1 $\hbar = c = 1$

2 Schwartz Problem 2.3

The GZK bound. In 1966 Greisen, Zarsepin, and Kuzmin argued that we should not see cosmic rays psigh energy protons) above a certain frequency because of interactions with the cosmic microwave background.

(a) The universe is a blackbody at 2.73 K. What is the mean energy of the photons in outer space?

In a blackbody cavity, the expected energy in each mode ω_n is:

$$\langle E_n \rangle = \frac{\omega_n}{e^{\omega_n \beta} - 1}$$

If we regard the universe as a blackbody cavity with infinte size, the discrete ω_n's become a continuous variable, and the average total energy for a given frequency becomes:

$$E(\omega) = \frac{\omega}{e^{\omega\beta} - 1}$$

Let's consider the case of a black body cube with side lengths L. If we integrate through momentum-space all values of frequency, the average total energy becomes:

$$E_t ot = \frac{L^3}{2\pi^2} \int_0^\infty d\omega \frac{\omega^3}{e^{\omega\beta} - 1}$$

If we divide by volume, we obtain an energy density:

$$d^3 E = \frac{1}{2\pi^2} \int_0^\infty d\omega \frac{\omega^3}{e^{\omega\beta} - 1}$$

3 Non-Relativistic Classical Field Theory

Consider the following non-relativistice Lagrangian density:

$$\mathcal{L} = \psi^* \left(i\partial_t + \nabla^2/2m \right) \psi - \frac{1}{2}(\psi^*\psi)^2$$

where ψ is a complex scalar field.

3.1 a

Consider varying the fields in $\mathcal{L}$ starting from some arbitrary ψ. Show that the resulting change in $\mathcal{L}$ can be written in the form:

$$\delta\mathcal{L} = \delta\psi A + \delta\psi^* B + i(\partial_t \delta\rho + \nabla \cdot \delta J) + \mathcal{O}\left(\delta\psi^2\right)$$

Give expressions for A, B, $\delta\rho$, and δJ in terms of ψ, ψ^*, $\delta\psi$, $\delta\psi^*$.

3.1.1 Transforming the Lagrangian into a more symmetric form

Since the lagrangian density is interpreted as always being under an integral, we are allowed to use integration by parts directly on the lagrangian.

We can expand the laplacian:

$$\mathcal{L} = i\psi^*\partial_t\psi + \psi^*\frac{\partial_x^2 + \partial_y^2 + \partial_z^2}{2m}\psi - \frac{1}{2}(\psi^*\psi)^2$$

And use the integration by parts rule to turn $\psi^*\partial_i^2\psi$ into $-(\partial_i\psi^*)(\partial_i\psi)$.

$$\mathcal{L} = i\psi^*\partial_t\psi - \frac{(\partial_x\psi^*)(\partial_x\psi) + (\partial_y\psi^*)(\partial_y\psi) + (\partial_z\psi^*)(\partial_z\psi)}{2m} - \frac{1}{2}(\psi^*\psi)^2$$

And finally rewrite with gradient notation:

$$\mathcal{L} = i\psi^*\partial_t\psi - (\nabla\psi^*)\cdot(\nabla\psi)/2m - \frac{1}{2}(\psi^*\psi)^2 \tag{1}$$

3.1.2 Varying the new Lagrangian

To the new gradient lagrangian, we can make the substitutions $\psi \to \psi + \delta\psi$ and evaluate.

$$\begin{aligned}
\delta\mathcal{L} &= i(\psi^* + \delta\psi^*)\partial_t(\psi + \delta\psi) - (\nabla(\psi^* + \delta\psi^*))\cdot(\nabla(\psi + \delta\psi))/2m - \frac{1}{2}(((\psi^* + \delta\psi^*)(\psi + \delta\psi))^2 - \mathcal{L} \\
&= i\psi^*\partial_t\delta\psi + i\delta\psi^*\partial_t\psi - \nabla\psi^*\cdot\nabla\delta\psi/2m - \nabla\delta\psi^*\cdot\nabla\psi/2m + \mathcal{O}\left(\delta\psi^2\right) \\
&= -i\partial_t\psi^*\delta\psi + i\delta\psi^*\partial_t\psi + \nabla^2\psi^*\delta\psi/2m + \delta\psi^*\nabla^2\psi/2m + \mathcal{O}\left(\delta\psi^2\right) \\
\delta\mathcal{L} &= \delta\psi\left(-i\partial_t\psi^* + \nabla^2\psi/2m\right) + \delta\psi^*\left(i\partial_t\psi + \nabla^2\psi/2m\right) + \mathcal{O}\left(\delta\psi^2\right)
\end{aligned}$$

Since the lagrangian is underneath an integral, and we make the assumption that boundary terms are zero, we can make the following substitution freely (Schwartz 3.13):

$$\mathcal{L} \to \mathcal{L} + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi\right]$$

We can split this up into spatial and time terms (remember this theory is non-relativistic).

$$\frac{\partial\mathcal{L}}{\partial(\partial_t\psi)} = i\psi^* \tag{2}$$

$$\frac{\partial\mathcal{L}}{\partial(\nabla\psi)} = -\nabla\psi^*/2m \tag{3}$$

Therefore, for this lagrangian the following substitution will be valid:

$$\mathcal{L} \to \mathcal{L} + \partial_t(i\psi^* \delta\psi) + \nabla \cdot (-\nabla\psi^* \delta\psi/2m)$$

We can conclude that the total variation in the lagrangian can now be written:

$$\delta\mathcal{L} = \delta\psi \left(-i\partial_t\psi^* + \nabla^2\psi^*/2m\right) + \delta\psi^* \left(i\partial_t\psi + \nabla^2\psi/2m\right) + i\left(\partial_t\left(\psi^*\right) + \nabla \cdot \left(-\nabla\psi^*/2m\right)\right) + \mathcal{O}\left(\delta\psi^2\right)$$

which is the form that the problem asked us to put the variation into.

If we define:

$$\begin{aligned} A &= -i\partial_t\psi^* + \nabla^2\psi^*/2m \\ B &= -i\partial_t\psi + \nabla\psi^*/2m \\ \rho &= \psi^*\delta\psi \\ J &= -\nabla\psi^*\delta\psi/2m \end{aligned}$$

then the above lagrangian is:

$$\delta\mathcal{L} = \delta\psi A + \delta\psi^* B + i(\partial_t\delta\rho + \nabla \cdot \delta J) + \mathcal{O}\left(\delta\psi^2\right)$$

as desired.

3.2 Classical Equations of Motion

The classical equations of motion for the theory:

$$\mathcal{L} = i\psi^*\partial_t\psi - (\nabla\psi^*) \cdot (\nabla\psi)/2m - \frac{1}{2}(\psi^*\psi)^2$$

are given by the euler lagrange equations:

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right] &= 0 \\ \frac{\partial\mathcal{L}}{\partial\psi^*} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^*)}\right] &= 0 \end{aligned}$$

We can compute the derivatives with respect to the lagrangian:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \psi} &= -(\psi^*)^2\psi \\
\frac{\partial \mathcal{L}}{\partial \psi^*} &= i\partial_t\psi - (\psi)^2\psi^* \\
\frac{\partial \mathcal{L}}{\partial(\partial_t\psi)} &= i\psi^* \\
\frac{\partial \mathcal{L}}{\partial(\partial_t\psi^*)} &= 0 \\
\frac{\partial \mathcal{L}}{\partial(\nabla\psi)} &= -\nabla\psi^*/2m \\
\frac{\partial \mathcal{L}}{\partial(\nabla\psi^*)} &= -\nabla\psi/2m
\end{aligned}
$$

We can substitute these derivatives into the euler-lagrange equations to obtain:

$$
\begin{aligned}
i\partial_t\psi &= -\nabla\psi/2m + G\psi^*\psi^2 \\
-i\partial_t\psi^* &= -\nabla\psi^*/2m + G(\psi^*)^2\psi
\end{aligned}
$$

3.3 Conserved Current

The lagrangian density is unchanged by the transformation:

$$
\psi \to e^{i\epsilon}\psi \qquad\qquad \psi^* \to e^{-i\epsilon}\psi^*
$$

for any real value of ϵ. What is the conserved current corresponding to this symmetry? What is the corresponding conserved quantity?

Since the lagrangian is unchanged under the above symmetry, we conclude that:

$$
\frac{\delta \mathcal{L}}{\delta \epsilon} = 0
$$

We can compute $\frac{\delta \mathcal{L}}{\delta \epsilon}$, to get (Schwartz 3.22):

$$
\begin{aligned}
\frac{\delta \mathcal{L}}{\delta \epsilon} = &\left[\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)}\right]\frac{\delta\psi}{\delta\epsilon} + \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)}\frac{\delta\psi}{\delta\epsilon}\right] \\
&+ \left[\frac{\partial \mathcal{L}}{\partial \psi^*} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi^*)}\right]\frac{\delta\psi^*}{\delta\epsilon} + \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi^*)}\frac{\delta\psi^*}{\delta\epsilon}\right]
\end{aligned}
$$

If the equations of motion are satisfied, and the symmetry is really a symmetry (which it is), the whole expression reduces into the form:

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \frac{\delta \psi}{\delta \epsilon} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} \frac{\delta \psi^*}{\delta \epsilon} \right] = 0$$

Let's compute the variations under the symmetry:

$$\frac{\delta \psi}{\delta \epsilon} = i\psi$$
$$\frac{\delta \psi^*}{\delta \epsilon} = -i\psi^*$$

There is a conserved 4-vector associated with the equations:

$$p = -\psi^* \psi$$
$$\vec{J} = -i\nabla\psi^*\psi/2m + i\nabla\psi\psi^*/2m$$

The conserved quantity is:

$$\int d^x p = \int d^x - \psi^* \psi$$

This has an interesting interpretation that if ψ is treated as a wave function as in quantum mechanics, the normalization is conserved. We should think about the conserved current therefore as a probability current.

3.4 d

Does this have a time-reversal symmetry?

Suppose $\psi(x,t)$ is a solution. Consider $\psi^*(x,-t)$.

$$i\partial_t(\psi^*(x,-t)) = -\nabla\psi^*(x,-t)/2m + G\psi(x,-t)(\psi^*(x,-t))^2$$
$$-i\partial_t\psi^*(x,-t) = -\nabla\psi^*(x,-t)/2m + G\psi(x,-t)(\psi^*(x,-t))^2$$

Which is the other equation of motion which is satisfied. Therefore, if $\psi(x,t)$ is a solution, $\psi^*(x,-t)$ is also a solution.