# 9/9: Homework #2 (due Wed 16 Sept, in class) P7651–2015

**Graders:** The graders for this course are John Stout (jes554@cornell.edu) and Simon Löwe (skl89@cornell.edu).

## 1. Lorentz Invariant Measure (5 points)

**a)** Show that

$$\int \frac{d^4k}{(2\pi)^4}\,\delta(k^2-m^2)\,\theta(k^0)\,f(k) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\; f(k)|_{k^0=\omega_{\mathbf{k}}} \tag{1.1}$$

where $\omega_{\mathbf{k}} \equiv \sqrt{\mathbf{k}^2+m^2}$.

**b)** As discussed in class, $d^4k$ is Lorentz invariant. Assuming this, why is

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \tag{1.2}$$

Lorentz invariant?

## 2. Spontaneous Symmetry Breaking (15 points)

First do parts (a)–(c) of Problem 3.5 (p. 43) in Schwartz. Then add part (d) below.

**d)** Now consider the analogous theory where $\psi(x)$ is complex:

$$\mathcal{L} = \partial\psi^* \cdot \partial\psi + m^2\psi^*\psi - \frac{\lambda}{2}(\psi^*\psi)^2. \tag{2.1}$$

Show that there are infinitely many solutions of the equations of motion for which $\psi = c$ is a (complex) constant. What symmetry is broken spontaneously in this case? Choose $c$ to be real, and re-express $\mathcal{L}$ in terms of fields $\eta(x)$ and $\theta(x)$ where

$$\psi(x) \equiv (c+\eta(x))\,\mathrm{e}^{i\theta(x)}. \tag{2.2}$$

Keep only terms that are linear or quadratic in $\eta$ and $\theta$ and their derivatives (that is, ignore interactions). What are the masses corresponding to the $\eta$ and $\theta$ fields. Why should we not be surprised by the appearance of a massless field?

## 3. Classical Approximation and Measurement (30 points)

Our starting point for quantizing a field was to assume that the field has a classical limit. The canonical quantization procedure is a recipe that gives a quantum theory whose solutions include ones resembling the classical theory. Here we examine what those solutions look like, starting first with a single harmonic oscillator.

Consider a single harmonic oscillator with Hamiltonian $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2q^2$ where $[q,p]=i$, and

$$q = \frac{1}{\sqrt{2\omega}}\left(a+a^\dagger\right) \qquad p = -i\sqrt{\frac{\omega}{2}}\left(a-a^\dagger\right) \tag{3.1}$$

with $[a,a^\dagger]=1$. The $n^{\text{th}}$ excited state of this system is $|n\rangle \equiv (a^\dagger)^n|0\rangle/\sqrt{n!}$, where $|0\rangle$ is the ground state. The commuation relation implies that $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.

A state $|\psi\rangle$ is classical if the uncertainties in $q$ and $p$ are small compared with their mean values:

$$\Delta q \ll q_{\text{cl}} \equiv \langle\psi|q|\psi\rangle \qquad \Delta p \ll p_{\text{cl}} \equiv \langle\psi|p|\psi\rangle. \tag{3.2}$$

That is, we want *approximate* eigenstates where

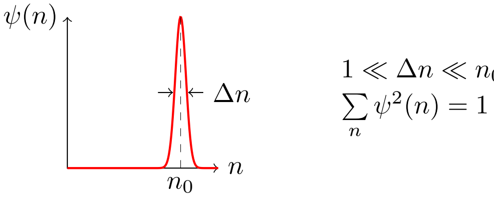
$$q\,|\psi\rangle \approx q_{\text{cl}}\,|\psi\rangle \quad p\,|\psi\rangle \approx p_{\text{cl}}\,|\psi\rangle. \tag{3.3}$$

$q_{\text{cl}}$ and $p_{\text{cl}}$ are the classical coordinate and momentum of the oscillator correponding to $|\psi\rangle$.

**a)** Consider a state

$$|\psi\rangle \equiv \sum_n \psi(n)\mathrm{e}^{i\theta n}|n\rangle \tag{3.4}$$

where real function $\psi(n)$ is sharply peaked around $n=n_0$, and the peak's width $\Delta n$ is much larger than 1 but much smaller than $n_0$:



Show that

$$q\,|\psi\rangle \approx q_{\rm cl}\,|\psi\rangle \qquad p\,|\psi\rangle \approx p_{\rm cl}\,|\psi\rangle \tag{3.5}$$

and find approximate expressions for $q_{\rm cl}$ and $p_{\rm cl}$ in terms of just $n_0$, $\omega$, and $\theta$.

**b)** What goes wrong with your analysis if $\Delta n \approx 1$? Or $\Delta n \approx n_0$?

**c)** Calculate the (approximate) energy of this state from $q_{\rm cl}$ and $p_{\rm cl}$. Does the energy make sense? How do $q_{\rm cl}$ and $p_{\rm cl}$ vary with time? Does this behavior make sense?

**d)** (Optional part: Try it if you have time; otherwise skip to part e) on Field Theory.) Show that the *coherent state* defined by $|z\rangle \equiv Ae^{za^\dagger}|0\rangle$, where $z$ is a complex number and $A$ is a chosen so $\langle z|z\rangle = 1$, is an example of a classical state provided $|z|$ is large. Do this by estimating $n_0$ and $\Delta n$ for this state. What are $q_{\rm cl}$ and $p_{\rm cl}$ for this state?

**e) Field Theory:** We now carry this analysis over to a field theory with $\mathcal{L} \equiv \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2$. To simplify matters we consider the field confined to a cavity of size $L$ with periodic boundary conditions in each direction. This restricts the momentum in each direction to be an integer multiple of $\Delta p = 2\pi/L$. The quantum field then is a sum over the allowed momenta:

$$\phi(\mathbf{x}) = \sum_{\mathbf{p}} \frac{1}{\sqrt{2E_{\mathbf{p}}V}} \left(a_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}\right) \mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}}, \tag{3.6}$$

$$\dot\phi(\mathbf{x}) = -i\sum_{\mathbf{p}} \sqrt{\frac{E_{\mathbf{p}}}{2V}} \left(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right) \mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}}, \tag{3.7}$$

where $V = L^3$ is the volume of the cavity, and the creation and annihilation operators have commutation relations

$$[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = \delta_{\mathbf{p},\mathbf{q}} \quad [a_{\mathbf{p}}, a_{\mathbf{q}}] = [a^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = 0 \tag{3.8}$$

with Kronecker deltas instead of delta functions (so that $[\phi(\mathbf{x}), \dot\phi(\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})$). A state with $n$ particles in a single mode is then $|n\mathbf{p}\rangle = (a^\dagger_{\mathbf{p}})^n|0\rangle/\sqrt{n!}$.

Focusing on a single cavity mode with $\mathbf{p} = \mathbf{p}_0$, an analogue of our classical harmonic oscillator state is

$$|\psi_{\mathbf{p}_0}\rangle \equiv \sum_n \psi(n)\,|n\mathbf{p}_0\rangle \tag{3.9}$$

where, again, $\psi(n)$ is sharply peaked about $n_0 \gg \Delta n \gg 1$. What are the classical fields $\phi_{\rm cl}(\mathbf{x})$ and $\dot\phi_{\rm cl}(\mathbf{x})$ associated with this state? How do they depend upon time? Do they satisfy the classical equations of motion?

**f)** Show that the uncertainty in the field is

$$\begin{aligned}(\Delta\phi(\mathbf{x}))^2 &\equiv \langle\psi_{\mathbf{p}_0}|\phi^2(\mathbf{x})|\psi_{\mathbf{p}_0}\rangle - \langle\psi_{\mathbf{p}_0}|\phi(\mathbf{x})|\psi_{\mathbf{p}_0}\rangle^2 \\ &= \infty.\end{aligned} \tag{3.10}$$

(Hint: try writing $\phi(\mathbf{x})|\psi_{\mathbf{p}_0}\rangle \approx \phi_{\rm cl}(\mathbf{x})|\psi_{\mathbf{p}_0}\rangle + |\delta\psi\rangle$ where $|\delta\psi\rangle$ comes from $\mathbf{p} \neq \mathbf{p}_0$.) The infinity is decidely non-classical. What has gone wrong?

**g)** Our formulation of the classical limit of a field theory is unrealistic in one important respect. Any realistic apparatus or probe of the $\phi$ field would have a finite spatial resultion. Call this $\Delta x \approx \sigma$. An example of an operator whose expectation values might correspond to field measurements by such a probe is (at $x^0 = 0$)

$$\phi^{(\sigma)}(\mathbf{x}) \equiv \int \frac{d^3\mathbf{y}}{(\sigma\sqrt{2\pi})^3}\, \mathrm{e}^{-(\mathbf{x}-\mathbf{y})^2/2\sigma^2}\, \phi(\mathbf{y}), \tag{3.11}$$

where we smear operator $\phi$ over a region of size $\sigma$. Show that the expectation values of $\phi^{(\sigma)}(\mathbf{x})$ and $\dot\phi^{(\sigma)}(\mathbf{x})$ in $|\psi_{\mathbf{p}_0}\rangle$ agree with the results you obtained above for $\phi_{\rm cl}$ and $\dot\phi_{\rm cl}$ provided $\sigma$ is sufficiently small. How small does $\sigma$ need to be?

**h)** Show that the uncertainty $\Delta\phi^{(\sigma)}(x)$ in $\phi^{(\sigma)}(x)$ is finite for state $|\psi_{\mathbf{p}_0}\rangle$. For what range of $\sigma$ is the uncertainty small compared with $\phi_{\rm cl}$? Under what conditions will there be a range of $\sigma$s such that the classical value is (approximately) unaffected by $\sigma$, while at the same time the uncertainty is small compared with the classical value? Ignore the mass compared with $|\mathbf{p}_0|$ when making this estimate. This is the condition under which the system is well described by classical equations.