

# Counting

## Chapter 6

With Question/Answer Animations

# Chapter Summary

- The Basics of Counting
- The Pigeonhole Principle
- Permutations and Combinations
- Binomial Coefficients and Identities
- Generalized Permutations and Combinations

# The Basics of Counting

Section 6.1



# Section Summary

- The Product Rule
- The Sum Rule
- The Subtraction Rule
- The Division Rule
- Examples, Examples, and Examples
- Tree Diagrams

# Basic Counting Principles: The Product Rule

**The Product Rule:** A procedure can be broken down into a sequence of two tasks. There are  $n_1$  ways to do the first task and  $n_2$  ways to do the second task. Then there are  $n_1 \cdot n_2$  ways to do the procedure.

**Example:** How many bit strings of length seven are there?

**Solution:** Since each of the seven bits is either a 0 or a 1, the answer is  $2^7 = 128$ .

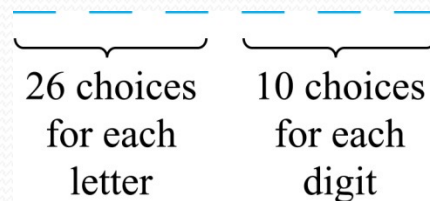


# The Product Rule

**Example:** How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

**Solution:** By the product rule,

there are  $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000$  different possible license plates.



# Counting Functions

**Counting Functions:** How many functions are there from a set with  $m$  elements to a set with  $n$  elements?

**Solution:** Since a function represents a choice of one of the  $n$  elements of the codomain for each of the  $m$  elements in the domain, the product rule tells us that there are  $n \cdot n \cdots n = n^m$  such functions.

**Counting One-to-One Functions:** How many one-to-one functions are there from a set with  $m$  elements to one with  $n$  elements?

**Solution:** Suppose the elements in the domain are  $a_1, a_2, \dots, a_m$ . There are  $n$  ways to choose the value of  $a_1$  and  $n-1$  ways to choose  $a_2$ , etc. The product rule tells us that there are  $n(n-1)(n-2)\cdots(n-m+1)$  such functions.



# Telephone Numbering Plan

**Example:** The *North American numbering plan* (NANP) specifies that a telephone number consists of 10 digits, consisting of a three-digit area code, a three-digit office code, and a four-digit station code. There are some restrictions on the digits.

- Let  $X$  denote a digit from 0 through 9.
- Let  $N$  denote a digit from 2 through 9.
- Let  $Y$  denote a digit that is 0 or 1.
- In the old plan (in use in the 1960s) the format was  $NYX\text{-}NNX\text{-}XXXX$ .
- In the new plan, the format is  $NXX\text{-}NXX\text{-}XXXX$ .

How many different telephone numbers are possible under the old plan and the new plan?

**Solution:** Use the Product Rule.

- There are  $8 \cdot 2 \cdot 10 = 160$  area codes with the format  $NYX$ .
- There are  $8 \cdot 10 \cdot 10 = 800$  area codes with the format  $NXX$ .
- There are  $8 \cdot 8 \cdot 10 = 640$  office codes with the format  $NNX$ .
- There are  $10 \cdot 10 \cdot 10 \cdot 10 = 10,000$  station codes with the format  $XXXX$ .

Number of old plan telephone numbers:  $160 \cdot 640 \cdot 10,000 = 1,024,000,000$ .

Number of new plan telephone numbers:  $800 \cdot 800 \cdot 10,000 = 6,400,000,000$ .



# Counting Subsets of a Finite Set

**Counting Subsets of a Finite Set:** Use the product rule to show that the number of different subsets of a finite set  $S$  is  $2^{|S|}$ . (*In Section 5.1, mathematical induction was used to prove this same result.*)

**Solution:** When the elements of  $S$  are listed in an arbitrary order, there is a one-to-one correspondence between subsets of  $S$  and bit strings of length  $|S|$ . When the  $i$ th element is in the subset, the bit string has a 1 in the  $i$ th position and a 0 otherwise.

By the product rule, there are  $2^{|S|}$  such bit strings, and therefore  $2^{|S|}$  subsets.

# Product Rule in Terms of Sets

- If  $A_1, A_2, \dots, A_m$  are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set.
- The task of choosing an element in the Cartesian product  $A_1 \times A_2 \times \dots \times A_m$  is done by choosing an element in  $A_1$ , an element in  $A_2$ , ..., and an element in  $A_m$ .
- By the product rule, it follows that:  
$$|A_1 \times A_2 \times \dots \times A_m| = |A_1| \cdot |A_2| \cdot \dots \cdot |A_m|.$$



# DNA and Genomes

- A *gene* is a segment of a DNA molecule that encodes a particular protein and the entirety of genetic information of an organism is called its *genome*.
- DNA molecules consist of two strands of blocks known as nucleotides. Each nucleotide is composed of bases: adenine (A), cytosine (C), guanine (G), or thymine (T).
- The DNA of bacteria has between  $10^5$  and  $10^7$  links (one of the four bases). Mammals have between  $10^8$  and  $10^{10}$  links. So, by the product rule there are at least  $4^{10^5}$  different sequences of bases in the DNA of bacteria and  $4^{10^8}$  different sequences of bases in the DNA of mammals.
- The human genome includes approximately 23,000 genes, each with 1,000 or more links.
- Biologists, mathematicians, and computer scientists all work on determining the DNA sequence (genome) of different organisms.

# Basic Counting Principles: The Sum Rule

**The Sum Rule:** If a task can be done either in one of  $n_1$  ways or in one of  $n_2$ , where none of the set of  $n_1$  ways is the same as any of the  $n_2$  ways, then there are  $n_1 + n_2$  ways to do the task.

**Example:** The mathematics department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student.

**Solution:** By the sum rule it follows that there are  $37 + 83 = 120$  possible ways to pick a representative.



# The Sum Rule in terms of sets.

- The sum rule can be phrased in terms of sets.

$|A \cup B| = |A| + |B|$  as long as  $A$  and  $B$  are disjoint sets.

- Or more generally,

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

when  $A_i \cap A_j = \emptyset$  for all  $i, j$ .

- The case where the sets have elements in common will be discussed when we consider the subtraction rule and taken up fully in Chapter 8.

# Combining the Sum and Product Rule

**Example:** Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels.

**Solution:** Use the product rule.

$$26 + 26 \cdot 10 = 286$$



# Counting Passwords

- Combining the sum and product rule allows us to solve more complex problems.

**Example:** Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

**Solution:** Let  $P$  be the total number of passwords, and let  $P_6$ ,  $P_7$ , and  $P_8$  be the passwords of length 6, 7, and 8.

- By the sum rule  $P = P_6 + P_7 + P_8$ .
- To find each of  $P_6$ ,  $P_7$ , and  $P_8$ , we find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters. We find that:

$$P_6 = 36^6 - 26^6 = 2,176,782,336 - 308,915,776 = 1,867,866,560.$$

$$P_7 = 36^7 - 26^7 =$$

$$78,364,164,096 - 8,031,810,176 = 70,332,353,920.$$

$$P_8 = 36^8 - 26^8 =$$

$$2,821,109,907,456 - 208,827,064,576 = 2,612,282,842,880.$$

Consequently,  $P = P_6 + P_7 + P_8 = 2,684,483,063,360$ .

# Internet Addresses

- Version 4 of the Internet Protocol (IPv4) uses 32 bits.

Bit Number	0	1	2	3	4	8	16	24	31		
Class A	0	netid					hostid				
Class B	1	0	netid					hostid			
Class C	1	1	0	netid					hostid		
Class D	1	1	1	0	Multicast Address						
Class E	1	1	1	1	0	Address					

- Class A Addresses:** used for the largest networks, a 0, followed by a 7-bit netid and a 24-bit hostid.
- Class B Addresses:** used for the medium-sized networks, a 10, followed by a 14-bit netid and a 16-bit hostid.
- Class C Addresses:** used for the smallest networks, a 110, followed by a 21-bit netid and a 8-bit hostid.
  - Neither Class D nor Class E addresses are assigned as the address of a computer on the internet. Only Classes A, B, and C are available.
  - 1111111 is not available as the netid of a Class A network.
  - Hostids consisting of all 0s and all 1s are not available in any network.



# Counting Internet Addresses

**Example:** How many different IPv4 addresses are available for computers on the internet?

**Solution:** Use both the sum and the product rule. Let  $x$  be the number of available addresses, and let  $x_A$ ,  $x_B$ , and  $x_C$  denote the number of addresses for the respective classes.

- To find,  $x_A$ :  $2^7 - 1 = 127$  netids.  $2^{24} - 2 = 16,777,214$  hostids.  
 $x_A = 127 \cdot 16,777,214 = 2,130,706,178$ .
- To find,  $x_B$ :  $2^{14} = 16,384$  netids.  $2^{16} - 2 = 16,534$  hostids.  
 $x_B = 16,384 \cdot 16,534 = 1,073,709,056$ .
- To find,  $x_C$ :  $2^{21} = 2,097,152$  netids.  $2^8 - 2 = 254$  hostids.  
 $x_C = 2,097,152 \cdot 254 = 532,676,608$ .
- Hence, the total number of available IPv4 addresses is

$$\begin{aligned}x &= x_A + x_B + x_C \\&= 2,130,706,178 + 1,073,709,056 + 532,676,608 \\&= 3,737,091,842.\end{aligned}$$

Not Enough Today !!

The newer IPv6 protocol solves the problem of too few addresses.

# Basic Counting Principles:

## Subtraction Rule

**Subtraction Rule:** If a task can be done either in one of  $n_1$  ways or in one of  $n_2$  ways, then the total number of ways to do the task is  $n_1 + n_2$  minus the number of ways to do the task that are common to the two different ways.

- Also known as, the *principle of inclusion-exclusion*:

$$|A \cup B| = |A| + |B| - |A \cap B|$$



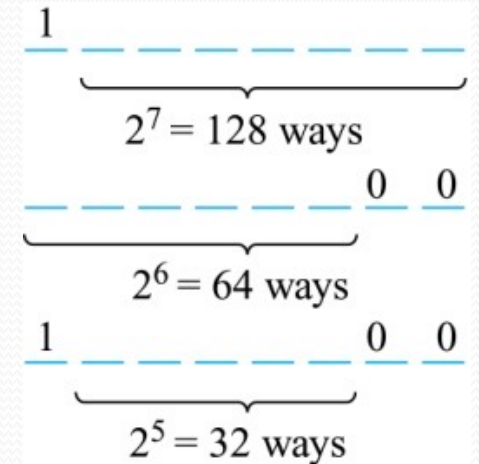
# Counting Bit Strings

**Example:** How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

**Solution:** Use the subtraction rule.

- Number of bit strings of length eight that start with a 1 bit:  $2^7 = 128$
- Number of bit strings of length eight that end with bits 00:  $2^6 = 64$
- Number of bit strings of length eight that start with a 1 bit and end with bits 00 :  $2^5 = 32$

Hence, the number is  $128 + 64 - 32 = 160$ .



# Basic Counting Principles: Division Rule

**Division Rule:** There are  $n/d$  ways to do a task if it can be done using a procedure that can be carried out in  $n$  ways, and for every way  $w$ , exactly  $d$  of the  $n$  ways correspond to way  $w$ .

- Restated in terms of sets: If the finite set  $A$  is the union of  $n$  pairwise disjoint subsets each with  $d$  elements, then  $n = |A|/d$ .
- In terms of functions: If  $f$  is a function from  $A$  to  $B$ , where both are finite sets, and for every value  $y \in B$  there are exactly  $d$  values  $x \in A$  such that  $f(x) = y$ , then  $|B| = |A|/d$ .

**Example:** How many ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left and right neighbor?

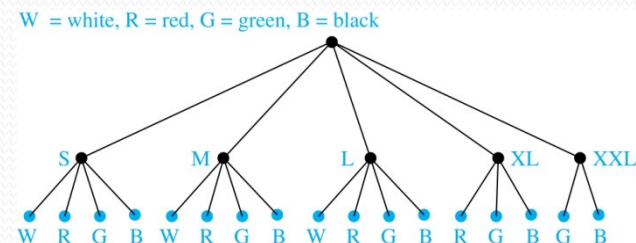
**Solution:** Number the seats around the table from 1 to 4 proceeding clockwise. There are four ways to select the person for seat 1, 3 for seat 2, 2, for seat 3, and one way for seat 4. Thus there are  $4! = 24$  ways to order the four people. But since two seatings are the same when each person has the same left and right neighbor, for every choice for seat 1, we get the same seating.

Therefore, by the division rule, there are  $24/4 = 6$  different seating arrangements.



# Tree Diagrams

- **Tree Diagrams:** We can solve many counting problems through the use of *tree diagrams*, where a branch represents a possible choice and the leaves represent possible outcomes.
- **Example:** Suppose that “I Love Discrete Math” T-shirts come in five different sizes: S,M,L,XL, and XXL. Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black. What is the minimum number of shirts that the campus book store needs to stock to have one of each size and color available?
- **Solution:** Draw the tree diagram.



- The store must stock 17 T-shirts.

# The Pigeonhole Principle

Section 6.2

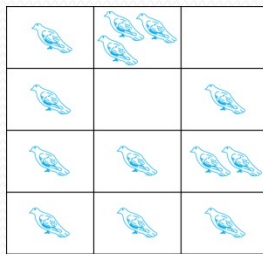


# Section Summary

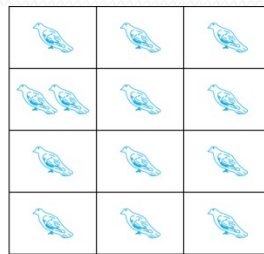
- The Pigeonhole Principle
- The Generalized Pigeonhole Principle

# The Pigeonhole Principle

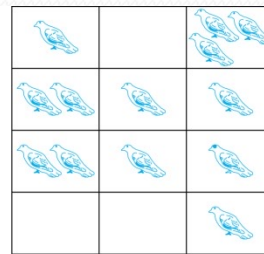
- If a flock of 20 pigeons roosts in a set of 19 pigeonholes, one of the pigeonholes must have more than 1 pigeon.



(a)



(b)



(c)

**Pigeonhole Principle:** If  $k$  is a positive integer and  $k + 1$  objects are placed into  $k$  boxes, then at least one box contains two or more objects.

**Proof:** We use a proof by contraposition. Suppose none of the  $k$  boxes has more than one object. Then the total number of objects would be at most  $k$ . This contradicts the statement that we have  $k + 1$  objects. ◀



# The Pigeonhole Principle

**Corollary 1:** A function  $f$  from a set with  $k + 1$  elements to a set with  $k$  elements is not one-to-one.

**Proof:** Use the pigeonhole principle.

- Create a box for each element  $y$  in the codomain of  $f$ .
- Put in the box for  $y$  all of the elements  $x$  from the domain such that  $f(x) = y$ .
- Because there are  $k + 1$  elements and only  $k$  boxes, at least one box has two or more elements.

Hence,  $f$  can't be one-to-one.



# Pigeonhole Principle

**Example:** Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

**Example (optional):** Show that for every integer  $n$  there is a multiple of  $n$  that has only 0s and 1s in its decimal expansion.

**Solution:** Let  $n$  be a positive integer. Consider the  $n + 1$  integers 1, 11, 111, ..., 11...1 (where the last has  $n + 1$  1s). There are  $n$  possible remainders when an integer is divided by  $n$ . By the pigeonhole principle, when each of the  $n + 1$  integers is divided by  $n$ , at least two must have the same remainder. Subtract the smaller from the larger and the result is a multiple of  $n$  that has only 0s and 1s in its decimal expansion.



# The Generalized Pigeonhole Principle

**The Generalized Pigeonhole Principle:** If  $N$  objects are placed into  $k$  boxes, then there is at least one box containing at least  $\lceil N/k \rceil$  objects.

**Proof:** We use a proof by contraposition. Suppose that none of the boxes contains more than  $\lceil N/k \rceil - 1$  objects. Then the total number of objects is at most

$$k \left( \left\lceil \frac{N}{k} \right\rceil - 1 \right) < k \left( \left( \frac{N}{k} + 1 \right) - 1 \right) = N,$$

where the inequality  $\lceil N/k \rceil < \lceil N/k \rceil + 1$  has been used. This is a contradiction because there are a total of  $n$  objects. ◀

**Example:** Among 100 people there are at least  $\lceil 100/12 \rceil = 9$  who were born in the same month.

# The Generalized Pigeonhole Principle

**Example:** a) How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

b) How many must be selected to guarantee that at least three hearts are selected?

**Solution:** a) We assume four boxes; one for each suit. Using the generalized pigeonhole principle, at least one box contains at least  $\lceil N/4 \rceil$  cards. At least three cards of one suit are selected if  $\lceil N/4 \rceil \geq 3$ . The smallest integer  $N$  such that  $\lceil N/4 \rceil \geq 3$  is  $N = 2 \cdot 4 + 1 = 9$ .

b) A deck contains 13 hearts and 39 cards which are not hearts. So, if we select 41 cards, we may have 39 cards which are not hearts along with 2 hearts. However, when we select 42 cards, we must have at least three hearts. (Note that the generalized pigeonhole principle is not used here.)



# Permutations and Combinations

Section 6.3

# Section Summary

- Permutations
- Combinations
- Combinatorial Proofs



# Permutations

**Definition:** A *permutation* of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of  $r$  elements of a set is called an  *$r$ -permutation*.

**Example:** Let  $S = \{1,2,3\}$ .

- The ordered arrangement 3,1,2 is a permutation of  $S$ .
- The ordered arrangement 3,2 is a 2-permutation of  $S$ .
- The number of  $r$ -permutations of a set with  $n$  elements is denoted by  $P(n,r)$ .
  - The 2-permutations of  $S = \{1,2,3\}$  are 1,2; 1,3; 2,1; 2,3; 3,1; and 3,2. Hence,  $P(3,2) = 6$ .

# A Formula for the Number of Permutations

**Theorem 1:** If  $n$  is a positive integer and  $r$  is an integer with  $1 \leq r \leq n$ , then there are

$$P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1)$$

$r$ -permutations of a set with  $n$  distinct elements.

**Proof:** Use the product rule. The first element can be chosen in  $n$  ways. The second in  $n - 1$  ways, and so on until there are  $(n - (r - 1))$  ways to choose the last element.

- Note that  $P(n, 0) = 1$ , since there is only one way to order zero elements.

**Corollary 1:** If  $n$  and  $r$  are integers with  $1 \leq r \leq n$ , then

$$P(n, r) = \frac{n!}{(n-r)!}$$



# Solving Counting Problems by Counting Permutations

**Example:** How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

**Solution:**

$$P(100,3) = 100 \cdot 99 \cdot 98 = 970,200$$

# Solving Counting Problems by Counting Permutations (*continued*)

**Example:** Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

**Solution:** The first city is chosen, and the rest are ordered arbitrarily. Hence the orders are:

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

If she wants to find the tour with the shortest path that visits all the cities, she must consider 5040 paths!



# Solving Counting Problems by Counting Permutations (*continued*)

**Example:** How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

**Solution:** We solve this problem by counting the permutations of six objects, *ABC*, *D*, *E*, *F*, *G*, and *H*.

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

# Combinations

**Definition:** An  $r$ -combination of elements of a set is an unordered selection of  $r$  elements from the set. Thus, an  $r$ -combination is simply a subset of the set with  $r$  elements.

- The number of  $r$ -combinations of a set with  $n$  distinct elements is denoted by  $C(n, r)$ . The notation  $\binom{n}{r}$  is also used and is called a *binomial coefficient*. (We will see the notation again in the binomial theorem in Section 6.4.)

**Example:** Let  $S$  be the set  $\{a, b, c, d\}$ . Then  $\{a, c, d\}$  is a 3-combination from  $S$ . It is the same as  $\{d, c, a\}$  since the order listed does not matter.

- $C(4, 2) = 6$  because the 2-combinations of  $\{a, b, c, d\}$  are the six subsets  $\{a, b\}$ ,  $\{a, c\}$ ,  $\{a, d\}$ ,  $\{b, c\}$ ,  $\{b, d\}$ , and  $\{c, d\}$ .



# Combinations

**Theorem 2:** The number of  $r$ -combinations of a set with  $n$  elements, where  $n \geq r \geq 0$ , equals

$$C(n, r) = \frac{n!}{(n-r)!r!}.$$

**Proof:** By the product rule  $P(n, r) = C(n, r) \cdot P(r, r)$ .  
Therefore,

$$C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{(n-r)!r!}.$$

# Combinations

**Example:** How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a deck of 52 cards?

**Solution:** Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$\begin{aligned} C(52, 5) &= \frac{52!}{5!47!} \\ &= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960 \end{aligned}$$

- The different ways to select 47 cards from 52 is

$$C(52, 47) = \frac{52!}{47!5!} = C(52, 5) = 2,598,960.$$

*This is a special case of a general result. →*



# Combinations

**Corollary 2:** Let  $n$  and  $r$  be nonnegative integers with  $r \leq n$ . Then  $C(n, r) = C(n, n - r)$ .

**Proof:** From Theorem 2, it follows that

$$C(n, r) = \frac{n!}{(n-r)!r!}$$

and

$$C(n, n - r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!}.$$

Hence,  $C(n, r) = C(n, n - r)$ . ◀

*This result can be proved without using algebraic manipulation. →*

# Combinatorial Proofs

- **Definition 1:** A *combinatorial proof* of an identity is a proof that uses one of the following methods.
  - A *double counting proof* uses counting arguments to prove that both sides of an identity count the same objects, but in different ways.
  - A *bijective proof* shows that there is a bijection between the sets of objects counted by the two sides of the identity.



# Combinatorial Proofs

- Here are two combinatorial proofs that

$$C(n, r) = C(n, n - r)$$

when  $r$  and  $n$  are nonnegative integers with  $r < n$ :

- *Bijective Proof*: Suppose that  $S$  is a set with  $n$  elements. The function that maps a subset  $A$  of  $S$  to  $\bar{A}$  is a bijection between the subsets of  $S$  with  $r$  elements and the subsets with  $n - r$  elements. Since there is a bijection between the two sets, they must have the same number of elements. ◀
- *Double Counting Proof*: By definition the number of subsets of  $S$  with  $r$  elements is  $C(n, r)$ . Each subset  $A$  of  $S$  can also be described by specifying which elements are not in  $A$ , i.e., those which are in  $\bar{A}$ . Since the complement of a subset of  $S$  with  $r$  elements has  $n - r$  elements, there are also  $C(n, n - r)$  subsets of  $S$  with  $r$  elements. ◀

# Combinations

**Example:** How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.

**Solution:** By Theorem 2, the number of combinations is

$$C(10, 5) = \frac{10!}{5!5!} = 252.$$

**Example:** A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission?

**Solution:** By Theorem 2, the number of possible crews is

$$C(30, 6) = \frac{30!}{6!24!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593,775 .$$



# Binomial Coefficients and Identities

Section 6.4

# Section Summary

- The Binomial Theorem
- Pascal's Identity and Triangle



# Powers of Binomial Expressions

**Definition:** A *binomial* expression is the sum of two terms, such as  $x + y$ . (More generally, these terms can be products of constants and variables.)

- We can use counting principles to find the coefficients in the expansion of  $(x + y)^n$  where  $n$  is a positive integer.
- To illustrate this idea, we first look at the process of expanding  $(x + y)^3$ .
- $(x + y)(x + y)(x + y)$  expands into a sum of terms that are the product of a term from each of the three sums.
- Terms of the form  $x^3$ ,  $x^2y$ ,  $xy^2$ ,  $y^3$  arise. The question is what are the coefficients?
  - To obtain  $x^3$ , an  $x$  must be chosen from each of the sums. There is only one way to do this. So, the coefficient of  $x^3$  is 1.
  - To obtain  $x^2y$ , an  $x$  must be chosen from two of the sums and a  $y$  from the other. There are  $\binom{3}{2}$  ways to do this and so the coefficient of  $x^2y$  is 3.
  - To obtain  $xy^2$ , an  $x$  must be chosen from one of the sums and a  $y$  from the other two. There are  $\binom{3}{1}$  ways to do this and so the coefficient of  $xy^2$  is 3.
  - To obtain  $y^3$ , a  $y$  must be chosen from each of the sums. There is only one way to do this. So, the coefficient of  $y^3$  is 1.
- We have used a counting argument to show that  $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ .
- Next we present the binomial theorem gives the coefficients of the terms in the expansion of  $(x + y)^n$ .

# Binomial Theorem

**Binomial Theorem:** Let  $x$  and  $y$  be variables, and  $n$  a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

**Proof:** We use combinatorial reasoning . The terms in the expansion of  $(x + y)^n$  are of the form  $x^{n-j}y^j$  for  $j = 0, 1, 2, \dots, n$ . To form the term  $x^{n-j}y^j$ , it is necessary to choose  $n-j$   $x$ s from the  $n$  sums. Therefore, the coefficient of  $x^{n-j}y^j$  is  $\binom{n}{n-j}$  which equals  $\binom{n}{j}$ . ◀



# Using the Binomial Theorem

**Example:** What is the coefficient of  $x^{12}y^{13}$  in the expansion of  $(2x - 3y)^{25}$ ?

**Solution:** We view the expression as  $(2x + (-3y))^{25}$ .  
By the binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j.$$

Consequently, the coefficient of  $x^{12}y^{13}$  in the expansion is obtained when  $j = 13$ .

$$\binom{25}{13} 2^{12} (-3)^{13} = -\frac{25!}{13!12!} 2^{12} 3^{13}.$$

# A Useful Identity

**Corollary 1:** With  $n \geq 0$ , 
$$\sum_{k=0}^n \binom{n}{k} = 2^n.$$

**Proof (using binomial theorem):** With  $x = 1$  and  $y = 1$ , from the binomial theorem we see that:

$$2^n = (1 + 1)^n = \sum_{k=0}^n \binom{n}{k} 1^k 1^{(n-k)} = \sum_{k=0}^n \binom{n}{k}.$$

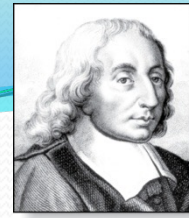
**Proof (combinatorial):** Consider the subsets of a set with  $n$  elements. There are  $\binom{n}{0}$  subsets with zero elements,  $\binom{n}{1}$  with one element,  $\binom{n}{2}$  with two elements, ..., and  $\binom{n}{n}$  with  $n$  elements. Therefore the total is

$$\sum_{k=0}^n \binom{n}{k}.$$

Since, we know that a set with  $n$  elements has  $2^n$  subsets, we conclude:

$$\sum_{k=0}^n \binom{n}{k} = 2^n.$$





# Pascal's Identity

**Pascal's Identity:** If  $n$  and  $k$  are integers with  $n \geq k \geq 0$ , then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

**Proof (combinatorial):** Let  $T$  be a set where  $|T| = n + 1$ ,  $a \in T$ , and  $S = T - \{a\}$ . There are  $\binom{n+1}{k}$  subsets of  $T$  containing  $k$  elements. Each of these subsets either:

- contains  $a$  with  $k - 1$  other elements, or
- contains  $k$  elements of  $S$  and not  $a$ .

There are

- $\binom{n}{k-1}$  subsets of  $k$  elements that contain  $a$ , since there are  $\binom{n}{k-1}$  subsets of  $k - 1$  elements of  $S$ ,
- $\binom{n}{k}$  subsets of  $k$  elements of  $T$  that do not contain  $a$ , because there are  $\binom{n}{k}$  subsets of  $k$  elements of  $S$ .

Hence,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$



*See Exercise 19  
for an algebraic  
proof.*

# Pascal's Triangle

The  $n$ th row in the triangle consists of the binomial coefficients  $\binom{n}{k}$ ,  $k = 0, 1, \dots, n$ .

$$\begin{array}{c}
 \binom{0}{0} \\
 \binom{1}{0} \binom{1}{1} \\
 \binom{2}{0} \binom{2}{1} \binom{2}{2} \\
 \binom{3}{0} \binom{3}{1} \binom{3}{2} \binom{3}{3} \\
 \binom{4}{0} \binom{4}{1} \binom{4}{2} \binom{4}{3} \binom{4}{4} \\
 \binom{5}{0} \binom{5}{1} \binom{5}{2} \binom{5}{3} \binom{5}{4} \binom{5}{5} \\
 \binom{6}{0} \binom{6}{1} \binom{6}{2} \binom{6}{3} \binom{6}{4} \binom{6}{5} \binom{6}{6} \\
 \binom{7}{0} \binom{7}{1} \binom{7}{2} \binom{7}{3} \binom{7}{4} \binom{7}{5} \binom{7}{6} \binom{7}{7} \\
 \binom{8}{0} \binom{8}{1} \binom{8}{2} \binom{8}{3} \binom{8}{4} \binom{8}{5} \binom{8}{6} \binom{8}{7} \binom{8}{8} \\
 \dots \\
 \text{(a)}
 \end{array}$$

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$\begin{array}{c}
 1 \\
 1 \quad 1 \\
 1 \quad 2 \quad 1 \\
 1 \quad 3 \quad 3 \quad 1 \\
 1 \quad 4 \quad 6 \quad 4 \quad 1 \\
 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
 1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
 1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
 \dots \\
 \text{(b)}
 \end{array}$$

By Pascal's identity, adding two adjacent binomial coefficients results in the binomial coefficient in the next row between these two coefficients.



# Generalized Permutations and Combinations

Section 6.5

# Section Summary

- Permutations with Repetition
- Combinations with Repetition
- Permutations with Indistinguishable Objects
- Distributing Objects into Boxes



# Permutations with Repetition

**Theorem 1:** The number of  $r$ -permutations of a set of  $n$  objects with repetition allowed is  $n^r$ .

**Proof:** There are  $n$  ways to select an element of the set for each of the  $r$  positions in the  $r$ -permutation when repetition is allowed. Hence, by the product rule there are  $n^r$   $r$ -permutations with repetition. ◀

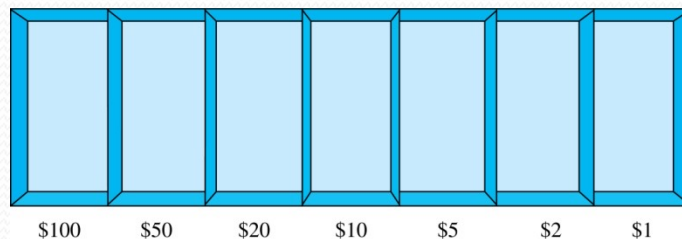
**Example:** How many strings of length  $r$  can be formed from the uppercase letters of the English alphabet?

**Solution:** The number of such strings is  $26^r$ , which is the number of  $r$ -permutations of a set with 26 elements.

# Combinations with Repetition

**Example:** How many ways are there to select five bills from a box containing at least five of each of the following denominations: \$1, \$2, \$5, \$10, \$20, \$50, and \$100?

**Solution:** Place the selected bills in the appropriate position of a cash box illustrated below:

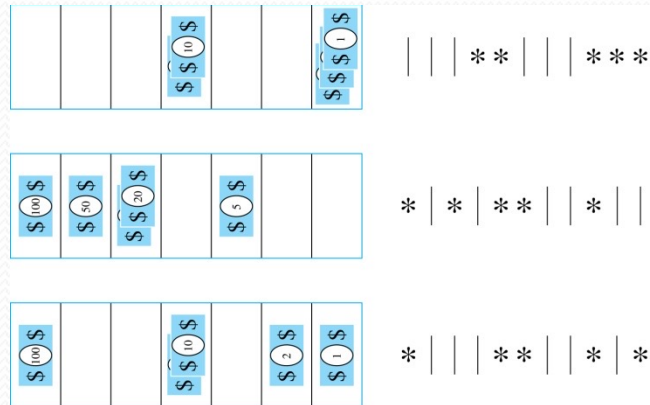


*continued* →



# Combinations with Repetition

- Some possible ways of placing the five bills:



- The number of ways to select five bills corresponds to the number of ways to arrange six bars and five stars in a row.
- This is the number of unordered selections of 5 objects from a set of 11. Hence, there are

$$C(11, 5) = \frac{11!}{5!6!} = 462$$

ways to choose five bills with seven types of bills.

# Combinations with Repetition

**Theorem 2:** The number of  $r$ -combinations from a set with  $n$  elements when repetition of elements is allowed is

$$C(n + r - 1, r) = C(n + r - 1, n - 1).$$

**Proof:** Each  $r$ -combination of a set with  $n$  elements with repetition allowed can be represented by a list of  $n - 1$  bars and  $r$  stars. The bars mark the  $n$  cells containing a star for each time the  $i$ th element of the set occurs in the combination.

The number of such lists is  $C(n + r - 1, r)$ , because each list is a choice of the  $r$  positions to place the stars, from the total of  $n + r - 1$  positions to place the stars and the bars. This is also equal to  $C(n + r - 1, n - 1)$ , which is the number of ways to place the  $n - 1$  bars.





# Combinations with Repetition

**Example:** How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

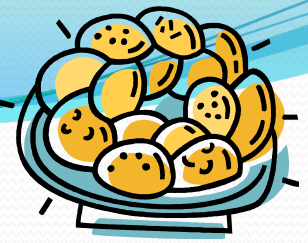
have, where  $x_1$ ,  $x_2$  and  $x_3$  are nonnegative integers?

**Solution:** Each solution corresponds to a way to select 11 items from a set with three elements;  $x_1$  elements of type one,  $x_2$  of type two, and  $x_3$  of type three.

By Theorem 2 it follows that there are

$$C(3 + 11 - 1, 11) = C(13, 11) = C(13, 2) = \frac{13 \cdot 12}{1 \cdot 2} = 78$$

solutions.



# Combinations with Repetition

**Example:** Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?

**Solution:** The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. By Theorem 2

$$C(9, 6) = C(9, 3) = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84$$

is the number of ways to choose six cookies from the four kinds.



# Summarizing the Formulas for Counting Permutations and Combinations with and without Repetition

**TABLE 1** Combinations and Permutations With and Without Repetition.

<i>Type</i>	<i>Repetition Allowed?</i>	<i>Formula</i>
$r$ -permutations	No	$\frac{n!}{(n-r)!}$
$r$ -combinations	No	$\frac{n!}{r!(n-r)!}$
$r$ -permutations	Yes	$n^r$
$r$ -combinations	Yes	$\frac{(n+r-1)!}{r!(n-1)!}$

# Permutations with Indistinguishable Objects

**Example:** How many different strings can be made by reordering the letters of the word *SUCCESS*.

**Solution:** There are seven possible positions for the three Ss, two Cs, one U, and one E.

- The three Ss can be placed in  $C(7,3)$  different ways, leaving four positions free.
- The two Cs can be placed in  $C(4,2)$  different ways, leaving two positions free.
- The U can be placed in  $C(2,1)$  different ways, leaving one position free.
- The E can be placed in  $C(1,1)$  way.

By the product rule, the number of different strings is:

$$C(7, 3)C(4, 2)C(2, 1)C(1, 1) = \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!} = \frac{7!}{3!2!1!1!} = 420.$$

*The reasoning can be generalized to the following theorem. →*



# Permutations with Indistinguishable Objects

**Theorem 3:** The number of different permutations of  $n$  objects, where there are  $n_1$  indistinguishable objects of type 1,  $n_2$  indistinguishable objects of type 2, ..., and  $n_k$  indistinguishable objects of type  $k$ , is:

$$\frac{n!}{n_1!n_2!\cdots n_k!} \cdot$$

**Proof:** By the product rule the total number of permutations is:

$C(n, n_1) C(n - n_1, n_2) \cdots C(n - n_1 - n_2 - \cdots - n_k, n_k)$  since:

- The  $n_1$  objects of type one can be placed in the  $n$  positions in  $C(n, n_1)$  ways, leaving  $n - n_1$  positions.
- Then the  $n_2$  objects of type two can be placed in the  $n - n_1$  positions in  $C(n - n_1, n_2)$  ways, leaving  $n - n_1 - n_2$  positions.
- Continue in this fashion, until  $n_k$  objects of type  $k$  are placed in  $C(n - n_1 - n_2 - \cdots - n_k, n_k)$  ways.

The product can be manipulated into the desired result as follows:

$$\frac{n!}{n_1!(n - n_1)!} \frac{(n - n_1)!}{n_2!(n - n_1 - n_2)!} \cdots \frac{(n - n_1 - \cdots - n_{k-1})!}{n_k!0!} = \frac{n!}{n_1!n_2!\cdots n_k!} \cdot$$



# Distributing Objects into Boxes

- Many counting problems can be solved by counting the ways objects can be placed in boxes.
  - The objects may be either different from each other (*distinguishable*) or identical (*indistinguishable*).
  - The boxes may be labeled (*distinguishable*) or unlabeled (*indistinguishable*).



# Distributing Objects into Boxes

- *Distinguishable objects and distinguishable boxes.*
  - There are  $n!/(n_1!n_2! \cdots n_k!)$  ways to distribute  $n$  distinguishable objects into  $k$  distinguishable boxes.
  - (See Exercises 47 and 48 for two different proofs.)
  - Example: There are  $52!/(5!5!5!5!32!)$  ways to distribute hands of 5 cards each to four players.
- *Indistinguishable objects and distinguishable boxes.*
  - There are  $C(n + r - 1, n - 1)$  ways to place  $r$  indistinguishable objects into  $n$  distinguishable boxes.
  - Proof based on one-to-one correspondence between  $n$ -combinations from a set with  $k$ -elements when repetition is allowed and the ways to place  $n$  indistinguishable objects into  $k$  distinguishable boxes.
  - Example: There are  $C(8 + 10 - 1, 10) = C(17, 10) = 19,448$  ways to place 10 indistinguishable objects into 8 distinguishable boxes.

# Distributing Objects into Boxes

- *Distinguishable objects and indistinguishable boxes.*
  - Example: There are 14 ways to put four employees into three indistinguishable offices (*see Example 10*).
  - There is no simple closed formula for the number of ways to distribute  $n$  distinguishable objects into  $j$  indistinguishable boxes.
  - See the text for a formula involving *Stirling numbers of the second kind*.
- *Indistinguishable objects and indistinguishable boxes.*
  - Example: There are 9 ways to pack six copies of the same book into four identical boxes (*see Example 11*).
  - The number of ways of distributing  $n$  indistinguishable objects into  $k$  indistinguishable boxes equals  $p_k(n)$ , the number of ways to write  $n$  as the sum of at most  $k$  positive integers in increasing order.
  - No simple closed formula exists for this number.