Scaling limits for random Schrödinger operators with unbounded potentials on weighted graphs

Bachelor Thesis

by

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Statement of Authorship

I hereby declare that I have independently and solely created the present work using only the literature and resources cited.

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Christof Friedrich Peter

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Abstract

We establish analytical estimates for Schrödinger Operators in the Random Conductance Model, operating within an ergodic environment with unbounded weights, subject to some integrability condition. These findings are especially important when examining the scaling limit of the square of the Gaussian Free Field. We therefore extend some of the findings by Andres, Deuschel and Slowik for Laplace operators $\mathcal{L}^\omega$ in [2] to Schrödinger operators $\mathcal{L}^\omega_{\lambda,V} := \mathcal{L}^\omega + \frac{\lambda}{n^2}V^\omega$ with unbounded non-deterministic potential V^ω. We start by employing a discrete version of Moser's iteration technique in Sobolev spaces to prove a maximal inequality for parabolic Schrödinger Poisson equations $\partial_t u - \mathcal{L}^\omega_{\lambda,V} u \leq \frac{1}{n^2} f$. Additionally, we derive bounds for the Laplace killed Green's function $g^\omega_{B(n)}$ with $\mathcal{L}^\omega g^\omega_{B(n)}(\cdot, y) = -\delta_y(\cdot)$ on $B(n)$ and $g^\omega_{B(n)} = 0$ else. Those bounds serve as a foundation for proving the Hölder-continuity of Schrödinger harmonic functions, where u satisfies $\mathcal{L}^\omega u + \frac{\lambda}{n^2}V^\omega u = 0$ on $B(n)$. Finally, we are able to prove Hölder continuity of the Schrödinger killed Green's function $g^\omega_{V,B(n)}$. Along the way, we will observe an interesting relation between the Schrödinger and Laplace killed Green's function, similar to [10], which also guarantees the existence of solutions to the Schrödinger equation.

Abstract

Wir zeigen analytische Abschätzungen für Schrödinger-Operatoren im Random-Conductance-Modell in einer ergodischen Umgebung mit unbeschränkten Gewichten unter einer Integriebarkeitsbedingung. Die Resultate sind unter anderem relevant für den Skalierungslimes des Quadrat des Gaußschen freien Felds. Wir erweitern daher einige der Ergebnisse von Andres, Deuschel und Slowik für Laplace-Operatoren $\mathcal{L}^\omega$ in [2] auf Schrödinger-Operatoren $\mathcal{L}^\omega_{\lambda,V} := \mathcal{L}^\omega + \frac{\lambda}{n^2} V^\omega$ mit unbeschränktem nicht-deterministischem Potential V^ω. Wir beginnen mit der Anwendung einer diskreten Version Moser's Iterationstechnik in Sobolev-Räumen, um eine maximale Ungleichung für parabolische Schrödinger-Poisson-Gleichungen $\partial_t u - \mathcal{L}^\omega_{\lambda,V} u \leq \frac{1}{n^2} f$ zu beweisen. Zusätzlich leiten wir Schranken für die "Laplace killed Green's Function" $g^\omega_{B(n)}$ her, mit $\mathcal{L}^\omega g^\omega_{B(n)}(\cdot, y) = -\delta_y(\cdot)$ auf $B(n)$ und $g^\omega_{B(n)} = 0$ sonst. Diese Schranken dienen als Grundlage für den Beweis der Hölder-Stetigkeit von Schrödinger-harmonischen Funktionen, wobei u die Gleichung $\mathcal{L}^\omega u + \frac{\lambda}{n^2} V^\omega u = 0$ auf $B(n)$ erfüllt. Schließlich können wir die Hölder-Stetigkeit der "Schrödinger killed Green's Function" $g^\omega_{V,B(n)}$. Auf dem Weg werden wir eine interessante Beziehung zwischen der Schrödinger und der "Laplace killed Green's Function" beobachten, ähnlich wie in [10], die auch die Existenz von Lösungen der Schrödinger-Gleichung garantiert.

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Contents

1 Introduction

The Gaussian Free Field emerges as a natural model for random fluctuations in diverse systems, ranging from statistical mechanics and random surfaces to quantum field theory and geometric analysis. To describe the scaling limit of the Gaussian Free Field, one explores its connection to the Random Conductance Model, a class of random walks in random environments characterized by reversible Markov chains. Here, one considers a continuous time random walk $\left(X_t\right)_{t\geq 0}$ on a Graph (V, E) endowed with positive weights $\omega(e)$ on Edges $e \in E$. These conductances dictate the walker's behavior, with higher conductances implying a greater probability of utilizing specific edges. The surprising and fascinating connection arises from interpreting the Gaussian Free Field as a Gaussian Process, with its covariance structure determined by the killed Green's function of the random walk generated by the well known *discrete Laplace operator* acting on bounded functions $f : V \to \mathbb{R}$

$$\left(\mathcal{L}^\omega f\right)(x) \;:=\; \sum_{x\sim y} \omega(x,y)\left(f(y)-f(x)\right).$$

This walk is called the *variable speed random walk* (VSRW), since is waits at position $x \in \mathbb{Z}^d$ for an exponential distributed time before choosing its next position y depending on its location. When exploring the square of the Gaussian Free Field (SGFF), the associated VSRW operates under a slightly altered dynamic. Specifically, we must contend with an additional potential $V^\omega : V \to \mathbb{R}$. Consequently, we introduce *Schrödinger Operators with unbounded random potentials* $\mathcal{L}^\omega_{\lambda,V}$, which are given by

$$\left(\mathcal{L}^\omega_{\lambda,V} f\right)(x) \;:=\; \left(\mathcal{L}^\omega f\right)(x) + \frac{\lambda}{n^2} V^\omega(x) f(x)$$

and thus take on a form similar to the operator in Erwin Schrödinger's renowned equation to characterize the behavior of quantum particles. A key step in exploring the scaling limit of the field is to prove a local central limit theorem for the killed Green's function. To accomplish this, Hölder continuity estimates for the killed Green's function serve as a crucial step in the proof.

Therefore, the primary objective of this mainly analytical thesis is to establish regularity conditions for $\mathcal{L}^\omega_{\lambda,V}$-harmonic functions within balls $B(n)$. Specifically, we seek functions $u > 0$ satisfying $\mathcal{L}^\omega_{\lambda,V} u = 0$ on $B(n)$, and then apply these conditions to the Schrödinger killed Green's function $g^\omega_{\lambda,V}$, which satisfies $\mathcal{L}^\omega_{\lambda,V} g^\omega_{V,B(n)}(\cdot, y) = -\delta_y(\cdot)$ on $B(n)$. While similar results have been derived for the Gaussian Free Field in the Laplace setting by Andres, Deuschel, and Slowik in [2], the inclusion of the potential in the Schrödinger Operator introduces new challenges. Consequently, we adopted a different approach, directly showing Hölder continuity by finding bounds for the Laplace Green's function instead of initially proving a Harnack inequality. Our method drew considerable inspiration from a PDE paper in a less general setting without random unbounded weights [9] by Bramanti.

2 The Random Conductance Model and Main Results

2.1 A short introduction to the Random Conductance Model

Assume that the graph, $G = (V, E)$ with vertex set V and edge set E, is endowed with positive weights, that is, we consider a family $\omega = \{\omega(e) : e \in E\} \in (0,\infty)^E$. To lighten notation, we set $\omega(x,y) = \omega(y,x) := \omega(\{x,y\})$ for all $\{x,y\} \in E$ and $\omega(x,y) := 0$ for all $\{x,y\} \notin E$. Let us further define measures μ^ω and ν^ω on V by

$$\mu^\omega(x) \;:=\; \sum_{x\sim y} \omega(x,y) \qquad \text{and} \qquad \nu^\omega(x) \;:=\; \sum_{x\sim y} \frac{1}{\omega(x,y)}. \tag{2.1}$$

The name derives from interpreting the weighted graph in this setup as a resistor network, where the weights correspond to the conductance or, more precisely, the inverse resistance of a wire connecting nodes [5, chapter 2]. By considering the corresponding random walker on the RCM, which is reversible, one can study analytical electrostatic problems using probabilistic methods and vice versa. What makes the model truly fascinating is its vast range of application fields that appear to be unrelated at first glimpse. Most importantly for us will be the application to random fields like the Gaussian Free Field as discussed by Biskup in [7].

We will define a path of length n between x and y as a sequence $\big(x_1, \ldots, x_n\big)$ with $x_0 = x$, $x_n = y$ and $\{x_i, x_{i+1}\} \in E$, i.e. all points are connected via an edge. Thus, we can define a distance $d(x,y)$ as the minimal length of a path between x and y. Additionally, $B(x,r) := \{y \in V \,|\, d(x,y) \le \lfloor r \rfloor\}$ denotes a closed ball with center x and radius r. We frequently denote by $B(n) := B(x_0, n)$ a ball with radius n with fixed $x_0 \in V$ as a center point. By $|A|$ we mean the counting measure, i.e. the number of elements in a set $A \in V$. Although the Euclidean lattice will be the main focus of interest for the application to the Gaussian Free Field, we are actually able to prove most of our analytical results for a larger class of graphs satisfying the following assumptions:

Assumption 2.1.1. *For some $d \ge 2$, the graph G satisfies the following conditions:*

(i) volume regularity of order d, that is, there exists $C_{\mathrm{reg}} \in (0,\infty)$ such that

$$C_{\mathrm{reg}}^{-1}\, r^d \;\le\; |B(x,r)| \;\le\; C_{\mathrm{reg}}\, r^d \qquad \forall\, x \in V,\; r \ge 1. \tag{2.2}$$

(ii) relative isoperimetric inequality, that is, there exists $C_{\mathrm{riso}} \in (0,\infty)$ such that for all $x \in V$ and $r \ge 1$

$$\frac{|\partial_{B(x,r)} A|}{|A|} \;\ge\; \frac{C_{\mathrm{riso}}}{r} \qquad \forall\, A \subset B(x,r) \;\; \textit{s.th.} \;\; |A| < \tfrac{1}{2}|B(x,r)|. \tag{2.3}$$

Remark 2.1.2. The Euclidean lattice, $(\mathbb{Z}^d, E_d)$, satisfies Assumption 2.1.1.

Remark 2.1.3. The following Sobolev inequality (S_1^d) holds, that is,

$$\left(\frac{1}{|B(x,n)|}\sum\nolimits_{y\in B(x,n)}|u(y)|^{\frac{d}{d-1}}\right)^{\frac{d-1}{d}} \;\leq\; \frac{C_{\mathrm{S}_1} n}{|B(x,n)|}\sum_{\substack{y\vee y'\in B(x,n)\\ \{y,y'\}\in E}}\big|u(y)-u(y')\big| \tag{2.4}$$

for all functions $u\colon V\to\mathbb{R}$ with finite support.

For each non-oriented edge $e \in E$, we specify out of its two endpoints one as its initial vertex e^- and the other one as its terminal vertex e^+. Nothing of what will follow depends on the particular choice. In the context of the square of the Gaussian Free Field, we will consider *discrete Schrödinger Operators with unbounded random potential* $V^\omega : V \to \mathbb{R}$, i.e.

$$\big(\mathcal{L}^\omega_{\lambda,V} f\big)(x) \;:=\; \big(\mathcal{L}^\omega f\big)(x) + \frac{\lambda}{n^2}V^\omega(x)f(x), \tag{2.5}$$

where given a weighted graph (V, E, ω), we define the *discrete Laplacian*, $\mathcal{L}^\omega$, acting on bounded functions $f\colon V\to\mathbb{R}$ by

$$\big(\mathcal{L}^\omega f\big)(x) \;:=\; \sum_{x\sim y}\omega(x,y)\,\big(f(y)-f(x)\big) \;=\; -\nabla^*(\omega\nabla f)(x). \tag{2.6}$$

Notice that the Laplacian gives rise to the variable speed random walk (VSRW) which is defined by the continuous time Markov chain $X = (X_t : t \geq 0)$ on $\mathbb{Z}^d$ with generator $\mathcal{L}^\omega$. This walk waits at position $x \in \mathbb{Z}^d$ an exponential distributed time with mean $\frac{1}{\mu^\omega(x)}$.

As the discrete analog of the continuous setting, define the operators ∇ and ∇^* by $\nabla f\colon E\to\mathbb{R}$ and $\nabla^* F\colon V\to\mathbb{R}$

$$\nabla f(e) \;:=\; f(e^+) - f(e^-), \qquad \text{and} \qquad \nabla^* F(x) \;:=\; \sum_{e:e^+=x}F(e) - \sum_{e:e^-=x}F(e)$$

for $f : V \to \mathbb{R}$ and $F : E \to \mathbb{R}$. Mind that ∇^* is the adjoint of ∇, that is, for all $f \in \ell^2(V)$ and $F \in \ell^2(E)$, it holds $\langle\nabla f, F\rangle_{\ell^2(E)} = \langle f, \nabla^* F\rangle_{\ell^2(V)}$. Notice that thereby we also get the self-adjointness $\mathcal{L}^\omega = \mathcal{L}^{\omega *}$ of the generator:

$$\begin{aligned}\big\langle f, \mathcal{L}^\omega g\big\rangle_{\ell^2(V)} &= \big\langle f, -\nabla^*(\omega\nabla g)\big\rangle_{\ell^2(V)} = \big\langle \nabla f, -\omega\nabla g\big\rangle_{\ell^2(E)}\\ &= \big\langle -\omega\nabla f, \nabla g\big\rangle_{\ell^2(E)} = \big\langle -\nabla^*(\omega\nabla f), g\big\rangle_{\ell^2(V)} = \big\langle \mathcal{L}^\omega f, g\big\rangle_{\ell^2(V)}.\end{aligned} \tag{2.7}$$

We will also need the discrete analog of the product rule which can be written as

$$\nabla(fg) \;=\; \mathrm{av}(f)\nabla g + \mathrm{av}(g)\nabla f, \tag{2.8}$$

where the average of a function on an edge $e \in E$ is defined to be

$$\mathrm{av}(f)(e) \;:=\; \frac{1}{2}\big(f(e^+)+f(e^-)\big).$$

Further, we get an integration by parts formula for the Laplace Operator by using again the adjointness of ∇ and ∇^*:

$$\big\langle f, -\mathcal{L}^\omega g\big\rangle_{\ell^2(V)} \;=\; \big\langle f, \nabla^*(\omega\nabla g)\big\rangle_{\ell^2(V)} \;=\; \big\langle \nabla f, \omega\nabla g\big\rangle_{\ell^2(E)}. \tag{2.9}$$

In contrast to the continuum setting, a discrete version of the chain rule cannot be established. However, we will use the following three substitutes to tackle the arising problems. First, by [1, Lemma A.1.] we get

$$|a^\alpha - b^\alpha| \;\leq\; \Big(1 \vee |\frac{\alpha}{\beta}|\Big)\, |a^\alpha - b^\beta|\, (a^{\alpha-\beta} + b^{\alpha-\beta})$$

for $a, b \in \mathbb{R}_{\geq 0}$ and $\alpha, \beta \neq 0$, which written with discrete gradients and choosing $a = |f(e_+)|$, $b = |f(e_-)|$ and $\beta = 1$ reads as:

$$|\nabla |f|^\alpha| \;\leq\; 2(1 \vee \alpha)\; \mathrm{av}(|f|^{\alpha-1}) \nabla(|f|). \tag{2.10}$$

Further, by [3, Lemma B.1-(i)] we have for $\alpha > 1/2$

$$(a^\alpha - b^\alpha)^2 \;\leq\; \left|\frac{\alpha^2}{2\alpha - 1}\right| (a-b)(a^{2\alpha-1} - b^{2\alpha-1}),$$

which yields

$$(\nabla f^\alpha)^2 \;\leq\; \left|\frac{\alpha^2}{2\alpha - 1}\right| \nabla f^{2\alpha-1}\, \nabla f. \tag{2.11}$$

Finally, by means of the following inequality from [3, Lemma B.1-(ii)]

$$|a^{2\alpha-1} b - a b^{2\alpha-1}| \;\leq\; \Big(1 - \frac{1}{\alpha}\Big) |a^{2\alpha} - b^{2\alpha}|$$

for $\alpha \geq 1$, we receive

$$|f(e_+)^{2\alpha-1} f(e_-) - f(e_+) f(e_-)^{2\alpha-1}| \;\leq\; \Big(1 - \frac{1}{\alpha}\Big) |\nabla f^{2\alpha}|. \tag{2.12}$$

The proof of the Sobolev inequality 4.1.1 relies on (2.10), while (2.11) and (2.12) are heavily used in the proof of Lemma 4.2.3.

In our setup, the *Dirichlet form* or *energy* associated to $\mathcal{L}^\omega$ will play a vital role and is defined by

$$\mathcal{E}^\omega(f,g) \;:=\; \big\langle f, -\mathcal{L}^\omega g \big\rangle_{\ell^2(V)} \;=\; \big\langle \nabla f, \omega \nabla g \big\rangle_{\ell^2(E)}, \qquad \mathcal{E}^\omega(f) \;\equiv\; \mathcal{E}^\omega(f,f). \tag{2.13}$$

Finally, for any nonempty, finite $A \subset V$ and $p \in [1, \infty)$, we introduce space-averaged ℓ^p-norms on functions $f \colon A \to \mathbb{R}$ by the usual formula

$$\|f\|_{p,A} \;:=\; \bigg(\frac{1}{|A|} \sum_{x \in A} |f(x)|^p\bigg)^{1/p} \quad \text{and} \quad \|f\|_{p,A,\mu^\omega} \;:=\; \bigg(\frac{1}{|A|} \sum_{x \in A} \mu^\omega(x)\, |f(x)|^p\bigg)^{1/p}.$$

In the application to the scaling limits we will approximate $\mathbb{R}^d$ by the discrete lattice $\mathbb{Z}^d$. Therefore, for any $x \in \mathbb{R}^d$ we set $[x]$ to be the closest point in $\mathbb{Z}^d$ with respect to the maximum norm. For $D \subset \mathbb{R}^d$ we define a discretization of D by $D(n) := nD \cap \mathbb{Z}^d$. Further, for any bounded $D \subset \mathbb{R}^d$ and $\delta > 0$ we define

$$D^\delta \;:=\; \{x \in D : \mathrm{dist}(x, \partial D) \geq \delta\},$$

where $\mathrm{dist}(x, A) := \inf_{y \in A} \|x - y\|_2$ is defined as the usual distance between a point and a set.

2.2 The killed Green's function

Like many other objects and methods in the analytical part of this thesis, the Green's function traces its origins back to PDE theory and can be viewed as a fundamental solution of a boundary value problem. Additionally, the killed Green's function satisfies the properties of a covariance function. In the upcoming chapter, we will delve into how this association forms the basis for the relationship between the GFF and the RCM. We will start things off by defining the killed Green's function using the first exit time of a stochastic process, followed by establishing its link to boundary value problems.

For a stochastic process $\zeta = \{\zeta_t : t \geq 0\}$ on a set V and a subset $D \subset V$ we define the first exit time of ζ of D by

$$\tau_D(\zeta) \;:=\; \inf\{t \geq 0 : \zeta_t \notin D\}.$$

Let $D \subset V$ be finite, and X be the VSRW. We define the killed Green's function by

$$g_D^\omega(x,y) \;:=\; \mathbb{E}_x^\omega\bigg[\int_0^{\tau_D} \mathbb{1}_{X_t=y}\,\mathrm{d}t\bigg] \;=\; \int_0^\infty \mathbb{P}_x^\omega[X_t = y; t < \tau_B]\,\mathrm{d}t \tag{2.14}$$

for all $x, y \in V$. Thus, the killed Green's function describes the expected amount of time X spent in y when starting in x before exiting D. Although the definition seems different in the probabilistic case, it is actually the same as in PDE theory, where it is defined as the fundamental solution to a Poisson problem:

Proposition 2.2.1. *Let $D \subset V$ be finite and $y \in D$. Then, the killed Green's function solves the following boundary value problem*

$$\begin{cases} \mathcal{L}^\omega g_D^\omega(\cdot,y) = -\delta_y(\cdot) & \text{on } D, \\ g_D^\omega(\cdot,y) = 0 & \text{on } D^c, \end{cases} \tag{2.15}$$

where δ_y denotes the Kronecker delta at y.

Proof. The second condition follows, since for $x \in D^c$ $\tau_D(X) = 0$ $\mathbb{P}_y^\omega$-a.s. and hence, $g_D^\omega(x,y) = 0$ for all $x \in D^c$. Now, let $x \in D$ and define by $T_1 = T_1(X) := \inf\{t > 0 : X_t \neq X_0\}$ the first jump-time of X. Then, we have

$$\begin{aligned}
(\mathcal{L}^\omega g_D^\omega(\cdot,y))(x) \;&=\; \sum_{z\sim x} \omega(x,z)\big(g_D^\omega(z,y) - g_D^\omega(x,y)\big) \\
&=\; \sum_{z\sim x} \omega(x,z)\mathbb{E}_z^\omega\bigg[\int_0^{\tau_D} \mathbb{1}_{X_t=y}\,\mathrm{d}t\bigg] \;-\; \mu^\omega(x)\,\mathbb{E}_x^\omega\bigg[\int_0^{\tau_D} \mathbb{1}_{X_t=y}\,\mathrm{d}t\bigg] \\
&=\; \mu^\omega(x)\mathbb{E}_x^\omega\bigg[\mathbb{E}_{X_{T_1}}^\omega\bigg[\int_0^{\tau_D} \mathbb{1}_{X_t=y}\,\mathrm{d}t\bigg]\bigg] \\
&\quad-\; \mu^\omega(x)\,\mathbb{E}_x^\omega\bigg[\int_0^{T_1} \mathbb{1}_{X_t=y}\,\mathrm{d}t\bigg] \;-\; \mu^\omega(x)\,\mathbb{E}_x^\omega\bigg[\int_{T_1}^{\tau_D} \mathbb{1}_{X_t=y}\,\mathrm{d}t\bigg] \\
&=\; -\mu^\omega(x)\,\mathbb{E}_x^\omega\bigg[\int_0^{T_1} \mathbb{1}_{X_t=y}\,\mathrm{d}t\bigg] \\
&=\; -\mu^\omega(x)\,\mathbb{E}_x^\omega[T_1]\cdot\delta_y(x) \\
&=\; -\delta_y(x),
\end{aligned}$$

where we used the Tower and strong Markov property in the fourth inequality in the following way:

$$\begin{aligned}\mathbb{E}_x^\omega\Big[\int_{T_1}^{\tau_D} \mathbb{1}_{X_t=y}\,\mathrm{d}t\Big] &= \mathbb{E}_x^\omega\Big[\mathbb{E}_x^\omega\Big[\int_{T_1}^{\tau_D} \mathbb{1}_{X_t=y}\,\mathrm{d}t|\mathcal{F}_{T_1}\Big]\Big] \\ &= \mathbb{E}_x^\omega\Big[\mathbb{E}_{X_{T_1}}^\omega\Big[\int_{0}^{\tau_D} \mathbb{1}_{X_t=y}\,\mathrm{d}t|\mathcal{F}_{T_1}\Big]\Big].\end{aligned}$$

□

As a next step, we want to prove the symmetry and positive semi-definiteness of the killed Green's function. This will come in handy in the proof of the Hölder continuity, where we will use the killed Green's function (2.14) and switch the roles of x and y, as in Bramanti's paper [9].

Proposition 2.2.2. *Let $D \subset V$ be finite. Then, we have*

(i) (Symmetric) For any $x, y \in V$

$$g_D^\omega(x,y) = g_D^\omega(y,x). \tag{2.16}$$

(ii) (Positive semi-definite) For any $f : V \to \mathbb{R}$ with finite support

$$\sum_{x,y\in V} g_D^\omega(x,y)f(x)f(y) \geq 0. \tag{2.17}$$

Proof. (i) For any x or $y \in D^c$ both sides are zero so the equality holds. Suppose $x, y \in D$, then by (2.15) and (2.7) we obtain

$$\begin{aligned}g_D^\omega(x,y) &= \big\langle g_D^\omega(\cdot,y),\delta_x\big\rangle_{\ell^2(V)} = -\big\langle g_D^\omega(\cdot,y),\mathcal{L}^\omega g_D^\omega(\cdot,x)\big\rangle_{\ell^2(V)} \\ &= -\big\langle \mathcal{L}^\omega g_D^\omega(\cdot,y), g_D^\omega(\cdot,x)\big\rangle_{\ell^2(V)} = \big\langle \delta_y, g_D^\omega(\cdot,x)\big\rangle_{\ell^2(V)} = g_D^\omega(y,x).\end{aligned}$$

Thus, the killed Green's function is symmetric.

(ii) First, we define $F : V \to \mathbb{R}$, $F(x) = \sum_{y\in V} g_D^\omega(x,y)f(y)$. Then $\operatorname{supp} F \subset D$ and we have

$$\mathcal{L}^\omega F(x) = \sum_{y\in V}(\mathcal{L}^\omega g_D^\omega(\cdot,y))(x)f(y) = -\sum_{y\in V}\mathbb{1}_{x=y}f(y) = -f(x).$$

Therefore, we get

$$\begin{aligned}\sum_{x,y\in V} g_D^\omega(x,y)f(x)f(y) &= \sum_{x\in V} F(x)f(x) = -\sum_{x\in V}F(x)\mathcal{L}^\omega F(x) \\ &= -\big\langle F,\mathcal{L}^\omega F\big\rangle_{\ell^2(V)} = \big\langle \nabla F,\omega\nabla F\big\rangle_{\ell^2(E)} \geq 0,\end{aligned}$$

where we used integration by parts (2.9) in the last step.

□

Consequently, we can interpret the killed Green's function as a covariance function. Analogous to the correspondence between the killed Green's function for the Laplace operator and the GFF, we observe a similar connection between the Schrödinger killed Green's function and the squared GFF. It is defined as the fundamental solution to the Schrödinger equation with boundary condition:

$$\begin{cases} \mathcal{L}^{\omega}_{\lambda,V} g^{\omega}_{V,D}(\cdot,y) = -\delta_y(\cdot) & \text{on } D, \\ g^{\omega}_{V,D}(\cdot,y) = 0 & \text{on } D^c. \end{cases} \tag{2.18}$$

Of course, the Schrödinger version of the killed Green's function also has a probabilistic interpretation with regard to the VSRW generated by the Schrödinger operator. However, as our focus primarily lies on exploring its properties from an analytical standpoint, we will also use the latter definition, which is similar to the definitions found in the extensively referenced PDE literature.

2.3 The Method and Main results

As a main result of this thesis, we establish Hölder continuity estimates for $\mathcal{L}^{\omega}_{\lambda,V}$-harmonic functions and the Schrödinger killed Green's function $g^{\omega}_{V,B(n)}$ on balls $B(n)$. We therefore extend results proven by Andres, Deuschel and Slowik for Laplace operators $\mathcal{L}^{\omega}$ in [2] to the Schrödinger case. There, the authors first establish an elliptic Harnack inequality by proving similar maximal inequalities as in Chapter 4 of this thesis and then apply the lemma by Bombieri and Giusti [8] to derive the Harnack inequality. A crucial step involves finding a maximal inequality for the inverse u^{-1} of a $\mathcal{L}$-harmonic function u. While this is straightforward in the Laplace case using Jensen's inequality as can be seen in [2, Corollary 3.3.], it only works for negative potentials V in the Schrödinger case. We thus had to explore alternative approaches. Initially, we attempted to apply a result by John and Nirenberg [16], but it wasn't applicable to our context due to the requirement of uniformly bounded weights. Fortunately, we found inspiration in a result by Bramanti [9] in the continuous setting, which required us to derive bounds for the Laplace killed Green's function (4.41) and then rewrite our boundary value problem by splitting the solution into two parts (4.45).

Let $(\Omega, \mathcal{F}) = (\mathbb{R}_+^E, \mathcal{B}(\mathbb{R}_+)^{\otimes E})$ be a measurable space, where the graph (V, E) is endowed with a configuration $\omega = \{\omega(e) \; : \; e \in E\} \in \Omega$ of conductances. Denote by $\mathbb{P}$ a probability measure on $(\Omega, \mathcal{F})$, and write $\mathbb{E}$ to denote the expectaion with respect to $\mathbb{P}$. For our results to hold on large scales, we have to impose the following ergodicity and integrability assumptions on the Random Conductance Model:

Assumption 2.3.1. *For some $p, q \in [1, \infty]$ and $d \geq 2$ with*

$$\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$$

assume that the following integrability condition holds

$$\mathbb{E}[\omega(e)^p] < \infty \qquad \text{and} \qquad \mathbb{E}[\omega(e)^{-q}] < \infty$$

Assumption 2.3.2. *Assume that $\mathbb{P}$ satisfies the following conditions:*

(i) $\mathbb{P}\big[0 < \omega(e) < \infty\big] = 1$ *and* $\mathbb{E}\big[\omega(e)\big] < \infty$ *for all* $e \in E_d$.

(ii) $\mathbb{P}$ *is ergodic with respect to translations of* $\mathbb{Z}^d$*, that is,* $\mathbb{P} \circ \tau_x^{-1} = \mathbb{P}$ *for all* $x \in \mathbb{Z}^d$ *and* $\mathbb{P}[A] \in \{0, 1\}$ *for any* $A \in \mathcal{F}$ *such that* $\tau_x(A) = A$ *for all* $x \in \mathbb{Z}^d$*,*

where we define a space shift by $z \in \mathbb{Z}^d$ *through the map* $\tau : \Omega \to \Omega$ *as* $(\tau_z\omega)_{x,y} := \omega_{x+z,y+z}$ *on edges* $\{x, y\} \in E_d$.

The p, q integrability condition in Assumption 2.3.1 repeatedly pops up in the analytical part, were we establish different maximal inequalities. One can show that the latter Assumptions imply the following Assumption. This formulation is taken from [2] and further discussed there:

Assumption 2.3.3. *Assume that*

$$\bar{\mu} \;=\; \sup_{x_0 \in V} \limsup_{n \to \infty} \big\|\mu^\omega\big\|_{p,B(x_0,n)} \;<\; \infty, \qquad \bar{\nu} \;=\; \sup_{x_0 \in V} \limsup_{n \to \infty} \big\|\nu^\omega\big\|_{q,B(x_0,n)} \;<\; \infty,$$

$$\bar{V} \;=\; \sup_{x_0 \in V} \limsup_{n \to \infty} \big\|V^\omega\big\|_{p,B(x_0,n)} \;<\; \infty.$$

In particular, Assumption 2.3.3 implies for the constants appearing in Hölder-Continuity Theorems 2.3.5 and 2.3.6 and in the Harnack Inequality Theorem 5.1.1 that $\tilde{C}_H^* := \tilde{C}_H(\bar{\mu}, \bar{\nu}) < \infty$, $C_H^* := C_H(\bar{V}, \bar{\mu}, \bar{\nu}) < \infty$ and $C_{EH}^* := C_{EH}(\bar{\mu}, \bar{\nu}) < \infty$ do not depend on x_0 and n. Furthermore, there exists $s^\omega(x_0) \geq 1$ such that

$$\sup_{n \geq s^\omega(x_0)} \tilde{C}_H\big(\big\|\mu^\omega\big\|_{p,B(x_0,n)}, \big\|\nu^\omega\big\|_{q,B(x_0,n)}\big) \;\leq\; \tilde{C}_H^* \;<\; \infty,$$

$$\sup_{n \geq s^\omega(x_0)} C_H\big(\big\|V^\omega\big\|_{p,B(x_0,n)}, \big\|\mu^\omega\big\|_{p,B(x_0,n)}, \big\|\nu^\omega\big\|_{q,B(x_0,n)}\big) \;\leq\; C_H^* \;<\; \infty$$

$$\sup_{n \geq s^\omega(x_0)} C_{EH}\big(\big\|\mu^\omega\big\|_{p,B(x_0,n)}, \big\|\nu^\omega\big\|_{q,B(x_0,n)}\big) \;\leq\; C_{EH}^* \;<\; \infty.$$

In other words, under Assumption 2.3.3 the results in Theorems 2.3.5, 2.3.6 and 5.1.1 are becoming effective for balls with radius n large enough.

Remark 2.3.4. The Euclidean lattice, $(\mathbb{Z}^d, E_d)$, satisfies Assumption 2.3.3, if the weights E_d are chosen such that Assumptions 2.3.1 and 2.3.2 are satisfied.

We now are state our main theorems:

Theorem 2.3.5 (Hölder Continuity Schrödinger)**.** *Let* $\delta \in (0, 1)$ *and suppose that* $u > 0$ *is a* $\mathcal{L}^\omega_{\lambda,V}$*-harmonic function, i.e.* $\mathcal{L}^\omega_{\lambda,V} u = 0$ *on* $B(x_0, n)$ *for some* $n \in \mathbb{N}$*. Then, for any* $x \in B(x_0, \lfloor \delta n \rfloor)$*,*

$$\big|u(x) - u(x_0)\big| \;\leq\; C_H^\omega \cdot \delta^\alpha \max_{B(x_0,n)} u, \tag{2.19}$$

where $\alpha = \frac{\gamma}{2-\gamma}\frac{1}{p_*} \vee \theta \in (0, 1)$ *and* $C_H^\omega := C_H(\|V^\omega\|_{p,B(x_0,n)}, \|\mu^\omega\|_{p,B(x_0,n)}, \|\nu^\omega\|_{q,B(x_0,n)})$.

Note again that the following theorem only becomes effective on large scales. This is encoded by imposing Assumption 2.3.3 in second part of the Theorem.

Theorem 2.3.6 (Hölder Continuity Schrödinger killed Green's function)**.** *Let $B_{z,r} := \{y \in \mathbb{R}^d : \|z-y\|_2 < r\}$ be a continuous ball around some $z \in \mathbb{R}$ with radius $r > 0$. Let $\delta > \varepsilon > 0$ and $x \in \mathbb{R}^d$, $x_0 \in B^{\delta}_{z,r}$ with $\|x - x_0\|_2 > \varepsilon$ be fixed. For any $n \in \mathbb{N}$ and $\eta < \varepsilon$*

$$|g^{\omega}_{V,B_{z,r}(n)}(y,[nx]) - g^{\omega}_{V,B_{z,r}(n)}(z,[nx])| \;\leq\; 2C^{\omega}_{H}\eta^{\alpha} \max_{B([nx_0],\eta n)} g_{V,D(n)}(\cdot,[nx]), \tag{2.20}$$

with α defined as above in Theorem 2.3.5.

Further, given Assumption 2.3.3 holds and λ explicitly chosen as

$$\lambda := \left(\frac{1}{4C^{*}_{EH}C_3}\right)^{\frac{2-\gamma}{2\kappa+\gamma}} \bar{V}\bar{\nu} < \infty, \tag{2.21}$$

we get for $\mathbb{P}$- a.e. ω

$$\limsup_{n\to\infty} n^{d-2} \sup_{z,y\in B([nx_0],\eta n)} |g^{\omega}_{V,B_{z,r}(n)}(z,[nx]) - g^{\omega}_{V,B_{z,r}(n)}(y,[nx])| \;\leq\; C_6\eta^{\alpha}. \tag{2.22}$$

Note that $B_{z,r}(n) := nB_{z,r} \cap \mathbb{Z}^d$ is the discretization of the continuous ball $B_{z,r}$.

3 The square of the Gaussian Free Field - Why Schrödinger Operators?

In the following, we will heuristically motivate the connection between the square of the Gaussian Free Field and the Schrödinger Operator. First, we will define the Gaussian Free Field and briefly discuss some of its properties. A more complete introduction to the GFF can be found in Chapter 8 of Friedli and Velenik's book about Statistical Mechanics [14] and in Sheffield's survey about the Gaussian Free Field for mathematicans [18].

3.1 The Gaussian Free Field

Definition 3.1.1. Let $\Lambda \subset \mathbb{Z}^d$ be finite. Then, the discrete weighted Gaussian Free Field (DGGF) in Λ is a process $\phi^\Lambda = \{\phi_x : x \in \mathbb{Z}^d\}$ indexed by the vertices in $\mathbb{Z}^d$ with the law given by

$$\mathbb{P}[\phi^\Lambda \in A] := \frac{1}{Z_\Lambda} \int_A e^{-\frac{1}{2}\sum_{\{x,y\}\in E^d(\Lambda)} \omega(x,y)(\phi(x)-\phi(y))^2} \prod_{x\in\Lambda} \mathrm{d}\phi_x \prod_{x\notin\Lambda} \delta_0(\mathrm{d}\phi_x) \tag{3.1}$$

for all measurable $A \in \mathbb{R}^{\mathbb{Z}^d}$. Here δ_0 is the Dirac point-mass at 0, and Z_Λ is a partition constant. By choosing $\omega(x,y) = \frac{1}{2}$ one gets to the standard Definition of the DGGF which can be found in the literature.

The partition constant is more explicitly given by

$$Z_\Lambda = \int_{\mathbb{R}^{\mathbb{Z}^d}} \exp(-\mathcal{H}_\Lambda(\phi)) \prod_{x\in\Lambda} \mathrm{d}\phi_x \prod_{x\notin\Lambda} \delta_0(\mathrm{d}\phi_x) < \infty,$$

where the Hamiltonian $\mathcal{H}_\Lambda(\phi)$ is defined as in (3.3). Since Λ is finite and the integrator can be estimated by a Gaussian density, Z_Λ is also finite and thus, the law of the DGGF is well defined.

Additionally, we notice that the law given in (3.1) is in the form of a Gibbs measure, i.e.

$$\mu_\Lambda(\mathrm{d}\phi) := \frac{1}{Z_\Lambda} e^{-\mathcal{H}_\Lambda(\phi)} \nu(\mathrm{d}\phi) \tag{3.2}$$

with Hamiltonian defined as

$$\mathcal{H}_\Lambda(\phi) = \sum_{e\in E^d(\Lambda)} \omega(e)(\nabla\phi)^2(e) = \sum_{\{x,y\}\in E^d(\Lambda)} \omega(x,y)(\phi(x)-\phi(y))^2 \tag{3.3}$$

and a priori measure

$$\nu(\mathrm{d}\phi) = \prod_{x\in\Lambda} \mathrm{d}\phi_x \prod_{x\notin\Lambda} \delta_0(\mathrm{d}\phi_x).$$

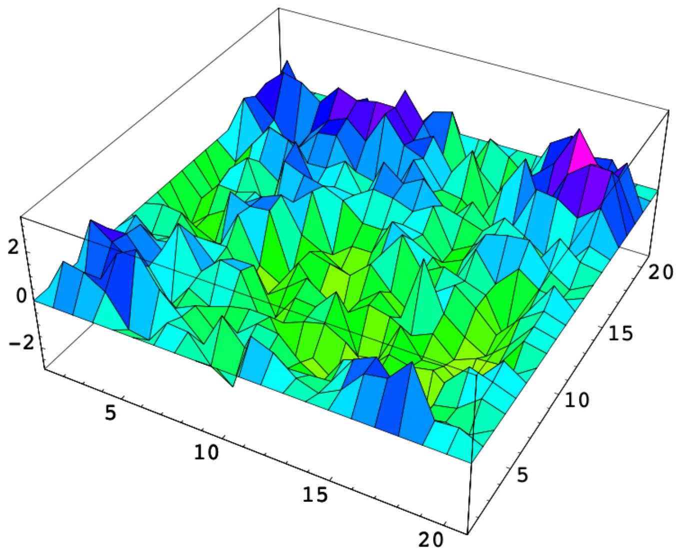


Figure 3.1: Discrete Gaussian free field on 20 by 20 grid with zero boundary conditions taken from [18]

The GFF is frequently used as a mathematical framework for random surfaces, as can be visually motivated by Figure 3.1, which shows a simulation of the field. We want to round up our very short introduction of the DGFF by showing the promised connection between the killed Green's function and the field.

Proposition 3.1.2. *Let $\Lambda \subset \mathbb{Z}^d$ be finite and ϕ^Λ be as in Definition 3.1.1. Then, ϕ is a multivariate Gaussian with*

$$\mathbb{E}[\phi_x^\Lambda] = 0 \qquad \text{and} \qquad \mathbb{E}[\phi_x^\Lambda \phi_y^\Lambda] = g_\Lambda^\omega(x,y) \tag{3.4}$$

for all $x, y \in \mathbb{Z}^d$, where g_Λ^ω defined as in (2.14). In short terms, we can write

$$\phi^\Lambda \sim \mathcal{N}(0, g_\Lambda^\omega).$$

Proof. First, we rewrite the Hamiltonian (3.3) in the exponent as

$$-\sum_{\{x,y\}\in E^d(\Lambda)} \omega(x,y)(\phi(x)-\phi(y))^2 = -\langle \nabla\phi, \omega\nabla\phi\rangle_{\ell^2(E_d)} = -\langle \phi, -\mathcal{L}^\omega\phi\rangle_{\ell^2(\mathbb{Z}^d)},$$

where we were able to use the integration by parts formula (2.9), since $\phi_x = 0$ for all $x \notin \Lambda$. This shows that, restricted to Λ, the ϕ_x are multivariate Gaussian with 0 mean and covariance given by the negative of $\mathcal{L}^\omega$ restricted to Λ. By (2.15), we obtain for $x, y \in \Lambda$

$$(-\mathcal{L}^\omega)^{-1}(x,y) := \langle \delta_x, (-\mathcal{L}^\omega)^{-1}\delta_y\rangle_{\ell^2(\mathbb{Z}^d)} = \langle \delta_x, g_\Lambda^\omega(\cdot,y)\rangle_{\ell^2(\mathbb{Z}^d)} = g_\Lambda^\omega(x,y).$$

□

3.2 The square of the Gaussian Free Field

In the following, we will motivate the occurrence of Schrödinger Operators in the context of the Squared Gaussian Free Field (SGGF) in a rather heuristic way. The derivation relies heavily on the so called Helffer-Sjöstrand representation. For more mathematically sophisticated discussions, we will frequently refer to Chapter 2.3 of lecture notes by Bauerschmidt [6] and to the last chapter in Part 3 of Deuschel's book [11].

The goal is to proof similar scaling limit as in [7, Corollary 6.4.]. Therefore, we first define the log moment-generating function of the SGGF for some $V : \mathbb{Z}^d \to \mathbb{R}$ as

$$K(s) := \log\left(\mathbb{E}[e^{s\left\langle V,\phi^2\right\rangle_{\ell^2(\mathbb{Z}^d)}}]\right). \tag{3.5}$$

For the scaling limit, one would consider the so called Wick-normalization $\phi_n(z) := n^{1/2}\phi_{\lfloor nz \rfloor}$, for $z \in \mathbb{R}^d$. By following similar arguments as in Deuschel and Rodriguez [13] one then arrives at a discretization of V defined as

$$V_n(x) := \frac{1}{n^2}\int_{\frac{x}{n}+[0,1)^d} V(\frac{z}{n})\mathrm{d}z. \tag{3.6}$$

This explains the $\frac{1}{n^2}$ scaling factor in our Definition of $\mathcal{L}^{\omega}_{\lambda,V}$, which also arises naturally from the Sobolev inequality in our analytical arguments in Chapter 4. Going forward, we will ignore this and make our calculations based on the definition of $K(s)$ in (3.5).

The next step is then to consider the scaling behaviour around zero. Therefore, one uses a second order Taylor approximation of the log moment-generating function. In the calculations of the corresponding first and second derivative, a new Gibbs-measure (3.9) will emerge which will deliver our path to the Schrödinger connection. Beforehand, we first prove a short Lemma about squared Gaussian covariances:

Lemma 3.2.1. *Let X, Y be multivariate centered Gaussian. Then, we have*

$$\mathbb{C}\mathrm{ov}(X^2, Y^2) = 2 \cdot \mathbb{C}\mathrm{ov}(X, Y)^2.$$

Proof. As a consequence of a famous result by Isserlis [15], we have

$$\mathbb{E}[X^2Y^2] = \mathbb{E}[X^2]\mathbb{E}[Y^2] + 2\mathbb{E}[XY]^2.$$

By the definition of the covariance, we thus get

$$\mathbb{C}\mathrm{ov}(X^2, Y^2) = \mathbb{E}[X^2Y^2] - \mathbb{E}[X^2]\mathbb{E}[Y^2] = 2 \cdot \mathbb{E}[XY]^2 = 2 \cdot \mathbb{C}\mathrm{ov}(X, Y)^2$$

$\square$

In order to show the convergence of the log-moment generating function K as defined in 3.5, we employ a second order Taylor approximation around 0 of the form:

$$K(s) = \underbrace{K(0)}_{=0} + \underbrace{K'(0)s}_{!=0} + \left(\int_0^1 (1-\theta)K''(\theta s)\,\mathrm{d}\theta\right)s^2. \tag{3.7}$$

To ensure that the second term vanishes, one has to center the field, which yields the so called Wick-renormalization

$$:\phi_n^2: := \phi_n^2 - \mathbb{E}[\phi_n^2], \tag{3.8}$$

which can also be found in [13]. In the following calculation one thus would have to use $:\phi_n:$ instead of ϕ to get $K'(0) = 0$. The relation to the convergence of the covariance as discussed below then follows by dominated convergence and the Taylor argument.

For the Taylor approximation, we now need to calculate the first two derivatives of K, where we use the above lemma as a key ingredient in the last equation:

$$
\begin{aligned}
K(0) &= \log(1) = 0 \\
K'(s) &= \frac{\mathbb{E}[\frac{\mathrm{d}}{\mathrm{d}s} e^{s\langle V,\phi^2\rangle_{\ell^2(\mathbb{Z}^d)}}]}{\mathbb{E}[e^{s\langle V,\phi^2\rangle_{\ell^2(\mathbb{Z}^d)}}]} = \mathbb{E}\Big[\langle V,\phi^2\rangle_{\ell^2(\mathbb{Z}^d)} \frac{e^{s\langle V,\phi^2\rangle_{\ell^2(\mathbb{Z}^d)}}]}{\mathbb{E}[e^{s\langle V,\phi^2\rangle_{\ell^2(\mathbb{Z}^d)}}]}\Big] \\
&= \mathbb{E}_{s,V}[\langle V,\phi^2\rangle_{\ell^2(\mathbb{Z}^d)}] \\
K'(0) &= \mathbb{E}[\langle V,\phi^2\rangle_{\ell^2(\mathbb{Z}^d)}] \\
K''(s) &= \mathbb{E}\Big[\langle V,\phi^2\rangle_{\ell^2(\mathbb{Z}^d)} \frac{\mathrm{d}}{\mathrm{d}s} \frac{e^{s\langle V,\phi^2\rangle_{\ell^2(\mathbb{Z}^d)}}]}{\mathbb{E}[e^{s\langle V,\phi^2\rangle_{\ell^2(\mathbb{Z}^d)}}]}\Big] \\
&= \mathbb{E}_{s,V}[\langle V,\phi^2\rangle^2_{\ell^2(\mathbb{Z}^d)}] - \mathbb{E}_{s,V}[\langle V,\phi^2\rangle_{\ell^2(\mathbb{Z}^d)}]^2 \\
&= \mathrm{Var}_{s,V}(\langle V,\phi^2\rangle_{\ell^2(\mathbb{Z}^d)}) \\
&= \sum_{x,y\in\mathbb{Z}^d} V(x)\,\mathbb{C}\mathrm{ov}_{s,V}(\phi^2(x),\phi^2(y))V(y) \\
&= \sum_{x,y\in\mathbb{Z}^d} V(x)2\,\mathbb{C}\mathrm{ov}_{s,V}(\phi(x),\phi(y))^2 V(y),
\end{aligned}
$$

where we defined $\mathbb{E}_{s,V} := \mathbb{E}_{\mu_\Lambda^{s,V}}$ as the expectation taken under the Gibbs-measure

$$
\begin{aligned}
\mu_\Lambda^{s,V}(\mathrm{d}\phi) &:= \mu_\Lambda(\mathrm{d}\phi)\cdot \frac{e^{-s\langle V,\phi^2\rangle_{\ell^2(\mathbb{Z}^d)}}}{\mathbb{E}[e^{s\langle V,\phi^2\rangle_{\ell^2(\mathbb{Z}^d)}}]} \\
&= \frac{1}{Z_\Lambda \mathbb{E}[e^{s\langle V,\phi^2\rangle_{\ell^2(\mathbb{Z}^d)}}]} e^{-\frac{1}{2}\sum_{\{x,y\}\in E^d(\Lambda)}(\phi(x)-\phi(y))^2 - s\sum_{x\in\mathbb{Z}^d} V(x)\phi^2(x)}\nu(\mathrm{d}\phi) \\
&= \frac{1}{Z_\Lambda^{s,V}} e^{-\mathcal{H}_\Lambda^{s,V}(\phi)}\nu(\mathrm{d}\phi)
\end{aligned}
\tag{3.9}
$$

with Hamiltonian

$$
\mathcal{H}_\Lambda^{s,V}(\phi) = \frac{1}{2}\sum_{\{x,y\}\in E^d(\Lambda)} (\phi(x)-\phi(y))^2 + s\sum_{x\in\mathbb{Z}^d} V(x)\phi^2(x).
$$

In the following, our object of interest will be the covariance under the new Gibbs measure, i.e. $\mathbb{C}\mathrm{ov}_{s,V}(\phi(x),\phi(y))$. This gives rise to a Markov-Process $(\{\Phi_s(x)\}_{x\in\mathbb{Z}^d} : t \geq 0)$ on ϕ^Λ

with generator

$$\begin{aligned}\big(L_{s,V}F\big)(\phi) &= e^{\mathcal{H}_\Lambda^{s,V}(\phi)} \sum_{x\in\mathbb{Z}^d} \partial_x\big(e^{-\mathcal{H}_\Lambda^{s,V}(\phi)}\partial_x F(\phi)\big) \\ &= -\sum_{x\in\mathbb{Z}^d} \partial_x \mathcal{H}_\Lambda^{s,V}(\phi)\partial_x F(\phi) + \partial_x^2 F(\phi), \end{aligned} \tag{3.10}$$

similar as in [11].

Next, we want to calculate the covariance by using a Helffer–Sjöstrand representation analogous to [6, chapter 2.3]. For $F, J : \mathbb{Z}^d \to \mathbb{R}$ with $\mathbb{E}[J(\phi)] = 0$, we choose $G : \mathbb{Z}^d \to \mathbb{R}$ such that $J = -L_{s,V}G$. We then use the Gauss Green Formula to get

$$\begin{aligned}\mathbb{E}_{s,V}[F(\phi)J(\phi)] &= \mathbb{E}_{s,V}[F(\phi)(-L_{s,V}G)(\phi)] = \mathbb{E}_{s,V}[\langle \nabla F(\phi), \nabla G(\phi)\rangle_{\ell^2(\mathbb{Z}^d)}] \\ &= \sum_{x\in\mathbb{Z}^d} \mathbb{E}_{s,V}[\partial_x F(\phi)\partial_x G(\phi)],\end{aligned}$$

since the Gibbs-measure is invariant. For the derivative of J, we get:

$$\begin{aligned}\partial_x J(\phi) &= -\partial_x\big(L_{s,V}G\big)(\phi) \\ &= \partial_x\Bigg(\sum_{y\in\mathbb{Z}^d} \partial_y \mathcal{H}_\Lambda^{s,V}(\phi)\partial_y G(\phi) - \partial_y^2 G(\phi)\Bigg) \\ &= \sum_{y\in\mathbb{Z}^d} \partial_y \mathcal{H}_\Lambda^{s,V}(\phi)\partial_y\partial_x G(\phi) - \partial_y^2\partial_x G(\phi) + \sum_{y\in\mathbb{Z}^d} \partial_x\partial_y \mathcal{H}_\Lambda^{s,V}(\phi)\partial_y G(\phi) \\ &= \big(-L_{s,V}\partial_x G\big)(\phi) + \sum_{y\in\mathbb{Z}^d} \partial_x\partial_y \mathcal{H}_\Lambda^{s,V}(\phi)\partial_y G(\phi)\end{aligned}$$

and for the Hamiltonian, we have

$$\begin{aligned}\partial_y \mathcal{H}_\Lambda^{s,V}(\phi) &= \partial_y\Bigg(\sum_{\{z,z'\}\in E^d(\Lambda)} \omega(z,z')(\phi(z)-\phi(z'))^2 + s\sum_{z\in\mathbb{Z}^d} V^\omega(z)\phi^2(z)\Bigg) \\ &= 2\sum_{z:z\sim y} \omega(z,y)(\phi(y)-\phi(z)) + 2sV(y)\phi(y),\end{aligned}$$

which yields

$$\begin{aligned}\partial_x J(\phi) &= \big(-L_{s,V}\partial_x G\big)(\phi) + 2\sum_{y\in\mathbb{Z}^d} \partial_x\Bigg(\sum_{z:z\sim y} \omega(z,y)\big(\phi(y)-\phi(z)\big) + sV(y)\phi(y)\Bigg)\partial_y G(\phi) \\ &= \big(-L_{s,V}\partial_x G\big)(\phi) + 2\Bigg(\sum_{y:y\sim x} \omega(x,y)\big(\partial_y G(\phi) - \partial_x G(\phi)\big) + sV(x)\partial_x G(\phi)\Bigg).\end{aligned}$$

We thus have

$$\nabla J = -\tilde{\mathcal{L}}_{s,V}\nabla G \iff \big(-\tilde{\mathcal{L}}_{s,V}\big)^{-1}\nabla J = \nabla G$$

where $\tilde{\mathcal{L}}_{s,V}$ is the Witten Laplacian acting on ∇G given by

$$\big(\tilde{\mathcal{L}}_{s,V}\nabla G\big)(\phi) := \big(L_{s,V} \otimes \mathrm{id}\big)(\nabla G) - D^2\nabla G - 2sV\nabla G(\phi)$$

as in [6, Chapter 2.3].

Thus, we have

$$\mathbb{E}_{s,V}[F(\phi)J(\phi)] = \mathbb{E}_{s,V}[\langle \nabla F(\phi), \nabla G(\phi)\rangle_{\ell^2(\mathbb{Z}^d)}] = \mathbb{E}_{s,V}[\langle \nabla F(\phi), \big(-\tilde{\mathcal{L}}_{s,V}\big)^{-1}\nabla J(\phi)\rangle_{\ell^2(\mathbb{Z}^d)}]$$
$$= \mathbb{E}_{s,V}[\langle \nabla F(\phi), \big(-\mathcal{L}_{s,V}\big)^{-1}\nabla J(\phi)\rangle_{\ell^2(\mathbb{Z}^d)}],$$

where the term $\big(L_{s,V} \otimes \mathrm{id}\big)$ vanishes in the last equation such that the resulting Witten Laplacian simplifies to

$$\big(\mathcal{L}_{s,V}\nabla G\big)(\phi) := -D^2\nabla G - 2sV\nabla G(\phi). \tag{3.11}$$

This last step is highly non-trivial an can be found in [11]. Finally, we get for the covariance, since the field is centered and by choosing $F(\phi) = \phi(x)$, $J(\phi) = \phi(y)$:

$$\begin{aligned}\mathbb{C}\mathrm{ov}_{s,V}(\phi(x),\phi(y)) = \mathbb{E}_{s,V}[\phi(x)\phi(y)] &= \mathbb{E}_{s,V}[\langle \delta_x, \big(-\mathcal{L}_{s,V}\big)^{-1}\delta_y\rangle_{\ell^2(\mathbb{Z}^d)}]\\ &= \big(-\mathcal{L}_{s,V}\big)^{-1}\delta_y(x)\\ &= g^{\Lambda}_{s,V}(x,y),\end{aligned} \tag{3.12}$$

since $\nabla\phi(x) = \delta_x$. Here $g^{\Lambda}_{s,V}$ is the Green's function associated with $\mathcal{L}_{s,V}$, i.e.

$$\begin{cases}\mathcal{L}^{\omega}_{s,V}g^{\Lambda}_{s,V}(\cdot,y) = -\delta_y(\cdot) & \text{on } \Lambda,\\ g^{\Lambda}_{s,V}(\cdot,y) = 0 & \text{on } \Lambda^c.\end{cases} \tag{3.13}$$

By substituting $s = \frac{\lambda}{2}$ and considering the discrete case with the rescaled discrete potential V_n (3.6), one can observe the immediate connection to our Definiton of $\mathcal{L}^{\omega}_{\lambda,V}$ in (2.5) and the corresponding killed Green's function.

4 Analytical estimates for elliptic and parabolic differential equations

In this chapter we will derive analytical estimates for the solutions u of some important parabolic and elliptic equations. Initially, we will employ Moser's iteration [17] to find a maximum inequality for solutions of the parabolic Schrödinger Poisson-equation, where $u\colon \mathbb{R}\times V \to \mathbb{R}$ satisfies $\partial_t u - \mathcal{L}^{\omega}_{\lambda,V} u \le \frac{1}{n^2} f$. Further, we will find a bound for the Laplace killed Green's function $g^{\omega}_{B(n)}(x,\cdot)$ as defined in (2.14). This bound will serve as a key element in proving one of our main results, the Hölder continuity of Schrödinger harmonic functions, i.e. $\mathcal{L}^{\omega}_{\lambda,V} u = 0$. We thereby extend several important results that were previously only proven for Laplace equations to the more general case of Schrödinger equations.

4.1 Sobolev Inequality

The analytical estimates in this section rely on a weighted Sobolev inequality derived below. Sobolev inequalities are a powerful tool from functional analysis and are used there, among other things, to establish regularity results. For our purposes, we will derive an inequality which will relate the ρ-Norm of a function with its Dirichlet energy. To state the inequality on the weighted graph, we define for some $q \ge 1$:

$$\rho \;=\; \rho(q,d) \;:=\; \frac{d}{(d-2)+d/q} \tag{4.1}$$

By Remark 2.1.3, we get a Sobolev inequality (S_1^d) on the unweighted graph, which reads as

$$\left(\frac{1}{|B(x,n)|}\sum\nolimits_{y\in B(x,n)} |u(y)|^{\frac{d}{d-1}}\right)^{\frac{d-1}{d}} \;\le\; \frac{C_{\mathrm{S}_1} n}{|B(x,n)|} \sum_{\substack{y\vee y'\in B(x,n)\\ \{y,y'\}\in E}} \big|u(y)-u(y')\big|$$

for all functions $u\colon V\to\mathbb{R}$ with support in $B(x,n)$. It is now our task to extend this result on the weighted graph:

Proposition 4.1.1 (Sobolev inequality)**.** *Suppose that the graph, $G = (V,E)$, satisfies Assumption 2.1.1 and let $B \subset V$ be finite and connected. Further, consider a non-negative function $\eta : V \to \mathbb{R}$ with*

$$\operatorname{supp}\eta \;\subset\; B, \qquad 0 \;\le\; \eta \;\le\; 1 \qquad \textit{and} \qquad \eta \;\equiv\; 0 \quad \textit{on} \quad \partial B.$$

Then, for any $q\in[1,\infty)$, there exists $C_{\mathrm{S}} \equiv C_{\mathrm{S}}(d,q) < \infty$ such that for any function $f : V \to \mathbb{R}$,

$$\big\|(\eta f)^2\big\|_{\rho,B} \;\le\; \frac{C_{\mathrm{S}}}{2}\,|B|^{\frac{2}{d}}\,\big\|\nu^{\omega}\big\|_{q,B}\,\frac{\mathcal{E}_{\omega}(\eta f)}{|B|} \tag{4.2}$$

Proof. We will follow a similar approach as in the proof of [1, Proposition 3.5]. Note that due to the cutoff function η it suffices to prove the statement for functions $f : V \to \mathbb{R}$ with $\operatorname{supp} f \subset B(n)$. Applying (2.10) with $\alpha = 2\rho\frac{d-1}{d}$ and (2.4) yields

$$\begin{aligned} \big\| |f|^\alpha \big\|_{\frac{d}{d-1},B(n)} &\leq \frac{C_{S_1} n}{|B(n)|} \big\| \nabla |f|^\alpha \big\|_{\ell^1(E)} \\ &\leq \frac{2C_{S_1} n (1 \vee \alpha)}{|B(n)|} \big\| \operatorname{av}(|f|^{\alpha-1}) \nabla |f| \big\|_{\ell^1(E)}. \end{aligned} \tag{4.3}$$

Next, we apply the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} \big\| \operatorname{av}(|f|^{\alpha-1}) \nabla |f| \big\|_{\ell^1(E)} &\leq \big\| \nabla |f| \big\|_{\ell^2(E)}^{\frac{1}{2}} \big\| \operatorname{av}(|f|^{\alpha-1}) \big\|_{\ell^2(E)}^{\frac{1}{2}} \\ &\leq \mathcal{E}^\omega(f)^{\frac{1}{2}} |B(n)|^{\frac{1}{2}} \big\| |f|^{2(\alpha-1)} \nu^\omega \big\|_{1,B(n)}^{\frac{1}{2}}, \end{aligned} \tag{4.4}$$

where we used that by Jensen's inequality, we have $\operatorname{av}(|f|)^2 \leq \operatorname{av}(|f|^2)$ and also $\mathcal{E}^\omega(|f|) \leq \mathcal{E}^\omega(f)$ in the last step. If $q = 1$, then $\rho = \frac{d}{2(d-1)}$ and thus $\alpha = 1$. In this case, the claim follows, after combining (4.3) and (4.4) and using volume regularity. If $q > 1$ we have to apply Hölder's inequality to obtain

$$\big\| |f|^{2(\alpha-1)} \nu^\omega \big\|_{1,B(n)}^{\frac{1}{2}} \leq \big\| |f|^{2(\alpha-1)} \big\|_{q_*,B(n)} \big\| \nu^\omega \big\|_{q,B(n)},$$

where q_* is the Hölder component of q. Due to the definition of $\rho(q,d)$, we have that $\alpha\frac{d}{d-1} = 2q_*(\alpha - 1) = 2\rho$. Thus combining (4.3) and (4.4), using volume regularity and solving for $\|f^2\|_{\rho,B(n)}$ gives the desired result. □

4.2 Maximum Inequalities for Elliptic and Parabolic Poisson equations

First, we establish a maximum equality for time-inhomogeneous Poisson equation without boundary condition. Namely, we consider a function $u \colon \mathbb{R} \times V \to \mathbb{R}$ solving

$$\partial_t u - \mathcal{L}^\omega_{\lambda,V} u \leq \frac{1}{n^2} f \tag{4.5}$$

on $Q(n) := I(n) \times B(n) = [t_0, t_0 + n^2] \times B(x_0, n)$ for some $f : \mathbb{R} \times V \to \mathbb{R}$. This is a very general case, since the time and space homogenous versions are special cases of the above. To simplify notation, we denote for $t \in I : u_t := u(t, \cdot)$.

The approach for utilizing Moser's iteration technique to establish maximal inequalities, as demonstrated by Andres, Deuschel, and Slowik in 2016 [2], involves first deriving an estimate for the Dirichlet energy and then using Sobolev's inequality to initiate the iteration process. Therefore, we begin by proving an estimate for the Dirichlet Energy:

Lemma 4.2.1. *Suppose that $Q = I \times B$, where $I = [s_1, s_2]$ is an interval and B is a finite, connected subset of V. Consider a function $\eta \colon V \to \mathbb{R}$ and a smooth function $\zeta \colon \mathbb{R} \to \mathbb{R}$ with*

$$\begin{array}{llll} \operatorname{supp}\eta \subset B, & 0 \leq \eta \leq 1 & \textit{and} & \eta \equiv 0 \ \textit{ on } \ \partial B, \\ \operatorname{supp}\zeta \subset I, & 0 \leq \zeta \leq 1 & \textit{and} & \zeta \equiv 0 \ \textit{ on } \ [s_2, \infty). \end{array}$$

Further, let $u > 0$ be such that $\partial_t u - \mathcal{L}^{\omega}_{\lambda,V} u \le \frac{1}{n^2} f$ on Q. Then, for all $\alpha \ge 1$,

$$\begin{aligned} &\max_{t\in I} \Big(\zeta(t)\, \big\|(\eta\, u_t^{\alpha})^2\big\|_{1,B}\Big) \;+\; \int_I \zeta(t)\, \frac{\mathcal{E}^{\omega}(\eta u_t^{\alpha})}{|B|}\, \mathrm{d}t \\ &\le\; 2\alpha \Big(\|\nabla\eta\|^2_{\ell^{\infty}(E)}\, \|\mu^{\omega}\|_{p,B} \;+\; \frac{\lambda}{n^2}\, \|V^{\omega}\|_{p,B} \;+\; \frac{1}{n^2}\, \|f\|_{p,B} \;+\; \|\zeta'\|_{L^{\infty}(I)}\Big) \\ &\qquad \times \int_I \|u_t\|^{\tilde{\gamma}}_{2\alpha p_*,B}\, \mathrm{d}t \end{aligned} \tag{4.6}$$

where

$$\tilde{\gamma}(\alpha, p_*, B) \;:=\; \begin{cases} 2\alpha & \text{if } \|u_t\|_{2\alpha p_*,B} \ge 1, \\ 2\alpha - 1 & \text{otherwise.} \end{cases} \tag{4.7}$$

Remark 4.2.2. In the case of the right hand side being zero, i.e. $f \equiv 0$, we can simply choose $\tilde{\gamma} = 2\alpha$.

Before we present the proof of the energy estimate, we prove a technical lemma, which will make the following proof much more readable:

Lemma 4.2.3. *Under the assumptions of Lemma 4.2.1, it holds for all $\alpha \ge 1$, that:*

$$\big\langle \nabla(\eta^2\, u_t^{2\alpha-1}), \omega\, \nabla u_t \big\rangle_{\ell^2(E)} \;\ge\; \frac{1}{\alpha}\Big(\mathcal{E}^{\omega}(\eta u_t^{\alpha}) \;-\; \big\langle (\nabla\eta)^2, \omega\ \mathrm{av}(u_t^{2\alpha}) \big\rangle_{\ell^2(E)}\Big) \tag{4.8}$$

Proof. Firstly, by (2.11)

$$\nabla u_t^{2\alpha-1} \nabla u_t \;\ge\; \frac{2\alpha-1}{\alpha^2} (\nabla u_t^{\alpha})^2. \tag{4.9}$$

Furthermore, by using that $\mathrm{av}(\eta)\nabla u_t^{\alpha} = \nabla(\eta u_t^{\alpha}) - \nabla\eta\ \mathrm{av}(u_t^{\alpha})$, we obtain

$$\big(\mathrm{av}(\eta)\nabla u_t^{\alpha}\big)^2 \;=\; \big(\nabla \eta u_t^{\alpha}\big)^2 - 2\nabla(\eta u_t^{\alpha})\nabla\eta\ \mathrm{av}(u_t^{\alpha}) + \big(\nabla\eta\ \mathrm{av}(u_t^{\alpha})\big)^2. \tag{4.10}$$

With (2.12) and $\nabla u_t^{2\alpha} = 2\ \mathrm{av}(u_t^{\alpha})\nabla u_t^{\alpha}$:

$$\begin{aligned} &\mathrm{av}(u_t^{2\alpha-1})(e)\nabla u_t(e) \\ &=\; \frac{1}{2}\big(u_t(e^+)^{2\alpha-1} + u_t(e^-)^{2\alpha-1}\big)\,\big(u_t(e^+) - u_t(e^-)\big) \\ &=\; \frac{1}{2}\big(u_t(e^+)^{2\alpha} - u_t(e^-)^{2\alpha}\big) - \frac{1}{2}\big(u_t(e^+)^{2\alpha-1}u_t(e^-) - u_t(e^-)^{2\alpha-1}u_t(e^+)\big) \\ &\ge\; \frac{1}{2}\nabla u_t^{2\alpha}(e) - \frac{1}{2}\,\frac{2(\alpha-1)}{\alpha}\ \mathrm{av}(u_t^{\alpha})(e)\,\nabla u_t^{\alpha}(e) \\ &=\; \big(1 - \frac{\alpha-1}{\alpha}\big)\ \mathrm{av}(u_t^{\alpha})(e)\,\nabla u_t^{\alpha}(e) \\ &=\; \frac{1}{\alpha}\ \mathrm{av}(u_t^{\alpha})(e)\,\nabla u_t^{\alpha}(e) \end{aligned} \tag{4.11}$$

for all $e \in E$ and, using the discrete product rule:

$$\mathrm{av}(\eta)\,\nabla u_t^{\alpha} \;=\; \nabla(\eta u_t^{\alpha}) - \nabla\eta\ \mathrm{av}(u_t^{\alpha}). \tag{4.12}$$

Recalling Young's inequality $ab \leq \frac{1}{2}\big(\varepsilon a^2 + \frac{1}{\varepsilon} b^2\big)$, we get:

$$\nabla\eta \ \mathrm{av}(u_t^\alpha)\,\nabla(\eta u_t^\alpha) \ \leq \ \frac{1}{2}\Big(\frac{1}{\varepsilon}\big(\nabla\eta u_t^\alpha\big)^2 + \varepsilon\big(\nabla\eta \ \mathrm{av}(u_t^\alpha)\big)^2\Big) \tag{4.13}$$

Combining the above results and choosing $\varepsilon = 1$, we now get the following lower bound for the gradient, by using Jensen's inequality which yields $\mathrm{av}(\eta^2) \geq \mathrm{av}(\eta)^2$:

$$\begin{aligned}
&\nabla\big(\eta^2 u_t^{2\alpha-1}\big)\nabla u_t \\
&= \mathrm{av}(\eta^2)\nabla u_t^{2\alpha-1}\nabla u_t + \mathrm{av}(u_t^{2\alpha-1})\nabla u_t \nabla\eta^2 \\
&\overset{4.9}{\geq} \frac{2\alpha-1}{\alpha^2}\big(\mathrm{av}(\eta)\nabla u_t^\alpha\big)^2 + \mathrm{av}(u_t^{2\alpha-1})\nabla u_t\nabla\eta^2 \\
&\overset{4.10}{=} \frac{2\alpha-1}{\alpha^2}\Big(\big(\big(\nabla\eta u_t^\alpha\big)^2 + \big(\nabla\eta \ \mathrm{av}(u_t^\alpha)\big)^2\big) - 2\nabla(\eta u_t^\alpha)\nabla\eta \ \mathrm{av}(u_t^\alpha)\Big) \\
&\quad + \mathrm{av}(u_t^{2\alpha-1})\nabla u_t \nabla\eta^2 \\
&\overset{4.11}{\geq} \frac{2\alpha-1}{\alpha^2}\Big(\big(\nabla\eta u_t^\alpha\big)^2 - \big(\nabla\eta \ \mathrm{av}(u_t^\alpha)\big)^2\Big) + 2 \ \mathrm{av}(\eta)\nabla\eta\left(\frac{1}{\alpha} - \frac{2\alpha-1}{\alpha^2}\right)\nabla u_t^\alpha \ \mathrm{av}(u_t^\alpha) \\
&= \frac{2\alpha-1}{\alpha^2}\Big(\big(\nabla\eta u_t^\alpha\big)^2 - \big(\nabla\eta \ \mathrm{av}(u_t^\alpha)\big)^2\Big) - 2\,\frac{\alpha-1}{\alpha^2}\,\nabla\eta \ \mathrm{av}(u_t^\alpha) \ \mathrm{av}(\eta)\,\nabla u_t^\alpha \\
&\overset{4.12}{=} \frac{2\alpha-1}{\alpha^2}\Big(\big(\nabla\eta u_t^\alpha\big)^2 - \big(\nabla\eta \ \mathrm{av}(u_t^\alpha)\big)^2\Big) - 2\,\frac{\alpha-1}{\alpha^2}\,\nabla\eta \ \mathrm{av}(u_t^\alpha)\,\Big(\nabla(\eta u_t^\alpha) \\
&\quad - \nabla\eta \ \mathrm{av}(u_t^\alpha)\Big) \\
&\overset{4.13}{\geq} \left(\frac{2\alpha-1}{\alpha^2} - \frac{\alpha-1}{\varepsilon\alpha^2}\right)\big(\nabla\eta u_t^\alpha\big)^2 - \left(\frac{2\alpha-1}{\alpha^2} + \frac{\alpha-1}{\alpha^2}\varepsilon - 2\,\frac{\alpha-1}{\alpha^2}\right)\big(\nabla\eta \ \mathrm{av}(u_t^\alpha)\big)^2 \\
&\geq \frac{1}{\alpha}\Big(\big(\nabla\eta u_t^\alpha\big)^2 - \big(\nabla\eta\big)^2 \ \mathrm{av}(u_t^{2\alpha})\Big),
\end{aligned}$$

where we used in the last step that by Hölders inequality $\mathrm{av}(u^\alpha)^2 \geq \mathrm{av}(u^{2\alpha})$. We now conclude the proof by using that $\mathcal{E}(\eta u^\alpha) = \big\langle \nabla\eta u^\alpha, \omega\nabla\eta u^\alpha\big\rangle_{\ell^2(E)}$, which yields:

$$\big\langle \nabla(\eta^2\, u_t^{2\alpha-1}), \omega\,\nabla u_t\big\rangle_{\ell^2(E)} \ \geq \ \frac{1}{\alpha}\Big(\mathcal{E}^\omega(\eta u_t^\alpha) \ - \ \big\langle(\nabla\eta)^2, \omega \ \mathrm{av}(u_t^{2\alpha})\big\rangle_{\ell^2(E)}\Big).$$

□

We will now proceed with proving the upper bound for the Dirichlet energy:

Proof of Lemma 4.2.1. Since $\partial_t u - \mathcal{L}^\omega_{\lambda,V} u \leq \frac{1}{n^2} f$ on Q, we get for every $t \in I$ using integration by parts,

$$\begin{aligned}
\frac{1}{2\alpha}\,\partial_t\langle\eta^2, u_t^{2\alpha}\rangle_{\ell^2(V)} &= \langle\eta^2\, u_t^{2\alpha-1}, \partial_t\, u_t\rangle_{\ell^2(V)} \\
&\leq \langle\eta^2\, u_t^{2\alpha-1}, \mathcal{L}^\omega_{\lambda,V}\, u_t + \frac{1}{n^2} f\rangle_{\ell^2(V)} \\
&= \langle\eta^2\, u_t^{2\alpha-1}, \mathcal{L}^\omega\, u_t\rangle_{\ell^2(V)} + \langle\eta^2\, u_t^{2\alpha-1}, \frac{\lambda}{n^2} V^\omega u_t + \frac{1}{n^2} f\rangle_{\ell^2(V)} \\
&= -\langle\nabla(\eta^2\, u_t^{2\alpha-1}), \omega\,\nabla u_t\rangle_{\ell^2(E)} + \langle\eta^2\, u_t^{2\alpha-1}, \frac{\lambda}{n^2} V^\omega u_t + \frac{1}{n^2} f\rangle_{\ell^2(V)}.
\end{aligned}$$

We can now use the estimate from Lemma 4.2.3 for the first term:

$$\frac{1}{2\alpha}\,\partial_t\langle\eta^2, u_t^{2\alpha}\rangle_{\ell^2(V)}$$
$$\leq\; -\frac{1}{\alpha}\,\mathcal{E}^\omega(\eta u_t^\alpha)\;+\;\frac{1}{\alpha}\,\big\langle(\nabla\eta)^2, \omega\;\mathrm{av}(u_t^{2\alpha})\big\rangle_{\ell^2(E)}+\langle\eta^2\,u_t^{2\alpha-1}, \frac{\lambda}{n^2}V^\omega u_t+\frac{1}{n^2}f\rangle_{\ell^2(V)}.$$

Hence, by using Hölders Inequality

$$\frac{1}{2}\,\partial_t\big\|(\eta\,u_t^\alpha)^2\big\|_{1,B}\;+\;\frac{\mathcal{E}^\omega(\eta u_t^\alpha)}{|B|}$$
$$\leq\;\Big(\|\nabla\eta\|^2_{\ell^\infty(E)}\;\|\mu^\omega\|_{p,B}\;+\;\alpha\frac{\lambda}{n^2}\;\|V^\omega\|_{p,B}\Big)\|u_t^{2\alpha}\|_{p_*,B}\;\;+\;\alpha\frac{1}{n^2}\;\|f\|_{p,B}\;\|u_t^{2\alpha-1}\|_{p_*,B}.$$

For $f\neq 0$, we have to consider two cases. If $\|u_t\|_{2\alpha p_*,B}\geq 1$, then $\|u_t\|^{-1}_{2\alpha p_*,B}\leq 1$ and thus $\|u_t^{2\alpha-1}\|_{p_*,B}\leq\|u_t\|^{2\alpha}_{2\alpha p_*,B}$. In the other case, we get $\|u_t\|^{-1}_{2\alpha p_*,B}\geq 1$ and thus $\|u_t^{2\alpha}\|_{p_*,B}\leq\|u_t\|^{2\alpha-1}_{2\alpha p_*,B}$. Together, we obtain

$$\frac{1}{2}\,\partial_t\big\|(\eta\,u_t^\alpha)^2\big\|_{1,B,\mu^\omega}\;+\;\frac{\mathcal{E}^\omega(\eta u_t^\alpha)}{|B|}$$
$$\leq\;\Big(\|\nabla\eta\|^2_{\ell^\infty(E)}\;\|\mu^\omega\|_{p,B}\;+\;\alpha\frac{\lambda}{n^2}\;\|V^\omega\|_{p,B}\;+\;\alpha\frac{1}{n^2}\;\|f\|_{p,B}\Big)\|u_t\|^{\tilde\gamma}_{2\alpha p_*,B}. \tag{4.14}$$

where $\tilde\gamma=\tilde\gamma(\alpha,p_*,B)$ is defined as

$$\tilde\gamma(\alpha,p_*,B)\;:=\;\begin{cases}2\alpha & \text{if } \|u_t\|_{2\alpha p_*,B}\geq 1,\\ 2\alpha-1 & \text{otherwise.}\end{cases}$$

Notice that in the case of $f=0$ we do not have to worry about the different exponents and can always choose $\tilde\gamma=2\alpha$.

For every $s\in[s_1,s_2)$ by the product rule and since $\zeta(s_2)=0$ we get

$$\int_s^{s_2}-\zeta(t)\;\partial_t\big\|(\eta u_t^\alpha)^2\big\|_{1,B}\,\mathrm{d}t\;=\;\int_s^{s_2}-\partial_t\zeta(t)\;\big\|(\eta u_t^\alpha)^2\big\|_{1,B}\;+\;\zeta'(t)\big\|(\eta u_t^\alpha)^2\big\|_{1,B}\,\mathrm{d}t$$
$$\geq\;\zeta(s)\,\big\|(\eta u_s^\alpha)^2\big\|_{1,B}\;-\;\|\zeta'\|_{L^\infty(I)}\int_I\|u_t\|^{2\alpha}_{2\alpha p_*,B}\,\mathrm{d}t.$$

Then, for any $s\in I$ we get by multiplying both sides of (4.14) with $\zeta(t)$ and integrating over I

$$\begin{aligned}
&\frac{1}{2}\,\zeta(s)\,\big\|(\eta\,u_s^\alpha)^2\big\|_{1,B} + \int_{s_1}^{s}\zeta(t)\,\frac{\mathcal{E}^\omega(\eta u_t^\alpha)}{|B|}\,\mathrm{d}t \\
&\quad\le\ \Big(\|\nabla\eta\|^2_{\ell^\infty(E)}\,\|\mu^\omega\|_{p,B} + \alpha\frac{\lambda}{n^2}\,\|V^\omega\|_{p,B} + \alpha\frac{1}{n^2}\,\|f\|_{p,B}\Big)\int_I \|u_t\|^{\tilde\gamma}_{2\alpha p_*,B}\,\mathrm{d}t \\
&\quad\quad + \frac{1}{2}\,\|\zeta'\|_{L^\infty(I)}\int_I \|u_t\|^{2\alpha}_{2\alpha p_*,B}\,\mathrm{d}t \\
&\quad\le\ \Big(\|\nabla\eta\|^2_{\ell^\infty(E)}\,\|\mu^\omega\|_{p,B} + \alpha\frac{\lambda}{n^2}\,\|V^\omega\|_{p,B} + \alpha\frac{1}{n^2}\,\|f\|_{p,B} + \|\zeta'\|_{L^\infty(I)}\Big) \\
&\quad\quad \times\int_I \|u_t\|^{\tilde\gamma}_{2\alpha p_*,B}\,\mathrm{d}t \\
&\quad\le\ \alpha\Big(\|\nabla\eta\|^2_{\ell^\infty(E)}\,\|\mu^\omega\|_{p,B} + \frac{\lambda}{n^2}\,\|V^\omega\|_{p,B} + \frac{1}{n^2}\,\|f\|_{p,B} + \|\zeta'\|_{L^\infty(I)}\Big) \\
&\quad\quad \times\int_I \|u_t\|^{\tilde\gamma}_{2\alpha p_*,B}\,\mathrm{d}t
\end{aligned}$$

where we used that $\|u_t\|^{2\alpha}_{2\alpha p_*,B} \le \|u_t\|^{\tilde\gamma}_{2\alpha p_*,B}$ in the second step and $\alpha \ge 1$ in the last step.

The claim now follows, since

$$\begin{aligned}
&\max_{t\in I}\Big(\zeta(t)\,\big\|(\eta\,u_t^\alpha)^2\big\|_{1,B}\Big) + \int_I \zeta(t)\,\frac{\mathcal{E}^\omega(\eta u_t^\alpha)}{|B|}\,\mathrm{d}t \\
&\quad\le\ 2\left(\frac{1}{2}\max_{t\in I}\Big(\zeta(t)\,\big\|(\eta\,u_t^\alpha)^2\big\|_{1,B}\Big) + \int_I \zeta(t)\,\frac{\mathcal{E}^\omega(\eta u_t^\alpha)}{|B|}\,\mathrm{d}t\right).
\end{aligned}$$

□

The control of the Dirichlet energy can now be combined with the Soboloev inequality to prove the maximal inequality. First, we will prove a technical lemma, which will be handy to apply the energy estimate in the following proof:

Lemma 4.2.4. *Let $u\colon \mathbb{R}\times V \to \mathbb{R}$ and $\alpha \ge 1$ depending on ρ and p_* as follows*

$$\alpha\ :=\ \alpha(\rho, p_*)\ =\ 1 - \frac{1}{\rho} + \frac{1}{p_*}. \tag{4.15}$$

Then on $Q = I\times B$ it holds

$$\|u\|_{\alpha p_*,\alpha,Q}\ \le\ \|u\|_{1,\infty,Q} + \|u\|_{\rho,1,Q}. \tag{4.16}$$

Proof. Let α be as in (4.15) and $\theta \in (0,1)$ a constant. We then have

$$\begin{aligned}
\|u\|_{\alpha p_*,\alpha,Q}\ &=\ \left(\frac{1}{|I|}\int_I \|u_t^\theta u_t^{1-\theta}\|^\alpha_{\alpha p_*}\,\mathrm{d}t\right)^{\frac{1}{\alpha}} \\
&\le\ \left(\frac{1}{|I|}\int_I \|u_t^\theta\|^\alpha_{\alpha p_*\beta}\|u_t^{1-\theta}\|^\alpha_{\alpha p_*\beta^*}\,\mathrm{d}t\right)^{\frac{1}{\alpha}} \\
&=\ \left(\frac{1}{|I|}\int_I \|u_t\|^{\theta\alpha}_{\theta\alpha p_*\beta}\|u_t\|^{(1-\theta)\alpha}_{(1-\theta)\alpha p_*\beta^*}\,\mathrm{d}t\right)^{\frac{1}{\alpha}},
\end{aligned}$$

where we choose β and β^* as Hölder conjugates such that we achieve the desired result, namely:

$$\beta \;=\; \frac{\rho}{\rho - p_*} \qquad \text{and} \qquad \beta^* \;=\; \frac{\rho}{p_*}.$$

Thus, choosing α as in (4.20) and $\theta = -\frac{1}{\alpha} + 1$ yields

$$\begin{aligned} \|u\|_{\alpha p_*, \alpha, Q} \;&\leq\; \|u\|_{1,\infty,Q}^{\theta}\, \|u\|_{\rho,1,Q}^{1-\theta} \\ &\leq\; \theta \|u\|_{1,\infty,Q} + (1-\theta)\|u\|_{\rho,1,Q} \\ &\leq\; \|u\|_{1,\infty,Q} + \|u\|_{\rho,1,Q}, \end{aligned}$$

where we used Young's inequality to split the product in the second step. □

Now we are able to prove the maximum inequality by using Moser's iteration technique:

Theorem 4.2.5. *Assume that $u > 0$ solves $\partial_t u - \mathcal{L}^{\omega}_{\lambda,V} u \leq \frac{1}{n^2} f$ on $Q(n)$. Then, for any $\Delta \in [0,1)$ and $p, q \in (1,\infty]$ satisfying*

$$\frac{1}{p} + \frac{1}{q} \;<\; \frac{2}{d} \tag{4.17}$$

there exist $N(\Delta) < \infty$, $\gamma \equiv \gamma(d,p,q) \in (0,1]$, $\kappa \equiv \kappa(d,p,q) \in (1,\infty)$ and $C_1 \equiv C(d) < \infty$ such that for all $n \geq \max\{2N_1(x_0), 2N_2(x_0), N(\Delta)\}$ and $1/2 \leq \sigma' < \sigma \leq 1$ with $\sigma - \sigma' > n^{-\Delta}$,

$$\max_{(t,x)\in Q(\sigma' n)} |u(t,x)| \;\leq\; C_1 \left(\frac{m^{\omega}(n)}{(\sigma-\sigma')^2}\right)^{\kappa} M_{\gamma}\big(\|u\|_{2p_*,2,Q(\sigma n)}\big), \tag{4.18}$$

with $M_\gamma(s) := s^\gamma \vee s$, $p_* := p/(p-1)$ and

$$m^{\omega}(n) \;:=\; \|1 \vee \nu^{\omega}\|_{q,B(n)}\, \|1 \vee \mu^{\omega}\|_{p,B(n)} \Big(1 + \lambda \|V^{\omega}\|_{p,B(n)} + \|f\|_{p,B(n)}\Big). \tag{4.19}$$

Remark 4.2.6. In the case $f \equiv 0$, we have by Remark 4.2.2 $\tilde{\gamma} = 2\alpha$ and thereby $\gamma = \frac{\tilde{\gamma}}{2\alpha} = 1$, where γ is defined as in (4.27) and thus $M_\gamma(s) = s$.

Proof. For any $p \in (1,\infty)$, let $p_* := p/(p-1)$ and be the Hölder conjugate of p. Further, set

$$\alpha \;:=\; 1 - \frac{1}{\rho} + \frac{1}{p_*} \qquad \text{and} \qquad \alpha_k \;:=\; \alpha^k, \tag{4.20}$$

where ρ is defined in (4.1). Notice that for any $p, q \in (1,\infty]$ for which (4.17) is satisfied, $\alpha > 1$ and therefore $\alpha_k \geq 1$ for every $k \in \mathbb{N}_0$. In particular, $\alpha > 1$ implies that Lemma 4.2.4 is applicable.

For some $\Delta \in [0,1)$, let $n \geq 2(N_1(x_0) \vee N_2(x_0)) \vee N(\Delta)$, where $N(\Delta) < \infty$ is such that $n^{1-\Delta}/(\ln n)^{(\ln 2)/(\ln \alpha)} \vee \mathrm{e}^{\alpha^2} \geq 2$ for all $n \geq N(\Delta)$. Set $K := \lfloor (\ln \ln n)/(\ln \alpha) \rfloor$. In the sequel, fix some $1/2 \leq \sigma' < \sigma \leq 1$ with $\sigma - \sigma' > n^{-\Delta}$, and consider a sequence $\{Q(\sigma_k n) : k \in \mathbb{N}_0\}$ of space-time cylinders, where

$$\sigma_k \;=\; \sigma' + 2^{-k}(\sigma - \sigma') \qquad \text{and} \qquad \tau_k \;=\; 2^{-k-1}(\sigma - \sigma'), \quad k \in \mathbb{N}_0.$$

In particular, we have that $\sigma_k = \sigma_{k+1} + \tau_k$ and $\sigma_0 = \sigma$. For abbreviation we write $I_k := [t_0, t_0 + \sigma_k n^2]$, $B_k := B(x_0, \sigma_k n)$ and $Q_k := I_k \times B_k$. Note that $|I_k|/|I_{k+1}| \leq 2$ and $|B_k|/|B_{k+1}| \leq C_{\mathrm{reg}}^2 2^d$.

Let us emphasise that $B_k \subsetneq B_{k-1}$ for any $k = 1, \dots, K$, since

$$(\sigma_{k-1} - \sigma_k)n \;=\; 2^{-k}(\sigma - \sigma')n \;>\; \frac{n^{1-\Delta}}{(\ln n)^{(\ln 2)/(\ln \alpha)}} \;\geq\; 2, \qquad \forall\, k \in \{1, \dots, K\}.$$

This is important, since if $B_k = B_{k-1}$, the cutoff functions in (4.21) would not be well defined on the boundaries. Hence, we can define a sequence $\{\eta_k : k \in \mathbb{N}_0\}$ of cut-off functions in space and a sequence $\{\zeta_k \in C^\infty(\mathbb{R}) : k \in \mathbb{N}_0\}$ of smooth cut-off functions in time having the following properties as in the energy estimate (Lemma 4.2.1):

$$\begin{aligned} &\operatorname{supp} \eta_k \subset B_k, && \eta_k \equiv 1 \text{ on } B_{k+1}, && \eta_k \equiv 0 \text{ on } \partial B_k, && \big\|\nabla \eta_k\big\|_{\ell^\infty(E)} \leq \frac{1}{\tau_k n} \\ &\operatorname{supp} \zeta_k \subset I, && \zeta_k \equiv 1 \text{ on } I_{k+1}, && \zeta_k \equiv 0 \text{ on } [t_0 + \sigma_k n^2, \infty), && \|\zeta_k'\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\tau_k^2 n^2}. \end{aligned} \tag{4.21}$$

The condition on the gradient of the cutoff functions is needed in (4.25), to cancel the n^2 factor from the Sobolev inequality.

Applying (4.16), we have that

$$\begin{aligned} \big\|u^{2\alpha_k}\big\|_{\alpha p_*, \alpha, Q_{k+1}} \;&\leq\; \left(\frac{|B_k|}{|B_{k+1}|}\right)^{\frac{1}{\alpha}} \big\|(\eta_k u^{\alpha_k})^2\big\|_{\alpha p_*, \alpha, I_{k+1} \times B_k} \\ &\leq\; \left(\frac{|B_k|}{|B_{k+1}|}\right)^{\frac{1}{\alpha}} \Big(\big\|(\eta_k u^{\alpha_k})^2\big\|_{1, \infty, I_{k+1} \times B_k} \;+\; \big\|(\eta_k u^{\alpha_k})^2\big\|_{\rho, 1, I_{k+1} \times B_k}\Big). \end{aligned} \tag{4.22}$$

For the first term, we know that since $|I_k| \leq cn^2$ for some $c > 0$

$$\max_{t \in I_{k+1}} \big\|(\eta_k u_t^{\alpha_k})^2\big\|_{1, B_k} \;\leq\; c\,n^2 \frac{1}{|I_k|} \max_{t \in I_k} \Big(\zeta_k(t)\, \big\|(\eta_k\, u_t^{\alpha_k})^2\big\|_{1, B_k}\Big). \tag{4.23}$$

By the definition of the space cutoff-function (4.21), $\eta_k u_t^{\alpha_k}$ has support only on B_k. Thus, we can apply Sobolev's inequality (4.2) to obtain

$$\begin{aligned} \big\|(\eta_k u^{\alpha_k})^2\big\|_{\rho, 1, I_{k+1} \times B_k} \;&\leq\; \frac{1}{|I_{k+1}|} \int_{I_k} \zeta_k(t)\, \big\|(\eta_k u_t^{\alpha_k})^2\big\|_{\rho, B_k}\, \mathrm{d}t \\ &\leq\; \frac{C_S}{2}\, n^2\, \big\|\nu^\omega\big\|_{q, B_k} \frac{|I_k|}{|I_{k+1}|} \frac{1}{|I_k|} \int_{I_k} \zeta_k(t)\, \frac{\mathcal{E}^\omega(\eta_k u_t^{\alpha_k})}{|B_k|}\, \mathrm{d}t. \end{aligned} \tag{4.24}$$

Since $|B_k|/|B_{k+1}| \leq C_{reg}^2 2^d$ and $|I_k|/|I_{k+1}| \leq 2$, there exists a constant $\tilde{c} > 0$ such that combining (4.23), (4.24) and using the energy estimate (4.6) with $\tilde{\gamma} = \tilde{\gamma}(\alpha_k, p_*, B_k)$ as in

(4.7) in the second line, we get

$$\begin{aligned}
&\big\|u^{2\alpha_k}\big\|_{\alpha p_*,\alpha,Q_{k+1}} \\
&\leq\ \tilde{c}\, n^2\, \big\|1 \vee \nu^\omega\big\|_{q,B_k}\, \frac{1}{|I_k|}\bigg(\max_{t\in I_k}\Big(\zeta_k(t)\ \big\|(\eta_k\, u_t^{\alpha_k})^2\big\|_{1,B_k}\Big) + \int_{I_k} \zeta_k(t)\ \frac{\mathcal{E}^\omega(\eta_k u_t^{\alpha_k})}{|B_k|}\,\mathrm{d}t\bigg) \\
&\leq\ \tilde{c}\, \frac{2\alpha_k}{\tau_k^2}\, 2\Big(\big\|1 \vee \nu^\omega\big\|_{q,B_k}\ \big\|1 \vee \mu^\omega\big\|_{p,B_k}\Big(1 + \lambda\big\|V^\omega\big\|_{p,B_k} + \big\|f\big\|_{p,B_k}\Big)\Big) \\
&\qquad \times \frac{1}{|I_k|}\int_{I_k} \ \|u_t\|_{2\alpha_k p_*,B_k}^{\tilde{\gamma}}\,\mathrm{d}t \\
&\leq\ c\frac{2^{2k}\alpha_k}{(\sigma-\sigma')^2} m^\omega(n)\ \|u\|_{2\alpha_k p_*,2\alpha_k,Q_k}^{\tilde{\gamma}}
\end{aligned} \tag{4.25}$$

where we used that $2\alpha_k \geq \tilde{\gamma}$ and Jensen's inequality in the last line. Since we have $\|u^{2\alpha_k}\|_{\alpha p_*,\alpha,Q_{k+1}} = \|u\|_{2\alpha_{k+1}p_*,2\alpha_{k+1},Q_{k+1}}^{2\alpha_k}$ we obtain the following estimate:

$$\|u\|_{2\alpha_{k+1}p_*,2\alpha_{k+1},Q_{k+1}} \ \leq\ \left(c\frac{2^{2k}\alpha_k}{(\sigma-\sigma')^2}m^\omega(n)\right)^{\frac{1}{2\alpha_k}} \|u\|_{2\alpha_k p_*,2\alpha_k,Q_k}^{\gamma_k}, \tag{4.26}$$

where we define

$$\gamma_k \ :=\ \frac{\tilde{\gamma}(\alpha_k, p_*, B_k)}{2\alpha_k} \ =\ \begin{cases} 1 & \text{if } \|u_t\|_{2\alpha_k p_*,B_k} \geq 1, \\ 1 - \frac{1}{2\alpha_k} & \text{otherwise.} \end{cases} \tag{4.27}$$

To initiate the iteration step, we need a similar upper bound for the time-maximum norm. Using that Norms are positive in the second and the estimate for (4.22) in the last step, we obtain:

$$\begin{aligned}
\big\|u^{2\alpha_k}\big\|_{1,\infty,Q_{k+1}} \ &\leq\ \frac{|B_k|}{|B_{k+1}|}\ \big\|(\eta_k u^{\alpha_k})^2\big\|_{1,\infty,I_{k+1}\times B_k} \\
&\leq\ \frac{|B_k|}{|B_{k+1}|}\ \Big(\big\|(\eta_k u^{\alpha_k})^2\big\|_{1,\infty,I_{k+1}\times B_k} + \big\|(\eta_k u^{\alpha_k})^2\big\|_{\rho,1,I_{k+1}\times B_k}\Big) \\
&\leq\ c\frac{2^{2k}\alpha_k}{(\sigma-\sigma')^2}m^\omega(n)\ \|u\|_{2\alpha_k p_*,2\alpha_k,Q_k}^{\tilde{\gamma}},
\end{aligned} \tag{4.28}$$

Now, we are able to initialize the Moser iteration by observing that $|B_K|^{1/2\alpha_{K-1}} \leq c < \infty$ uniformly in n. Hence, an application of (4.28) yields

$$\begin{aligned}
&\max_{(t,x)\in Q(\sigma' n)} |u(t,x)| \\
&\leq\ \max_{(t,x)\in Q_K} |u(t,x)| \ \leq\ |B_K|^{\frac{1}{2\alpha_{K-1}}}\ \big\|\tilde{u}^{2\alpha_{K-1}}\big\|_{1,\infty,Q_K}^{\frac{1}{2\alpha_{K-1}}} \\
&\leq\ \left(c\frac{2^{2(K-1)}\alpha_{K-1}}{(\sigma-\sigma')^2}m^\omega(n)\right)^{\frac{1}{2\alpha_{K-1}}}\ \|u\|_{2\alpha_{K-1}p_*,2\alpha_{K-1},Q_{K-1}}^{\gamma_{K-1}}.
\end{aligned}$$

Due to the discreteness of the space, it is sufficient to stop the iteration after a finite number of steps. Thus, by iterating the inequality (4.26) $K-1$ times, we get

$$\max_{(t,x)\in Q(\sigma' n)} |u(t,x)| \\ \leq C \left(\frac{m^\omega(n)}{(\sigma-\sigma')^2} \right)^{\sum_{k=0}^{K-1} \frac{1}{2\alpha_k} \prod_{i=k}^{K-1} \gamma_{i+1}} \|u\|_{2\alpha_0 p_*, 2\alpha_0, Q_0}^{\prod_{k=0}^{K-1} \gamma_k},$$

for some $C < \infty$ independent of k. Notice, that since $\gamma_k \geq 1 - \frac{1}{2\alpha_k}$ and $\sum_{k=0}^\infty \frac{1}{2\alpha_k} < \infty$ we have that

$$0 < \prod_{k=1}^{\infty} 1 - \frac{1}{2\alpha_k} =: \gamma \leq 1 \quad \text{and} \quad \sum_{k=0}^{K-1} \frac{1}{2\alpha_k} \prod_{i=k}^{K-1} \gamma_{i+1} \leq \sum_{k=0}^{K-1} \frac{1}{2\alpha_k} =: \kappa < \infty, \tag{4.29}$$

where the infinite product is not zero, since the series over the $1/2\alpha_k$ converges. Together, this yields the claim in its final form:

$$\max_{(t,x)\in Q(\sigma' n)} |u(t,x)| \leq C_1 \left(\frac{m^\omega(n)}{(\sigma-\sigma')^2} \right)^\kappa M_\gamma\big(\|u\|_{2p_*,2,Q(\sigma n)}\big),$$

since $\alpha_0 = 1$. □

Throughout the following proofs, we will continuously exploit the following properties of the function M_γ:

Lemma 4.2.7. *For any $\gamma \in (0,1]$, $\theta \geq 0$ and $s,t \geq 0$, the function $M_\gamma(s) := s^\gamma \vee s$ fulfills the following properties:*

$$M_\gamma(st) \leq M_\gamma(s) M_\gamma(t), \quad M_\gamma(s+t) \leq M_\gamma(s) + M_\gamma(t) \quad \textit{and} \quad M_\gamma(s^\theta) \leq M_\gamma(s)^\theta. \tag{4.30}$$

Proof. The short proof mainly relies on case distinction:

1. To prove that M_γ is submultiplicative, we have to distinguish four different cases:

   (i) $s \geq 1, t \geq 1$: Since $st \geq 1$ and $\gamma \in (0,1]$ we get $M_\gamma(st) = st$. On the other side, we have $M_\gamma(s) = s$ and $M_\gamma(t) = t$. Therefore we even have an equality in this case.

   (ii) $s \geq 1, t < 1$: Since $\gamma \in (0,1]$, we have $M_\gamma(s) = s$ and $M_\gamma(t) = t^\gamma$. First consider the case, where $st \geq 1$, then $M_\gamma(st) = st$ and thus $M_\gamma(st) = st \leq st^\gamma = M_\gamma(s)M_\gamma(t)$. If on the other side $st < 1$, then $M_\gamma(st) = (st)^\gamma$ and thus $M_\gamma(st) = s^\gamma t^\gamma \leq st^\gamma = M_\gamma(s)M_\gamma(t)$. Here we used that since $\gamma \in (0,1]$ we have $s^\gamma \leq s$ and $t \leq t^\gamma$.

   (iii) $s < 1, t \geq 1$: Follows by switching the roles of s and t in the above case.

   (iv) $s < 1, t < 1$: Since $st < 1$, $M_\gamma(st) = (st)^\gamma$, which yields $M_\gamma(st) = s^\gamma t^\gamma = M_\gamma(s)M_\gamma(t)$.

2. Using Hölders inequality applied to the standard p-norms, we obtain since $\gamma \in (0,1]$:

   $$(s+t)^\gamma = (s^{\gamma \frac{1}{\gamma}} + t^{\gamma \frac{1}{\gamma}})^\gamma = \|(s^\gamma, t^\gamma)\|_{\frac{1}{\gamma}} \leq \|(s^\gamma, t^\gamma)\|_1 = s^\gamma + t^\gamma.$$

   By distinguishing cases as done above, we get $M_\gamma(s+t) \leq M_\gamma(s) + M_\gamma(t)$.

3. Follows from the submultiplicativity.

□

In the upcoming section, we will require a similar maximal inequality for solutions of the following elliptic Poisson equations with Dirichlet boundary conditions, corresponding to both the Laplace and Schrödinger operators, respectively:

$$\begin{cases} \mathcal{L}^\omega_{\lambda,V} u \geq -\frac{1}{n^2} f & \text{on } B(n), \\ u = 0 & \text{on } B(n)^c \end{cases} \qquad \begin{cases} \mathcal{L}^\omega u \geq -\frac{1}{n^2} f & \text{on } B(n), \\ u = 0 & \text{on } B(n)^c. \end{cases} \tag{4.31}$$

In the proof of the following theorem, we will observe that the boundary condition simplifies the energy estimate, as $\operatorname{supp} u \subset B(n)$ enables us to apply Sobolev's inequality directly without the use of a cutoff function η. Another distinction from the Moser iteration for the parabolic equation is that we now iterate infinitely many steps. While we could employ the same approach as previously, given the discrete nature of the space, the method below aligns more closely with Moser's original idea in the continuous setting, where the iteration does not conclude after a finite number of steps.

Proposition 4.2.8. *Let $u \geq 0$ be a solution to (4.31), i.e. $\mathcal{L}^\omega_{\lambda,V} u \geq -\frac{1}{n^2} f$ on $B(n)$ and $u = 0$ on $B(n)^c$. Then, for any $p, q \in (1, \infty]$ satisfying*

$$\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$$

there exist $C_2 < \infty$ such that for all $n \geq \max\{N_1(x_0), N_2(x_0)\}$ and $\beta \in [2p_*, \infty]$

$$\|u\|_{\beta,B(n)} \leq C_2 \Big(\|\nu^\omega\|_{q,B(n)} \Big(\|f\|_{p,B(n)} + \lambda \|V^\omega\|_{p,B(n)} \Big) \Big)^\kappa M_\gamma(\|u\|_{2p_*,B(n)}), \tag{4.32}$$

and in the special case of $\mathcal{L}^\omega u \geq -\frac{1}{n^2} f$, we get

$$\|u\|_{\beta,B(n)} \leq C_2 \Big(\|\nu^\omega\|_{q,B(n)} \, \|f\|_{p,B(n)} \Big)^\kappa \|u\|^\gamma_{2p_*,B(n)}, \tag{4.33}$$

where $\kappa < \infty$ and $\gamma \in (0, 1]$ are defined as in (4.29) and only depend on α as in (4.34).

Proof. We will only state the proof for the $\mathcal{L}^\omega_{\lambda,V}$ operator, since (4.33) follows analogous.

First, we again want to find an estimate for the Dirichlet energy. For $\alpha > 0$, it holds by (2.11) that

$$\begin{aligned} \mathcal{E}^\omega(u^\alpha) &= \langle \nabla u^\alpha, \omega \nabla u^\alpha \rangle_{\ell^2(E)} \\ &\leq \frac{\alpha^2}{2\alpha - 1} \langle \nabla u^{2\alpha-1}, \omega \nabla u \rangle_{\ell^2(E)} \\ &= \frac{\alpha^2}{2\alpha - 1} \langle u^{2\alpha-1}, -\mathcal{L}^\omega u \rangle_{\ell^2(V)} \\ &\leq \frac{\alpha^2}{2\alpha - 1} \langle u^{2\alpha-1}, \frac{1}{n^2} f + \frac{\lambda}{n^2} V^\omega u \rangle_{\ell^2(V)} \\ &\leq \frac{\alpha^2}{2\alpha - 1} \frac{1}{n^2} |B(n)| \Big(\|f\|_{p,B(n)} \|u^{2\alpha-1}\|_{p_*,B(n)} + \lambda \|V^\omega\|_{p,B(n)} \|u^{2\alpha}\|_{p_*,B(n)} \Big), \end{aligned}$$

where we used integration by parts in the first and third , [2, Lemma A.1-(ii)] in the second and that $\mathcal{L}^\omega u + V^\omega u \geq -\frac{1}{n^2} f$ in the fourth step.

Since u only has support on $B(n)$, using a cutoff-function is redundant and the Sobolev inequality (4.2) simplifies to

$$\left\| u^{2\alpha} \right\|_{\rho,B(n)} \leq \frac{C_{\mathrm{S}}}{2} \left| B(n) \right|^{\frac{2}{d}} \left\| \nu^\omega \right\|_{q,B(n)} \frac{\mathcal{E}_\omega(u^\alpha)}{|B(n)|}.$$

Now, we choose

$$\alpha := \frac{\rho}{p_*} > 1 \qquad \text{and} \qquad \alpha_k := \alpha^k. \tag{4.34}$$

Using volume regularity (2.1.1) and $\alpha p_* = \rho$, we obtain

$$\begin{aligned} \left\| u^{2\alpha_k} \right\|_{\alpha p_*,B(n)} &\leq c\, n^2 \left\| \nu^\omega \right\|_{q,B(n)} \frac{\mathcal{E}_\omega(u^{\alpha_k})}{|B(n)|} \\ &\leq c \frac{\alpha_k^2}{2\alpha_k - 1} \left\| \nu^\omega \right\|_{q,B(n)} \\ &\quad \times \left(\left\| f \right\|_{p,B(n)} \left\| u^{2\alpha_k - 1} \right\|_{p_*,B(n)} + \lambda \left\| V^\omega \right\|_{p,B(n)} \left\| u^{2\alpha_k} \right\|_{p_*,B(n)} \right) \end{aligned}$$

Rearranging the terms and using Jensen's Inequality yields:

$$\left\| u \right\|_{2\alpha_{k+1} p_*,B(n)} \leq \left(c \frac{\alpha_k^2}{2\alpha_k - 1} \left\| \nu^\omega \right\|_{q,B(n)} \left(\left\| f \right\|_{p,B(n)} + \lambda \left\| V^\omega \right\|_{p,B(n)} \right) \right)^{\frac{1}{2\alpha_k}} \left\| u \right\|_{2\alpha_k p_*,B(n)}^{\gamma_k},$$

where γ_k is defined as in (4.27). This is sufficient to initialize the Moser iteration:

$$\begin{aligned} \max_{x \in B(n)} u(x) &= \lim_{K \to \infty} \left\| u \right\|_{2\alpha_{K+1} p_*,B(n)} \\ &\leq \left(C \left\| \nu^\omega \right\|_{q,B(n)} \left(\left\| f \right\|_{p,B(n)} + \lambda \left\| V^\omega \right\|_{p,B(n)} \right) \right)^{\kappa} M_\gamma(\left\| u \right\|_{2p_*,B(n)}), \end{aligned} \tag{4.35}$$

where γ and κ are defined as in (4.29). Notice, that since $\left(\frac{\alpha_k^2}{2\alpha_k - 1} \right)^{1/(2\alpha_k)} \leq \alpha_k^{1/(2\alpha_k)} \leq e^{e^{-1}/2}$ the existence of the constant $C_2 < \infty$ follows.

Since for $\|u\|_{\beta,B(n)} \leq \max_{x \in B(n)} u(x)$ for $\beta \in [2p_*, \infty]$, the claim follows. □

Remark 4.2.9. Of course, the inequality also holds for $\beta \in (0, 2p_*)$. But in this case, we can simply use Jensen's Inequality to get an even better estimate.

In the proof of the Hölder continuity of $\mathcal{L}^\omega_{\lambda,V}$-harmonic functions, we will need an upper bound for solutions v of the following boundary value problem:

$$\begin{cases} \mathcal{L}^\omega v = -\frac{\lambda}{n^2} \left| V^\omega(y) \delta_{B_1} \right| & \text{on } B(n), \\ v = 0 & \text{on } B(n)^c, \end{cases} \tag{4.36}$$

where $B_1 \subset B(n)$.

This can be established by applying (4.33) with right hand side $f = \lambda \left| V^\omega(y) \delta_{B_1} \right|$:

Proposition 4.2.10. *Let $v \geq 0$ be a solution to (4.36), i.e. $\mathcal{L}^\omega v = -\frac{\lambda}{n^2}\left|V^\omega(y)\delta_{B_1}\right|$ on $B(n)$ and $v = 0$ on $B(n)^c$. Then, for any $p, q \in (1, \infty]$ satisfying*

$$\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$$

there exist $C_3 < \infty$ such that for all $n \geq \max\{N_1(x_0), N_2(x_0)\}$

$$\max_{x \in B(n)} v(x) \;\leq\; C_3 \left(\lambda \left\|V^\omega\right\|_{p,B(n)} \left\|\nu^\omega\right\|_{q,B(n)}\right)^{\frac{2\kappa+\gamma}{2-\gamma}} \left(\frac{|B_1|}{|B(n)|}\right)^{\frac{\gamma}{2-\gamma}\frac{1}{p_*}}, \tag{4.37}$$

where $\kappa < \infty$ and $\gamma \in (0, 1]$ are defined as in (4.29) and only depend on α as in (4.34).

Proof. We apply the maximal inequality for elliptic Poisson equations (4.33) with right side $f = \lambda|V^\omega \delta_{B_1}|$ and $\beta = \infty$. Since $\rho > p_*$ we can apply Jensen's inequality and, together with Sobolev's inequality we get:

$$\begin{aligned}
\max_{x \in B(n)} v(x) \;&\leq\; c \left(\lambda \left\|V^\omega \delta_{B_1}\right\|_{p,B(n)} \left\|\nu^\omega\right\|_{q,B(n)}\right)^{\kappa} \left\|v\right\|^{\gamma}_{2p_*,B(n)} \\
&\leq\; c \left(\lambda \left\|V^\omega\right\|_{p,B(n)} \left\|\nu^\omega\right\|_{q,B(n)}\right)^{\kappa} \left\|v^2\right\|^{\frac{\gamma}{2}}_{\rho,B(n)} \\
&\leq\; c \left(\lambda \left\|V^\omega\right\|_{p,B(n)} \left\|\nu^\omega\right\|_{q,B(n)}\right)^{\kappa} \left(n^2 \left\|\nu^\omega\right\|_{q,B(n)} \frac{\mathcal{E}^\omega(v)}{|B(n)|}\right)^{\frac{\gamma}{2}}
\end{aligned}$$

We can easily find an estimate for the energy using Hölders inequality

$$\begin{aligned}
\mathcal{E}^\omega(v) \;=\; \left\langle v, -\mathcal{L}^\omega v\right\rangle_{\ell^2(V)} \;&=\; \left\langle v, \frac{\lambda}{n^2}\left|V^\omega \delta_{B_1}\right|\right\rangle_{\ell^2(V)} \\
&\leq\; \frac{\lambda}{n^2}|B_1| \left\|V^\omega\right\|_{p,B_1} \left\|v\right\|_{p_*,B_1} \\
&\leq\; \frac{\lambda}{n^2}|B_1| \left(\frac{|B(n)|}{|B_1|}\right)^{\frac{1}{p}} \left\|V^\omega\right\|_{p,B(n)} \max_{x \in B(n)} v(x),
\end{aligned}$$

which together yields:

$$\max_{x \in B(n)} v(x) \;\leq\; c \left(\lambda \left\|V^\omega\right\|_{p,B(n)} \left\|\nu^\omega\right\|_{q,B(n)}\right)^{\kappa+\frac{\gamma}{2}} \left(\frac{|B_1|}{|B(n)|}\right)^{\frac{\gamma}{2}\left(1-\frac{1}{p}\right)} \max_{x \in B(n)} v(x)^{\frac{\gamma}{2}}.$$

Now rearranging the terms and using that $1 - \frac{1}{p} = \frac{1}{p_*} > 0$ we obtain

$$\max_{x \in B(n)} v(x) \;\leq\; c \left(\lambda \left\|V^\omega\right\|_{p,B(n)} \left\|\nu^\omega\right\|_{q,B(n)}\right)^{\frac{2\kappa+\gamma}{2-\gamma}} \left(\frac{|B_1|}{|B(n)|}\right)^{\frac{\gamma}{2-\gamma}\frac{1}{p_*}}.$$

Notice for the second exponent that $\frac{\gamma}{2-\gamma}\frac{1}{p_*} \in (0, 1)$, since $\gamma \in (0, 1]$ and $p_* > 1$. □

Andres, Deuschel, and Slowik established the following maximal inequality for $\mathcal{L}^\omega$-harmonic functions in [2], which we will utilize in the next chapter to derive an upper bound for the Green's function. We will extend this result to the Schrödinger case, i.e. $\mathcal{L}^\omega_{\lambda,V}$-harmonic functions. The proof involves iterating (4.18) further in the context of the elliptic and harmonic special case.

Proposition 4.2.11. *Consider a non-negative u such that $\mathcal{L}^{\omega}_{\lambda,V}\, u \geq -\frac{1}{n^2} f$ on $B(n)$. Then, for all $\beta \in (0,\infty)$ and $1/2 \leq \sigma' < \sigma \leq 1$, there exists $C_4 = C(\beta) < \infty$ such that*

$$\max_{x \in B(\sigma' n)} u(x) \;\leq\; C_4\, M_\gamma \left(\frac{m^{\omega}(n)}{(\sigma - \sigma')^2} \right)^{\kappa'} M_\gamma\big(\|u\|_{\beta, B(\sigma n)} \big) \tag{4.38}$$

with $m^{\omega}(n)$ defined as in (4.19) and in the special case of $\mathcal{L}^{\omega} u = 0$, we have

$$\max_{x \in B(\sigma' n)} u(x) \;\leq\; C_4 \left(\frac{1 \vee \|\mu^{\omega}\|_{p,B(n)}\, \|\nu^{\omega}\|_{q,B(n)}}{(\sigma - \sigma')^2} \right)^{\kappa'} \|u\|_{\beta, B(\sigma n)} \tag{4.39}$$

where $\kappa' = \big(1 \vee \frac{2\alpha p_*}{\beta}\big)\kappa > 1$.

Proof. We will again only state the proof for the $\mathcal{L}^{\omega}_{\lambda,V}$ operator, since (4.39) follows analogous.

First, notice that $\mathcal{L}^{\omega}_{\lambda,V}\, u \geq -\frac{1}{n^2} f$ is a special case of (4.5), by choosing $u(t,x) = u(x)$, the elliptic version without time component. Thus, for all $1/2 \leq \sigma' < \sigma \leq 1$, the maximal inequality (4.18) reads as:

$$\max_{x \in B(\sigma' n)} u(x) \;\leq\; C \left(\frac{m^{\omega}(n)}{(\sigma - \sigma')^2} \right)^{\kappa} M_\gamma\big(\|u\|_{2\alpha p_*, B(\sigma n)} \big). \tag{4.40}$$

For any $\beta \geq 2\alpha p_*$, we get by Jensen's inequality $\|u\|_{2\alpha p_*, B(\sigma n)} \leq \|u\|_{\beta, B(\sigma n)}$. Thus, in view of (4.40), the claim follows in this case.

It remains to consider the case $\beta \in (0, 2\alpha p_*)$. For arbitrary but fixed $1/2 \leq \sigma' < \sigma \leq 1$, set $\sigma_k = \sigma - 2^{-k}(\sigma - \sigma')$ for any $k \in \mathbb{N}_0$. By Hölder's inequality, we have for any $\beta \in (0, 2\alpha p_*)$

$$\begin{aligned} \|u\|_{2\alpha p_*, B(\sigma_k n)} \;&\leq\; \|u\|^{\theta}_{\theta 2\alpha p_*, B(\sigma_k n)}\, \|u\|^{1-\theta}_{\infty, B(\sigma_k n)} \\ &\leq\; \|u\|^{\theta}_{\beta, B(\sigma_k n)}\, \|u\|^{1-\theta}_{\infty, B(\sigma_k n)}, \end{aligned}$$

where $\theta = \frac{\beta}{2\alpha p_*}$. Thus, by (4.30):

$$M_\gamma\big(\|u\|_{2\alpha p_*, B(\sigma_k n)} \big) \;\leq\; M_\gamma\big(\|u\|_{\beta, B(\sigma_k n)} \big)^{\theta}\, M_\gamma\big(\|u\|_{\infty, B(\sigma_k n)} \big)^{1-\theta}.$$

Hence, in view of (4.40) and the volume regularity which implies that $|B(\sigma n)|/|B(\sigma' n)| \leq C_{\mathrm{reg}}^2 2^d$, we obtain

$$\max_{x \in B(\sigma_{k-1} n)} u(x) \;\leq\; 2^{2\kappa k} M_\gamma(J)\, M_\gamma\big(\|u\|_{\beta, B(\sigma_k n)} \big)^{\theta}\, M_\gamma\big(\|u\|_{\infty, B(\sigma_k n)} \big)^{1-\theta},$$

where we introduced for the readers convenience

$$J = \left(\frac{m_{\omega}(n)}{(\sigma - \sigma')^2} \right)^{\kappa}.$$

Iterating the equation i-times, gives us

$$\begin{aligned} &\max_{x \in B(\sigma_{k-1} n)} u(x) \\ &\leq\; 2^{2\kappa \sum_{k=0}^{i-1} (k+1)(1-\theta)^k} \Big(M_\gamma(J)\, M_\gamma\big(\|u\|_{\beta, B(\sigma_k n)} \big)^{\theta} \Big)^{\sum_{k=0}^{i-1} (1-\theta)^k} M_\gamma\big(\|u\|_{\infty, B(\sigma_k n)} \big)^{(1-\theta)^i}. \end{aligned}$$

In the limit, we thus get (4.38), since by convergence of geometric sums and $\theta \in (0,1)$

$$\sum_{k=0}^{\infty}(k+1)(1-\theta)^k = \frac{1}{\theta^2} \qquad \text{and} \qquad \sum_{k=0}^{\infty}(1-\theta)^k = \frac{1}{\theta}.$$

□

4.3 Bounds for the Laplace killed Green's function

In the proof of our first main result, the Hölder continuity of $\mathcal{L}^{\omega}_{\lambda,V}$ - harmonic functions, the control of the killed Green's function will play a crucial role. In this section we will derive an estimate on balls $B(z,r)$ with center $z \in B(n)$ and $x \notin B(z,r)$.

Proposition 4.3.1. *Let $g^{\omega}_{B(n)}(x,\cdot)$ be the killed Green's function as in (2.14). Further, let $z \in B(n)$ and $r < n$ be such that there exists $b > 1$ with $x \notin B(z,br)$. Notice, that overlapping balls, i.e. $B(z,r) \not\subset B(n)$ are possible. Then there exists $C_5 < \infty$ such that*

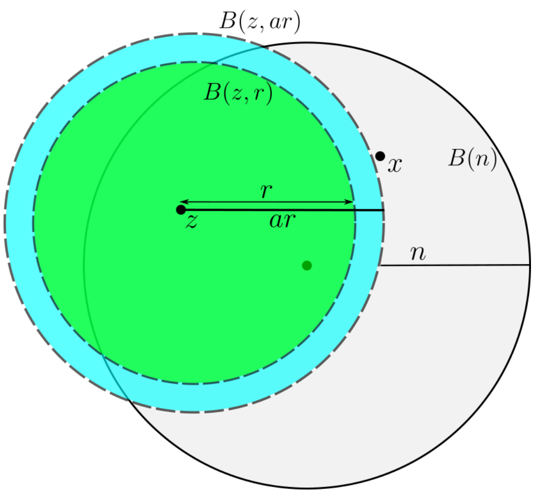
$$\max_{y \in B(z,r)} g^{\omega}_{B(n)}(x,y) \le C_5\, \mathcal{A}^{\omega}(2n)^{\kappa' \frac{2\kappa+\gamma}{2-\gamma}} n^{2-d}, \tag{4.41}$$

where

$$\mathcal{A}^{\omega}(n) := 1 \vee \left\|\mu^{\omega}\right\|_{p,B(n)} \left\|\nu^{\omega}\right\|_{q,B(n)} \tag{4.42}$$

is a random constant depending on $n \in \mathbb{N}$.

Proof. First, we want to cover $B := B(z,r)$ by a slightly larger ball, to then apply (4.39). Therefore, choose $a := \sup\{b > 1 : x \notin B(z,br)\}$ and define $\tilde{B} := B(z,ar)$ the largest ball with center z covering B and not containing x. Notice that for any $y \in B$, the radius of the cover ball $\tilde{B}$ is $ar < n - r_* < n$ and thus $\tilde{B} \subset B(2n)$, where we defined $r_* := \min\{n \in \mathbb{N} : x \in B(n)\}$ as in the proof of Theorem 4.4.1.



Next, we have to show that $g^{\omega}_{B(n)}$ is $\mathcal{L}^{\omega}$-subharmonic on $\tilde{B} \subset B(2n)$. First, consider points $y \in \tilde{B} \cap B(n)$. Then, since $x \notin \tilde{B}$ and $y \in \tilde{B}$, we have $\mathcal{L}^{\omega} g^{\omega}_{B(n)}(x,y) = -\delta_x(y) = 0$. In the second case, where $y \in \tilde{B} \cap (B(2n) \setminus B(n))$, we are lucky, because using $g^\omega_{B(n)}(x,y) = 0$ for $y \in B(n)^c$ in the first and by (2.17) $g^\omega_{B(n)}(x,\cdot) \geq 0$ in the second step, we get

$$\begin{aligned}\big(\mathcal{L}^\omega g^\omega_{B(n)}(x,\cdot)\big)(y) \;&=\; \sum_{y' \sim y} \omega(y,y')\big(g^\omega_{B(n)}(x,y') - g^\omega_{B(n)}(x,y)\big)\\ &=\; \sum_{y' \sim y} \omega(y,y') g^\omega_{B(n)}(x,y') \;\geq\; 0.\end{aligned}$$

Thus, indeed $\mathcal{L}^\omega g^\omega_{B(n)}(x,\cdot) \geq 0$ on $\tilde{B}$ and we can apply (4.39), choosing $\sigma' = a^{-1}$, $\sigma = 1$ and $\beta = 1$:

$$\max_{y \in B} g^\omega_{B(n)}(x,y) \;\leq\; \left(C \left(\frac{a}{a-1}\right)^2 \left(1 \vee \left\|\mu^\omega\right\|_{p,B(2n)} \left\|\nu^\omega\right\|_{q,B(2n)}\right)\right)^{\kappa'} \left\|g^\omega_{B(n)}(x,\cdot)\right\|_{1,\tilde{B}} \tag{4.43}$$

Since $g^\omega_{B(n)}(x,\cdot) = 0$ on $B(n)^c$ we get by volume regularity (2.1.1) for some $\tilde{c} < \infty$

$$\begin{aligned}\left\|g^\omega_{B(n)}(x,\cdot)\right\|_{1,\tilde{B}} \;&=\; \frac{1}{|\tilde{B}|}\left(\sum_{y \in \tilde{B}} g^\omega_{B(n)}(x,y) - \sum_{y \in B(n)^c} g^\omega_{B(n)}(x,y)\right)\\ &=\; \frac{1}{|\tilde{B}|}\left(\sum_{y \in \tilde{B} \cap B(n)} g^\omega_{B(n)}(x,y)\right)\\ &\leq\; \frac{|B(n)|}{|\tilde{B}|} n^2 \left\| \frac{1}{n^2} g^\omega_{B(n)}(x,\cdot)\right\|_{1,B(n)}\\ &\leq\; \tilde{c}\,\frac{n^2}{|B(n)|} w(x).\end{aligned}$$

As a next step, we notice that by reversibility $w(x) := \sum_{y \in B(n)} \frac{1}{n^2}\, g^\omega_{B(n)}(y,x) \geq 0$ is a solution to the elliptic Poisson boundary value problem (4.31) with right side $f = \delta_{B(n)}(\cdot)$, since $w(x) = 0$ for $x \in B(n)^c$ and for $x \in B(n)$

$$\mathcal{L}^\omega w(x) \;=\; \sum_{y \in B(n)} \mathcal{L}^\omega \frac{1}{n^2}\, g^\omega_{B(n)}(x,y) \;=\; \sum_{y \in B(n)} -\frac{1}{n^2}\delta_y(x) \;=\; -\frac{1}{n^2}\delta_{B(n)}(x),$$

i.e. $\mathcal{L}^\omega w = -\frac{1}{n^2}\delta_{B(n)}(\cdot)$ on $B(n)$ and $w = 0$ else. Thus, we can apply (4.32) with $\beta = \infty$ and obtain a bound for the maximum of w analogous to the proof of Proposition 4.2.10:

$$\begin{aligned}\max_{x \in B(n)} w(x) \;&\leq\; c\left\|\nu^\omega\right\|^\kappa_{q,B(n)} \left\|w^2\right\|^{\frac{\gamma}{2}}_{\rho,B(n)}\\ &\leq\; c\left\|\nu^\omega\right\|^\kappa_{q,B(n)} \left(n^2 \left\|\nu^\omega\right\|_{q,B(n)} \frac{\mathcal{E}^\omega(w)}{|B(n)|}\right)^{\frac{\gamma}{2}}\\ &\leq\; c\left\|\nu^\omega\right\|^\kappa_{q,B(n)} \left(\left\|\nu^\omega\right\|_{q,B(n)} \left\|w\right\|_{1,B(n)}\right)^{\frac{\gamma}{2}}\\ &\leq\; c\left\|\nu^\omega\right\|^{\kappa+\frac{\gamma}{2}}_{q,B(n)} \max_{x \in B(n)} w(x)^{\frac{\gamma}{2}},\end{aligned}$$

where we used (4.2.8) in the first, Sobolev's inequality in the second and that $\mathcal{E}^\omega(w) = \langle w, -\mathcal{L}^\omega w\rangle_{\ell^2(V)} \leq \langle w, \frac{1}{n^2}\delta_{B(n)}(\cdot)\rangle_{\ell^2(V)} \leq \frac{1}{n^2}\,|B(n)| \|w\|_{1,B(n)}$ in the third step. Rearranging the terms gives for some $c < \infty$:

$$\max_{x \in B(n)} w(x) \;\leq\; c \left\| \nu^\omega \right\|_{q,B(n)}^{\frac{2\kappa+\gamma}{2-\gamma}}. \tag{4.44}$$

Combining (4.43) and (4.44), we obtain the following bound for the killed Green's function:

$$\begin{aligned}
\max_{y \in B} g^\omega_{B(n)}(x,y) \;&\leq\; c \left(1 \vee \left\|\mu^\omega\right\|_{p,B(2n)} \left\|\nu^\omega\right\|_{q,B(2n)}\right)^{\kappa'} \frac{n^2}{|B(n)|} \max_{x \in B(n)} w(x) \\
&\leq c \left(1 \vee \left\|\mu^\omega\right\|_{p,B(2n)} \left\|\nu^\omega\right\|_{q,B(2n)}\right)^{\kappa'} \left\|\nu^\omega\right\|_{q,B(2n)}^{\frac{2\kappa+\gamma}{2-\gamma}} \frac{n^2}{|B(n)|} \\
&\leq C_5 \, \mathcal{A}^\omega(2n)^{\kappa' \frac{2\kappa+\gamma}{2-\gamma}} n^{2-d}.
\end{aligned}$$

□

Remark 4.3.2. In the application in the following proof of Theorem 2.3.5, we will need to bound the killed Green's function $g^\omega_{B(n)}(x, \cdot)$ on a Torus $B_2 \subset B(n)$ with $x_0, x \notin B_2$, i.e. x_0, x will be located in the hole in the middle. To apply the latter result, we will use a finite cover $\{B_2^i\}_{i=1,\ldots,N} := \{B(y_i, r_i)\}_{i=1,\ldots,N}$ of balls with center $y_i \in B_2$ and radius r_i chosen such that there exists positive $a_i := \sup\{a > 1 : x \notin B(y_i, ar_i)\} > 1$, i.e.

$$B_2 \subset \bigcup_{i=1}^N B_2^i.$$

A more detailed discussion on the existence of such a sequence can be found in [12, Lemma 2.10].

4.4 Hölder Continuity of Elliptic Schrödinger Operators

We now will prove the Hölder Continuity for solutions of the Schrödinger equation $\mathcal{L}^\omega_{\lambda,V} u = 0$. Our approach will be similar to the 1992 result by Bramanti [9] in the continuous setting. The trick is to rewrite our boundary problem and then use the continuity of solutions to the Laplace equation $\mathcal{L}^\omega u = 0$, which was proven by Andres, Deuschel and Slowik in [2].

Let $u > 0$ be a solution to $\mathcal{L}^\omega_{\lambda,V} u = 0$ on $B(n) := B(x_0, n)$ and $u = 0$ on $B(n)^c$. Then for u_1, u_2 non negative such that

$$\begin{cases} \mathcal{L}^\omega u_1 = 0 & \text{on } B(n), \\ u_1 = u & \text{on } B(n)^c \end{cases} \qquad\qquad \begin{cases} \mathcal{L}^\omega u_2 = -\frac{\lambda}{n^2} V^\omega u & \text{on } B(n), \\ u_2 = 0 & \text{on } B(n)^c \end{cases}$$

we have

$$u = u_1 + u_2. \tag{4.45}$$

This can be easily seen, since $\mathcal{L}^\omega_{\lambda,V} u = \mathcal{L}^\omega_{\lambda,V}(u_1+u_2) = \mathcal{L}^\omega u_1 + \frac{\lambda}{n^2} V^\omega u_1 + \mathcal{L}^\omega u_2 + \frac{\lambda}{n^2} V^\omega u_2 = \frac{\lambda}{n^2} V^\omega (u_1+u_2) - \frac{\lambda}{n^2} V^\omega u = 0$ on $B(n)$ and $u = u_1 + u_2$ on $B(n)^c$.

By [2, Proposition 3.8], we already know that u_1 is Hölder continuous on $B(n)$:

Theorem 4.4.1. *Let $x_0 \in V$ and C^{ω}_{EH} be as in [2, Proposition 3.8]. Further, let $\delta \in (0,1)$ and suppose that $u > 0$ is a $\mathcal{L}^{\omega}$-harmonic function, i.e. $\mathcal{L}^{\omega} u = 0$ on $B(x_0, n)$ for some $n \geq 2\lfloor \delta n \rfloor$. Then, for any $x \in B(x_0, \lfloor \delta n \rfloor) \geq s(x_0)$,*

$$\big|u(x) - u(x_0)\big| \;\leq\; \tilde{C}^{\omega}_H \cdot \delta^{\theta} \max_{B(x_0, n/2)} u,$$

where $\theta = \ln\big(2C^{\omega}_{\mathrm{EH}}/(2C^{\omega}_{\mathrm{EH}} - 1)\big)/\ln 2$ and $\tilde{C}^{\omega}_H := \tilde{C}_H(\|\mu^{\omega}\|_{p,B(x_0,n)}, \|\nu^{\omega}\|_{q,B(x_0,n)})$.

Proof. The statement follows directly from [2, Proposition 3.8], by chosing $R = \lfloor \delta n \rfloor$, $R_0 = n$, $x_1 = x_0$ and $x_2 = x$. □

Now, we have all ingredients, to prove the Hölder continuity of $\mathcal{L}^{\omega}_{\lambda,V}$-harmonic functions:

Proof of Theorem 2.3.5. First, notice that by the triangle inequality and by (4.45)

$$\big|u(x) - u(x_0)\big| \;\leq\; \big|u_1(x) - u_1(x_0)\big| + \big|u_2(x) - u_2(x_0)\big|.$$

Since by Theorem 4.4.1, u_1 is Hölder continuous, it suffices to prove the Hölder continuity for u_2.

We rewrite u_2 using the killed Green's function as defined in (2.14):

$$\begin{aligned}
u_2(x) \;=\; \big\langle u_2, \delta_x(\cdot)\big\rangle_{\ell^2(V)} \;&=\; \big\langle u_2, -\mathcal{L}^{\omega}\, g^{\omega}_{B(n)}(x,\cdot)\big\rangle_{\ell^2(V)} \\
&= \big\langle \mathcal{L}^{\omega}\, u_2, -g^{\omega}_{B(n)}(x,\cdot)\big\rangle_{\ell^2(V)} \\
&= \big\langle -\frac{\lambda}{n^2} V^{\omega} u, -g^{\omega}_{B(n)}(x,\cdot)\big\rangle_{\ell^2(V)} \\
&= \sum_{y \in B(n)} g^{\omega}_{B(n)}(x,y)\, \frac{\lambda}{n^2} V^{\omega}(y) u(y)\,,
\end{aligned}$$

where we used in the second line that by (2.7) $\mathcal{L}^{\omega} = \mathcal{L}^{\omega *}$ is a self adjoint operator. Further, we get:

$$\begin{aligned}
\big|u_2(x) - u_2(x_0)\big| \;&\leq\; \max_{x \in B(n)} u(x) \sum_{y \in B(n)} \big|g^{\omega}_{B(n)}(x,y) - g^{\omega}_{B(n)}(x_0,y)\big|\, \frac{\lambda}{n^2} \big|V^{\omega}(y)\big| \\
&= \max_{x \in B(n)} u(x) \Bigg(\sum_{y \in B_1} \big|g^{\omega}_{B(n)}(x,y) - g^{\omega}_{B(n)}(x_0,y)\big|\, \frac{\lambda}{n^2} \big|V^{\omega}(y)\big| \\
&\quad + \sum_{y \in B_2} \big|g^{\omega}_{B(n)}(x,y) - g^{\omega}_{B(n)}(x_0,y)\big|\, \frac{\lambda}{n^2} \big|V^{\omega}(y)\big| \Bigg) \\
&= \max_{x \in B(n)} u(x)\, (I + II)
\end{aligned}$$

with $B_1 := B(x_0, 2\lceil \delta n \rceil)$ where $\delta := \inf\big\{\delta > 0 : x \in B(x_0, \lceil \delta n \rceil)\big\}$ and $B_2 = B(n) \setminus B_1$. Thus, we split the ball such that $x, x_0 \in B_1$ and $x, x_0 \notin B_2$.

We will now rewrite the first term again in terms of another boundary problem. Since by (2.17), the killed Green's function is non negative, we get

$$
\begin{aligned}
I &= \sum_{y\in B_1} \left|g^\omega_{B(n)}(x,y) - g^\omega_{B(n)}(x_0,y)\right| \frac{\lambda}{n^2}\left|V^\omega(y)\right| \\
&\leq \sum_{y\in B} g^\omega_{B(n)}(x,y) \frac{\lambda}{n^2}\left|V^\omega(y)\delta_{y\in B_1}\right| + \sum_{y\in B} g^\omega_{B(n)}(x_0,y) \frac{\lambda}{n^2}\left|V^\omega(y)\delta_{y\in B_1}\right| \\
&= v(x) + v(x_0)
\end{aligned}
$$

Notice that

$$
v(x) := \sum_{y\in B} g^\omega_{B(n)}(x,y) \frac{\lambda}{n^2}\left|V^\omega(y)\delta_{B_1}\right|
$$

is a solution to (4.36), since for $x \notin B(n)$ we have $g^\omega_{B(n)}(x,y) = 0 \; \forall y \in B$ and for $x \in B(n)$, we obtain

$$
\begin{aligned}
\left(\mathcal{L}^\omega v\right)(x) &= \sum_{y\in B} \omega(x,y)(v(y) - v(x)) \\
&= \sum_{y\in B} \omega(x,y) \left(\sum_{z\in B} g^\omega_{B(n)}(y,z) \frac{\lambda}{n^2}\left|V^\omega(z)\delta_{z\in B_1}\right| \right. \\
&\left. - \sum_{z\in B} g^\omega_{B(n)}(x,z) \frac{\lambda}{n^2}\left|V^\omega(z)\delta_{z\in B_1}\right| \right) \\
&= \sum_{z\in B} \frac{\lambda}{n^2}\left|V^\omega(z)\delta_{z\in B_1}\right| \sum_{y\in B} \omega(x,y)\left(g^\omega_{B(n)}(z,y) - g^\omega_{B(n)}(z,x)\right) \\
&= \sum_{z\in B} \frac{\lambda}{n^2}\left|V^\omega(z)\delta_{z\in B_1}\right| \mathcal{L}^\omega g^\omega_{B(n)}(z,x) \\
&= -\frac{\lambda}{n^2} \sum_{z\in B} \delta_{x=z}\left|V^\omega(z)\delta_{z\in B_1}\right| \\
&= -\frac{\lambda}{n^2}\left|V^\omega(y)\delta_{y\in B_1}\right|,
\end{aligned}
$$

where we used in the third step that by (2.16) $g^\omega_{B(n)}(y,z) = g^\omega_{B(n)}(z,y)$. Thus, we can apply (4.37) and define

$$
c^\omega_I := 2c\left(\lambda \left\|V^\omega\right\|_{p,B(n)} \left\|\nu^\omega\right\|_{q,B(n)}\right)^{\frac{2\kappa+\gamma}{2-\gamma}},
$$

to finally obtain

$$
\begin{aligned}
I &\leq 2 \max_{x \in B(n)} v(x) \\
&\leq 2c \left(\lambda \left\| V^\omega \right\|_{p,B(n)} \left\| \nu^\omega \right\|_{q,B(n)} \right)^{\frac{2\kappa+\gamma}{2-\gamma}} \left(\frac{|B_1|}{|B(n)|} \right)^{\frac{\gamma}{2-\gamma}\frac{1}{p_*}} \\
&\leq 2c \left(\lambda \left\| V^\omega \right\|_{p,B(n)} \left\| \nu^\omega \right\|_{q,B(n)} \right)^{\frac{2\kappa+\gamma}{2-\gamma}} \delta^{\frac{\gamma}{2-\gamma}\frac{1}{p_*}} \\
&\leq c_I^\omega \, \delta^{\frac{\gamma}{2-\gamma}\frac{1}{p_*}},
\end{aligned}
$$

where we used in the third step that by volume regularity (2.2), we get $|B_1|/|B(n)| \leq (\delta n)^d / n^d = \delta^d \leq \delta \in (0,1)$. Notice that the second exponent $\frac{\gamma}{2-\gamma}\frac{1}{p_*} \in (0,1)$ is bounded.

Now, we will come to the second term. Since $x, x_0 \notin B_2$ and by definition $B_1 \cap B_2 = \emptyset$, we have $\mathcal{L}^\omega g^\omega_{B(n)}(y,x) = -\delta_y(x) = 0$ for all $x \in B_1$ and $y \in B_2$. Thus, for all $y \in B_2$, $g^\omega_{B(n)}(y,\cdot) > 0$ is $\mathcal{L}^\omega$-harmonic on B_1. Hence, by Theorem 4.4.1, $g^\omega_{B(n)}(y,\cdot)$ is Hölder continuous on B_1, and since $x, x_0 \in B_1 = B(x_0, 2\lfloor \delta n \rfloor)$, we have

$$
\left| g^\omega_{B(n)}(y,x) - g^\omega_{B(n)}(y,x_0) \right| \leq c\, \delta^\theta \max_{z \in B(x_0, n/2)} g^\omega_{B(n)}(y,z). \tag{4.46}
$$

Using the symmetry of the Green's function (2.16) $g^\omega_{B(n)}(x,y) = g^\omega_{B(n)}(y,x)$ and (4.46), we obtain:

$$
\begin{aligned}
II &\leq \sum_{y \in B_2} \left| g^\omega_{B(n)}(y,x) - g^\omega_{B(n)}(y,x_0) \right| \frac{\lambda}{n^2} \left| V^\omega(y) \right| \\
&\leq \frac{\lambda}{n^2} \sum_{y \in B_2} \left| V^\omega(y) \right| \max_{y \in B_2} \left| g^\omega_{B(n)}(x,y) - g^\omega_{B(n)}(x_0,y) \right| \\
&\leq \frac{\lambda}{n^2} \left| B(n) \right| \left\| V^\omega \right\|_{p,B(n)} \max_{y \in B_2} \left| g^\omega_{B(n)}(y,x) - g^\omega_{B(n)}(y,x_0) \right| \\
&\leq \tilde{C}^\omega_H \lambda \, n^{d-2} \left\| V^\omega \right\|_{p,B(n)} \delta^\theta \max_{y \in B_2, z \in B(x_0, n/2)} g^\omega_{B(n)}(y,z).
\end{aligned}
$$

Since B_2 is a torus and not a ball, we cannot simply apply most of our previous results. As discussed in Remark 4.3.2, we can avoid the arising problems, by using a finite cover consisting of balls. Thus, let $\{B_2^i\}_{i=1,\dots,N} := \{B(y_i, r_i)\}_{i=1,\dots,N}$ be a sequence of balls with center $y_i \in B_2$ and radius $r_i < \frac{n-2r_*}{2} < n$ covering B_2, i.e.

$$
B_2 \subset \bigcup_{i=1}^N B_2^i
$$

and $x, x_0 \notin B_2^i$ for all $i \in \{i = 1, \dots, N\}$. Using again the symmetry and applying (4.41), we get for all $z \in B(x_0, \lfloor \delta n \rfloor)$

$$
\begin{aligned}
\max_{y \in B_2} g^\omega_{B(n)}(y,z) &= \max_{y \in B_2} g^\omega_{B(n)}(z,y) \\
&\leq \sum_{i=1}^N \max_{y \in B_2^i} g^\omega_{B(n)}(z,y) \\
&\leq cN \, \mathcal{A}^\omega(2n)^{\kappa' \frac{2\kappa+\gamma}{2-\gamma}} n^{2-d}.
\end{aligned} \tag{4.47}
$$

Since $N \in \mathbb{N}$ is finite and independent of n, we can define

$$c_{II}^{\omega} := c\,\tilde{C}_H^{\omega} N \lambda \left\| V^{\omega} \right\|_{p,B(n)} \mathcal{A}^{\omega}(2n)^{\kappa' \frac{2\kappa+\gamma}{2-\gamma}}.$$

Notice, that the bound in (4.47) is also uniform in z. Thus, the maximum vanishes, and we finally get:

$$II \le c\,\tilde{C}_H^{\omega} N \lambda \left\| V^{\omega} \right\|_{p,B(n)} \mathcal{A}^{\omega}(2n)^{\kappa' \frac{2\kappa+\gamma}{2-\gamma}} \delta^{\theta} \le c_{II}^{\omega}\,\delta^{\theta}.$$

Plugging the latter results together, we obtain

$$\begin{aligned} |u_2(x) - u_2(x_0)| &\le \max_{x \in B(n)} u(x) \left(c_I^{\omega}\,\delta^{\frac{\gamma}{2-\gamma}\frac{1}{p*}} + c_{II}^{\omega} \delta^{\theta} \right) \\ &\le (c_I^{\omega} + c_{II}^{\omega}) \delta^{\frac{\gamma}{2-\gamma}\frac{1}{p*} \vee \theta} \max_{x \in B(n)} u(x) \end{aligned}$$

and thus

$$\begin{aligned} |u(x) - u(x_0)| &\le \tilde{C}_H^{\omega}\,\delta^{\theta} \max_{B(n/2)} u(x) + (c_I^{\omega} + c_{II}^{\omega})\,\delta^{\frac{\gamma}{2-\gamma}\frac{1}{p*} \vee \theta} \max_{x \in B(n)} u(x) \\ &\le C_H^{\omega} \delta^{\frac{\gamma}{2-\gamma}\frac{1}{p*} \vee \theta} \max_{x \in B(n)} u(x), \end{aligned}$$

where $C_H^{\omega} := \tilde{C}_H^{\omega} + c_I^{\omega} + c_{II}^{\omega}$. □

5 Analytical estimates for the Schrödinger killed Green's function

In this chapter we will present a proof for our final result, which establishes the Hölder continuity on balls $B(n)$ of the killed Green's function $g^{\omega}_{V,B(n)}$ associated to the Schrödinger operator $\mathcal{L}^{\omega}_{\lambda,V}$. Along the way, we observe an interesting relation between the Schrödinger killed Green's function $g_{V,B(n)}$ and its Laplace counterpart $g_{B(n)}$ inspired by results in Cranston, Michael and Fabes [10] in the PDE setting, which also guarantees the existence of the killed Green's function for specific λ (5.8).

5.1 The Relation between the Laplace and Schrödinger killed Green's function

The following proof heavily relies on a Harnack inequality for $\mathcal{L}^{\omega}$-harmonic functions, which was proven by Andres, Deuschel and Slowik in [2]:

Theorem 5.1.1 (Elliptic Harnack inequality). *Suppose that Assumption 2.3.3 holds. For any $x_0 \in V$ and $n \geq 1$ let $B(n) = B(x_0, n)$. Suppose that $u > 0$ is harmonic on $B(n)$, i.e. $\mathcal{L}^{\omega} u = 0$ on $B(n)$. Then, for any $p, q \in (1, \infty)$ with*

$$\frac{1}{p} + \frac{1}{q} \;<\; \frac{2}{d}$$

there exists $C^{\omega}_{\mathrm{EH}} \equiv C^{\omega}_{\mathrm{EH}}(n, p, q) > 0$ such that

$$\max_{x \in B(n/2)} u(x) \;\leq\; C^{\omega}_{\mathrm{EH}} \min_{x \in B(n/2)} u(x). \tag{5.1}$$

Proof. The proof can be found in [2, Theorem 1.3]. □

We are now able to prove the following theorem, which is also inspired by the results stated in Bramanti [9]. We will first show that we can express the Schrödinger Green's function as a solution of the integral equation $(T - \mathrm{id}) g^{\omega}_{V,B(n)} = g^{\omega}_{B(n)}$. We will then use a Neumann Series argument on a Banachspace $\mathcal{B}$ defined in (5.12) to solve this equation to then apply the Harnack principle, i.e. $g^{\omega}_{B(n)}(z, y)/g^{\omega}_{B(n)}(x, y) \leq C_{EH}$, to achieve our result. This argument is a substitute for the so called 3G-Theorem in the continuous setting, stated by Cranston, Michael and Fabes in [10]:

Before we start the proof, we first have to investigate an apriori lower bound for the Laplace heat kernel, which we define as

$$p^{\omega,B(n)}(t, x, y) \;:=\; \mathbb{P}^{\omega}_x[X_t = y; t < \tau_B]. \tag{5.2}$$

Proposition 5.1.2. *Suppose that $d \geq 2$ and Assumption 2.3.3 holds. Then for any $1/2 \leq \sigma' < 1$ there exists an $N_3 \in \mathbb{N}$ and $\delta_0 \in (0,1)$ such that for any $n \geq N_3$ and $\delta \in (0, \delta_0]$, there exists a constant $C_6 \in (0,1)$ such that*

$$p^{\omega, B(n)}(t,x,y) \geq C_6 \, t^{-d/2} \tag{5.3}$$

for any $x, y \in B(\sigma' n)$ and $\delta n^2 \leq t \leq \delta^{-1} n^2$.

Proof. The proof can be found in [5, Theorem 6.24] and [4, Proposition 5.1] □

This leads us to the following lower bound for the Laplace killed greens function on the slightly smaller ball $B(\sigma' n)$:

Corollary 5.1.3. *Suppose that $d \geq 2$ and Assumption 2.3.3 holds. Then for any $1/2 \leq \sigma' < 1$ there exists an $N_3 \in \mathbb{N}$ such that for any $n \geq N_3$ and constant $C_7 \equiv C_7(d,q,\delta,\sigma) > 0$ such that*

$$g^{\omega}_{B(n)}(x,y) \geq C_7 \, n^{2-d} \tag{5.4}$$

for any $x, y \in B(\sigma' n)$ (where we set $n^0 = 1$).

Proof. For $x, y \in B(\sigma' n)$ the latter proposition implies $p^{\omega, B(n)}(t,x,y) \geq C_6 \, t^{-d/2}$ for any $\delta n^2 \leq t \leq \delta^{-1} n^2$. Thus we get

$$\begin{aligned} g^{\omega}_{B(n)}(x,y) &= \int_0^{\infty} p^{\omega, B(n)}(t,x,y) \, \mathrm{d}t \\ &\geq \int_{\delta n^2}^{\delta^{-1} n^2} p^{\omega, B(n)}(t,x,y) \, \mathrm{d}t \\ &\geq \int_{\delta n^2}^{\delta^{-1} n^2} C_6 \, t^{-d/2} ; \mathrm{d}t. \end{aligned} \tag{5.5}$$

Computing the integral:

$$\begin{aligned} \int_{\delta n^2}^{\delta^{-1} n^2} C_6 \, t^{-d/2} ; \mathrm{d}t &= C_6 \begin{cases} \ln t \Big|_{\delta n^2}^{\delta^{-1} n^2} & , d = 2 \\ \frac{2}{2-d} t^{\frac{2-d}{2}} \Big|_{\delta n^2}^{\delta^{-1} n^2} & , d > 2 \end{cases} \\ &= C_6 \begin{cases} \ln \delta^{-2} & , d = 2 \\ \frac{2}{2-d} \left(\delta^{\frac{d-2}{2}} - \delta^{\frac{2-d}{2}} \right) n^{2-d} & , d > 2. \end{cases} \end{aligned} \tag{5.6}$$

Since $\delta \in (0,1)$ we have $\ln \delta^{-2} > 0$ in the first case and also $\frac{2}{2-d} \left(\delta^{\frac{d-2}{2}} - \delta^{\frac{2-d}{2}} \right) > 0$ in the second case where $d > 2$. Now choosing

$$C_7(d,q,\delta,\sigma) := C_6 \, \min\{ \ln \delta^{-2}, \frac{2}{2-d} \left(\delta^{\frac{d-2}{2}} - \delta^{\frac{2-d}{2}} \right) n^{2-d} \} > 0 \tag{5.7}$$

yields the claim. □

We are now able to prove the estimate for the Schrödinger killed Green's function for all points x, y in $B(\sigma' n)$ for any $1/2 \leq \sigma' < 1$ on sufficiently large scales.

Proposition 5.1.4. *Let $g^{\omega}_{V,B(n)}(x,\cdot)$ be the Schrödinger killed Green's function as in (2.18) and assume that Assumption 2.3.3 holds. Assume that for any $p,q \in (1,\infty)$ with*

$$\frac{1}{p}+\frac{1}{q} \;<\; \frac{2}{d}$$

it holds that

$$\lambda \equiv \lambda^{\omega}(n,p,q) \;:=\; \left(\frac{1}{4\,c^{\omega}(n)C_3}\right)^{\frac{2-\gamma}{2\kappa+\gamma}} \|V^{\omega}\|_{p,B(n)}\,\|\nu^{\omega}\|_{q,B(n)} < \infty. \tag{5.8}$$

Then, for any $x,y \in B(\sigma' n)$ with $1/2 \le \sigma' < 1$ and any $\beta \in (0,1)$ there exists an $N_3 \in \mathbb{N}$ such that

$$g^{\omega}_{V,B(n)}(x,y) \;\le\; \frac{1}{1-\beta}\, g^{\omega}_{B(n)}(x,y). \tag{5.9}$$

Remark 5.1.5. The latter Proposition further implies the existence of solutions $u>0$ with $\mathcal{L}^{\omega}_{\lambda,V}u = f$ on $B(n)$, with u explicitly given by

$$u(x) = \sum_{y\in B(n)} g^{\omega}_{V,B(n)}(x,y)f(y) \tag{5.10}$$

for $\lambda = \lambda^{\omega}(n,p,q) < \infty$.

Proof. To start, we rewrite the Schrödinger killed Green's function in a similar way as in the proof of the Hölder continuity (4.45). By observing that

$$\begin{cases} \mathcal{L}^{\omega} g^{\omega}_{B(n)}(x,\cdot) = -\delta_x(\cdot) & \text{on } B(n), \\ g^{\omega}_{B(n)}(x,\cdot) = 0 & \text{on } B(n)^c \end{cases} \qquad \begin{cases} \mathcal{L}^{\omega} w = -\frac{\lambda}{n^2} V^{\omega} g^{\omega}_{V,B(n)}(x,\cdot) & \text{on } B(n), \\ w = 0 & \text{on } B(n)^c, \end{cases}$$

we have

$$g^{\omega}_{V,B(n)}(x,\cdot) = g^{\omega}_{B(n)}(x,\cdot) + w,$$

since

$$\begin{aligned} \mathcal{L}^{\omega}_{\lambda,V}(g^{\omega}_{B(n)}(x,\cdot)+w) &= -\delta_x(\cdot) + \frac{\lambda}{n^2}V^{\omega} g^{\omega}_{B(n)}(x,\cdot) + (-\frac{\lambda}{n^2}V^{\omega} g^{\omega}_{V,B(n)}(x,\cdot)) + \frac{\lambda}{n^2}V^{\omega} w \\ &= -\delta_x(\cdot). \end{aligned}$$

More explicitly, we can express w as:

$$\begin{aligned} w(y) \;=\; \langle w, \delta_y(\cdot)\rangle_{\ell^2(V)} \;&=\; \langle w, -\mathcal{L}^{\omega}\; g^{\omega}_{B(n)}(y,\cdot)\rangle_{\ell^2(V)} \\ &= \big\langle -\mathcal{L}^{\omega}\, w,\, g^{\omega}_{B(n)}(y,\cdot)\big\rangle_{\ell^2(V)} \\ &= \big\langle \frac{\lambda}{n^2} V^{\omega} g_{V,B(n)}(x,\cdot),\, g^{\omega}_{B(n)}(y,\cdot)\big\rangle_{\ell^2(V)} \\ &= \frac{\lambda}{n^2} \sum_{z\in B(n)} g^{\omega}_{V,B(n)}(x,z)V^{\omega}(z)g^{\omega}_{B(n)}(z,y), \end{aligned}$$

which means, that we can rewrite the Schrödinger killed Green's function as

$$g^{\omega}_{V,B(n)}(x,\cdot) = g^{\omega}_{B(n)}(x,\cdot) + \frac{\lambda}{n^2} \sum_{z\in B(n)} g^{\omega}_{V,B(n)}(x,z) V^{\omega}(z) g^{\omega}_{B(n)}(z,\cdot). \tag{5.11}$$

From now on we will only consider points x, y in $B(\sigma' n)$ in order to apply the lower bound for the Laplace killed Green's function (5.4) which only holds on $B(\sigma' n)$. Notice that for (5.11) to hold we still need to sum over all $z \in B(n)$. This will however not pose any problems for us in the proof of the 3G-Theorem, which will become clear later on.

The calculations above give rise to the following Banachspace of functions on $B(\sigma' n) \times B(\sigma' n)$

$$\mathcal{B} := \{f : B(\sigma' n) \times B(\sigma' n) \to \mathbb{R}, \|f\|_{\mathcal{B}} := \max_{x\in B(\sigma' n)} \sum_{y\in B(\sigma' n)} |f(x,y)| < \infty\}. \tag{5.12}$$

Further, we define an operator T on $\mathcal{B}$ as

$$Tf(x,y) = \frac{\lambda}{n^2} \sum_{z\in B(n)} f(x,z) V^{\omega}(z) g^{\omega}_{B(n)}(z,y)$$

and notice that the Schrodinger Green's function is a solution to the following integral equation:

$$(\mathrm{id} - T) g^{\omega}_{V,B(n)} = g^{\omega}_{B(n)}. \tag{5.13}$$

Thus, our strategy will now be to use a Neumann Series to invert the operator on the left side.

First, we realize that $T \in \mathcal{L}(\mathcal{B})$ is indeed a continuous linear operator since

$$\begin{aligned}
\|Tf\|_{\mathcal{B}} &= \max_{x\in B(\sigma' n)} \sum_{y\in B(\sigma' n)} \frac{\lambda}{n^2} \Big| \sum_{z\in B(n)} f(z,y) V^{\omega}(z) g^{\omega}_{B(n)}(x,z) \Big| \\
&\le \|f\|_{\mathcal{B}} \max_{x\in B(n)} \sum_{z\in B(n)} \frac{\lambda}{n^2} |V^{\omega}(z)| g^{\omega}_{B(n)}(x,z) \\
&\le \|f\|_{\mathcal{B}}\, C_3 \left(\lambda \|V^{\omega}\|_{p,B(n)} \|\nu^{\omega}\|_{q,B(n)}\right)^{\frac{2\kappa+\gamma}{2-\gamma}} \\
&\le \|f\|_{\mathcal{B}}\, \beta,
\end{aligned}$$

where we used in the third step, that $v(x) := \sum_{z\in B(n)} \frac{\lambda}{n^2} |V^{\omega}(z)| g^{\omega}_{B(n)}(x,z)$ solves (4.36) and thus, we can apply (4.37). In the last step, we used the explicit choice of λ in (5.8) , which yields that for $\beta < 1$, we have that T is indeed a continuous linear operator:

$$\|T\|_{\mathcal{B}} = \sup_{\|f\|_{\mathcal{B}}=1} \|Tf\|_{\mathcal{B}} \le \beta < 1.$$

Now, we have satisfied the requirements to apply Neumann series, which yields, that $(\mathrm{id} - T)^{-1} = \sum_{k=0}^{\infty} T^k$ and thus solve the integral equation (5.13) to obtain:

$$g^{\omega}_{V,B(n)} = (\mathrm{id} - T)^{-1} g^{\omega}_{B(n)} = \sum_{k=0}^{\infty} T^k g^{\omega}_{B(n)}$$

with convergence in $\mathcal{L}(\mathcal{B})$.

Our goal is to find a discrete analogon to the 3-G Theorem in [10], to bound $\frac{g^\omega_{B(n)}(z,y)}{g^\omega_{B(n)}(x,y)}$ where $x, y \in B(\sigma' n)$ and $z \in B(n)$. We will split $B(\sigma' n)$ into four parts and treat each case on its own. Therefore, define $r_x := \sup\{r < n : B(x,r) \in B(\sigma' n) \text{ and } y \notin B(x,r)\}$ and $r_y := \sup\{r < n : B(y,r) \in B(\sigma' n) \text{ and } x \notin B(y,r)\}$.

Firstly, we consider points in $B(x, r_x)$ and $B(y, r_y)$. Since we chose $r_x, r_y < n$ such that $x \notin B(y, r_y)$ and $y \notin B(x, r_x)$, we know that $\mathcal{L}^\omega g^\omega_{B(n)}(\cdot, y) = 0$ on $B(x, r_x)$ and $\mathcal{L}^\omega g^\omega_{B(n)}(x, \cdot) = 0$ on $B(y, r_y)$. Applying the Harnack inequality (Theorem 5.1.1) yields:

$$\frac{g^\omega_{B(n)}(\cdot, y)}{g^\omega_{B(n)}(x,y)} \le C^\omega_{EH} \text{ on } B(x, \frac{r_x}{2}) \qquad \text{and} \qquad \frac{g^\omega_{B(n)}(x, \cdot)}{g^\omega_{B(n)}(x,y)} \le C^\omega_{EH} \text{ on } B(y, \frac{r_y}{2}) \qquad .$$

Secondly, we treat the remaining points in $\tilde{B}_x := B(n) \setminus B(x, r_x/2)$ and $\tilde{B}_y := B(n) \setminus B(y, r_y/2)$. Since $x \notin \tilde{B}_x$, we can find a finite cover of balls $\tilde{B}_x \subset \bigcup_{i=1}^{N_x} B(z^i, r^i)$ not containing x similar as in 4.3.2 to apply 4.3.1 in order to find an upper bound for the numerator $g^\omega_{B(n)}(x,z)$ for every $z \in \tilde{B}_x$:

$$g^\omega_{B(n)}(x,z) \le cN_x\, \mathcal{A}^\omega(2n)^{\kappa' \frac{2\kappa+\gamma}{2-\gamma}} n^{2-d}. \tag{5.14}$$

Similarly, we can find an upper bound for $g^\omega_{B(n)}(z,y)$ for each $z \in \tilde{B}_y$. Now we treat the denominator $g^\omega_{B(n)}(x,y)$ for $x, y \in B(\sigma' n)$. Here, we can apply apply our last corollary 5.1.3, which yields

$$g^\omega_{B(n)}(x,z) \ge C_7 n^{2-d}. \tag{5.15}$$

Combining the upper estimate for the numerator and the lower estimate for the denominator gives

$$\frac{g^\omega_{B(n)}(x, \cdot)}{g^\omega_{B(n)}(x,y)} \le \tilde{c}^\omega \text{ on } \tilde{B}_x \qquad \text{and} \qquad \frac{g^\omega_{B(n)}(\cdot, y)}{g^\omega_{B(n)}(x,y)} \le \tilde{c}^\omega \text{ on } \tilde{B}_y \qquad .$$

where $c^\omega := c \max\{N_x, N_y\}\, \mathcal{A}^\omega(2n)^{\kappa' \frac{2\kappa+\gamma}{2-\gamma}} C_7^{-1}$.

Thus, by splitting the sum and cleverly multiplying with $1 = g^\omega_{B(n)}(x,y)/g^\omega_{B(n)}(x,y)$ in the second inequality, we get defining $c^\omega := \max\{C^\omega_{EH}, \tilde{c}^\omega\}$

$$
\begin{aligned}
&|Tg^\omega_{B(n)}(x,y)| \\
&\le \frac{\lambda}{n^2} \sum_{z\in B(n)} \frac{g^\omega_{B(n)}(x,y)}{g^\omega_{B(n)}(x,y)} g^\omega_{B(n)}(z,y)|V^\omega(z)|g^\omega_{B(n)}(x,z) \\
&\le \frac{\lambda}{n^2}\Bigg(\sum_{z\in B(x,\frac{r_x}{2})} \frac{g^\omega_{B(n)}(z,y)}{g^\omega_{B(n)}(x,y)}|V^\omega(z)|g^\omega_{B(n)}(x,z) + \sum_{z\in B(y,\frac{r_y}{2})} g^\omega_{B(n)}(z,y)|V^\omega(z)|\frac{g^\omega_{B(n)}(x,z)}{g^\omega_{B(n)}(x,y)} \\
&+ \sum_{z\in\tilde{B}_x} \frac{g^\omega_{B(n)}(z,y)}{g^\omega_{B(n)}(x,y)}|V^\omega(z)|g^\omega_{B(n)}(x,z) + \sum_{z\in\tilde{B}_y} g^\omega_{B(n)}(z,y)|V^\omega(z)|\frac{g^\omega_{B(n)}(x,z)}{g^\omega_{B(n)}(x,y)}\Bigg) \\
&\times g^\omega_{B(n)}(x,y) \\
&\le c^\omega\Bigg(2\sum_{z\in B(n)} \frac{\lambda}{n^2}|V^\omega(z)|g^\omega_{B(n)}(x,z) + 2\sum_{z\in B(n)} \frac{\lambda}{n^2}|V^\omega(z)|g^\omega_{B(n)}(y,z)\Bigg) \\
&\times g^\omega_{B(n)}(x,y) \\
&\le 4\,c^\omega \max_{z\in B(n)} V^\omega(z)\, g^\omega_{B(n)}(x,y) \\
&\le 4\,c^\omega\, C_3 \left(\lambda\, \|V^\omega\|_{p,B(n)}\, \|\nu^\omega\|_{q,B(n)}\right)^{\frac{2\kappa+\gamma}{2-\gamma}} g^\omega_{B(n)}(x,y) \\
&\le \beta g^\omega_{B(n)}(x,y)
\end{aligned}
$$

where we again applied (4.37) in the second to last step. Iterating the equation now yields

$$|T^k g^\omega_{B(n)}(x,y)| \;\le\; \beta^k g^\omega_{B(n)}(x,y),$$

which together with the Neumann Series argument above and the convergence of geometric sums gives the final result:

$$g^\omega_{V,B(n)}(x,y) \;=\; \sum_{k=0}^{\infty} T^k g^\omega_{B(n)}(x,y) \;\le\; \frac{1}{1-\beta} g^\omega_{B(n)}(x,y)$$

□

5.2 Hölder continuity of the Schrödinger killed Green's function

Since Proposition 4.3.1 in Chapter 4 delivered us upper bounds for the Laplace killed Green's function, we can now apply Proposition 5.1.4 to transfer those bounds to the Schrödinger killed Green's function:

Corollary 5.2.1. *Let $g^\omega_{V,B(n)}(x,\cdot)$ be the Schrödinger killed Green's function as in (2.18) and assume that Assumption 2.3.3 holds. Assume that for any $p,q\in(1,\infty)$ with*

$$\frac{1}{p}+\frac{1}{q} \;<\; \frac{2}{d}$$

it holds that

$$\lambda \equiv \lambda^{\omega}(n,p,q) := \left(\frac{1}{4\,c^{\omega}(n)C_3}\right)^{\frac{2-\gamma}{2\kappa+\gamma}} \|V^{\omega}\|_{p,B(n)}\,\|\nu^{\omega}\|_{q,B(n)} < \infty. \tag{5.16}$$

Further, let $z \in B(\sigma' n)$ *and* $r < \sigma' n$ *be such that* $x \notin B(z,r)$*. Then, for any* $x, y \in B(\sigma' n)$ *with* $1/2 \le \sigma' < 1$ *and any* $\beta \in (0,1)$ *there exists* $N_3 \in \mathbb{N}$ *and* $C_8 < \infty$ *such that*

$$\max_{y\in B(z,r)} g^{\omega}_{V,B(n)}(x,y) \le C_8\,\mathcal{A}^{\omega}(n)^{\kappa' \frac{2\kappa+\gamma}{2-\gamma}} n^{2-d}, \tag{5.17}$$

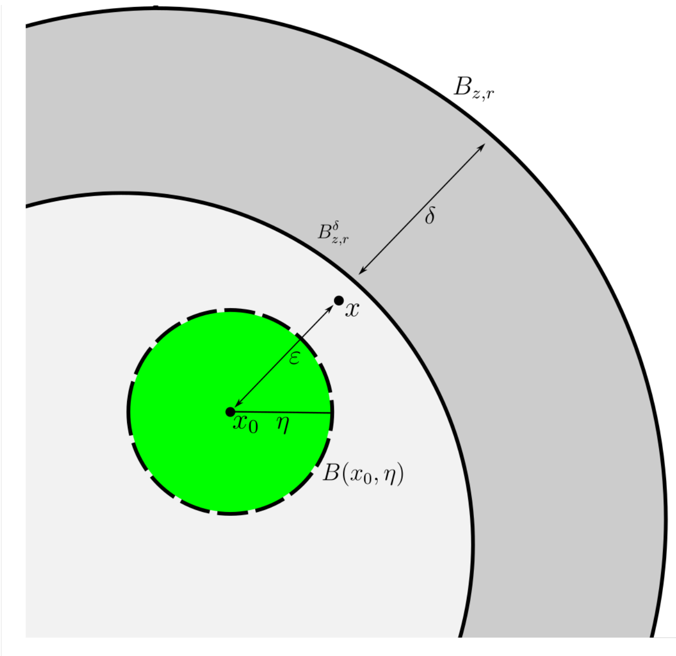
for all $x, y \in B^*$*, where*

$$\mathcal{A}^{\omega}(n) := 1 \vee \|\mu^{\omega}\|_{p,B(n)}\,\|\nu^{\omega}\|_{q,B(n)}.$$

Proof. The statement follows by combining Proposition 4.3.1 with Proposition 5.1.4 and choosing $C_8 = C_5/(1-\delta)$. □

We now combine the latter result and the Hölder continuity for $\mathcal{L}^{\omega}_{\lambda,V}$-harmonic functions to prove our final result, which will be the Hölder continuity of the Schrödinger killed Green's function.

Proof of Theorem 2.3.6. Since by (2.18) $\mathcal{L}^{\omega}_{\lambda,V}\, g^{\omega}_{V,B_{z,r}(n)}(x,\cdot) = -\delta_x(\cdot)$, the Schrödinger killed Green's function is harmonic on $B_{z,r}(n)\setminus\{x\}$. Further, since we chose $\eta < \varepsilon$ and $\|x - x_0\|_2 > \varepsilon > \eta$, we observe that $x \notin B([nx_0], \eta\, n)$ for n large enough.



Thus, $g^{\omega}_{V,B_{z,r}(n)}(x,\cdot)$ is harmonic on $B([nx_0], \eta\, n)$. Therefore, we can apply the Hölder continuity for $\mathcal{L}^{\omega}_{\lambda,V}$-harmonic functions (2.19) to obtain for each $x, y \in B([nx_0], \eta\, n)$:

$$\begin{aligned}
&|g^{\omega}_{V,B_{z,r}(n)}(y,[nx]) - g^{\omega}_{V,B_{z,r}(n)}(z,[nx])| \\
&\le |g^{\omega}_{V,B_{z,r}(n)}(y,[nx]) - g^{\omega}_{V,B_{z,r}(n)}(x_0,[nx])| + |g^{\omega}_{V,B_{z,r}(n)}(z,[nx]) - g^{\omega}_{V,B_{z,r}(n)}(x_0,[nx])| \\
&\le 2C^{\omega}_{H}\eta^{\alpha} \max_{B([nx_0],\eta n)} g_{V,B_{z,r}(n)}(\cdot,[nx]),
\end{aligned} \tag{5.18}$$

which yields the first part of the claim.

Now assume that Assumption 2.3.3 holds. Then for n large enough, Assumption 2.3.3 holds, which means that λ chosen in (2.21) is finite. Further, we can estimate $\mathcal{A}^\omega(n)$ by a constant $c > 0$ and C_H^ω by $C_H^* < \infty$, which yields

$$\begin{aligned}
\limsup_{n\to\infty} n^{d-2} &\sup_{z,y\in B([nx_0],\eta n)} |g^\omega_{V,B_{z,r}(n)}(z,[nx]) - g^\omega_{V,B_{z,r}(n)}(y,[nx])| \\
&\le \limsup_{n\to\infty} 2C_H^* \eta^\alpha \, C_8 \, \mathcal{A}^\omega(2n)^{\kappa' \frac{2\kappa+\gamma}{2-\gamma}} \\
&\le 2C_H^* \eta^\alpha \, C_6 \, c^{\kappa' \frac{2\kappa+\gamma}{2-\gamma}} . \qquad (5.19)
\end{aligned}$$

Choosing $C_8 := 2C_H^* \, C_6 \, c^{\kappa' \frac{2\kappa+\gamma}{2-\gamma}} < \infty$ gives the second part of the claim. $\square$

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