1 Logistic Model and Data

1.1 Interpreting the Coefficients

One of the simplest models of population growth is the logistic equation:

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}) \tag{1}$$

1.1.1 Significance of K

The constant K in the model represents the "carrying capacity", the equilibrium population. This model (which only makes sense for positive N) assumes that there is a certain constant, K, below which the population will always increase, and above which the population will decrease. The value of K determines at what population N the population is in a stable equilibrium. At that population, $\frac{dN}{dt}$ is zero.

1.1.2 Significance of r

The constant r in the model represents the speed of population movement. The value of r doesn't change any of the qualitative aspects of the model. The time axis of the solution should be interpreted in units of $\frac{1}{r}$.

1.2 Responding to Data

To try to evaluate how good this model is, we should compare its predictions based on earlier data with the values of later data. First, we should solve the differential equation.

1.2.1 Solution and Interpretation of the Differential Equation

The reader can check that for r, K constants, the function below satisfies the differential equation in question:

$$N(t) = K\left(\frac{Ae^{rt}}{1 + Ae^{rt}}\right) \tag{2}$$

The integration constant A is a free parameter which we can use to make the line intersect a chosen point.

Suppose we gather a data point that the population is N_0 at time t_0. Then, what is A?

$$N(t_0) = K\left(\frac{Ae^{rt_0}}{1 + Ae^{rt}}\right)$$

So therefore:

$$\frac{N_0}{K} = \frac{Ae^{rt_0}}{1 + Ae^{rt}}$$

from which we deduce:

$$A = \frac{N_0}{K - N_0} e^{-rt_0}$$

If we plug this into the original model, we obtain:

$$N(t) = K \left(\frac{N_0 e^{r(t-t_0)}}{(K - N_0) + N_0 e^{r(t-t_0)}} \right) \tag{3}$$

Suppose at time t_1, the population

2 Marriage age

2.1 Differential Equation for Marriage

Assume the following factors affect a person's likelihood of marrying in a time interval Δt:

1. The likelihood is proportional to the length of the time interval Δt
2. The likelihood is proportional to the fraction of people in the person's age group who are already married, $m(t)$.

2.1.1 Derivation of Differential Equation

Suppose there are N people. The likelihood that one individual marries is proportional to $\Delta t m(t)$, based on the above assumptions. Since there are $N(1 - m(t))$ unmarried people, in a large enough sample size the expected number of marriages in a time interval Δt is $N(\Delta t)m(t)(1 - m(t))$ Therefore, the fraction of people married in a time interval Δt, which I will denote Δm, is proportional to $(\Delta t)m(t)(1 - m(t))$. Taking the limit of difference quotients, we arrive at the following differential equation:

$$\frac{dm}{dt} = cm(t)\left(1 - m(t)\right) \tag{4}$$

where c is a constant of proportionality.

2.1.2 Solution of Differential Equation

We begin with:

$$\frac{dm}{dt} = cm(1-m)$$

We first split the derivative into two differntials and bring them to opposite sides:

$$\frac{dm}{m(1-m)} = cdt$$

We want to integrate these two differentials. We should express the left hand side in terms of a sum of two fractions.

Suppose, for unknown A and B, the following equation is an identity:

$$\frac{A}{m} + \frac{B}{1-m} = \frac{1}{m(1-m)}$$

Then by multiplying, the following equation should be an identity for all m.

$$A(1-m) + Bm = 1$$

Specifically, for m= 1, we deduce:

$$B = 1$$

And for $m = 0$, we deduce:

$$A = 1$$

Huh. We probably should have guessed. Oh well.

Anyways, the differential equation can now be written:

$$\frac{dm}{m} + \frac{dm}{1-m} = cdt$$

Integrating these seperately, we obtain:

$$log(m) - log(1-m) = cT + C$$

Taking the exponential of both sides, we obtain:

$$\frac{m}{1-m} = \exp cT \exp C$$

For convenience, we define $A = \exp C$ to be a positive number.

This equation implies:

$$m = (1-m)A \exp cT$$

We then bring the m's over to the left side:

$$m(1 + A \exp cT) = A \exp cT$$

Finally, we achieve an expression for $m(t)$:

$$m(t) = \frac{Ae^{cT}}{1 + Ae^{cT}} \tag{5}$$

2.2 Criticism of the Model

This model is extremely simplistic and only attempts to capture peer behavior. Notably, the model completely ignores age. It assumes that everybody has the same peer group, and are equally influenced by every pressure. However, the model does capture the central characteristics of marriage based on peer influence, which are widely observed and documented. The model may have a limited range of applicability: perhaps the model becomes effective among a certain population, for example people between the ages of 25 and 35 with relatively large social circles. Group pressure may influence these people to formalize their relationships. However, the model will not be effective when there are comparatively few marriages. When there are many marriages at once, though, the model predicts that their distribution may look somewhat like the logistic curve.

2.3 Time-varying Proportionality Constant

Suppose the constant of proportionality c is changing in time. The first thing we should do is add an additional assumption:

3. The likelihood of an individual marrying is proportional to a function of time $c(t)$.

2.3.1 Discussion of the Significance of the Time-Varying Constant

The time-varying constant can help the model by speeding up or slowing down the rates of marriage at certain times. For example, if the model is applied to a localized age cohort, the function $c(t)$ could be proportional to salaries wihin the age cohort, or otherwise somehow related to external factors which may cause marriage. It would be an interesting population-modeling result to fit marriage rates within a social cohort to, for example, a logistic model with $c(t)$ proportional to mean standard of living within the cohort. This would be a quantitative way of saying that the two most significant factors which influenced marriage within this age cohort are standard of living and peer pressure.

2.3.2 Solution the Time-Varying Constant in the Differential Equation

We repeat the analysis above, but with one change.

Above, we integrated

$$cdt$$

to obtain

$$cT + C$$

Here, if $c = c(t)$, we integrate

$$c(t)dt$$

to obtain

$$\int_0^t c(t)dt + c(0)$$

If we repeat the rest of the analysis, the overall solution becomes:

$$m(t) = \frac{Ae^{\int_0^t c(t)dt}}{1 + Ae^{\int_0^t c(t)}} \tag{6}$$

2.3.3 Predicting Marriage Fractions in my Age Class

I might observe in popular culture that the standard ages for marriage are between 25 and 40. I propose the following $c(t)$:

$$c(t) = \left\{ \begin{array}{cc} 0 & 0 \leq t \leq 25 \\ 0.04 & 25 < t \leq 40 \\ 0 & 40 < t \end{array} \right\}$$

Some small fraction m_0 of people are married at 25. To find out A from this, we note:

$$m(0) = \frac{A}{1 + A}$$

And from this we deduce:

$$A = \frac{1}{1 - m_0}$$

Suppose 5 percent of people are married at age 25. Then, $A \approx 1.05$ Therefore, by the time $t = 40$,

$$m(40) = \frac{Ae^{0.04*25}}{1 + Ae^{0.04*25}} \approx 0.74$$

This prediction is contingent on choice of c and assumption that 5 percent of people marry by 25.

2.4 Properties of a good $c(t)$

A good $c(t)$ will explain several observations:

1. Nobody marries before the age of 18.
2. Very few people marry before the age of 23.
3. Marriages seem to peak between late twenties and late thirties.
4. There are very few marriages after the forties.

The $c(t)$ should therefore be zero for $t \leq 18$, increase slowly until it reaches its peak in the mid thirties, and then decrease until it is near zero in the late forties.

2.5 Initial Conditions

Since $m(t)$ was zero at age 18, naively the model predicts no marriages. However, this shouldn't be taken seriously. The model is only applicable when there are enough marriages in a population for there to be significant peer pressure to marry. Therefore, we should choose a small value of m and declare that the model doesn't apply below it, that other factors predominantly influence marriage rates. Then, once those other factors predict the marriages to rise significantly, we should start using our logistic model.

2.6 Modeling Differences in Personality

Can we use this model to describe a certain fraction of the population who does not want to marry? We cannot, because as long as the integral of $c(t)$ increases, which it will as long as c is postive, the marriage fraction will increase to 1. Therefore, we should treat m as the fraction of marriagable people who are married.

Can we use this model to describe a range of preferences in what ages people want to marry at? If instead of being sharply peaked, $c(t)$ is nonzero over a long range, this seems to capture this behavior. The marriage fraction will be steadily but slowly increasing as more people reach the different ages at which they want to marry.

2.7 Transformation of the Axis

Some statisticians found that if we make a linear transformation of the age axis and a scale transformation of the proportion married axis,

$$y = e^{-e^{-x}}$$

fits the data very closely.

This fits in with the previous discussion because the fraction married exponentially approaches the measured fraction at infinity. I don't see any motivation for this model, however.

3 Stochastic Modeling of Logistic Model

3.1 Gossip Propagation

I want to make a stochastic model to model gossip propagation. Gossip starts as something that N_0 people know, where N_0 is a positive integer between 0 and K (representing the total population size). We want a rule to stochastically update a variable N, which represents the number of people who know the gossip, in discrete time steps. We make the following assumptions:

1. Nobody ever forgets the knowledge, so N never decreases.
2. People who do not know the gossip must learn it from somebody who does.
3. Everybody has an equal chance of meeting everybody else each round. Nobody meets themself.
4. There are an even number of people, so everybody is in exactly one two-person meeting per round.
5. During a meeting, every person who knows the information reveals it to the other person.

Each time step, each member of the population will meet a single other member of the population. The only meetings which spread gossip are meetings between people who know the gossip and people who do not. If there are K total people and N know, we ask what is the probability of exactly a meetings between knowledgables and ignorants? This probability will be equal to the number of ways that a meetings between N knowledgables and $K-N$ ignorants can occur, divided by the total number of possible pairings.

3.2 Total Number of Pairings in K people

How many ways p_K are there for K people to arrange themselves into pairs? There is no way if K is odd, so let us assume K is even.

If we first divide K people into pairs, order the pairs, and order the individuals in each pair, we have ordered K individuals. There are $K!$ ways of ordering K individuals, and since there are $K/2$ pairs there are $(K/2)!$ ways of ordering the pairs. Since there are 2 orders each pair can be in, and $K/2$ pairs, there are $2^{K/2}$ possible orderings of the individuals within the pair. Therefore:

$$p_K(K/2)!2^{K/2} = K!$$

or equivalently:

$$p_K = \frac{K!}{(K/2)!2^{K/2}} \tag{7}$$

3.3 Total number of ways to form a pairs with one individual one of N individuals and the other individual one of $K-N$ individuals

We need to find out how many ways there are to form a pairs between the N knowledgeables and the $K-N$ ignorants.

Before we begin, let's note that if there are an even number of knowledgables, there is no way that a can be odd because one knowledgable would be unpaired. Similarly, if there is an odd number of knowledgables there can't be an even number of pairs.

Let's select a knowledgables and a ignorants. Since the two are chosen independently, the total number of ways of doing this is:

$$\binom{N}{a}\binom{K-N}{a}$$

We now have to choose which knowledgeables match with which ignoramuses. We can arbitrarily label the a knowledgeables, and then the number of possible matchings will be the number of ways of ordering a elements, which is $a!$. The total number of ways of pairing up N knowledgables with $N-K$ ignoramuses is thus:

$$\binom{N}{a}\binom{K-N}{a}a!$$

But wait, there's more! We still need to pair up the unpicked knowledgeables and ignoramuses with each other. Fortunately we have done this calculation already. Since there are $N-a$ unpaired knowledgables and $K-N-a$ unpaired ignoramuses, and they are paired independently, the total number of ways of pairing them together will be the product of the number of ways of pairing them off individually. The total number of ways of pairing all K individuals, with exactly a connections between N knowledgables and $K-N$ ignoramuses is thus:

$$n_{N,K,a} = \frac{(N-a)!}{2^{(N-a)/2}((N-a)/2)!}\binom{N}{a}a!\binom{K-N}{a}\frac{(K-N-a)!}{2^{\frac{K-N-a}{2}}(\frac{K-N-a}{2})!} \tag{8}$$

3.4 Turning $n_{N,K,a}$ into a normalized probability there there are a connections

To turn (2) into a probability that there are a connections, we need to divide by the total number of configurations. We have already computed this, it is (1). After much algebra, the normalized form becomes:

$$p_{N,K,a} = \frac{(K/2)!}{\binom{K}{N}}\frac{2^a}{\left(\frac{N-a}{2}\right)!\left(\frac{K-N-a}{2}\right)!a!} \tag{9}$$

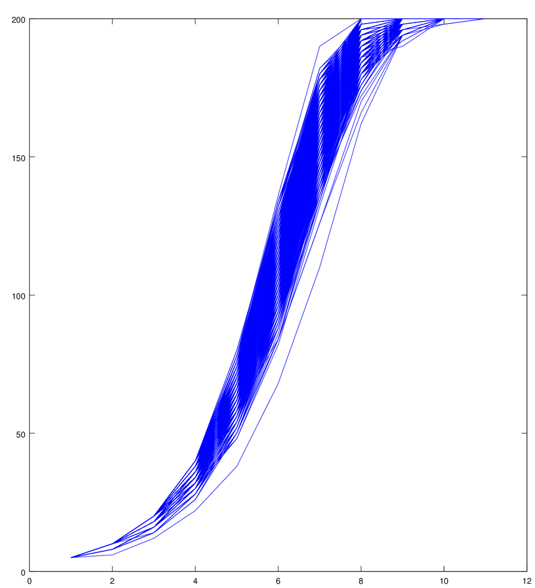
Since only some of this depends on a, we can search for a recurrence relation to simplify computation. After more algebra and separation, I find:

$$p_{N,K,a+2} = \frac{(N-a)(K-N-a)}{(a+1)(a+2)} p_{N,K,a} \tag{10}$$

Note that the numerator means that it is not possible to form more than N or $K-N$ connections.

3.5 Implementation of a Stochastic Model

To implement a stochastic model based on these probabilities, I need to choose an even K and a starting population N_0. Then each round, I create a probability distribution over the different values of a and sample from it. The a ignoramuses who were selected now become knowledgables, so N is incremented by a. The process then repeats.



This is a stochastic model.

3.6 Extension to coefficients c

In order to have more parameters, we could introduce a parameter c which represents the probability that two people who meet decide to share their knowledge. Since we have determined the probability that a people meet, we can sample that distribution and then find out how many meetings there are beween knowledgables and ignoramuses. Then we can sample from a binomial distribution in c ranging from 0 to a to get the expected number of ignoramuses converted into knowledgables.

4 Stochastic Modeling of Population Growth

To model population growth, we can re-use a significant portion of the previous analysis. Suppose we have a population of N members, with N_m males and N_f females. Each round, every member of the population randomly meets another. Then, at the end of the round, natality and mortality happen. For every meeting between creatures of the opposite sex, there is a probability p of the female becoming pregnant. The next round, the female will give birth to a creature which is equally likely to be male or female.

At the same time, there is a death rate d. Each round, every creature might die with probability d. Since the births take place at the end of the round, we should compute deaths before births.

To update the numbers N_m and N_f for each round, we need to compute the following:

1. Begin with N_m males and N_m females.
2. Determine the number of meetings between creatures of the opposite sex. As discussed before, the normalized probability of a meetings is:

   $$M_{N_m,N_f,a} = \frac{\left(\frac{N_m+N_f}{2}\right)!}{\binom{N_m+N_f}{N_m}} \frac{2^a}{\left(\frac{N_m-a}{2}\right)!\left(\frac{N_f-a}{2}\right)!a!}$$

   First, we should obtain the total number of meetings, M, by sampling the distribution.
3. Next, we should sample from a binomial distribution on M letters with probability p, to get the number of pregnancies N_p.
4. Sample from a binomial distribution of N_m elements with probability d, to obtain the D_m, the number of dead males. Similarly obtain D_{nf}, the number of dead females who are not pregnant, and D_{pf}, the number of dead pregnant females.
5. Finally we need to determine the number of births of each sex. The total number of male babies and of female babies is determined by sampling a binomial distribution on $N_p - D_{pf}$ with even probability. This gives us a number of male babies B_m, and a number of female babies B_f
6. Update the populations:

   $$N_m(t+1) = N_m(t) - D_m + B_m$$
   $$N_f(t+1) = N_f(t) - D_{nf} - D_{pf} + B_f$$