Introduction to Converter Sampled-Data Modeling

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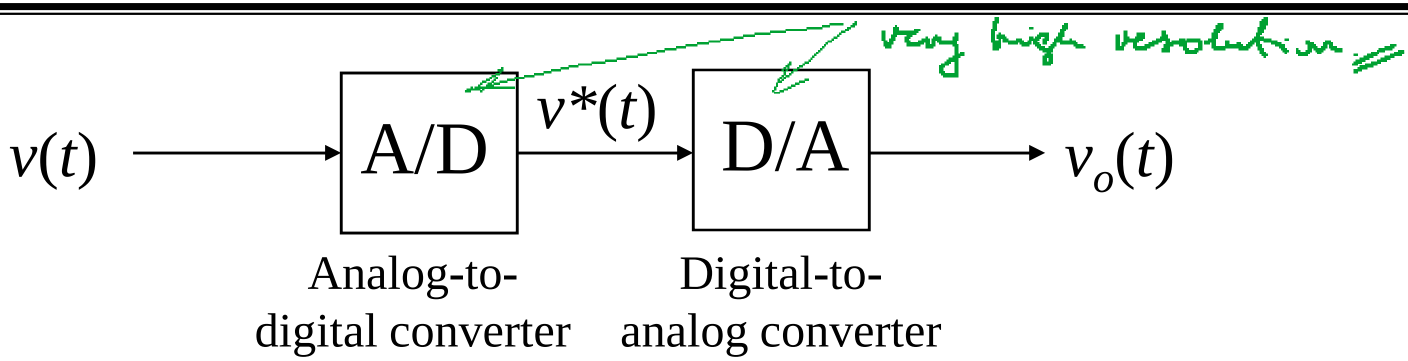
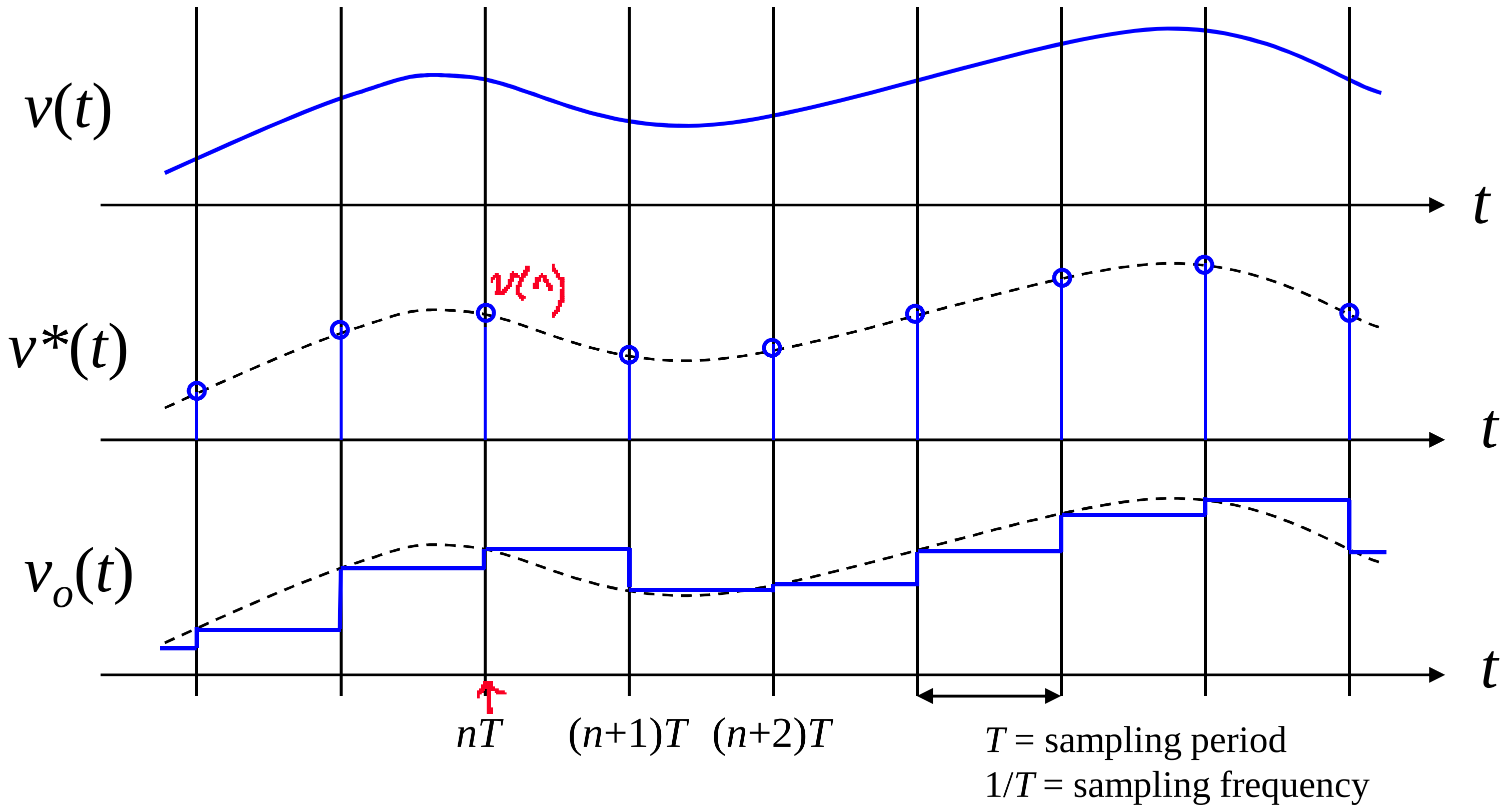
Objectives

- Better understanding of converter small-signal dynamics, especially at high frequencies
- Applications
  - DCM high-frequency modeling
  - Current mode control
  - Digital control

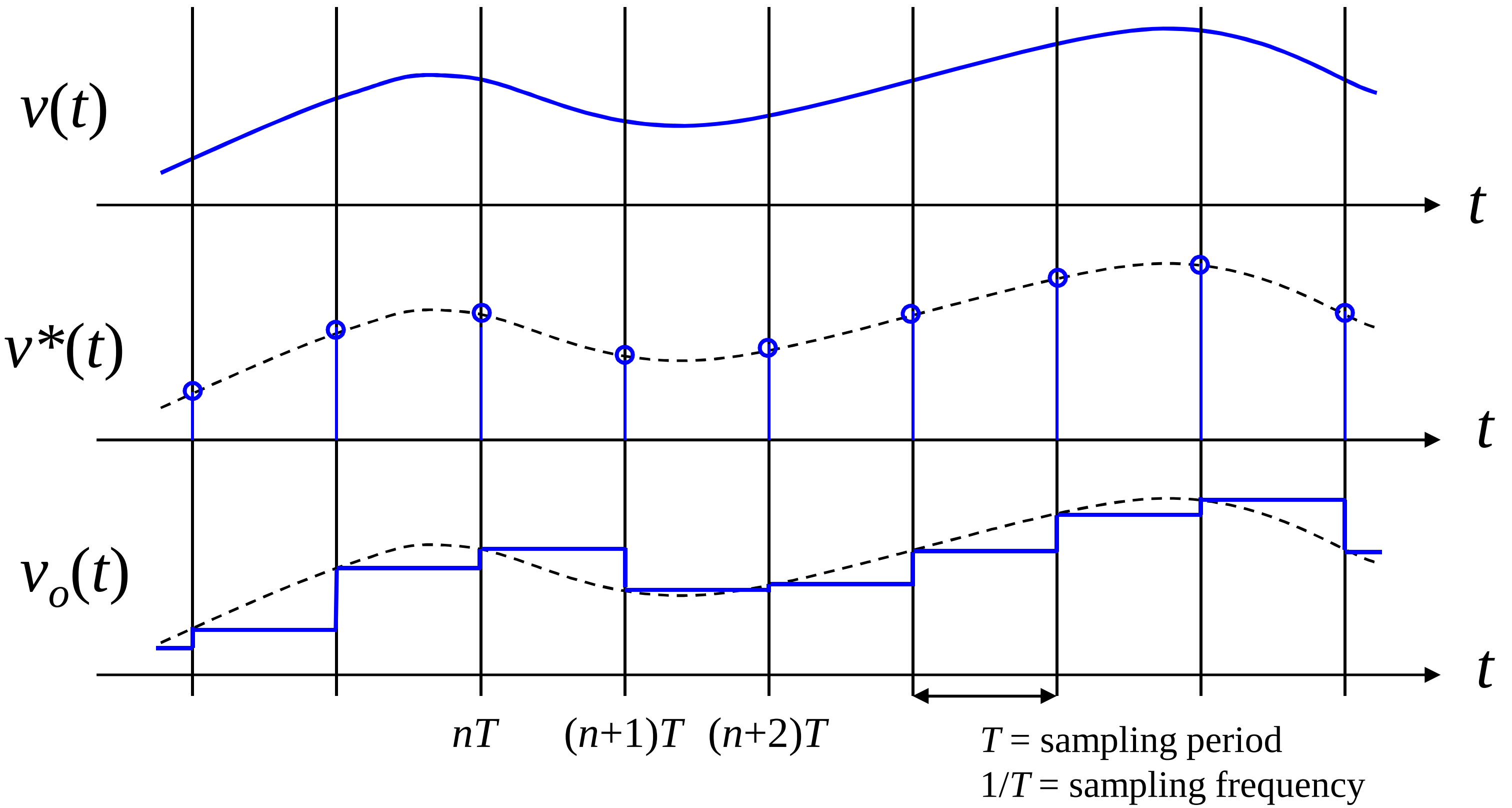
Example: A/D and D/A conversion



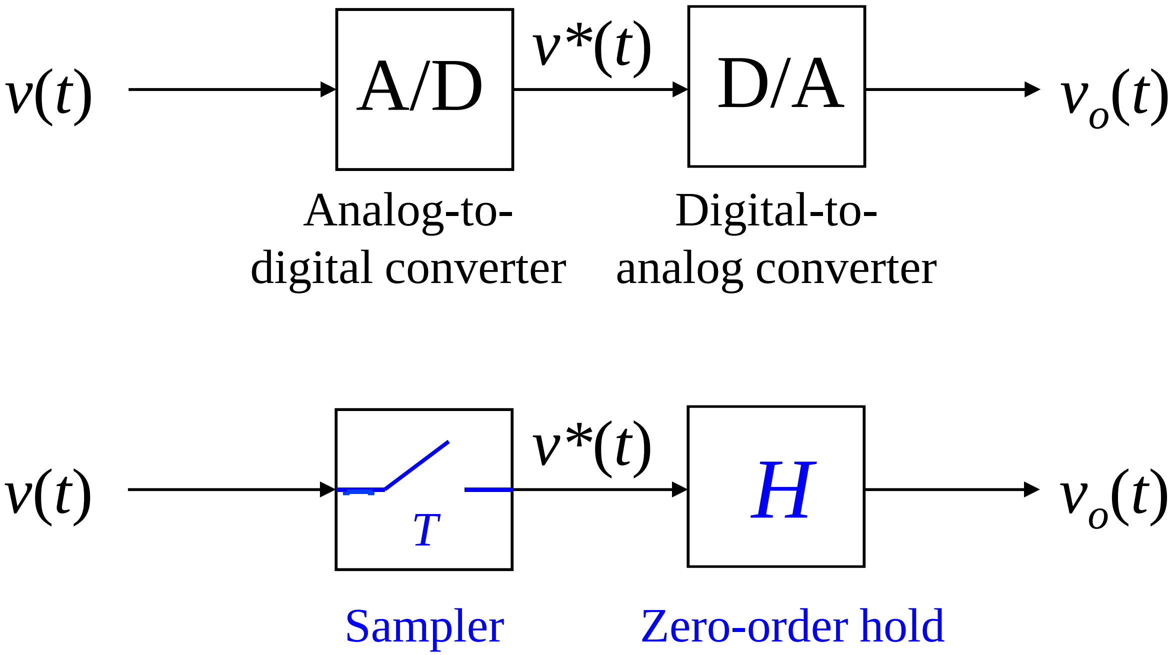


Modeling objectives

- Relationships: v to v^* to v_o
  - *Time domain:* $v(t)$ to $v^*(t)$ to $v_o(t)$
  - *Frequency domain:* $v(s)$ to $v^*(s)$ to $v_o(s)$

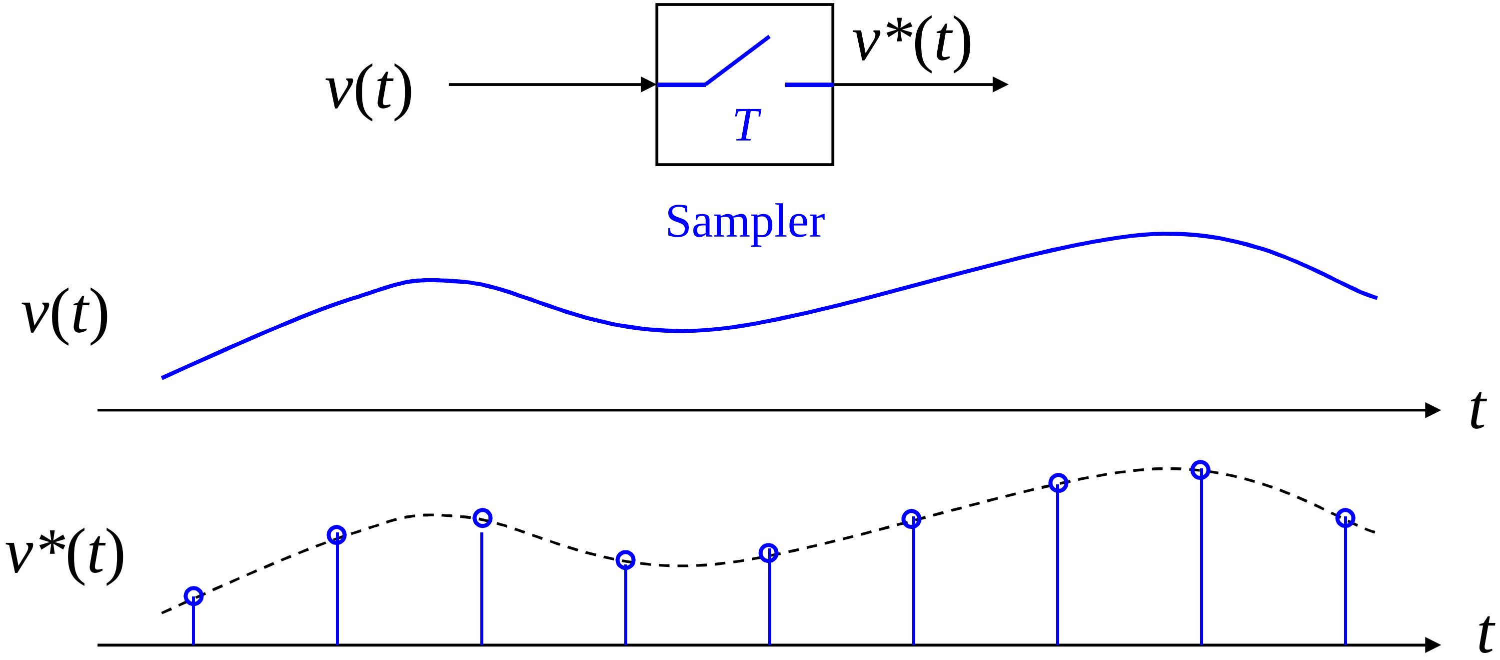


Model



v(t)
A/D
v*(t)
D/A
v_o(t)
Analog-to-digital converter
Digital-to-analog converter
v(t)
T
v*(t)
H
v_o(t)
Sampler
Zero-order hold

Sampling



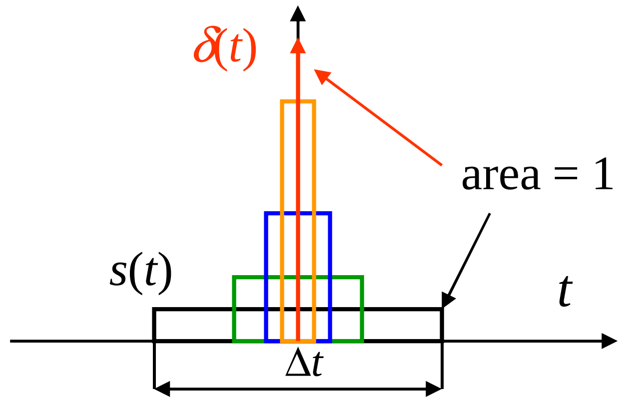
$$v^*(t) = v(t)\sum_{-\infty}^{+\infty} \delta(t - nT)$$

Unit impulse (Dirac)

Unit impulse

$$s(t) \xrightarrow[\Delta t \to \emptyset]{} \delta(t)$$

area = 1



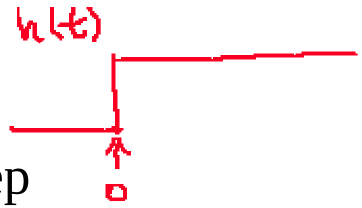
Properties

$$\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$$

$$\int_{-\infty}^{+\infty} v(t)\,\delta(t - t_s)\,dt = v(t_s)$$

$$\int_{-\infty}^{t} \delta(\tau)\,d\tau = h(t)$$

unit step



Laplace transform

$$\mathcal{L}(\delta(t)) = \int_{-\infty}^{+\infty} \delta(t)\,e^{-st}\,dt = 1$$

Sampling in frequency domain

$$v^*(t) = v(t)\sum_{-\infty}^{+\infty}\delta(t-nT)$$

$$v(s) = \int_{-\infty}^{+\infty} v(t)e^{-st}dt$$

$$v^*(s) = \int_{-\infty}^{+\infty} v^*(t)e^{-st}dt$$

$$\boxed{v^*(s) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} v(s-jk\omega_s)}$$

Sampling in frequency domain: derivation

$$v^*(t) = v(t)\boxed{\sum_{-\infty}^{+\infty} \delta(t-nT)}$$

periodic string of unit pulses

$$v^*(s) = \int_{-\infty}^{+\infty} v^*(t) e^{-st}\, dt$$

$\mathcal{L}(v^*(t))$

period $= T$

Fourier series

$$\sum_{-\infty}^{+\infty} \delta(t-nT) = \sum_{k=-\infty}^{+\infty} C_k e^{jk\omega_s t} \qquad \omega_s = \frac{2\pi}{T} = 2\pi f_s$$

$$C_k = \frac{1}{T} \int_{-T/2}^{T/2} \left(\sum_{n=-\infty}^{n=+\infty} \delta(t-nT) \right) e^{-jk\omega_s t}\, dt = \frac{1}{T}$$

$$\sum_{-\infty}^{+\infty} \delta(t-nT) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} e^{jk\omega_s t}$$

Sampling in frequency domain: derivation

$$v^*(s) = \frac{1}{T}\int_{-\infty}^{+\infty} v(t)\left(\sum_{k=-\infty}^{+\infty} e^{jk\omega_s t}\right) e^{-st}\,dt$$

$$v^*(s) = \frac{1}{T}\sum_{k=-\infty}^{+\infty}\int_{-\infty}^{+\infty} v(t)\, e^{-st}\, e^{jk\omega_s t}\,dt$$

$$= \frac{1}{T}\sum_{k=-\infty}^{+\infty}\underbrace{\int_{-\infty}^{+\infty} v(t)\, e^{-(s-jk\omega_s)t}\,dt}_{v(s-jk\omega_s)}$$

$$v(s) = \mathcal{L}(v(t)) = \int_{-\infty}^{+\infty} v(t)\, e^{-st}\,dt$$

$$v^*(s) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} v(s-jk\omega_s)$$

Aliasing

$V(s)$ = input

$\|V\|$

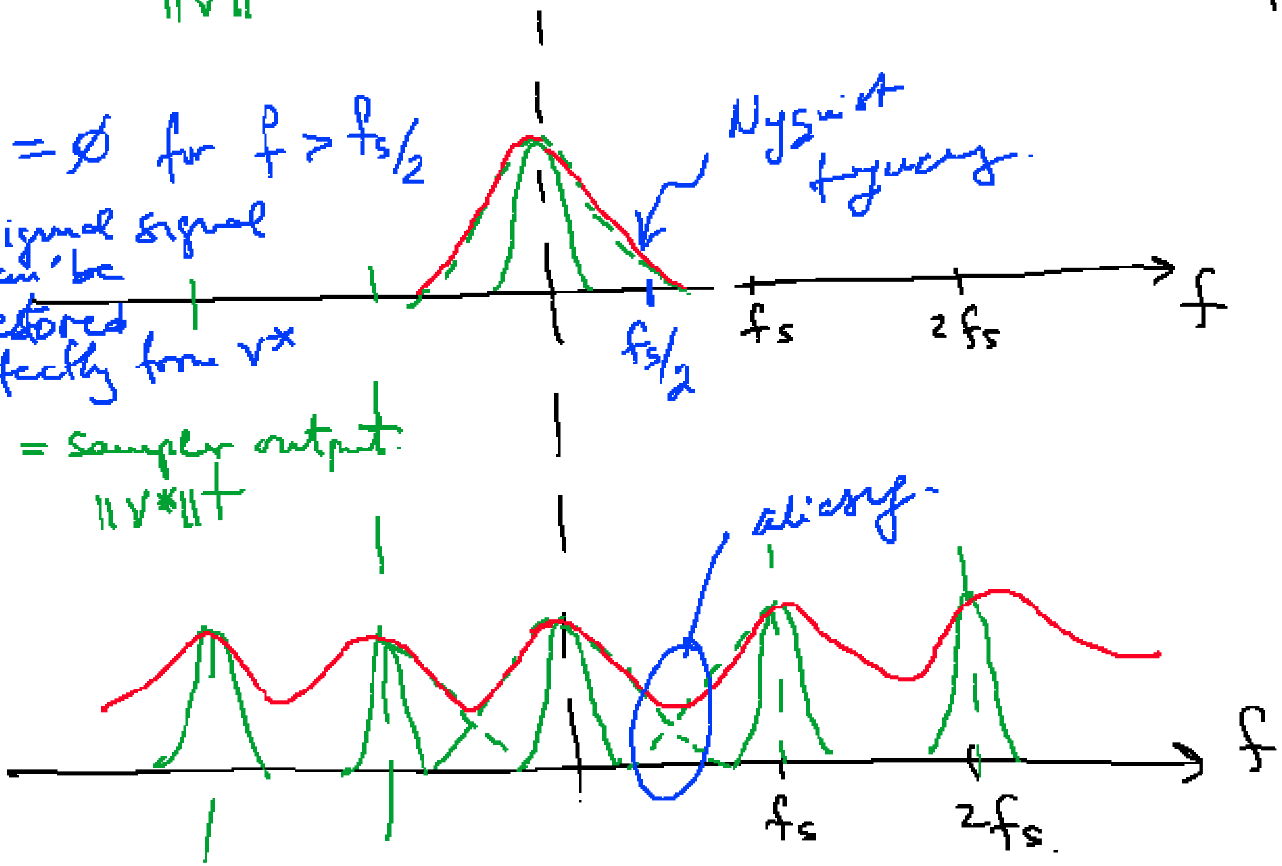
$f_s = \frac{1}{T}$ = sampling frequency

$\|V\| = \varnothing$ for $f > f_s/2$

⇒ original signal can't be restored perfectly from V^*

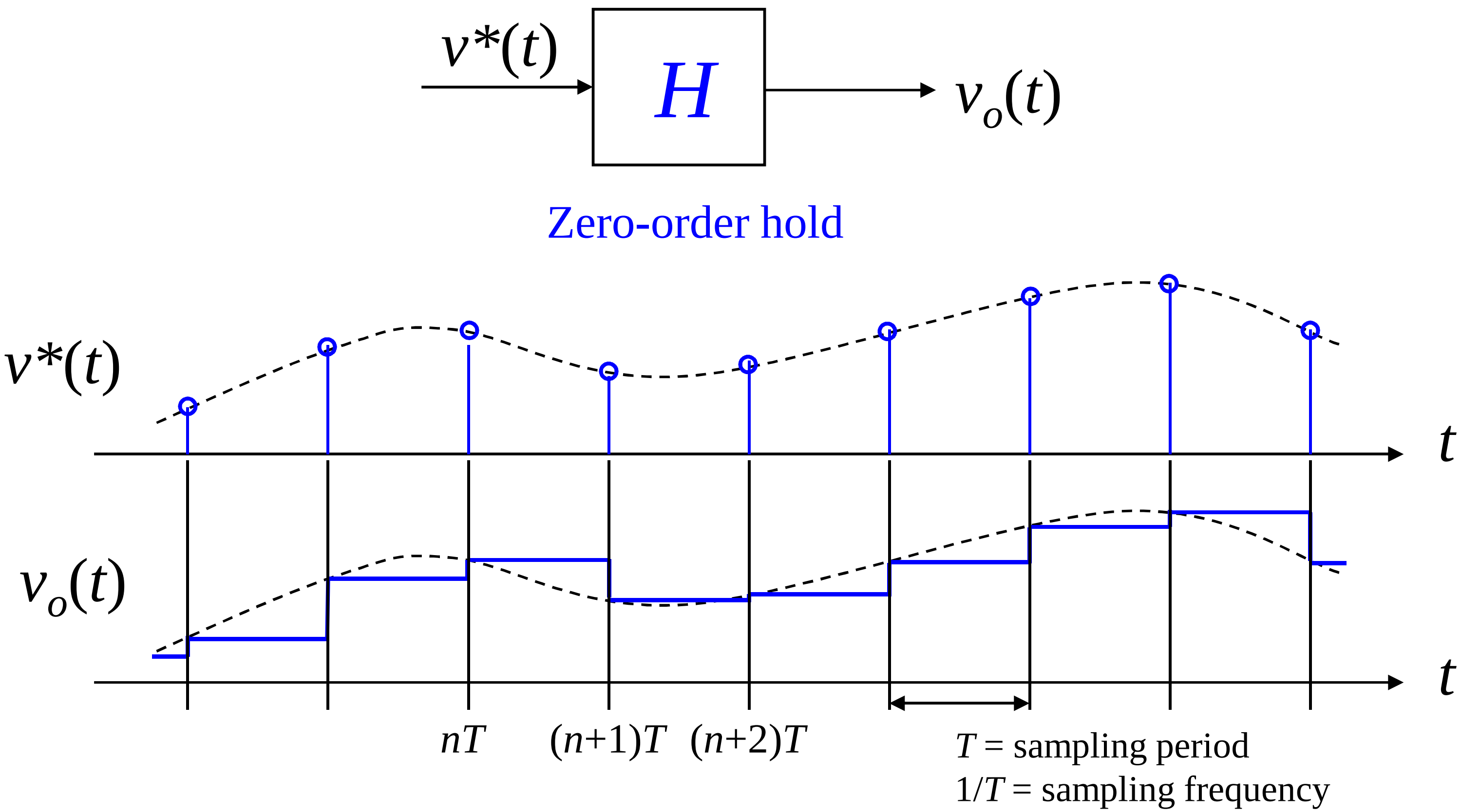
$V^*(s)$ = sampler output

$\|V^*\|$



$$V^*(s) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} V(s - jk\omega_s)$$

Zero-order hold

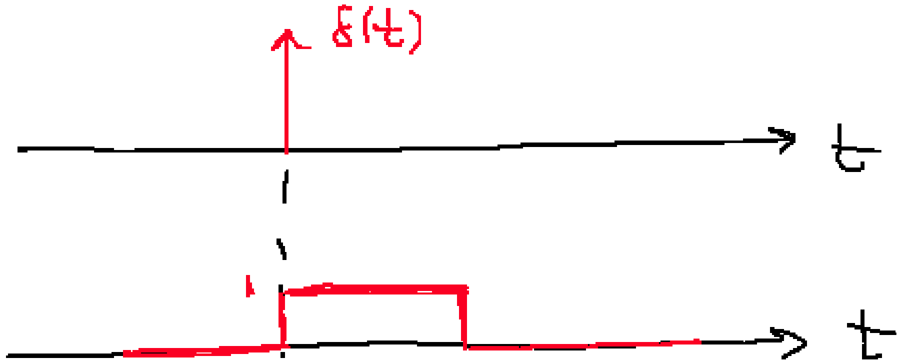


Zero-order hold: time domain

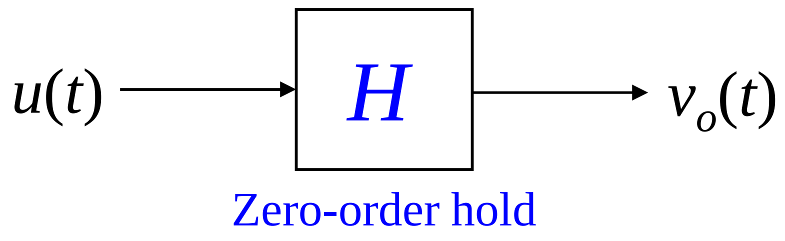
$$\delta(t) \longrightarrow \boxed{H} \longrightarrow v_o(t)$$

Zero-order hold

$$v_o(t) = \int_{t-T}^{t} \delta(\tau)\, d\tau \longrightarrow \text{Freq. domain.}$$



Zero-order hold: frequency domain



$$v_o(t) = \int_{t-T}^{t} u(\tau)\,d\tau$$

$\mathcal{L}$ $\qquad \dfrac{dv_o(t)}{dt} = u(t) - u(t-T)$

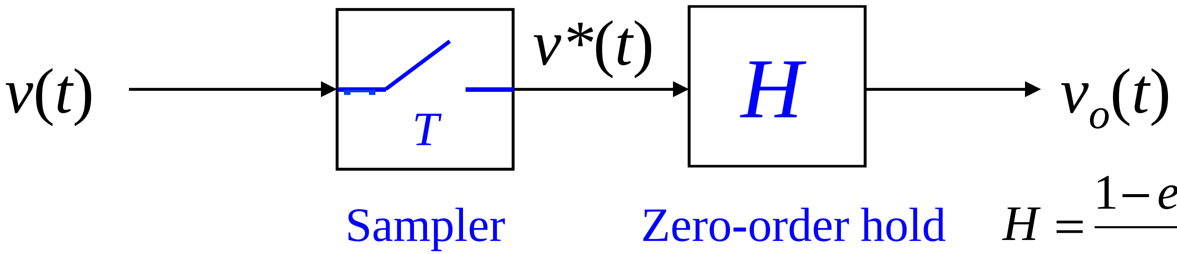
$$s\,V_o(s) = u(s) - u(s)\,e^{-sT} = \underline{u(s)\left(1 - e^{-sT}\right)}$$

$H = \dfrac{V_o}{u} =$

$$\boxed{H = \frac{1 - e^{-sT}}{s}}$$

Transf funct. of a zero-order-hold.

Sampled-data system example: frequency domain



$$H = \frac{1 - e^{-sT}}{s}$$

$$v^*(s) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} v(s - jk\omega_s)$$

$$v_o(s) = \frac{1 - e^{-sT}}{s} v^*(s) = \frac{1 - e^{-sT}}{sT} \sum_{k=-\infty}^{+\infty} v(s - jk\omega_s)$$

Consider only low-frequency signals: $v_o(s) \approx \dfrac{1 - e^{-sT}}{sT} v(s)$

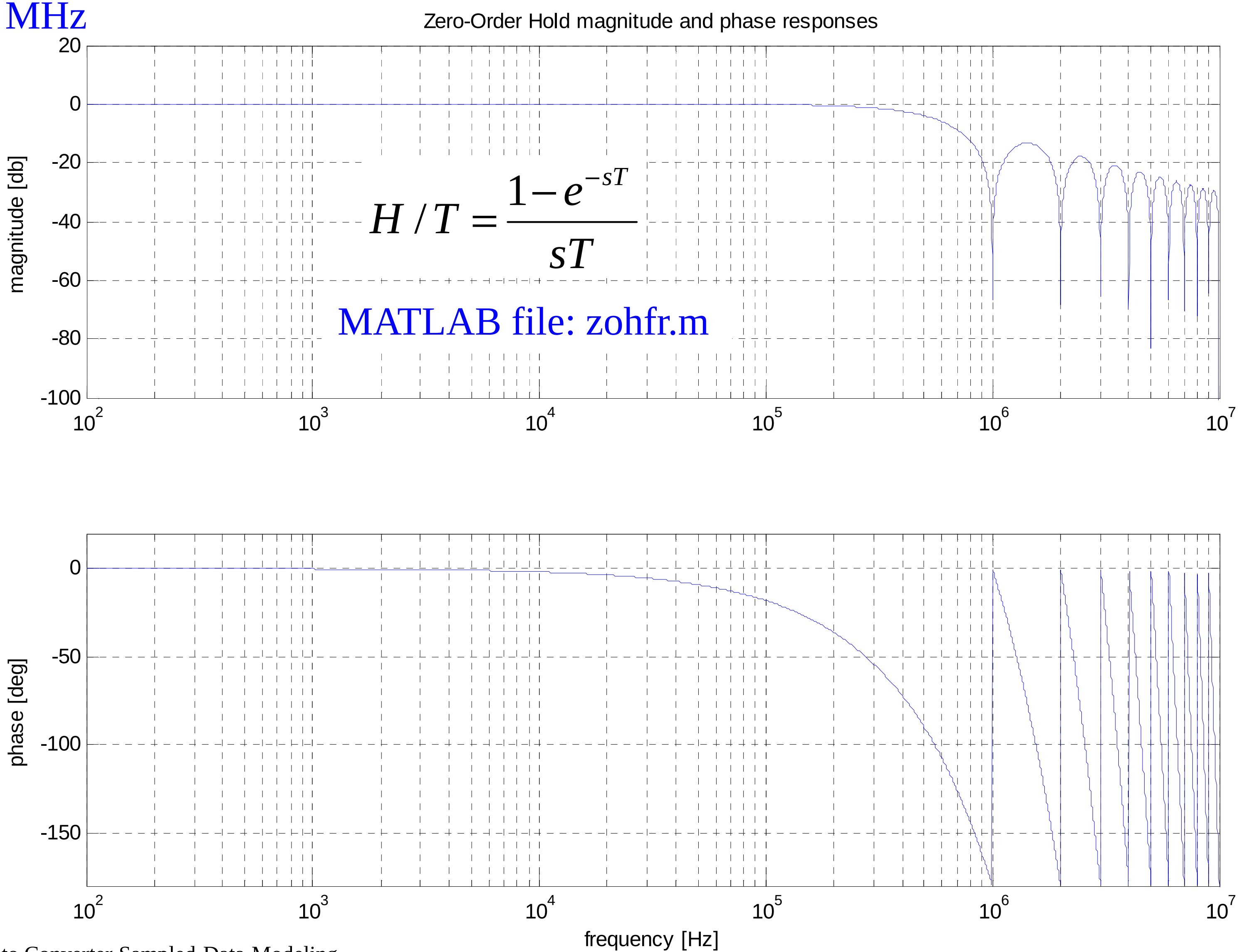
System “transfer function” = $\dfrac{v_o}{v} = \dfrac{1 - e^{-sT}}{sT}$

Zero-order hold: frequency responses

$$\frac{1-e^{-j\omega T}}{j\omega T}=e^{-j\omega T/2}\,\frac{e^{j\omega T/2}-e^{-j\omega T/2}}{2j}\,\frac{1}{\omega T/2}=\frac{\sin(\omega T/2)}{\omega T/2}e^{-j\omega T/2}=\mathrm{sinc}(\omega T/2)e^{-j\omega T/2}$$

Zero-order hold: frequency responses

f_s = 1 MHz



$$H/T = \frac{1-e^{-sT}}{sT}$$

MATLAB file: zohfr.m

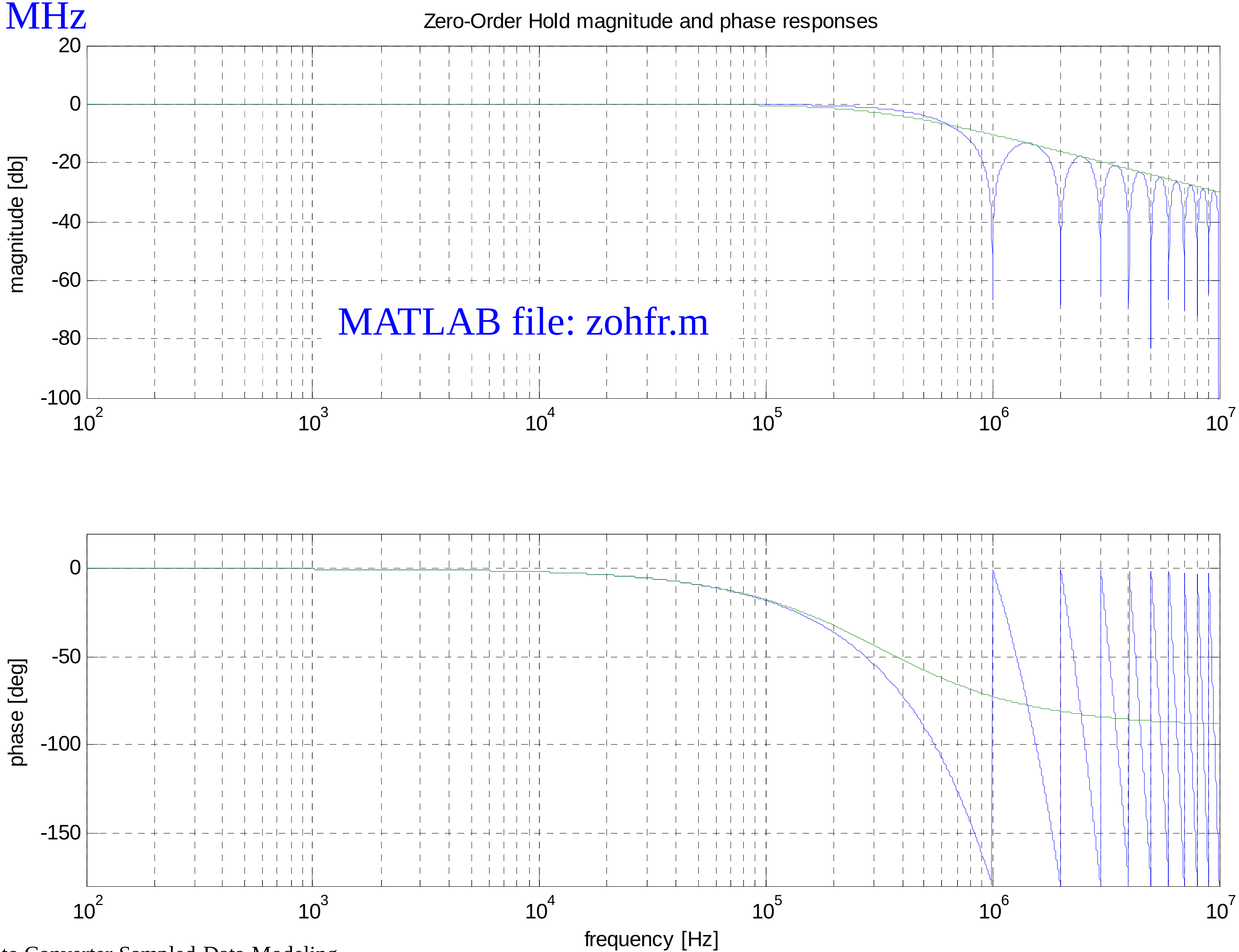
Zero-order hold: 1st-order approximation

$$e^{-sT} \approx \frac{1-\frac{s}{\omega_p}}{1+\frac{s}{\omega_p}}$$ 1st-order Pade approximation

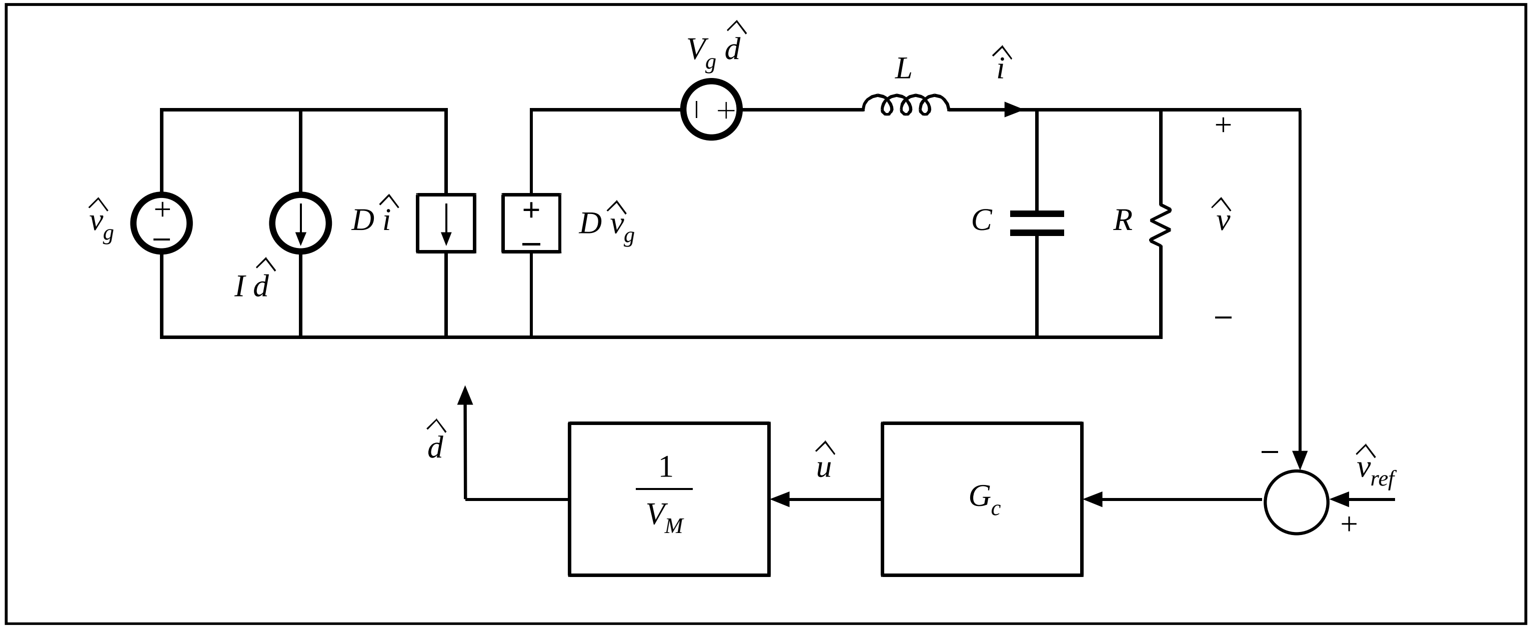
$$\boxed{\frac{1-e^{-sT}}{sT} \approx \frac{1}{1+\frac{s}{\omega_p}} \qquad \omega_p = \frac{2}{T} \qquad f_p = \frac{1}{T\pi} = \frac{f_s}{\pi}}$$

Zero-order hold: frequency responses

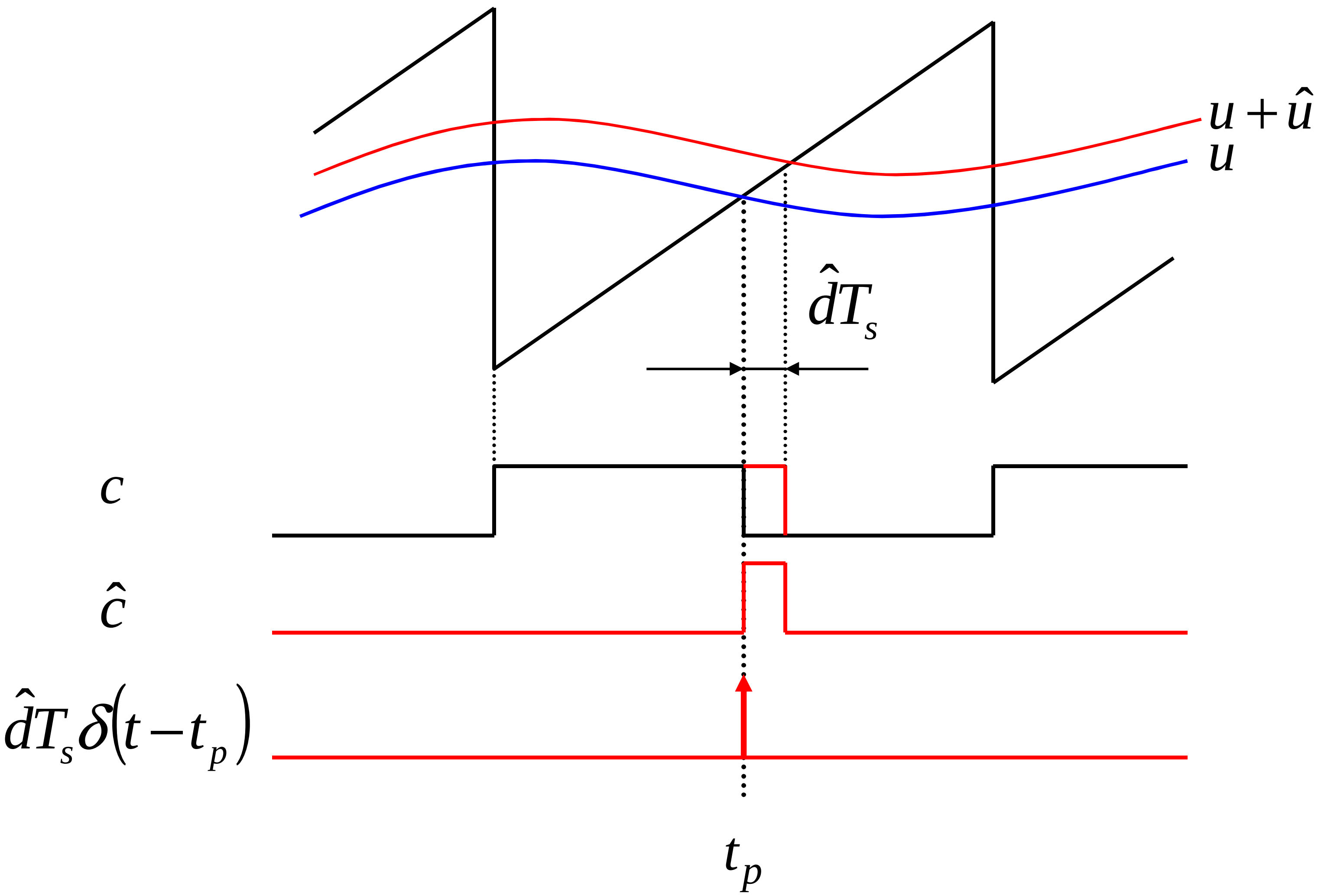
f_s = 1 MHz



How does any of this apply to converter modeling?

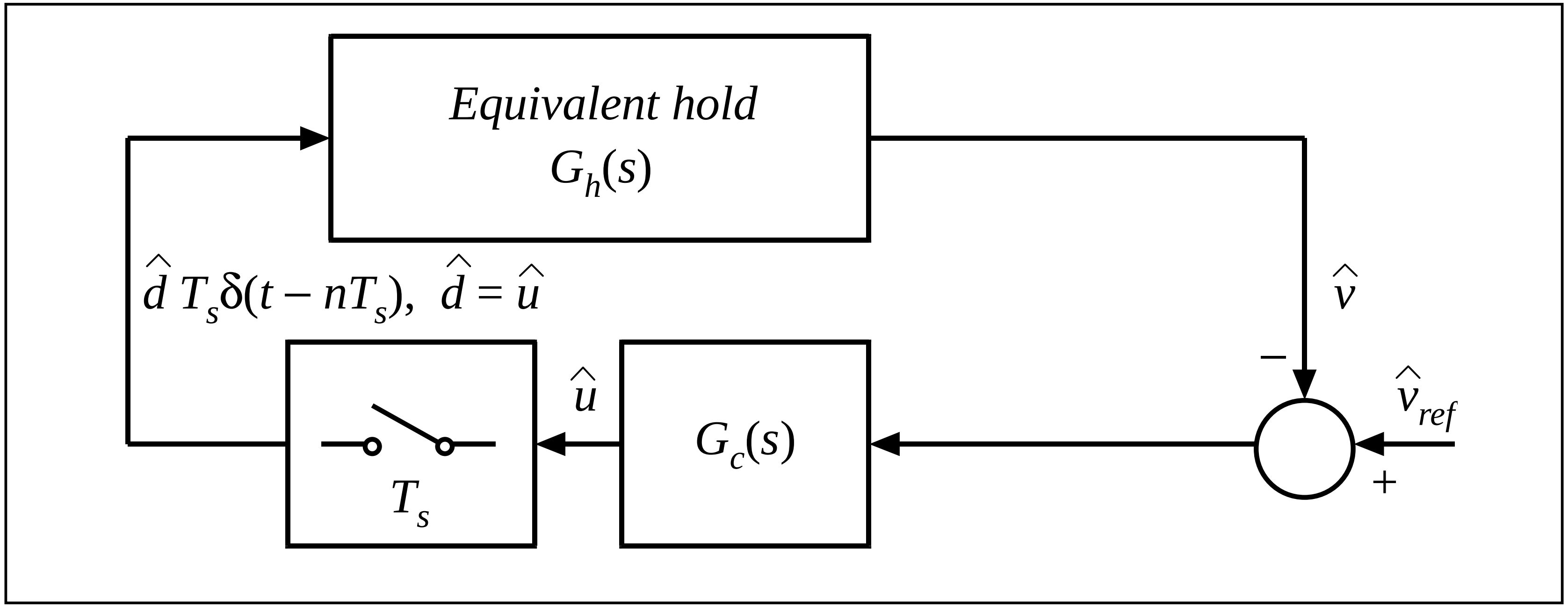


PWM is a small-signal sampler!



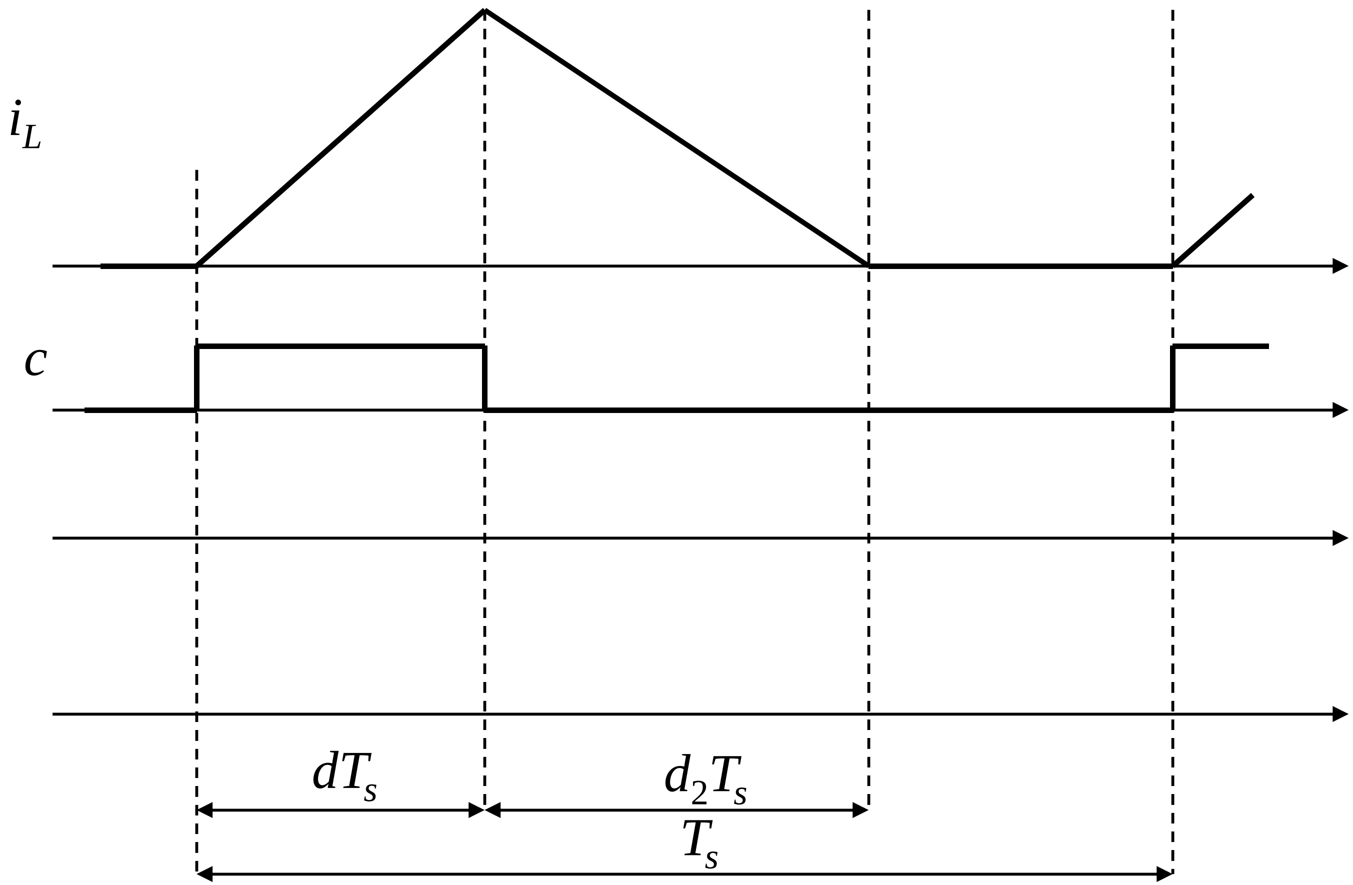
PWM sampling occurs at t_p (*i.e.* at dT_s, periodically, in each switching period)

General sampled-data model

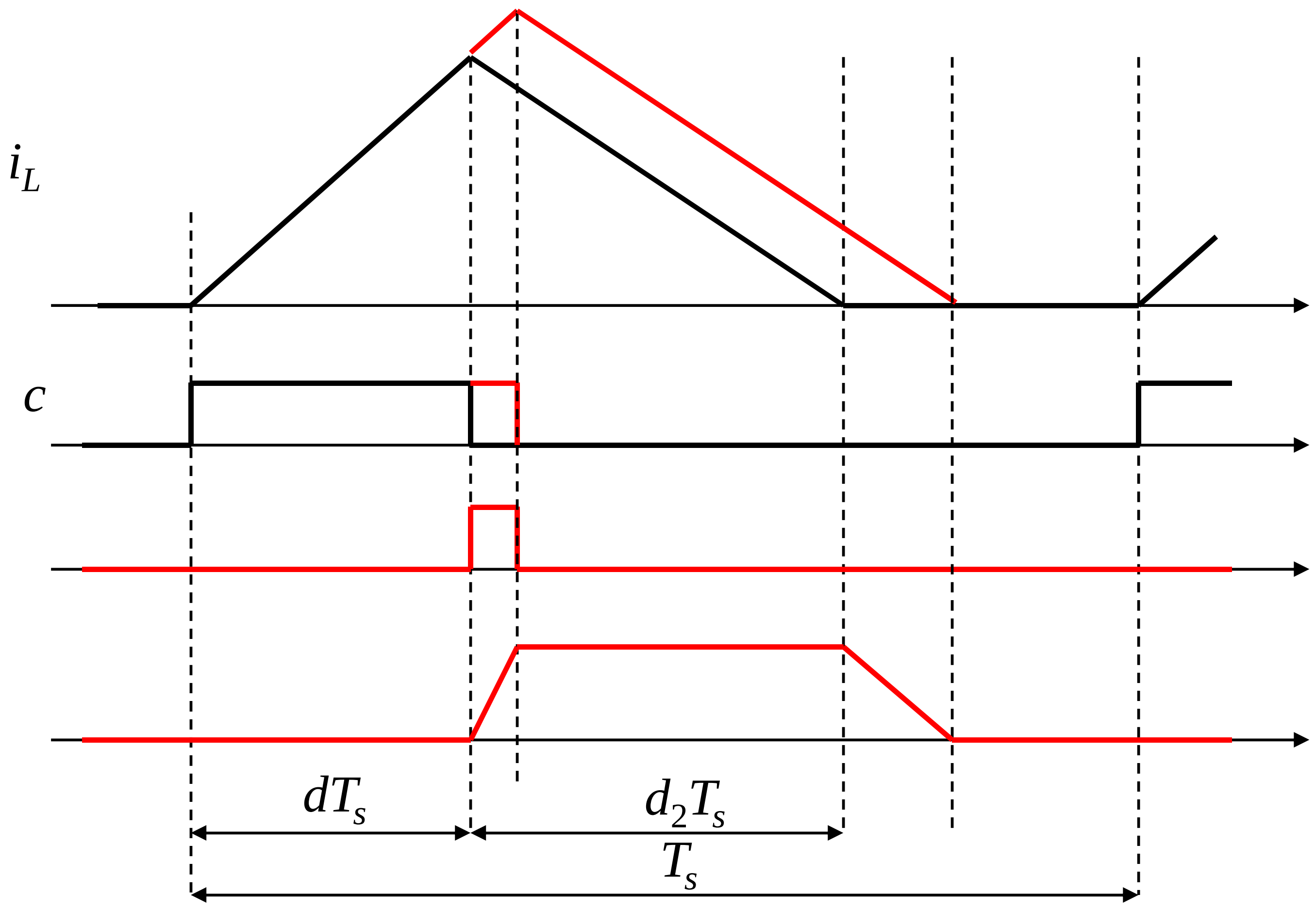


- Sampled-data model valid at all frequencies
- *Equivalent hold* describes the converter small-signal response to the sampled duty-cycle perturbations [Billy Lau, PESC 1986]
- State-space averaging or averaged-switch models are low-frequency continuous-time approximations to this sampled-data model

Application to DCM high-frequency modeling



Application to DCM high-frequency modeling



DCM inductor current high-frequency response

$$\hat{i}_L(s) = \frac{V_1 + V_2}{L} T_s \frac{1 - e^{-sD_2T_s}}{s} \hat{d}^*(s) = \frac{V_1 + V_2}{L} T_s \frac{1 - e^{-sD_2T_s}}{s} \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} \hat{d}(s - jk\omega_s)$$

$$\hat{i}_L(s) \approx \frac{V_1 + V_2}{L} D_2 T_s \frac{1 - e^{-sD_2T_s}}{D_2 T_s s} \hat{d}(s)$$

$$\frac{\hat{i}_L(s)}{\hat{d}(s)} \approx \frac{V_1 + V_2}{L} D_2 T_s \frac{1}{1 + \frac{s}{\omega_2}} \qquad \omega_2 = \frac{2}{D_2 T_s}$$

$$f_2 = \frac{f_s}{\pi D_2}$$

High-frequency pole due to the inductor current dynamics in DCM, see (11.77) in Section 11.3

Conclusions

- PWM is a small-signal sampler
- Switching converter is a sampled-data system
- Duty-cycle perturbations act as a string of impulses
- Converter response to the duty-cycle perturbations can be modeled as an equivalent hold
- Averaged small-signal models are low-frequency approximations to the equivalent hold
- In DCM, at high frequencies, the inductor-current dynamic response is described by an equivalent hold that behaves as zero-order hold of length $D_2 T_s$
- Approximate continuous-time model based on the DCM sampled-data model correlates with the analysis of Section 11.3: the same high-frequency pole at $f_s/(\pi D_2)$ is obtained
- Next: current-mode control (Chapter 12)