1 Why does the Dirac Field need left and right handed components?

Suppose we want to describe a massive particle with spin $\frac{1}{2}$. We need a mass term, which will have no derivatives.

Let's assume that our theory has one field, ψ_L. The handedness is irrelevant. Then, one candidate mass term is $\psi_L^\dagger\psi$. However, this is not lorentz invariant, which I will now show:

Under infinitesimal transformations, the fields ψ_L and ψ_R transform with the following properties:

$$\delta\psi_R = \frac{1}{2}(i\theta_j + \beta_j)\sigma_j\psi_R \tag{1}$$

$$\delta\psi_L = \frac{1}{2}(i\theta_j - \beta_j)\sigma_j\psi_L \tag{2}$$

$$\delta\psi_R^\dagger = \frac{1}{2}(-i\theta_j + \beta_j)\psi_R^\dagger\sigma_j \tag{3}$$

$$\delta\psi_L^\dagger = \frac{1}{2}(-i\theta_j - \beta_j)\psi_L^\dagger\sigma_j \tag{4}$$

So let's try $\psi_L^\dagger\psi_L$.

$$\begin{aligned}\delta(\psi_L^\dagger\psi_L) &= \psi_L^\dagger\frac{1}{2}(i\theta_j - \beta_j)\sigma_j\psi_L + \frac{1}{2}(-i\theta_j - \beta_j)\psi_L^\dagger\sigma_j\psi_L \\ &= -\beta_j\psi_L^\dagger\sigma_j\psi_L \end{aligned}\tag{5}$$

Therefore, $\psi_L^\dagger\psi_L$ is not lorentz-invariant, and it cannot be used as a mass term.

Let's write $\psi_L = \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix}$.

Mass terms are bilinear in the fields so therefore, for A an unknown matrix with complex entries, a mass term might be $\psi_L^\dagger A\psi_L$. Then:

$$\begin{aligned}\delta(\psi_L^\dagger A\psi_L) &= \psi_L^\dagger\frac{1}{2}(i\theta_j - \beta_j)\sigma_j A\psi_L + \frac{1}{2}(-i\theta_j - \beta_j)\psi_L^\dagger A\sigma_j\psi_L \\ &= \frac{i}{2}\theta_j\psi_L^\dagger[\sigma_j, A]\psi_L - \frac{1}{2}\beta_j\psi_L^\dagger\{\sigma_j, A\}\psi_L \end{aligned}\tag{6}$$

Therefore, for $\psi_L^\dagger A\psi_L$ is only generically lorentz-invariant if $[\sigma_j, A] = 0$ and $\{\sigma_j, A\} = 0$, for each j from 1 to 3. Let's see what conditions this puts on A:

$$\begin{aligned}
[\sigma_1, A] &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} c - a & d - b \\ a - c & b - d \end{pmatrix} \\
[\sigma_2, A] &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= i \begin{pmatrix} -c - b & a - d \\ a - c & b + c \end{pmatrix} \\
[\sigma_3, A] &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & b + b \\ -c - c & 0 \end{pmatrix}
\end{aligned}$$

Therefore, since each of these commutators must be zero, $b = c = 0$. Then, the first relation reduces to:

$$[\sigma_1, A] = \begin{pmatrix} -a & d \\ a & -d \end{pmatrix}$$

Therefore, lorentz invariance of $\psi_L^\dagger A \psi_L$ implies that $A = 0$.

The only other scalar quantity (up to a multiple) that we can make out of ψ_L is $\psi_L^T A \psi_L$. This transforms as:

$$\delta \psi_L^T = \frac{1}{2}(i\theta_j - \beta_j)\psi_L^T \sigma_j^T \tag{7}$$

$$\delta \psi_R^T = \frac{1}{2}(i\theta_j + \beta_j)\psi_R^T \sigma_j^T \tag{8}$$

Therefore,

$$\begin{aligned}
\delta(\psi_L^T A \psi_L) &= \frac{1}{2}(i\theta_j - \beta_j)\psi_L^T \sigma_j^T A \psi_L + \psi_L^T A \frac{1}{2}(i\theta_j - \beta_j)\sigma_j \psi_L \\
&= \frac{1}{2}(i\theta_j - \beta_j)\psi_L^T(\sigma_j^T A + A\sigma_j)\psi_L
\end{aligned} \tag{9}$$

If $\psi_L^T A \psi_L$ is lorentz-invariant, then for j ranging from 1 to 3, $\sigma_j^T A + A\sigma_j$ should be zero.

$$
\begin{aligned}
\sigma_1^T A + A\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} c+b & d+a \\ a+d & b+c \end{pmatrix} \\
\sigma_2^T A - A\sigma_2 &= \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= i \begin{pmatrix} c+b & d-a \\ -a+d & -b-c \end{pmatrix} \\
\sigma_3^T A - A\sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 2a & 0 \\ 0 & 2d \end{pmatrix}
\end{aligned}
$$

From the third equation, $a = d = 0$. From the first equation, $c + b = 0$, so therefore $c = -b$. The second equation adds no new constraints. Therefore, if $\psi_L^T A \psi_L$ is lorentz-invariant, then for some arbitrary complex α, A is of the form:

$$
A = \alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{10}
$$

This is the form of σ_2, so therefore, any mass term of ψ_L is proportional to $\psi_L^T \sigma_2 \psi_L$. This is known as the *Majorana Mass*.

This is not a triumph: we need to remember that the spinor is a two-component vector. If we write $\psi_L = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, then $\psi_L^T \sigma_2 \psi_L$ is just $i(\psi_2\psi_1 - \psi_1\psi_2)$. If ψ_1 and ψ_2 are regular numbers, then we cannot use the majorana mass. If, however, they anticommute for some reason, or have some other more complicated commutation relation, then we can still use the majorana mass. Therefore a massive particle which does not couple to the electromagnetic field can freely use only one component.

1.1 Coupling to the Photon Field

Suppose we want to describe a massive spin-$\frac{1}{2}$ particle which couples nontrivially to the electromagnetic field. Since the coupling could only happen through a term such as: $A_\mu \psi_L^\dagger \bar{\sigma}_\mu \psi_L$, or perhaps $A_\mu \psi_L^\dagger \partial_\mu \psi_L$, Then, the field needs to have a nontrivial gauge transformation to cotransform with the field. It doesn't have any additional degrees of freedom, and it is already complex. Therefore, a single Weyl spinor cannot couple to the electromagnetic field.

A single Weyl spinor can describe a massless particle through the kinetic term $i\psi_L^\dagger \bar{\sigma}_\mu \partial_\mu \psi_L$, or a massive particle through the additional mass term $\psi_L^T \sigma_2 \psi_L$.

A single Weyl spinor cannot describe a particle which interacts with the electromagnetic field.

1.2 Two-Component Particle

Now that we've shown that a single Weyl spinor cannot descibe a massive spin-$\frac{1}{2}$ particle which couples to the electromagnetic field, Multiple spinors are thus required. I still need to show that a left-handed spinor and a right-handed spinor are the required ones.

2 Components

ψ_L has two independant components, because it's the $(\frac{1}{2},0)$ representation of $so(1,3)$. Therefore it has $2J+1 = 2\frac{1}{2}+1 = 2$ independent components.

ψ is in $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$. Each one has two independant components, so therefore ψ has four independant components.

Finally, $\bar{\psi}\gamma_5\psi$ has one component, because $\bar{\psi}$ is a row matrix(as a spinor), γ_5 is a square matrix (as a spinor), and ψ is a column matrix(as a spinor). Therefore their product has one component.

3 Spinor Transformations

$$\begin{aligned}\bar{\psi} &\to \psi^\dagger \Lambda_s^\dagger \\ \psi &\to \Lambda_s \psi\end{aligned}$$

Note that in 10.86 of schwartz, it is shown that:

$$(\gamma^0 \Lambda_s \gamma^0)^\dagger = \Lambda_s^{-1} \tag{11}$$

Therefore,

$$\psi^\dagger \gamma^0 \psi \to \psi^\dagger \Lambda_s^\dagger \gamma^0 \psi$$

Since $\gamma^0\gamma^0 = 2g^{00} = 1$, and ${\gamma^0}^\dagger = \gamma^0$,

$$\begin{aligned} &\to \psi^\dagger \gamma^0 \gamma^0 \Lambda_s^\dagger \gamma^0 \Lambda_s \psi \\ &\to \psi^\dagger \gamma^0 (\gamma^0 \Lambda_s \gamma^0)^\dagger \Lambda_s \psi \\ &\to \psi^\dagger \gamma^0 \Lambda_s^{-1} \Lambda_s \psi \\ \psi^\dagger \gamma^0 \psi &\to \psi^\dagger \gamma^0 \psi \end{aligned}$$

Similarly,

$$
\begin{aligned}
\psi^\dagger \gamma^0 \gamma^\mu \psi &\to \psi^\dagger \Lambda_s^\dagger \gamma^0 \gamma^\mu \Lambda_s \psi \\
&\to \psi^\dagger \gamma^0 \Lambda_s^{-1} \gamma^\mu \Lambda_s^{-1} \psi \\
\psi^\dagger \gamma^0 \gamma^\mu \psi &\to \psi^\dagger \gamma^0 (\Lambda_V)^{\mu\nu} \gamma^\nu \psi
\end{aligned}
$$

Therefore $\psi^\dagger \gamma^0 \gamma^\mu \psi$ transforms as a vector.
Similarly:

$$
\begin{aligned}
\psi^\dagger \gamma^0 \sigma^{\mu\nu} \psi &\to \psi^\dagger \Lambda_s^\dagger \gamma^0 \left(\frac{i}{2} [\gamma^\mu, \gamma^\nu] \right) \Lambda_s \psi \\
&\to \psi^\dagger \gamma^0 \Lambda_s^{-1} \left(\frac{i}{2} \gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu \right) \Lambda_s \psi \\
&\to \psi^\dagger \Lambda_s^\dagger \gamma^0 (\Lambda_V)^{\sigma\mu} (\Lambda_V)^{\rho\nu} \left(\frac{i}{2} [\gamma^\mu, \gamma^\nu] \right) \psi \\
&\to \psi^\dagger \Lambda_s^\dagger \gamma^0 (\Lambda_V)^{\sigma\mu} (\Lambda_V)^{\rho\nu} \sigma^{\mu\nu} \psi
\end{aligned}
$$

Therefore it transforms as a lorentz tensor.
Finally:

$$
\begin{aligned}
\psi^\dagger \gamma^0 \gamma^5 \psi &\to \psi^\dagger \Lambda_s^\dagger \gamma^0 \gamma^5 \Lambda_s \psi \\
&\to i \psi^\dagger \gamma^0 \Lambda_s^{-1} \gamma^0 \gamma^1 \gamma^2 \gamma^3 \Lambda_s \psi \\
&\to i \psi^\dagger \gamma^0 \Lambda_s^{-1} \gamma^0 \Lambda_s \Lambda_s^{-1} \gamma^1 \Lambda_s \Lambda_s^{-1} \gamma^2 \Lambda_s \Lambda_s^{-1} \gamma^3 \Lambda_s \psi \\
&\to i \psi^\dagger \gamma^0 \Lambda_V^{0\mu} \gamma^\mu \Lambda_V^{1\nu} \gamma^\nu \Lambda_V^{2\rho} \gamma^\rho \Lambda_V^{3\lambda} \gamma^\lambda \psi \\
&\to i \psi^\dagger \gamma^0 \Lambda_V^{0\mu} \Lambda_V^{1\nu} \Lambda_V^{2\rho} \Lambda_V^{3\lambda} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\lambda \psi
\end{aligned}
$$

4 Eigenstates

Value	C Eigenvalue	P Eigenvalue	T Eigenvalue
$\bar{\psi}\psi$	1	1	1
$i\bar{\psi}\gamma^5\psi$	1	-1	-1
$\bar{\psi}\gamma^\mu\psi$	-1	1	-1
$\bar{\psi}\gamma^\mu\gamma^5\psi$	1	-1	-1
$\bar{\psi}\sigma^{\mu\nu}\psi$	-1	1	-1