Computational Statistics (732A90) Lab03

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Question 2

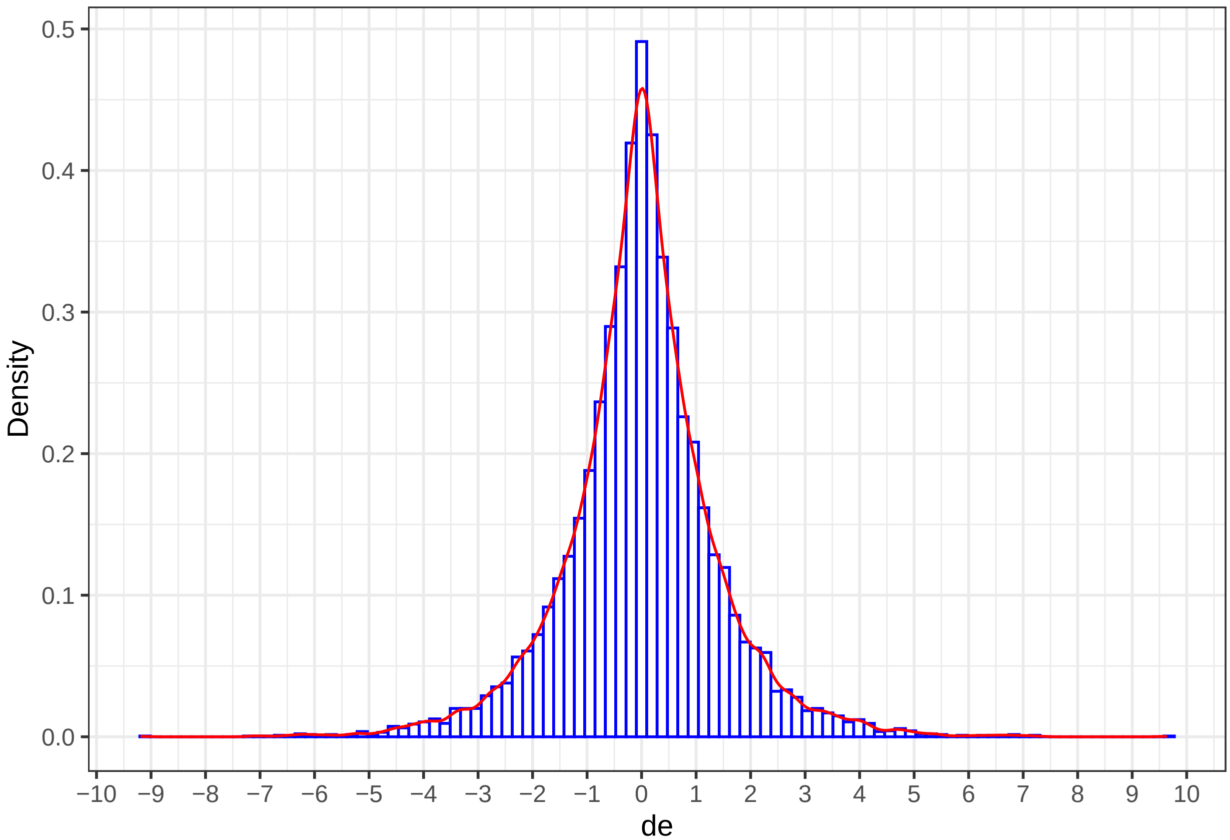
```
set.seed(12345)

x <- runif(n=10000, min=0, max=1)
data <- data.frame(unif=x)
```

```
de_distribution = function(m, a, u){
  result <- m-(1/a)*sign(u-0.5)*log(1-2*abs(u-0.5))
  #result <- m-sign(u-0.5)*1/a*log(1+sign(u-0.5)-sign(u-0.5)*2*u)
  return(result)
}

data_de = data.frame(de = de_distribution(0,1, x))

ggplot(data = data_de, aes(x = de)) +
  geom_histogram(bins = 100, color = "blue", fill = "white", aes(y=..density..))+
  geom_density(colour = "red")+
  ylab("Density")+
  scale_x_continuous(breaks = -10:10)+
  theme_bw()
```



Question 2

Task 1

First of all one needs the CDF of DE(x).

$$
\begin{aligned}
DE(x) &= 0.5 \cdot \exp\left(-|x|\right) \\
F_X(x) &:= \int_{-\infty}^{x} DE(\tilde{x}) d\tilde{x} \\
&= \int_{-\infty}^{x} 0.5 \cdot \exp\left(-|\tilde{x}|\right) d\tilde{x} && \text{For } x > 0 \\
&\overset{x \geq 0}{=} 0.5 + \int_{0}^{x} 0.5 \cdot \exp\left(-\tilde{x}\right) d\tilde{x} \\
&= 0.5 \left(1 + \int_{0}^{x} \exp\left(-\tilde{x}\right) d\tilde{x}\right) \\
&= 0.5 \left(1 + \left[-\exp\left(-\tilde{x}\right)\right]_0^x\right) \\
&= 1 - 0.5 \cdot \exp\left(-x\right) \\
& && \text{For } x < 0 \\
F_X(x) &\overset{x \leq 0}{=} \int_{-\infty}^{x} 0.5 \cdot \exp\left(\tilde{x}\right) d\tilde{x} \\
&= \left[0.5 \cdot \exp\left(\tilde{x}\right)\right]_{-\infty}^{x} \\
&= 0.5 \cdot \exp\left(x\right)
\end{aligned}
$$

Then the inverse of the CDF is needed.

$$\begin{aligned}
&\text{For } x > 0 \\
1 - 0.5 \cdot exp\big(-F_X^{-1}(y)\big) &= y \\
2 - 2y &= exp\big(-F_X^{-1}(y)\big) \\
-\ln(2 - 2y) &= F_X^{-1}(y) \\
&\text{For } x < 0 \\
0.5 \cdot exp\big(F_X^{-1}(y)\big) &= y \\
F_X^{-1}(y) &= \ln(2y)
\end{aligned}$$

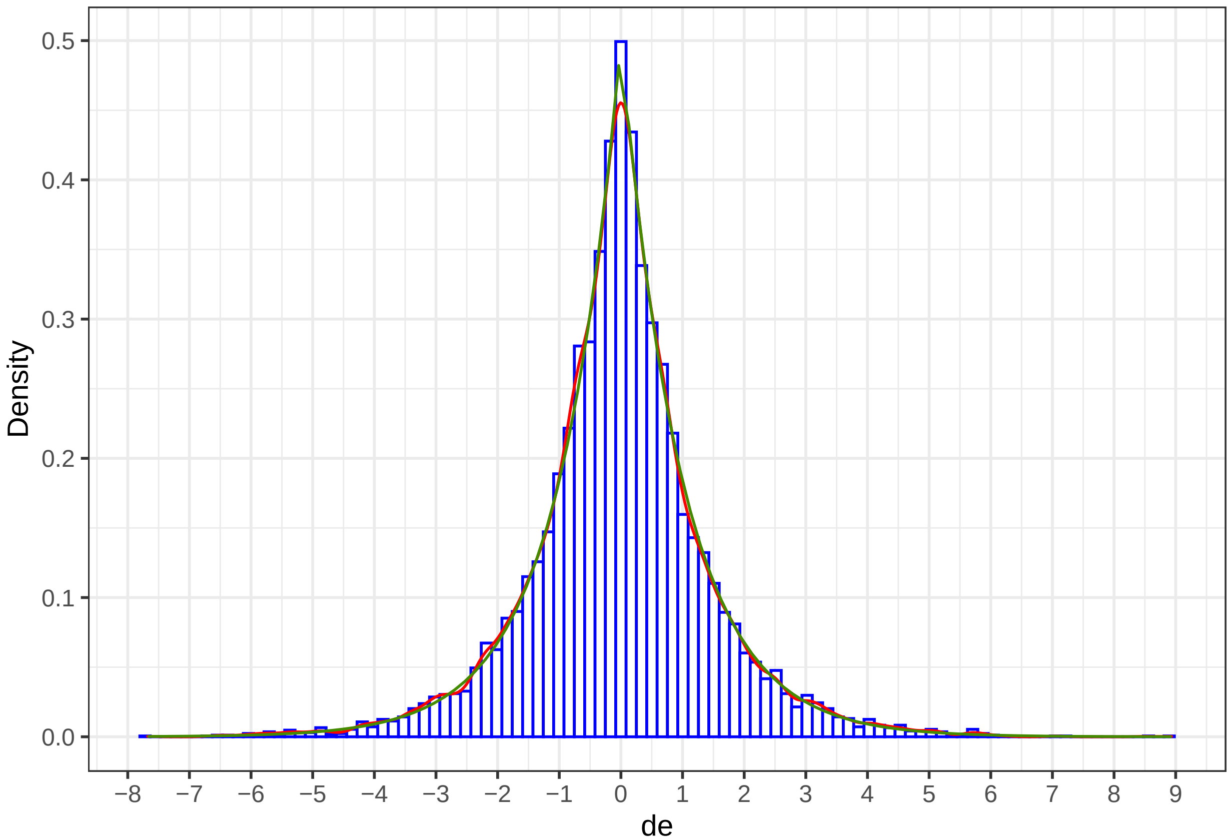
Then for $U \sim \text{Unif}(0, 1)$

$$X = F_X^{-1}(U) \sim \text{double exponential } (0,\ 1)$$

```
DE <- function(x) 0.5 * exp(-abs(x))
DE_CDF <- function(x){
  if (x > 0) return(1 - 0.5 * exp(1)^(-x))
  else return(0.5 * exp(1)^x)
}
DE_CDF_invers <- function(y){
  res_pos <- -log(2-2*y)
  if(res_pos > 0) return(res_pos)
  else return(log(2*y))
}
CDF_mehtod <- function(n){
  U <- runif(n)
  return(sapply(U, DE_CDF_invers))
}

data_de = data.frame(de = CDF_mehtod(10000))

ggplot(data = data_de, aes(x = de)) +
  geom_histogram(bins = 100, color = "blue", fill = "white", aes(y=..density..))+
  geom_density(colour = "red")+
  stat_function(fun = DE, color = "chartreuse4") +
  ylab("Density")+
  scale_x_continuous(breaks = -10:10)+
  theme_bw()
```



Task 2

To use the acceptance rejection method the functions have to fulfilll the following condition:

$$
\begin{aligned}
m &\geq \frac{\mathcal{N}(0,1)}{DE(0,1)} \\
&= \frac{2\exp\left(0.5 \cdot x^2\right)}{\sqrt{2\pi} \cdot \exp(-|x|)} \\
&= \frac{2}{\sqrt{2\pi}} \cdot \exp\left(0.5 \cdot x^2 - |x|\right) \\
&\geq \frac{2}{\sqrt{2\pi}} \cdot \exp\left(0.5\right)
\end{aligned}
$$

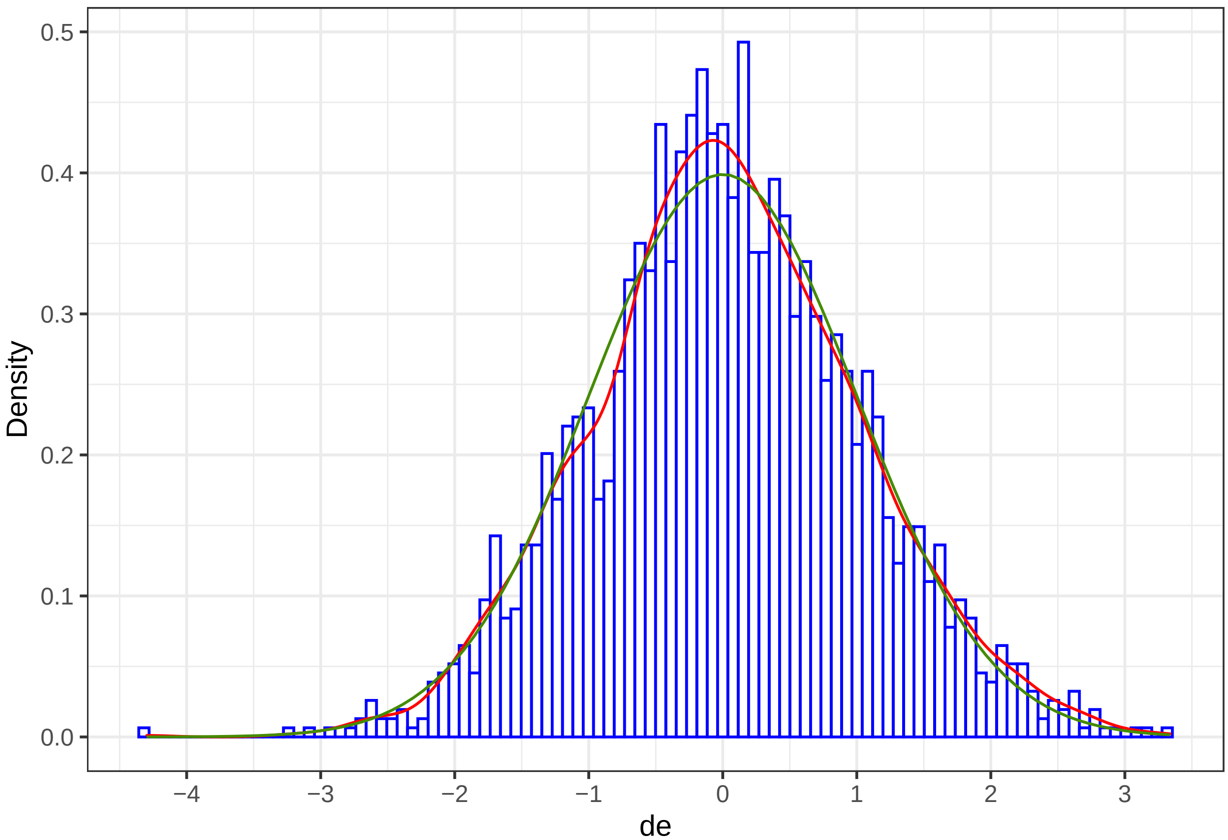
So the majorising constant can be chosen to be $m = \frac{2}{\sqrt{2\pi}} \cdot \exp\left(0.5\right)$.

```
rnorm_acc_rej <- function(n){
  DE <- function(x) 0.5 * exp(-abs(x))
  m <- 2 / sqrt(2 * pi) * exp(0.5)
  X <- c()
  counter <- 0
  while (length(X) != n) {
    counter <- counter + 1
    U <- runif(1)
    Y <- CDF_mehtod(1)
    if (U <= dnorm(Y) / (m * DE(Y))) {
      X <- append(X, Y)
    }
```

```
  }
  return(list("rnumbers" = X, "rej_rate" = 1 - (n / counter)))
}

data_de = data.frame(de = rnorm_acc_rej(2000)$rnumbers)

ggplot(data = data_de, aes(x = de)) +
  geom_histogram(bins = 100, color = "blue", fill = "white", aes(y=..density..))+
  geom_density(colour = "red")+
  stat_function(fun = dnorm, color = "chartreuse4") +
  ylab("Density")+
  scale_x_continuous(breaks = -10:10)+
  theme_bw()
```



The expected rejection rate ER is

$$
\begin{aligned}
ER &= 1 - \frac{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-0.5 \cdot x^2} dx}{m \cdot \int_{-\infty}^{\infty} 0.5 \cdot e^{-|x|} dx} \\
&= 1 - \frac{1}{m} \\
&= 1 - \frac{1}{\frac{2}{\sqrt{2\pi}} \cdot e^{(0.5)}} \\
&\approx 23.98\%
\end{aligned}
$$

The average actual rejection rate for $n = 50000$ samples is

```r
paste0(as.character(round(rnorm_acc_rej(50000)$rej_rate * 100, 2)), "%")
```

```
## [1] "24.09%"
```