Math 3610 (Fall 2015) Homework 3 – Part I

Please solve the following two problems from an old (but excellent) textbook

"An Introduction to Mathematical Modeling" by E.A. Bender (1978)

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8. One of the simplest models of population growth is the logistic equation $dN/dt = rN(1 - N/K)$.
   - (*a*) Interpret r and K. Discuss the model.
   - (*b*) Suppose you were given census data for a population (i.e., a table of date versus population size). How could you test the fit of the logistic model to the data? Remember that r and K are *not* given.
   - (*c*) E. G. Leigh (1971, p. 124) quotes the following data from the U.S. Census Bureau on the growth of the U.S. population and from Gause on the growth of a population of the one-celled animal *Paramecium aurelia*. How well does the logistic model fit the data?

Year	$N \times 10^{-6}$	Day	N
1790	3.93	1	2
1810	7.24	2	7
1830	12.87	3	25
1850	23.19	4	68
1870	39.82	5	168
1890	62.95	6	138
1910	91.97	7	190
1930	122.78	10	122
1950	150.70	11	280
1970	208.0	12	260
		13	300

   - (*d*) Can you suggest better models for the growth of the two populations given above? "Better" is a vague word. It could mean simpler, fitting the data more accurately, having a firmer biological and sociological foundation, and so on.

Leigh, E. G. Jr. (1971) Adaptation and diversity: natural history and the mathematics of evolution.

4. At what age are your friends going to be marrying most rapidly? 15? 20? 25? 30? What factors cause people to marry? Sociologists and psychologists generally believe that peer group behavior plays a major role. Can we model this? The following attempt is adapted from G. Hernes (1972).

(a) It is assumed that a person's chances of marrying in some small time interval Δt are proportional to Δt and to the fraction of people in the person's age group that are already married $m(t)$. This is based on the idea that there is overt and covert peer group pressure to marry. Show that this leads to the differential equation

$$m' = cm(1 - m).$$

Solve the equation.

(*b*) The model may be criticized for a variety of reasons; for example, it assumes that all people feel the same pressure to marry regardless of individual and age as long as the fraction of the peer group that is married is the same. Discuss the model critically.

(*c*) Suppose $c = c(t)$. How can this help the model? What is the solution to the differential equation? In terms of properties of $c(t)$, determine what fraction of people in your age class will eventually marry.

(*d*) Hernes finds that

$$\log [c(t)] = ab^t \log k, \qquad b < 1,$$

gives a rather good fit, but a variety of other forms for $c(t)$ may do just as well. Can you suggest properties a good $c(t)$ is likely to have?

(*e*) We have ignored the problem caused by the fact that, since $m(t)$ was zero when your peer group was younger, the differential equation predicts that it will remain zero. How can we get around this? Remember that we are trying to provide a model that will roughly fit the situation.

(*f*) Discuss how to handle the fact that people are not identical. Can this be incorporated in $c(t)$ somehow? (We could expect the average value of c to decrease with time as those who are more likely to marry do so.)

(*g*) A. J. Coale (1971) found that, by making a linear transformation of the age axis, $x = at - b$, and a scale transformation of the proportion married axis, $y = m/m(\infty)$, a curve was obtained that was closely fitted by

$$y = \exp(-e^{-x})$$

How does this fit in with the previous discussion? (Coale used data from a variety of countries; Hernes used data from a U.S. census.) K. C. Land (1971) discusses a Poisson model for divorce.

Coale, A. J. (1971). Age Patterns of Marriage. *Popul. Stud. (London)* 25: 193-214.

Hernes, G. (1972). The process of entry into first marriage. *Amer. Sociol. Rev.* 37: 173-182.

Land, K. C. (1971). Some exhaustible Poisson process models of divorce by marriage cohort. *J. Math. Sociol.* 1: 213-232.