Math Modeling Assignment 8

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1 Rate Changes

Suppose b_i for $1 \leq i \leq n$ represents the expected number of females birthed in a given year by a single i-year old female, and s_i for $1 \leq i \leq n-1$ represents the expected fraction of the i-year-old population to survive to age $i+1$ the next year [1] Then, if $\vec{p} \in \mathbb{R}^n$ is a vector representing the population one year, then for A defined in (1), the expected population is given by Ap.

$$A = \begin{pmatrix} b_1 & b_2 & \dots & \dots & b_{n-1} & b_n \\ s_1 & 0 & \dots & \dots & 0 & 0 \\ 0 & s_2 & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & s_{n-2} & 0 & 0 \\ 0 & \dots & \dots & 0 & s_{n-1} & 0 \end{pmatrix} \tag{1}$$

We are provided with data for natality and mortality rates from the US census, and construct an A matrix corresponding to this data. The A matrix is constructed in part 4 of the code appendix. The "stationary age distribution" is the eigenvector corresponding to the largest eigenvalue of the matrix. The others will be exponentially suppressed, and the stationary distribution will not depend on initial conditions, as long as there is a single largest eigenvalue.

We rescaled the census data by factors ranging from 0.1 to 1.0, and plotted the resulting stationary age distributions.

1.1 Steady Population

As long as the largest eigenvalue is positive, *at large times*, the population will not be shrinking. [2] We want to figure out by how much of a factor US natality

[1] This assumes that there is an age, n, which is the maximum age and cannot be exceeded.

[2] Initial conditions can always be chosen so that the population is shrinking: for example, if the entire population is placed at a large age where reproduction is unlikely. At first, the age distribution will decrease, but eventually, the few children who the elderly have will reproduce and we will recover the stationary distribution *at large times in the future*

can decrease without the population shrinking. We found that the natality can be multiplied by a factor of 0.488666 to produce a population which is stable. Any value of r above this will produce a population whose size is asymptotically increasing. This example is section 2 in the code appendix. [3]

2 Maximize Working Age

From looking at the graph of age distributions, we observe that a larger proportion of the people are older as r is decreased, and a larger proportion are younger as r is increased. Suppose we want to maximize the number of working-age adults, from ages 15 to 64. Which values of r should we choose?

As shown in Plot 2, to achieve a population which is 60% working age, r must be between 0.533371 and 0.71044. The code for plot 2 is given in section 3 of the code appendix. The reason that there is such a narrow interval is because if r is too small, the population skews too old, while if r is too large, the population skews too young.

3 Code Appendix

Listing 1: Code to plot the asymptotic population distributions for values of r from 0.1 to 1.0

```
function [eigvec, eigval] = eigvec(r)
        A = setup_a(r);

        [v,d]=eig(A);
        [eigval,j] = max(abs(diag(d)));
        eigvec = v(:,j)/sum(v(:,j));
end

hold on
for r = 0.1:0.1:1.0
        [r_eigvec, r_eigval] = eigvec(r);
        plot(r_eigvec, 'color', [r 0 1-r], sprintf(';r=%f, eigval=%f;',r,r_eigva
end
legend()
title('Population Distributions for Scaled Natality Rates');
hold off
print('population-distributions.png')
pause
```

[3]Sascha Hernandez pointed out that the population could initially decrease, for any choice of r, and that it was necessary to specify what it meant for the population to be "non-decreasing"

Listing 2: Code to find a value for r such that the largest eigenvalue of A is 1

```
function eigval = eigval(r)
        A = setup_a(r);

        [v,d]=eig(A);
        [eigval,j] = max(abs(diag(d)));
end

r_steady_population = fzero(@(r) eigval(r) - 1.0, 0.5);
printf('R Value for non-shrinking population: %f\n', r_steady_population)
```

Listing 3: Code to find the values of r for which the working age (15-64) population is greater than 60% of the total population

```
function between = percentage_between_ages(r, min_age, max_age)
        A = setup_a(r);

        [v,d] = eig(A);
        [eigval, j] = max(abs(diag(d)));
        ages = v(:,j);
        between = sum(ages(min_age:max_age))/sum(ages);
end

r_between = fzero(@(r) percentage_between_ages(r, 15, 64) - 0.6, 0.5);
r_between_above = fzero(@(r) percentage_between_ages(r, 15, 64) - 0.6, 0.71);

r_values = 0.0:0.01:1.0;
percentage_working_age_values = zeros(1,length(0.0:0.01:1.0));
for i = 1:length(r_values)
        percentage_working_age(i) = percentage_between_ages(r_values(i),15,64);
end

hold on
plot(r_values, percentage_working_age, ';Percentage Working Age;');
plot(r_values, 0.6*ones(1,length(r_values)), sprintf(';60 percent cutoff. Minimum
scatter([r_between, r_between_above], [percentage_between_ages(r_between, 15, 64

xlabel('r');
ylabel('Working-Age Percentage of the Population');
title('Natality Coefficient Effectos on Working-Age Percentage of the Population
hold off

printf('R must be between %f and %f', r_between, r_between_above)

pause
```

Listing 4: Code to setup the A matrix, with inline US census data

```
function A = setup_a(r)
    %The goal is to find the stable age distribution
    %and the rate of population growth based on
    %the known rates of birth/death for various age groups.


    %birth-rate by age
    % source:   http://www.cdc.gov/nchs/data/statab/t991x07.pdf
    % data taken from the year 1999.
    %br(i) is the chance that a woman of age (i-1) will give birth this year
    br = 1e-3*[
    0
    0
    0
    0
    0
    0
    0
    0
    0
    0
    0.9
    0.9
    0.9
    0.9
    0.9
    28.7
    28.7
    28.7
    80.3
    80.3
    111
    111
    111
    111
    111
    117.8
    117.8
    117.8
    117.8
    117.8
    89.6
    89.6
    89.6
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89.6
89.6
38.3
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38.3
38.3
38.3
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0];

%death-rate by age
% source:  http://www.ssa.gov/oact/STATS/table4c6.html
% data taken for females from the year 2009.
%dr(i) is the chance that a woman of age (i-1) will die this year
```

```
dr = [
0.005728
0.000373
0.000241
0.000186
0.00015
0.000133
0.000121
0.000112
0.000104
0.000098
0.000094
0.000098
0.000114
0.000143
0.000183
0.000229
0.000274
0.000314
0.000347
0.000374
0.000402
0.000431
0.000458
0.000482
0.000504
0.000527
0.000551
0.000575
0.000602
0.00063
0.000662
0.000699
0.000739
0.00078
0.000827
0.000879
0.000943
0.00102
0.001114
0.001224
0.001345
0.001477
0.001624
0.001789
0.001968
```

0.002161
0.002364
0.002578
0.0028
0.003032
0.003289
0.003559
0.003819
0.004059
0.004296
0.004556
0.004862
0.005222
0.005646
0.006136
0.006696
0.007315
0.007976
0.008676
0.009435
0.010298
0.011281
0.01237
0.013572
0.014908
0.01644
0.018162
0.020019
0.022003
0.024173
0.026706
0.029603
0.032718
0.036034
0.039683
0.043899
0.048807
0.054374
0.060661
0.067751
0.075729
0.084673
0.094645
0.105694
0.117853
0.131146

```
        0.145585
        0.161175
        0.17791
        0.195774
        0.213849
        0.231865
        0.249525
        0.266514
        0.282504
        0.299455
        0.317422
        0.336467
        0.356655
        0.378055
        0.400738
        0.424782
        0.450269
        0.477285
        0.505922
        0.536278
        0.568454
        0.602561
        0.638715
        0.677038
        0.71766
        0.76072
        0.806363
        0.851378
        0.893947
        ];
        %now let's form a Leslie matrix
        A = zeros(120, 120);

        %births
        A(1,:) = r*br';
        %ageing & dying
        A = A + diag(1-dr(1:end-1), -1);
        %assume the same rate of dying beyond 119 years
        A(120,120) = 1-dr(120);
end
```