Going from spin-0 to spin-1

1. To make a relativistic quantum field theory, the field operator needs to be lorentz invariant. If we want our theory to have a single field with multiple degrees of freedom, it cannot be a lorentz scalar. The next lorentz-covariant object is a lorentz vector or one-form. If our space is of dimension n, then lorentz-invariant objects can only have n^i degrees of freedom, for some i, because each "index" needs four degrees of freedom. Therefore, if we want objects with a certain number of degrees of freedom, we will have to impose additional constraints.

2. A lagrangian must be a lorentz scalar. How many lorentz scalars can we make with a single vector field A^μ?

   (a) $A_\mu A^\mu$
   (b) $\partial_\mu A^\mu$
   (c) $A_\mu \partial_\nu \partial^\nu A^\mu$
   (d) $A_\mu \partial_\nu \partial^\mu A^\nu$
   (e) Terms with more than two derivatives.

   To see this, we see that any zero-derivative lorentz-invariant quantity must only be a function of $A_\mu A^\mu$. Any one-derivative lorentz-invariant quantity must have the derivative contract with something. Since there's only one field, it has to contract with that.

   Any two derivative quantity can have the two derivatives contracted with the same index or bearing different indices. If they bear different indices, then eventually integration by parts will push one of the fields contracting one of the indices to the left of the derivatives. The same thing will happen with the same-index case. Therefore, the following quantities are the only things that lorentz scalars can be built from. This is restrictive.

3. Lagrangians do not necessarily describe physical theories with positive energy densities. Even lagrangians which containt non-scalars as their fields don't necessarily produce theories which behave as non-scalar theories.

   Consider the following lagrangian:

$$\mathcal{L} = \frac{1}{2} A_\mu \partial_\nu \partial^\nu A^\mu + \frac{1}{2} m^2 A_\mu A^\mu$$

   When varied, its equations of motion are:

$$\left(m^2 + \partial_\nu \partial^\nu\right) A_\mu = 0$$

   A derivative fixes the constraint to be:

$$\left(m^2 + (a-b)\partial_n u \partial^\nu\right)\left(\partial_\mu A^\mu\right) = 0$$

Therefore, if we choose $a = b$, then this constraint says $\partial_\mu A^\mu = 0$ *because* $m \neq 0$. This constraint is lorentz-invariant and removes one of the four degrees of freedom. Therefore our field has three degrees of freedom as desired.

4. We need to determine what the possible polarizations are. For massive scalar field theories, there are three polarizations. From the quantum perspective, they correspond to the spin-0, spin-1, and spin-(-1) states for a spin-1 particle, which we expect.

From Massive to Massless

1. We need to fix a gauge. Removing the mass term adds an extra symmetry to the lagrangian:

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x)$$

   where $\alpha(x)$ is a scalar.

   We need to chose a specific gauge for A_μ, which means imposing an additional constraint. One such constraint is the Coulumb gauge. This constraint removes a degree of freedom from the previously mentioned three, so now there are only two.

2. There are only two polarizations.

   The interpretation of this is that massless vector fields cannot have spin zero, they must have spin -1 or 1.