1 Reservoir

1.1 Problem Statement

In this section, I model a lake system with concentrations of a chemical. There are n lakes, which each have volume V_i (see Figure 1). Each lake has a concentration c_i.

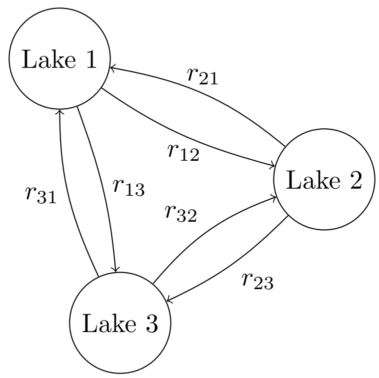


Figure 1: A Lake System, with flow rates

The rate of flow between lake i and lake j is given by the difference in their concentrations times the rate of flow. So, the flow into any one lake is given by the following matrix:

$$R = \begin{bmatrix} -(r_{21}+r_{31}) & r_{21} & r_{31} \\ r_{12} & -(r_{12}+r_{32}) & r_{32} \\ r_{13} & r_{23} & -(r_{23}+r_{13}) \end{bmatrix} \tag{1}$$

If C is a vector of concentrations, the model predicts the following flow rates:

$$\frac{dC}{dt} = RC \tag{2}$$

1.2 Asymptotic Concentrations

We ask the question, what will the asymptotic concentrations be? Since the system of differential equations is linear, we can solve it eigenvector-by-eigenvector.

We can get some insight by analyzing the eigenvalues of R. Consider the following eigenvector:

$$\begin{bmatrix} -(r_{21}+r_{31}) & r_{21} & r_{31} \\ r_{12} & -(r_{12}+r_{32}) & r_{32} \\ r_{13} & r_{23} & -(r_{23}+r_{13}) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} r_{21}+r_{31}-(r_{21}+r_{31}) \\ r_{12}+r_{32}-(r_{12}+r_{32}) \\ r_{23}+r_{13}-(r_{23}+r_{13}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The vector $[0,0,0]^T$ is an eigenvector with eigenvalue zero. If there are positive eigenvalues of R, then there will not be asymptotic concentrations, because as time goes to infinity, those eigenvectors will blow up in magnitude. Therefore, it is only meaningful to talk about asymptotic concentrations if there are no positive eigenvalues.

As time goes to infinity, every negative eigenvalue will see its corresponding eigenvector's magnitude decrease to zero. Therefore, only eigenvalues which have no negative real part will survive going to infinity.

The eigenvector $[1,1,1]^T$ is one such eigenvector, with eigenvalue zero. This eigenvector will always be present. However, depending on the values of r_{ij}, there may be other eigenvectors with eigenvalues with zero real part. Only if $[1,1,1]^T$ is the only eigenvector with eigenvalue of real part equal to zero will generic choices of initial concentrations converge to a constant concentration accross the lakes.

1.3 Monotonicity of Convergence

We next ask the question, will the concentrations converge monotonically to their final values? This will happen if there is no cyclic behavior. Cyclic behavior might happen in a three-reservoir system if the system is closely approximated by a cycle, with much larger flow rates in one direction than the other. One such cycle is shown in Figure 2. Intuitively, if a large amount of the substance is dumped in one of the lakes, most of the substance will make a circuit around the lakes. The concentration might rise and fall in each reservoir until it settles down.

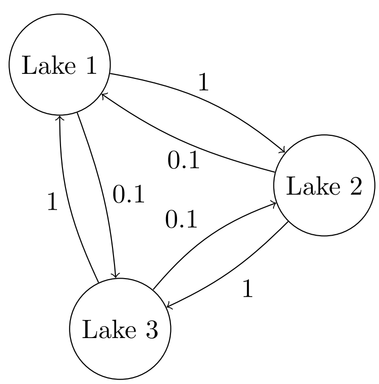


Figure 2: A Lake System, with cyclic flow dominating

In this case, the R matrix is:

$$R = \begin{bmatrix} -1.1 & 0.1 & 1 \\ 1 & -1.1 & 0.1 \\ 0.1 & 1 & -1.1 \end{bmatrix}$$

which has eigenvalues $\lambda_1 = 0, \lambda_2 = -1.65 + 0.77942i, \lambda_3 = -1.65 - 0.77942i$ and eigenvectors:

$$\lambda_1 = 0 \qquad v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -1.65 + 0.77942i \qquad v_2 = \begin{bmatrix} -0.28868 - 0.5i \\ -0.28868 + 0.5i \\ 0.57735 \end{bmatrix}$$

$$\lambda_3 = -1.65 - 0.77942i \qquad v_3 = \begin{bmatrix} -0.28868 + 0.5i \\ -0.28868 - 0.5i \\ 0.57735 \end{bmatrix}$$

Notice that these eigenvalues are *complex*. The answers will still be real, because they and their eigenvectors are complex conjugates of each other, and thus add to a real number, but their complex-ness can lead to the non-monotone behavior that intuition has predicted.

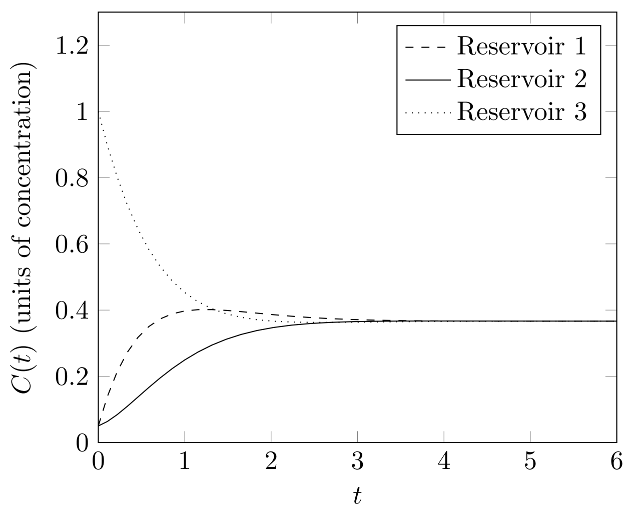


Figure 3: Substance Concentrations, in lake system from Figure 2

In Figure 3, the Reservoir 2's trajectory goes above the equilibrium value, and then relaxes to it. The concentration in each of the lakes is therefore not monotonic.

I conjecture that the flow in the lake with the highest initial concentration is monotonic.

1.4 Monotonicity of Convergence in Lakes with Symmetric Flow

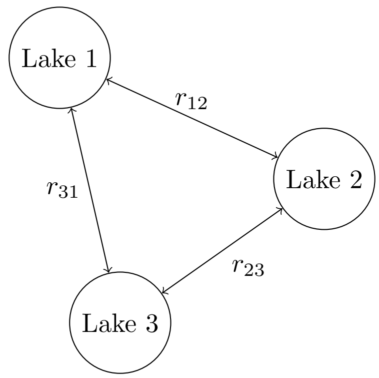


Figure 4: A Lake System, with symmetric flow rates

Suppose we have a lake system where the rates of flow between lakes are the same in both directions (Figure 4). Then, the R matrix is symmetric. Since symmetric matrices have real eigenvalues, then there will be no cyclic behavior. The concentration will therefore converge monotonically in this special case.

1.5 Water Purifying Plants

Suppose there is a water purifying plant in lake i. This plant will purify the water at a rate α_i. This will add negative terms to the diagonal of the R-matrix:

$$R = \begin{bmatrix} -(r_{21}+r_{31})-\alpha_1 & r_{21} & r_{31} \\ r_{12} & -(r_{12}+r_{32})-\alpha_2 & r_{32} \\ r_{13} & r_{23} & -(r_{23}+r_{13})-\alpha_3 \end{bmatrix} \tag{3}$$

Notice that $[1,1,1]^T$ is no longer an eigenvector of the R-matrix. Since we have assumed that there is an equilibrium limit, the eigenvalues of R are nonpositive. Therefore, the real part of each eigenvalue is either zero or negative. If the parameters are perturbed slightly, then the eigenvalues will also change slightly. Therefore, generically, the real parts of all of the eigenvalues are negative.

1.5.1 Example

The following matrix is provided in the example. In the example, the matrix is named A, but we will rename it R to keep with the conventions of this paper. This is the matrix before we introduce the purification term:

$$R = \begin{bmatrix} -0.01 & 0 & 0 & 0 & 0 & 0.01 \\ 0.01 & -0.01 & 0 & 0 & 0 & 0 \\ 0 & 0.01 & -0.01 & 0 & 0 & 0 \\ 0 & 0 & 0.01 & -0.01 & 0 & 0 \\ 0 & 0 & 0 & 0.01 & -0.01 & 0 \\ 0 & 0 & 0 & 0 & 0.01 & -0.01 \end{bmatrix}$$

$$c_0 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Suppose Reservoir 1 has a volume of 1 km^3, and a purification plant processes the water at a rate of 0.005 km^3 per day. We assume that the plant removes all of the chemical from the water it processes. Therefore, the concentration at lake 1 is decreased at a rate of 0.005 per day. The new R matrix is:

$$R = \begin{bmatrix} -0.015 & 0 & 0 & 0 & 0 & 0.01 \\ 0.01 & -0.01 & 0 & 0 & 0 & 0 \\ 0 & 0.01 & -0.01 & 0 & 0 & 0 \\ 0 & 0 & 0.01 & -0.01 & 0 & 0 \\ 0 & 0 & 0 & 0.01 & -0.01 & 0 \\ 0 & 0 & 0 & 0 & 0.01 & -0.01 \end{bmatrix} \tag{4}$$

$$c_0 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

Without the purification plant, the concentrations follow the following profile:

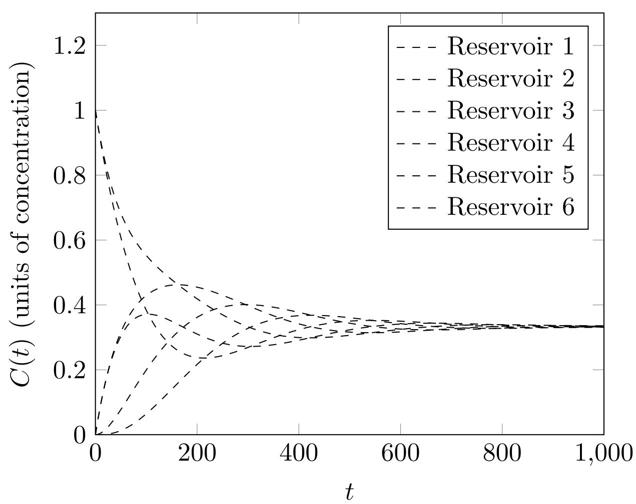


Figure 5: Six lake system with no purification plant

When the purification plant is added, the concentrations now follow this profile:

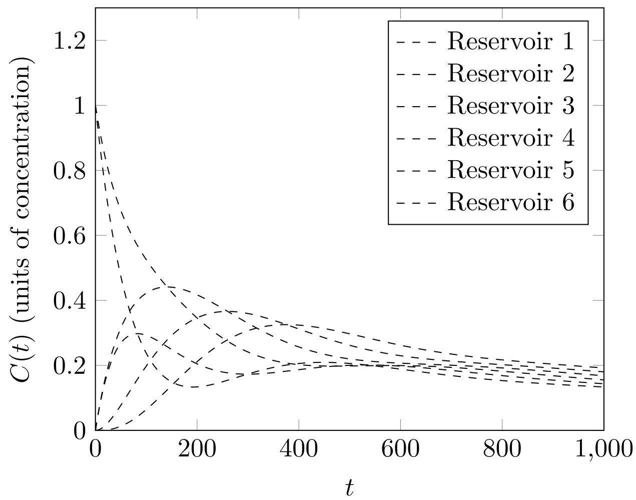


Figure 6: Six lake system with purification plant

The last lake concentration falls below 0.2 after approximately 931.74 days.

1.6 Optimizing Chemical Distribution

The third matlab script contains a slight modification to the system. The rates of diffusion are time-varying. The new differential equations are shown below, where 6 is the old, time-independant differential equation and 7 is the new, time-dependant differential equation. Notice that this differential equations varies on a timescale of about 100 days.

$$\frac{dC}{dt} = RC \tag{6}$$

$$\frac{dC}{dt} = \left(2 + a\sin\left(\frac{2\pi t}{100}\right)\right) RC \tag{7}$$

The goal for this problem is to minimize the total amount of reactant mass required for the concentration in lake 6 to be exactly 0.4 at $t = 300$, and for the concentrations of all the other lakes to fall into the interval $[0.3, 0.4]$.

First I'm going to determine what it means to "minimize the total amount of reactant mass". I'm going to assume that the concentrations are by a ratio of mass of reactant to total volume of water. Therefore, a mass of reactant m put into a lake with volume V will yield a concentration proportional to $\frac{m}{V}$.

$$f = \begin{bmatrix} V_1 & \dots & V_n \end{bmatrix} \tag{8}$$

Therefore, to minimize the reactant mass, one should try to minimize the product of the initial concentrations in each lake with the one-form 8:

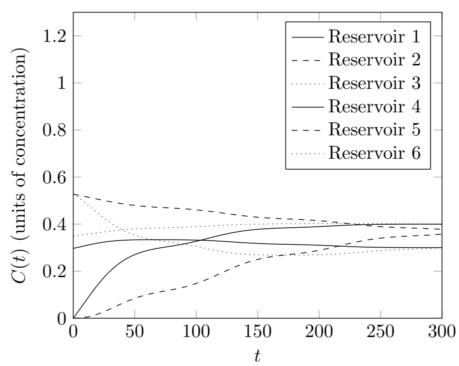
Next, I need to determine how to relate initial concentrations to final concentrations. Since the equations of motion are linear, the function relating initial states to final states is a linear transformation. Therefore, it can be completely determined by its values on a basis of the intitial concentrations space. I will use the standard basis to determine the linear transformation. Once I have a linear transformation relating the initial concentrations to the final concentrations, I can use that linear transformation to impose linear constraints on the final concentrations at $t = 300$.

$$c_0 = \begin{bmatrix} 0 & 0 & 0.52835 & 0.29671 & 0.52700 & 0.34956 \end{bmatrix}^T \tag{9}$$

$$m_0 = \begin{bmatrix} 0 & 0 & 1.58506 & 1.18686 & 2.63502 & 2.09736 \end{bmatrix}^T \tag{10}$$

$$x_f = \begin{bmatrix} 0.4 & 0.35754 & 0.3 & 0.3 & 0.37784 & 0.4 \end{bmatrix}^T \tag{11}$$

I can then use the objective function f to solve a linear programming problem which yields the initial concentrations 9, which minimize initial reactant mass 10. Note that the final concentrations 11 satisfy the desired constraints.



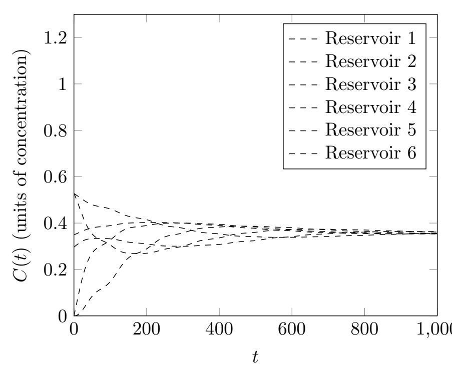


Figure 7: Results of Linear Optimization to Minimize Chemical Mass

Notice that the asymptotic concentrations will lie within the interval, for each lake.

2 Cell Phone Tower Placement

Suppose you want to place a cell phone tower within a space. The reception from the tower is given by Eq. (12) and shown on the right in Fig (8.

$$q(x, y \mid x_c, y_c) = \begin{cases} 20 - 5((x - x_c)^2 + (y - y_c)^2) & \text{if } (x - x_c)^2 + (y - y_c)^2 < 4 \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

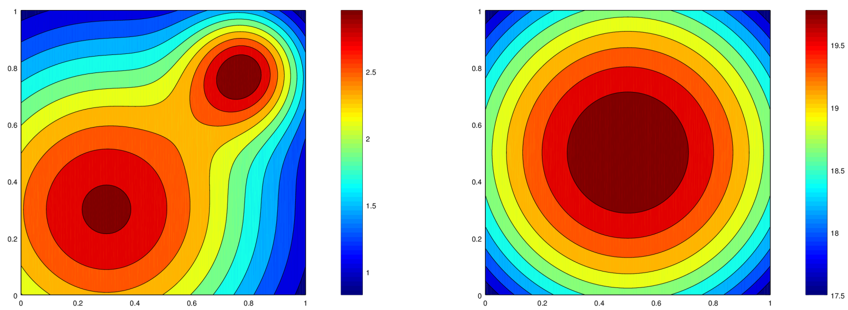


Figure 8: Left: Population Density. Note that Optimal Tower Placement will be along the diagonal, on the axis of symmetry. Right: Tower Reception, if the Tower is in the center

Suppose the population is distributed according to the plot on the left of Fig (8). Then, what is the optimal place to place the tower, to maximize the average reception?

If $p(x, y)$ denotes the spatial population density, the average reception is:

$$\langle q \rangle = \frac{\int p(x,y)q(x,y)d^2x}{\int p(x,y)d^2x} \tag{13}$$

Since the population distribution is symmetric about the diagonal, the tower should be placed on the center line. Furthermore, since the population is supported on $[0,1] \times [0,1]$, the optimum tower placement occurs between $(0,0)$ and $(1,1)$ on the line connecting them. Therefore, we have a univariate function 14, whose graph is shown in 9:

$$f(t) = \frac{\int p(x,y)q(x,y \mid t,t)d^2}{\int p(x,y)d^2x} \tag{14}$$

The maximum of f is at 0.47950.

3 Cell Phone Walk

Suppose a cell tower generates the following reception profile (15).

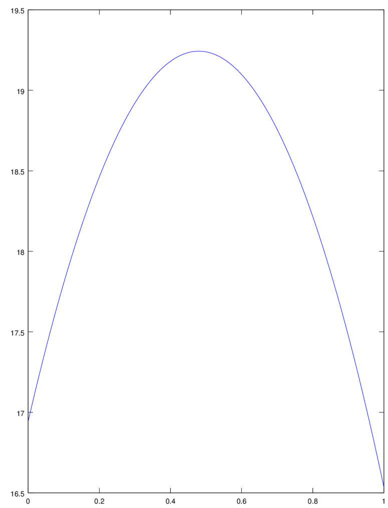


Figure 9: Average Reception as a function of Tower Location along the diagonal

$$s(x,y) = \begin{cases} 20 - 5(x^2+y^2) & \text{if } x^2+y^2 < 4 \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

Suppose the caller needs to walk from point a to point b in at least T minutes, and the caller's speed of motion cannot exceed V. What path through the plane has the best average reception?

Let's say that a path is a function γ which satisfies:

$$\gamma : [0,T] \to \mathbb{R}^2$$
$$\gamma(0) = a$$
$$\gamma(T) = b$$

Let's define $x(t)$, $y(t)$ by:

$$\gamma(t) = (x(t), y(t)) \tag{16}$$

3.1 Extremal Cases

Since s has a maximum at $(0,0)$, if we have enough time to go to the maximum and back, this will have the best average reception. Going to the maximum from $(-2,0)$ is 2 distance units away, and going from the maximum to $(0,-2)$

is also 2 distance units away. Therefore, if we have the time to cover 4 distance units, we can go straight to the maximum. We can travel VT distance units total if we always move at our maximum speed. Therefore, if $VT \geq 4$, then we will be able to go right to the maximum.

Since we can travel only VT distance units total, if this is smaller than the distance from a to b, then there will be no way to move in the desired way. The distance from $(-2, 0)$ to $(0, 2)$ is $2\sqrt{2}$ distance units, so therefore, if $VT < 2\sqrt{2}$, then there will be no solution.

The interesting case is therefore when $2\sqrt{2} < VT < 4$, when a solution exists, but it is not possible to go directly to the maximum.

3.2 Minimizing a Functional (Cartesian Coordinates)

I first derive the differential equations for a minimized functional in cartesian coordianates to analyze singularities and demonstrate the validity of the method. I will then re-derive them in polar coordiantes, and use those.

Suppose $2\sqrt{2} < VT < 4$. Then, we need to determine the shape of the path γ to maximize the average reception (17):

$$\langle s \rangle = \frac{\int_0^T s(x(t), y(t))dt}{\int_0^T dt} \tag{17}$$

If we have a path, and on some subsection of that path, we are not moving at the maximum speed, we can increase the total average reception by traversing this path at full speed, and then going towards the center and turning around. Therefore, all paths which do not move at the maximum allowed speed are not optimal.

Therefore, we conclude that:

$$\sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2} = V \tag{18}$$

Suppose $y(t) = f(x(t))$. This is a non-trivial assumption. It is assuming that optimal paths go only one way in the x direction. It is true in this case because if a path does go backwards in the x direction, one can make a better path simply by waiting where you are and then skipping ahead. Then,

$$\begin{aligned} V &= \sqrt{\frac{dx}{dt}^2 + \frac{df}{dx}^2 \frac{dx}{dt}^2} \\ V &= \frac{dx}{dt}\sqrt{1 + \frac{df}{dx}^2} \\ dt &= \frac{1}{V}\sqrt{1 + \frac{df}{dx}^2}\, dx \end{aligned} \tag{19}$$

Therefore we can rewrite the expression for $\langle s \rangle$:

$$\langle s \rangle = \int_{x_a}^{x_b} \frac{s(x, f(x))}{VT} \sqrt{1 + \frac{df}{dx}^2} dx \tag{20}$$

Now that we're integrating over x, we don't automatically have the constraint on the path length anymore. We want to maximize $\langle s \rangle$, but subject to the constraint that the path length is VT. This constraint can be written as a functional of f:

$$VT = \int_{x_a}^{x_b} \sqrt{1 + \frac{df}{dx}^2} dx \tag{21}$$

To find the f which extremizes $\langle s \rangle$ with the desired arclength, we use a lagrange multiplier λ, and look for optimal solutions to the following functional.

$$S[f] = \frac{1}{VT} \int_{x_a}^{x_b} (s(x, f(x)) + \lambda) \sqrt{1 + \frac{df}{dx}^2} dx \tag{22}$$

Let $n(x, f(x)) = s(x, f(x)) + \lambda)$. This functional is of the form of an integral $\int_a^b \mathcal{L}(x, f(x), f'(x))dx$, so therefore every f which extremizes S satisfies the following instance of the euler-lagrange equation:

$$\mathcal{L} = n(x, f(x)) \sqrt{1 + \frac{df}{dx}^2} \tag{23}$$

$$\frac{d}{dx} \left(\frac{\partial \mathcal{L}}{\partial \frac{df}{dx}} \right) = \frac{\partial \mathcal{L}}{\partial f}$$

In this case, this becomes:

$$\frac{d}{dx}\left(\frac{n(x,f(x))\frac{df}{dx}}{\sqrt{1+\frac{df}{dx}^2}}\right) = \frac{\partial n}{\partial y}\sqrt{1+\frac{df}{dx}^2}$$

$$\frac{\partial n}{\partial y}\sqrt{1+\frac{df}{dx}^2} = \frac{\frac{\partial n}{\partial y}\frac{df}{dx} + \frac{\partial n}{\partial x} + n(x,f(x))\frac{\partial^2 f}{\partial x^2}}{\sqrt{1+\frac{df}{dx}^2}} - \frac{n(x,f(x))\frac{df}{dx}^2\frac{d^2 f}{dx^2}}{\left(1+\frac{df}{dx}^2\right)^{\frac{3}{2}}}$$

$$\frac{\partial n}{\partial y}\left(1+\frac{df}{dx}^2\right)^2 = \left(\frac{\partial n}{\partial y}\frac{df}{dx} + \frac{\partial n}{\partial x} + n(x,f(x))\frac{d^2 f}{dx^2}\right)\left(1+\frac{df}{dx}^2\right) - n(x,f(x))\frac{df}{dx}^2\frac{d^2 f}{dx^2}$$

$$\frac{d^2 f}{dx^2} = \frac{\frac{\partial n}{\partial y}}{n}\left(1+\frac{df}{dx}^2\right)^2 - \left(\frac{\frac{\partial n}{\partial y}}{n}\frac{df}{dx} + \frac{\frac{\partial n}{\partial x}}{n}\right)\left(1+\frac{df}{dx}^2\right)$$

$$\frac{d^2 f}{dx^2} = \left(1+\frac{df}{dx}^2\right)\left(\frac{\frac{\partial n}{\partial y}}{n}\left(1-\frac{df}{dx}+\frac{df}{dx}^2\right) - \frac{\frac{\partial n}{\partial x}}{n}\right)$$

Since there are two boundary conditions, for each choice of λ there exists a unique f such that $f(x_b) = y_b$ and $f(x_a) = y_a$. Furthermore, we know that $n = s(x, f(x)) - \lambda$ should never be zero within any area where we expect the path to be.

3.3 Minimizing a Functional (Polar Coordinates)

Suppose $2\sqrt{2} < VT < 4$. We can also express $\gamma(t) = (r(t), \theta(t))$ in ploar coordinates. Then, we need to determine the shape of the path γ to maximize the average reception (24):

$$\langle s \rangle = \frac{\int_0^T s(r(t),\theta(t))dt}{\int_0^T dt} \tag{24}$$

Since, as before, we are always moving at the maximum speed, we conclude that:

$$\sqrt{\frac{dr}{dt}^2 + r^2\frac{d\theta}{dt}^2} = V \tag{25}$$

Suppose $r(t) = f(\theta(t))$. Then,

$$
\begin{aligned}
V &= \sqrt{\frac{df}{d\theta}^2 \frac{d\theta}{dt}^2 + f(\theta)^2 \frac{d\theta}{dt}^2} \\
V &= \frac{d\theta}{dt} \sqrt{f(\theta)^2 + \frac{df}{d\theta}^2} \\
dt &= \frac{1}{V} \sqrt{f(\theta)^2 + \frac{df}{d\theta}^2} dx
\end{aligned}
\tag{26}
$$

Therefore we can rewrite the expression for $\langle s \rangle$:

$$
\langle s \rangle = \int_{\theta_a}^{\theta_b} \frac{s(\theta, r(\theta))}{VT} \sqrt{f(\theta)^2 + \frac{df}{d\theta}^2} dx \tag{27}
$$

Now that we're integrating over θ, we don't automatically have the constraint on the path length anymore. We want to maximize $\langle s \rangle$, but subject to the constraint that the path length is VT. This constraint can be written as a functional of f:

$$
VT = \int_{\theta_a}^{\theta_b} \sqrt{f(\theta)^2 + \frac{df}{d\theta}^2} dx \tag{28}
$$

To find the f which extremizes $\langle s \rangle$ with the desired arclength, we use a lagrange multiplier λ, and look for optimal solutions to the following functional.

$$
S[f] = \frac{1}{VT} \int_{\theta_a}^{\theta_b} (s(\theta, f(\theta)) + \lambda) \sqrt{f(\theta)^2 + \frac{df}{d\theta}^2} d\theta \tag{29}
$$

Let $n(\theta, f(\theta)) = s(\theta, f(\theta)) + \lambda)$. This functional is of the form of an integral $\int_a^b \mathcal{L}(x, f(x), f'(x)) dx$, so therefore every f which extremizes S satisfies the following instance of the euler-lagrange equation:

$$
\mathcal{L} = n(\theta, f(\theta)) \sqrt{f(\theta)^2 + \frac{df}{d\theta}^2} \tag{30}
$$

$$
\frac{d}{d\theta} \left(\frac{\partial \mathcal{L}}{\partial \frac{df}{d\theta}} \right) = \frac{\partial \mathcal{L}}{\partial f} \tag{31}
$$

In this case, this becomes (note that since $s(\theta, r) = 20 - 5r^2$, $\frac{\partial s}{\partial \theta} = 0$):

$$
\begin{aligned}
\frac{\partial n}{\partial r}\sqrt{f(\theta)^2+\frac{df}{d\theta}^2} &= \frac{d}{d\theta}\left(\frac{n(\theta,f(\theta))\frac{df}{d\theta}}{\sqrt{f(\theta)^2+\frac{df}{d\theta}^2}}\right)\\
\frac{\partial n}{\partial r}\sqrt{f(\theta)^2+\frac{df}{d\theta}^2} &= \frac{\frac{\partial n}{\partial r}\frac{df}{d\theta}+n(\theta,f(\theta))\frac{\partial^2 f}{\partial\theta^2}}{\sqrt{f(\theta)^2+\frac{df}{d\theta}^2}}-\frac{n(\theta,f(\theta))\frac{df}{d\theta}\left(f(\theta)\frac{df}{d\theta}+\frac{df}{d\theta}\frac{d^2f}{d\theta^2}\right)}{\left(f(\theta)^2+\frac{df}{d\theta}^2\right)^{\frac{3}{2}}}\\
\frac{\partial n}{\partial r}\left(f(\theta)^2+\frac{df}{d\theta}^2\right)^2 &= \left(\frac{\partial n}{\partial r}\frac{df}{d\theta}+n(\theta,f(\theta))\frac{\partial^2 f}{\partial\theta^2}\right)\left(f(\theta)^2+\frac{df}{d\theta}^2\right)\\
&\quad -n(\theta,f(\theta))\frac{df}{d\theta}\left(f(\theta)\frac{df}{d\theta}+\frac{df}{d\theta}\frac{d^2f}{d\theta^2}\right)\\
\frac{\frac{\partial n}{\partial r}}{n}\left(f(\theta)^2+\frac{df}{d\theta}^2\right)^2 &= \frac{\frac{\partial n}{\partial r}}{n}\frac{df}{d\theta}\left(f(\theta)^2+\frac{df}{d\theta}^2\right)-\frac{df}{d\theta}^2 f(\theta)+f(\theta)^2\frac{d^2f}{d\theta^2}\\
f(\theta)^2\frac{d^2f}{d\theta^2} &= \frac{\frac{\partial n}{\partial r}}{n}\left(f(\theta)^2+\frac{df}{d\theta}^2\right)^2-\frac{\frac{\partial n}{\partial r}}{n}\frac{df}{d\theta}\left(f(\theta)^2+\frac{df}{d\theta}^2\right)+\frac{df}{d\theta}^2 f(\theta)\\
\frac{d^2f}{d\theta^2} &= \frac{\frac{\partial n}{\partial r}}{n}\left(1+\left(\frac{\frac{df}{d\theta}}{f(\theta)}\right)^2\right)\left(f(\theta)^2-\frac{df}{d\theta}+\frac{df}{d\theta}^2\right)+\frac{df}{d\theta}^2\frac{1}{f(\theta)}
\end{aligned}
\tag{32}
$$

Since there are two boundary conditions, for each choice of λ there exists a unique f such that $f(x_b)=y_b$ and $f(x_a)=y_a$.