a) Explain Schwartz's stragtegy for quantizing the ϕ field. Specifically why does the harmonic oscillator come up, and why does it make sense to express non-interactiong quantum fields and their Hamiltonians in terms of harmonic oscillator creation and annhilation operators $a_p^\dagger$ and a_p?

Schwartz first defines the ϕ field classically: according to him, it staisfies the simplest possible lorentz-invariant equation of motion:

$$\partial_\mu \partial^\mu \phi = 0$$

Schwartz then breaks this expression down into $\left(\partial_t^2 - \vec{\nabla}^2\right)\phi = 0$ (using the particle physics sign convention). He then considers solutions of the form:

$$\phi(\vec{x}, t) = a_p(t) e^{i\vec{p}\cdot\vec{x}}$$

and asks what the dynamics of $a_p(t)$ must be.

We can derive these dynamics:

$$\begin{aligned}
\left(\partial_t^2 - \vec{\nabla}^2\right)\phi &= 0 \\
\left(\ddot{a_p}(t) + \vec{p}\cdot\vec{p}\right) e^{i\vec{p}\cdot\vec{x}} &= 0 \\
\ddot{a_p}(t) &= -p^2 a_p(t)
\end{aligned}$$

Note that the differential equation describing the dynamics of $a_p(t)$ is identical to the differential equation describing a harmonic oscillator with angular frequency $|p|$. Next we will try to quantize these hamiltonians.

In order to quantize a harmonic oscillator hamiltonian with frequency ω, we can construct creation and annihilation operators a and $a^\dagger$ which obey the commutation relation $[a, a^\dagger] = 1$. We can then express the hamiltonian as:

$$H = \omega \left(a^\dagger a + \frac{1}{2}\right)$$

Suppose we want to associate a hamiltonian with each of the pure-momentum plane wave solutions to the wave equation. We have already seen that the amplitude coefficients of a plane wave with momentum $\vec{p}$ behave as if they are a harmonic oscillator of angular frequency $|\vec{p}|$. All we have left to do is to define creation and annihilation operators. If we have these creation and annihilation operators already, the dynamics of a plane wave of momentum $\vec{p}$ are given by:

$$H_p = \omega_p \left(a_p^\dagger a_p + \frac{1}{2}\right)$$

To get the full hamiltonian, we can integrate over all possible values of $\vec{p}$.

$$H_0 = \int d^p \omega_p \left(a_p^\dagger a_p + \frac{1}{2}\right)$$

b Why are there two terms in the Fourier transform of $\phi(x)$ in Eq 2.78? Why is there a factor of $1/\sqrt{2\omega_p}$?

c The state $|0>$