Computational Statistics (732A90) Lab04

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2021-11-25

Question 1

Question 2

Task 1

The formula of the Bayes Theorem is given by $P(\mu|Y) = \frac{P(Y|\mu)P(\mu)}{\int P(Y|\mu)P(\mu)d\mu} = \frac{P(Y|\mu)P(\mu)}{P(Y)}$. The $P(Y)$ is the model evidence, which it does not depend on μ, thus we have the following relation, $P(\mu|Y) \propto P(Y|\mu)P(\mu)$. In the following steps we are going to calculate the likelihood $P(Y|\mu)$ and the prior $P(\mu)$.

The likelihood formula of the Normal distribution is given by:

$$
\begin{aligned}
L(\mu_i, \sigma^2; y_1, y_2, ..., y_n) &= \prod_{i=1}^{n} f_Y(y_j; \mu_i, \sigma^2) \\
&= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} exp^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu_i)^2} \\
&= \frac{1}{\sqrt[n]{2\pi\sigma^2}} exp^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu_i)^2} \Leftrightarrow \\
P(Y|\mu) &= \frac{1}{\sqrt[n]{2\pi\sigma^2}} exp^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu_i)^2}
\end{aligned}
$$

The general prior formula is given by $p(\mu) = p(\mu_1)p(\mu_{i+1}|\mu_i).....p(\mu_n|\mu_{n-1})$.

In our case, the prior formula is given by:

$$
\begin{aligned}
p(\mu) &= p(\mu_1)p(\mu_2|\mu_1)p(\mu_2)p(\mu_3|\mu_2).....p(\mu_n|\mu_{n-1}) \\
&= 1p(\mu_2|\mu_1)p(\mu_2)p(\mu_3|\mu_2).....p(\mu_n|\mu_{n-1}) \\
&= p(\mu_2|\mu_1)p(\mu_2)p(\mu_3|\mu_2).....p(\mu_n|\mu_{n-1}) \\
&= \prod_{i=2}^{n} p(\mu_i|\mu_{i-1}) \\
&= \frac{1}{\sqrt[n-1]{2\pi\sigma^2}} exp^{-\frac{1}{2\sigma^2}\sum_{i=2}^{n}(\mu_i-\mu_{i-1})^2}
\end{aligned}
$$

Task 2

Bayes' Theorem is used to get the posterior up to a constant proportionality.

$$
\begin{aligned}
P(\mu|Y) \propto P(Y|\mu)P(\mu) &= \frac{1}{\sqrt[n]{2\pi\sigma^2}} exp^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu_i)^2} \frac{1}{\sqrt[n-1]{2\pi\sigma^2}} exp^{-\frac{1}{2\sigma^2}\sum_{i=2}^{n}(\mu_i-\mu_{i-1})^2} \\
&= \frac{1}{\sqrt[\frac{2n-1}{n(n-1)}]{2\pi\sigma^2}} exp^{-\frac{1}{2\sigma^2}(\sum_{i=1}^{n}(y_i-\mu_i)^2+\sum_{i=2}^{n}(\mu_i-\mu_{i-1})^2)}
\end{aligned}
$$

To separate the above formula the Hint A, Hint B and Hint C were used.

$$
\begin{aligned}
p(\mu_1|\vec{\mu}_{-1},\vec{Y}) &= \frac{p(\vec{\mu},\vec{Y})}{p(\vec{\mu}_{-1},\vec{Y})} \\
&\propto \frac{1}{\sqrt[\frac{2n-1}{n(n-1)}]{2\pi\sigma^2}} exp^{(-\frac{(y_1-\mu_1)^2+(\mu_2-\mu_1)^2}{2\sigma^2})} \\
&\propto \frac{1}{\sqrt[\frac{2n-1}{n(n-1)}]{2\pi\sigma^2}} exp^{(-\frac{(\mu_1-(y_1+\mu_2)/2)^2}{\frac{2\sigma^2}{2}})}
\end{aligned}
$$

$$
\begin{aligned}
p(\mu_i|\vec{\mu}_{-i},\vec{Y}) &= \frac{p(\vec{\mu},\vec{Y})}{p(\vec{\mu}_{-i},\vec{Y})} \\
&\propto \frac{1}{\sqrt[\frac{2n-1}{n(n-1)}]{2\pi\sigma^2}} exp^{(-\frac{(y_i-\mu_i)^2+(\mu_{i+1}-\mu_i)^2+(\mu_i-\mu_{i-1})^2}{2\sigma^2})} \\
&\propto \frac{1}{\sqrt[\frac{2n-1}{n(n-1)}]{2\pi\sigma^2}} exp^{(-\frac{(\mu_i-(y_i+\mu_{i-1}+\mu_{i+1})/3)^2}{\frac{2\sigma^2}{3}})}
\end{aligned}
$$

$$
\begin{aligned}
p(\mu_n|\vec{\mu}_{-n},\vec{Y}) &= \frac{p(\vec{\mu},\vec{Y})}{p(\vec{\mu}_{-n},\vec{Y})} \\
&\propto \frac{1}{\sqrt[\frac{2n-1}{n(n-1)}]{2\pi\sigma^2}} exp^{(-\frac{(y_n-\mu_n)^2+(\mu_n-\mu_{n-1})^2}{2\sigma^2})} \\
&\propto \frac{1}{\sqrt[\frac{2n-1}{n(n-1)}]{2\pi\sigma^2}} exp^{(-\frac{(\mu_n-(y_n+\mu_{n-1})/2)^2}{\frac{2\sigma^2}{2}})}
\end{aligned}
$$

$$
(\mu_i|\vec{\mu}_{-i},\vec{Y}) \sim \begin{cases} N(\frac{y_1+\mu_2}{2},\frac{\sigma^2}{2}) & i=1 \\ N(\frac{y_i+\mu_{i-1}+\mu_{i+1}}{3},\frac{\sigma^2}{3}) & Otherwise \\ N(\frac{y_n+\mu_{n-1}}{2},\frac{\sigma^2}{2}) & i=n \end{cases}
$$

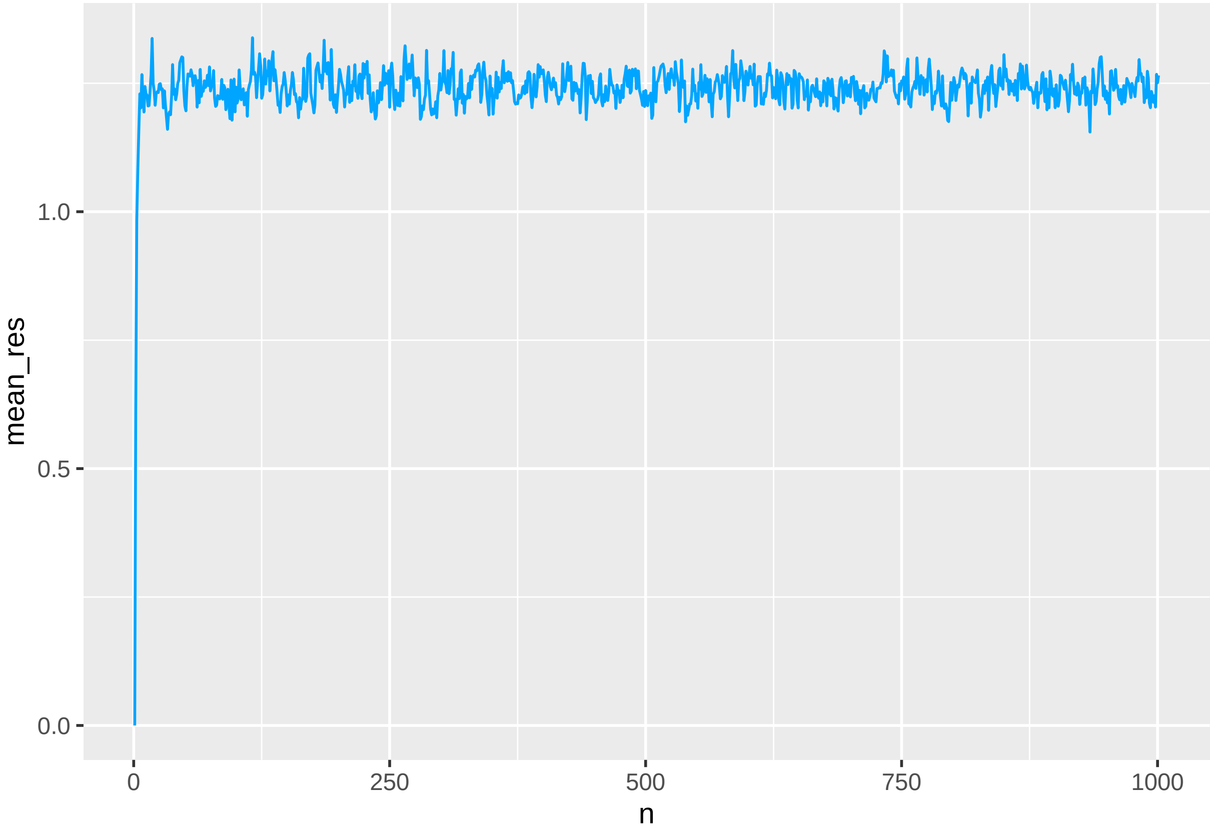
Task 3

```
load("./chemical.RData")
n = 1000
dim = length(Y)
mu_init = rep(0, dim)
sigma = 0.2

gibbs_sampler = function(n, dim, data, mu_init, sigma) {

    res = matrix(0, nrow = n + 1, ncol = dim)
    res[1, ] = mu_init

    for (i in 2:nrow(res)) {
        res[i, 1] = rnorm(1, (data[1] + res[i - 1, 2])/2, sqrt(sigma^2/2))
        for (j in 2:(dim - 1)) {
            res[i, j] = rnorm(1, (data[j] + res[i, j - 1] + res[i - 1, j + 1])/3,
                sqrt(sigma^2/3))
        }
        res[i, dim] = rnorm(1, (data[dim] + res[i, dim - 1])/2, sqrt(sigma^2/2))
    }

    return(res)
}

res = gibbs_sampler(n, dim, Y, mu_init, sigma)
```

```r
library(ggplot2)

mean_res = colMeans(res)

data = data.frame(X = X, Y = Y, Gibbs = mean_res)

ggplot(data) + geom_line(aes(x = X, y = Y), color = "#d1495b") + geom_line(aes(x = X,
    y = Gibbs), color = "#00A5FF")
```

Task 4

```r
data2 = data.frame(n = 1:1001, mean_res = rowMeans(res))
# data2 = data.frame('n' =1:50, 'mean_res' = res[1001,])

ggplot(data2, aes(x = n, y = mean_res)) + geom_line(color = "#00A5FF")
```


mean_res
1.0
0.5
0.0
0
250
500
750
1000
n