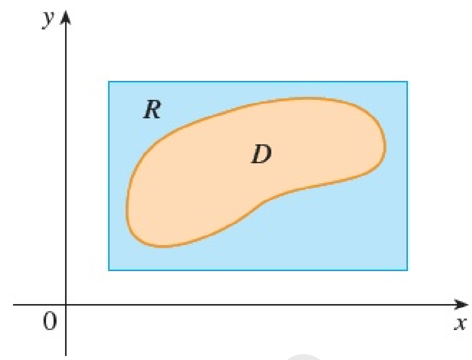
At the end of this topic you are expected to:

- evaluate double integrals over general region
- evaluate double integrals in polar coordinates

15.1 Double Integrals Over General Regions

For single integrals, the region over which we integrate is always an interval. But for double integrals, we want to be able to integrate a function f not just over rectangles but also over regions D of more general shape, such as the one illustrated below





We suppose that D is a bounded region, which means that D can be enclosed in a rectangular region R as shown above (in blue rectangle). Then we define a new function F with domain R by

$$F(x,y=)\begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases}$$

If F is integrable over R, then we define the **double integral of f over D** by

$$\iint_D f(x,y)\, dA = \iint_R F(x,y)\, dA$$

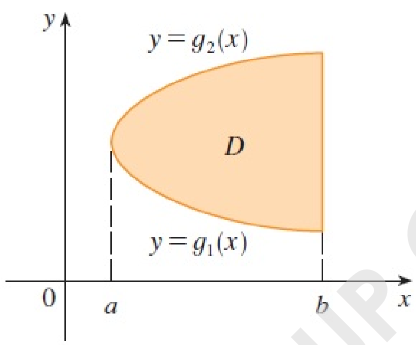
This definition makes sense because R is a rectangle and so $\iint_R F(x,y)\, dA$ has been previously defined. The procedure that we have used is reasonable because the values of $F(x,y)$ are 0 when (x,y) lies outside D and so they contribute nothing to the integral. This means that it doesn't matter what rectangle R we use as long as it contains D.

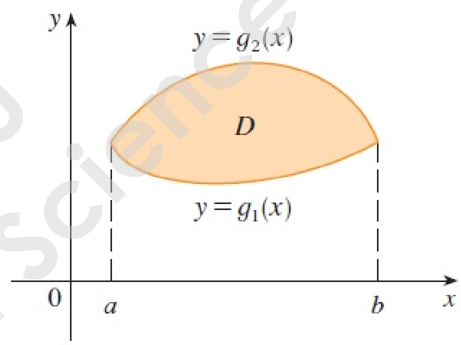
In the case where $f(x,y) \geq 0$, we can still interpret $\iint_D f(x,y)\, dA$ as the volume of the solid that lies above D and under the surface $z = f(x,y)$ (the graph of f).

A plane region D is said to be of **Type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x,y)\,|\,a \le x \le b,\ g_1(x) \le y \le g_2(x)\}.$$

Below are sketches of regions of type I.





Theorem 15.1.1 If f is continuous on a type I region D such that

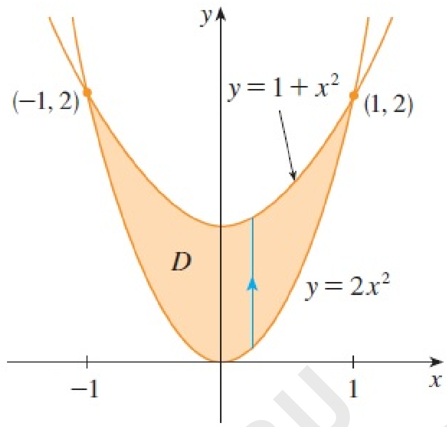
$$D = \{(x,y)\,|\,a \le x \le b,\ g_1(x) \le y \le g_2(x)\},$$

then

$$\iint\limits_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

■ **Example 15.1** Evaluate $\iint\limits_D (x+2y)\,dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Solution: The parabolas intersect when $2x^2 = 1 + x^2$, that is $x^2 = 1$ which means that $x = \pm 1$. Then we sketch region D



We can see that D is a region of type I but not of type II and we can write D as

$$D = \{(x,y) \mid -1 \le x \le 1,\ 2x^2 \le y \le 1 + x^2\}.$$

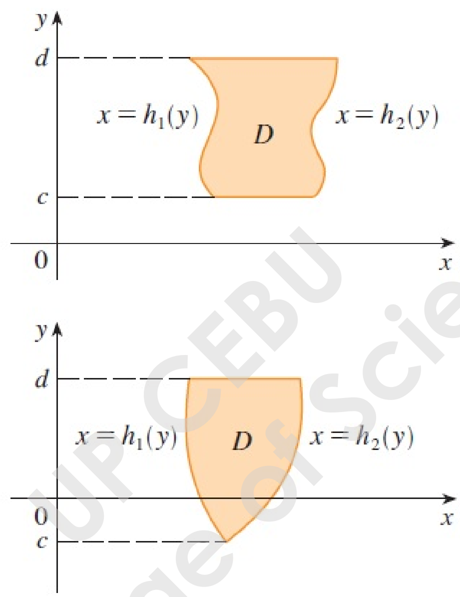
Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$, we have

$$\begin{aligned}
\iint\limits_D f(x,y)\, dA &= \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx \\
&= \int_{-1}^{1} \Big[xy + y^2 \Big]_{y=2x^2}^{y=1+x^2} dx \\
&= \int_{-1}^{1} \Big[x(1+x^2) + (1+x^2)^2 - \Big(x(2x^2) + (2x^2)^2 \Big) \Big] dx \\
&= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx \\
&= \left(-\frac{3}{5}x^5 - \frac{x^4}{4} + \frac{2}{3}x^3 + \frac{x^2}{2} + x \right) \Bigg|_{-1}^{1} \\
&= \frac{32}{15}.
\end{aligned}$$

A plane region D is said to be of **Type II** if it lies between the graphs of two continuous functions of y, that is,

$$D = \{(x,y)\,|\, c \le y \le d,\ h_1(y) \le x \le h_2(y)\}.$$

Below are sketches of regions of type II.



Theorem 15.1.2 If f is continuous on a type II region D such that

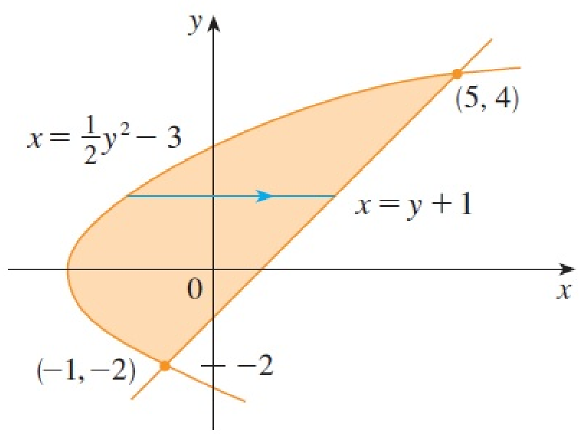
$$D = \{(x,y)\,|\, c \le y \le d,\ h_1(y) \le x \le h_2(y)\},$$

then

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dy\,dx.$$

Example 15.2 Evaluate $\iint_D xy\,dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$

Solution: Let us sketch first the region D.



Here, D is both type I and type II, but the description of D as a type I region is more complicated because the lower boundary consists of two parts. Therefore, we prefer to express D as a type II region:

$$D = \{(x,y) \mid -2 \le y \le 4,\ \tfrac{1}{2}y^2 - 3 \le x \le y+1\}$$

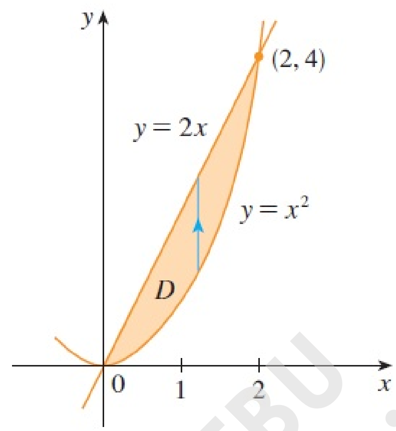
Then

$$\begin{aligned}
\iint_D xy\, dA &= \int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\, dx\, dy = \int_{-2}^{4}\left[\frac{x^2}{2}y\right]_{x=\frac{1}{2}y^2-3}^{x=y+1} dy \\
&= \tfrac{1}{2}\int_{-2}^{4} y\left[(y+1)^2 - (\tfrac{1}{2}y^2-3)^2\right] dy \\
&= \tfrac{1}{2}\int_{-2}^{4}\left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\right) dy \\
&= \frac{1}{2}\left[-\frac{y^6}{24} + y^4 + \frac{2}{3}y^3 - 4y^2\right]_{-2}^{4} \\
&= 36.
\end{aligned}$$

▪ **Example 15.3** Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Solution: (D as region of type I)

Sketching the D, we have



Here, our D can be described as

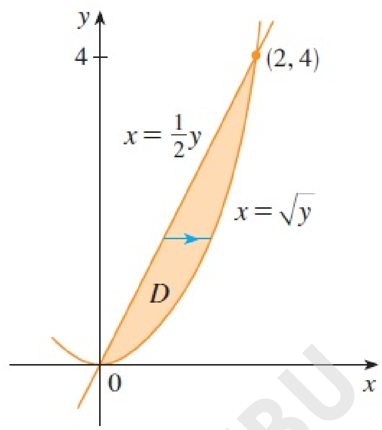
$$D = \{(x,y) \,|\, 0 \leq x \leq 2,\ x^2 \leq y \leq 2x\}.$$

Therefore, the volume under $z = x^2 + y^2$ and above D is

$$\begin{aligned}
V = \iint_D (x^2+y^2)\,dA &= \int_0^2 \int_{x^2}^{2x} (x^2+y^2)\,dy\,dx \\
&= \int_0^2 \left[x^2y + \frac{y^3}{3}\right]_{y=x^2}^{y=2x} dx \\
&= \int_0^2 \left[x^2(2x) + \frac{(2x)^3}{3} - x^2x^2 - \frac{(x^2)^3}{3}\right] dx \\
&= \int_0^2 \left(-\frac{1}{3}x^6 - x^4 + \frac{14}{3}x^3\right) dx \\
&= \left(-\frac{1}{21}x^7 - \frac{1}{5}x^5 + \frac{7}{6}x^4\right)\Bigg|_0^2 \\
&= \frac{216}{35}.
\end{aligned}$$

Solution: (D as region of type II)

Sketching the D, we have



Here, our D can be described as

$$D = \{(x,y)\,|\,0 \le y \le 4,\ \tfrac{1}{2}y \le x \le \sqrt{y}\}.$$

Therefore another expression for V is

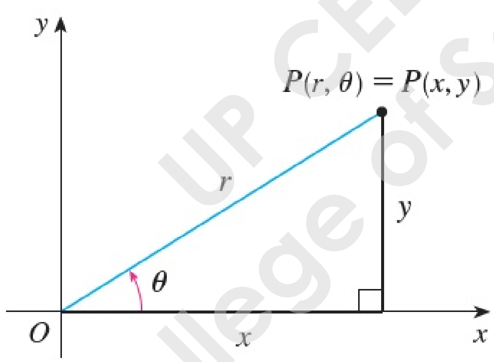
$$\begin{aligned} V = \iint_D (x^2+y^2)\,dA &= \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} (x^2+y^2)\,dx\,dy \\ &= \int_0^4 \left[\frac{x^3}{3} + y^2 x\right]_{x=\frac{1}{2}y}^{x=\sqrt{y}} dy \\ &= \int_0^4 \left(\frac{1}{3}y^{3/2} + y^{5/2} - \frac{1}{24} - \frac{1}{2}y^3\right) dy \\ &= \left(\frac{2}{15}y^{5/2} + \frac{2}{7}y^{7/2} - \frac{13}{96}y^4\right)\Bigg|_0^4 \\ &= \frac{216}{35}. \end{aligned}$$

15.2 Double Integrals in Polar Coordinates

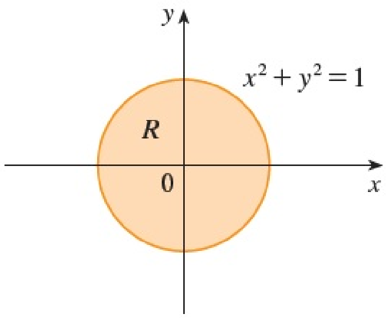
Suppose that we want to evaluate a double integral $\iint_R f(x,y)\,dA$ over a circular region or a region between two concentric circles. In either case the description of R in terms of rectangular coordinates is rather complicated, but R is easily described using polar coordinates.

Recall that the polar coordinates (r,θ) of a point are related to the rectangular coordinates (x,y) by the equations

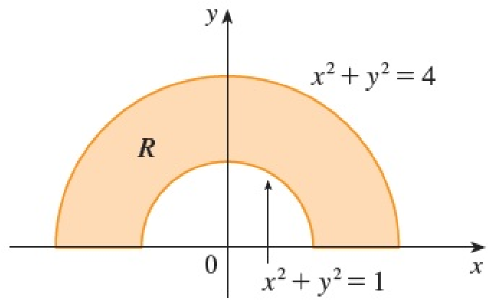
$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad x = r\sin\theta$$



For instance the circle whose rectangular equation is $x^2 + y^2 = 4$ is equivalent to $r = 2$ in polar coordinates while the circle centered at $(0,2)$ with radius 2 has equation $x^2 + (y-2)^2 = 4$ is equivalent to $r = 4\sin\theta$ in polar coordinates.



(a) $R=\{(r,\theta)\mid 0\leqslant r\leqslant 1, 0\leqslant\theta\leqslant 2\pi\}$

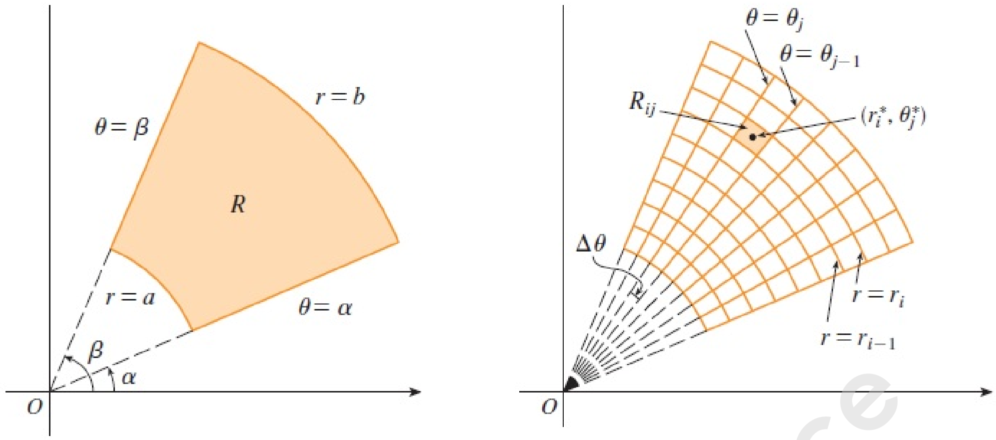


(b) $R=\{(r,\theta)\mid 1\leqslant r\leqslant 2, 0\leqslant\theta\leqslant \pi\}$

The regions shown above are special cases of a **polar rectangle**

$$R=\{(r,\theta)\,|\,a\leq r\leq b,\ \alpha\leq\theta\leq\beta\}$$

In order to compute the double integral $\iint_R f(x,y)\,dA$, where R is a polar rectangle, we divide the interval $[a,b]$ into m subintervals $[r_{i-1},r_i]$ of equal width $\Delta r=(b-a)/m$ and we divide the interval $[\alpha,\beta]$ into n subintervals $[\theta_{i-1},\theta_i]$ of equal width $\Delta\theta=(\beta-\alpha)/n$. Then the circles $r=r_i$ and the rays $\theta=\theta_j$ divide the polar rectangle R into the small polar rectangles R_{ij} shown below.



The "center" of the polar subrectangle

$$R_{ij} = \{(r, \theta) \,|\, r_{i-1} \leq r \leq r_i,\ \theta_{j-1} \leq \theta \leq \theta_j\}$$

has polar coordinates

$$r_i^* = \tfrac{1}{2}(r_{i-1} + r_i) \qquad \theta_j^* = \tfrac{1}{2}(\theta_{i-1} + \theta_i)$$

We compute the area of R_{ij} using the fact that the area of a sector of a circle with radius r and central angle θ is $\frac{1}{2}r^2\theta$. Subtracting the areas of two such sectors, each of which has central angle $\Delta\theta = \theta_j - \theta_{j-1}$, we find that the area of R_{ij} is

$$\begin{aligned}\Delta A_i &= \tfrac{1}{2}r_i^2\Delta\theta - \tfrac{1}{2}r_{i-1}^2\Delta\theta = \tfrac{1}{2}(r_i^2 - r_{i-1}^2)\,\Delta\theta \\ &= \tfrac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta = r_i^*\Delta r\,\Delta\theta.\end{aligned}$$

The rectangular coordinates of the center of of R_{ij} are $(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)$, so the typical Riemann sum is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\,\Delta A_i = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\,r_i^*\Delta r\,\Delta\theta$$

If we write $g(r,\theta) = rf(r\cos\theta\, r\sin\theta)$, then the above Riemann sum can be written as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*)\,\Delta r\,\Delta\theta$$

which is a Riemann sum for the double integral

$$\int_\alpha^\beta \int_a^b g(r,\theta)\,dr\,d\theta.$$

Therefore, we have the following theorem.

Theorem 15.2.1 Change to Polar Coodinates in a Double Integral

If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

■ **Example 15.4** Evaluate $\iint_R (3x+4y^2)\,dA$, where R is the is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Solution: The region R is clearly half-ring and in polar coordinates it is given by $1 \le r \le 2$, $0 \le \theta \le \pi$. Then by Theorem 15.2.1, we have

$$\begin{aligned}
\iint_R (3x+4y^2)\,dA &= \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\,r\,dr\,d\theta \\
&= \int_0^\pi \int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\,dr\,d\theta \\
&= \int_0^\pi \left[r^3\cos\theta + r^4\sin^2\theta\right]_{r=1}^{r=2} d\theta \\
&= \int_0^\pi (7\cos\theta + 15\sin^2\theta)\,d\theta \\
&= \int_0^\pi \left[7\cos\theta + \tfrac{15}{2}(1-\cos 2\theta)\right] d\theta \\
&= \left(7\sin\theta + \frac{15}{2}\theta - \frac{15}{4}\sin 2\theta\right)\Big|_0^\pi \\
&= \frac{15}{2}\pi.
\end{aligned}$$

What we have done so far can be extended to the more complicated type of region. It's similar to the type II rectangular regions that we have considered before. Thus, the next theorem will give us the following formula.

Theorem 15.2.2 If f is continuous on a polar region D of the form

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}.$$

Then

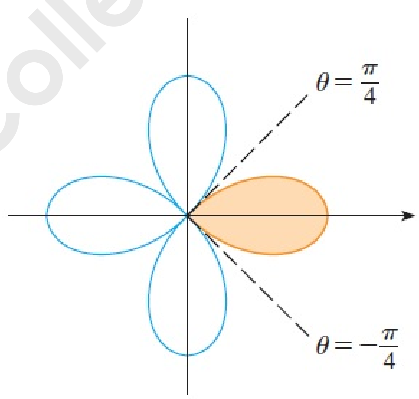
$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

In particular, taking $f(x,y) = 1$, $h_1(\theta) = 0$, and $h_2(\theta) = h(\theta)$ in this formula, we see that the area of region D bounded by $\theta = \alpha$, $\theta = \beta$, and $r = h(\theta)$ is

$$A(D) = \iint_D 1\, dA = \int_\alpha^\beta \int_0^{h(\theta)} r\, dr\, d\theta.$$

Example 15.5 Use double integral to find the area of the region enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

Solution: The graph of $r = \cos 2\theta$ is shown below.



Here, we see that

$$D = \{(r,\theta) \mid 0 \le r \le 2\cos 2\theta,\ -\pi/4 \le \theta \le \pi/4\}.$$

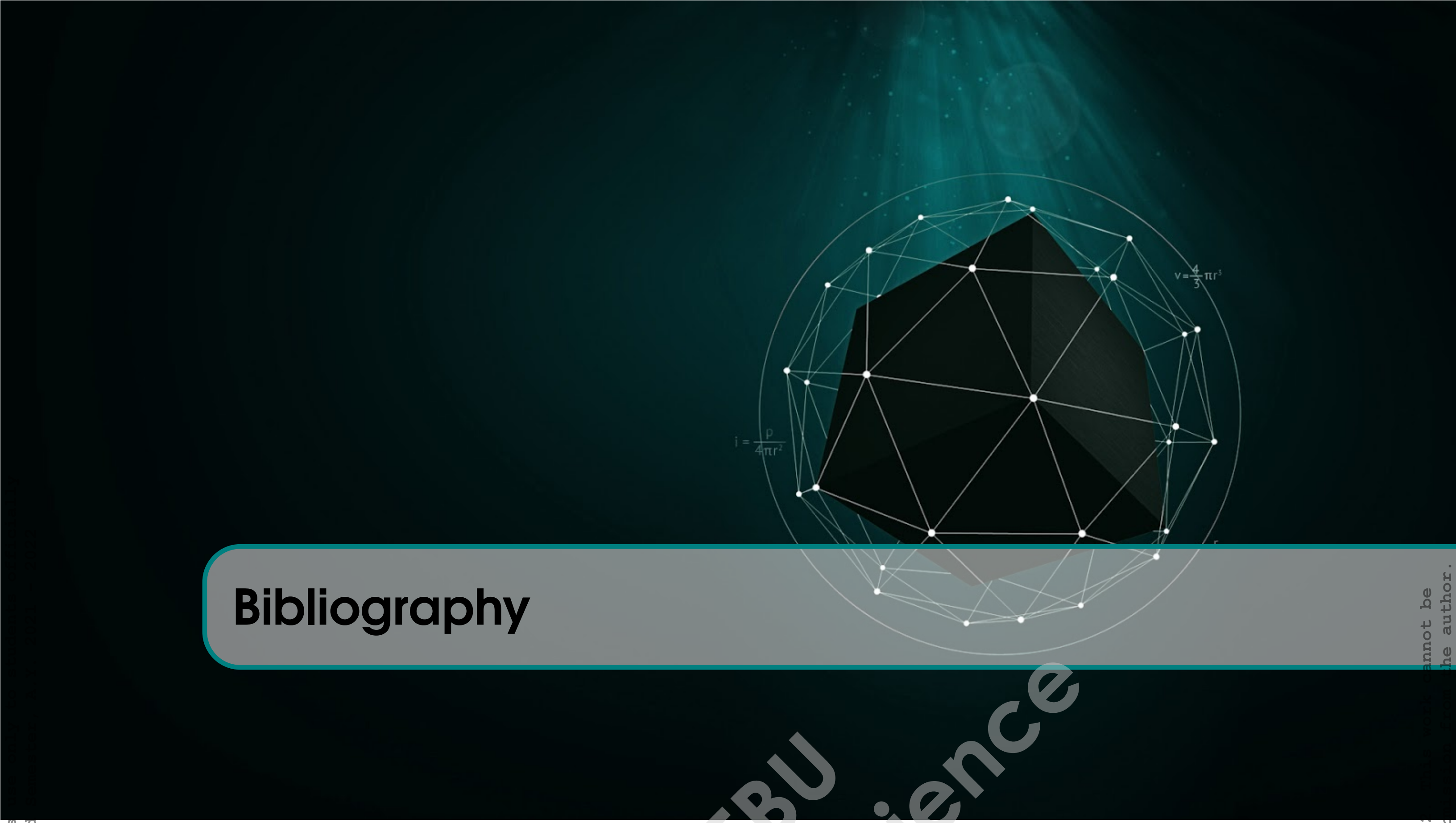
So the area of one loop is denoted by $A(D)$ is

$$\begin{aligned}
A(D) = \iint_D dA &= \int_{-\pi/4}^{\pi/4} \int_0^{\cos\theta} r\,dr\,d\theta \\
&= \int_{-\pi/4}^{\pi/4} \left[\frac{1}{2}r^2\right]_0^{\cos\theta} d\theta \\
&= \tfrac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta\,d\theta \\
&= \tfrac{1}{4}\int_{-\pi/4}^{\pi/4} (1+\cos 4\theta)\,d\theta \\
&= \frac{1}{4}\left[\theta + \frac{1}{4}\sin 4\theta\right]_{-\pi/4}^{\pi/4} \\
&= \frac{\pi}{8}.
\end{aligned}$$

15.3 Exercises

Exercise 15.1 Do as directed.

1. Evaluate the following double integrals.
   (a) $\iint_D \frac{y}{x^2+1}\,dA, \quad D = \{(x,y) \mid 0 \le x \le 4,\ 0 \le y \le \sqrt{x}\}$
   (b) $\iint_D (x^2+2y)\,dA, \quad D$ is bounded by $y = x,\ y = x^3,\ x \ge 0$
2. Evaluate the given integral by changing to polar coordinates.
   (a) $\iint_D (x^2y)\,dA,$ where D is the top half of the disk with center the at the origin and radius 5.
   (b) $\iint_R (2x-y)\,dA,$ where R is the region in the first quadrant enclosed by the circle $x^2+y^2=4$ and the lines $x=0$, $y=x$



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