# Wavelets

Christof Seiler

Stanford University, Spring 2016, Stats 205

# Introduction

What we've seen so far

- Nonparametric regression using smoothers
- Different types of smoothers: e.g. kernel and local polynomial

Today

- Construct basis functions that are
  - Multiscale
  - Adaptive
- Find sparse set of coefficients for a given basis

# Introduction

- In nonparametric regression we estimated the unkonwn function $f$ directly
- With wavelets we use a orthogonal series representation of $f$
- This shifts the estimation problem
  - from directly estimating $f$
  - to estimating a set of scalar coefficients that represents $f$
- Wavelets are used in the image file format JPEG 2000 to compress data

# Sparsity

- Wavelet methods are closely related to the concept of sparity
- A function

  $$f(x) = \sum_j \theta_j \psi_j(x)$$

  is sparse in a basis $\psi_1, \psi_2, \ldots$ if most of the $\theta_j$ are zero (or close it zero)
- Sparsity is not captured well by the $L_2$ norm but it is capture by the $L_1$ norm

# Sparsity

- For example,

$$a = (1, 0, \ldots, 0) \qquad b = (1/\sqrt{n}, \ldots, 1/\sqrt{n})$$

- then both have the same $L_2$ norm

$$\|a\|_2 = \sqrt{1 + 0 + \cdots + 0} = 1$$
$$\|b\|_2 = \sqrt{1/n + \cdots + 1/n} = \sqrt{n \times 1/n} = 1$$

- but with $L_1$ norm

$$\|a\|_1 = 1 + 0 + \cdots + 0 = 1$$
$$\|b\|_1 = 1/\sqrt{n} + \cdots + 1/\sqrt{n} = n \times 1/\sqrt{n} = \sqrt{n}$$

# Assumptions

- Obervations

$$Y_i = f(x_i) + \epsilon_i \qquad i = 1, \ldots, n$$

- The $\epsilon_i$ are iid
- The function $f$ is square integrable $\int f^2 < \infty$
- Defined on a close interval $[a, b]$

# Basis Function

- A set of functions $\Psi = \{\psi_1, \psi_2, \ldots\}$ is called a basis for a class of functions $\mathcal{F}$
- If any function $f \in \mathcal{F}$ can be represented as a linear combination of the basis functions $\psi_i$
- Written as

$$f(x) = \sum_{i=1}^{\infty} \theta_i \psi_i(x)$$

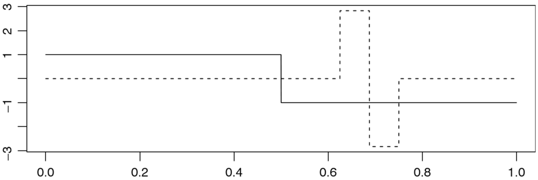
  with $\theta_i$ are scalar constants refered to as coefficients
- The constants $\theta_i$ are inner products of the function $f$ and the basis functions $\psi_i$

$$\theta_i = \langle f, \psi_i \rangle = \int f(x)\psi_i(x)\, dx$$

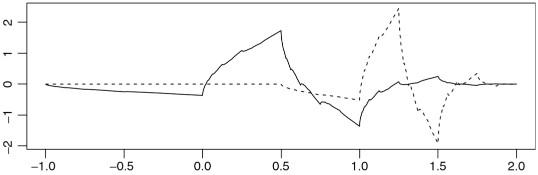
- The basis is orthogonal if $\langle \psi_i, \psi_j \rangle = 0$ for $i \neq j$
- The basis is orthonormal if orthogonal and $\langle \psi_i, \psi_j \rangle = 1$

# Basis Function

- Many sets of basis functions
- We consider orthonomal wavelet bases
- A simple wavelet function was first introduced by Haar in 1910




- More flexible and powerful wavelets were developed by Daubechies in 1992 and many others



# Multiresolution Analysis

- If $\psi$ is a wavelet function, then the collection of functions

$$\Psi = \{\psi_{jk} : j, k \text{ integers}\}$$

with

$$\psi_{jk} = 2^{j/2}\psi(2^j x - k)$$

forms a basis for square-integrable functions
- $\Psi$ is a collection of translations and dilations of $\psi$
- The $\psi$ is constructed to ensure the the set $\Psi$ is orthonormal
- The property $\int \psi_i^2 = 1$ implies that the value of $\psi$ is near 0 except over a small range
- This property combined with the constrution above means that as $j$ increases $\psi_{jk}$ becomes increasingly localized

# Multiresolution Analysis

- A careful construction of $\psi$ leads to a multiresolution analysis
- It provides an interpretation of the wavelet representation $f$ in terms of location and scale by rewritting

$$f(x) = \sum_{i=1}^{\infty} \theta_i \psi_i(x)$$

in terms of translation $k$ and scaling $j$ as ($\mathbb{Z}$ is set of integers)

$$f(x) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \theta_{jk} \psi_{jk}(x)$$

- This can be intepreted as approximation at different scale $j$
- Here scale $j$ is the same as frequency
- For a fixed $j$ the index $k$ represents behavior of $f$ at resolution $j$ and a particular location

# Multiresolution Analysis

- Consider the approximation

$$f_J(x) = \sum_{j<J} \sum_{k \in \mathbb{Z}} \theta_{jk} \psi_{jk}(x)$$

- As $J$ increases $f_J$ is able to model smaller scales (higher frequncy) behavior of $f$
- Corresponds to changes that occur over smaller interval of the $x$-axis
- As $J$ deceases $f_J$ models larger scale (lower frequency) behavior of $f$
- Adding gloabl scaling term

$$f_J(x) = \sum_{k \in \mathbb{Z}} \xi_{j_0 k} \phi_{j_0 k}(x) + \sum_{j_0<j<J} \sum_{k \in \mathbb{Z}} \theta_{jk} \psi_{jk}(x)$$

# Multiresolution Analysis

- Consider a simple example

$$f(x) = x, \qquad x \in [0, 1)$$

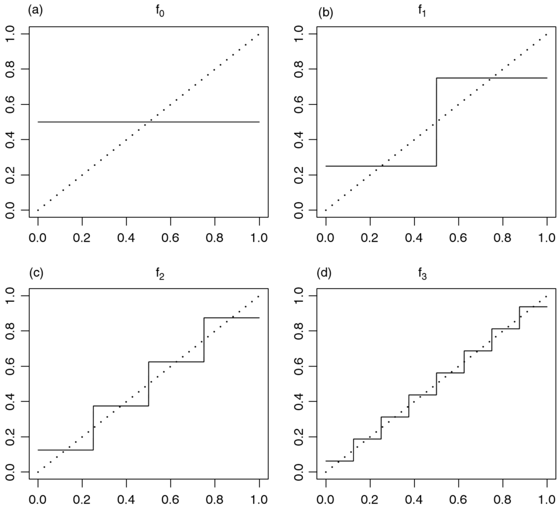
- The Haar wavelet functions are defined as

$$\psi(x) = \begin{cases} 1, & x \in [0, 1/2), \\ -1, & x \in [1/2, 1) \end{cases}$$

- and

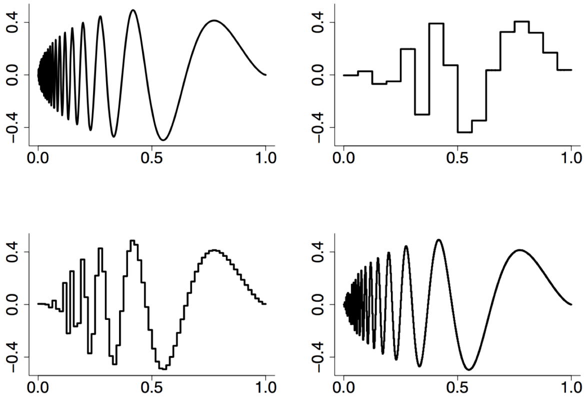
$$\phi(x) = 1, \qquad x \in [0, 1)$$

# Linear Example




Source: Hollander, Wolfe, and Chicken (2013)

# Doppler Example




Source: Wasserman (2006)

# Discrete Wavelet Transform

- The simple linear function example has exact solution to determine coefficients
- Usually this is not the case and numerical approximations are necessary to estimate coefficients
- One numerical methods is called the **cascade algorithm** by Mallat 1989
- It works for if the sample size is a power of 2

$$n = 2^J$$

  for some positive integer $J$
- Using this algorithm restricts the upper level of summation to $J - 1$ with

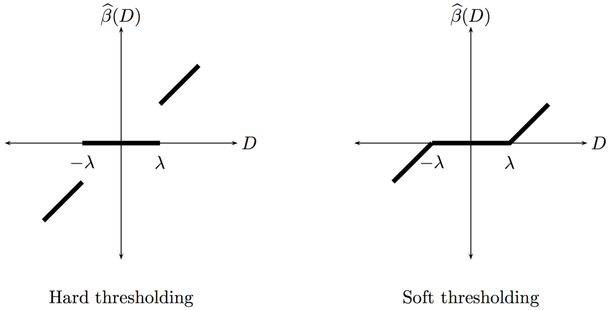
$$J = \log_2(n)$$

# Wavelet Thresholding

TODO

# Wavelet Thresholding

- Hard and soft thresholding




Source: Wasserman (2006)

# Other Important Topics

- Practical, simultaneous confidence bands for wavelet estimators are not available (Wasserman 2006)
- Standard wavelet basis functions are not invariant to translation and rotations
- Recent work by Mallat (2012) and Bruna & Mallat (2013) extend wavelets to handle these kind of invariances
- This provides a promising new direction for the theory of convolutional neural network

# References


- Hollander, Wolfe, and Chicken (2013). Nonparametric Statistical Methods
- Wasserman (2006). All of Nonparametric Statistics
- Mallat (2012). Group Invariant Scattering
- Bruna and Mallat (2013). Invariant Scattering Convolution Networks