1 Lagrange Interpolating Polynomial

We are trying to write a polynomial which, if we are considering the points $x_0, \ldots, x_n$, is equal to one at x_i, and equal to zero for all x_j, with $j \neq i$. We exhibit this polynomial below.

$$f_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} \tag{1}$$

Note that $f_i(x_i) = \prod_{j \neq i} \frac{x_i - x_j}{x_i - x_j}$, and since all of the numerators and denominators are equal, we can see that $f_i(x_i) = 1$. For $f_i(x_k) = \prod_{j \neq i} \frac{x_k - x_j}{x_i - x_j}$, if $k \neq i$, since j ranges over all indices except i, one of the j's will be equal to k. That will make the numerator zero, and thus the whole product will be zero. Therefore, $f_i(x_k) = \delta_{ik}$.

If we want our polynomial to have the value y_i at each point x_i, we can sum several of these polynomials, so that the resultant polynomial has the characteristics which we desire:

$$\begin{aligned} f(x) &= \sum_i y_i f_i(x) && (2) \\ &= \sum_i y_i \left(\prod_{j \neq i} \frac{x - x_j}{x_i - x_j} \right) && (3) \end{aligned}$$

This polynomial has the required values at each point.

2 Differentiating a Lagrange Interpolating Polynomial

Given the definition of $f_i(x)$ given above, we can compute the derivative of $f_i(x)$ with respect to x.

$$\frac{df_i(x)}{dx} = \sum_{\substack{k=0 \\ k \neq i}}^{n} \frac{1}{x_i - x_k} \left(\prod_{\substack{j=0 \\ j \neq k \\ j \neq i}}^{j=n} \frac{x - x_j}{x_i - x_j} \right) \tag{4}$$

Therefore, for the whole approximating function $f(x)$, we have:

$$\frac{df}{dx}(x) = \sum_{i=0}^{n} y_i \sum_{\substack{k=0 \\ k \neq i}}^{n} \frac{1}{x_i - x_k} \left(\prod_{\substack{j=0 \\ j \neq k \\ j \neq i}}^{j=n} \frac{x - x_j}{x_i - x_j} \right) \tag{5}$$

This can be interpeted as a dot product. If we consider the vector $\vec{D}(x)$ given by:

$$D_i(x_l) = \sum_{\substack{k=0 \\ k \neq i}}^{n} \frac{1}{x_i - x_k} \left(\prod_{\substack{j=0 \\ j \neq k \\ j \neq i}}^{j=n} \frac{x - x_j}{x_i - x_j} \right) \tag{6}$$

Then if we consider the vector $\vec{y} = \{y_0, \ldots, y_n\}$, then $\frac{df}{dx}(x) = \vec{D}(x) \cdot \vec{y}$.

2.1 Computing the Derivative at the Stencil Points $x_0, \ldots x_n$

Suppose $x = x_l$ is one of the coordinates. Then, $D_i(x_l)$ is given by:

$$D_i(x_l) = \sum_{\substack{k=0 \\ k \neq i}}^{n} \frac{1}{x_i - x_k} \left(\prod_{\substack{j=0 \\ j \neq k \\ j \neq i}}^{j=n} \frac{x_l - x_j}{x_i - x_j} \right) \tag{7}$$

However, this can be greatly simplified. If $x_l = x_i$, then we see that the product terms all drop out, since the numerators and denomiators are all equal. If $x_l \neq x_i$, then the product would only be nonzero for the case where $k = l$. Therefore, only that term in the sum survives. We show the special cases for $D_i(x_l)$ below:

$$D_i(x_i) = \sum_{\substack{k=0 \\ k \neq i}} \frac{1}{x_i - x_k} \tag{8}$$

$$\underset{l \neq i}{D_i(x_l)} = \frac{1}{x_i - x_l} \prod_{\substack{j=0 \\ j \neq l \\ j \neq i}}^{j=n} \frac{x_l - x_j}{x_i - x_j} \tag{9}$$

3 Placeholder Title for Problem 3

4 Placeholder Title for Problem 4

5 Computation of mesh