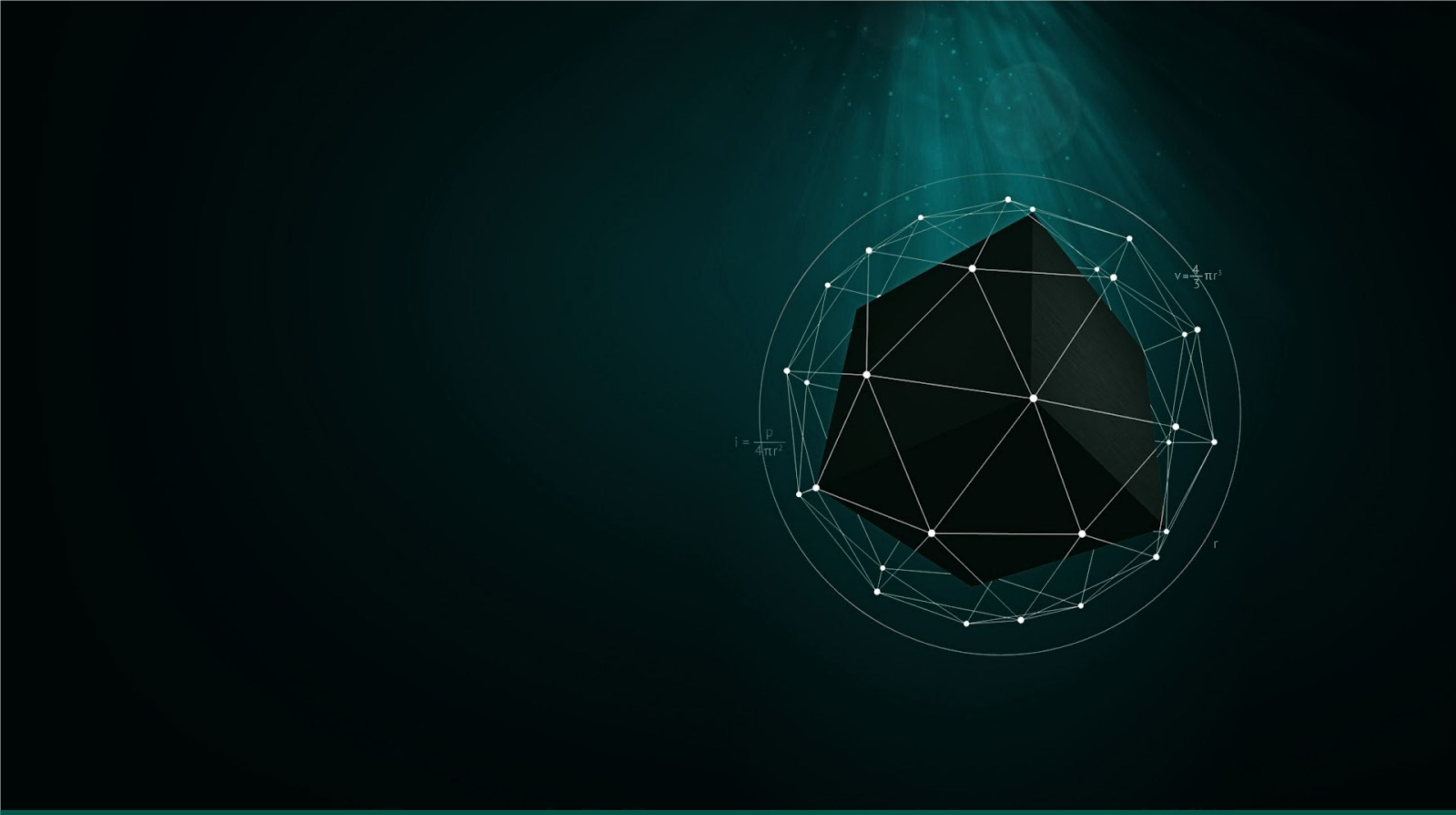


Elementary Analysis III

M A T H 5 5

Learner's Module

This module is prepared by Jimboy R. Albaracin, Ramises G. Manzano Jr. and Recson G. Canton

Contents

Vectors and the Geometry of Space

1. Three-Dimensional Coordinate Systems

1.1 3D space

To locate a point in a plane, we need two numbers. We know that any point in the plane can be represented as an ordered pair (a,b) numbers, where a is the x-coordinate and b is the y-coordinate. For this reason, a plane is called two-dimensional. To locate a point in space, three numbers are required. We represent any point in space by an ordered triple (a,b,c) of real numbers.

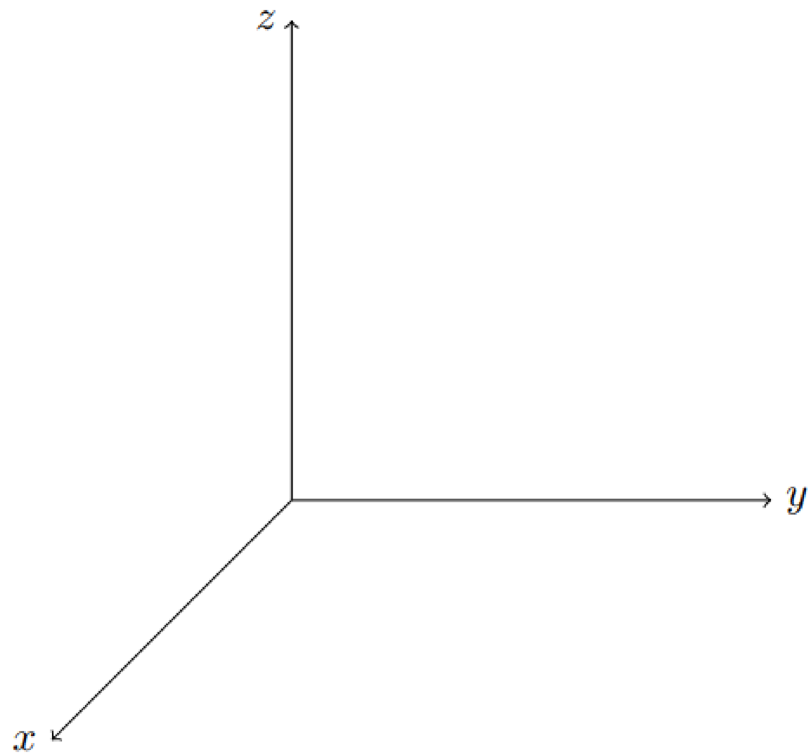


Figure 1.1

Because many people have some difficulty visualizing diagrams of three-dimensional figures, you may find it helpful to do the following [Figure 1.2]. Look at any bottom corner of a room and call the corner the origin. The wall on your left is in the *xz*-plane, the wall on your right is in the *yz*-plane, and the floor is in the *xy*-plane. The x-axis runs along the intersection of the floor and the left wall. The y-axis runs along the intersection of the floor and the right wall. The z-axis runs up from the floor toward the ceiling along the intersection of the two walls. You are situated in the first octant, and you can now imagine seven other rooms situated in the other seven octants [Figure 1.3] (three on the same floor and four on the floor below), all connected by the common corner point O.

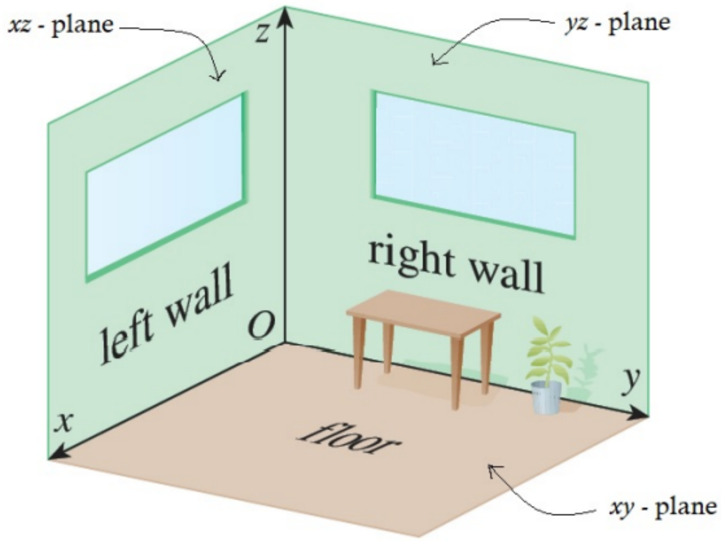


Figure 1.2: A model of *xy*-, *xz*- and *yz*- plane.

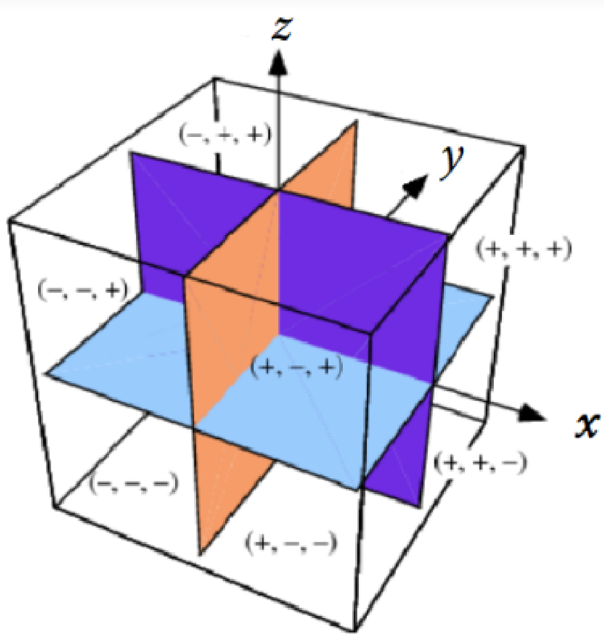


Figure 1.3: A 3-D model the octants

Usually we think of x- and y-axes as being horizontal and the z-axis as being vertical. The direction of the z-axis is determined by the right-hand rule as illustrated below [Figure

1.4]: If you curl the fingers of your right hand around the z-axis in the direction of a 90° counterclockwise rotation from the positive x-axis to the positive y-axis, then your thumb points in the positive direction of the z-axis

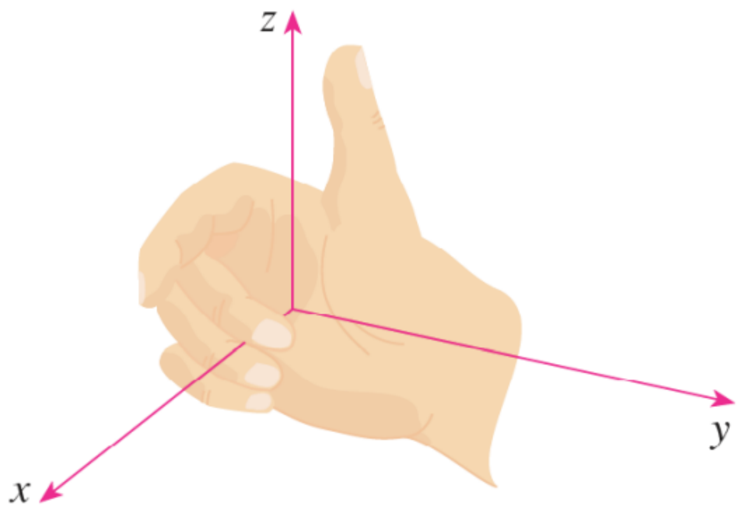


Figure 1.4: The Right Hand Rule

The point $P(a,b,c)$ determines a rectangular box as in Figure 1.5 . If we drop a perpendicular from P to the xy-plane, we get a point Q with coordinates $(a,b,0)$ called the projection of P onto the xy-plane. Similarly, $R(0,b,c)$ and $S(a,0,c)$ are the projections of P onto the yz-plane and xz-plane, respectively

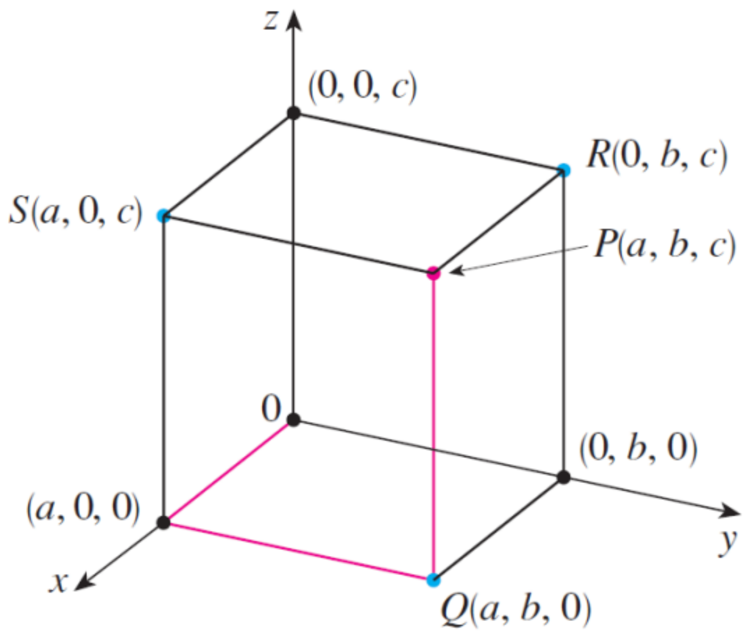
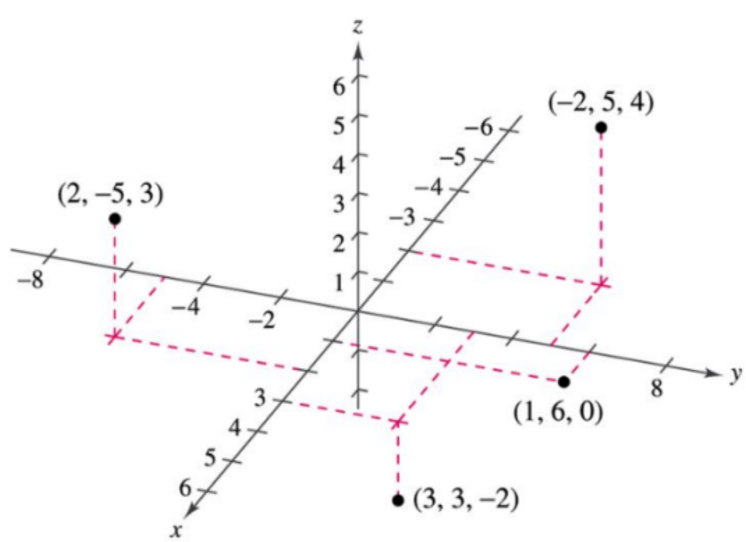


Figure 1.5

■ **Example 1.1** Plot the points $(1,\ 6,\ 0)$, $(3,\ 3,\ -2)$, $(-2,\ 5,\ 4)$ and $(2,\ -5,\ 3)$.

Solution:



The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,\ y,\ z) \mid x,\ y,\ z \in \mathbb{R}\}$ is the set of all ordered triples of real numbers and is denoted by $\mathbb{R}^3$. We have given a one-to-one correspondence between points P in space and ordered triples $(a\, b,\ c)$ in $\mathbb{R}^3$. It is called a three-dimensional rectangular coordinate system. Notice that, in terms of coordinates, the first octant can be described as the set of points whose coordinates are all positive.

1.2 Surfaces

In two-dimensional analytic geometry, the graph of an equation involving x and y is a curve in $\mathbb{R}^2$. In three-dimensional analytic geometry, an equation in x, y, and z represents a surface in $\mathbb{R}^3$.

■ **Example 1.2** What surfaces in $\mathbb{R}^3$ are represented by the following equations?

(a) $z = 3$

(b) $y = 5$

Solution:

(a) : The equation $z = 3$ represents the set $\{(x,\ y,\ z)|z = 3\}$, which is the set of all points in $\mathbb{R}^3$ whose z-coordinate is 3 (x and y can be any value). This is simply a horizontal plane that is parallel to the xy-plane and 3 units above it.

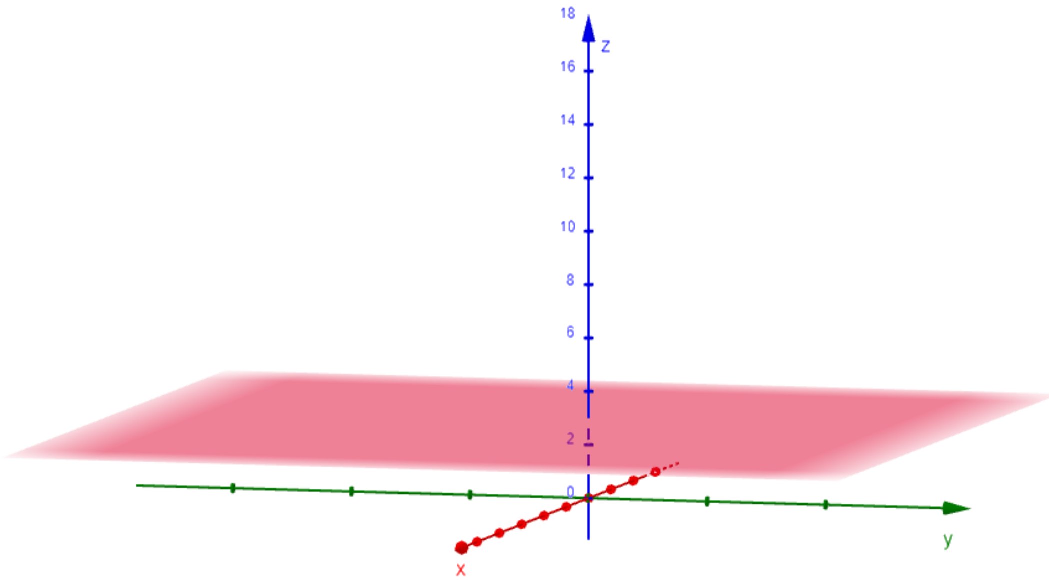


Figure 1.6: $z = 3$, a plane in $\mathbb{R}^3$

(b) : The equation $y = 5$ represents the set of all points in $\mathbb{R}^3$ whose y-coordinate is 5. This is the vertical plane that is parallel to the xz-plane and five units to the right of it.

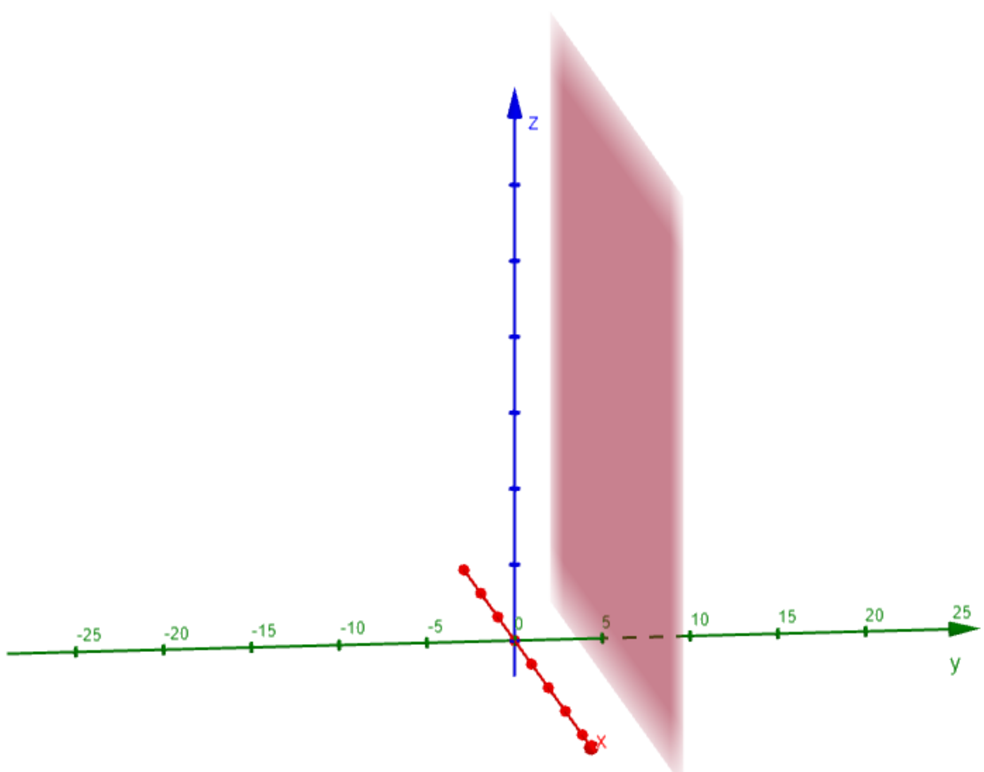


Figure 1.7: $y = 5$, a plane in $\mathbb{R}^3$

Note: When an equation is given, we must understand from the context whether it represents a curve in $\mathbb{R}^2$ or a surface in $\mathbb{R}^3$. For instance the equation $x = 1$ represents a plane in $\mathbb{R}^3$ but $x = 1$ also represents a line in $\mathbb{R}^2$ when we are dealing with two-dimensional analytic geometry.

Example 1.3 What surfaces in $\mathbb{R}^3$ are represented by the following equations?

(a) Which points (x, y, z) satisfy the equations

$$x^2 + y^2 = 1 \text{ and } z = 3?$$

(b) What does the equation $x^2 + y^2 = 1$ represents as a surface in $\mathbb{R}^3$?

Solution:

(a) : Because $z = 3$, the set of points lie in the horizontal plane $z = 3$ and because $x^2 + y^2 = 1$, the set of points also lie on the circle with radius 1 and center on the z-axis [Figure 1.8].

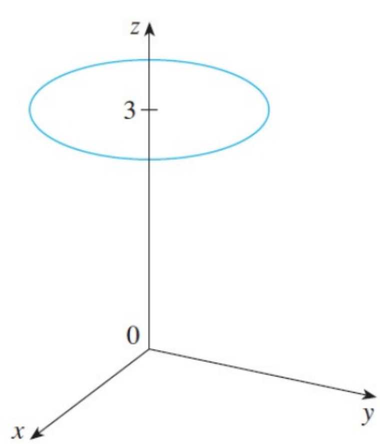


Figure 1.8: The circle $x^2 + y^2 = 1$, $z = 3$.

(b) : Given that $x^2 + y^2 = 1$ and z with no restrictions, we see that the point $(x,\ y,\ z)$ could lie on a circle in any horizontal plane $z = k$ where k is any real number. So the surface

$x^2+y^2=1$ consists of all possible horizontal circles $x^2+y^2=1, z=k$ and is therefore the circular cylinder with radius 1 whose axis is the z-axis [Figure 1.9].

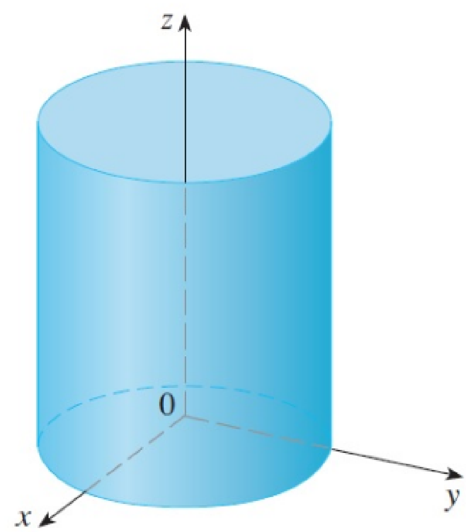


Figure 1.9: The cylinder $x^2+y^2=1$

■ **Example 1.4** Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $y=x$.

Solution: The equation represents the set of all points in $\mathbb{R}^3$ whose x- and y-coordinate are equal, that is, $\{(x,\ x,\ z)|x,z\in R\}$. This is a vertical plane that intersects the xy-plane in the line $y=x$, $z=0$. A portion of this plane is that lies in the first octant is seen if the graph below [Figure 1.10].

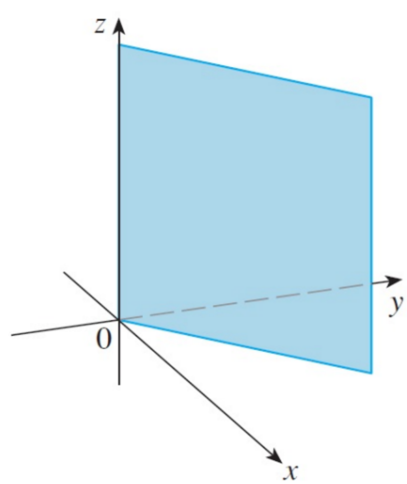


Figure 1.10: The plane $y = x$.

1.3 Distance and Spheres

The familiar formula for the distance between two points in a plane is easily extended to the following three-dimensional formula.

Definition 1.3.1 (Distance Formula in $\mathbb{R}^3$)

The distance $|P_1P_2|$ between the points $P_1(x_1,\ y_1,\ z_1)$ and $P_2(x_2,\ y_2,\ z_2)$ is given by

$$|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.$$

To see why this formula is true, we construct a rectangular box as seen below [Figure 1.11], where P_1 and P_2 are opposite vertices and the faces of the of the box are parallel to the coordinate planes. If $A(x_2,\ y_1,\ z_1)$ and $B(x_2,\ y_2,\ z_1)$ are the vertices of the box indicated in the figure, then

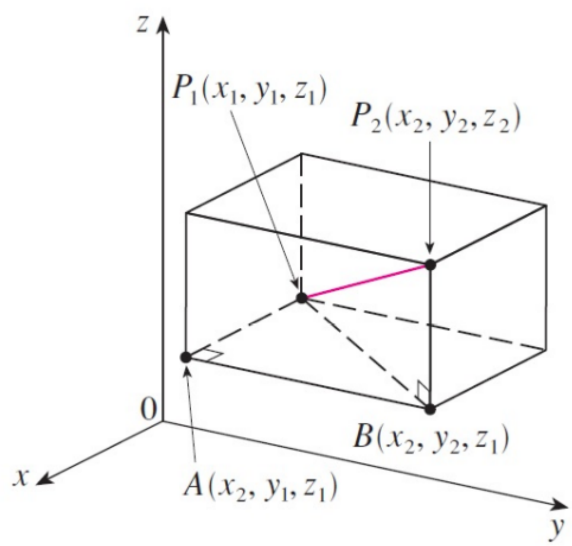


Figure 1.11

$$|P_1A| = |x_2 - x_1|$$
$$|AB| = |y_2 - y_1|$$
$$|BP_2| = |z_2 - z_1|$$

Because triangles P_1BP_2 and P_1AB are both right-angled, two applications of Pythagorean

Theorem give

$$|P_1P_2|^2 = |P_1B|^2 + |BP_2|^2$$
$$\text{and } |P_1B|^2 = |P_1A|^2 + |AB|^2$$

Combining these equations, we get

$$\begin{aligned} |P_1P_2|^2 &= |P_1A|^2 + |AB|^2 + |BP_2|^2 \\ &= |x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \end{aligned}$$

Therefore,

$$|P_1P_2|^2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Example 1.5 The distance from the point $P(2, -1, 7)$ to the point $Q(1, -3, 5)$ is

$$|PQ| = \sqrt{(1-2)^2 + (-3+1)^2 + (5-7)^2} = \sqrt{1+4+4} = \sqrt{9} = 3.$$

Example 1.6 Find an equation of a sphere with radius r and center $C(h,\ k,\ l)$.

By definition, a sphere is the set of all points $P(x,\ y,\ z)$ whose distance from C is r [Figure 1.12].

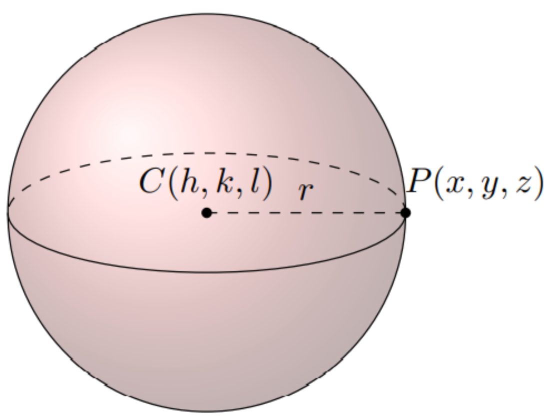


Figure 1.12

Thus, P is on the sphere if and only if $|PC| = r$. Squaring both sides, we have $|PC|^2 = r^2$ or

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2.$$

The result of this example is worth remembering.

Definition 1.3.2 (Equation of a Sphere) An equation of a sphere with center (h,k,l) and radius r is

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2.$$

In particular, In particular, if the center is at the origin O, then an equation of the sphere is

$$x^2+y^2+z^2=r^2.$$

■ **Example 1.7** Show that

$$x^2+y^2+z^2+4x-6y+2z+6=0$$

is an equation of a sphere, and find its center and radius.

Solution: We can rewrite the given equation in the form of an equation of a sphere if we complete squares:

$$(x^2+4x+4)+(y^2-6y+9)+(z^2+2z+1)=-6+4+9-1$$
$$(x+2)^2+(y-3)^2+(z+1)^2=8$$

Therefore, the center of the sphere is at $(-2,3,-1)$ with radius $2\sqrt{2}$.

■ **Example 1.8** What region in $\mathbb{R}^3$ is represented by the following inequalities?

$$1\leq x^2+y^2+z^2\leq 4, \quad z\leq 0$$

Solution: The inequalities

$$1\leq x^2+y^2+z^2\leq 4$$

can be written as

$$1\leq \sqrt{x^2+y^2+z^2}\leq 2.$$

They represent the points (x,y,z) whose distance from the origin is at least 1 and at most 2. But we are also given that $z\leq 0$, so the points lie on or below the xy-plane. Thus the given inequalities represent the region that lies between (or on) the spheres

$$x^2+y^2+z^2=1 \quad \text{and} \quad x^2+y^2+z^2=4,$$

and beneath (on or) the xy-plane. The graph is sketched below [Figure 1.12a]

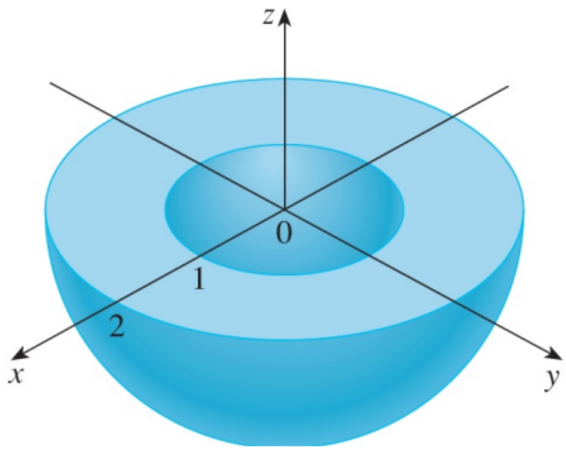


Figure 1.12a

The term **vector** is used by scientists to indicate a quantity (such as displacement or velocity or force) that has both magnitude and direction. A vector is often represented by an arrow or a directed line segment. The length of the arrow represents the magnitude of the vector and the arrow points in the direction of the vector. We denote a vector by printing a letter in boldface (**v**) or by putting an arrow above the letter $\overrightarrow{v}$.

For instance, suppose a particle moves along a line segment from point A to point B. The corresponding **displacement vector v**, shown below [Figure 1.13], has initial point A (the tail) and terminal point B (the tip) and we indicate this by writing $v = \overrightarrow{AB}$. Notice that the vector $u = \overrightarrow{CD}$ has the same length and the same direction as **v** even though it is in a different position. We say that **u** and **v** are **equivalent** (or equal) and we write **u=v**. The **zero vector**, denoted by **0**, has length 0. It is the only vector with no specific direction

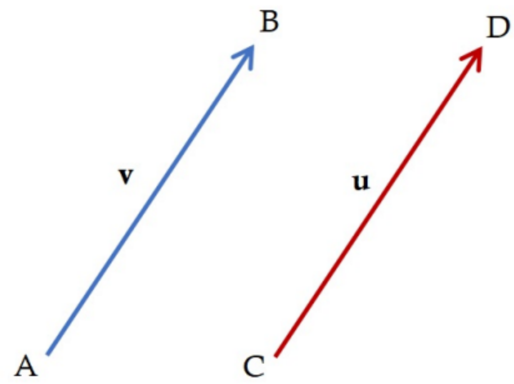


Figure 1.13 Equivalent vectors

1.4 Combining Vectors

Suppose a particle moves from A to B, so its displacement vector is $\overrightarrow{AB}$. Then the particle changes direction and moves from B to C, with displacement vector $\overrightarrow{AB}$ [Figure 1.14]. The combined effect of these displacements is that the particle has moved from A to C. The resulting displacement vector $\overrightarrow{AC}$ is called the sum of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ and we write

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}.$$

Definition 1.4.1 ((Vector Addition) If **u** and **v** are vectors positioned so the initial point of **v** is at the terminal point of **u**, then the sum **u+v** is the vector from the initial point of **u** to the terminal point of **v**.

The following figures below show the sum of vectors **u** and **v** and that **u+v=v+u**

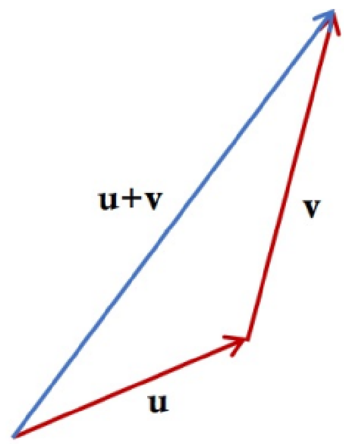


Figure 1.14 The Triangle Law

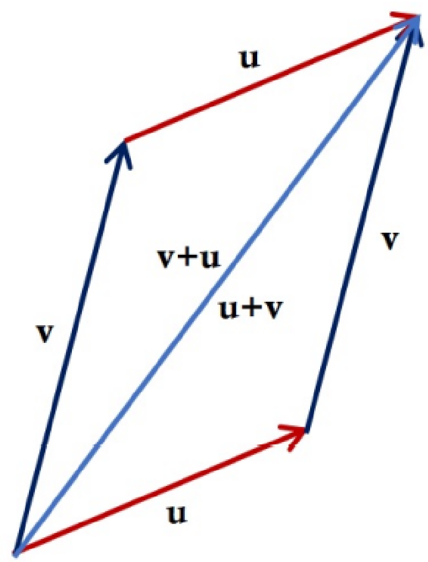


Figure 1.15 The Parallelogram Law

Definition 1.4.2 (Scalar Multiplication) If c is a scalar and **v** is a vector, then scalar multiple $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of **v** and whose direction is the same as **v** if $c > 0$ and is opposite to **v** if $c < 0$. If $c = 0$ or v=0, then $c\mathbf{v}$ = 0.

This definition is illustrated below [Figure 1.16]. We see that real numbers work like scaling factors here; that is why we call them scalars. Notice that two nonzero vectors are parallel if they are scalar multiples of one another. In particular, the vector -**v**=-1(**v** has the same length as **v** but points in the opposite direction. We call it the negative of **v**.

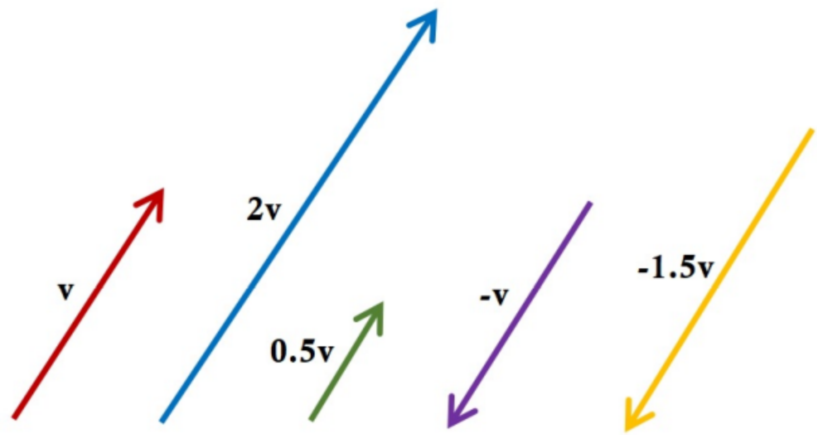


Figure 1.16 Scalar multiples of **v**.

By the difference **u-v** of two vectors we mean

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$

The figure below shows the difference of **u** and **v**.

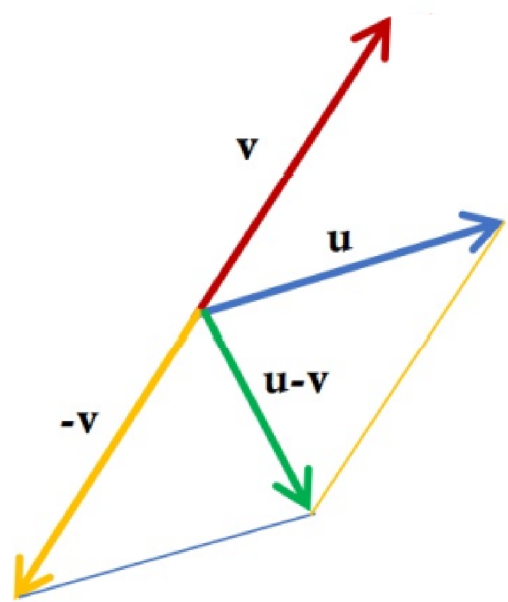


Figure 1.17 The difference of **u** and **v**.

■ **Example 1.9** Given the vectors **a** and **b** below, draw **a** - 3**b**.

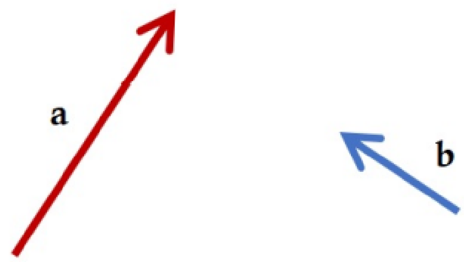
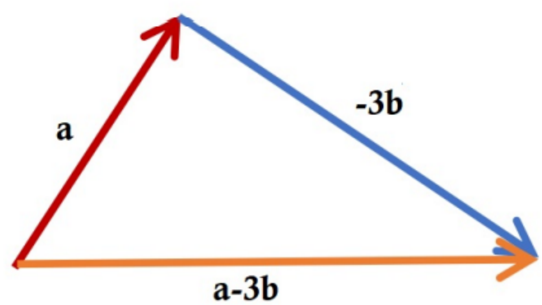


Illustration:



1.5 Components

If we place the initial point of a vector a at the origin of a rectangular coordinate system, then the terminal point of a has coordinates of the form (a,b) or (a,b,c), depending on whether our coordinate system is two- or three dimensional [Figure 1.18]. These coordinates are called **components** of **a**, and we write

$$a = \langle a_1, a_2 \rangle \text{ or } a = \langle a_1, a_2, a_3 \rangle.$$

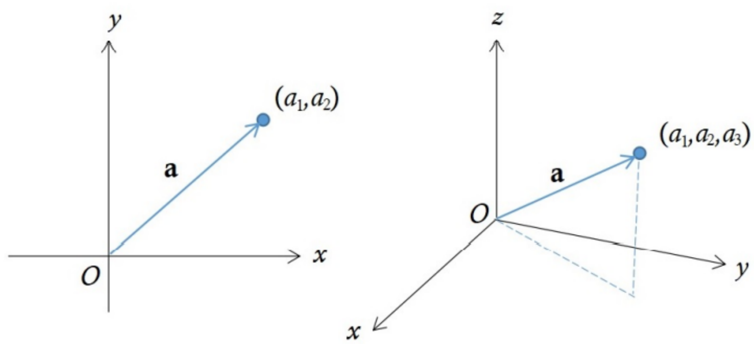


Figure 1.18: $\langle a_1, a_2 \rangle$ and $\langle a_1, a_2, a_3 \rangle$ respectively.

The notation $\langle a_1, a_2 \rangle$ is for the ordered pair that refers to a vector should not be confused with $(a_1,\ a_2)$ which is a point in the plane. Consider the following vectors below [Figure 1.19]

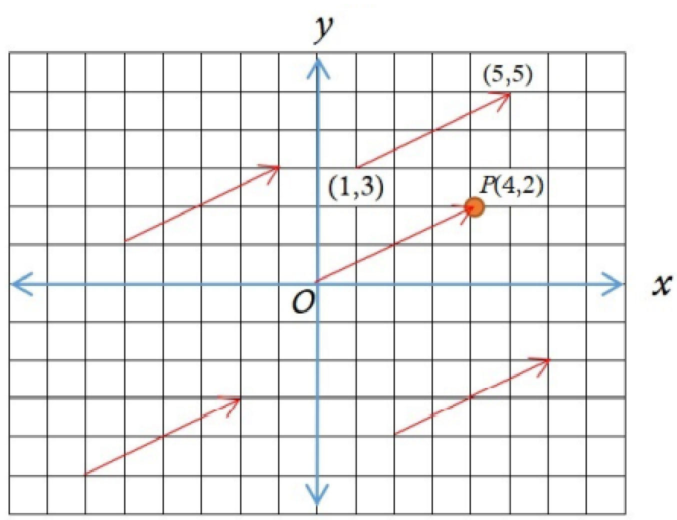


Figure 1.19: Equivalent vectors.

The vectors shown above are all equivalent. These geometric vectors are **representations** of the algebraic vector **a**$= \langle 4, 2 \rangle$. The particular representation $\overrightarrow{OP}$ from the origin to point $P(4,2)$ is called the **position vector** of the point P.

In three dimensions, the vector a $= \overrightarrow{OP} = \langle a_1, a_2, a_3 \rangle$ is the **position vector** of the point $P(a_1, a_2, a_3)$ [Figure 1.20], where $x_2 = x_1 + a_1, y_2 = y_1 + a_2$ and $z_2 = z_1 + a_3$.

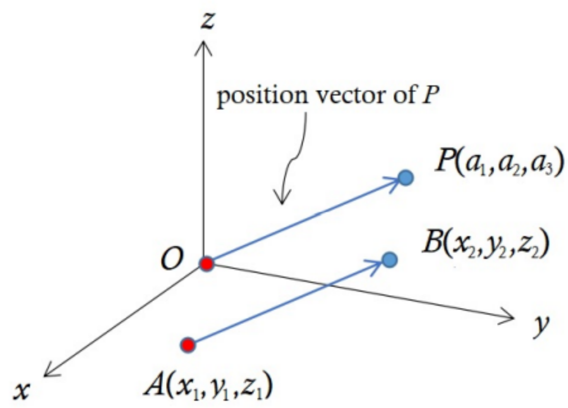


Figure 1.20: Representations of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$.

Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with representation $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

■ **Example 1.10** Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with representation $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

Solution: By the above boxed result, the vector a corresponding to $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle 2 - 1, 4 - 3, 1 - (-4) \rangle = \langle 1, 3, 5 \rangle.$$

The **magnitude** or **length** of the vector $\mathbf{v}$ is the length of any of its representations and is denoted by the symbol $|\mathbf{v}|$ or $\|\mathbf{v}\|$. By using the distance formula to compute the length of a segment, we obtain the following formulas.

The length of the two-dimensional vector **a**$= \langle a_1, a_2 \rangle$ is

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector **a**$= \langle a_1, a_2, a_3 \rangle$ is

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

How do we add or subtract two vectors algebraically? To add algebraic vectors we add corresponding components. Similarly, to subtract vectors we subtract corresponding components.

If **a**$= \langle a_1, a_2 \rangle$ and **b**$= \langle b_1, b_2 \rangle$, then

$$\mathbf{a}\text{+}\mathbf{b} \;=\; \langle\, a_1 + b_1,\; a_2 + b_2 \rangle \qquad \mathbf{a}\text{-}\mathbf{b} \;=\; \langle a_1 - b_1,\; a_2 - b_2 \rangle$$

$$c\mathbf{a} \;=\; \langle ca_1,\; ca_2 \rangle.$$

Similarly,

$$\langle\, a_1,\; a_2,\; a_3 \rangle + \langle\, b_1,\; b_2,\; b_3 \rangle = \langle a_1 + b_1,\; a_2 + b_2,\; a_3 + b_3 \rangle$$

$$\langle\, a_1,\; a_2,\; a_3 \rangle - \langle\, b_1,\; b_2,\; b_3 \rangle = \langle a_1 - b_1,\; a_2 - b_2,\; a_3 - b_3 \rangle$$

$$c\langle\, a_1,\; a_2,\; a_3 \rangle = \langle ca_1,\; ca_2,\; ca_3 \rangle$$

Example 1.11 If $\mathbf{a} = \langle 4,\ 0,\ 3 \rangle$ and $\mathbf{b} = \langle -2,\ 1,\ 5 \rangle$, find $\|\mathbf{a}\|$ and the vectors $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $3\mathbf{b}$ and $2\mathbf{a} + 5\mathbf{b}$.

Solution:

$$
\begin{aligned}
\|\mathbf{a}\| &= \sqrt{4^2+0^2+3^3} = \sqrt{25} = 5 \\
\mathbf{a}+\mathbf{b} &= \langle 4,\ 0,3\rangle + \langle -2,\ 1,5\rangle \\
&= \langle 4+(-2),\ 0+1,3+5\rangle = \langle 2,\ 1,8\rangle \\
\mathbf{a}-\mathbf{b} &= \langle 4,\ 0,3\rangle - \langle -2,\ 1,5\rangle \\
&= \langle 4-(-2),\ 0-1,3-5\rangle = \langle 6,\ -1,-2\rangle \\
3\mathbf{b} &= 3\langle -2,\ 1,5\rangle = \langle 3(-2),\ 3(1),3(5)\rangle = \langle -6,\ 3,15\rangle \\
2\mathbf{a}+5\mathbf{b} &= 2\langle 4,\ 0,3\rangle + 5\langle -2,\ 1,5\rangle \\
&= \langle 8,\ 0,6\rangle + \langle -10,\ 5,25\rangle = \langle -2,5,31\rangle.
\end{aligned}
$$

We denote by V_2 the set of all two-dimensional vectors and V_3 the set of all three-dimensonal vectors. In general we denote by V_n all n-dimensional vectors. An n-dimensional vector is an ordered n-tuple:

$$\mathbf{a} = \langle a_1, a_2, \ldots, a_n\rangle$$

where $a_1, a_2, \ldots, a_n$ are real numbers that are called the components of $\mathbf{a}$. Addition and scalar multiplication are defined in terms of components just as for the cases $n = 2$ and $n = 3$.

If $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are vectors in V_n and c and d scalars, then

1. $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$
2. $\mathbf{a}+(\mathbf{b}+\mathbf{c}) = (\mathbf{a}+\mathbf{b})+\mathbf{c}$
3. $\mathbf{a}+0 = \mathbf{a}$
4. $\mathbf{a}+(-\mathbf{a}) = 0$
5. $c(\mathbf{a}+\mathbf{b}) = c\mathbf{a}+c\mathbf{b}$
6. $(c+d)\mathbf{a} = c\mathbf{a}+d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$
8. $1\mathbf{a} = \mathbf{a}$.

Standard Basis Vectors

$$\mathbf{i} = \langle 1,0,0 \rangle \quad \mathbf{j} = \langle 0,1,0 \rangle \quad \mathbf{k} = \langle 0,0,1 \rangle.$$

These vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are called the **standard basis vectors**. They have length 1 and point in the directions of the positive x-, y- and z-axes. Similarly in two-dimensions, we define $\mathbf{i} = \langle 1,0 \rangle$ and $\mathbf{j} = \langle 0,1 \rangle$

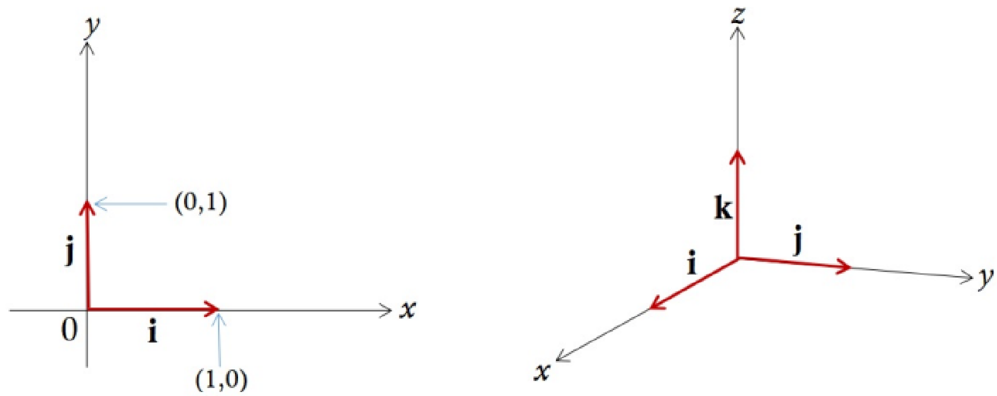


Figure 1.21: Standard basis vectors in V_2 and V_3 respectively

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then we can write

$$\begin{aligned}\mathbf{a} &= \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \\ &= a_1 \langle 1,0,0 \rangle + a_2 \langle 0,1,0 \rangle + a_3 \langle 0,0,1 \rangle \\ \mathbf{a} &= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}\end{aligned}$$

Thus, any vector in V_3 can be expressed in terms of **i,i** and **k**. For instance

$$\langle 2, -3, 1 \rangle = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}.$$

In two dimensions, we can write

$$\mathbf{a} = \langle a_1, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}.$$

Example 1.5.1 If $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 5\mathbf{k}$, express the vector $3\mathbf{a} + 2\mathbf{b}$ in terms of **i, j** and **k**,

$$\begin{aligned} 3\mathbf{a} + 2\mathbf{b} &= 3(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) + 2(4\mathbf{i} + 4\mathbf{k}) \\ &= 3\mathbf{i} - 6\mathbf{j} + 9\mathbf{k} + 8\mathbf{i} + 10\mathbf{k} \\ &= 11\mathbf{i} - 6\mathbf{j} + 19\mathbf{k} \end{aligned}$$

A **unit vector** is a vector whose length is 1. The vectors **i, j** and **k** are all unit vectors. If $\mathbf{a} \neq 0$, the the unit vector $\mathbf{u}$ with same directon as $\mathbf{a}$ is

$$\mathbf{u} = \frac{1}{\|\mathbf{a}\|}\mathbf{a} = \frac{\mathbf{a}}{\|\mathbf{a}\|}.$$

In order to verify this, we simple let $c = \dfrac{1}{\|\mathbf{a}\|}$. Note that c here is a positive scalar. Then the vector $\mathbf{u} = c\mathbf{a}$ is a vector same direction as $\mathbf{a}$. Now,

$$\|\mathbf{u}\| = \|c\mathbf{a}\| = \|c\|\|\mathbf{a}\| = c\|\mathbf{a}\| = \frac{1}{\|\mathbf{a}\|}\|\mathbf{a}\| = 1.$$

Example 1.5.2 Find the unit vector in the direction of the vector **v**= 3**i** - **j** + 2**k**.

Solution: We will find first $\|\mathbf{v}\|$. In our case,

$$\|\mathbf{v}\| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14}.$$

Thus,

$$\|\mathbf{u}\| = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{14}}(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \frac{3}{\sqrt{14}}\mathbf{i} - \frac{1}{\sqrt{14}}\mathbf{j} + \frac{2}{\sqrt{14}}\mathbf{k}.$$

1.6 The Dot Product

From the previous section we have learned how to add two vectors and to multiply a vector by a scalar. The question is: is it possible to multiply two vectors so that their product is a useful quantity? One such product is the dot product, whose definition follows.

Definition 1.6.1 (Dot Product)

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of **a** and **b** is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Thus, the result of the dot product is a real number, that is, a scalar. For this reason, the dot product is sometimes called the **scalar product** (or **inner product**). For two-dimensional vectors, the dot product is defined in a similar manner:

$$\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2.$$

Example 1.6.1

$$\langle 1,2 \rangle \cdot \langle 4,-1 \rangle = 1(4) + 2(-1) = 2$$

$$\langle -1,2,3 \rangle \cdot \langle 1,-1,0 \rangle = -1(1) + 2(-1) + 3(0) = -3$$

$$(\mathbf{i} - 6\mathbf{j} + 5\mathbf{k}) \cdot (\mathbf{j} - 2\mathbf{k}) = 1(0) + (-6)(1) + 5(-2) = -16$$

Properties of the Dot Product

If **a**, **b**, and **c** are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $0 \cdot \mathbf{a} = 0.$

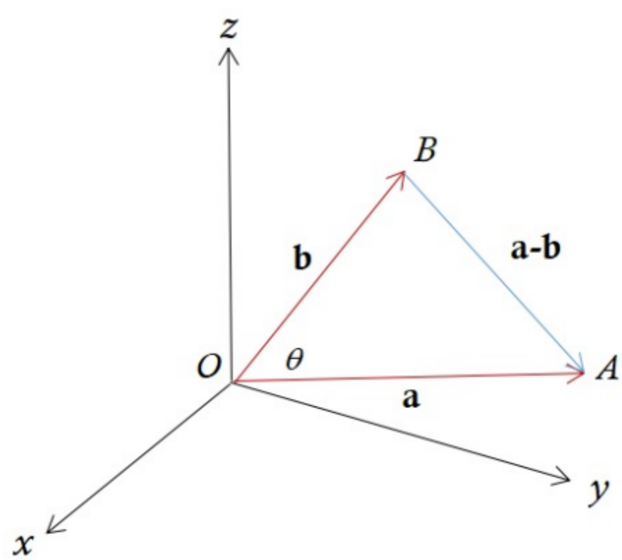


Figure 1.22

Theorem 1.6.1 If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$$

We shall leave the proof of the above theorem as an exercise.

Example 1.6.2 If vectors $\mathbf{a}$ and $\mathbf{b}$ have lengths 3 and 4 respectively, and the angle between them is $\dfrac{\pi}{3}$, find $\mathbf{a}\cdot\mathbf{b}$.

Solution:

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\left(\frac{\pi}{3}\right) = (3)(4)\left(\frac{1}{2}\right) = 6.$$

Corollary 1.6.2 If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}.$$

Example 1.6.3 Find the angle between the vectors $\mathbf{a} = \langle 2, 2, -1\rangle$ and $\mathbf{b} = \langle 2, 1, 3\rangle$.

Solution: Let us compute first the lengths of $\mathbf{a}$ and $\mathbf{b}$ together with the dot product of $\mathbf{a}$ and $\mathbf{b}$

$$\|\mathbf{a}\| = \sqrt{(2)^2+(2)^2+(-1)^2} = 3 \text{ and } \|\mathbf{b}\| = \sqrt{(2)^2+(1)^2+(3)^2} = \sqrt{14}$$

$$\mathbf{a}\cdot\mathbf{b} = 2(2)+2(1)+(-1)(3) = 3.$$

From the above corollary,

$$\cos\theta = \frac{3}{3\sqrt{14}} = \frac{1}{\sqrt{14}}.$$

Thus,

$$\theta = \cos^{-1}\frac{1}{\sqrt{14}} \approx 74.5^\circ.$$

Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are **perpendicular** or **orthogonal** if the angle between them is $\dfrac{\pi}{2}$. Thus,

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\left(\frac{\pi}{2}\right) = 0.$$

On the other hand if $\mathbf{a}\cdot\mathbf{b} = 0$, then $\cos\theta = 0$, so, $\theta = \frac{\pi}{2}$. The result below gives the criterion in determining the orthogonality of two vectors.

Two vectors **a** and **b** are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

The zero vector **0** is considered to be orthogonal to any vectors.

Example 1.6.4 Show that **i-j-k** is orthogonal to 2**i**+ **j**+**k**.

Solution:

$$(\mathbf{i} - \mathbf{j} - \mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} + \mathbf{k})) = 1(2) + (-1)(1) + (-1)(1) = 0.$$

Therefore, the two vectors are orthogonal.

1.7 Direction Angles and Direction Cosines

The **direction angles** of a nonzero vector **a** are the angles α, β, and γ in the interval $[0, \pi]$ that **a** makes with the positive $x-, y-$, and z-axis respectively [Figure 1.23]

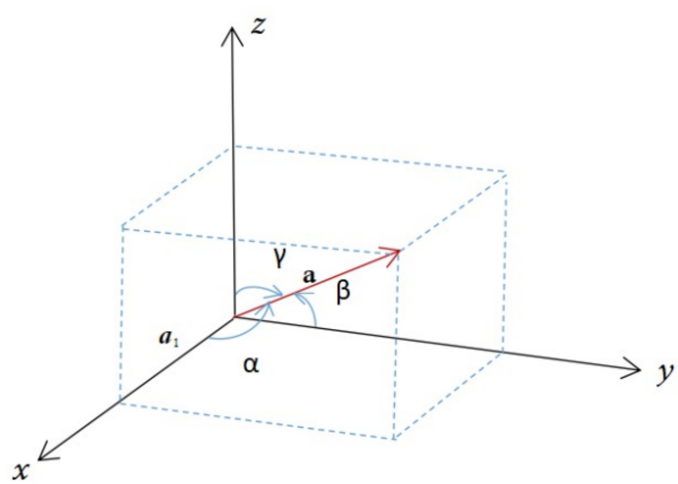


Figure 1.23

The cosines of these direction angles, $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the **direction cosines** of a vector **a**. Thus,

$$\cos\alpha = \frac{\mathbf{a} \cdot \mathbf{i}}{||\mathbf{a}|| \cdot ||\mathbf{i}||} = \frac{a_1}{||\mathbf{a}||}$$

Similarly, we have

$$\cos\beta = \frac{a_2}{||\mathbf{a}||} \quad \text{and} \quad \cos\gamma = \frac{a_3}{||\mathbf{a}||}.$$

Squaring these three equations and adding we get

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Moreover, from the above equations,

$$\begin{aligned}\mathbf{a} = \langle a_1, a_2, a_3\rangle &= \langle \|\mathbf{a}\|\cos\alpha, \|\mathbf{a}\|\cos\beta, \|\mathbf{a}\|\cos\gamma\rangle \\ &= \|\mathbf{a}\|\langle \cos\alpha, \cos\beta, \cos\gamma\rangle\end{aligned}$$

Therefore,

$$\frac{a}{\|\mathbf{a}\|} = \langle \cos\alpha, \cos\beta, \cos\gamma\rangle$$

which mean that the direction of cosines of a **a** are the components of the unit vector in the direction of **a**.

Example 1.7.1 Find the direction cosines and direction angles (to the nearest degree) of a vector $\mathbf{a} = \langle 1, -2, -3\rangle$.

Solution: Note that

$$\|\mathbf{a}\| = \sqrt{1^2 + (-2)^2 + (-3)^2} = \sqrt{14}.$$

Hence,

$$\cos\alpha = \frac{1}{\sqrt{14}} \quad \cos\beta = \frac{-2}{\sqrt{14}} \quad \cos\gamma = \frac{-3}{\sqrt{14}}$$

and so,

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) \approx 74^\circ \quad \beta = \cos^{-1}\left(\frac{-1}{\sqrt{14}}\right) \approx 122^\circ$$
$$\gamma = \cos^{-1}\left(\frac{-3}{\sqrt{14}}\right) \approx 143^\circ.$$

1.7.1 Projection

Consider the representations $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ of two vectors **a** and **b** with P as their common initial point [Figure 1.24]. The vector $\overrightarrow{PS}$ is called the **vector projection** of **b** onto **a** and is denote by $\text{proj}_{\mathbf{a}}\mathbf{b}$. You can think of this projection as the shadow of **b** onto **a**.

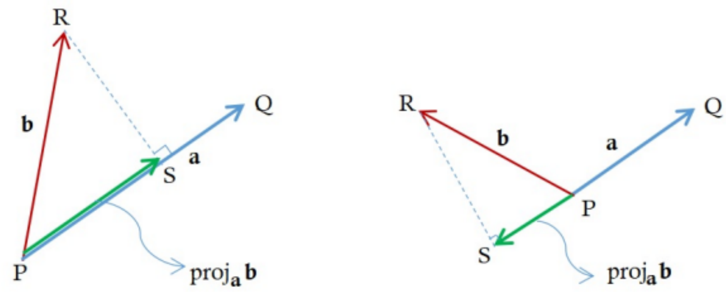


Figure 1.24: Vector Projections

The **scalar projection** of **b** onto **a** (also called the **component of b along a**) is defined to be signed magninuted of the vector projection [Figure 1.25], which is teh number $\|\mathbf{b}\| \cos\theta$, where θ is the angle between **a** and **b**. This is denoted by $\text{comp}_{\mathbf{a}}\mathbf{b}$.

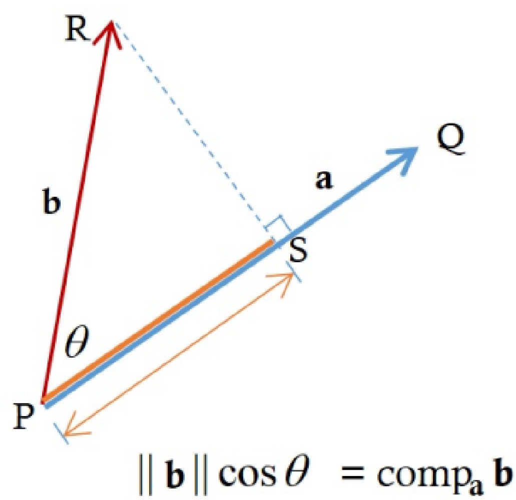


Figure 1.24a: Scalar projection

Thus, we have the following formulas

Scalar projection of **b** onto **a** : $\text{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}$

Vector projection of **b** onto **a** : $\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\right)\dfrac{\mathbf{a}}{\|\mathbf{a}\|} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}$

Example 1.7.2 Find the scalar projection and vector projection of $\mathbf{b} = \langle 1,1,2\rangle$. onto $\mathbf{a} = \langle -3,2,1\rangle$. *Solution*: Note that

$$\mathbf{a} = \sqrt{(-3)^2+2^2+1^2} = \sqrt{14}.$$

Thus,

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{(-3)1+2(1)+1(2)}{\sqrt{14}} = \frac{1}{\sqrt{14}}$$

The vector projection is just the scalar projection multiplied by the unit vector in the direction of **b** . So,

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{1}{\sqrt{14}}\frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{14}\mathbf{a} = \left\langle \frac{-3}{14}, \frac{1}{7}, \frac{1}{14}\right\rangle$$

1.8 The Cross Product

Suppose that $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ are nonzero vectors. If $\mathbf{c} = \langle c_1, c_2, c_3\rangle$ is a nonzero vector such that $\mathbf{a}\cdot\mathbf{c} = 0$ and $\mathbf{b}\cdot\mathbf{c} = 0$, then

$$a_1c_1 + a_2c_2 + a_3c_3 = 0 \qquad (1)$$

$$b_1c_1 + b_2c_2 + b_3c_3 = 0 \qquad (2)$$

Eliminating c_3 by multiplying (1) by b_3 and (2) by a_3 and subtracting we get:

$$(a_1b_3 - a_3b_1)c_1 + (a_2b_3 - a_3b_2)c_2 = 0$$

This equation is of the form $pc_1 + qc_2 = 0$ which $c_1 = q$ and $c_2 = -p$ can be a solution. In this case,

$$c_1 = a_2b_3 - a_3b_2 \qquad c_2 = a_3b_1 - a_1b_3$$

Substituting the value of c_1 and c_2 to (1) and (2), we get

$$c_3 = a_1b_2 - a_2b_1$$

Thus, a perpendicular vector to both **a** and **b** is

$$\langle c_1, c_2, c_3\rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle.$$

Definition 1.8.1 (Cross Product) If $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, then the **cross product** of **a** and **b** is the vector

$$\mathbf{a}\times\mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle.$$

An eas way in computing the cross product of two vectrs is by means of the symbolic determinant of a 3 by 3 matrix

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Expanding it in the first row as an ordinatry determinant, we have

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

Example 1.8.1 $\mathbf{a} = \langle 2, -1, 3\rangle$ and $\mathbf{a} = \langle 4, -1, 5\rangle$, find $\mathbf{a}\times\mathbf{b}$.

Solution:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 4 & -1 & 5 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ -1 & 5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ 4 & -1 \end{vmatrix}\mathbf{k}$$

$$= (-5-(-3))\mathbf{i} - (10-12)\mathbf{j} + (-2-(-4))\mathbf{k}$$

$$-2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

Example 1.8.2 For any vector **a** in V_3, $\mathbf{a}\times\mathbf{a} = \mathbf{0}$.

Solution:

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ a_1 & a_2 & a_3\end{vmatrix}=\begin{vmatrix}a_2 & a_3\\ a_2 & a_3\end{vmatrix}\mathbf{i}-\begin{vmatrix}a_1 & a_3\\ a_1 & a_3\end{vmatrix}\mathbf{j}+\begin{vmatrix}a_1 & a_2\\ a_1 & a_2\end{vmatrix}\mathbf{k}$$

$$=(a_2a_3-a_2a_3)\mathbf{i}+(a_1a_3-a_1a_3)\mathbf{j}+(a_1a_2-a_1a_2)\mathbf{k}$$

$$=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0}.$$

We have constructed $\mathbf{a}\times\mathbf{b}$ in such a way that it would be perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ Thus, the following theorem follows.

Theorem 1.8.3 The vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

The proof of this theorem will be left as an exercise. This simply done by showing that these dot products $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}$ and $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}$ are both equal to zero.

The cross product $\mathbf{a}\times\mathbf{b}$ points in a direction perpendicular to the plane through a and b. This direction is given by the right-hand rule, that is, if the fingers of your right hand curl in the direction of a rotation (through an angle less than 180°) from $\mathbf{a}$ to $\mathbf{b}$, then your thumb is pointing in the direction of $\mathbf{a}\times\mathbf{b}$ [Figure 1.25].

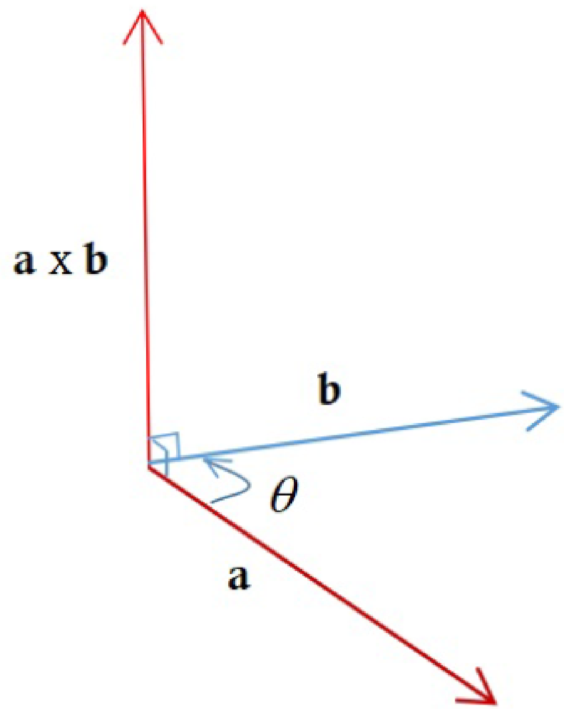


Figure 1.25: The direction of $\mathbf{a} \times \mathbf{b}$

After knowing the direction of $\mathbf{a} \times \mathbf{b}$, we need to know the geometric interpretation of $||\mathbf{a} \times \mathbf{b}||$. This is given in the next theorem below.

Theorem 1.8.4 If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ such that $0 \leq \theta \leq \pi$, then

$$||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}||\,||\mathbf{b}||\sin\theta.$$

If $\mathbf{a}$ and $\mathbf{b}$ are represented by directed line segments with the same initial point, they determine a parallelogram with base $||\mathbf{a}||$ and altitude $||\mathbf{b}||\sin\theta$ [Figure 1.26].

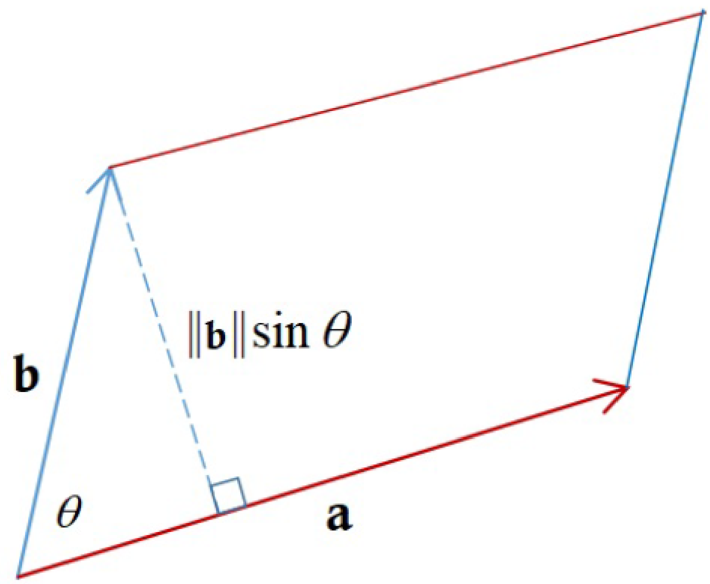


Figure 1.26

So, area of this paralleogram is

$$A = base \times height = \|\mathbf{a}\|(\|\mathbf{b}\|\sin\theta) = \|\mathbf{a}\times\mathbf{b}\|.$$

The length of the cross product $\mathbf{a}\times\mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

Corollary 1.8.5 Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if

$$\mathbf{a}\times\mathbf{b} = \mathbf{0}.$$

Proof: Two nonzero vectors are parellel if and only if the angle θ between them is either 0 or π. In either case, $\sin\theta = 0$. Thus, $\|\mathbf{a}\times\mathbf{b}\| = \mathbf{0}$. Therefore, $\mathbf{a}\times\mathbf{b} = \mathbf{0}$.

Example 1.8.6 Find a vector that is perpendicular to the plane containing the points $P(1,\ 2,\ 1)$, $Q(0,\ 3,\ -1)$ and $R(2,\ 3,\ 5)$.

Solution: Let us find first the representations of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. The resulting cross product $\overrightarrow{PQ}\times\overrightarrow{PR}$ will be a vector perpendicular to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and thus perpendicular to the plane. Now,

$$\overrightarrow{PQ} = (0-1)\mathbf{i} + (3-2)\mathbf{j} + (-1-1)\mathbf{k} = -\mathbf{i}+\mathbf{j}-2\mathbf{k}.$$

$$\overrightarrow{PR} = (2-1)\mathbf{i} + (3-2)\mathbf{j} + (5-1)\mathbf{k} = \mathbf{i} + \mathbf{j} + 4\mathbf{k}.$$

This time, we will compute their cross product

$$\begin{aligned}
\overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -2 \\ 1 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 1 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & -2 \\ 1 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} \mathbf{k} \\
&= (4-(-2))\mathbf{i} - (-4-(-2))\mathbf{j} + (-1-1)\mathbf{k} \\
&= 6\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}
\end{aligned}$$

Therefore, the vector $\langle 6, 2, -2 \rangle$ is perpedicular to the given plane. Note that the vector $\langle 3, 1, -1 \rangle$ is also perpendicular to the given plane and so are the other nonzero scalar multiple of this vector.

Example 1.8.7 Find the area of the triangle with vertices $P(1,2,1)$, $Q(0,3,-1)$ and $Q(2,3,5)$.

Solution:

From the previous example, we have computed $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 6, 2, -2 \rangle$ Thus, the area of the parallelogram with adjacent sides $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is

$$\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \sqrt{6^2 + 2^2 + (-2)^2} = \sqrt{44} = 2\sqrt{11}.$$

Since we are looking for the area of ΔPQR, we just divide the area of the parallelogram by 2. Therefore, the area of ΔPQR is $\frac{1}{2} 2\sqrt{11} = \sqrt{11}$. If we apply the cross product to the standard basis vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, we obtain the following:

Product of Standard Basis Vectors

$$\begin{aligned}
\mathbf{i} \times \mathbf{j} &= \mathbf{k} & \qquad \mathbf{j} \times \mathbf{i} &= -\mathbf{k} \\
\mathbf{j} \times \mathbf{k} &= \mathbf{i} & \qquad \mathbf{k} \times \mathbf{j} &= -\mathbf{i} \\
\mathbf{k} \times \mathbf{i} &= \mathbf{j} & \qquad \mathbf{i} \times \mathbf{k} &= \mathbf{j}
\end{aligned}$$

1.9 Triple Products

The product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is called the **scalar triple product** of vectors **a**, **b** and **c**. In terms of derterminant, can write the scalar triple product as

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The triple product is used in obtaining the volume of a **parallelepiped**–a solid in which all six faces are all parallelograms [Figure 1.27].

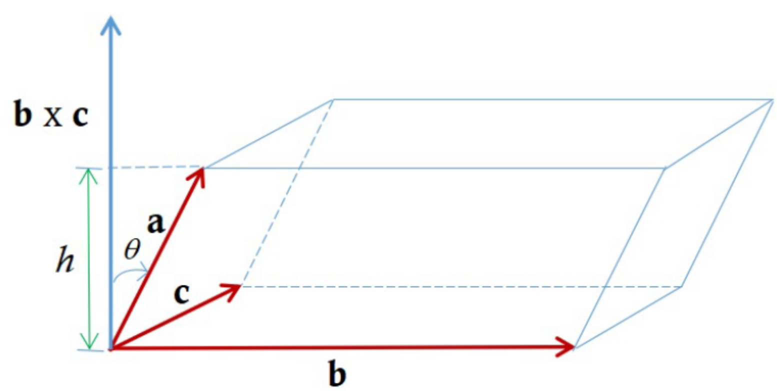


Figure 1.27

The base parallelogram has area $A = \|\mathbf{b}\times\mathbf{c}\|$. The height h of the parallelepiped is $h = \|a\||\cos\theta|$. The absolute value for $\cos\theta$ is used to maintain nonnegativity in cases where $\theta > 90^\circ$. Thus, the volume of the parallelepiped is

$$V = Ah = \|b\times c\|\|a\||\cos\theta| = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|.$$

Therefore, we have this result.

The volume of the parallelepiped determined by the vectors **a**, **b** and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a}\cdot(b\times\mathbf{c})|.$$

Example 1.9.1 Find the volume of the parallelepiped whose coterminous edges are represented by the vectors $\mathbf{a} = \langle 1, -2, 3 \rangle$, $\mathbf{b} = \langle 2, 1, -1 \rangle$ and $\mathbf{c} = \langle 0, 1, 1 \rangle$

Solution: Let us compute the cross product of **b** and **c**.

$$\begin{aligned}
\mathbf{b} \times \mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{k} \\
&= (1 - (-1))\mathbf{i} - (-2 - 0)\mathbf{j} + (2 - 0)\mathbf{k} \\
&= 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}
\end{aligned}$$

Now,

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot (2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \\
&= 1(2) + (-2)(-2) + 3(2) \\
&= 2 + 4 + 6 \\
&= 12
\end{aligned}$$

Thus the volume of the parallelepiped is

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |12| = 12$$

Another solution for this is simply computing direclty the determinant of representation of the scalar triple product. That is,

$$\begin{aligned}
a \cdot (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - (-2) \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} + 3 \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \\
&= 1(1 - (-1)) + 2(-2 - 0) + 3(2 - 0) \\
&= 1(2) + 2(2) + 3(2) \\
&= 12
\end{aligned}$$

Therefore, $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = 12$ which is just the same as the volume obtained in the other solution.

If scalar triple product of vectors **a**, **b** and **c** is equal to 0, then these three vectors are coplanar (lying on the same plane).

Example 1.9.2 Show that the vectors $\mathbf{a} = \langle -4, -6, -2\rangle$, $\mathbf{b} = \langle -1, 4, 3\rangle$, and $\mathbf{c} = \langle -8, -1, 3\rangle$, are coplanar

Solution: Computing the scalar triple product of this vector, we get

$$
\begin{aligned}
a \cdot (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} = -4 \begin{vmatrix} 4 & 3 \\ -1 & 3 \end{vmatrix} - (-6) \begin{vmatrix} -1 & 3 \\ -8 & 3 \end{vmatrix} - 2 \begin{vmatrix} -1 & 4 \\ -8 & -1 \end{vmatrix} \\
&= -4(12 - (-3)) + 6(-3 - (-24)) - 2(1 - (-32)) \\
&= -60 + 126 - 66 \\
&= 0.
\end{aligned}
$$

Since $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$, the three given vectors are coplanar.

1.10 Equation of Lines and Planes

1.10.1 Lines

In a two-dimensional coordinate system, a point and a slope is enough to determine an equation of a line. In a three-dimensional space it is more convenient to use vectors to determine an equation of a line.

Consider the line L through the point $P(x_1, y_1, z_1)$ and parallel to vector $\mathbf{v} = \langle a, b, c\rangle$ [Figure 1.27a]. This vector is the **direction vector** of line L with a, b and c as **direction numbers**.

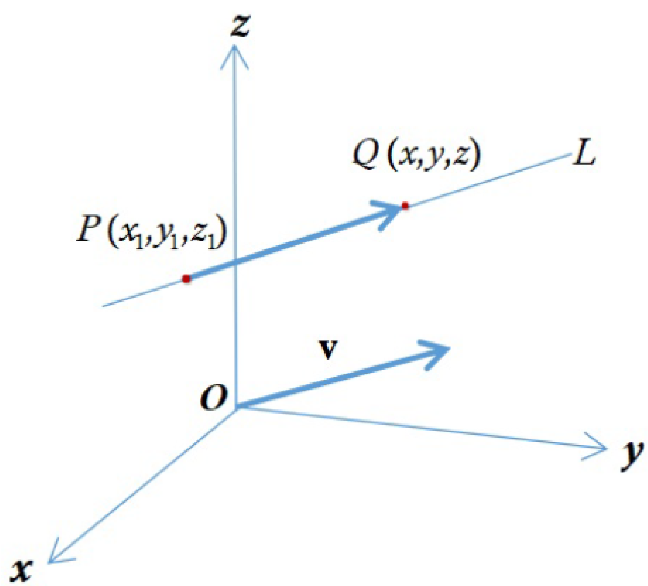


Figure 1.27a

Note that $\overrightarrow{PQ}$ is parallel to $\mathbf{v}$ and so $\overrightarrow{PQ}$ is a scalar multiple of $\mathbf{v}$, say $\overrightarrow{PQ} = t\mathbf{v}$, where t is a scalar (a real number.) Thus,

$\overrightarrow{PQ} = \langle x-x_1, y-y_1, z-z_1\rangle = \langle at, bt, ct\rangle = t\mathbf{v}$

Equating the corresponding components of $\overrightarrow{PQ}$, we obtain the **parametric equations** of the line.

Theorem 1.10.1 (Parametric Equations of a Line in Space) A line L parallel to vector $\mathbf{v} = \langle a, b, c\rangle$ and passing through the point $P(x_1, y_1, z_1)$ is represented by the parametric equations

$$x = x_1 + at, \quad y = y_1 + bt, \quad z = z_1 + ct$$

Example 1.10.2 Find a set of parametric equations of the line L that passes through the point $(-1, -2, 5)$ and parallel to vector $\mathbf{v}\langle 2, 4, -3\rangle$. Find also two other points on the line.

Solution: From the given we have $x_1 = -1$, $x_2 = -2$ and $z_1 = 5$ together with the direction numbers $a = 2, b = 4$ and $c = -3$ Thus, the desired parametric equations are

$$x = -1 + 2t, \quad y = -2 + 4t, \quad z = 5 - 3t$$

To find a point on the line we just assign value to the parameter t. If we set $t = 2$, then we have $x = 3, y = 6$ and $z = -1$. Thus, $(3, 6, 1)$ is on the line. If $t = -1$, we have $x = -1, y = -6$ and

$z = 84$. Thus, $(-1,-3,8)$ is also a point on the line. It is important to note that parametric equations of the line is not unique. For instance, from the above example, if we pick the point (3, 6, 1) which is on the line as our reference point with same direction number we have another set of parametric equations of the same line

$$x = 3 + 2t, \quad y = 6 + 4t \quad z = \quad 1 - 3t$$

Suppose that the direction numbers a, b and c are all nonzero, then we can eliminate the parameter t and obtain the symmetric equations of the line.

(Symmetric Equations of a Line) A line L with direction numbers a, b and c and passing through the point $P(x_1, y_1, z_1)$ is represented by the symmetric equations

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Example 1.10.3 (a) Find parametric and symmetric equations of the line that passes through the point $P(1,2,1)$ and $Q(-1,6,3)$. (b) at what point does this line intersects the xy-plane?

Solution: (a) We will find first the direction vector for the line through P and Q.

$$\mathbf{v} = \overrightarrow{PQ} = \langle -1-1, 6-2, 3-1 \rangle = \langle -2, 4, 2 \rangle = \langle a, b, c \rangle$$

For parametric equations of the line with P as our reference point, we have

$$x = 1 - 2t, \quad y = 2 + 4t \quad z = 1 + 2t$$

For symmetric equations of the line with P as our reference point, we have

$$\frac{x-1}{-2} = \frac{y-2}{4} = \frac{z-1}{2}$$

(b) To find the point of intersection of the line and the xy-plane, we set $z = 0$. Thus,

$$\frac{x-1}{-2} = \frac{y-2}{4} = -\frac{1}{2}$$

Solving for the values of x and y, we get $x = 2$ and $y = 0$. Therefore, the line intersects the xy-plane at the point $(2,0,0)$ In the symmetric equations of the line if one of the direction numbers is zero say $a = 0$, we can still elimnate t and write the equations as

$$x = x_1, \quad \frac{y - y_1}{b} = \frac{z - z_1}{c},$$

This line lies in the vertical plane $x = x_1$.

In a plane with two distinct lines, there are only two possiblities: either the two lines are parallel or the two lines intersect. In three-dimensional space, it is possible that the two lines are not parallel (noncoplanar) and at the same time do not intersect. Any pair of such line with the described characteristics are called **skew lines**.

The lines L_1 and L_2 illustrated below [Figure 1.28] are skew lines

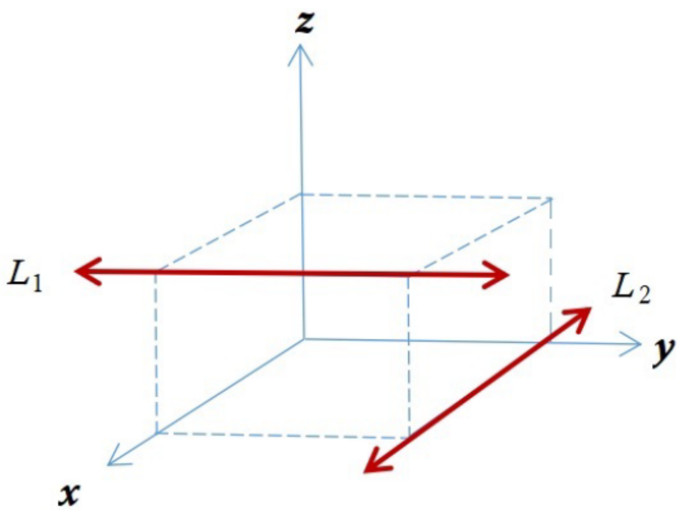


Figure 1.28

Example 1.10.4

$$(a) \quad L_1 : x = 1+t \qquad y = 1+3t \qquad z = 5-t$$
$$L_2 : x = 2s \qquad y = 3+s \qquad z = -3+s$$

$$(b) \quad L_1 : x = 2+t \qquad y = 1+2t \qquad z = 5+3t$$
$$L_2 : x = 1+s \qquad y = 2-s \qquad z = 1+4s$$

Solution: It is obvious that these two lines are not parallel since the directions vectors $\langle 1,3,-1\rangle$ are not parallel (components are not proportional). Let us verify whether L_1 and L_2 would intersect or not. Assuming that these lines intersect, then there would be s and t satisfying

$$1+t = 2s$$
$$1+3t = 3+s$$
$$5-t = -3+s$$

Solving for s and t from the first and second equation, we have $s = 1$ and $t = 1$. These values do not satisfy the third equation. Hence, there is no such values s and t that satsify the three equations simultaneously. Therefore, L_1 and L_2 do not intersect and are skew lines.

The lines L_1 and L_2 are also not parallel since the direction vectors $\langle 1,2,3\rangle$ and $\langle 1,-1,4\rangle$ are not parallel (not proportional). Let us verify whether L_1 and L_2 would intersect or not. Assuming that these lines intersect, then there would be s and t satisfying

$$2+t = 1+s$$
$$1+2t = 2-s$$
$$5+3t = 1+4s$$

Just like in (a), solving for s and t from the first two equations we have, $s = 1$ and $t = 0$. Note also that these values also satisfy the third equation, that is, $5+3(0) = 1+4(1)$.

Thus, L_1 and L_2 are intersecting lines. To find the coordinates of the point of intersection, we just substitute the value of s in L_2 or the value of t in L_1. In either case, $x = 2, y = 1$ and $z = 5$. Therefore, L_1 and L_2 interesect at the point $(2,1,5)$. The graphs of the the pair of lines in (a) and (b) in the above example are shown below [Figure 1.29] and [Figure 1.30] respectively.

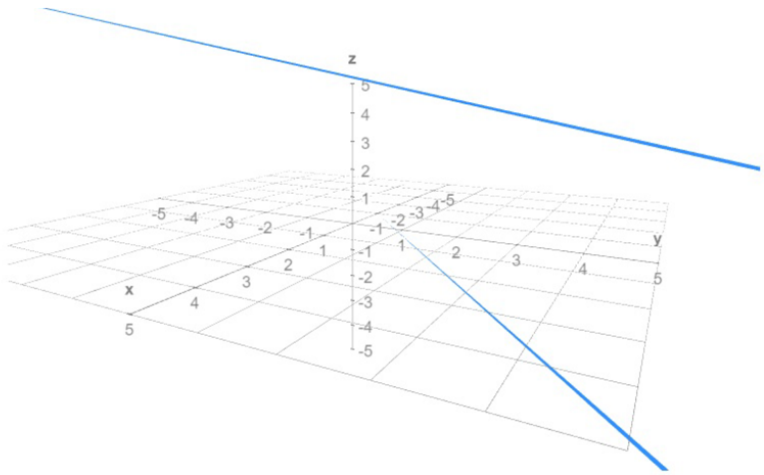


Figure 1.29

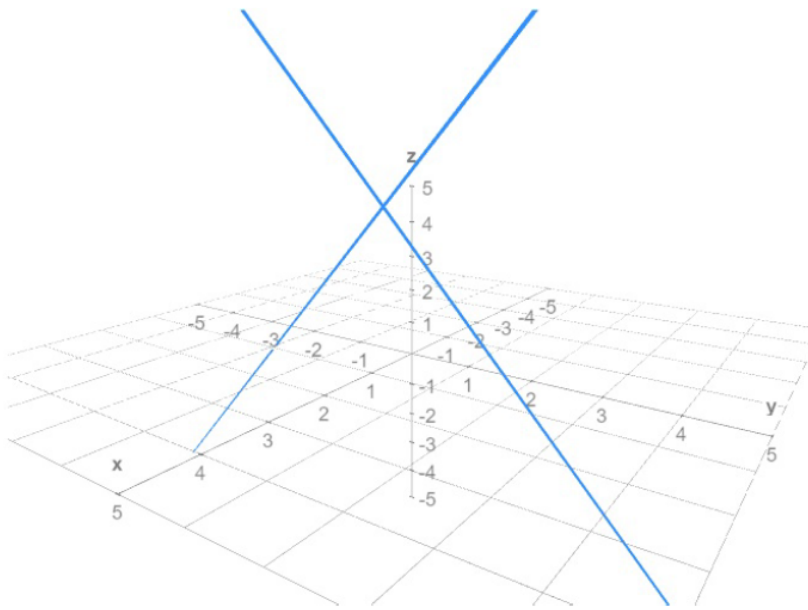


Figure 1.30

1.10.2 Planes

We have learned that an equation of a line in space can be determined by point and a vector parallel to it. One vector parallel to a plane cannot solely determine the direction of the plane but a vector perpendicular to that plane will completely specify its direction.

Consider the plane containing point $P(x_1, x_2, x_3)$ with nonzero normal vector $\mathbf{n} = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle$ [Figure 1.30a].

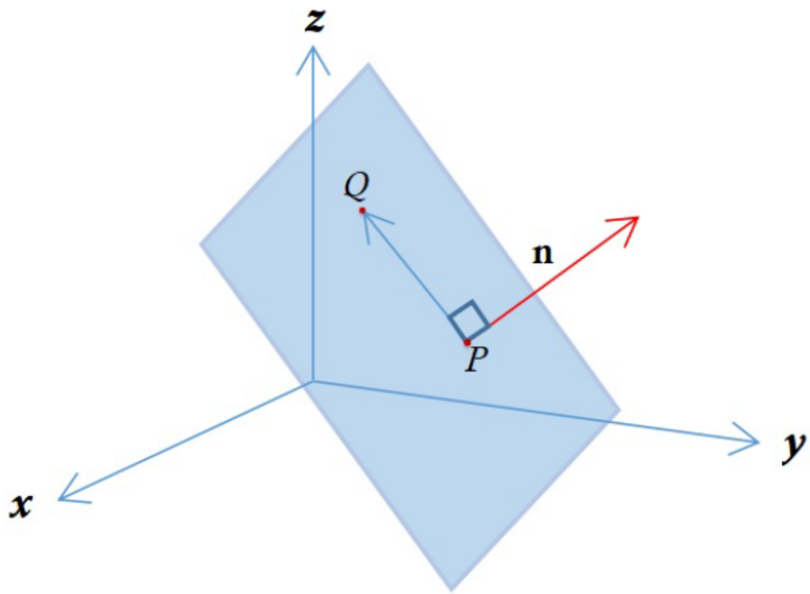


Figure 1.30a

This plane consists of all points $Q(x,y,z)$ such that the vector $\overrightarrow{PQ}$ is orthogonal to $\mathbf{n}$. Thus,

$$\mathbf{n}\cdot\overrightarrow{PQ} = 0$$
$$\langle a,b,c\rangle \cdot \langle x-x_1, y-y_1, z-z_1\rangle = 0$$
$$a(x-x_1)+b(y-y_1)+c(z-z_1) = 0$$

The third equation is called the **standard form** of the equation of a plane.

The plane containing the point (x_1,y_1,z_1) and having normal vector $\mathbf{n} = \langle a,b,c\rangle$can be represented by the standard form of the equation of a plane

$$a(x-x_1)+b(y-y_1)+c(z-z_1) = 0$$

Given the general form of the equation of a plane, it is easy to find a normal vector to the plane. Simply use the coefficients of x,y and z and write $\mathbf{n} = \langle a,b,c\rangle$.

In space, it takes three noncollinear points to determine an equation of a plane. If the three points are given you may substitute the each point to the general form and obtain a system consisting of three equations and then substituting back the values obtained to the general form. But there is an easy and simple way in obtain the equation of the plane when three points are given, and that is with the help of vectors.

Example 1.10.5 Find an equation of the plane passing through the points $P(1,0,0)$, $Q(2,0,1)$ and $R(1,1,1)$.

Solution: Substituting each of the coordinates in points P(1, 0, 0), Q(2, 0, 1) and R(1, 1, 1) in the general form, we obtain the system

$$\begin{cases} a+ \quad + \quad +d=0 & (1) \\ 2a+ \quad +c+d=0 & (2) \\ a+b+c+d=0 & (3) \end{cases}$$

From (1), $a=-d$. Substituting $a=-d$ to (2), we get $c=d$. Finally substituting $a=-d$ and $c=d$ to (3), we have $b=-d$. Now substituting these values to the general form, we obtain

$$-dx-dy+dz+d=0$$

or equivalently,

$$-x-y+z+1=0.$$

Therefore, the desired equation of the plane is $x+y-z-1=0$.

Let us try another solution for the same example using vectors.

Solution: Let us construct the vectors $\mathbf{u}=\overrightarrow{PQ}$ and $\mathbf{v}=\overrightarrow{PR}$. From the given points, we have

$$\mathbf{u}=\overrightarrow{PQ}=\langle 1,0,0\rangle \text{ and } \mathbf{v}=\overrightarrow{PR}.$$

The cross product of $\mathbf{u}$ and $\mathbf{v}$ be the normal vector $\mathbf{n}$ which is perpendicular to the plane that will tell us teh direction of the plant. Now,

$$\mathbf{n}=\mathbf{u}\times\mathbf{v}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}=(0-1)\mathbf{i}-(1-0)\mathbf{j}+(1-0)\mathbf{k}=-\mathbf{i}-\mathbf{j}+\mathbf{k}.$$

Hence, $\mathbf{n}=\langle -1,-1,1\rangle$.

Picking the point $P(1,0,0)$ together with $\mathbf{n}$ and substituting to the standard form of the equation of a plane, we have

$$-1(x-1)-1(y-0)+1(z-0)=0$$

or equivalently,

$$-x-y+z+1=0$$

which is precisely the same as the first solution that we have.

Two distinct planes in space are either parallel or intersect in line. For two planes that intersect, you can determine the angle $(0 \leq \theta \leq 90^\circ)$ between them from the angle in their normal vector [Figure 1.30b]

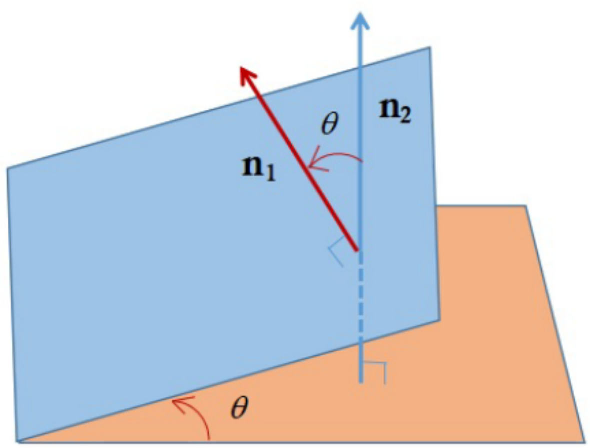


Figure 1.30b

Specifically, if vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are normal to two intersecting planes, then the angle θ between normal vectors in equal to the angle between the two planes and

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|},$$

Consequently, two planes with normal vector $\mathbf{n}_1$ and $\mathbf{n}_2$ are 1. perpendicular if $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$ 2. parallel when $\mathbf{n}_1$ is a scalar multiple of $\mathbf{n}_2$.

Example 1.10.6 Find the angle θ between the two planes $x-2y+z=0$ and $2x+3y-2z=0$. Then find the parametric equations of the line of intersection.

Solution: The normal vectors for the planes are $\mathbf{n}_1 = \langle 1, -2, 1\rangle$ and $\mathbf{n}_2 = \langle 2, 3, -2\rangle$. Thus,

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{|1(2) - 2(3) + 1(-2)|}{\sqrt{1+4+1}\sqrt{4+9+4}} = \frac{|-6|}{\sqrt{6}\sqrt{17}} = \frac{6}{\sqrt{102}}$$

Hence,

$$\theta = \cos^{-1}\frac{6}{\sqrt{102}} \approx 53.55^\circ.$$

You can find the line intersection of the two planes by simultaneously solving the two linear equations representing the planes.

$$\begin{cases} x - 2y + z = 0 \quad (1) \\ 2x + 3y - 2z = 0 \quad (2) \end{cases}$$

If we multiply equation (1) by -2 and adding to equation (2), we have $7y - 4z = 0$ for which $y = \frac{4}{7}z$. Substituting $y = \frac{4}{7}z$ to either (1) or (2), we have $x = \frac{1}{7}$. Letting $t = \frac{z}{7}$ would mean that $z = 7t$. Thus,

$$x = t, \quad y = 4t, \quad z = 7t$$

are the parametric equations of the line of intersection. Since the line of intersection lies on both planes, it is perpendicular to a normal vector of the two planes. Therefore, the direction of the line intersection can be determined by simply computing the cross product of the normal vectors of the planes.

Example 1.10.7 Find the symmetric equations of the lines of intersection of the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.

Solution: We only need here two things. One is any point on the line and the one is the direction numbers for the line. By setting $z = 0$ to both equations, we can find the point where the line intersect the xy-plane. In our case, after setting $z = 0$, we have the system

$$\begin{cases} x + y = 1 \\ x - 2y = 1 \end{cases}$$

Solving this system gives us $x = 1$ and $y = 0$. Thus, the points $(1,0,0)$ is on the line. Now,

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}.$$

Therefore, the symmetric equations of the line is

$$\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}.$$

1.11 Distances

Let us derive the formula for the distance D from the point $P_1(x_1, y_1, z_1)$ to a plane

$$ax + by + cz + d = 0.$$

The distance from a point to a plane is the length of the shortest line segment conneecting that point to the plane.

Let $P_0(x_0, y_0, z_0)$ be any point on the plane and let $\mathbf{b}$ be the vector corresponding to $\overrightarrow{P_0P_1}$ [Figure 1.31]

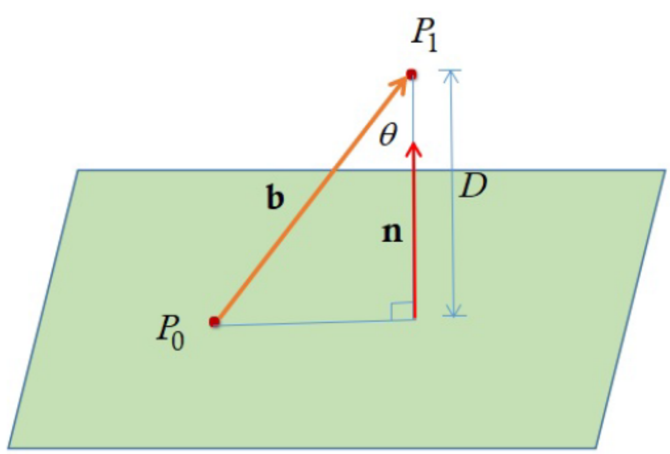


Figure 1.31

Then $\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$ The Length D is simply the absolute value of the scalar projetion of **b** onto the direction of the normal vector $\mathbf{n} = \langle a, b, c\rangle$. Thus,

$$\begin{aligned} D = \text{comp}_{\mathbf{n}}\mathbf{b} = \frac{|\mathbf{n}\cdot\mathbf{b}|}{\|\mathbf{n}\|} \\ = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2+b^2+c^2}} \\ = \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2+b^2+c^2}} \end{aligned}$$

Since $P_0(x_0, y_0, z_0)$ is a point on the plane, it must satisfy the equation of the plane $ax_0 + by_0 + cz_0 + d = 0$ or $ax_0 + by_0 + cz_0 = -d$. Therefore, we arrived at the desired formula.

The distance D from the point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is given by

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}}.$$

Example 1.11.1 Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

Solution: The two given planes are parallel because their normal vectors $\langle 10, 2, -2\rangle$ and $\langle 5, 1, -1\rangle$ are parallel. The next thing to do is to pick an arbitrary point on the plane and find its distance to the other plane.

If we set $y = z = 0$ in the first equation we obtain $x = \dfrac{1}{2}$. Thus, $(\frac{1}{2}, 0, 0)$ is a point on the plane $10x + 2y - 2z = 5$.

Hence, the distance from the point $\left(\frac{1}{2}, 0, 0\right)$ to the line $5x + y - z = 1$ is

$$D = \frac{|5(\frac{1}{2}) + 1(0) - 1(0) - 1|}{\sqrt{25+1+1}} = \frac{(3/2)}{\sqrt{27}} = \frac{\sqrt{3}}{6}$$

Example 1.11.2 find the distance between the skew lines

$$L_1 : x = 1 + t \quad y = -2 + 3t \quad z = 4 - t$$

$$L_2 : x = 2s \quad y = 3 + s \quad z = -3 + 4s$$

Since the two lines L_1 and L_2 are skew, they can be viewed as lines lying on two parallel planes P_1 and P_2. The distance between L_1 and L_2 is the same as the distance between P_1 and

P_2. The cross product of the vector representations of the direction of the two lines is both normal to P_1 to P_2. So,

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = 13\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}$$

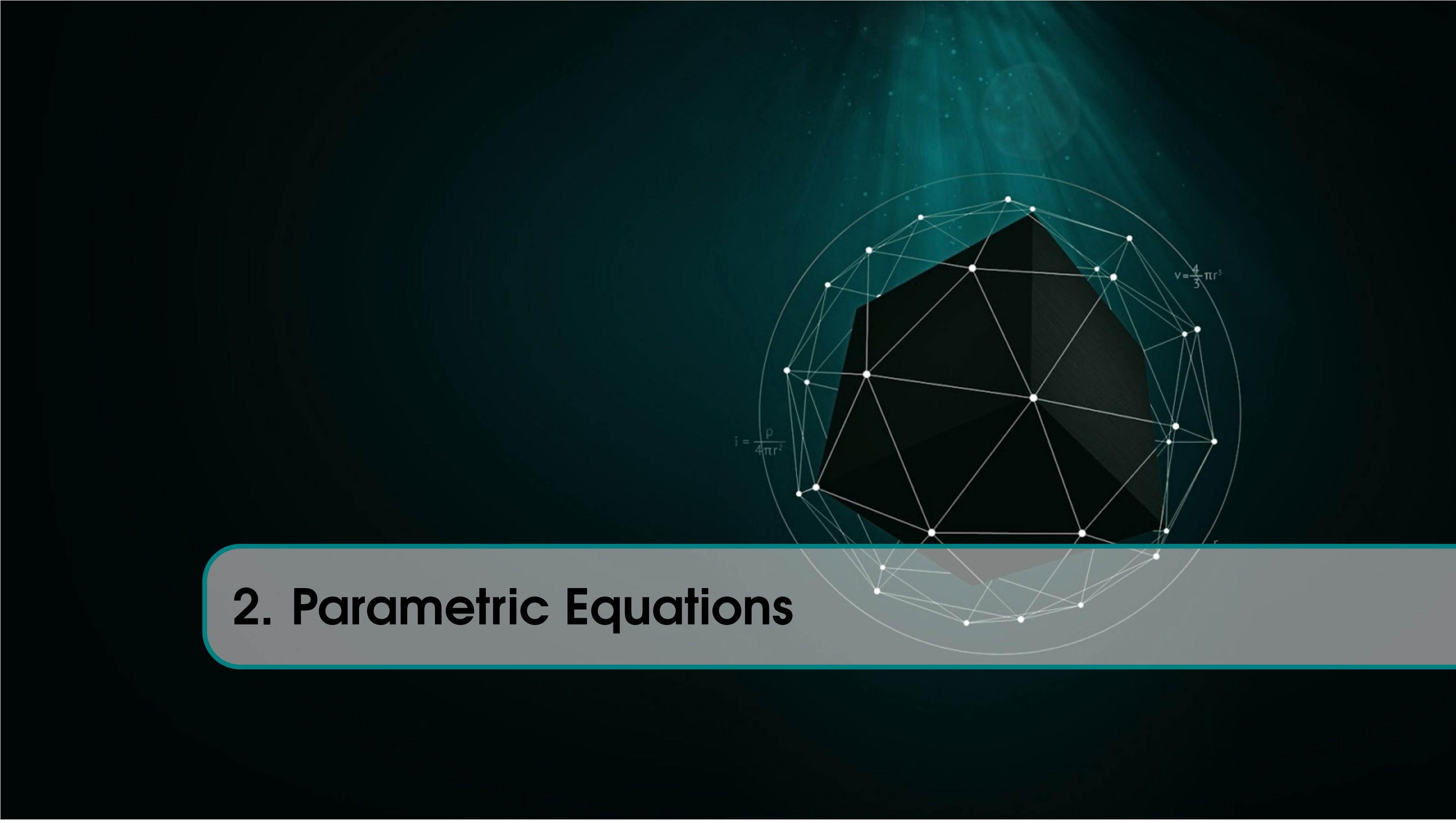
Now, we look for a point in L_2. Set the parameter $s = 0$ and obtain the point $(0, 3, -3)$. So an equation of the plane containing L_2 and normal to $\mathbf{n}$ is

$$13(x-0) - 6(y-3) - 5(z+3) = 0 \text{ or } 13x - 6y - 5z + 3 = 0.$$

We now pick a point on L_1. By setting $t = 0$, we obtain the point $(1, -2, 4)$. The distance from this point to the plane $13x - 6y - 5z + 3 = 0$ is the distance between the skwe lines, that is

$$D = \frac{13(1) - 6(-2) - 5(4) + 3}{\sqrt{169 + 36 + 25}} = \frac{8}{\sqrt{230}}.$$

Therefore, the distance between the skew lines is $\dfrac{8}{\sqrt{230}}$.



2. Parametric Equations

2.1 Curves Defined by the Parametric equations

Imagine that a particle moves along the curve C shown in the following figure:

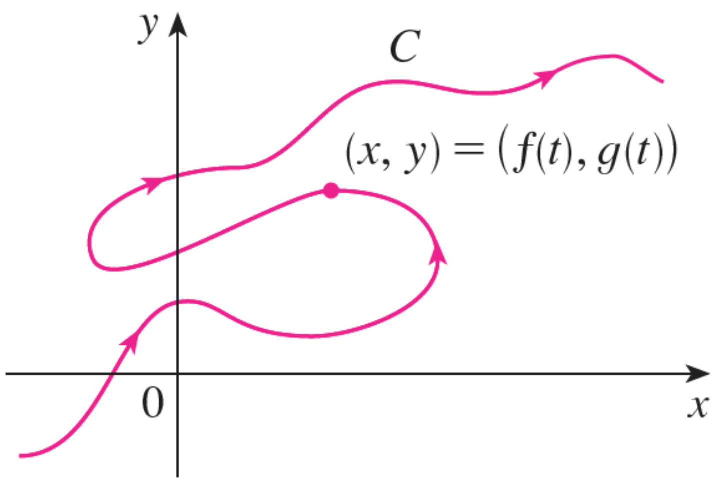


Figure 2.1

It is impossible to describe C by an equation of the form $y = f(x)$ because C fails the Vertical Line Test. But the $x-$ and $y-$ coordinates of the particle are functions of time and so we can write $x = f(t)$ and $y = g(t)$. Such a pair of equations is often a convenient way of describing a curve and gives rise to the following definition.

Suppose that x and y are both given as functions of a third variable t (called a **parameter**) by the equations

$$x = f(t) \qquad y = g(t)$$

(called **parametric equations**) Each value of t determines a point (x,y) which we can plot in a coordinate plane. As t varies, the point $(x,y) = (f(t), g(t))$ varies and traces out a curve C, which we call a parametric curve. The parameter t does not necessarily represent time and, in fact, we could use a letter other than t for the parameter. But in many applications of parametric curves, t does denote time and therefore we can interpret $(x,y) = (f(t), g(t))$ as the position of a particle at time t.

Example 2.1.1 Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \quad y = t + 1$$

Solution: Each value of t gives a point on the curve, as shown in the table. For instance, if $t = 0$, then $x = 0$, $y = 1$ and so the corresponding point is $(0,1)$. In Figure 2 we plot the points (x,y) determined by several values of the parameter and we join them to produce a curve.

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5

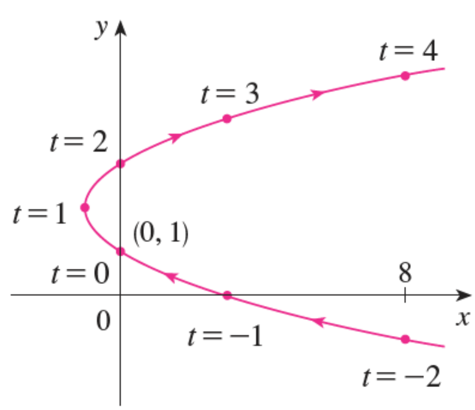


Figure 2.2

A particle whose position is given by the parametric equations moves along the curve in the direction of the arrows as t increases. Notice that the consecutive points marked on the curve appear at equal time intervals but not at equal distances. That is because the particle slows down and then speeds up as t increases. It appears from Figure 2.2 that the curve traced out by the particle may be a parab ola. This can be confirmed by eliminating the parameter t as follows. We obtain $t = y-1$ from the second equation and substitute into the first equation. This gives

$$x = t^2 - 2t = (y-1)^2 - 2(y-1) = y^2 - 4y + 3$$

and so the curve representad by the given parametric equations is the parabola $x = y^2 - 4y + 3$.

No restriction was placed on the parameter t in the Example 2.1.1, so we assumed that t could be any real number. But sometimes we restrict t to lie in a finite interval. For instance, the parametric curve

$$x = t^2 - 2t \quad y = t+1 \qquad 0 \leq t \leq 4.$$

shown in Figure 2.3 is the part of the parabola in Example 2.1.1 that starts at the point $(0,1)$ and ends at the point $(8,5)$. The arrowhead indicates the direction in which the curve is traced as t increases from 0 to 4. In general, the curve with parametric equations

$$x = f(t) \quad y = g(t) \qquad a \leq t \leq b$$

has **initial point** $(f(a), g(a))$ and **terminal point** $(f(b), g(b))$.

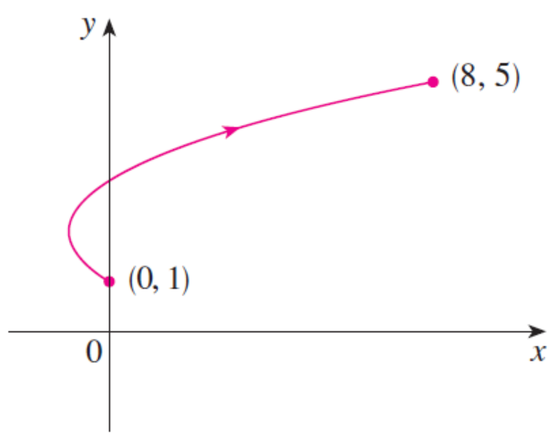


Figure 2.3

Example 2.1.2 What curve is represented by the following parametric equations?

$$x = \cos t \qquad y = \sin t \qquad 0 \leq t \leq 2\pi.$$

Solution: If we plot points, it appears that the curve is a circle. We can confirm this impression by eliminating t. Observe that

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1.$$

Thus the point (x, y) moves on the unit circle $x^2 + y^2 = 1$. Notice that in this example the parameter t can be interpreted as the angle (in radians) shown in Figure 2.3.4.

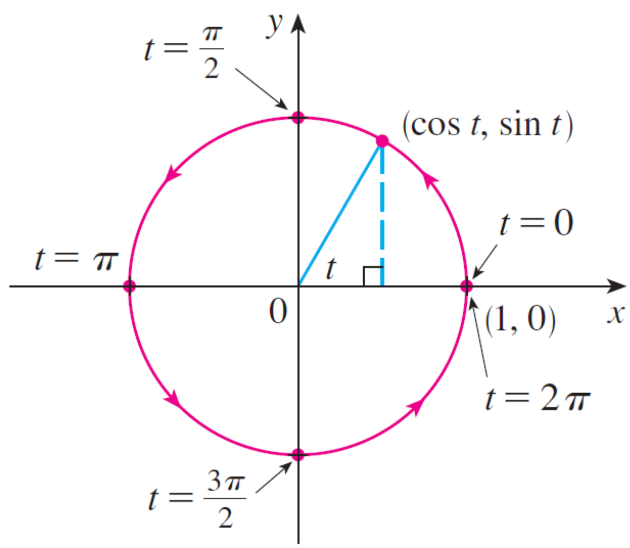


Figure 2.4

As t increases from 0 to 2π, the point $(x, y) = (\cos t, \sin t)$ moves once around the circle in the counterclockwise direction starting from the point $(1, 0)$.

Example 2.1.3 What curve is represented by the given parametric equations?

$$x = \sin 2t \quad y = \cos 2t \quad 0 \leq t \leq 2\pi.$$

Solution: Again we have

$$x^2 + y^2 = \sin^2 2t + \cos^2 2t = 1$$

so the parametric equations again represent the unit circle $x^2 + y^2 = 1$. But as t increases from 0 to 2π, the point $(x, y) = (\sin 2t, \cos 2t)$ starts at $(0, 1)$ and moves twice around the circle in the clockwise direction as indicated in Figure 2.5.

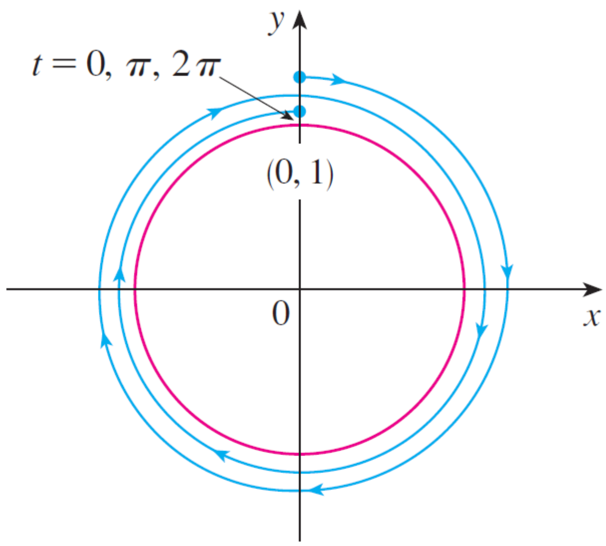


Figure 2.5

Examples 2.1.2 and 2.1.3 show that different sets of parametric equations can represent the same curve. Thus we distinguish between a *curve*, which is a set of points, and a *parametric curve*, in which the points are traced in a particular way.

Example 2.1.4 Find parametric equations for the circle with center (h,k) and radius r.

Solution: If we take the equations of the unit circle in Example 2.1.2 and multiply the expressions for x and y by r, we get $x = r\cos t$, $y = r\sin t$. You can verify that these equations represent a circle with radius r and center the origin traced counterclockwise. We now shift h units in the x-direction and k units in the y-direction and obtan parametric equations of circle (Figure 6)

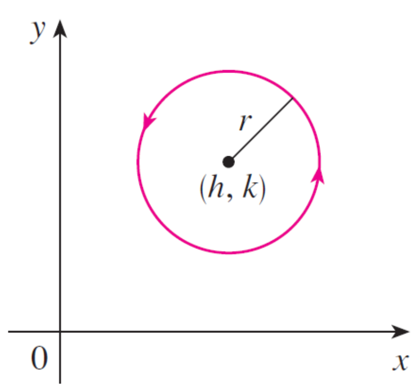


Figure 2.6 $x = h + r\cos t \quad y = k + r\sin t$

with center (h,k) and radius r.

$$x = h + r\cos t \quad y = k + r\sin t \quad 0 \le t \le 2\pi$$

Example 2.1.5 Sketch the curve with parametric equations $x = \sin t$, $y = \sin^2 t$

Solution: Observe that $y = (\sin t)^2 = x^2$ and so the point (x,y) moves on the parabola $y = x^2$. But note also that, since $-1 \le \sin t \le 1$, we have $-1 \le x \le 1$, so the parametric equations represent only the part of the parabola for which $-1 \le x \le 1$. Since t is periodic, the point $(x,y) = (\sin t, \sin^2 t)$ move back and forth infinitely often along the parabola from $(-1,1)$ to $(1,1)$. (See Figure 2.7)

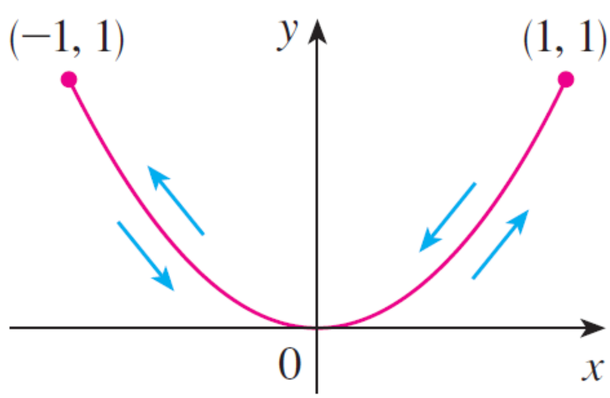


Figure 2.7

2.1.1 Graphing Devices

Most graphing calculators and other graphing devices can be used to graph curves defined by parametric equations. In fact, it's instructive to watch a parametric curve being drawn by a graphing calculator because the points are plotted in order as the corresponding parameter values increase.

Example 2.1.6 Use a graphing device to graph the curve $x = y^4 - 3y^2$.

Solution: If we let the parameter be $t = y$, then we have the equations

$$x = t^4 - 3t^2 \qquad y = t.$$

Using these parametric equations to graph curve, we obtain Figure 2.8.

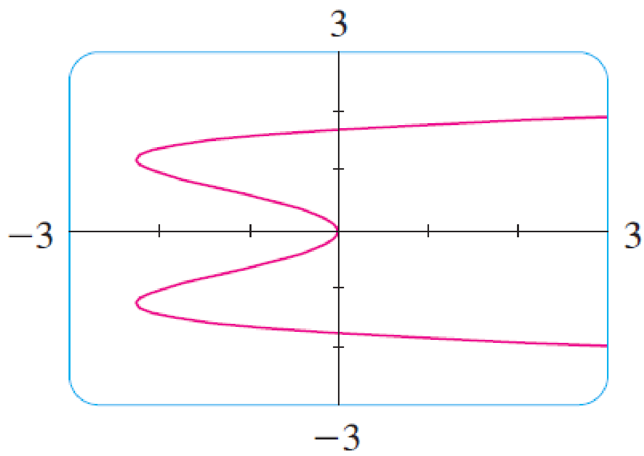


Figure 2.8

It would be possible to solve the given equation $(x = y^4 - 3y^2)$ for y as four functions of x and graph them individually, but the parametric equations provide a much easier method.

In general, if we need to graph an equation of the form $x = g(y)$, we can use the parametric equations

$$x = g(t) \qquad y = t.$$

Notice also that curves with equations $y = f(x)$ (the ones we are most familiar with graphs of functions) can also be regarded as curves with parametric equations

$$x = t \quad y = f(t)$$

Graphing devices are particularly useful for sketching complicated parametric curves. For instance, the curves shown in Figure 2.9, 2.10, 2.11 would be virtually impossible to produce by hand.

Example 2.1.7 The graph given by the parametric equations

$$x = t + \sin 5t \quad y = t + \sin 6t :$$

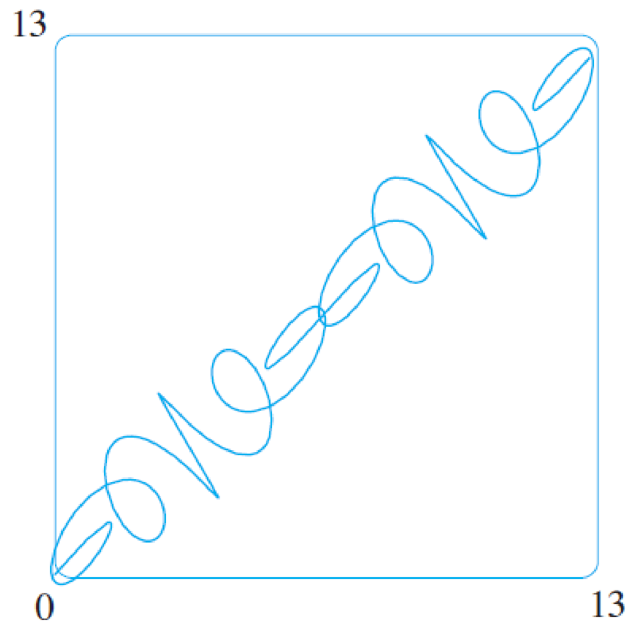


Figure 2.9 $x = t + \sin 5t \quad y = t + \sin 6t$

Example 2.1.8 The graph given by the parametric equations

$$x = \sin 9t \quad y = \sin 10t :$$

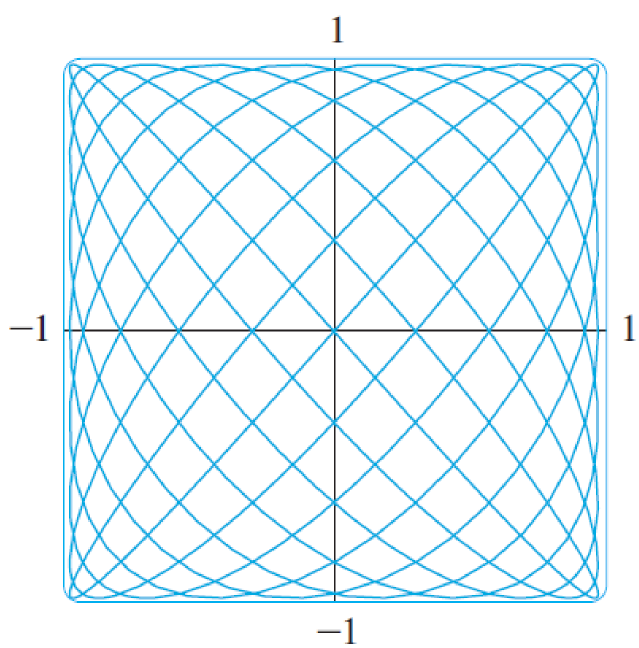


Figure 2.10 $x = \sin 9t \quad y = \sin 10t$

Example 2.1.9 The graph given by the parametric equations

$$x = 2.3\cos 10t + \cos 23t \quad y = 2.3\sin 10t - \sin 23t:$$

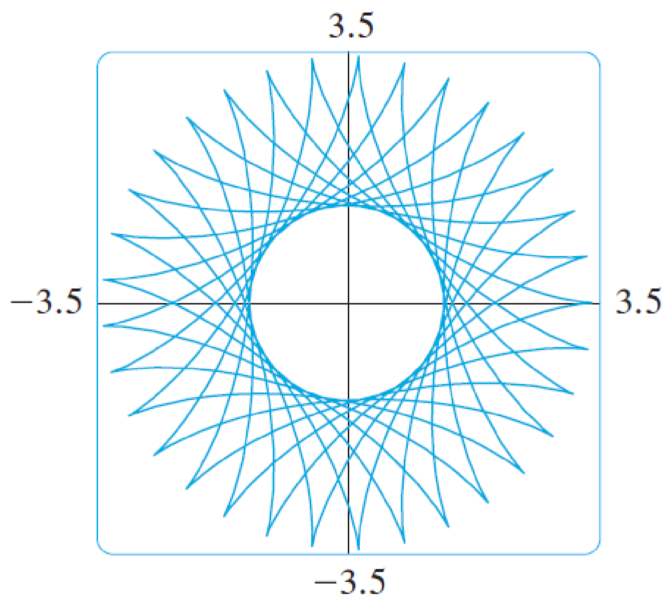


Figure 2.11 $x = 2.3\cos 10t + \cos 23t \quad y = 2.3\sin 10t - \sin 23t$

Example 2.1.10 (The Cycloid) The curve traced out by a point P on the circumference of a circle as circle rolls along a straight line is called a cycloid (see Figure 2.12).

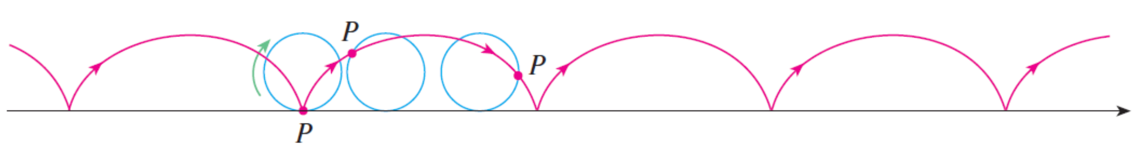


Figure 2.12

If the circle has radius and rolls along the x-axis and if one position of P is the origin, find parametric equations for the cycloid.

Solution: We choose as parameter the angle of rotation θ of the circle ($\theta = 0$ when P is at the origin). Suppose the circle has rotated through θ radians. Because the circle has been in contact with the line (see from Figure 2.13)

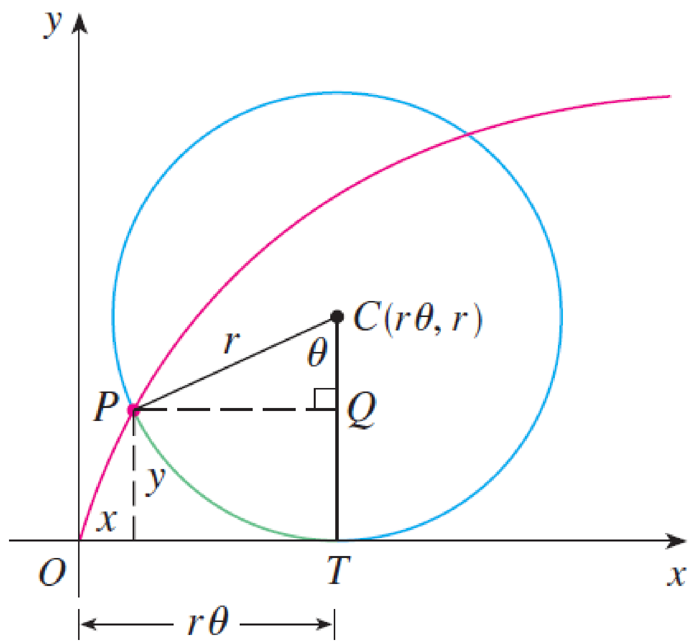


Figure 2.13

the distance it has rolled from the origin is

$$|OT| = \text{arc } PT = r\theta.$$

Therefore the center of the circle is $C(r\theta, r)$. Let the coordinates of P be (x, y). Then from Figure 2.13 we see that

$$x = |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta)$$

$$y = |TC| - |QC| = r - r\cos\theta = r(1 - \cos\theta)$$

Therefore, parametric equation of the cycloid are

$$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta) \quad \theta \in \mathbb{R}$$

Example 2.1.11 Family of Parametric Curves Investigate the family of curves with parametric equations

$$x = a + \cos t \qquad y = a\tan t + \sin t.$$

What do these curves have in common? How does the shape changes as a increases?

Solution: We use a graphing device to product the graphs for the cases $a = -2, -1, -\frac{1}{2}, -\frac{1}{5}, 0, \frac{1}{2}$ and 2 shown in Figure 2.14.

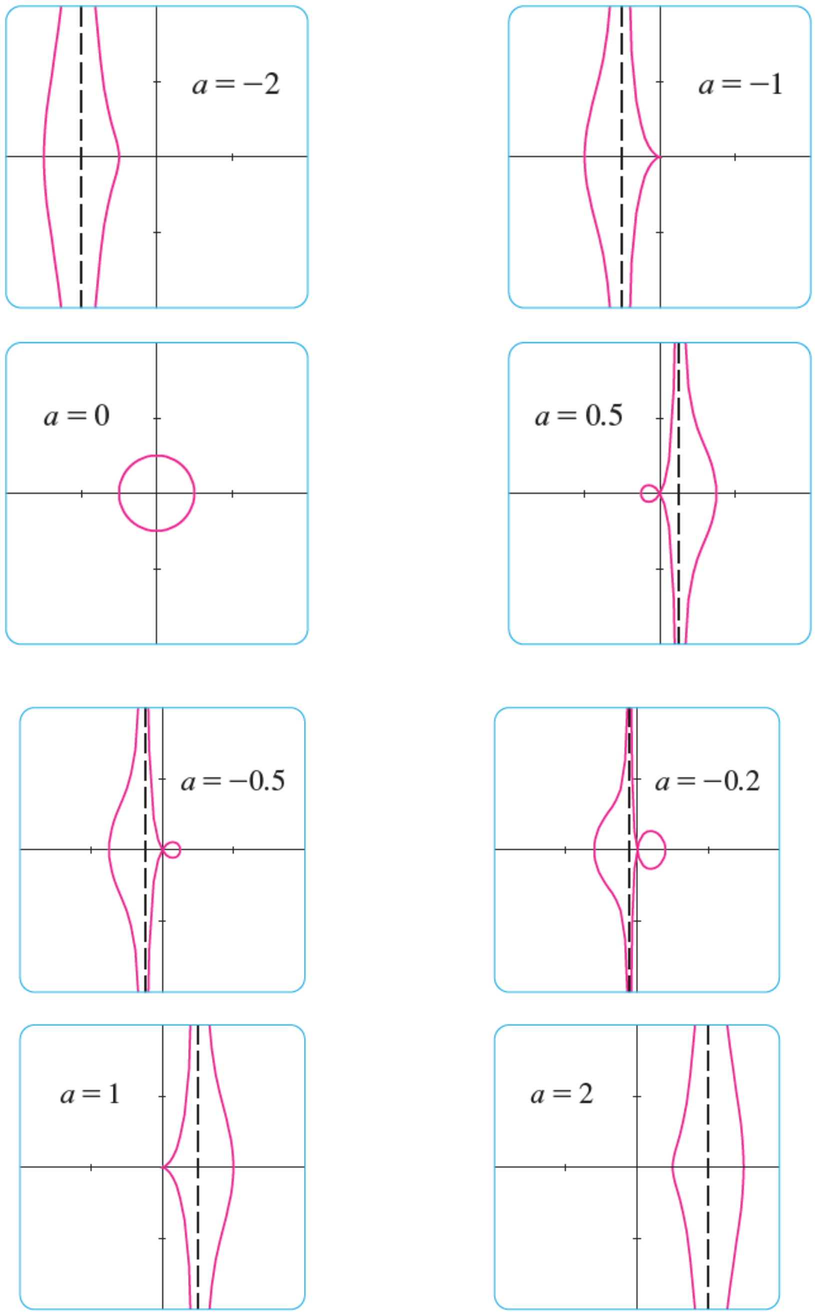


Figure 2.14

Notice that all of these curves (except the case $a = 0$) have two branches, and both branches approach the vertical asymptote $x = a$ as x approaches a from the left or right. When $a < 21$, both branches are smooth; but when a reaches -1, the right branch acquires a sharp point, called a *cusp*.

For a between -1 and 0 the cusp turns into a loop, which becomes larger as a approaches 0. When $a = 0$, both branches come together and form a circle (see Example 2.1.2). For a between 0 and 1, the left branch has a loop, which shrinks to become a cusp when $a = 1$. For $a > 1$, the branches become smooth again, and as a increases further, they become less curved. Notice that the curves with a positive are reflections about the y-axis of the corresponding curves with a negative.

These curves are called **conchoids of Nicomedes** after the ancient Greek scholar Nicomedes. He called them conchoids because the shape of their outer branches resembles that of a conch shell or mussel shell

2.2 Tangents

Suppose f and g are differentiable functions and we want to find tangent line at a point on the parametric curve $x = f(t)$, $y = g(t)$, where y is also a differentiable function of x. Then the Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

If $\dfrac{dx}{dt} \neq 0$, we can solve for $\dfrac{dy}{dx}$:

$$(1) \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{if} \quad \frac{dx}{dt} \neq 0$$

Equation 1 (which you can remember by thinking of cancelling the dt's) enable us to find the slope $\dfrac{dy}{dx}$ of the tangent to a parametric curve without having to eliminate the parameter t. We see from (1) that the curve has a horizontal tangent when $\dfrac{dy}{dt} = 0$ (provided that $\dfrac{dx}{dt} \neq 0$) and it has a vertical tangent when $\frac{dx}{dt} = 0$ (provided that $\dfrac{dy}{dt} \neq 0$.) This information is useful for sketching parametric curves.

As we know from Chapter 4, it is also useful to consider $\dfrac{d^2y}{dx^2}$. This can be found by replacing y by $\dfrac{dy}{dx}$ in Equation (1):

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}$$

Remark 2.2.1

$$\frac{d^2y}{dx^2} \neq \frac{\dfrac{d^2y}{dt^2}}{\dfrac{d^2x}{dt^2}}.$$

Also, if we think of the curve as being traced out by a moving particle, then $\dfrac{dy}{dt}$ and $\dfrac{dx}{dt}$ are vertical and horizontal velocities of the particle respectively and formula (1) says that the slope of the tangent is the **ratio** of these velocities.

Example 2.2.2 A curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.

a) Show that C has two tangents at the point $(3,0)$ and find their equations.

b) Find the points on C where the tangent is horizontal or vertical

c) Determine where the curve is concave upward or downward

e) Sketch the curve.

Solution: (a) : Notice that $y = t^3 - 3t = t(t^2 - 3) = 0$ when $t = 0$ or $t = \pm\sqrt{3}$. Therefore, the points $(3,0)$ on C arises from two values of the parameter, $t = \sqrt{3}$ and $t = -\sqrt{3}$. This indicates that C crosses itself at $(3,0)$. Since

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{3t^2-3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right)$$

the slope of the tangent when $t = \pm\sqrt{3}$ is $\dfrac{dy}{dx} = \pm\dfrac{6}{2\sqrt{3}} = \pm\sqrt{3}$, so equations of the tangent lines at $(3,0)$ are

$$y = \sqrt{3}(x-3) \quad \text{and } y = -\sqrt{3}(x-3)$$

(b) C has a horizontal tangent when $\dfrac{dy}{dx} = 0$, that is, when $\dfrac{dy}{dt} = 0$ and $\dfrac{dx}{dt} = 0$. Since $\dfrac{dy}{dt} = 3t^2 - 3$, this happens when $t^2 = 1$, that is, $t = \pm 1$, The corresponding points on C are

$(1,-2)$ and $(1,2)$. C has a vertical tangent when $\frac{dx}{dt} = 2t = 0$, that is, $t = 0$. (Note that $\frac{dy}{dt} \neq 0$ there.) The corresponding point on C is $(0,0)$.

(c) To determine concavity we calculate the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{3}{2}\left(1+\frac{1}{t^2}\right)}{2t} = \frac{3(t^2+1)}{4t^3}.$$

Thus the curve is concave upward when $t > 0$ and concave downward when $t < 0$.

(d) Using the informaltion from parts (b) and c, we sketch C in Figure 2.15

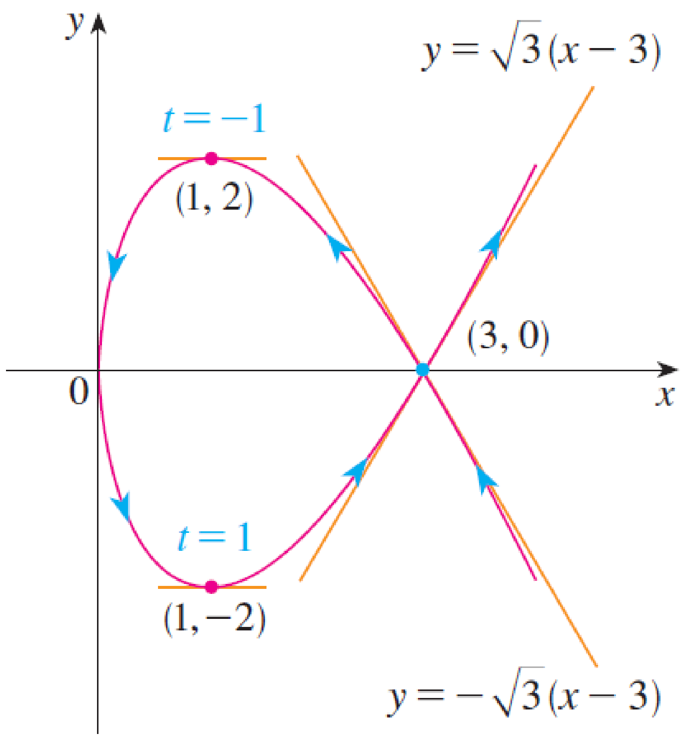


Figure 2.15

Example 2.2.3 (a) Find the tangent to the cycloid $x = r(\theta - \sin\theta)$, $y = r(1-\cos\theta)$ at the point $\theta = \frac{\pi}{3}$. (b) At what points is the tangent horizontal? When it is vertical?

Solution: (a) The slope of the tangent line is

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r\sin\theta}{r(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}.$$

When $\theta = \dfrac{\pi}{3}$, we have

$$x = r\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right) = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) \quad y = r\left(1 - \cos\frac{\pi}{3}\right) = \frac{r}{2}.$$

and

$$\frac{dy}{dx} = \frac{\sin\left(\frac{\pi}{3}\right)}{1 - \cos\left(\frac{\pi}{3}\right)} = \frac{\frac{\sqrt{3}}{2}}{1 - \frac{1}{2}} = \sqrt{3}$$

Therefore the slope of the tangent is $\sqrt{3}$ and its equation is

$$y - \frac{r}{2} = \sqrt{3}\left(x - \frac{r\pi}{3} + \frac{r\sqrt{3}}{2}\right) \quad \text{or} \quad x - y = r\left(\frac{\pi}{\sqrt{3}} - 2\right)$$

The tangent is sketched in Figure 2.16.

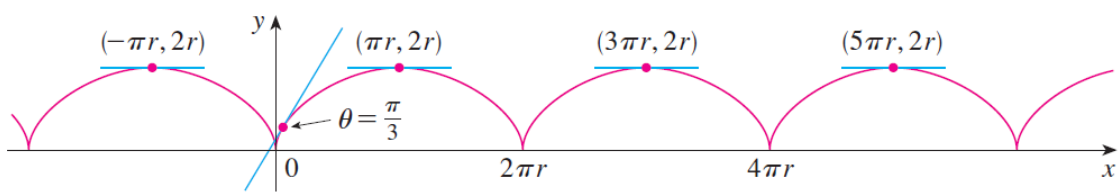


Figure 2.16

(b) The tangent is horizontal when $\dfrac{dy}{dx} = 0$, which occurs when $\sin\theta = 0$ and $1 - \cos\theta \neq 0$, that is, $(2n-1)\pi$, n an integer. The corresponding point on the cycloid is $((2n-1)\pi r, 2r)$.

When $\theta = 2n\pi$, both $\dfrac{dx}{d\theta}$ and $\dfrac{dy}{d\theta}$ are 0. It appears from the graph that there are vertical tangents at those points. We can verify this by using l'Hospital's Rule as follows:

$$\lim_{\theta \to 2n\pi^+} \frac{dy}{dx} = \lim_{\theta \to 2n\pi^+} \frac{\sin\theta}{1 - \cos\theta} = \lim_{\theta \to 2n\pi^+} \frac{\cos\theta}{\sin\theta} = \infty.$$

A similar computation shows that $\dfrac{dy}{dx} \to -\infty$ as $\theta \to 2n\pi^+$, so indeed there are vertical tangents when $\theta = 2n\pi$, that is, when $x = 2n\pi r$.

3. Vector Functions

3.1 Vector Function and Space Curves

There is a close connection between continuous vector functions and space curves. Suppose that f, g and h are continuous real-valued functions on an interval I. Then the set C of all points (x,y,z) in space, where

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

and t varies throughout the interval I, is called a space curve.The equation (2) are called **parametric equations of** C and t is called a **parameter**. We can think of C as being traced out by a moving particle whose position at time t is $(f(t),g(t),h(t))$. If we now consider the vector function $r(t) = \langle f(t),g(t),h(t)\rangle$, then $r(t)$ is the position vector of the point $P(f(t),g(t),h(t))$ on C. Thus any continuous vector function r defines a space curve c that is traced out by the tip of the moving vector $r(t)$, as shown in Figure 3.1.

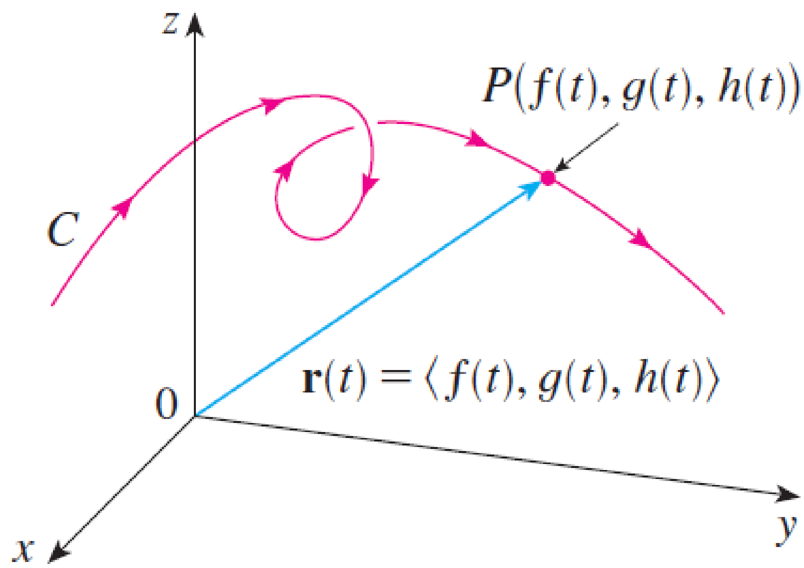


Figure 3.1

Example 3.1.1 Describe the curve defined by the vector function

$$\mathbf{r}(t) = \langle 1+t, 2+5t, -1+6t \rangle$$

Solution: The corresponding parametric equations are

$$x = 1+t \qquad y = 2+5t \qquad z = -1+6t$$

which we recognize parametric equations of a line passing through the point $(1,2,-1)$ and parallel to the vector $\langle 1,5,6 \rangle$. Alternatively, we could observe that the function can be written as $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, where $\mathbf{r}_0 = \langle 1,2,-1 \rangle$ and $\mathbf{v} = \langle 1,5,6 \rangle$ and this is the vector equation of a line.

Plane curves can also be represented in vector notation. For instance, the curve given by the parametric equations $x = t^2 - 2t$ and $y = t+1$ (see Example 10.1.1) could also be described by the vector equation

$$\mathbf{r}(t) = \langle t^2 - 2t, t+1 \rangle = (t^2 - 2t)\mathbf{i} + (t+1)\mathbf{j}.$$

where $\mathbf{i} = \langle 1,0 \rangle$ and $\mathbf{j} = \langle 0,1 \rangle$.

Example 3.1.2 Sketch the curve whose vector equation is

$$\cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k}$$

Solution: The parametric equation for this curve are

$$x = \cos t \qquad y = \sin t \qquad z = t$$

Since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$ for all values of t, the curve must lie on the circular cylinder $x^2 + y^2 = 1$. The point (x, y, z) lies directly above the point $(x, y, 0)$, which moves counterclockwise around the circle $x^2 + y^2 = 1$ in the xy-plane. (The projection of the curve onto the xy-plane has vector equation $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$. See Example 10.1.2.) Since $z = t$, the curve spirals upward around the cylinder as t increases. The curve, shown in Figure 3.2, is called a **helix**.

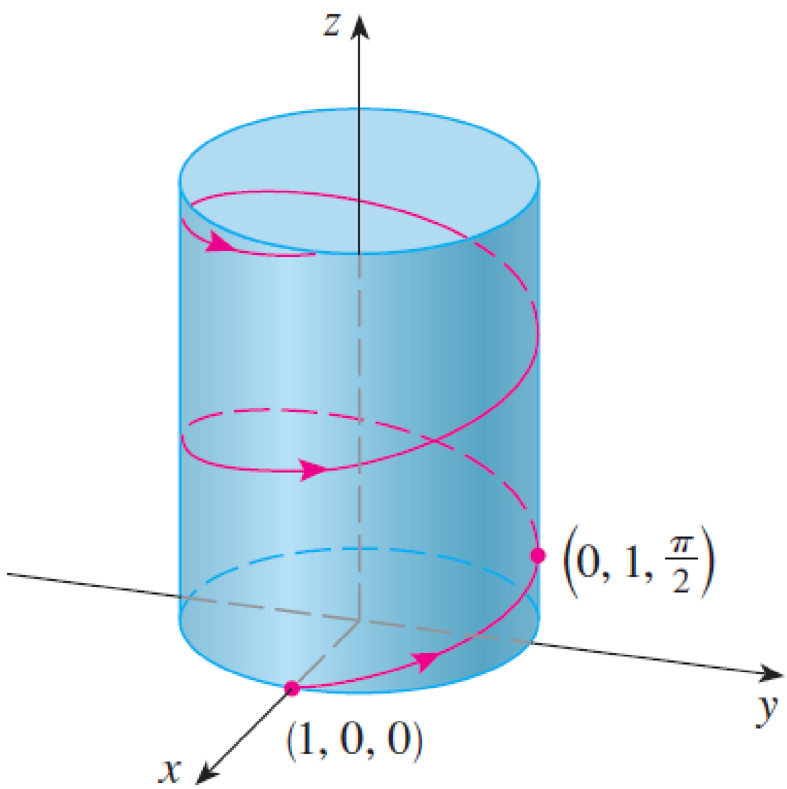


Figure 3.2

The corkscrew shape of the helix in Example 3.1.2 is familiar from its occurrence in coiled springs. It also occurs in the model of DNA (deoxyribonucleic acid, the genetic material of living cells). In 1953 James Watson and Francis Crick showed that the structure of the DNA molecule is that of two linked, parallel helixes that are intertwined as in Figure 3.3. In

Examples 3.1.1 and 3.1.1 we were given vector equations of curves and asked for a geometric description or sketch. In the next two examples we are given a geometric description of a curve and are asked to find parametric equations for the curve.

Figure 3.3. Double Helix

Remark 3.1.3 The vector equation for the line segment that joins the tip of the vector r_0 to the tip of the vector $\mathbf{r}_1$ is given by

$$\mathbf{r}(t) = (1-t)r_0 + t\mathbf{r}_1 \quad 0 \leq t \leq 1.$$

Example 3.1.4 Find a vector equation and parametric equations for the line segment that joins the point $P(1,3,-2)$ to the point $Q(2,-1,3)$.

Solution: From previous remark, the vector equation for the line segment that joins the tip of the vector r_0 to the tip of the vector $\mathbf{r}_1$:

$$\mathbf{r}(t) = (1-t)r_0 + t\mathbf{r}_1 \quad 0 \leq t \leq 1.$$

$$\mathbf{r}(t) = (1-t)\langle 1,3,-2\rangle + t\langle 2,-1,3\rangle \quad 0 \leq t \leq 1$$

or

$$\mathbf{r}(t) = \langle 1+t, 3-4t, -2+5t\rangle \quad 0 \leq t \leq 1$$

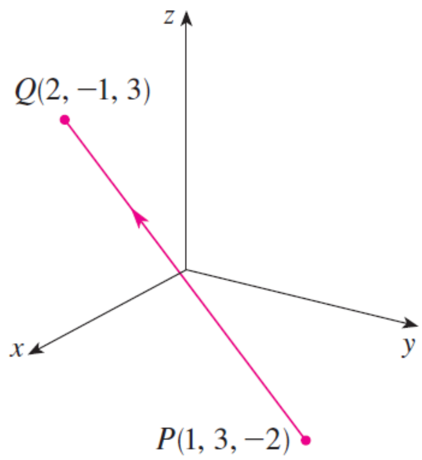


Figure 3.4

The corresponding parametric equations are

$$x = 1 + t \qquad y = 3 - 4t \qquad z = -2 + 5t$$

Example 3.1.5 Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

Solution: Figure 3.5 shows how the plane and the cylinder intersect, and Figure 3.6 shows the curve of intersection C, which is an ellipse.

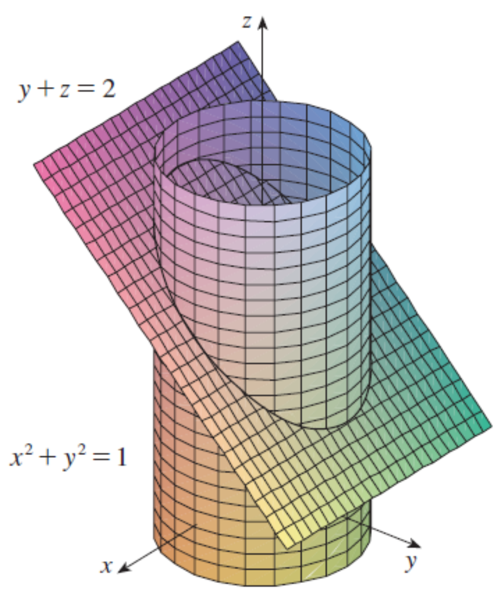


Figure 3.5

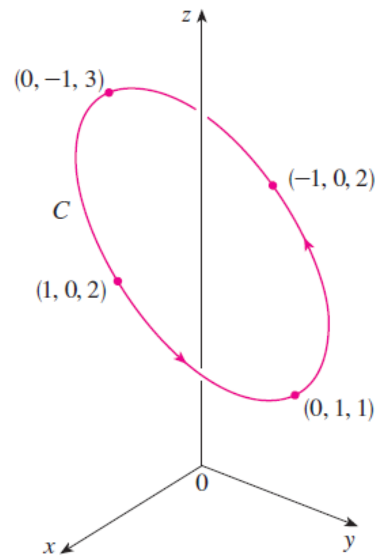


Figure 3.6

The projection of C onto the xy-plane is the circle $x^2 + y^2 = 1$, $z = 0$. So we know from Example 10.1.2 the we can write

$$x = \cos t \quad y = \sin t \quad 0 \leq t \leq 2\pi.$$

From the equation of the plane, we have

$$z = 2 - y = 2 - \sin t.$$

So we can write parametric equations for C as

$$x = \cos t \quad y = \sin t \quad z = 2 - \sin t \quad 0 \leq t \leq 2\pi.$$

The corresponding vector equation is

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + (2 - \sin t)\mathbf{k} \quad 0 \leq t \leq 2\pi.$$

This equation is called a *parametrization* of the curve C. The arrows in the Figure 3.6 indicate the direction in which C is traced as the parameter t increases.

3.2 Using Computer to Draw Space Curves

Space curves are inherently more difficult to draw by hand than plane curves; for an accurate representation we need to use technology. For instance, Figure 3.7 shows a computer-generated graph of the curve with parametric equations

$$x = (4 + \sin 20t)\cos t \quad y = (4 + \sin 20t)\sin t \quad z = \cos 20t$$

It's called **toroidal spiral** because it lies on a toros. Another interesting curve, the **trefoil knot**, with equations

$$x = (2 + \cos 1.5t)\cos t \quad y = (2 + \cos 1.5t)\sin t \quad z = \sin 1.5t$$

is graphed in Figure 3.8. It wouldnt be easy to plot either of these curves by hand.

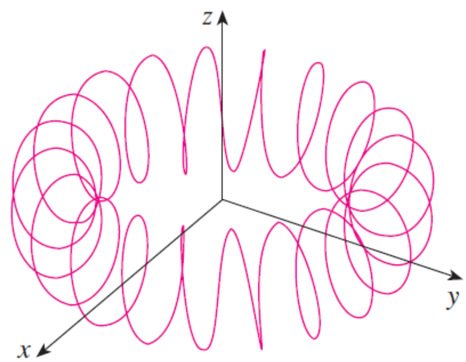


Figure 3.7 Toroidal Spiral

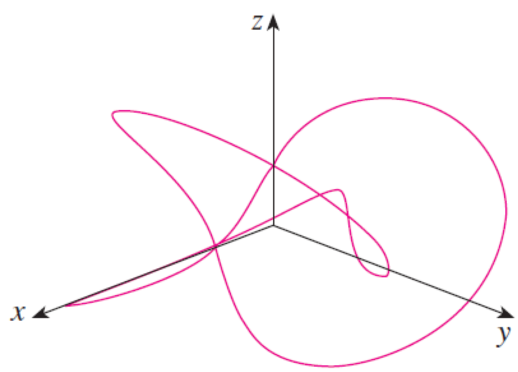


Figure 3.8 A trefoil knot

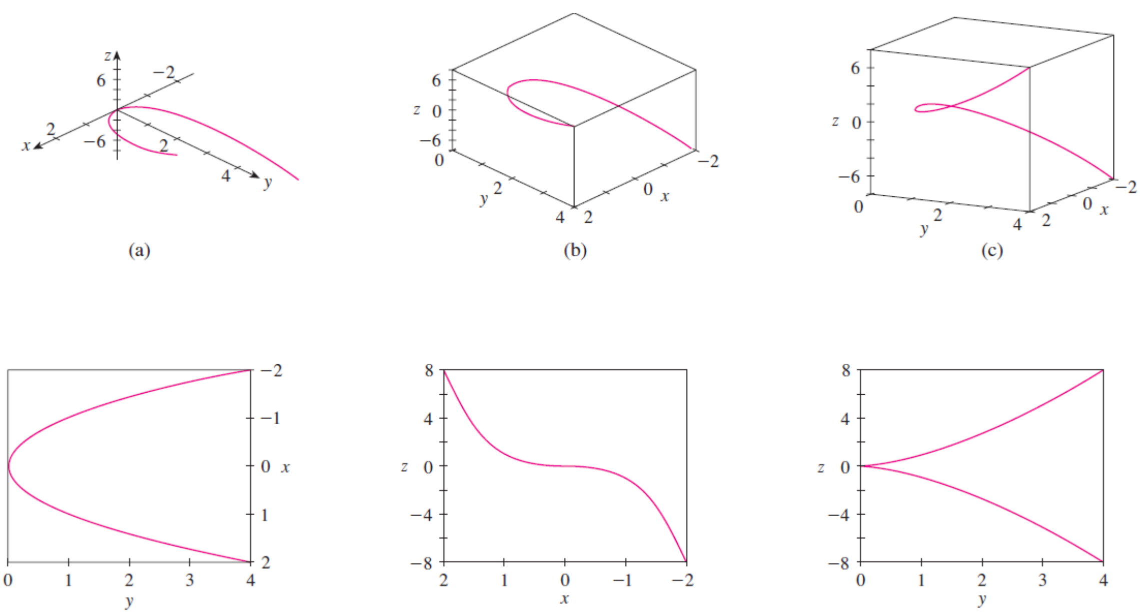
Even when a computer is used to draw a space curve, optical illusions make it difficult to get a good impression of what the curve really looks like. (This is especially true in Figure 3.8. See Exercise 52 [1] .) The next example shows how to cope with this problem.

Example 3.2.1 Use a computer to draw the curve with vector equation $\langle t, t^2, t^3 \rangle$ This curve is called a **twisted cubic.**

Solution: We start by using the computer to plot the curve with parametric equations

$$x = t \quad y = t^2 \quad z = t^3 \quad -2 \leq t \leq 2.$$

The result is show in Figure 9(a), but it's hard to see the true nature of the curve from that graph alone. Most three-dimensional computer graphing programs allow user to enclose a curve or surface in a box instead of displaying the coordinate axes. When we look at the same curve in a box in Figure 9(b) we see that it climbs from a lower corner of the box to the upper corner nearest us, and it twists as it climbs.

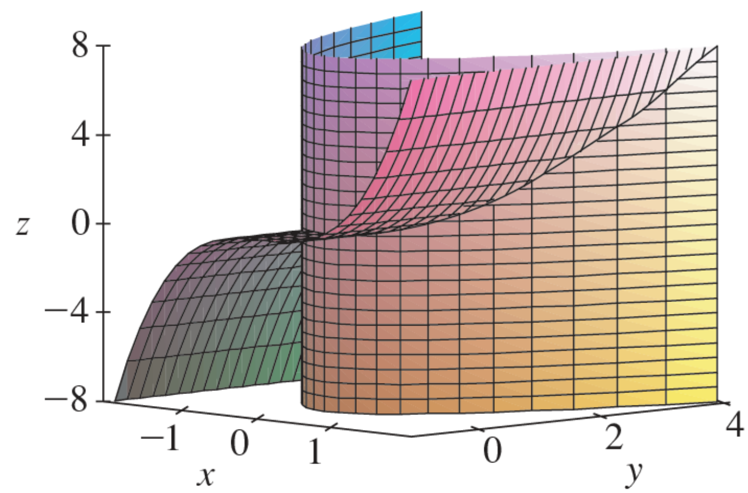


(a) (b) (c) (d) (e) (f)

We get an even better idea of the curve when we view it from different vantage points. Part (c) shows the result of rotating the box to give another viewpoint. Parts (d), (e), and (f) show the views we get when we look directly at a face of the box. In par - ticular, part (d) shows the view from directly above the box. It is the projection of the curve onto the xy-plane, namely, the parabola $y = x^2$. Part (e) shows the projection onto the xz-plane, the cubic curve $z = x^3$. It's now obvious why the given curve is called a twisted cubic. Another method of visualizing a space curve is to draw it on a surface. For instance, the twisted cubic in Example 3.2.1

lies on the parabolic cylinder $y = x^2$. (Eliminate the parameter from the first two parametric equations, $x = t$ and $y = t^2$.) Figure 10 shows both the cylinder and the twisted cubic, and we see that the curve moves upward from the origin along the surface of the cylinder. We also used this method in Example 3.1.2 to visualize the helix lying on the circular cylinder (see Figure 3.2).

A third method for visualizing the twisted cubic is to realize that it also lies on the cylinder $z = x^3$. So it can be viewed as the curve of intersection of the cylinders $y = x^2$ and $z = x^3$. (See Figure 11.)



3.3 Derivatives and Integrals of Vector Functions

3.3.1 Derivatives

The derivative $\mathbf{r}'$ of the vector function $\mathbf{r}$ will be defined just like the derivatives for real-valued functions that we have in our previous calculus courses.

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

A similar geometric interpretation to the derivative of a function in one variable is exhibited by the derivative of a vector function. That is, $\mathbf{r}'(t)$ is the tangent called the **tangent vector** to the curve C defined by $\mathbf{r}$ at the point P, provided that $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq 0$.

The **tangent line** to the curve C at the point P on the curve is the line through P and parallel to the vector $\mathbf{r}'(t)$.

The **unit tangent vector** is the vector $\mathbf{T}(t)$ given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\left\|\mathbf{r}'(t)\right\|}.$$

Theorem 3.3.1 If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

Example 3.3.2

1. Find the derivative of $\mathbf{r}(t) = (1+t^3)\,\mathbf{i} + te^{-t}\,\mathbf{j} + \sin 2t\,\mathbf{k}$.
2. Find the unit tangent vector at the point where t=0.

Solution:

1. Differentiating the components of $\mathbf{r}$ component by component, we have
$$\mathbf{r}'(t) = 3t^2\,\mathbf{i} + (1-t)e^{-t}\,\mathbf{j} + 2\cos 2t\,\mathbf{k}.$$
2. Since $\mathbf{r}(0) = \mathbf{i}$ and $\mathbf{r}'(0) = \mathbf{j} + 2\mathbf{k}$, the unit tangent vector at $(1,0,0)$ is
$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{\mathbf{j}+2\mathbf{k}}{\sqrt{1+4}} = \frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\,\mathbf{k}.$$

Example 3.3.3 Find the parametric equations for the tangent line to the helix with parametric equations

$$x = 2\cos t \qquad y = \sin t \qquad z = t$$

at the point $(0, 1, \pi/2)$

Solution: The vector equation of the helix is $\mathbf{r}(t) = \langle 2\cos t, \sin t, t \rangle$ Hence,

$$\mathbf{r}'(t) = \langle -2\sin t, \cos t, 1 \rangle$$

The value of the parameter corresponding to the point $(0,1,\pi/2)$ is $t=\pi/2$, so the tangent vector is

$$\mathbf{r}'(\pi/2)=\langle -2,0,1\rangle$$

Therefore, the equation of the tangent line through $(0,1,\pi/2)$ and parallel to vector $\langle -2,0,1\rangle$ is

$$x=-2t \qquad y=1 \qquad z=\frac{\pi}{2}+t.$$

3.4 Integrals

The **definite integral** of a continuous vector function r(t) can be defined in much the same way as for real-valued functions except that the integral is a vector.

$$\int_a^b \mathbf{r}(t)dt = \lim_{n\to\infty}\sum_{i=1}^{n}\mathbf{r}(t_i^*)\Delta t$$

$$\left[\lim_{n\to\infty}\sum_{i=1}^{n} f(t_i^*)\Delta t\right]\mathbf{i}+\left[\lim_{n\to\infty}\sum_{i=1}^{n} g(t_i^*)\Delta t\right]\mathbf{j}+\left[\lim_{n\to\infty}\sum_{i=1}^{n} h(t_i^*)\Delta t\right]\mathbf{k}$$

Hence,

$$\int_a^b \mathbf{r}(t)dt=\int_a^b f(t)dt\,\mathbf{i}+\int_a^b g(t)dt\,\mathbf{j}+\int_a^b h(t)dt\,\mathbf{k}$$

Example 3.4.1 Evaluate the integral $\int_0^{\pi/2}\mathbf{r}(t)dt$, where

$$\mathbf{r}(t)=2\cos t\,\mathbf{i}+\sin t\,\mathbf{j}+2t\,\mathbf{k}.$$

Solution: Let us simplify the evaluation by taking first the indefinite integral of $\mathbf{r}(t)$.

$$\begin{aligned}\int \mathbf{r}(t)dt &= \left(\int 2\cos t dt\right)\mathbf{i}+\left(\int \sin t dt\right)\mathbf{j}+\left(\int 2t dt\right)\mathbf{k}\\ &= 2\sin t\,\mathbf{i}-\cos t\,\mathbf{j}+t^2\,\mathbf{k}+\mathbf{C}\end{aligned}$$

where $\mathbf{C}$ is the vector constant of integration. Thus,

$$\int_0^{\pi/2} \mathbf{r}(t)dt = \Big[2\sin t\ \mathbf{i} - \cos t\ \mathbf{j} + t^2\ \mathbf{k}\Big]_0^{\pi/2} = 2\mathbf{i} + \mathbf{j} + \frac{\pi^2}{4}\ k.$$

Example 3.4.2 Find the antiderivative of

$$\mathbf{r}'(t) = \cos t\ \mathbf{i} - \sin t\ \mathbf{j} + \frac{1}{1+t^2}\ \mathbf{k}$$

that satisfies the initial condition

$$\mathbf{r}(0) = 3\ \mathbf{i} - 2\ \mathbf{j} + \ \mathbf{k}.$$

Solution:

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{r}'(t)dt \\ &= \Big(\int \cos t dt\Big)\ \mathbf{i} + \Big(\int -\sin t dt\Big)\ \mathbf{j} + \Big(\int \frac{1}{1+t^2} dt\Big)\ \mathbf{k} \\ &= (\sin t + \mathbf{C}_1)\ \mathbf{i} + (\cos t + \mathbf{C}_2)\ \mathbf{j} + (\arctan t + \mathbf{C}_3)\ \mathbf{k}\end{aligned}$$

At $t = 0$, we have

$$\mathbf{r}(0) = (0 + C_1)\ \mathbf{i} + (1 + C_2)\ \mathbf{j} + (0 + C_3)\ \mathbf{k} = 3\ \mathbf{i} - 2\ \mathbf{j} + \ \mathbf{k}$$

After equating coefficients of the respective components, we have

$$C_1 = 3 \qquad C_2 = -3 \qquad C_3 = 1$$

Therefore, the antiderivative that satisfies the given initial condition is

$$\mathbf{r}(t) = (\sin t + 3)\ \mathbf{i} + (\cos t - 3)\ \mathbf{j} + (\arctan t + 1)\ \mathbf{k}.$$

3.5 Arc Length and Curvature

3.5.1 Length of a Curve.

A smooth plane curve can be represented by the parametric equations

$$x = f(t), \qquad y = g(t) \qquad a \le t \le b$$

has length L is given by

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

The formula for the arc length of a plane curve has a natural extension to a smooth curve in space, as stated in the next theorem.

Theorem 3.5.1 Arc Length of a Space Curve

If C is a smooth curve given by $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ on a closed interval $[a,b]$, then the length L of the arc C on the interval is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt = \int_a^b \|\mathbf{r}'(t)\|\, dt.$$

Example 3.5.2 Find the length of the arc of the circular helix with vector equation is

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$$

from the point $(1,0,0)$ to the point $(1,0,2\pi)$.

Solution: Taking the derivative of the vector function $\mathbf{r}$, we get

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \,\mathbf{k}$$

Hence,

$$\|\mathbf{r}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

Note that the arc is being described by the parameter $0 \le t \le 2\pi$. Thus,

$$L = \int_0^{2\pi} \|\mathbf{r}'(t)\|\, dt = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi.$$

Example 3.5.3 Find the length of the curve defined by the vector-valued function

$$\mathbf{r}(t) = \sqrt{2}t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}$$

where $0 \le t \le 1$.

Solution: Taking the derivative of the given function $\mathbf{r}$, we get

$$\mathbf{r}'(t) = \sqrt{2}\,\mathbf{i} + e^t\,\mathbf{j} - e^{-t}\,\mathbf{k}$$

Therefore,

$$
\begin{aligned}
L &= \int_0^1 \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2}\,dt = \int_0^1 \sqrt{2 + e^{2t} + e^{-2t}}\,dt \\
&= \int_0^1 \sqrt{(e^t + e^{-t})^2}\,dt = \int_0^1 (e^t + e^{-t})^2\,dt \\
&= (e^t + e^{-t})\Big|_0^1 \\
&= \frac{e^2 - 1}{e}.
\end{aligned}
$$

3.5.2 Curvature

Suppose that C is a curve given by the vector function

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}, \quad a \le t \le b$$

where $\mathbf{r}$ is continuous and C is traversed exacty once as t increases from a to b, we define the **arc length function** s by

$$s(t) = \int_a^t \|\mathbf{r}'(u)\|\,du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\,du.$$

Thus, $s(t)$ is the length of the part of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$. If we derive both sides of the above equation, we obtain

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\|$$

If C is a smooth curve defined by the vector function $\mathbf{r}$, recall that the unit tangent vector $\mathbf{T}(t)$ is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

and indicates the direction of the curve. Because the unit tangent vector has a constant length, only chances direction contribute to the rate of change of $\mathbf{T}$.

The **curvature** of a curve is given by

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|$$

where $\mathbf{T}$ is the unit tangent vector and s is the arc length function.

The curvature is easier to compute if it is express in terms of the parameter t instead of s. By chain rule,

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} \text{ and } \kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \left\| \frac{d\mathbf{T}/dt}{ds/dt} \right\|$$

Note that

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\|$$

so κ in the definition can now be express as a function of t.

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

Example 3.5.4 Show that the curvature of a circle with radius a is $\dfrac{1}{a}$.

Solution: The circle centered at the origin with radius a has the parameterization

$$\mathbf{r}(t) = a\cos t\ \mathbf{i} + a\sin t\ \mathbf{j}$$

Thus,

$$\mathbf{r}'(t) = -a\sin t\ \mathbf{i} + a\cos t\ \mathbf{j}$$

and

$$\|\mathbf{r}'(t)\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2} = \sqrt{a^2(\sin^2 t + \cos^2 t)} = a.$$

Now,

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\sin t\ \mathbf{i} + \cos t\ \mathbf{j}.$$

This means that

$$\mathbf{T}'(t) = -\cos t\ \mathbf{i} - \sin t\ \mathbf{j}.$$

Hence,

$$\|\mathbf{T}'(t)\| = 1$$

Therefore,

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{1}{a}$$

Although the formula above can be used in all cases to compute the curvature, the formula given by the following theorem is often more convenient to apply.

Theorem 3.5.5 The curvature of the curve given by the vector $\mathbf{r}$ is given by

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

Example 3.5.6 Find the curvature of the twisted cubic $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at a general point at $(0,0,0)$

Solution: Let us compute the necessary quantities in the formula

$$\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle \text{ and } \mathbf{r}''(t) = \langle 0, 2, 6t \rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{1 + 4t^2 + 9t^4}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 6t^2\,\mathbf{i} - 6t\,\mathbf{j} + 2\,\mathbf{k}$$

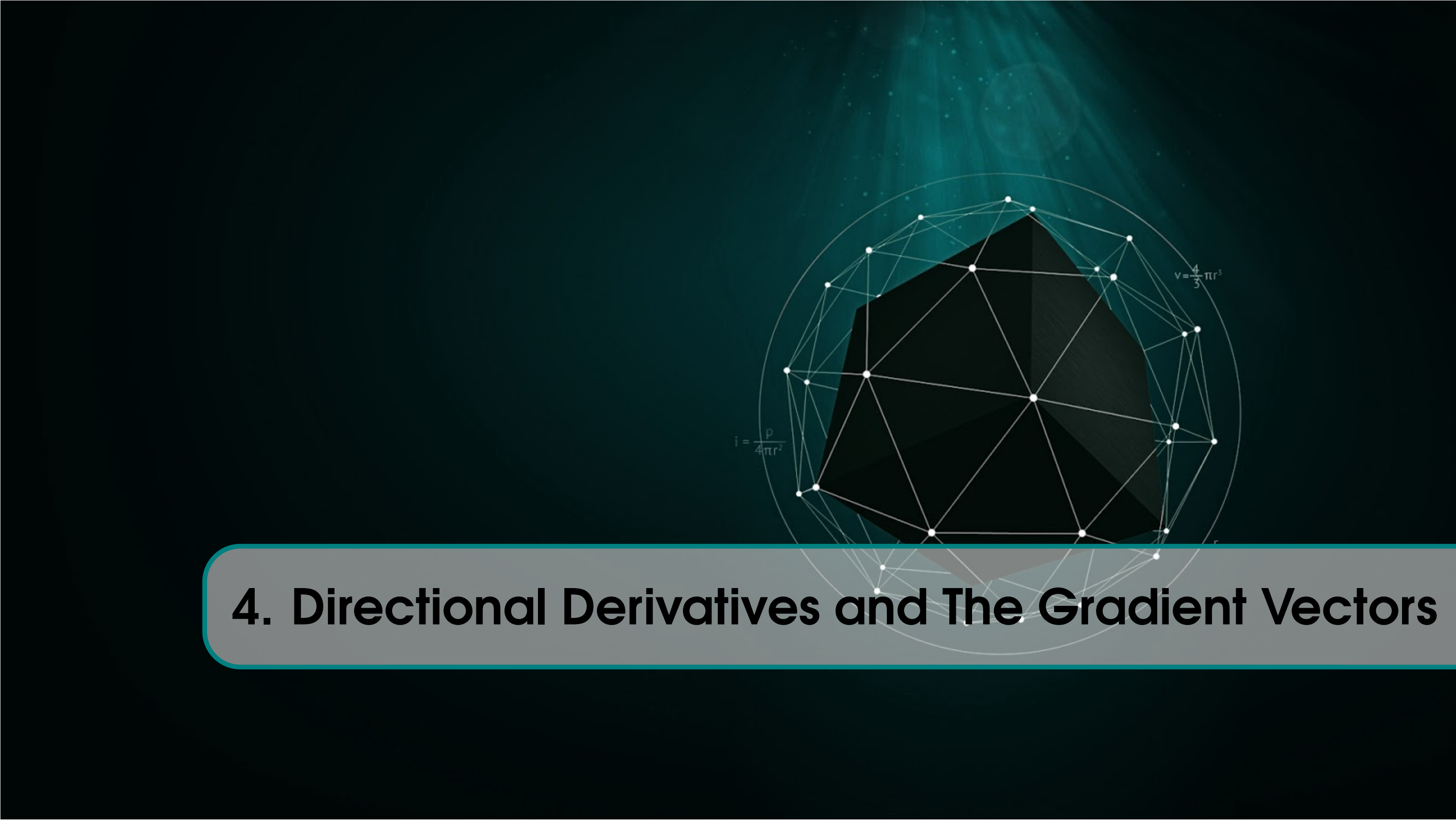
$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = 2\sqrt{9t^4 + 9t^2 + 1}$$

Thus,

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{2\sqrt{9t^4 + 9t^2 + 1}}{(1 + 4t^2 + 9t^4)^{3/2}}$$

at $(0,0,0)$, that is when $t = 0$, we have

$$\kappa(0) = 2.$$



4. Directional Derivatives and The Gradient Vectors

4.1 Directional Derivatives

We have learned that if $z = f(x,y)$, the partial derivatrives f_x and f_y represent the change of z in the $x-$ and $y-$ directions or in the direction of the unit vectors **i** and **j**. This time, we will learn another type of derivative, called the **directional derivative**. This derivative is used if we want to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b \rangle$

The **directional derivative** of f at (x_0, y_0) in the direction of the unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Theorem 4.1.1 If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b$$

If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis, then we can write the formula in the above theorem as

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$

Example 4.1.2 Find the directional derivative of

$$f(x,y) = 4 - x^2 - \frac{1}{4}y^2$$

at $(1,2)$ in the direction of

$$\mathbf{u} = \left(\cos\frac{\pi}{3}\right)\mathbf{i} + \left(\sin\frac{\pi}{3}\right)\mathbf{j}$$

Solution: We will find the first f_x and f_y. In our case, we have

$$f_x(x,y) = -2x \text{ and } f_y(x,y) = -\frac{1}{2}y$$

Note that these partial derivatives are continuous, hence, f is differentiable. Thus,

$$D_{\mathbf{u}}f(x,y) = (-2x)\cos\theta + \left(-\frac{y}{2}\right)\sin\theta$$

Evaluating $D_{\mathbf{u}}f$ using the values $\theta = \dfrac{\pi}{3}$, $x = 1$ and $y = 2$, we have

$$D_{\mathbf{u}}f(1,2) = (-2)\left(\frac{1}{2}\right) + (-1)\left(\frac{\sqrt{3}}{2}\right) = -\frac{2+\sqrt{3}}{2}.$$

Example 4.1.3 Find the directional derivative of

$$f(x,y) = x^2\sin 2y$$

at $(1,\pi/2)$ in the direction of

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}.$$

Solution: Note that f is differentiable since $f_x(x,y) = 2x\sin 2y$ and $f_y(x,y) = x^2\cos 2y$ are continuous Now, let us find a unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$.

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} - 4\mathbf{j}}{\sqrt{9+16}} = \frac{3}{5}\,\mathbf{i} - \frac{4}{5}\,\mathbf{j} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$$

Hence,

$$D_{\mathbf{u}}f(x,y) = (2x\sin 2y)\cos\theta + (x^2\cos 2y)\sin\theta$$

and so,

$$\begin{aligned} D_{\mathbf{u}} f(1,\pi/2) &= (2\sin\pi)\left(\frac{3}{5}\right) + (2\cos\pi)\left(-\frac{4}{5}\right) \\ &= (0)(3/5) + (-2)(-4/5) \\ &= \frac{8}{5} \end{aligned}$$

4.2 The Gradient Vectors

The directional derivative of a differentiable function can be written as the dot product of two vectors:

$$\begin{aligned} D_{\mathbf{u}} f(x,y) &= f_x(x,y)\ a + f_y(x,y)\ b \\ &= \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle a,b\rangle \\ &= \langle f_x(x,y), f_y(x,y)\rangle \cdot \mathbf{u} \end{aligned}$$

The first vector in the dot product occurs oftentimes and is given a special name. This is called the **gradient** of f. This is denoted by **grad** f or ∇f (read as "del of f").

If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}.$$

Example 4.2.1 Find the gradient of

$$f(x,y) = y\ln x + xy^2$$

at the point $(1,2)$

Solution: We will compute first f_x and f_y. In our case,

$$f_x(x,y) = \frac{y}{x} + y^2 \text{ and } f_y(x,y) = \ln x + 2xy$$

Thus,

$$\nabla f(x,y) = \left(\frac{y}{x} + y^2\right)\mathbf{i} + (\ln x + 2xy)\,\mathbf{j}$$

At the point $(1,2)$,

$$\nabla f(1,2) = \left(\frac{2}{1} + 2^2\right)\mathbf{i} + (\ln 1 + 2(1)(2))\,\mathbf{j} = 6\,\mathbf{i} + 4\,\mathbf{j}$$

Example 4.2.2 Find the directional derivative of

$$f(x,y) = 3x^2 - 2y^2$$

at $(-\frac{3}{4},0)$ in the direction from $P(-\frac{3}{4},0)$ to $Q(0,1)$

Solution: It can be seen right away that f is differentiable since the partial derivatives are continuous. The vector in the specified direction is

$$\overrightarrow{PQ} = \left(0 + \frac{3}{4}\right)\mathbf{i} + (1-0)\,\mathbf{j} = \frac{3}{4}\,\mathbf{i} + \mathbf{j}$$

A unit vector $\mathbf{u}$, in this direction is

$$\mathbf{u} = \frac{\overrightarrow{PQ}}{||\overrightarrow{PQ}||} = \frac{3}{5}\,\mathbf{i} + \frac{4}{5}\,\mathbf{j}.$$

Now,

$$\nabla f(x,y) = f_x(x,y)\,\mathbf{i} + f_y(x,y)\,\mathbf{j} = 6x\mathbf{i} - 4y\,\mathbf{j}$$

and so

$$\nabla f\left(-\frac{3}{4},0\right) = -\frac{9}{2}\,\mathbf{i} + 0\,\mathbf{j}.$$

Therefore, the directional derivative at $(-3/2,0)$ is

$$\begin{aligned} D_{\mathbf{u}} f\left(-\frac{3}{4},0\right) &= \nabla f\left(-\frac{3}{4},0\right)\cdot\mathbf{u} \\ &= \left(-\frac{9}{2}\,\mathbf{i} + 0\,\mathbf{j}\right)\cdot\left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\,\mathbf{j}\right) \\ &= -\frac{27}{10} \end{aligned}$$

5. Vector Calculus

5.1 Vector Fields

In this section we will be learning another type of vector-valued function, called **vector fields**. These functions assign a vector to a point in a plane or in space.

A vector field over a plane region R is a function $\mathbf{F}$ that assigns a vector $\mathbf{F}(x,y)$ to each point in R.

A vector field over a solid region Q is a function $\mathbf{F}$ that assigns a vector $\mathbf{F}(x,y,z)$ to each point in Q.

Example 5.1.1 A vector field in $\mathbb{R}^2$ defined by $\mathbf{F}(x,y) = -y\,\mathbf{i} + x\,\mathbf{j}$. Describe $\mathbf{F}$ by sketching some of the vectors $\mathbf{F}(x,y)$. by sketching some of the vectors $\mathbf{F}(x,y)$.

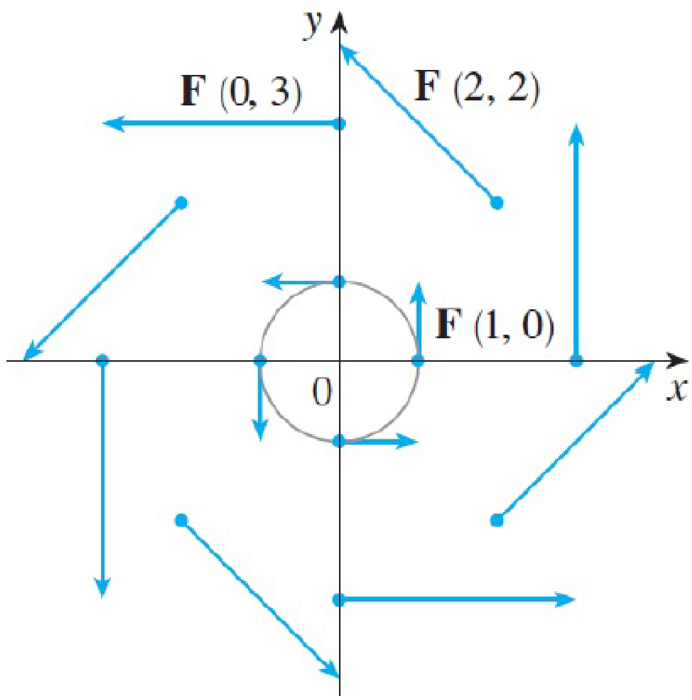
Solution: Let us recall first the impotant unit vectors in the plane: $\mathbf{i} = \langle 0,1\rangle$ and $\mathbf{j} = \langle 0,1\rangle$. Consider the point $(1,0)$ in the plane. To find where is this point assigned to, just substitute it to $\mathbf{F}$. In our case, $\mathbf{F}(1,0) = \ \mathbf{j}$. Hence, the point $(0,1)$ is assigned to vector $\mathbf{j} = \langle 0,1\rangle$. To sketch this on the plane, just make a copy of vector $\mathbf{j}$ whose initial point is the point $(1,0)$.

Similarly, for any point in the plane there corresponds a unique vector for that point

The table below shows some of the representatives of the vector field.

(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$
$(1, 0)$	$\langle 0, 1\rangle$	$(-1, 0)$	$\langle 0, -1\rangle$
$(2, 2)$	$\langle -2, 2\rangle$	$(-2, -2)$	$\langle 2, -2\rangle$
$(3, 0)$	$\langle 0, 3\rangle$	$(-3, 0)$	$\langle 0, -3\rangle$
$(0, 1)$	$\langle -1, 0\rangle$	$(0, -1)$	$\langle 1, 0\rangle$
$(-2, 2)$	$\langle -2, -2\rangle$	$(2, -2)$	$\langle 2, 2\rangle$
$(0, 3)$	$\langle -3, 0\rangle$	$(0, -3)$	$\langle 3, 0\rangle$

We sketch some of the representatives of the field below.



There are many physical examples that exhibits vector fields. Among them are velocity fields, magnetic fields, gravitational fields and electric force fields. Vector fields are best done on a computer algebra system, for this reason, we will just limit our discussion here. Some examples and illustrations will be given on a separate online references.

5.2 Gradient Fields

The gradient of f,

$$\nabla f(x,y) = f_x(x,y)\,\mathbf{i} + f_y(x,y)\,\mathbf{j}$$

is a vector fiend on $\mathbb{R}^2$ and is called a gradient vector field. Similarly,

$$\nabla f(x,y,z) = f_x(x,y,z)\,\mathbf{i} + f_y(x,y,z)\,\mathbf{j} + f_z(x,y,z)\,\mathbf{k}$$

is a gradient vector field on $\mathbb{R}^3$.

Example 5.2.1 Find the gradient vector of $f(x,y) = x^2y - y^3$.

Solution: By simple computation, we obtain

$$\nabla f(x,y) = 2xy\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}.$$

5.3 Conservative Vector Fields

Some vector fields can be represented as the gradients of differentiable functions and some cannot. Those that can be represented are called **conservative vector fields.**

A vector field $\mathbf{F}$ is called **conservative** when there exists a differentiable function f such that $\mathbf{F} = \nabla f$. The function f is called a **potential function** for $\mathbf{F}$.

Example 5.3.1 The vector field given by

$$\mathbf{F}(x,y) = 3x^2\,\mathbf{i} + y\,\mathbf{j}$$

is conservative since there is a function

$$f(x,y) = x^3 + \frac{1}{2}y^2$$

in which

$$\nabla f(x,y) = 3x^2\,\mathbf{i} + y\,\mathbf{j} = \mathbf{F}(x,y)$$

Theorem 5.3.2 **(Test for Conservative Vector Field in the Plane)**
Let M and N have continuous first partial derivatives on an open disk R. The vector field $\mathbf{F}(x,y) = M\,\mathbf{i} + N\,\mathbf{j}$ is conservative if and only if

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Example 5.3.3 Determine whether the vector field given by $\mathbf{F}$ is conservative or not

1. $\mathbf{F}(x,y) = x^3y\,\mathbf{i} + xy\,\mathbf{j}$
2. $\mathbf{F}(x,y) = 2x\,\mathbf{i} + y\,\mathbf{j}$

Solution:

1. Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}[x^3y] = x^3$ and $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}[xy] = y$,

   the given vector fiend is not conservative.

1. Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}[2x] = 0$ and $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}[y] = 0$,

   the given vector fiend is conservative.

Theorem 5.3.4 **(Test for Conservative Vector Field in Space)**

Suppose M, N, and P have continuous first partial derivatives on an open sphere Q. The vector field

$$\mathbf{F}(x,y) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k} \text{ is conservative}$$

if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

Example 5.3.5 Consider $\mathbf{F}(x,y) = \dfrac{x}{x^2+y^2+z^2}\,\mathbf{i} + \dfrac{y}{x^2+y^2+z^2}\,\mathbf{j} + \dfrac{z}{x^2+y^2+z^2}\,\mathbf{k}$

Solution:

$$\frac{\partial P}{\partial y} = \frac{(x^2+y^2+z^2)(0) - z(2y)}{(x^2+y^2+z^2)^2} = \frac{-2yz}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial N}{\partial z} = \frac{(x^2+y^2+z^2)(0) - y(2z)}{(x^2+y^2+z^2)^2} = \frac{-2yz}{(x^2+y^2+z^2)^2}.$$

So,

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}.$$

Next,

$$\frac{\partial P}{\partial x} = \frac{(x^2+y^2+z^2)(0) - z(2x)}{(x^2+y^2+z^2)^2} = \frac{-2xz}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial M}{\partial z} = \frac{(x^2+y^2+z^2)(0) - x(2z)}{(x^2+y^2+z^2)^2} = \frac{-2xz}{(x^2+y^2+z^2)^2}.$$

Consequently,

$$\frac{\partial P}{\partial x} = \frac{\partial N}{\partial z}.$$

Finally,

$$\frac{\partial N}{\partial x} = \frac{(x^2+y^2+z^2)(0)-y(2x)}{(x^2+y^2+z^2)^2} = \frac{-2xy}{(x^2+y^2+z^2)^2}$$
$$\frac{\partial M}{\partial y} = \frac{(x^2+y^2+z^2)(0)-x(2y)}{(x^2+y^2+z^2)^2} = \frac{-2xy}{(x^2+y^2+z^2)^2}.$$

Hence,

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

By Theorem 5.3.4, F is conservative, i.e., the existence of the potential function f for which $F = \nabla f$ is justified.

We may note that the potential function which corresponds to **F** in Example 5.3.5 is of the form by

$$f(x,\ y,\ z) = \frac{1}{2}\ln|x^2+y^2+z^2| + C,$$

where C an arbitrary constant (**How?**).

It is noticable that

$$f_x(x,\ y,\ z) = M = \frac{x}{x^2+y^2+z^2},$$

can be integrated both sides with respect to the variable x. Doing so yields,

$$f(x,\ y,\ z) = \frac{1}{2}\ln|x^2+y^2+z^2| + g(y,z)$$

the additional term $g(y,z)$ refers to the constant term which vanishes when differentiating with respect to the variable x. Thus, differentiating $f(x,\ y,\ z)$ with respect to y gives us,

$$f_y(x,\ y,\ z) = \frac{y}{x^2+y^2+z^2} + g'(y,z)$$

It follows that,

$$\frac{y}{x^2+y^2+z^2} + g'(y,z) = \frac{y}{x^2+y^2+z^2}$$

This implies

$$g'(y,z) = 0$$

Consequently, integrate both sides of $g'(y,z)$ with respect to y gives

$$g(y,z) = C_1 + h(z)$$

for some constant C_1. So,

$$f(x,\ y,\ z) = \frac{1}{2}\ln|x^2+y^2+z^2| + h(z) + C_1,$$

Derive this equation with respect to z gives

$$f_z(x,\ y,\ z) = \frac{z}{x^2+y^2+z^2} + h'(z)$$

This means

$$\frac{z}{x^2+y^2+z^2} + h'(z) = \frac{z}{x^2+y^2+z^2}$$

and so, $h'(z) = 0$, consequently, $h(z) = C_2$ for some constant C_2. Combining, we have

$$f(x,\ y,\ z) = \frac{1}{2}\ln|x^2+y^2+z^2| + C$$

where $C = C_1 + C_2$.

5.4 Line Integrals

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval $[a,b]$, we integrate over a curve C. These integrals are called **line integrals**.

Definition. (Line Integral) If f is defined on a smooth curve C given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad a \le t \le b$$

then the line integral of f along C is

$$\int_C f(x,y)ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

if this limit exists.

We know that the length of the arc C is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

and in terms of definite integral, we can write the arc length formula as

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

or

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt$$

The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

Example 5.4.1 Evaluate

$$\int_C (3+x^2y)\,ds,$$

where C is the upper half of a unit circle $x^2+y^2=1$

Solution: We need to transform the equation of a circle into its parametric form. That is,

$$x=\cos t \qquad y=\sin t \qquad 0\le t\le \pi.$$

Then by the above formula, we have

$$\begin{aligned}
\int\limits_C (3+x^2y)\,ds &= \int\limits_0^{\pi} (3+\cos^2 t\sin t)\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt \\
&= \int\limits_0^{\pi} (3+\cos^2 t\sin t)\sqrt{(-\sin t)^2+(\cos t)^2}\,dt \\
&= \int\limits_0^{\pi} (3+\cos^2 t\sin t)\sqrt{\sin^2 t+\cos^2 t}\,dt \\
&= \int\limits_0^{\pi} (3+\cos^2 t\sin t)\,dt \\
&= \left[3t-\frac{1}{3}\cos^3 t\right]_0^{\pi} \\
&= \frac{9\pi+2}{3}.
\end{aligned}$$

Example 5.4.2 Evaluate $\int\limits_C 2x\,ds$, where C consists of the arcs C_1 of the parabola $y=x^2$ from $(0,0)$ to $(1,1)$ followed by C_2 which is the vertical line segment from $(1,1)$ to $(1,2)$

Solution: Let us begin with the first segment which is C_1. When can either parameterize C_1 in terms of x or y. Since that parabola is given as a function of x, we will just use x as our parameter and so we have,

$$x=x \quad y=x^2 \quad 0\le x\le 1$$

Hence,

$$\begin{aligned}
\int\limits_{C_1} 2x\,ds &= \int\limits_0^{1} 2x\sqrt{\left(\frac{dx}{dx}\right)^2+\left(\frac{dy}{dx}\right)^2}\,dx \\
&= \int\limits_0^{1} 2x\sqrt{1+4x^2}\,dx \\
&= \frac{1}{6}\left[1+4x^2\right]_0^1 \\
&= \frac{5\sqrt{5}-1}{6}
\end{aligned}$$

For C_2, we will parameterize it in terms of y so we have

$$x = 1 \qquad y = y \quad 1 \leq y \leq 2$$

Thus,

$$\begin{aligned}
\int_{C_2} 2x\, ds &= \int_1^2 2(1)\sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\, dy \\
&= \int_1^2 2\sqrt{0+1}\, dy \\
&= \int_1^2 \; dy \\
&= 2y \Big|_1^2 \\
&= 2.
\end{aligned}$$

5.5 Line integrals in Space

Suppose C is a smooth space curve given by the parametric equations

$$x = x(t) \quad y = y(t) \quad z = z(t) \quad a \leq t \leq b$$

or by a vector equation

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}.$$

If f is a function of three variables that is continuous on some region containing C, then we define the **line integral of f along C** by

$$\int_C f(x,y,z)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta s_i$$

Upon evaluation, we arrive at

$$\int_C f(x,y,z)\, ds = \int_a^b f(x(t),\; y(t),\; z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

Example 5.5.1 Evaluate $\int_C y\sin z\, ds$, where C is the circular helix given by the equation

$$x = \cos t \qquad y = \sin t \qquad 0 \le t \le 2\pi.$$

Solution:

$$\begin{aligned}
\int_C y\sin z\, ds &= \int_0^{2\pi} (\sin t)\sin t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\
&= \int_0^{2\pi} \sin^2 t\sqrt{\sin^2 t + \cos^2 t + 1}\, dt \\
&= \sqrt{2}\int_0^{\pi} \sin^2 t\, dt \\
&= \sqrt{2}\int_0^{\pi} \frac{1}{2}(1 - \cos 2t)\, dt \\
&= \frac{\sqrt{2}}{2}\left[t - \frac{1}{2}\sin t\right]_0^{2\pi} \\
&= 2\pi.
\end{aligned}$$

Example 5.5.2 Evaluate

$$\int_C (ydx + zdy + xdz)$$

where C consist of the line segment C_1 from $(2,\ 0,\ 0)$ to $(3,\ 4,\ 5)$ followed by a vertical line segment C_2 from $(3,\ 4,\ 5)$ to $(3,4,0)$.

Solution: We begin with C_1. We can write C_1 in parametric form as

$$x = 2 + t \qquad y = 4t \qquad z = 5t \qquad 0 \le t \le 1.$$

Thus,

$$\begin{aligned}
\int_{C_1} (y\,dx + z\,dy + x\,dz) &= \int_0^1 (4t)\,dt + (5t)4\,dt + (2+t)5\,dt \\
&= \int_0^1 (10+29t)\,dt \\
&= \left[10t + \frac{29}{2}t^2\right]_0^1 \\
&= \frac{49}{2}
\end{aligned}$$

For C_2, we can write it in parametric form as

$$x = 3 \qquad y = 4 \qquad z = 5 - 5t \qquad 0 \le t \le 1.$$

Hence,

$$\int_{C_2} (y\,dx + z\,dy + x\,dz) = \int_0^1 3(-5)\,dt = -15$$

Therefore,

$$\int_C (y\,dx + z\,dy + x\,dz) = \frac{49}{2} + (-15) = \frac{19}{2}.$$

5.6 The Fundamental Theorem for Line Integrals

Recall from the Fundamental Theorem of Calculus that

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

with F' continuous on $[a,b]$. If we think of the gradient vector ∇f of a function f as a sort of derivative of f , then the following theorem can be regarded as a version of the Fundamental Theorem for line integrals.

Theorem 5.6.1 **(Fundamental Theorem of Line Integrals)** Suppose C is a piecewise smooth curve lying in an open region R and given

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} \qquad a \le t \le b.$$

If $\mathbf{F}(x,y) = P\,\mathbf{i} + Q\,\mathbf{j}$ is conservative in R, P and Q continuous on R, then

$$\int_C \mathbf{F}\cdot\, d\mathbf{r} = \int_C \nabla f\cdot\, d\mathbf{r} = f(x(b),y(b)) - f(x(a),y(a))$$

where f is a potential function $\mathbf{F}$. That is, $\mathbf{F}(x,y) = \nabla f(x,y)$.

Proof:

Since $\mathbf{F}$ is conservative,

$$\mathbf{F}(x,y) = \nabla f(x,y) = f_x(x,y)\,\mathbf{i} + f_y(x,y)\,\mathbf{j}$$

and so,

$$\begin{aligned}
\int_C \mathbf{F}\cdot\, d\mathbf{r} &= \int_a^b \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt \\
&= \int_a^b \left[f_x(x,y)\,\frac{dx}{dt} + f_y(x,y)\,\frac{dy}{dt}\right] dt \\
&= \int_a^b \frac{d}{dt}[f(x(t),y(t))]\,dt \\
&= f(x(b),y(b)) - f(x(a),y(a)).
\end{aligned}$$

In space, the Fundamental Theorem of Line Integrals takes the following form.

$$\int_C \mathbf{F}\cdot\, d\mathrm{r} = \int_C \nabla f\cdot\, d\mathrm{r} = f(x(b),y(b),z(b)) - f(x(a),y(a),z(a))$$

The Fundamental Theorem of Line Integrals states that if the vector field $\mathbf{F}$ is conservative, then the line integral between any two points is simply the difference in the values of the potential function f at these points.

Example 5.6.2 Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is a smooth curve from $(1,4)$ to $(1,2)$ and

$$\mathbf{F}(x,y) = 2xy\,\mathbf{i} + (x^2 - y)\,\mathbf{j}.$$

Solution: Since
$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$
$\mathbf{F}$ is conservative. Now it is time to look for a potential function f.

$$f_x(x,y) = M = 2xy \quad \text{and} \quad f_y(x,y) = N = x^2 - y$$

Integrating both sides of f_x with respect to x we obtain

$$f(x,y) = x^2y + g(y)$$

Differentiating this equation with respect to y gives us

$$f_y(x,y) = x^2 + g'(y)$$

Note that expression on the is equal to N. This means that

$$x^2 + g'(y) = x^2 - y$$

which implies that
$$g'(y) = -y$$

Integrating this with respect to y, we obtain
$$g(y) = -\frac{1}{2}y^2 + K$$

Hence,
$$f(x,y) = x^2y - \frac{1}{2}y^2 + K.$$

We can easily verify that $\mathbf{F}(x,y) = \nabla f(x,y)$. Therefore, by the Fundamental Theorem of Line Integral,

$$\begin{aligned}\int_C \mathbf{F}\cdot\, d\mathbf{r} = \int_C \nabla f\cdot\, d\mathbf{r} &= f(1,2) - f(1,4)\\ &= 0 - (-4)\end{aligned}$$

Remark 5.6.3 The Fundamental Theorem for line integrals defies all possible complications given by piecewise smooth curves as the integral value is path independent, precisely, the value desired is simply the difference of the endpoints evaluation of the potential function

Example 5.6.4 Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is a smooth curve from $(1,1,0)$ to $(0,2,3)$ and

$$\mathbf{F}(x,y,z) = 2xy\,\mathbf{i} + (x^2+z^2)\,\mathbf{j} + 2yz\,\mathbf{k}$$

Solution: The details for the verification that $\mathbf{F}$ is conservative will be left as an exercise.

If f is a function such that $\mathbf{F}(x,y,z) = \nabla f(x,y,z)$, then

$$f_x(x,y,z,) = 2xy, \qquad f_y(x,y,z) = x^2+z^2, \qquad f_z(x,y,z) = 2yz,$$

and integrating with respect to x, y, and z separately produces

$$\begin{aligned} f(x,y,z) &= \int M\,dx = \int 2xy\,dx = x^2y + g(y,z) \\ f(x,y,z) &= \int N\,dy = \int (x^2+z^2)\,dx = x^2y + yz^2 + h(x,z) \\ f(x,y,z) &= \int P\,dz = \int 2xy\,dx = yz^2 + k(x,y) \end{aligned}$$

Comparing these three versions of $f(x,y,z)$, you can conclude that

$$g(y,z) = yz^2 + K_1, \qquad h(x,z) = K_2, \qquad \text{and} \quad k(x,y) = x^2y + K_3.$$

So, f(x,y,z) is given by

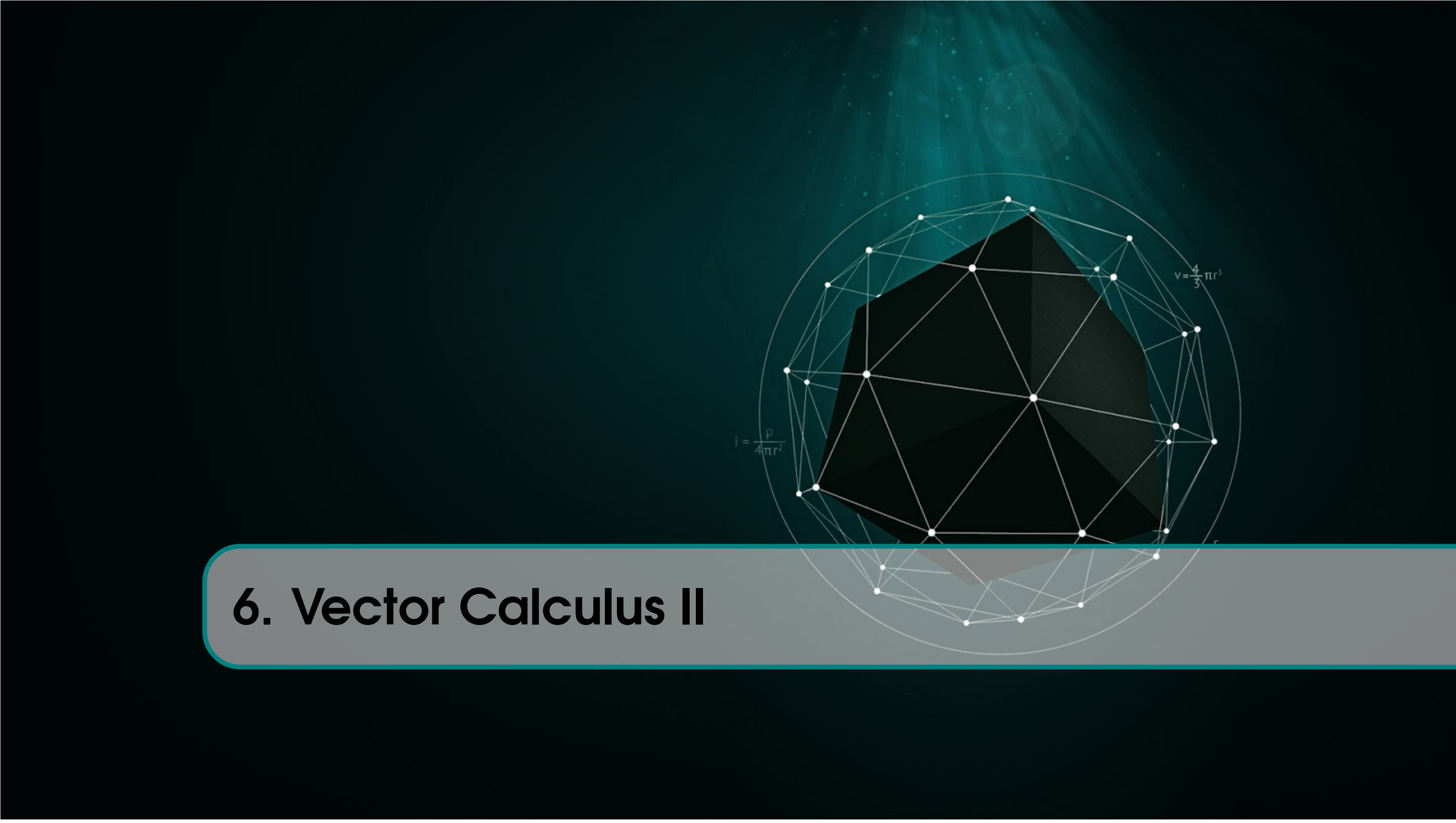
$$f(x,y,z) = x^2y + yz^2 + K. \qquad K = K_1 + K_2 + K_3$$

The function

$$f(x,y,z) = x^2y + yz^2 + K$$

is potential function that we are looking for. By the Fundamental Theorem of Line Integrals,

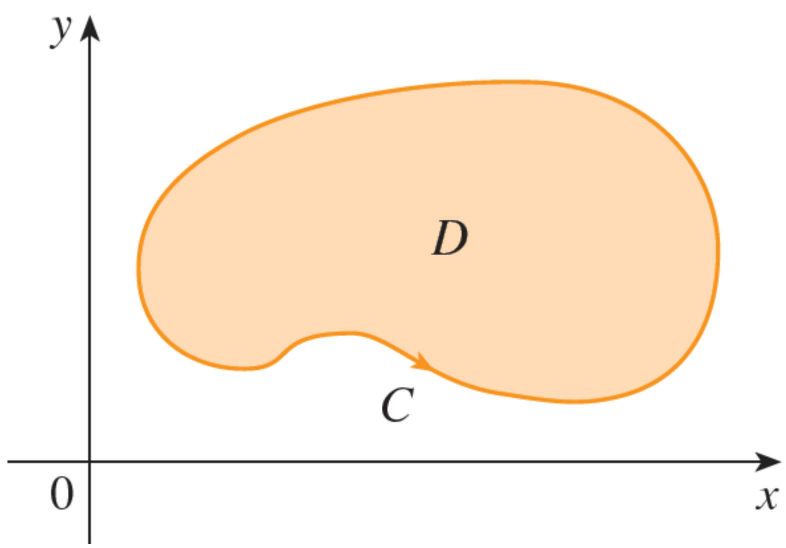
$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(0,\ 2,\ 3) - f(1,\ 1,\ 0) = 18 - 1 = 17.$$



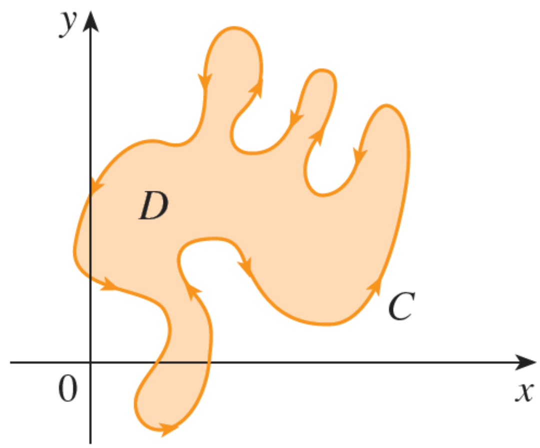
6. Vector Calculus II

6.1 Green's Theorem

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C. (See Figure 6.1.

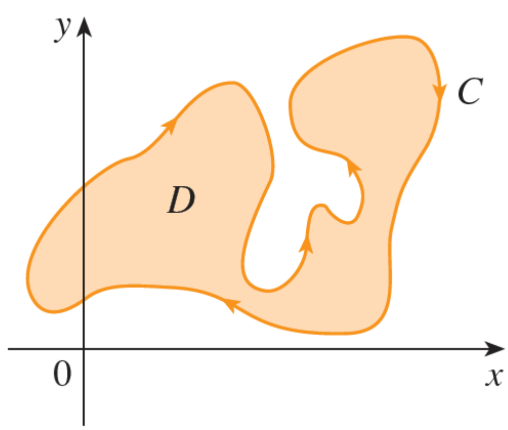


We assume that D consists of all points inside C as well as all points on C.) In stating Green's Theorem we use the convention that the positive orientation of a simple closed curve C refers to a single counterclockwise traversal of C. Thus if C is given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$, then the region D is always on the left as the point $\mathbf{r}(t)$ traverses C (See Figures 6.2a and 6.2b)



(a) Positive orientation

Figure 6.2 a



(b) Negative orientation

Figure 6.2 b

Theorem 6.1.1 **(Green's Theorem)** Let C be positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

We may also make use of the following notation

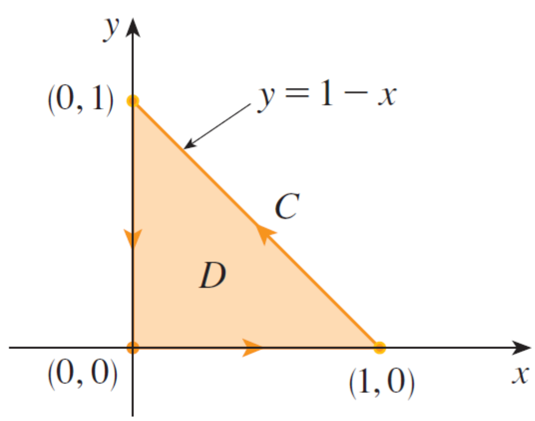
$$\oint_C P\,dx + Q\,dy \qquad \text{or} \qquad \ointctrclockwise_C P\,dx + Q\,dy$$

is sometimes used to indicate that the line integral is calculated using the positive orientation of the closed curve C. Another notation for the positively oriented boundary curve of D is ∂D, so the equation in Green's Theorem can be written as

$$\iint_{\partial D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D P\,dx + Q\,dy.$$

Example 6.1.2 Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$, and from $(0,1)$ to $(0,0)$.

Illustration:



Solution: Although the given line integral could be evaluated as usual by the method introduced in previous section, that would involve setting up three separate integrals along the three sides of the triangle, so let's use Green's Theorem instead. Notice that the region D enclosed by C is simple and C has a positive orientation (see Figure 6.3). If we let $P(x,y) = x^4$ and $Q(x,y) = xy$, then we have

$$\int\limits_C x^4\, dx + xy\, dy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int\limits_0^1 \int\limits_0^{1-x} (y-0)\, dy\, dx$$
$$= \int\limits_0^1 \left[\frac{1}{2}y^2\right]_{y=0}^{y=1-x} dx = \frac{1}{2}\int\limits_0^1 (1-x)^2\, dx$$
$$= -\frac{1}{6}(1-x)^3\Big]_0^1 = \frac{1}{6}.$$

Example 6.1.3

$$\oint\limits_C (3y - e^{\sin x})\, dx + (7x + \sqrt{y^4+1})\, dy,$$

where C is the circle $x^2 + y^2 = 9$.

Solution: The region D bounded by C is the disk $x^2 + y^2 \leq 9$, so let's change to polar coordinates after applying Green's Theorem:

$$\oint\limits_C (3y - e^{\sin x})\, dx + (7x + \sqrt{y^4+1})\, dy$$
$$= \iint\limits_D \left[\frac{\partial}{\partial x}(7x + \sqrt{y^4+1}) - \frac{\partial}{\partial y}(3y - e^{\sin x})\right] dA$$
$$= \int\limits_0^{2\pi} \int\limits_0^3 (7-3)r\, dr\, d\theta = 4\int\limits_0^{2\pi} d\theta \int\limits_0^3 r\, dr = 36\pi$$

In Examples 6.1.2 and 6.1.3 we found that the double integral was easier to evaluate than the line integral. (Try setting up the line integral in Example 2 and you'll soon be convinced!) But sometimes it's easier to evaluate the line integral, and Green's Theorem is used in the reverse direction. For instance, if it is known that $P(x,y) = Q(x,y) = 0$ on the curve C, then Green's Theorem gives

$$\int\limits_C P\, dx + Q\, dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 0$$

no matter what values P and Q assume in the region D.

Another application of the reverse direction of green's Theorem is in computing areas.

Since the area D is $\iint\limits_D 1\,dA$, we wish to choose P and Q so that

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=1,$$

There are several possibilities:

$$P(x,y)=0 \qquad P(x,y)=-y \qquad P(x,y)=-\frac{1}{2}y$$

$$Q(x,y)=x \qquad Q(x,y)=0 \qquad Q(x,y)=-\frac{1}{2}x$$

The Green's Theorem gives the following formulas for the area of D:

$$A=\oint\limits_C x\,dy=-\oint\limits_C y\,dx=\frac{1}{2}\oint\limits_C x\,dy-y\,dx$$

Example 6.1.4 Find the area enclosed by the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Solution: The ellipse hs parametric equations $x=a\cos t$ and $y=b\sin t$, where $0\le t\le 2\pi$. Using the third formula in the previous equation, we have

$$\begin{aligned}A&=\frac{1}{2}\int\limits_C x\,dy-y\,dx\\&=\frac{1}{2}\int_0^{2\pi}(a\cos t)(b\cos t)\,dt-(b\ \sin t)(-a\ \sin t)\,dt\\&=\frac{ab}{2}\int_0^{2\pi}\ dt=\pi ab.\end{aligned}$$

6.2 Curl and Divergence

In this section we define two operations that can be performed on vector fields and that play a basic role in the applications of vector calculus to fluid flow and electricity and magnetism. Each operation resembles differentiation, but one produces a vector field whereas the other produces a scalar field

If $F=P\,\mathbf{i}+Q\,\mathbf{j}+R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P,Q, and R all exist, then the curl of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\text{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

An easy way to remember above definition is to use the symbolic expression

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F},$$

where $\nabla = \mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z}$.

Example 6.2.1 If $\mathbf{F}(x,y,z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find the curl of F.

Solution: Exercise! Answer: $-y(2+x)\,\mathbf{i} + x\,\mathbf{j} + yz\,\mathbf{k}$.

Remark 6.2.2 We may also note that if $\mathbf{F}$ is conservative, then curl $\mathbf{F} = 0$.

We also have divergence of a function F defined as follows: If If $F = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of $\dfrac{\partial P}{\partial x}$, $\dfrac{\partial Q}{\partial y}$ and $\dfrac{\partial R}{\partial z}$ all exist, then the curl of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

A similar symbolic counterpart of the curl is also given for the divergence of the function $\mathbf{F}$,

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F},$$

where $\nabla = \mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z}$.

Example 6.2.3 If $\mathbf{F}(x,y,z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find div $\mathbf{F}$.

Solution: By the definition of divergence we have,

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) = z + xz.$$

6.3 Surface Area

6.3.1 Parametric Surfaces

In much the same way that we describe a space curve by a vector function $\mathbf{r}(t)$ of a single parameter t, we can describe a surface by a vector function $\mathbf{r}(u,v)$ of two parameters u and v. We suppose that

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}.$$

is a vector-valued function defined on a region D in the uv-plane. So x, y, and z, the component functions of $\mathbf{r}$, are functions of the two variables u and v with domain D. The set of all points (x,y,z) in $\mathbb{R}^3$ such that

$$x = x(u,v) \qquad y = y(u,v) \qquad z = z(u,v)$$

and (u,v) varies throughout D, is called a **parametric surface** S and the latter equations are called **parametric equations of** S. Each choice of u and v gives a point on S; by making all choices, we get all of S. In other words, the surface S is traced out by the tip of the position vector $\mathbf{r}(u,v)$ as (u,v) moves throughout the region D. (See the Figure 6.4)

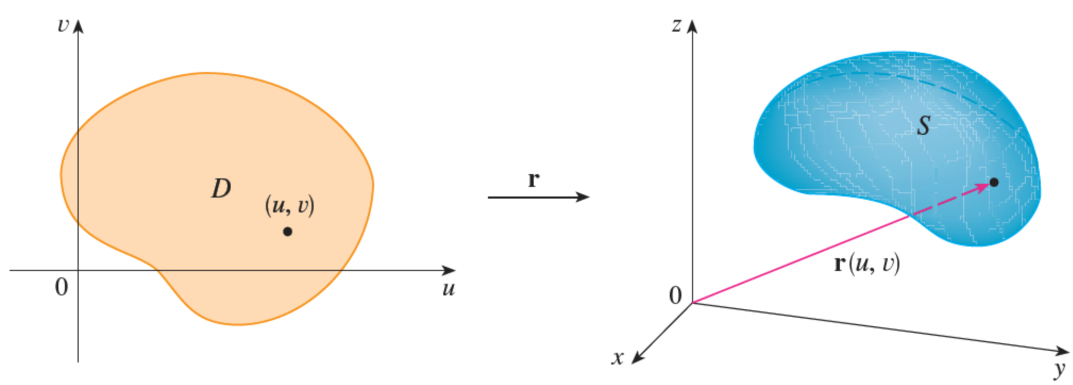


Figure 6.4

6.4 Surface Integrals

Now we define the surface area of a general parametric surface. For simplicity we start by considering a surface whose parameter domain D is a rectangle, and we divide it into subrectangles R_{ij}. Let's choose (u^*, v^*) to be the lower left corner of R_ij. (See Figure 6.5)

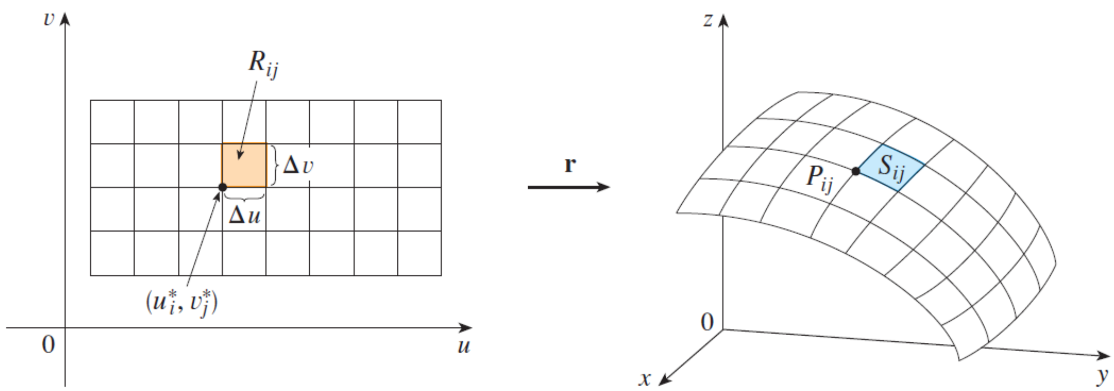


Figure 6.5

The part S_{ij} of the surface S that corresponds to R_{ij} is called a patch and has the point P_{ij} with position vector $\mathbf{r}(u^*, v^*)$ as one of its corners. Let

$$\mathbf{r}_u^* = \mathbf{r}_u(u_i^*, v_j^*) \qquad \text{and} \;\; \mathbf{r}_v^* = \mathbf{r}_v(u_i^*, v_j^*)$$

be the tangent vectors at P_{ij}

Figure 6.6(a) shows how the two edges of the patch that meet at P_{ij} can be approximated by vectors.

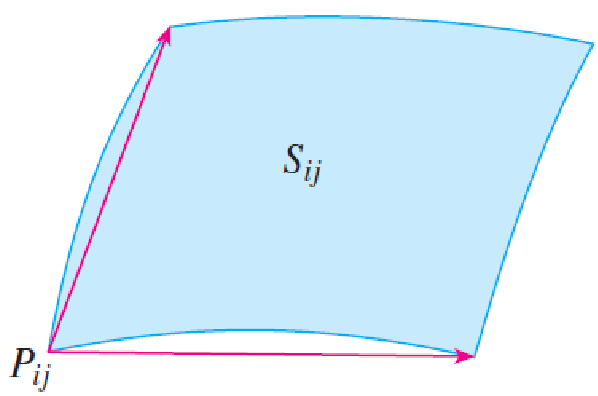


Figure 6.6 (a)

These vectors, in turn, can be approximated by the vectors $\Delta u \mathbf{r}_u^*$ and $\Delta v \mathbf{r}_v^*$ because partial derivatives can be approximated by difference quotients. So we approximate S_{ij} by the parallelogram determined by the vectors $\Delta u\mathbf{r}_u^*$ and $\Delta v\mathbf{r}_v^*$. This parallelogram is shown in Figure 6.6(b) and lies in the tangent plane to S at P_{ij}.

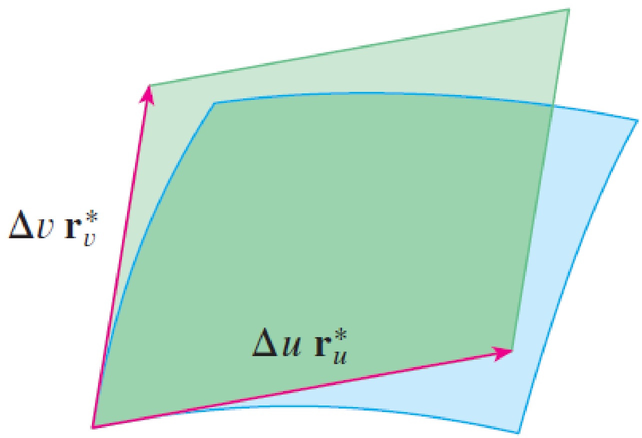


Figure 6.6 (b)

The area of this parallelogram is

$$|(\Delta\mathbf{r}_u^*) \times (\Delta v\mathbf{r}_v^*)| = |(\mathbf{r}_u^*) \times (\mathbf{r}_v^*)|\Delta u\Delta v,$$

which can be viewed as the approximation

$$\Delta S_{ij} \approx |r_u \times r_v|\ \Delta u\ \Delta v,$$

and so an approximation to the area of S is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} |(\mathbf{r}_u^*) \times (\mathbf{r}_v^*)|\Delta u\Delta v$$

Our intuition tells us that this approximation gets better as we increase the number of subrectangles, and we recognize the double sum as a Riemann sum for the double integral

$$\iint_D |r_u \times r_v|\ du\ dv.$$

Example 6.4.1 For the special case of a surface S with equation z=f(x,y), where (x,y) lies in D and f has continuous partial derivatives, we take x and y as parameters. The parametric equations are

$$x = x \quad y = y \quad z = f(x,y)$$

.

Solution: Here,

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y}\right)\mathbf{k}$$

and

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$$

Thus, we have

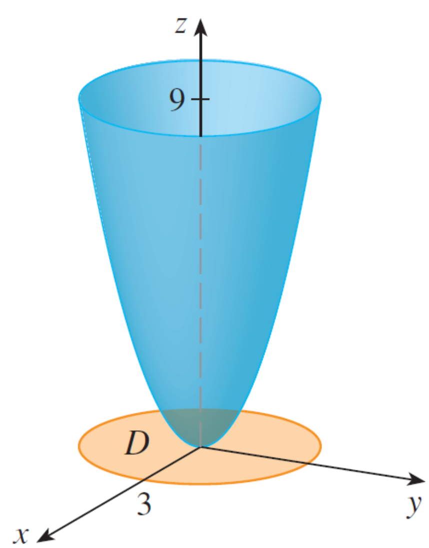
$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

and the surface area formula becomes

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA.$$

Example 6.4.2 Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

Solution: The plane intersects the paraboloid in the circle $x^2 + y^2 = 9, z = 9$. Therefore the given surface lies above the disk D with center the origin and radius 3. See the following illustration



Using the derived formula for $A(S)$ gives

$$\begin{aligned} A &= \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA \\ &= \iint_D \sqrt{1+(2x)^2+(2y)^2}\, dA \\ &= \iint_D \sqrt{1+4(x^2+y^2)}\, dA. \end{aligned}$$

Converting to polar coordinates, we obtain

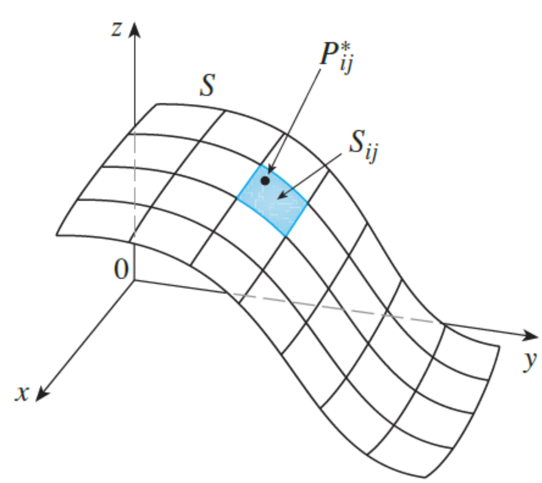
$$\begin{aligned} A &= \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}r dr\, d\theta = \int_0^{2\pi} d\theta \int_0^3 \sqrt{1+4r^2} r dr \\ &= 2\pi(1/8)(2/3)(1+4r^2)^{3/2}\Big]_0^3 = \frac{\pi}{6}(37\sqrt{37}-1). \end{aligned}$$

We now consider a function f defined over the surface S: If we evaluate f at a point P_{ij}^* in each patch, multiply by the area ΔS_{ij} of the patch, and form the Riemann sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}.$$

Then we take the limit as the number of patches increases. So, that the **surface integral of f over the surface S** is given by

$$\iint_S f(x,y,z)\, dS = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}.$$



We can rewrite latter equation as follows:

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))|r_u \times r_v|\,dA.$$

Example 6.4.3 Any surface S with equation $z = g(x,y)$, can be regarded as parametric surface with parametric equations

$$x = x \qquad y = y \qquad z = g(x,y)$$

so we have

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial g}{\partial x}\right)\mathbf{k} \qquad\qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial g}{\partial y}\right)\mathbf{k}$$

Thus,

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \dfrac{\partial g}{\partial x} \\ 0 & 1 & \dfrac{\partial g}{\partial y} \end{vmatrix} = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}$$

Hence, we have

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

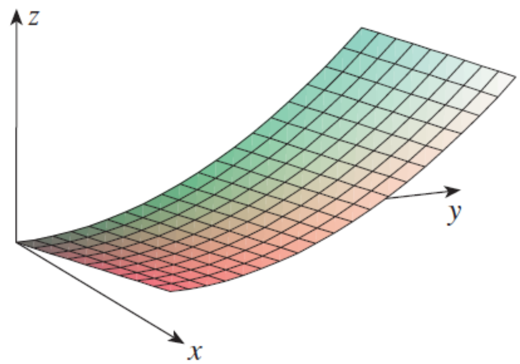
Therefore,

$$\iint_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA.$$

Similar formulas apply when it is more convenient to project S onto the yz-plane or xz-plane. For instance, if S is a surface with equation $y = h(x,z)$ and D is its projection onto the xz-plane, then

$$\iint_S f(x,y,z)dS = \iint_D f(x,h(x,z),z)\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\,dA.$$

Example 6.4.4 Evaluate $\int_S y\,dS$, where S is the surface $z = x + y^2$, $0 \le x \le 1, 0 \le y \le 2$. (See the following illustration)



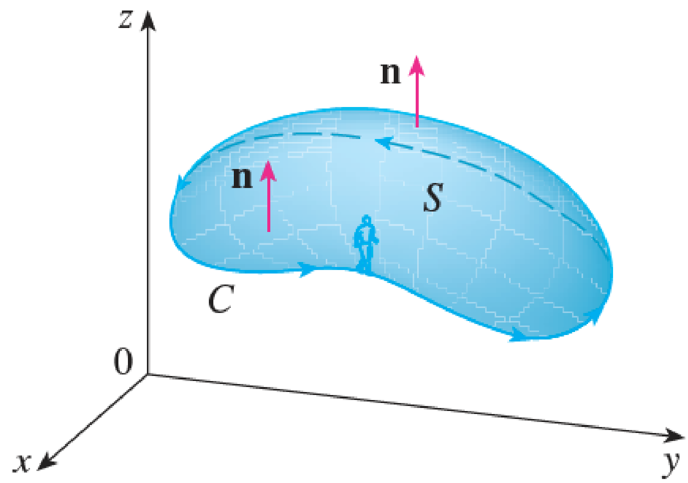
Solution: Since

$$\frac{\partial z}{\partial x} = 1 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = 2y.$$

$$\begin{aligned}
\iint_S y\,dS &= \iint_D y\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA \\
&= \int_0^1\int_0^2 y\sqrt{1+1+4y^2}dy\,dx \\
&= \int_0^1 dx\sqrt{2}\int_0^2 y\sqrt{1+2y^2}\,dy \\
&= \sqrt{2}(1/4)(2/3)(1+2y^2)^{3/2}\Big]_0^2 = \frac{13\sqrt{2}}{3}.
\end{aligned}$$

6.5 Stoke's Theorem

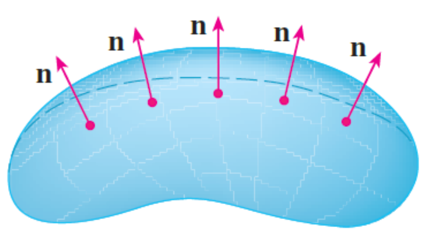
Stokes' Theorem can be regarded as a higher-dimensional version of Green's Theorem. Whereas Green's Theorem relates a double integral over a plane region D to a line integral around its plane boundary curve, Stokes' Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S (which is a space curve). Consider figure 6.10

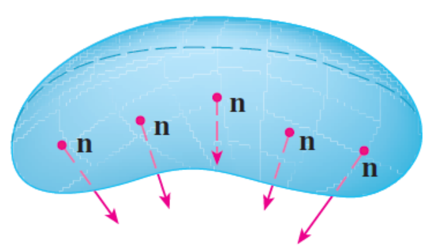


which shows an oriented surface with unit normal vector **n** which is given by

$$n = \frac{r_u \times r_v}{|r_u \times r_v|}$$

as in the figure





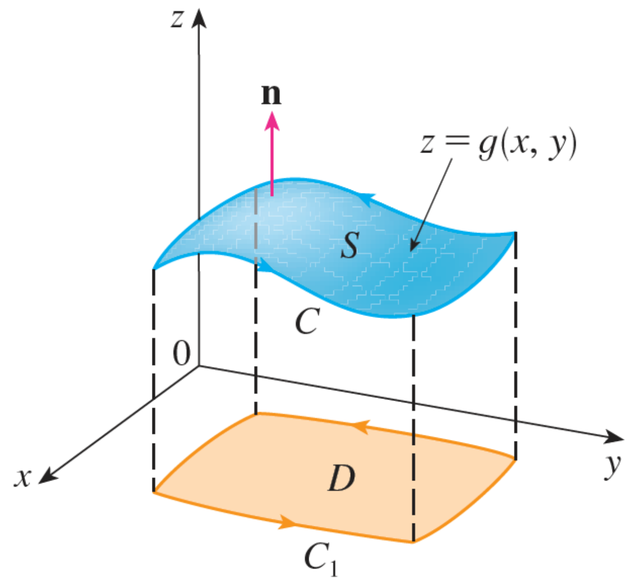
The orientation of S induces the **positive orientation of the boundary curve** C shown in the figure. This means that if you walk in the positive direction around C with your head pointing in the direction of **n**, then the surface will always be on your left

Theorem 6.5.1 (**Stokes' Theorem**)Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed piecewise-smooth boundary curve C with positive orientation.

Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

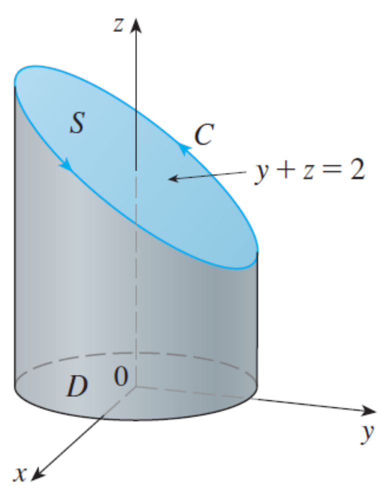
$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl }\ \mathbf{F}\cdot d\mathbf{S}.$$

The surface illustration:



Example 6.5.2 Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = -y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ and C is the curve of intersection of the plane $y+z=2$ and cylinder $x^2+y^2=1$. (Orient C to be counterclockwise when viewed from above.)

Solution: The curve C (an ellipse) is shown in following figure.



Although $\int\limits_C \mathbf{F}\cdot dr$ could be evaluated directly, it's easier to use Stokes' Theorem. We first compute

$$\text{curl }\ \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (1+2y)\ \mathbf{k}$$

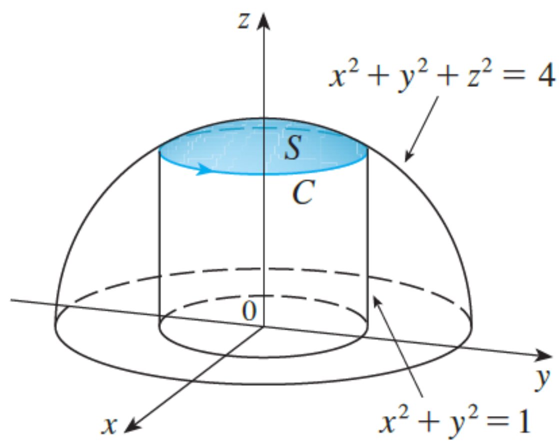
Although there are many surfaces with boundary C, the most convenient choice is the elliptical region S in the plane $y+z=2$ that is bounded by C. If we orient S upward, then C has the induced positive orientation. The projection D of S onto the xy-plane is the disk $x^2+y^2 \leq 1$ and considering $z = g(x,y) = 2-y$, we have

$$\begin{aligned}
\int\limits_C \mathbf{F}\cdot\ d\mathbf{r} &= \iint\limits_S \text{curl } \mathbf{F}\cdot d\mathbf{S} = \iint\limits_D (1+2y)\ dA \\
&= \int\limits_0^{2\pi}\int\limits_0^1 (1+2r\sin\theta)r\ dr\ d\theta \\
&= \int\limits_0^{2\pi} \left[\frac{r^2}{2} + 2\frac{r^3}{3}\sin\theta\right] d\theta = \int\limits_0^{2\pi} \left(\frac{1}{2}+\frac{1}{2}\sin\theta\right)\ d\theta \\
&= \frac{1}{2}(2\pi)+0 = \pi
\end{aligned}$$

Example 6.5.3 Use Stoke's Theorem to compute the integral $\int\limits_S \text{curl } \mathbf{F}\cdot d\mathbf{S}$, where

$$\mathbf{F}(x,y,z) = xz\ \mathbf{i} + yz\ \mathbf{j} + xy\ \mathbf{k}$$

and S is the part of the sphere $x^2+y^2+z^2 = 4$ that lies inside the cylinder $x^2+y^2 = 1$ and above the xy-plane. The surface illustration:



Solution:

To find the boundary curve C we solve the equations $x^2+y^2+z^2=4$ and $x^2+y^2=1$. Subtracting, we get $z^2=3$ and so $z=\sqrt{3}$ (since $z>0$). Thus, C is the circle given by equations $x^2+y^2=1$, $z=\sqrt{3}$. A vector equation of C is

$$r(t)=\cos t\mathbf{i}+\sin t\,\mathbf{j}+\sqrt{3}\,\mathbf{k}\quad 0\leq t\leq 2\pi$$

or

$$r'(t)=-\sin t\,\mathbf{i}+\cos t\,\mathbf{j}.$$

Also, we have

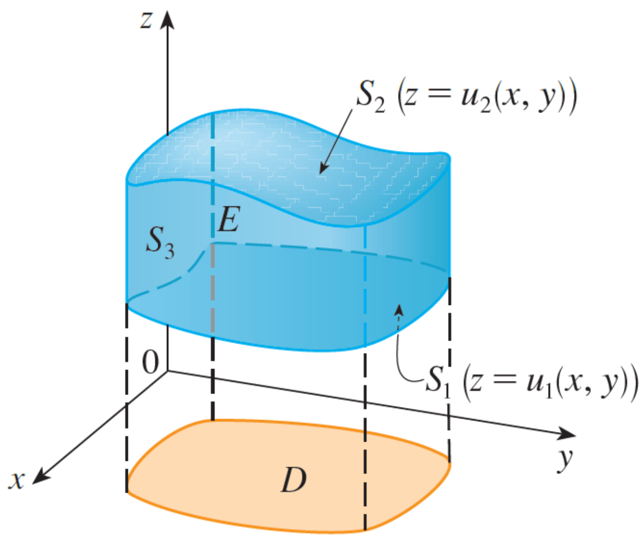
$$\mathbf{F}(\mathbf{r}(t))=\sqrt{3}\cos t\,\mathbf{i}+\sqrt{3}\sin t\,\mathbf{j}+\cos t\sin t\,\mathbf{k}.$$

Therefore, by Stoke's Theorem,

$$\begin{aligned}\int_S \text{curl}\,\mathbf{F}\cdot\,d\mathbf{S}=\int_C \mathbf{F}\cdot\,d\mathbf{r}&=\int_0^{2\pi}\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt\\&=\int_0^{2\pi}(-\sqrt{3}\cos t\sin t+\sqrt{3}\sin t\;cost)\,dt\\&=\sqrt{3}\int_0^{2\pi}0\,dt=0\end{aligned}$$

6.6 The Divergence Theorem

The Divergence Theorem is sometimes called Gauss's Theorem after the great German mathe matician Karl Friedrich Gauss (1777–1855), who discovered this theorem during his investigation of electrostatics. In Eastern Europe the Divergence Theorem is known as Ostrogradsky's Theorem after the Russian mathe matician Mikhail Ostrogradsky (1801–1862), who published this result in 1826. Now, considering the surface illustration:



We have the Divergence Theorem stated as follows:

Theorem 6.6.1 Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

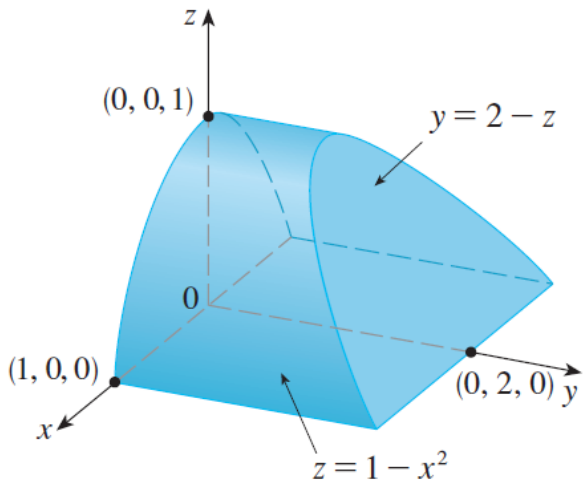
$$\iint_S \mathbf{F}\cdot\, d\mathbf{S} = \iiint_E \text{div}\, \mathbf{F}\, dV.$$

Example 6.6.2 Evaluate $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S}$, where

$$\mathbf{F}(x,y,z) = xy\,\mathbf{i} + (y^2 + e^{xz^2})\,\mathbf{j} + \sin(xy)\,\mathbf{k}$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and planes $z = 0, y = 0$, and $y + z = 2$

Surface illustration:



Solution: It would be extremely difficult to evaluate the given surface integral directly. (We would have to evaluate four surface integrals corresponding to the four pieces of S.) Furthermore, the divergence of $\mathbf{F}$ is much less complicated than $\mathbf{F}$ itself:

$$\mathrm{div}\mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y^2 + e^{xz^2}) + \frac{\partial}{\partial z}(\sin(xy)) = y + 2y = 3y$$

Therefore we use the Divergence Theorem to transform the given surface integral into a triple integral. The easiest way to evaluate the triple integral is to express E as a type 3 region:

$$E = \left\{ (x,y,z) \middle| -1 \le x \le 1, \quad 0 \le z \le 1 - x^2, \quad 0 \le y \le 2 - z \right\}.$$

Then we have

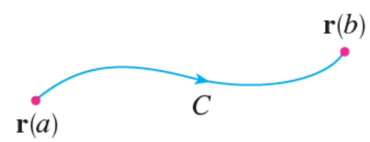
$$\begin{aligned}
\iint \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \mathrm{div}\, \mathbf{F}\, dV = \iiint_E 3y dV \\
&= 3 \int_{-1}^{1} \int_{0}^{1-x^2} \int_{0}^{2-z} y\, dy\, dz\, dx = 3 \int_{-1}^{1} \int_{0}^{1-x^2} \frac{(2-z)^2}{2} dz\, dx \\
&= \frac{3}{2} \int_{-1}^{1} \left[-\frac{(2-z)^3}{3} \right]_0^{1-x^2} dx = -\frac{1}{2} \int_{-1}^{1} [(x^2+1)^3 - 8]\, dx \\
&= -\int_{0}^{1} (x^6 + 3x^4 + 3x^2 - 7)\, dx = \frac{184}{35}.
\end{aligned}$$

Fundamental Theorem of Calculus

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

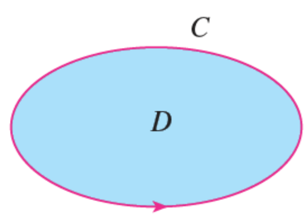
Fundamental Theorem for Line Integrals

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$



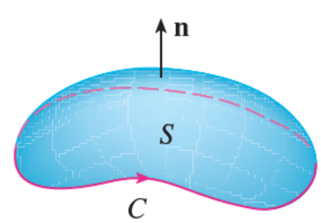
Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$$



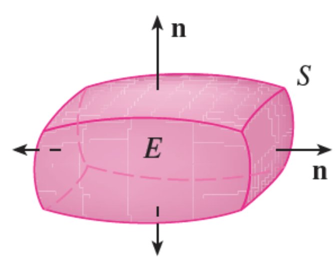
Stokes' Theorem

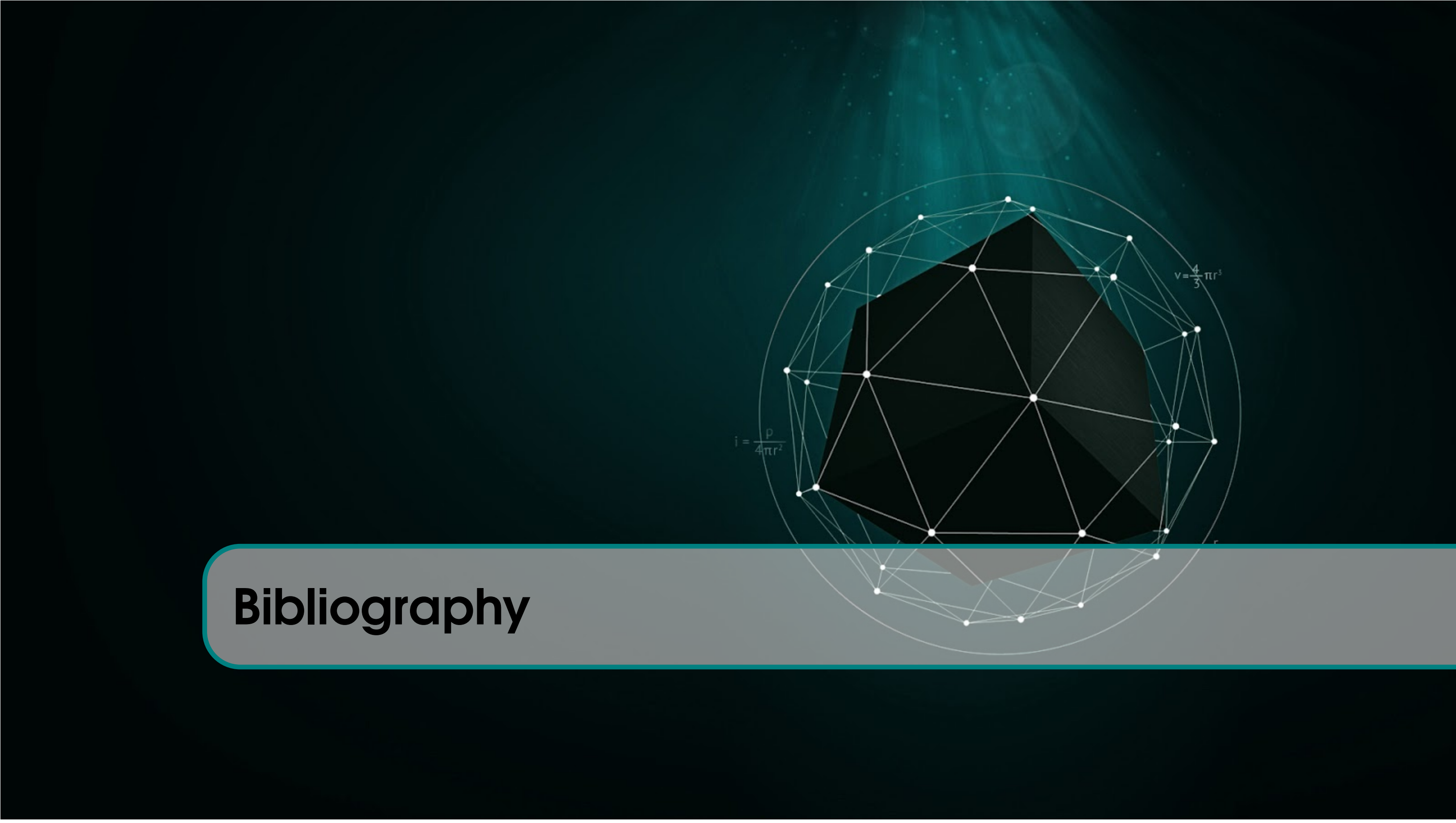
$$\iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$



Divergence Theorem

$$\iiint_E \text{div}\,\mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$





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