1 Brachistochrone Curves

A cycloid is parametrically described by the equations ($0 \leq s \leq 2\pi$):

$$x(s) = \frac{C}{2}(s - \sin s) \tag{1}$$

$$y(s) = \frac{C}{2}(\cos s - 1) \tag{2}$$

The brachistochrone curve relating two points A and B is a cycloid. Let $A = (0,0)$, and $B = (\bar{x}, \bar{y})$. I will show how to find C so that the brachistochrone curve joints A and B.

We must treat the cases $\bar{y} = 0$ and $\bar{y} \neq 0$ seperately.

1.1 The case $\bar{y} = 0$

If $\bar{y} = 0$, then either $C = 0$ or $\cos s = 1$. If C is zero, then $x(s)$ is zero for all s, and thus cannot be equal to $\bar{x}$ for any choice of s. Therefore, $\cos s = 1$. Thus, $s = 0$ or $s = 2\pi$. If $s = 0$, then $x(s) = 0$. If $\bar{x} = 0$, we are done. However, if $\bar{x} \neq 0$, then by process of elimination, if there is a solution, then $s = 2\pi$.

We can now solve for C:

$$\begin{aligned} \bar{x} &= \frac{C}{2}(2\pi - \sin(2\pi)) \\ &= C\pi \\ C &= \frac{\bar{x}}{\pi} \end{aligned} \tag{3}$$

1.2 The case $\bar{y} \neq 0$

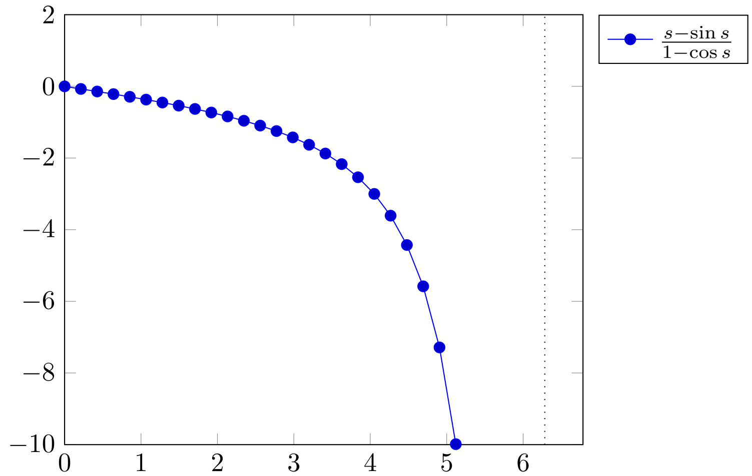
If $\bar{y} \neq 0$, then neither C nor $1 - \cos s$ is zero. Therefore, we can write:

$$\frac{C}{2} = \frac{\bar{y}}{1 - \cos s} \tag{4}$$

If we combine this with the expression for $x(s)$, we can obtain the expression:

$$\frac{\bar{x}}{\bar{y}} = \frac{s - \sin s}{1 - \cos s} \tag{5}$$

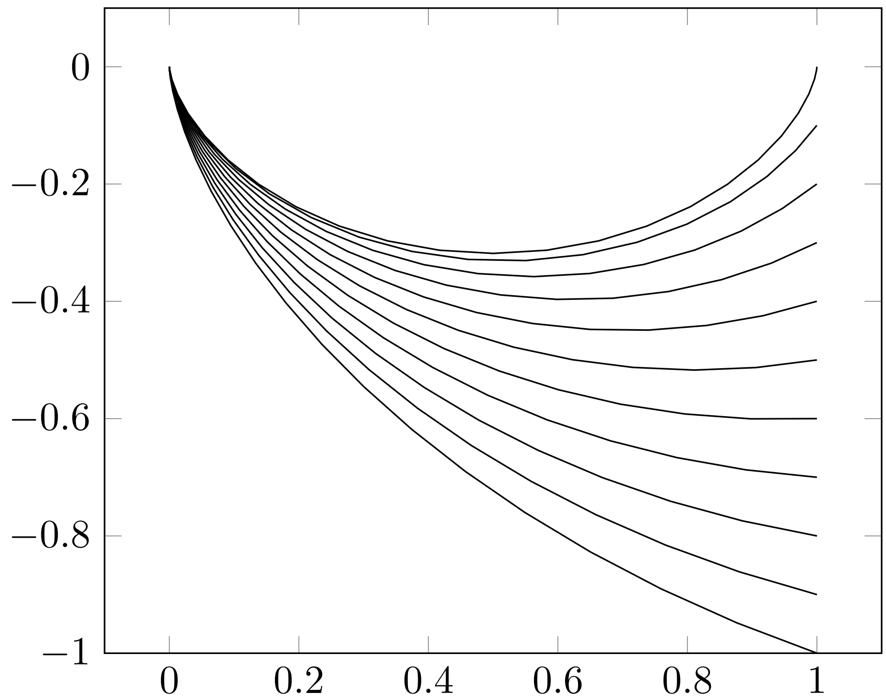
If we can solve this expression for s, then we can substitute it into the expression for $\bar{y}$ to obtain the value of C



We solve for s numerically when $\bar{y} \neq 0$:

$\bar{x}$	$\bar{y}$	s	C
1	0	π	$\frac{2}{\pi-1}$
1	0.1	5.1198	0.3312
1	0.2	4.5946	0.3579
1	0.3	4.1763	0.3971
1	0.4	3.8197	0.4497
1	0.5	3.5084	0.5172
1	0.6	3.2340	0.6013
1	0.7	2.9910	0.7040
1	0.8	2.7753	0.8275
1	0.9	2.5833	0.9740
1	1.0	2.4120	1.1458

We plot the corresponding brachistochrone cycloids.



2 Beachgoers Problem (feat. Sascha Hernandez)

Consider the problem of finding the best location for relaxing on a beach. Some parts of the beach are naturally more attractive than others, because of scenic views, proximity to parking, and lack of seagulls. Additionally, the higher the density of people in a certain area, the less attractive it becomes.

We assume that the "innate attractiveness" of the beach at x is measured by a function $N(x)$, and to find the actual attractiveness, $N(x)$ is scaled by a crowding penalty $C(q(x))$, which depends on $q(x)$, the total number of beachgoers at x. For this example, we will use the following values of N and C:

$$N(x) = e^{-\left(x-\frac{1}{2}\right)^2} \tag{6}$$

$$C(q) = \frac{1}{1+q^2} \tag{7}$$

The attractiveness is thus given by:

$$A(x) = N(x)C(q(x)) \tag{8}$$

2.1 Individual Choice

If each beachgoer chooses their spot individually, and only sees the segments of beach immediately surrounding them, then if the distribution of people along the beach is in equilibrium, the attractiveness must be constant everywhere. Therefore, for a choice of attractiveness A, the density of people on the beach is given by:

$$q(x) = C^{-1}\left(\frac{A}{N(x)}\right) \tag{9}$$

if C^{-1} exists.

2.1.1 Finding C^{-1}

A function is C^{-1} *at* q if $C(C^{-1}(q)) = q$. We can therefore try to find functions which are C^{-1} for positive q.

$$\begin{aligned} q &= C(C^{-1}(q)) \\ &= \frac{1}{1 + (C^{-1}(q))^2} \\ 1 + \left(C^{-1}(q)\right)^2 &= \frac{1}{q} \\ C^{-1}(q) &= \pm\sqrt{\frac{1}{q} - 1} \end{aligned} \tag{10}$$

We choose the positive branch of the square root in this case, because the population density at x must be positive. Note that there is a unique inverse, so we can proceed.

We can now find $q(x)$ for a given value of A:

$$q(x) = \sqrt{\frac{N(x)}{A} - 1} \tag{11}$$

Note that this is real everywhere because $A = N(x)C(q(x))$ everywhere, so $\frac{A}{N(x)} = C(q(x))$, and we have chosen $C(q(x))$ so that it is less than or equal to one eveywhere. Therefore, $\frac{N(x)}{A}$ is greater than or equal to one everywhere.

2.1.2 Finding the Equilibrium Distribution for a Fixed Population

Suppose we want the population size to be fixed, i.e., we require that for a constant P, $\int_0^1 q(x)dx = P$. We will first rephrase our expression for $q(x)$ so that it is in terms of $q(0)$ and not A. Since the value of A is constant, and $A(0) = N(0)C(q(0))$, then

$$A = \frac{N(0)}{1 + q(0)^2} \tag{12}$$

Substituting this into the expression for $q(x)$ yields:

$$q(x) = \sqrt{\frac{N(x)}{N(0)}\left(1 + q(0)^2\right) - 1} \tag{13}$$

2.1.3 Deriving a Differential Equation

In situations when C^{-1} is not well-behaved, it may be worthwhile to find a differential equation for $q(x)$. For individual equilibrium to hold, the derivative of the attractivness must be zero everywhere.

$$\begin{aligned}
0 &= \frac{d}{dx} A(x) \\
0 &= \frac{d}{dx}\left(N(x)C(q(x))\right) \\
0 &= \frac{dN}{dx}(x)C(q(x)) + N(x)C'(q(x))\frac{dq}{dx}
\end{aligned}$$

Assuming that $N(x) \neq 0$ everywhere, and $C'(q(x)) \neq 0$ everywhere, then $q(x)$ must satisfy the following differential equation:

$$\frac{dq}{dx}(x) = -\frac{\frac{dN}{dx}(x)C(q(x))}{N(x)C'(q(x))} \tag{14}$$

2.2 Finding the Distribution for a fixed population

Since we know the equation for $q(x)$, for a given initial population density $q(0)$, we can vary $q(0)$ in order to get a desired population. We show a plot of the total population P as a function of the population at the boundary, $q(0)$. At an initial population of zero, the total popualation is 0.412104. Obviously, the initial population cannot go below zero. Therefore this is the lowest population I could get it to work at.

2.2.1 Extension to Lower Populations

Obviously, our model should predict what happens on the beach when the total population is smaller than 0.412104. We need our population to be nonnegative. Our basic assumption is that people move to regions of greater desirability. If there are not enough people, however, then some regions may not be as desirable as others no matter how empty they are. These regions should be empty. The boundary between these regions and the populated regions will be when $q(x) = 0$, so that there is a continuous transition between the populated and unpopulated regions.

For our case, $q(x)$ is given by:

$$
\begin{aligned}
q(x) &= \sqrt{\frac{N(x)}{A} - 1} \\
N(x) &= e^{-(x-0.5)^2}
\end{aligned}
$$

Therefore, there is a boundary between an inhabited and an uninhabited zone when $q(x) = 0$, or when:

$$
\begin{aligned}
0 &= q(x) \\
&= \sqrt{\frac{N(x)}{A} - 1} \\
N(x) &= A
\end{aligned}
\tag{15}
$$

To determine at what points the population starts and stops, we need to invert $N(x)$.

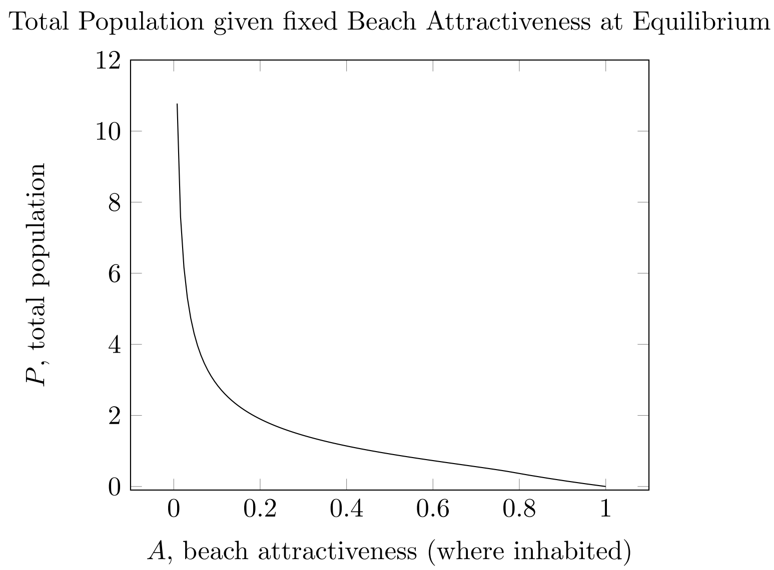
$$
N^{-1}(A) = \frac{1}{2} \pm \sqrt{-\log A} \tag{16}
$$

To determine the total population, we need to integrate between 0 and 1 if they both do not have population density zero, or between the two boundary points if 0 and 1 have population density zero. To determine the value of A at which we must switch strategies, we can compute the value of A at which $N(0) = N(1) = A$. This value of A is $e^{-\frac{1}{4}}$. Therefore, when $A < e^{-\frac{1}{4}}$, we can assume that the population is nonzero all the way up to the boundary. However, when $A > e^{-\frac{1}{4}}$, we must only integrate between $\frac{1}{2} - \sqrt{-\log A}$ and $\frac{1}{2} + \sqrt{-\log A}$.

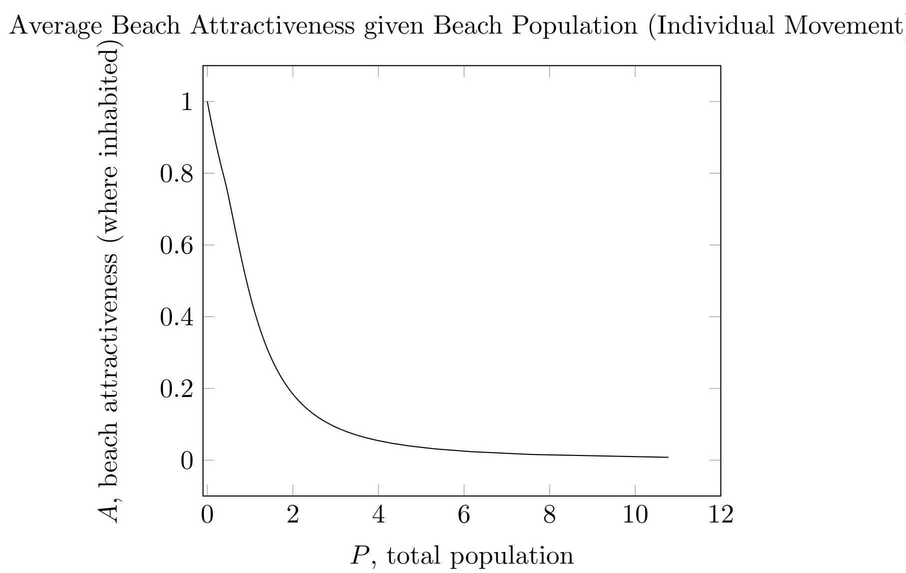
We can summarize this with a function for total population as a function of the attractiveness of the inhabited areas:

$$
P(A) = \begin{cases} \int_0^1 \sqrt{\frac{N(x)}{A} - 1} & A \le e^{-\frac{1}{4}} \\ \int_{\frac{1}{2}-\sqrt{-\log A}}^{\frac{1}{2}+\sqrt{-\log A}} \sqrt{\frac{N(x)}{A} - 1} & A > e^{-\frac{1}{4}} \end{cases} \tag{17}
$$

We can plot $P(A)$, which was found from numerical integration in OCTAVE:



To see how varying the population of the beach affects the overall happiness, we plot $A(P)$, the inverse.



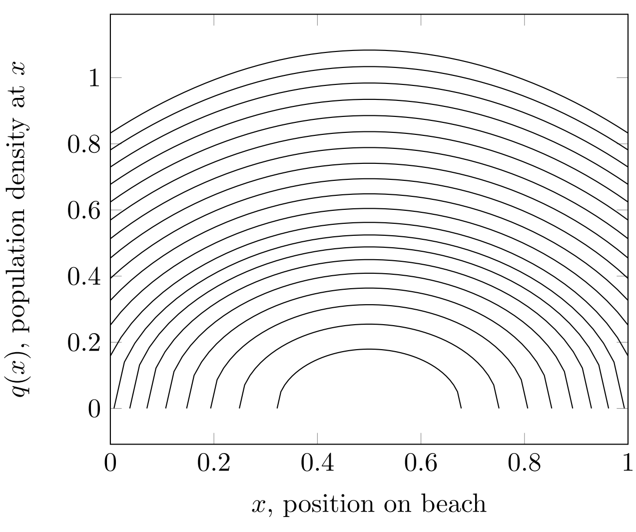
Notice that the beach becomes dramatically less attractive the more crowded it becomes.

We have numerically solved for the enjoyment A associated with some given values of P, which we present below:

Population	Enjoyment (A)
0.05	0.9689
0.10	0.9390
0.15	0.9105
0.20	0.8831
0.25	0.8568
0.30	0.8316
0.35	0.8074
0.40	0.7842
0.45	0.7597
0.50	0.7323
0.55	0.7037
0.60	0.6746
0.65	0.6454
0.70	0.6165
0.75	0.5881
0.80	0.5605
0.85	0.5338
0.90	0.5081
0.95	0.4835
1.00	0.4600

To illustrate some typical population densities, we plot $q(x)$ for each value of A in the above table. The lower curves are for the higher satisfaction A.

Population Densities under Free Movement for different total populations



2.3 Average Satisfaction

It is natural to define the average satisfaction for a beachgoer for each particular distribution:

$$S[q] = \frac{\int_0^1 N(x)C(q(x))q(x)dx}{\int_0^1 q(x)dx} \tag{18}$$

Notice that if $q(x)$ is such that $A(x)$ is constant where q is nonzero, then:

$$\begin{aligned} S[q] &= \frac{\int_0^1 Aq(x)dx}{\int_0^1 q(x)dx} \\ &= A \end{aligned} \tag{19}$$

Therefore, the average happiness is A.

2.4 A Benevolent Dictator maximizes Average Happiness

Suppose a benevolent dictator is attempting to maximize average happiness. Then, if $\bar{A}(x) = N(x)C(q(x))q(x)$ is the density of average happiness everywhere, then $\bar{A}(x)$ must be a constant. Therefore, for some constant B,

$$B = N(x)C(q(x))q(x) \tag{20}$$

From this we can derive a differential equation:

$$0 = \frac{d}{dx}\left[N(x)C(q(x))q(x)\right] \tag{21}$$

This differential equation can be put in standard form:

$$q'(x) = -\left(\frac{N'(x)}{N(x)}\right)\frac{C(q(x))q(x)}{C'(q(x))q(x) + C(q(x))} \tag{22}$$

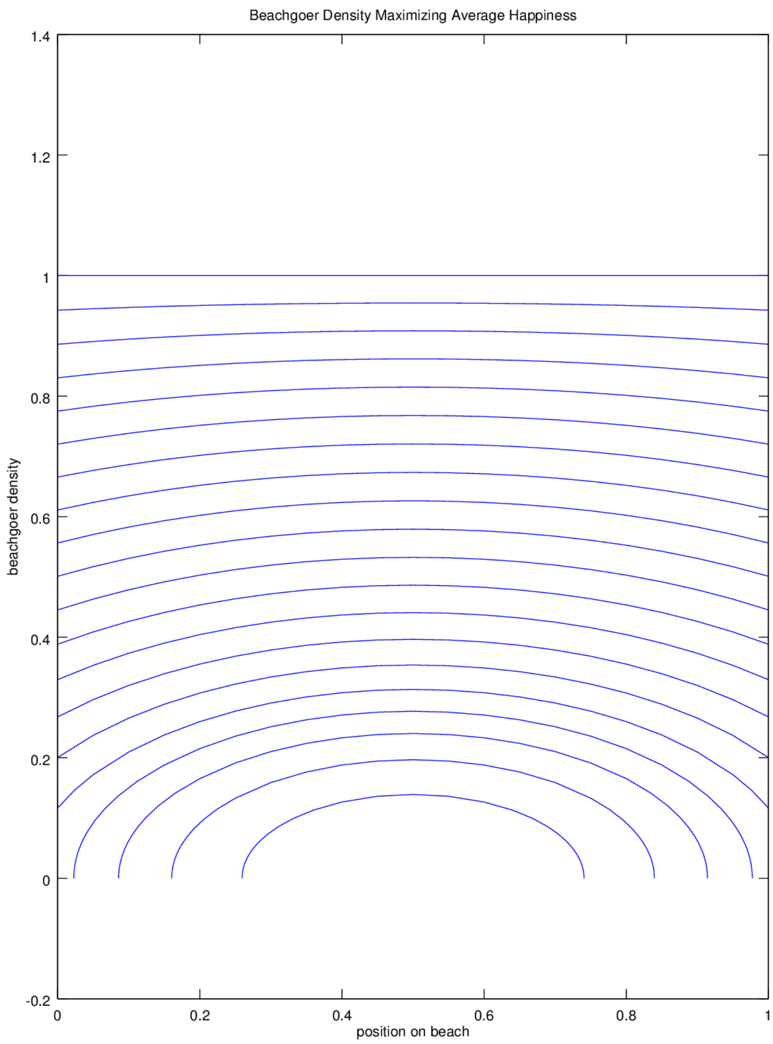
(This isn't yours? I'm not sure how to get the one given)

2.5 Derivation

I skipped this section.

2.6 Solutions

The differential equation was integrated for total populations ranging from 0.05 to 1.0



Population	Dictatorial Satisfaction	Individual Choice Satisfaction
0.05	0.969551949716576	0.9689
0.1	0.9410909123993225	0.9390
0.15	0.9148666447945552	0.9105
0.2	0.8897254121698162	0.8831
0.25	0.8650746522441468	0.8568
0.3	0.8403522597511855	0.8316
0.35	0.8146672860405727	0.8074
0.4	0.7875000139626287	0.7842
0.45	0.7591273355506913	0.7523
0.5	0.7299151727699459	0.7323
0.55	0.7002400142580769	0.7037
0.6000000000000001	0.6704534714129022	0.6746
0.6500000000000001	0.6408657300334135	0.6454
0.7000000000000001	0.6117388969061799	0.6165
0.7500000000000001	0.5832860501461835	0.5881
0.8	0.5556736577186869	0.5605
0.8500000000000001	0.5290258017829101	0.5338
0.9000000000000001	0.5034296789487028	0.5081
0.9500000000000001	0.4789408404669266	0.4835
1	0.4555887321040328	0.4600

Obviously something is wrong, since the dictatorial satisfaction is not always larger than the individual choice satisfaction!

3 Cell Phone Walk

(Reprint from earlier in the semester) Suppose a cell tower generates the following reception profile (23).

$$s(x,y) = \begin{cases} 20 - 5(x^2+y^2) & \text{if } x^2+y^2 < 4 \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

Suppose the caller needs to walk from point a to point b in at least T minutes, and the caller's speed of motion cannot exceed V. What path through the plane has the best average reception?

Let's say that a path is a function γ which satisfies:

$$\gamma : [0,T] \to \mathbb{R}^2$$
$$\gamma(0) = a$$
$$\gamma(T) = b$$

Let's define $x(t)$, $y(t)$ by:

$$\gamma(t) = (x(t), y(t)) \tag{24}$$

3.1 Extremal Cases

Since s has a maximum at $(0, 0)$, if we have enough time to go to the maximum and back, this will have the best average reception. Going to the maximum from $(-2, 0)$ is 2 distance units away, and going from the maximum to $(0, -2)$ is also 2 distance units away. Therefore, if we have the time to cover 4 distance units, we can go straight to the maximum. We can travel VT distance units total if we always move at our maximum speed. Therefore, if $VT \geq 4$, then we will be able to go right to the maximum.

Since we can travel only VT distance units total, if this is smaller than the distance from a to b, then there will be no way to move in the desired way. The distance from $(-2, 0)$ to $(0, 2)$ is $2\sqrt{2}$ distance units, so therefore, if $VT < 2\sqrt{2}$, then there will be no solution.

The interesting case is therefore when $2\sqrt{2} < VT < 4$, when a solution exists, but it is not possible to go directly to the maximum.

3.2 Minimizing a Functional (Cartesian Coordinates)

I first derive the differential equations for a minimized functional in cartesian coordianates to analyze singularities and demonstrate the validity of the method. I will then re-derive them in polar coordiantes, and use those.

Suppose $2\sqrt{2} < VT < 4$. Then, we need to determine the shape of the path γ to maximize the average reception (25):

$$\langle s \rangle = \frac{\int_0^T s(x(t), y(t))dt}{\int_0^T dt} \tag{25}$$

If we have a path, and on some subsection of that path, we are not moving at the maximum speed, we can increase the total average reception by traversing this path at full speed, and then going towards the center and turning around. Therefore, all paths which do not move at the maximum allowed speed are not optimal.

Therefore, we conclude that:

$$\sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2} = V \tag{26}$$

Suppose $y(t) = f(x(t))$. This is a non-trivial assumption. It is assuming that optimal paths go only one way in the x direction. It is true in this case because if a path does go backwards in the x direction, one can make a better path simply by waiting where you are and then skipping ahead. Then,

$$
\begin{aligned}
V &= \sqrt{\frac{dx}{dt}^2 + \frac{df}{dx}^2 \frac{dx}{dt}^2} \\
V &= \frac{dx}{dt}\sqrt{1 + \frac{df}{dx}^2} \\
dt &= \frac{1}{V}\sqrt{1 + \frac{df}{dx}^2}\, dx
\end{aligned} \tag{27}
$$

Therefore we can rewrite the expression for $\langle s \rangle$:

$$
\langle s \rangle = \int_{x_a}^{x_b} \frac{s(x, f(x))}{VT}\sqrt{1 + \frac{df}{dx}^2}\, dx \tag{28}
$$

Now that we're integrating over x, we don't automatically have the constraint on the path length anymore. We want to maximize $\langle s \rangle$, but subject to the constraint that the path length is VT. This constraint can be written as a functional of f:

$$
VT = \int_{x_a}^{x_b} \sqrt{1 + \frac{df}{dx}^2}\, dx \tag{29}
$$

To find the f which extremizes $\langle s \rangle$ with the desired arclength, we use a lagrange multiplier λ, and look for optimal solutions to the following functional.

$$
S[f] = \frac{1}{VT}\int_{x_a}^{x_b} (s(x, f(x)) + \lambda)\sqrt{1 + \frac{df}{dx}^2}\, dx \tag{30}
$$

Let $n(x, f(x)) = s(x, f(x)) + \lambda)$. This functional is of the form of an integral $\int_a^b \mathcal{L}(x, f(x), f'(x))dx$, so therefore every f which extremizes S satisfies the following instance of the euler-lagrange equation:

$$
\begin{aligned}
\mathcal{L} &= n(x, f(x))\sqrt{1 + \frac{df}{dx}^2} \\
\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \frac{df}{dx}}\right) &= \frac{\partial \mathcal{L}}{\partial f}
\end{aligned} \tag{31}
$$

In this case, this becomes:

$$\frac{d}{dx}\left(\frac{n(x,f(x))\frac{df}{dx}}{\sqrt{1+\frac{df}{dx}^2}}\right) = \frac{\partial n}{\partial y}\sqrt{1+\frac{df}{dx}^2}$$

$$\frac{\partial n}{\partial y}\sqrt{1+\frac{df}{dx}^2} = \frac{\frac{\partial n}{\partial y}\frac{df}{dx}+\frac{\partial n}{\partial x}+n(x,f(x))\frac{\partial^2 f}{\partial x^2}}{\sqrt{1+\frac{df}{dx}^2}} - \frac{n(x,f(x))\frac{df}{dx}^2\frac{d^2 f}{dx^2}}{\left(1+\frac{df}{dx}^2\right)^{\frac{3}{2}}}$$

$$\frac{\partial n}{\partial y}\left(1+\frac{df}{dx}^2\right)^2 = \left(\frac{\partial n}{\partial y}\frac{df}{dx}+\frac{\partial n}{\partial x}+n(x,f(x))\frac{d^2 f}{dx^2}\right)\left(1+\frac{df}{dx}^2\right) - n(x,f(x))\frac{df}{dx}^2\frac{d^2 f}{dx^2}$$

$$\frac{d^2 f}{dx^2} = \frac{\frac{\partial n}{\partial y}}{n}\left(1+\frac{df}{dx}^2\right)^2 - \left(\frac{\frac{\partial n}{\partial y}}{n}\frac{df}{dx}+\frac{\frac{\partial n}{\partial x}}{n}\right)\left(1+\frac{df}{dx}^2\right)$$

$$\frac{d^2 f}{dx^2} = \left(1+\frac{df}{dx}^2\right)\left(\frac{\frac{\partial n}{\partial y}}{n}\left(1-\frac{df}{dx}+\frac{df}{dx}^2\right)-\frac{\frac{\partial n}{\partial x}}{n}\right)$$

Since there are two boundary conditions, for each choice of λ there exists a unique f such that $f(x_b) = y_b$ and $f(x_a) = y_a$. Furthermore, we know that $n = s(x, f(x)) - \lambda$ should never be zero within any area where we expect the path to be.

3.3 Minimizing a Functional (Polar Coordinates)

Suppose $2\sqrt{2} < VT < 4$. We can also express $\gamma(t) = (r(t), \theta(t))$ in ploar coordinates. Then, we need to determine the shape of the path γ to maximize the average reception (32):

$$\langle s\rangle = \frac{\int_0^T s(r(t),\theta(t))dt}{\int_0^T dt} \tag{32}$$

Since, as before, we are always moving at the maximum speed, we conclude that:

$$\sqrt{\frac{dr}{dt}^2 + r^2\frac{d\theta}{dt}^2} = V \tag{33}$$

Suppose $r(t) = f(\theta(t))$. Then,

$$
\begin{aligned}
V &= \sqrt{\frac{df}{d\theta}^2 \frac{d\theta}{dt}^2 + f(\theta)^2 \frac{d\theta}{dt}^2} \\
V &= \frac{d\theta}{dt} \sqrt{f(\theta)^2 + \frac{df}{d\theta}^2} \\
dt &= \frac{1}{V} \sqrt{f(\theta)^2 + \frac{df}{d\theta}^2}\, dx
\end{aligned}
\tag{34}
$$

Therefore we can rewrite the expression for $\langle s \rangle$:

$$
\langle s \rangle = \int_{\theta_a}^{\theta_b} \frac{s(\theta, r(\theta))}{VT} \sqrt{f(\theta)^2 + \frac{df}{d\theta}^2}\, dx \tag{35}
$$

Now that we're integrating over θ, we don't automatically have the constraint on the path length anymore. We want to maximize $\langle s \rangle$, but subject to the constraint that the path length is VT. This constraint can be written as a functional of f:

$$
VT = \int_{\theta_a}^{\theta_b} \sqrt{f(\theta)^2 + \frac{df}{d\theta}^2}\, dx \tag{36}
$$

To find the f which extremizes $\langle s \rangle$ with the desired arclength, we use a lagrange multiplier λ, and look for optimal solutions to the following functional.

$$
S[f] = \frac{1}{VT} \int_{\theta_a}^{\theta_b} (s(\theta, f(\theta)) + \lambda) \sqrt{f(\theta)^2 + \frac{df}{d\theta}^2}\, d\theta \tag{37}
$$

Let $n(\theta, f(\theta)) = s(\theta, f(\theta)) + \lambda)$. This functional is of the form of an integral $\int_a^b \mathcal{L}(x, f(x), f'(x))dx$, so therefore every f which extremizes S satisfies the following instance of the euler-lagrange equation:

$$
\mathcal{L} = n(\theta, f(\theta)) \sqrt{f(\theta)^2 + \frac{df}{d\theta}^2} \tag{38}
$$

$$
\frac{d}{d\theta}\left(\frac{\partial \mathcal{L}}{\partial \frac{df}{d\theta}}\right) = \frac{\partial \mathcal{L}}{\partial f} \tag{39}
$$

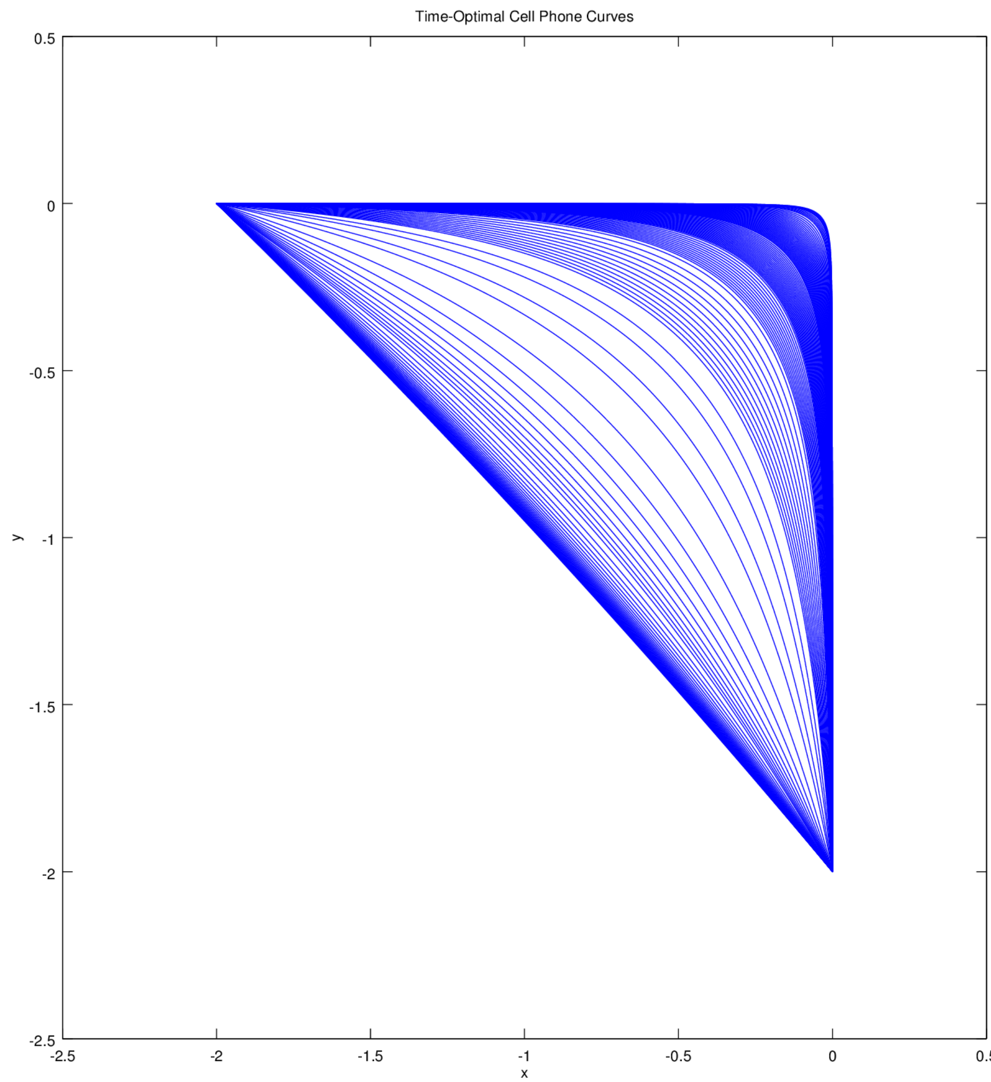
In this case, this becomes (note that since $s(\theta, r) = 20 - 5r^2$, $\frac{\partial s}{\partial \theta} = 0$):

$$
\begin{aligned}
\frac{\partial n}{\partial r}\sqrt{f(\theta)^2+\frac{df}{d\theta}^2} &= \frac{d}{d\theta}\left(\frac{n(\theta,f(\theta))\frac{df}{d\theta}}{\sqrt{f(\theta)^2+\frac{df}{d\theta}^2}}\right)\\
\frac{\partial n}{\partial r}\sqrt{f(\theta)^2+\frac{df}{d\theta}^2} &= \frac{\frac{\partial n}{\partial r}\frac{df}{d\theta}+n(\theta,f(\theta))\frac{\partial^2 f}{\partial\theta^2}}{\sqrt{f(\theta)^2+\frac{df}{d\theta}^2}} - \frac{n(\theta,f(\theta))\frac{df}{d\theta}\left(f(\theta)\frac{df}{d\theta}+\frac{df}{d\theta}\frac{d^2f}{d\theta^2}\right)}{\left(f(\theta)^2+\frac{df}{d\theta}^2\right)^{\frac{3}{2}}}\\
\frac{\partial n}{\partial r}\left(f(\theta)^2+\frac{df}{d\theta}^2\right)^2 &= \left(\frac{\partial n}{\partial r}\frac{df}{d\theta}+n(\theta,f(\theta))\frac{\partial^2 f}{\partial\theta^2}\right)\left(f(\theta)^2+\frac{df}{d\theta}^2\right)\\
&\quad - n(\theta,f(\theta))\frac{df}{d\theta}\left(f(\theta)\frac{df}{d\theta}+\frac{df}{d\theta}\frac{d^2f}{d\theta^2}\right)\\
\frac{\frac{\partial n}{\partial r}}{n}\left(f(\theta)^2+\frac{df}{d\theta}^2\right)^2 &= \frac{\frac{\partial n}{\partial r}}{n}\frac{df}{d\theta}\left(f(\theta)^2+\frac{df}{d\theta}^2\right)-\frac{df}{d\theta}^2 f(\theta)+f(\theta)^2\frac{d^2f}{d\theta^2}\\
f(\theta)^2\frac{d^2f}{d\theta^2} &= \frac{\frac{\partial n}{\partial r}}{n}\left(f(\theta)^2+\frac{df}{d\theta}^2\right)^2-\frac{\frac{\partial n}{\partial r}}{n}\frac{df}{d\theta}\left(f(\theta)^2+\frac{df}{d\theta}^2\right)+\frac{df}{d\theta}^2 f(\theta)\\
\frac{d^2f}{d\theta^2} &= \frac{\frac{\partial n}{\partial r}}{n}\left(1+\left(\frac{\frac{df}{d\theta}}{f(\theta)}\right)^2\right)\left(f(\theta)^2-\frac{df}{d\theta}+\frac{df}{d\theta}^2\right)+\frac{df}{d\theta}^2\frac{1}{f(\theta)}
\end{aligned}
\tag{40}
$$

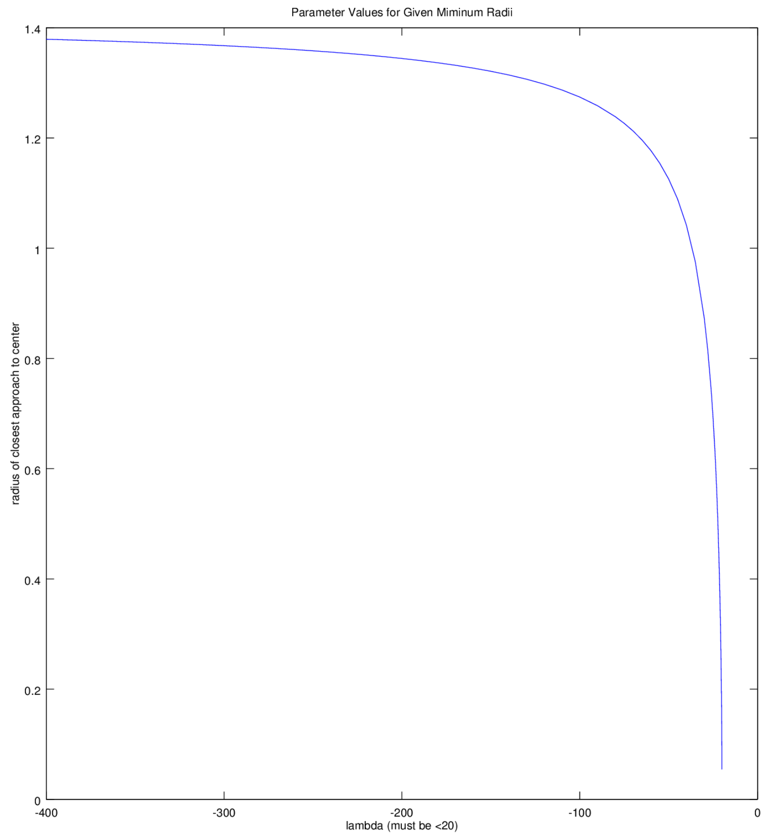
Since there are two boundary conditions, for each choice of λ there exists a unique f such that $f(x_b)=y_b$ and $f(x_a)=y_a$. (End Reprint)

3.4 Numerical Solutions

We solve these equations numerically:



We also plot the inner radius found for the parameter lambda. We found that this parameter was extremely sensitive, and required a very accurate initial guess for the solver to work. We therefore interpolated the values of lambda, which significantly sped up the process of varying lambda so that the desired arclength was correct.



This appears to be some sort of function, which may be fitted. We present it as emperical data.