Bounding quantum capacities via partial orders and complementarity

Christoph Hirche[1] and Felix Leditzky[2]

[1]Zentrum Mathematik, Technical University of Munich, 85748 Garching, Germany
[1]Centre for Quantum Technologies, National University of Singapore, Singapore
[2]Department of Mathematics & IQUIST, University of Illinois Urbana-Champaign, ,
Urbana, IL, 61801 USA

Abstract

Quantum capacities are fundamental quantities that are notoriously hard to compute and can exhibit surprising properties such as superadditivity. Thus a vast amount of literature is devoted to finding close and computable bounds on these capacities. We add a new viewpoint by giving operationally motivated bounds on several capacities, including the quantum capacity and private capacity of a channel and the one-way distillable entanglement and private key of a bipartite state. Our bounds themselves are generally given by certain capacities of the complementary channel or state. As a tool to obtain these bounds we discuss partial orders on quantum channels, such as the less noisy and the more capable order. Our bounds help to further understand the interplay between different capacities and give operational limitations on superadditivity properties and the difference between capacities. They can also be used as a new approach towards numerically bounding capacities, as discussed with some examples.

1 Overview and main results

Capacities give the optimal rate at which a certain information theoretic task can be achieved. As such, they play a fundamental role whenever we are trying to understand the capabilities we get from using a specific resource, for example a given quantum channel or a quantum state. A wide array of tasks are of interest and they range from public or private information transmission over a channel to the distillation of, for example, maximally entangled or private states. In many cases we even know of mathematical formulas that exactly determine these capacities. Those could already be the end of our journey, however to really understand or even numerically evaluate these quantities still remains an extremely challenging task. Two typical questions are as follows. One, while we know from operational arguments that for example the rate at which we can transmit private classical information over a quantum channel is never smaller than the rate at which we can send quantum information over the same channel, it is often unknown how much more exactly of the former can be send. Two, for both of the mentioned examples the capacity is given by a regularized formula, meaning it has to be evaluated on n copies of the channel under the limit of n going to infinity. This makes numerical evaluation generally intractable. It is again easy to see that the regularized quantity can never be smaller than the single-copy version it is based on, but it is also a priori unclear how much bigger the regularized quantity can become.

Due to these challenges, a significant part of the quantum information literature strives to find better and better bounds on quantum capacities that help us to narrow down their numerical value and hence give a better understanding of their properties. A small collection of recent results includes for example [33, 9, 11, 8, 18, 34].

Naturally, a main focus in this area is to find bounds that give a good approximation of the capacity but importantly are also easy to evaluate numerically. However, it can often be difficult to assign any operational understanding to these bounds. In this work, on the other hand, we take a somewhat different approach and aim to find bounds on capacities that themselves have an operational interpretation, ideally also via capacities, looking for a better understanding of the information theoretic structures that allow for phenomena such as superadditivity.

An important concept in this work will be that of complementarity. It is well known that one can think of any quantum channel $\mathcal{N}$ as an isometry onto a larger space followed by discarding the additional system usually referred to as environment. The complement of that channel is then denoted $\mathcal{N}^c$ and can be found by keeping the environment while discarding the original output system. Note that while the complement of a given channel is not unique, all choices are information-theoretically equivalent.

As a starting point, consider the class of degradable channels. Those are channels for which the receiver can apply another channel $\mathcal{D}$ to simulate the complementary channel, i.e. $\mathcal{N}^c = \mathcal{D} \circ \mathcal{N}$. Intuitively this implies that the channel $\mathcal{N}$ should generally never be worse at information transmission than $\mathcal{N}^c$. It is known that as a consequence some capacities of the degradable channel $\mathcal{N}$ simplify, in particular the private capacity $P(\mathcal{N})$ and the quantum capacity $Q(\mathcal{N})$ [6, 27]:

$$P(\mathcal{N}) = Q(\mathcal{N}) = Q^{(1)}(\mathcal{N}) = P^{(1)}(\mathcal{N}), \tag{1.1}$$

where the channels coherent information $Q^{(1)}(\mathcal{N})$ and private information $P^{(1)}(\mathcal{N})$ are the corresponding, non-regularized, single-copy quantities. This comes with some understanding that the relationship between a channel and its complement determines properties of their capacities. Later, Watanabe made this idea more precise by translating the classical concept of less noisy and more capable channels to the quantum setting. Both of these classes had previously proven useful in classical information theory, but they gain a new meaning when applied to a quantum channel and its complement. Namely, we call a channel regularized less noisy when the private capacity of its complement is zero, i.e. $P(\mathcal{N}^c) = 0$, and regularized more capable when its complements quantum capacity is zero, i.e. $Q(\mathcal{N}^c) = 0$. Note that degradability is a stricter requirement and implies the others. Watanabe showed that the former case is still sufficient for Equation (1.1) to hold and the latter, which is even weaker because $P(\mathcal{N}^c) \geq Q(\mathcal{N}^c)$, still implies

$$P(\mathcal{N}) = Q(\mathcal{N}). \tag{1.2}$$

Naturally, it is desirable to see what we can learn from these results for general channels. To this end, Sutter et. al. introduced the concept of approximately degradable channels [32], showing that the relations in Equation (1.1) still hold approximately when a channel is close to being degradable. This lead to some of the best capacity bounds available which are even efficiently computable as the optimal approximation constant is given by a convex optimization problem. Recently [12] introduced approximately less noisy and more capable classes, leading to potentially tighter bounds, however at the cost of generally loosing the efficient computability. Here, we remedy this disadvantage by showing that the approach can be used to give bounds with operational meaning and extend on the previously achieved results.

We will now discuss the main results of this work, while referring to the later sections for technical definitions, statements and proofs. In particular, the technical sections include new results on connections between classes of channels and partial orders that might be of independent interest beyond the capacity bounds presented here.

As a warm-up to the structure of our results we give bounds on the classical capacity $C(\mathcal{N})$ and the entanglement assisted classical capacity $C_E(\mathcal{N})$ in Theorem 2.3, including

$$Q(\mathcal{N}) \leq C(\mathcal{N}) \quad \leq Q(\mathcal{N}) + C(\mathcal{N}^c) \tag{1.3}$$

$$2Q^{(1)}(\mathcal{N}) \leq C_E(\mathcal{N}) \leq 2Q^{(1)}(\mathcal{N}) + C_E(\mathcal{N}^c), \tag{1.4}$$

where the first is easy to see and the second follows by showing equivalence of two classes of channels in Lemma 2.2. Note that different to the other capacity formulas discussed here, $C_E(\mathcal{N})$ does not require regularization and can be efficiently computed [10].

Next, we get back to the private and quantum capacity. Therefore we extend the results in [12] that help us to argue that the quantum capacity of the complementary channel limits how different the private and quantum capacity can be,

$$Q^{(1)}(\mathcal{N}) \leq P^{(1)}(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + Q^{(1)}(\mathcal{N}^c), \tag{1.5}$$

$$Q(\mathcal{N}) \leq P(\mathcal{N}) \quad \leq Q(\mathcal{N}) \quad + Q(\mathcal{N}^c), \tag{1.6}$$

and the entanglement assisted private information $P_E(\mathcal{N}^c)$, as defined later, limits the increase due to regularization,

$$Q^{(1)}(\mathcal{N}) \leq Q(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + P_E(\mathcal{N}^c), \tag{1.7}$$

$$P^{(1)}(\mathcal{N}) \leq P(\mathcal{N}) \leq P^{(1)}(\mathcal{N}) + Q(\mathcal{N}^c) + P_E(\mathcal{N}^c), \tag{1.8}$$

see Corollary 2.5. The entanglement assisted private information $P_E(\mathcal{N})$ was proven in [22] to equal the entanglement assisted private capacity of degradable channels. Here it should be remarked that this extends a result in [4] where it was essentially shown that, translated to our notation, if $P_E(\mathcal{N}^c) = 0$ then $Q^{(1)}(\mathcal{N}) = Q(\mathcal{N})$. Originally the result was formulated in terms of a partial order called informational degradability to which we will later refer to as fully quantum less noisy.

The above bounds give an operationally meaningful, quantitative version of the results by Watanabe. Furthermore they make the intuition precise that also for general channels the properties of the complementary channel limit the possibility of having superadditivity or a higher private capacity than quantum capacity in a fundamental way.

Section 3 then slightly changes focus from investigating channels to quantum states. Approximate degradable quantum states were defined in [17] and used there to give bounds on the one-way distillable entanglement $D_{\rightarrow}(\rho_{AB})$. Additionally we consider here the one-way distillable private key $K_{\rightarrow}(\rho_{AB})$. We define new partial orders based on these two quantities and let them lead us to results similar to the channel setting. First, we define the complementary state ρ^c_{AB} of a state ρ_{AB} as $\rho^c_{AB} := \rho_{AE} = \operatorname{Tr}_B \Psi_{ABE}$ where Ψ_{ABE} is a purification of ρ_{AB}. Now, we show that the one-way distillable entanglement of the complementary state limits the difference between distillable key and entanglement, see Theorem 3.4,

$$D^{(1)}_{\rightarrow}(\rho_{AB}) \leq K^{(1)}_{\rightarrow}(\rho_{AB}) \leq D^{(1)}_{\rightarrow}(\rho_{AB}) + D^{(1)}_{\rightarrow}(\rho^c_{AB}) \tag{1.9}$$

$$D_{\rightarrow}(\rho_{AB}) \leq K_{\rightarrow}(\rho_{AB}) \ \leq D_{\rightarrow}(\rho_{AB}) + D_{\rightarrow}(\rho^c_{AB}), \tag{1.10}$$

and the complements one-way distillable key limits the increase due to regularization, see Theorem 3.3 and Corollary 3.5,

$$D^{(1)}_{\rightarrow}(\rho_{AB}) \leq D_{\rightarrow}(\rho_{AB}) \leq D^{(1)}_{\rightarrow}(\rho_{AB}) + K_{\rightarrow}(\rho^c_{AB}) \tag{1.11}$$

$$K^{(1)}_{\rightarrow}(\rho_{AB}) \leq K_{\rightarrow}(\rho_{AB}) \leq K^{(1)}_{\rightarrow}(\rho_{AB}) + K_{\rightarrow}(\rho^c_{AB}) + D_{\rightarrow}(\rho^c_{AB}). \tag{1.12}$$

Together, these results show that a similar intuition as for channels also holds for states, namely that the possibility of extracting certain resources from the complementary state determines properties of the capacities of the state itself.

Finally in Section 4 we discuss symmetric side-channel assisted capacities and how superactivation is directly related to the question whether the sets of degradable and regularized less noisy channels are actually different, i.e. we show the implication

$$P(\cdot) \text{ can be superactivated} \quad \Rightarrow \quad DEG \subsetneq LN_{\infty}. \tag{1.13}$$

While the bounds in this work might not directly help to numerically nail down the exact value of a certain capacity, they provide new operational insides in how different capacities play together and especially what limits they have naturally set. By themselves our bounds are competitive to the best bounds in the literature, in particular they are at least as good as those one gets from approximate degradability [32]. Furthermore, because of their form, they can be combined with other capacity bounds to also give numerical bounds on the discussed capacities. Notably, not many bounds for the private capacity and the one-way distillable key are known, however our bounds allow to e.g. apply quantum capacity bounds to the private capacity. As an example we consider the Horodecki channel $\mathcal{N}_H$ [14] for which one can use the quantum capacity bound from [34] to get

$$P(\mathcal{N}_H) \leq Q(\mathcal{N}_H) + Q(\mathcal{N}^c_H) \leq 0.7284. \tag{1.14}$$

Therefore our presented bounds can also provide new ways for numerical investigation of capacities. We end by discussing some open problems in Section 5 that we hope will inspire further research in this direction.

Name	Entropic formulation	Capacity formulation
ϵ-less noisy	$I(U:E) \leq I(U:B) + \epsilon \quad \forall \rho_{UA}$	$P^{(1)}(\mathcal{N}^c) \leq \epsilon$
ϵ-regularized less noisy	$I(U:E^n) \leq I(U:B^n) + n\epsilon \quad \forall \rho_{UA^n}$	$P(\mathcal{N}^c) \leq \epsilon$
ϵ-fully quantum less noisy	$I(A:E) \leq I(A:B) + \epsilon \quad \forall \rho_{AA'}$	$P_E(\mathcal{N}^c) \leq \epsilon$
ϵ-more capable	$I(X:E) \leq I(X:B) + \epsilon \quad \forall \rho_{XA}$	$Q^{(1)}(\mathcal{N}^c) \leq \epsilon$
ϵ-regularized more capable	$I(X:E^n) \leq I(X:B^n) + n\epsilon \quad \forall \rho_{XA^n}$	$Q(\mathcal{N}^c) \leq \epsilon$
ϵ-fully quantum more capable	$I(A:E) \leq I(A:B) + \epsilon \quad \forall \Psi_{AA'}$	$Q_E(\mathcal{N}^c) \leq \epsilon$

Table 1: Definitions of the main sets of channels discussed. The entropic formulations are easily generalized to an arbitrary pair of channels $\mathcal{N}$ and $\mathcal{M}$.

2 Partial orders on channels and their implications

2.1 Definitions and properties

Some quick notation: Systems are denoted by capital letters, generally A, B, E denote quantum systems and U, T, X denote classical systems. All entropic quantities and capacities are defined in the usual way, see e.g. [36].

Well known orderings on quantum channels include the degrading, less noisy and more capable partial orders that can be found at several points in the literature. Here we consider a relaxed form: approximate partial orders. For an overview and recent results regarding these orders see [12]. An order that is particularly often used when bounding channel capacities is ϵ-degradabiltiy introduced by Sutter et. al. [32]. We will sometimes use it for comparison and therefore state the definition here. Note that this order implies all the other orders discussed in this section as can be seen from standard data-processing arguments.

Definition 2.1 ([32])**.** *A channel $\mathcal{N}$ is said to be an ϵ-degraded version of $\mathcal{M}$ if there exists a channel Θ such that $\|\mathcal{N} - \Theta \circ \mathcal{M}\|_\diamond \leq \epsilon$.*

In particular in classical information theory also the more capable and less noisy orders play an important role [24]. These are generally defined based on an entropic condition on the output states of the channels that needs to hold for a specified set of inputs. As so often, there is more than one way to translate the classical concept to the quantum setting. Out of these, the first to be introduced to the quantum world were the regularized more capable and less noisy orders by Watanabe [35] who also realized that, whenever the second channel is fixed to be the complementary of the first, they have a natural formulation in terms of the capacities of the complementary channel.

In this work we will consider three types of quantum generalizations of approximate orders which are summarized in Table 1, see also [12]. Generally speaking, the less noisy orders are based on mixed states, i.e. either the set of mixed quantum states $\rho_{AA'}$ or the classical-quantum states with mixed ensemble

$$\rho_{UA} = \sum_u p(u)|u\rangle\langle u| \otimes \rho_A^u, \tag{2.1}$$

while the more capable orders are based on pure states, i.e. either the set of pure states $\Psi_{AA'}$ or the classical-quantum states with pure ensemble

$$\rho_{XA} = \sum_u p(x)|x\rangle\langle x| \otimes \Psi_A^x. \tag{2.2}$$

We will now discuss the technical definitions of the capacity formulas used in this section. First, the classical capacity $C(\mathcal{N})$ of a quantum channel $\mathcal{N}$ is given by [13, 23]

$$C(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n}\chi(\mathcal{N}^{\otimes n}), \tag{2.3}$$

$$\chi(\mathcal{N}) = \sup_{\rho_{XA}} I(X:B), \tag{2.4}$$

where $\chi(\mathcal{N})$ is also called the Holevo quantity and its optimization is over the states defined in Equation (2.2). The entanglement assisted classical capacity $C_E(\mathcal{N})$ is [2]

$$C_E(\mathcal{N}) = \sup_{\Psi_{AA'}} I(A:B). \tag{2.5}$$

Further, we need the quantum capacity $Q(\mathcal{N})$ given by [21, 26, 5]

$$Q(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} Q^{(1)}(\mathcal{N}^{\otimes n}), \tag{2.6}$$

$$Q^{(1)}(\mathcal{N}) = \sup_{\rho_{XA}} I(X:B) - I(X:E) \tag{2.7}$$

$$= \sup_{\Psi_{AA'}} I(A\rangle B), \tag{2.8}$$

where $I(A\rangle B)$ is the coherent information, and the private capacity $P(\mathcal{N})$ given by [3, 5]

$$P(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} P^{(1)}(\mathcal{N}^{\otimes n}), \tag{2.9}$$

$$P^{(1)}(\mathcal{N}) = \sup_{\rho_{UA}} I(U:B) - I(U:E), \tag{2.10}$$

where the supremum in the last line is over states of the from in Equation (2.1). Finally we will need two quantities that are not strictly speaking capacities, at least not for all $\mathcal{N}$, namely the entanglement assisted private information [22]

$$P_E(\mathcal{N}) = \sup_{\rho_{AA'}} I(A:B) - I(A:E) \tag{2.11}$$

and its restriction to pure states

$$Q_E(\mathcal{N}) = \sup_{\Psi_{AA'}} I(A:B) - I(A:E). \tag{2.12}$$

Note that it was shown in [22] that for degradable channels $P_E(\mathcal{N}) = Q_E(\mathcal{N})$ and both then correspond to the entanglement assisted private capacity of the degraded channel $\mathcal{N}$. We will further expand on this comment at the end of this section. Also, if one desires an upper bound that has an operational interpretation for all channels, observe that

$$P^{(1)}(\mathcal{N}^c) \le P_E(\mathcal{N}^c) \le 2Q_{ss}(\mathcal{N}^c), \tag{2.13}$$

where the last quantity is the quantum capacity with symmetric side channel assistance that will be further discussed later.

We are now almost set to start exploring the desired capacity bounds, but will before make a useful observation regarding the fully quantum more capable order and its associated capacity formula.

Lemma 2.2. *Let $\mathcal{N}$ be a quantum channel, we have*

$$Q_E(\mathcal{N}) = 2Q^{(1)}(\mathcal{N}) \tag{2.14}$$

and therefore for $\mathcal{N}$

$$\epsilon\text{-more capable} \Leftrightarrow 2\epsilon\text{-fully quantum more capable.} \tag{2.15}$$

Proof. Let V be a Stinespring isometry of $\mathcal{N}$, and $\Psi_{ABE} = V\Psi_{AA'}V^\dagger$ for an arbitrary pure state $\Psi_{AA'}$. We have

$$I(A\rangle B) = H(B) - H(AB) \tag{2.16}$$

$$= \frac{1}{2}H(B) + \frac{1}{2}H(AE) - \frac{1}{2}H(AB) - \frac{1}{2}H(E) \tag{2.17}$$

$$= \frac{1}{2}I(A:B) - \frac{1}{2}I(A:E). \tag{2.18}$$

Since this holds for every pure state $\Psi_{AA'}$ it proves the first statement. The second then follows by definition of the orders. $\square$

2.2 Capacity bounds

We start by discussing the classical capacities of a quantum channel as a warm-up.

Theorem 2.3. *For a quantum channel $\mathcal{N}$, we have*

$$
\begin{aligned}
Q^{(1)}(\mathcal{N}) \leq \chi(\mathcal{N}) &\leq Q^{(1)}(\mathcal{N}) + \chi(\mathcal{N}^c) && (2.19)\\
Q(\mathcal{N}) \leq C(\mathcal{N}) &\leq Q(\mathcal{N}) + C(\mathcal{N}^c) && (2.20)\\
2Q^{(1)}(\mathcal{N}) \leq C_E(\mathcal{N}) &\leq Q_E(\mathcal{N}) + C_E(\mathcal{N}^c) = 2Q^{(1)}(\mathcal{N}) + C_E(\mathcal{N}^c) && (2.21)
\end{aligned}
$$

Proof. For each statement the first inequality is well known and is meant for comparison. The second inequality in the first statement follows by picking the state optimal state ρ_{XA} for $\chi(\mathcal{N})$ and simply noting

$$
\begin{aligned}
\chi(\mathcal{N}) &= I(X:B) && (2.22)\\
&= I(X:B) - I(X:E) + I(X:E) && (2.23)\\
&\leq Q^{(1)}(\mathcal{N}) + \chi(\mathcal{N}^c). && (2.24)
\end{aligned}
$$

The second statement follows from the first by regularizing. The third statement follows similarly to the first, using Lemma 2.2 for the last equality. □

To make the connection to partial orders, one can note as a direct consequence that e.g. if a channel $\mathcal{N}$ is anti-more capable we immediately have

$$
\begin{aligned}
\chi(\mathcal{N}) &\leq \chi(\mathcal{N}^c) && (2.25)\\
C_E(\mathcal{N}) &\leq C_E(\mathcal{N}^c) && (2.26)
\end{aligned}
$$

and similarly, if $\mathcal{N}$ is anti-regularized more capable,

$$
C(\mathcal{N}) \leq C(\mathcal{N}^c). \qquad (2.27)
$$

This gives our first simple bounds and also solidifies the intuition that capacities are limited by the usefulness of the channels complement.

We will now consider the more interesting case of quantum capacities of quantum channels. In [12] the approximate partial orders defined at the beginning of the section were used to proof a quantitative version of the previous results by Watanabe [35]. Those results will serve as starting point.

Theorem 2.4 ([12])**.** *Let $\mathcal{N}$ be a quantum channel.*

(i) If $\mathcal{N}$ is ϵ-more capable, then $Q^{(1)}(\mathcal{N}) \leq P^{(1)}(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + \epsilon$.

(ii) If $\mathcal{N}$ is ϵ-regularized more capable, then $Q(\mathcal{N}) \leq P(\mathcal{N}) \leq Q(\mathcal{N}) + \epsilon$.

(iii) If $\mathcal{N}$ is ϵ-fully quantum less noisy, then $Q^{(1)}(\mathcal{N}) \leq Q(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + \epsilon$.

(iv) If $\mathcal{N}$ is ϵ-fully quantum less noisy and ϵ-regularized more capable, then $P^{(1)}(\mathcal{N}) \leq P(\mathcal{N}) \leq P^{(1)}(\mathcal{N}) + 2\epsilon$.

Here, we record the following simple but important observation: for any quantum channel $\mathcal{N}$, approximate partial orders can always be satisfied when considering the approximation parameters in terms of capacities of the complementary channel $\mathcal{N}^c$. For example, every channel is ϵ-regularized more capable if we choose $\epsilon = Q(\mathcal{N}^c)$ and similar for the other orders. This immediately leads us to the following result. More precisely, we have the following:

Corollary 2.5. *Let $\mathcal{N}$ be a quantum channel, we have*

$$
\begin{aligned}
Q^{(1)}(\mathcal{N}) \leq P^{(1)}(\mathcal{N}) &\leq Q^{(1)}(\mathcal{N}) + Q^{(1)}(\mathcal{N}^c), && (2.28)\\
Q(\mathcal{N}) \leq P(\mathcal{N}) &\leq Q(\mathcal{N}) + Q(\mathcal{N}^c), && (2.29)\\
Q^{(1)}(\mathcal{N}) \leq Q(\mathcal{N}) &\leq Q^{(1)}(\mathcal{N}) + P_E(\mathcal{N}^c), && (2.30)\\
P^{(1)}(\mathcal{N}) \leq P(\mathcal{N}) &\leq P^{(1)}(\mathcal{N}) + Q(\mathcal{N}^c) + P_E(\mathcal{N}^c). && (2.31)
\end{aligned}
$$

Proof. They are a direct consequence of Theorem 2.4 and the previous observations. □

The corollary gives operationally meaningful bounds on the maximal difference between the private and the quantum capacity and the possible advantage to be gained from regularizing.

Although we are mostly concerned with upper bounds in this work, we want to state here that a similar idea can conversely sometimes a also be used to detect differences between the capacities. To this extend [35] proved that a channel being more capable is often also a necessary condition for the private information and and the coherent information of a channel to be equal. The following is essentially [35, Proposition 2] restated in the language of this work.

Corollary 2.6. *Let ρ_A^* be the optimizer of $Q^{(1)}(\mathcal{N})$. If ρ_A^* is full rank and $Q^{(1)}(\mathcal{N}^c) > 0$, then*

$$P^{(1)}(\mathcal{N}) > Q^{(1)}(\mathcal{N}). \tag{2.32}$$

If $|A| = 2$ and $P^{(1)}(\mathcal{N}) > 0$, then $Q^{(1)}(\mathcal{N}^c) = 0$ if and only if

$$P^{(1)}(\mathcal{N}) = Q^{(1)}(\mathcal{N}). \tag{2.33}$$

Proof. If $Q^{(1)}(\mathcal{N}^c) > 0$ then there exists at least one state ρ for which $I(A\rangle E)_\rho > 0$ and equivalently $I(A\rangle B)_\rho < 0$. With that the conditions for [35, Proposition 2] are fulfilled which then gives directly the first statement. The second is a direct translation of the second part of [35, Proposition 2]. □

In [35] it was furthermore shown that a channel being less noisy is equivalent to the channels coherent information being concave. We now give an approximate version of this observation leading to "approximate" concavity and convexity results for general quantum channels.

Lemma 2.7. *For ρ_A^i quantum states and a probability distribution $p(i)$ define $\rho_A = \sum_i p(i)\rho_A^i$. A channel $\mathcal{N}$ being ϵ-approximate less noisy is equivalent to the statement*

$$\sum_i p(i) I(A\rangle B)_{\rho_i} \leq I(A\rangle B)_\rho + \epsilon, \tag{2.34}$$

where $I(A\rangle B)_\rho$ is evaluated on the state $\mathcal{N}(\Psi_{AA'})$ with $\Psi_{AA'}$ a purifications of ρ_A. Similarly, a channel $\mathcal{N}$ being ϵ-approximate anti-less noisy is equivalent to

$$\sum_i p(i) I(A\rangle B)_{\rho_i} \geq I(A\rangle B)_\rho - \epsilon. \tag{2.35}$$

For an arbitrary quantum channel $\mathcal{N}$ we have

$$I(A\rangle B)_\rho - P^{(1)}(\mathcal{N}) \leq \sum_i p(i) I(A\rangle B)_{\rho_i} \leq I(A\rangle B)_\rho + P^{(1)}(\mathcal{N}^c). \tag{2.36}$$

Proof. The first statement follows by adjusting the proof of [35, Proposition 3] using the approximate order and similarly for the second statement. The third is then a direct consequence of the fact that every channel is ϵ-approximate less noisy with $\epsilon = P^{(1)}(\mathcal{N}^c)$ and ϵ-approximate anti-less noisy with $\epsilon = P^{(1)}(\mathcal{N})$. □

At this point it might be worth noting that while we use the fully quantum less noisy order in Theorem 2.4, it was shown in [35] that for $\epsilon = 0$ the same can be proven using the regularized less noisy order. In order to do the the authors take a detour using an alternative partial order based on relative entropies. We will for now define the following auxiliary quantities.

Definition 2.8. *For a quantum channel $\mathcal{N}$ we define the following quantities,*

$$R^{(1)}(\mathcal{N}) = \sup_{\rho_A, \sigma_A} D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) - D(\mathcal{N}^c(\rho)\|\mathcal{N}^c(\sigma)), \tag{2.37}$$

$$R(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} R^{(1)}(\mathcal{N}^{\otimes n}). \tag{2.38}$$

Going back to the work of [35] one can find the following inequality by adjusting their proof,

$$Q^{(1)}(\mathcal{N}) \leq Q(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + R(\mathcal{N}^c). \tag{2.39}$$

The quantity $R(\mathcal{N}^c)$ is interesting here because in [35] it was shown that the condition $R(\mathcal{N}^c) = 0$ for a channel $\mathcal{N}$ is equivalent to $P(\mathcal{N}^c) = 0$, i.e. the partial orders induced by the two quantities are the same. Unfortunately the same does not hold true for values other than 0, i.e. the approximate partial orders, see Appendix A for an example, otherwise we could prove the conjecture stated below. In summary we can prove the following result.

Theorem 2.9. *For a quantum channel* $\mathcal{N}$ *we have*

$$Q^{(1)}(\mathcal{N}) \leq Q(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + M(\mathcal{N}^c) \tag{2.40}$$

$$P^{(1)}(\mathcal{N}) \leq P(\mathcal{N}) \leq P^{(1)}(\mathcal{N}) + Q(\mathcal{N}^c) + M(\mathcal{N}^c), \tag{2.41}$$

where $M(\mathcal{N}^c) = \min\{R(\mathcal{N}^c), P_E(\mathcal{N}^c)\} \leq 2Q_{ss}(\mathcal{N}^c)$.

As announced, motivated by the above discussion we make the following conjecture.

Conjecture 2.10. *For a quantum channel* $\mathcal{N}$ *we have*

$$Q^{(1)}(\mathcal{N}) \leq Q(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + P(\mathcal{N}^c) \tag{2.42}$$

$$P^{(1)}(\mathcal{N}) \leq P(\mathcal{N}) \leq P^{(1)}(\mathcal{N}) + Q(\mathcal{N}^c) + P(\mathcal{N}^c) \leq P^{(1)}(\mathcal{N}) + 2P(\mathcal{N}^c). \tag{2.43}$$

Finally, we want bounds that relate the quantities P_E and Q_E.

Theorem 2.11. *Let* $\mathcal{N}$ *be an* ϵ*-fully quantum less noisy quantum channel, then*

$$Q_E(\mathcal{N}) \leq P_E(\mathcal{N}) \leq Q_E(\mathcal{N}) + \epsilon. \tag{2.44}$$

Therefore we have for any quantum channel $\mathcal{N}$

$$Q_E(\mathcal{N}) \leq P_E(\mathcal{N}) \leq Q_E(\mathcal{N}) + P_E(\mathcal{N}^c) \tag{2.45}$$

and equivalently

$$Q^{(1)}(\mathcal{N}) \leq \frac{1}{2}P_E(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + \frac{1}{2}P_E(\mathcal{N}^c). \tag{2.46}$$

Proof. The first inequality in Equation (2.44) follows be definition of the two quantities. We now prove the second inequality. Let $\rho_{AA'}$ the state the achieves the optimal value of $P_E(\mathcal{N})$ and let $\Psi_{ABER} = \mathcal{N}(\Psi_{AA'R})$ where $\Psi_{AA'R}$ is a purification of $\rho_{AA'}$. Observe the following,

$$I(A:B) - I(A:E) \tag{2.47}$$

$$= H(B) - H(AB) - H(E) + H(AE) \tag{2.48}$$

$$= H(B) - H(ER) - H(E) + H(BR) - H(RA) + H(RA) - H(RAB) + H(E) + H(RAE) - H(B) \tag{2.49}$$

$$= I(RA:B) - I(RA:E) + I(R:E) - I(R:B) \tag{2.50}$$

$$\leq Q_E(\mathcal{N}) + \epsilon, \tag{2.51}$$

where the second equality uses several times the purity of Ψ_{ABER} and the inequality follows the system AR purifies the channel input and because $\mathcal{N}$ is ϵ-fully quantum less noisy.

The next statement of the theorem follow because every channel is ϵ-fully quantum less noisy with $\epsilon = P_E(\mathcal{N}^c)$. The last statement follows from Lemma 2.2. □

Note that as a special case of the above, if $\mathcal{N}$ is fully quantum less noisy one finds

$$P_E(\mathcal{N}) = Q_E(\mathcal{N}) = 2Q(\mathcal{N}) \tag{2.52}$$

which should be compared to [22] where the first equality was shown for the potentially smaller set of degradable channels, but on he other hand in the more general setting of broadcast channels.

2.3 Comparison to approximate degradability bounds

In the introduction we briefly discussed the capacity bounds given in [32] and boldly claimed that our bounds are always at least as good as theirs. Here we will prove this statement and also show that we can sometimes slightly improve the results in [32]. A similar discussion can be found in [12], but we will add a few new elements leading to slightly improved constants and a simpler derivation. We start by defining the following two functions,

$$f_1(|E|,\epsilon) = \frac{\epsilon}{2}\log(|E|-1) + h\left(\frac{\epsilon}{2}\right), \tag{2.53}$$

$$f_2(|E|,\epsilon) = \epsilon\log|E| + \left(1+\frac{\epsilon}{2}\right)h\left(\frac{\epsilon}{2+\epsilon}\right), \tag{2.54}$$

where $h(x)$ is the binary entropy. Note that $f_1(|E|,\epsilon) \leq f_2(|E|,\epsilon)$. As defined in [32], a channel is called ϵ-degradable if there exists a channel $\mathcal{D}$ such that

$$\|\mathcal{N}^c - \mathcal{D}\circ\mathcal{N}\|_\diamond \leq \epsilon. \tag{2.55}$$

The main results of [32] can be stated as follows.

Theorem 2.12 (Theorem 3.4 in [32])**.** *Let $\mathcal{N}$ be ϵ-degradable, then*

$$Q^{(1)}(\mathcal{N}) \leq Q(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + f_1(|E|,\epsilon) + f_2(|E|,\epsilon), \tag{2.56}$$

$$P^{(1)}(\mathcal{N}) \leq P(\mathcal{N}) \leq P^{(1)}(\mathcal{N}) + f_1(|E|,\epsilon) + 3f_2(|E|,\epsilon), \tag{2.57}$$

$$Q^{(1)}(\mathcal{N}) \leq P^{(1)}(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + f_1(|E|,\epsilon) + f_2(|E|,\epsilon). \tag{2.58}$$

One of the main tools used to prove this result were continuity bounds from [1, 39] which we state here adding a thirds for classical-quantum states recently proven in [37]. For two states ρ and σ with $\frac{1}{2}\|\rho-\sigma\|_1 \leq \epsilon$ it holds that,

$$|H(A)_\rho - H(A)_\sigma| \leq f_1(|A|,2\epsilon), \tag{2.59}$$

$$|H(A|X)_\rho - H(A|X)_\sigma| \leq f_1(|A|,2\epsilon), \tag{2.60}$$

$$|H(A|B)_\rho - H(A|B)_\sigma| \leq f_2(|A|,2\epsilon). \tag{2.61}$$

It was also shown in [32] that if $\mathcal{N}$ is ϵ-anti degradable then

$$Q(\mathcal{N}) \leq P(\mathcal{N}) \leq f_1(|B|,\epsilon) + f_2(|B|,\epsilon). \tag{2.62}$$

Similarly we can easily see the following.

Lemma 2.13. *If $\mathcal{N}$ is ϵ-anti degradable, then*

$$P_E(\mathcal{N}) \leq f_1(|B|,\epsilon) + f_2(|B|,\epsilon), \tag{2.63}$$

$$Q^{(1)}(\mathcal{N}) \leq P^{(1)}(\mathcal{N}) \leq 2f_1(|B|,\epsilon). \tag{2.64}$$

Proof. This follows directly from data-processing and the above mentioned continuity bounds. □

Combining this with our new capacity bounds one easily gets the following result.

Corollary 2.14. *Let $\mathcal{N}$ be ϵ-degradable, then*

$$Q^{(1)}(\mathcal{N}) \leq Q(\mathcal{N}) \quad \leq Q^{(1)}(\mathcal{N}) + f_1(|E|,\epsilon) + f_2(|E|,\epsilon), \tag{2.65}$$

$$P^{(1)}(\mathcal{N}) \leq P(\mathcal{N}) \quad \leq P^{(1)}(\mathcal{N}) + 2f_1(|E|,\epsilon) + 2f_2(|E|,\epsilon), \tag{2.66}$$

$$Q^{(1)}(\mathcal{N}) \leq P^{(1)}(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + 2f_1(|E|,\epsilon), \tag{2.67}$$

$$Q(\mathcal{N}) \leq P(\mathcal{N}) \quad \leq Q(\mathcal{N}) + f_1(|E|,\epsilon) + f_2(|E|,\epsilon). \tag{2.68}$$

As the bounds in Theorem 2.12 are also primarily based on continuity bounds, our improvements are mostly due to the different proof technique, the fact that our bounds allow the use of the improved continuity bound from [37] and a clean regularization for the final inequality. Since the approximate degradability bounds above follow directly by applying further bounds to our operational capacity bounds it becomes clear that our bounds must be at least as good as those in [32].

2.4 Examples

While operationally meaningful, the bounds presented above are generally still hard to compute. The complementary channel of the depolarizing channel was considered in [19], showing that for $\eta > 0$ one has $Q^{(1)}(\mathcal{N}_D^c) > 0$, in line with the well established superadditivity of the quantum capacity for the depolarizing channel. Examples for the side-channel assisted quantum capacity have been e.g. given in [29].

A particular channel that we would like to discuss here is the Horodecki channel $\mathcal{N}_H$ [14] which is a particular entanglement-binding channel [15, 16]. It is known that this channel has zero quantum capacity, $Q(\mathcal{N}_H) = 0$, but strictly positive private capacity, however little is known in terms of upper-bounds on its private capacity. Using our Corollary 2.5 together with a quantum capacity bound from [34] we find

$$P(\mathcal{N}_H) \leq Q(\mathcal{N}_H) + Q(\mathcal{N}_H^c) = Q(\mathcal{N}_H^c) \leq 0.7284. \tag{2.69}$$

Therefore our presented bounds can also provide new ways for numerical investigation of capacities. Additional examples can be found in the forthcoming long version of this work.

3 Partial orders on quantum states

Recently the concept of (approximate) degradabiity was transfered to quantum states in [17]. We consider a bipartite quantum state ρ_{AB} which has a purification Φ_{ABE}. The state ρ_{AB} is called degradable if there exists a channel $\mathcal{D}_{B\to E}$ such that

$$\rho_{AE} = \mathcal{D}_{B\to E}(\rho_{AB}), \tag{3.1}$$

where $\rho_{AE} = \operatorname{Tr}_B \Phi_{ABE}$. From now on we will sometimes use the notation $\rho_{AE} =: \rho_{AB}^c$ to emphasize the role of ρ_{AE} as the complementary state.

It seems natural now to define new partial orders on states motivated by operational quantities. We pick the one-way distillable entanglement and secret key as our quantities of choice. Devetak and Winter showed [7] that the one-way distillable secret key is given by

$$K_{\to}(\rho_{ABE}) = \lim_{n\to\infty} \frac{1}{n} K_{\to}^{(1)}(\rho_{ABE}^{\otimes n}) \tag{3.2}$$

with

$$K_{\to}^{(1)}(\rho_{ABE}) = \max_{Q,T|X} I(X:B|T) - I(X:E|T) \tag{3.3}$$

evaluated on

$$\omega_{TXBE} = \sum_{t,x} R(t|x)P(x)|t\rangle\langle t|_T \otimes |x\rangle\langle x|_X \otimes \operatorname{Tr}_A(\rho_{ABE}(Q_x \otimes \mathbb{I}_{BE})) \tag{3.4}$$

and the one-way distillable entanglement by

$$D_{\to}(\rho_{AB}) = \lim_{n\to\infty} \frac{1}{n} D_{\to}^{(1)}(\rho_{AB}^{\otimes n}) \tag{3.5}$$

with

$$D_{\to}^{(1)}(\rho_{AB}) = \max_{\mathcal{T}} I(A'\rangle BX), \tag{3.6}$$

evaluated on $\mathcal{T}_{A\to A'X}(\rho_{AB})$ where $\mathcal{T}_{A\to A'X}$ is a quantum instrument.

Remark on instruments: Generally a quantum instrument is a channel $\mathcal{T}_{A\to A'X}(\cdot) = \sum_x T_x(\cdot) \otimes |x\rangle\langle x|_X$ with each map T_x being CP and such that $\sum_x T_x$ is also TP. It was shown in [7] that when considering the one-way distillable entanglement it is sufficient to optimize over instruments where each T_x is described by only one Kraus operator, i.e. $T_x(\cdot) = K_x \cdot K_x^\dagger$. Additionally they showed that one can further restrict to the case where $K_x \geq 0$, i.e. $K_x = \sqrt{K_x^\dagger K_x}$. With these observations it follows that every considered instrument is equivalently described by a POVM $\{K_x^2\}_x$. For the remainder of this work all instruments will be of this restricted form and this will allow us to discuss secret key and entanglement on equal footing.

Next, for the purpose of this work we shall specify a setting that brings both above quantities closer together. We want to consider a state ρ_{AB} with purification Φ_{ABE}. When distilling secret key we give the full environment system to the eavesdropper and define $K_\to(\rho_{AB}) := K_\to(\Phi_{ABE})$. Instead of the measurement Q we can optimize over an instrument, of the form as just discussed, and discard the output quantum state. We can further generalize by considering an isometric extension of the instrument $V := V_{A\to A'X\bar{X}}$ as e.g. in [17]. Taking all this together we define the following pure quantum state,

$$\Psi_{X\bar{X}A'BE} = V\Phi_{ABE}V^\dagger = \sum_{x,y}\sqrt{P(x)P(y)}|x\rangle\langle y|_X \otimes |x\rangle\langle y|_{\bar{X}} \otimes K_x\Phi_{ABE}K_y^\dagger, \tag{3.7}$$

for which

$$\operatorname{Tr}_{\bar{X}} \Psi_{X\bar{X}A'BE} = \mathcal{T}_{A\to A'X}(\Phi_{ABE}) = \sum_x P(x)|x\rangle\langle x|_X \otimes K_x\Phi_{ABE}K_x^\dagger, \tag{3.8}$$

is exactly the state we optimize over in the distillable entanglement. Now applying a classical channel to the system X we get the following state,

$$\omega_{TXA'BE} = \sum_{t,x} R(t|x)P(x)|t\rangle\langle t|_T \otimes |x\rangle\langle x|_X \otimes T_x(\Phi_{ABE}), \tag{3.9}$$

which is the state we optimize over when considering the distillable secret key. It follows that we can evaluate $K_\to^{(1)}(\rho_{AB})$ and $D_\to^{(1)}(\rho_{AB})$ both on essentially the same state.

It is now time to define the partial orders we will discuss here.

Definition 3.1. *For a quantum state ρ_{AB} we say, ρ_{AB} is ϵ-regularized more secret*[1] *if*

$$K_\to(\rho^c_{AB}) \leq \epsilon, \tag{3.10}$$

ρ_{AB} is ϵ-more secret if

$$K_\to^{(1)}(\rho^c_{AB}) \leq \epsilon, \tag{3.11}$$

ρ_{AB} is ϵ-regularized more informative if

$$D_\to(\rho^c_{AB}) \leq \epsilon, \tag{3.12}$$

ρ_{AB} is ϵ-more informative if

$$D_\to^{(1)}(\rho^c_{AB}) \leq \epsilon. \tag{3.13}$$

For $\epsilon = 0$ we drop the ϵ in the name. We define the corresponding anti-orders by exchanging ρ^c_{AB} with ρ_{AB}.

From the definition it is clear that e.g. ϵ-anti regularized more secret implies small distillable secret key, i.e. $K_\to(\rho_{AB}) \leq \epsilon$, and similar for the others.

Our next goal is to rephrase the partial orders in terms of entropic inequalities. $K_\to^{(1)}$ already has a convenient form for that, but we need to work a bit on $D_\to^{(1)}$. Note that we can evaluate $D_\to^{(1)}$ on the pure state $\Psi_{X\bar{X}A'BE}$ defined above. We then get

$$\begin{aligned} I(A'\rangle BX) &= H(BX) - H(A'BX) && (3.14)\\ &= H(BX) - H(\bar{X}E) && (3.15)\\ &= H(BX) - H(XE) && (3.16)\\ &= H(B|X) - H(E|X), && (3.17) \end{aligned}$$

[1] Names are a work in progress...

allowing us to write

$$D^{(1)}_{\rightarrow}(\rho_{AB}) = \max_{\mathcal{T}} H(B|X) - H(E|X). \tag{3.18}$$

We are now ready to give the following equivalences.

Lemma 3.2. *The state ρ_{AB} is, ϵ-regularized more secret iff for all $n \geq 1$, classical channels R and quantum instruments $\mathcal{T}$ applied to $\rho_{AB}^{\otimes n}$, we have*

$$I(X : E^n|T) \leq I(X : B^n|T) + n\epsilon, \tag{3.19}$$

ϵ-more secret iff for all R, $\mathcal{T}$ applied to ρ_{AB}, we have

$$I(X : E|T) \leq I(X : B|T) + \epsilon, \tag{3.20}$$

ϵ-regularized more informative iff for all $n \geq 1$ and $\mathcal{T}$ applied to $\rho_{AB}^{\otimes n}$, we have

$$H(E^n|X) \leq H(B^n|X) + n\epsilon, \tag{3.21}$$

ϵ-more informative iff for all $\mathcal{T}$ applied to ρ_{AB}, we have

$$H(E|X) \leq H(B|X) + \epsilon. \tag{3.22}$$

Proof. Follows from the above considerations. □

Although not immediately obvious from the entropic formulation, we have that $D^{(1)}_{\rightarrow}(\rho_{AB}) \leq K^{(1)}_{\rightarrow}(\rho_{AB})$. That means that more secret implies more informative and the same for the corresponding regularizations.

Note that ϵ-regularized more secret also implies the weaker condition,

$$I(X : E^n) \leq I(X : B^n) + n\epsilon, \tag{3.23}$$

to hold for every instrument $\mathcal{T}$. This follows simply by considering the special case where the classical map $\mathcal{R}$ is trivial.

We now come to the first application.

Theorem 3.3. *If the state ρ_{AB} is ϵ-regularized more secret, then we have that*

$$D^{(1)}_{\rightarrow}(\rho_{AB}) \leq D_{\rightarrow}(\rho_{AB}) \leq D^{(1)}_{\rightarrow}(\rho_{AB}) + \epsilon, \tag{3.24}$$

and therefore for every state ρ_{AB},

$$D^{(1)}_{\rightarrow}(\rho_{AB}) \leq D_{\rightarrow}(\rho_{AB}) \leq D^{(1)}_{\rightarrow}(\rho_{AB}) + K_{\rightarrow}(\rho^c_{AB}) \tag{3.25}$$

Proof. We start by proving the first claim. Note that $D^{(1)}_{\rightarrow}(\rho_{AB}) \leq D_{\rightarrow}(\rho_{AB})$ holds by definition. Next we show that $D^{(1)}_{\rightarrow}(\rho_{AB})$ is approximately additive if ρ_{AB} is ϵ-regularized more secret. We have,

$$\begin{aligned}
D^{(1)}_{\rightarrow}(\rho_{AB}) &= \max_{\mathcal{T}} H(B|X)_{\Psi} - H(E|X)_{\Psi} && (3.26)\\
&\leq \max_{\mathcal{T}} H(B)_{\Psi} - H(E)_{\Psi} + \epsilon && (3.27)\\
&= H(B)_{\Phi} - H(E)_{\Phi} + \epsilon && (3.28)\\
&= H(B)_{\Phi} - H(AB)_{\Phi} + \epsilon && (3.29)\\
&= I(A\rangle B)_{\Phi} + \epsilon && (3.30)
\end{aligned}$$

where the inequality follows from the more secret condition and the second equality is because $\Psi_{BE} = \Phi_{BE}$. The other steps are straight forward. We can now do the same with $\rho_{AB}^{\otimes n}$ and get

$$D^{(1)}_{\rightarrow}(\rho_{AB}^{\otimes n}) = I(A^n\rangle B^n)_{\Phi^{\otimes n}} + n\epsilon = nI(A\rangle B)_{\Phi} + n\epsilon, \tag{3.31}$$

because the coherent information is additive on product sates. Regularizing finishes the proof of the first statement. Because, by definition of the order, every state is ϵ-regularized more secret with $\epsilon = K_{\rightarrow}(\rho^c_{AB})$ the second claim follows directly. □

Note that Equation (3.42) is similar to one of the main results in [17], but with a weaker requirement on the state. We also remark that it seems like we haven't used the full power of the more secret ordering by only applying the weaker condition in Equation 3.23, which could itself be seen as a partial order on quantum states.

Let's show our next result.

Theorem 3.4. *If the state ρ_{AB} is ϵ-more informative, then we have that*

$$D^{(1)}_{\rightarrow}(\rho_{AB}) \leq K^{(1)}_{\rightarrow}(\rho_{AB}) \leq D^{(1)}_{\rightarrow}(\rho_{AB}) + \epsilon, \tag{3.32}$$

If the state ρ_{AB} is ϵ-regularized more informative, then we have that

$$D_{\rightarrow}(\rho_{AB}) \leq K_{\rightarrow}(\rho_{AB}) \leq D_{\rightarrow}(\rho_{AB}) + \epsilon, \tag{3.33}$$

and therefore for every state ρ_{AB},

$$D^{(1)}_{\rightarrow}(\rho_{AB}) \leq K^{(1)}_{\rightarrow}(\rho_{AB}) \leq D^{(1)}_{\rightarrow}(\rho_{AB}) + D^{(1)}_{\rightarrow}(\rho^{c}_{AB}) \tag{3.34}$$

$$D_{\rightarrow}(\rho_{AB}) \leq K_{\rightarrow}(\rho_{AB}) \leq D_{\rightarrow}(\rho_{AB}) + D_{\rightarrow}(\rho^{c}_{AB}) \tag{3.35}$$

Proof. Fix the measurement and channel that achieve the maximum in $K^{(1)}_{\rightarrow}(\rho_{AB})$, let ω be the corresponding output state. We have,

$$\begin{aligned}
K^{(1)}_{\rightarrow}(\rho_{AB}) &= I(X:B|T) - I(X:E|T) && (3.36)\\
&= I(XT:B) - I(XT:E) + I(T:E) - I(T:B) && (3.37)\\
&= H(E|XT) - H(B|XT) + H(B|T) - H(E|T) && (3.38)\\
&= H(E|X) - H(B|X) + H(B|T) - H(E|T) && (3.39)\\
&\leq \epsilon + H(B|T) - H(E|T) && (3.40)\\
&\leq \epsilon + D^{(1)}_{\rightarrow}(\rho_{AB}), && (3.41)
\end{aligned}$$

where the first three equalities are by definition of the involved quantities. The final equality is because we have $A \rightarrow X \rightarrow T$ and therefore T does not provide additional information over X. The first inequality follow by definition of the more informative partial order. The second inequality because the remaining entropies are independent of X and we can absorb the channel $X \rightarrow T$ into the choice of instrument and by the definition of $D^{(1)}_{\rightarrow}(\rho_{AB})$. This proves the first claim.

The second statement follows in the same way by considering $\rho_{AB}^{\otimes n}$ using the assumption that ρ_{AB} is ϵ-regularized more informative, followed by regularizing the resulting inequality.

The final claim follows easily from the previous two, noticing that every state fulfills the needed condition for appropriately large ϵ. □

Finally, we can combine the previous results to get one more corollary.

Corollary 3.5. *If the state ρ_{AB} is ϵ-regularized more secret, then we have that*

$$K^{(1)}_{\rightarrow}(\rho_{AB}) \leq K_{\rightarrow}(\rho_{AB}) \leq K^{(1)}_{\rightarrow}(\rho_{AB}) + 2\epsilon, \tag{3.42}$$

and for every state ρ_{AB},

$$\begin{aligned}
K^{(1)}_{\rightarrow}(\rho_{AB}) \leq K_{\rightarrow}(\rho_{AB}) &\leq K^{(1)}_{\rightarrow}(\rho_{AB}) + K_{\rightarrow}(\rho^{c}_{AB}) + D_{\rightarrow}(\rho^{c}_{AB}) && (3.43)\\
&\leq K^{(1)}_{\rightarrow}(\rho_{AB}) + 2K_{\rightarrow}(\rho^{c}_{AB}). && (3.44)
\end{aligned}$$

Proof. This follows simply by combining the previous two theorems. □

4 Partial orders with symmetric side channel assistance

In [29, 27] versions of the discussed capacities assisted by a symmetric side channel are introduced. Since the assistance can only help, they are naturally an upper bound on the respective capacities. The symmetric side channel by itself has zero quantum and private capacity and is both degradable and anti-degradable. The assisted capacities are particularly interesting as they were proven in [29, 27] to be additive, i.e. $Q_{ss}^{(1)}(\mathcal{N}) = Q_{ss}(\mathcal{N})$ and $P_{ss}^{(1)}(\mathcal{N}) = P_{ss}(\mathcal{N})$. One can now define *side channel assisted* partial orders based on these quantities, analog to the ones previously discussed, and because of the additivity it is not necessary to consider regularizations. Note here that we have

$$P_{ss}^{(1)}(\mathcal{N}) = P_{ss}(\mathcal{N}) \geq Q_{ss}^{(1)}(\mathcal{N}) = Q_{ss}(\mathcal{N}) \geq \frac{1}{2}P(\mathcal{N}), \tag{4.1}$$

where the final inequality was proven in [30]. This implies in particular that

$$Q_{ss}^{(1)}(\mathcal{N}^c) = 0 \quad \Rightarrow \quad P(\mathcal{N}^c) = 0, \tag{4.2}$$

which could provide us with an easier (non-regularized) condition to determine whether a channel is regularized less noisy. On the other hand, we still lack an example of a channel that is regularized less noisy but not degradable. We know that the quantum capacity is superadditive and in particular that it can be superactivated, i.e. there exists a channel for which $Q(\mathcal{N}) = 0$ but $Q_{ss}(\mathcal{N}) > 0$, see [30]. Note that if we had a channel with $P(\mathcal{N}) = 0$, but

$$P_{ss}^{(1)}(\mathcal{N}^c) > 0 \quad \Rightarrow \quad \mathcal{N} \text{ not degradable}, \tag{4.3}$$

it would give us the desired example. It seems intuitive that a similar construction works more generally and there is a deeper connection between superactivation of the private capacity and such examples. We can make this more precise in the following observation.

Corollary 4.1. *If the private capacity can be superactivated then degradable channels are a strict subset of regularized less noisy channels, i.e.*

$$P(\cdot) \text{ can be superactivated} \quad \Rightarrow \quad DEG \subsetneq LN_\infty. \tag{4.4}$$

Proof. Let us assume that the private capacity can be superactivated, meaning there exist channels $\mathcal{N}$ and $\mathcal{M}$ such that $P(\mathcal{N}) = P(\mathcal{M}) = 0$, but $P(\mathcal{N} \otimes \mathcal{M}) > 0$. We observe that if $\mathcal{N}$ and $\mathcal{M}$ are anti-degradable than also is $\mathcal{N} \otimes \mathcal{M}$ and we would have $P(\mathcal{N} \otimes \mathcal{M}) = 0$. Therefore, by assumption, not both channels can be anti-degradable. However, again by assumption, their complements $\mathcal{N}^c$ and $\mathcal{M}^c$ are regularized less noisy. It follows that at leasts one channel, $\mathcal{N}^c$ or $\mathcal{M}^c$ is regularized less noisy but not degradable which concludes the proof. □

The question whether the private capacity can be superactivated is interestingly still open despite significant effort to find an answer [20, 28, 31]. Of course the above also implies that if all regularized less noisy channels were also degradable, the private capacity could not be superactivated.

Before ending this section, we briefly observe a result similar to the main theme of this work and state some bounds on the symmetric side channel assisted capacity.

Corollary 4.2. *Let $\mathcal{N}$ be a quantum channel, we have*

$$Q_{ss}^{(1)}(\mathcal{N}) \leq P_{ss}^{(1)}(\mathcal{N}) \leq Q_{ss}^{(1)}(\mathcal{N}) + Q_{ss}^{(1)}(\mathcal{N}^c) \tag{4.5}$$

$$Q_{ss}(\mathcal{N}) \leq P_{ss}(\mathcal{N}) \leq Q_{ss}(\mathcal{N}) + Q_{ss}(\mathcal{N}^c). \tag{4.6}$$

Proof. This follows because according to [29, 27] we can write

$$Q_{ss}^{(1)}(\mathcal{N}) = \sup_d Q^{(1)}(\mathcal{N} \otimes \mathcal{A}_d) \tag{4.7}$$

$$P_{ss}^{(1)}(\mathcal{N}) = \sup_d P^{(1)}(\mathcal{N} \otimes \mathcal{A}_d), \tag{4.8}$$

as well as noticing that $(\mathcal{N} \otimes \mathcal{A}_d)^c = \mathcal{N}^c \otimes \mathcal{A}_d^c = \mathcal{N}^c \otimes \mathcal{A}_d$. Finally, combining both observations with Corollary 2.5 leads to the first result. The second follows by regularizing. □

5 Outlook and open problems

A particular interesting (but surely harder) case would be to look at distillable entanglement and distillable key under LOCC or maybe PPT-preserving operations. Also channel capacities with e.g. two-way assistance might obey similar bounds. Overall it might be interesting to investigate whether there is a more general framework in which such results can be proven. Going in the other direction, it would be interesting to look at more restricted scenarios, e.g. quantum capacities with energy restrictions.

We haven't really discussed how good our presented bounds are. However, note that approximately degradable implies approximately less noisy. Our bounds are therefore at least as good as the ones from naively using approximate degradability. However they have the big disadvantage, that is they can in most cases not be computed efficiently, but one can use any other capacity bound and combine it with ours to get numerical results.

Acknowledgments

Mark M. Wilde for suggesting to apply the ideas to one-way distillable secret key and energy-constrained capacities.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie Grant Agreement No. H2020-MSCA-IF-2020-101025848.

References

[1] K. M. Audenaert. A sharp continuity estimate for the von neumann entropy. *Journal of Physics A: Mathematical and Theoretical*, 40(28):8127, 2007.

[2] C. H. Bennett, P. W. Shor, J. A. Smolin, and A. V. Thapliyal. Entanglement-assisted classical capacity of noisy quantum channels. *Physical Review Letters*, 83(15):3081, 1999.

[3] N. Cai, A. Winter, and R. W. Yeung. Quantum privacy and quantum wiretap channels. *Problems of Information Transmission*, 40(4):318–336, 2004.

[4] A. Cross, K. Li, and G. Smith. Uniform additivity in classical and quantum information. *Physical review letters*, 118(4):040501, 2017.

[5] I. Devetak. The private classical capacity and quantum capacity of a quantum channel. *IEEE Transactions on Information Theory*, 51(1):44–55, 2005.

[6] I. Devetak and P. W. Shor. The capacity of a quantum channel for simultaneous transmission of classical and quantum information. *Communications in Mathematical Physics*, 256(2):287–303, 2005.

[7] I. Devetak and A. Winter. Distillation of secret key and entanglement from quantum states. *Proceedings of the Royal Society A: Mathematical, Physical and engineering sciences*, 461(2053):207–235, 2005.

[8] K. Fang and H. Fawzi. Geometric rényi divergence and its applications in quantum channel capacities. *Communications in Mathematical Physics*, pages 1–63, 2021.

[9] M. Fanizza, F. Kianvash, and V. Giovannetti. Quantum flags and new bounds on the quantum capacity of the depolarizing channel. *Physical Review Letters*, 125(2):020503, 2020.

[10] H. Fawzi and O. Fawzi. Efficient optimization of the quantum relative entropy. *Journal of Physics A: Mathematical and Theoretical*, 51(15):154003, 2018.

[11] O. Fawzi, A. Shayeghi, and H. Ta. A hierarchy of efficient bounds on quantum capacities exploiting symmetry. In *2021 IEEE International Symposium on Information Theory (ISIT)*, pages 272–277. IEEE, 2021.

[12] C. Hirche, C. Rouzé, and D. França. On contraction coefficients, partial orders and approximation of capacities for quantum channels. *arXiv:2011.05949*.

[13] A. S. Holevo. The capacity of the quantum channel with general signal states. *IEEE Transactions on Information Theory*, 44(1):269–273, 1998.

[14] K. Horodecki, L. Pankowski, M. Horodecki, and P. Horodecki. Low-dimensional bound entanglement with one-way distillable cryptographic key. *IEEE Transactions on Information Theory*, 54(6):2621–2625, 2008.

[15] M. Horodecki, P. Horodecki, and R. Horodecki. Separability of mixed states: necessary and sufficient conditions. *Physics Letters A*, 223(1):1–8, 1996.

[16] P. Horodecki. Separability criterion and inseparable mixed states with positive partial transposition. *Physics Letters A*, 232(5):333–339, 1997.

[17] F. Leditzky, N. Datta, and G. Smith. Useful states and entanglement distillation. *IEEE Transactions on Information Theory*, 64(7):4689–4708, 2017.

[18] F. Leditzky, D. Leung, and G. Smith. Quantum and private capacities of low-noise channels. *Physical Review Letters*, 120(16):160503, 2018.

[19] D. Leung and J. Watrous. On the complementary quantum capacity of the depolarizing channel. *Quantum*, 1:28, 2017.

[20] K. Li, A. Winter, X. Zou, and G. Guo. Private capacity of quantum channels is not additive. *Physical review letters*, 103(12):120501, 2009.

[21] S. Lloyd. Capacity of the noisy quantum channel. *Physical Review A*, 55(3):1613, 1997.

[22] H. Qi, K. Sharma, and M. M. Wilde. Entanglement-assisted private communication over quantum broadcast channels. *Journal of Physics A: Mathematical and Theoretical*, 51(37):374001, 2018.

[23] B. Schumacher and M. D. Westmoreland. Sending classical information via noisy quantum channels. *Physical Review A*, 56(1):131, 1997.

[24] C. E. Shannon. A note on a partial ordering for communication channels. *Information and control*, 1(4):390–397, 1958.

[25] M. E. Shirokov. On the energy-constrained diamond norm and its application in quantum information theory. *Problems of Information Transmission*, 54(1):20–33, 2018.

[26] P. W. Shor. The quantum channel capacity and coherent information. In *lecture notes, MSRI Workshop on Quantum Computation*, 2002.

[27] G. Smith. Private classical capacity with a symmetric side channel and its application to quantum cryptography. *Physical Review A*, 78(2):022306, 2008.

[28] G. Smith and J. A. Smolin. Extensive nonadditivity of privacy. *Physical review letters*, 103(12):120503, 2009.

[29] G. Smith, J. A. Smolin, and A. Winter. The quantum capacity with symmetric side channels. *IEEE Transactions on Information Theory*, 54(9):4208–4217, 2008.

[30] G. Smith and J. Yard. Quantum communication with zero-capacity channels. *Science*, 321(5897):1812–1815, 2008.

[31] S. Strelchuk. Parrondo's paradox and superactivation of classical and quantum capacity of communication channels with memory. *Physical Review A*, 88(3):032311, 2013.

[32] D. Sutter, V. B. Scholz, A. Winter, and R. Renner. Approximate degradable quantum channels. *IEEE Transactions on Information Theory*, 63(12):7832–7844, 2017.

[33] X. Wang. Optimizing the fundamental limits for quantum and private communication. *arXiv preprint arXiv:1912.00931*, 2019.

[34] X. Wang, K. Fang, and R. Duan. Semidefinite programming converse bounds for quantum communication. *IEEE Transactions on Information Theory*, 65(4):2583–2592, 2018.

[35] S. Watanabe. Private and quantum capacities of more capable and less noisy quantum channels. *Physical Review A*, 85(1):012326, 2012.

[36] M. M. Wilde. *Quantum information theory.* Cambridge University Press, 2013.

[37] M. M. Wilde. Optimal uniform continuity bound for conditional entropy of classical–quantum states. *Quantum Information Processing*, 19(2):1–9, 2020.

[38] M. M. Wilde and H. Qi. Energy-constrained private and quantum capacities of quantum channels. *IEEE Transactions on Information Theory*, 64(12):7802–7827, 2018.

[39] A. Winter. Tight uniform continuity bounds for quantum entropies: conditional entropy, relative entropy distance and energy constraints. *Communications in Mathematical Physics*, 347(1):291–313, 2016.

[40] A. Winter. Energy-constrained diamond norm with applications to the uniform continuity of continuous variable channel capacities. *arXiv preprint arXiv:1712.10267*, 2017.

A On relationships between approximate partial orders

The relationships between different partial orders are a fundamental problem and have been investigated at several points in the literature. In particular, the relationships between most of the orders discussed in this work have been recently investigated in [12], however only in the case $\epsilon = 0$. Focusing on approximate orders it is worth noting that the picture becomes substantially more complicated when $\epsilon > 0$.

We first start with a simple example motivated by Section 4. Note that we have the following implication,

$$P_{ss}^{(1)}(\mathcal{N}^c) = 0 \quad \Rightarrow \quad Q_{ss}^{(1)}(\mathcal{N}^c) = 0 \quad \Rightarrow \quad P(\mathcal{N}^c) = 0. \tag{A.1}$$

However, when allowing an approximation, the best we can currently show is

$$P_{ss}^{(1)}(\mathcal{N}^c) \leq \epsilon \quad \Rightarrow \quad Q_{ss}^{(1)}(\mathcal{N}^c) \leq \epsilon \quad \Rightarrow \quad P(\mathcal{N}^c) \leq 2\epsilon, \tag{A.2}$$

although we also have

$$P_{ss}^{(1)}(\mathcal{N}^c) \leq \epsilon \quad \Rightarrow \quad P(\mathcal{N}^c) \leq \epsilon. \tag{A.3}$$

Therefore the implied exact partial orders have a simpler relationship than the more general approximate versions.

We now discuss the main part of this section. Recall the definition of $Q_{rel}^{(1)}(\mathcal{N})$, which is, similar to the other quantities defined in this work, related to a partial order. For two quantum channels $\mathcal{N}$ and $\mathcal{M}$ we denote $\mathcal{N} \succeq_{\mathrm{rel}}^{\epsilon} \mathcal{M}$ if

$$D(\mathcal{M}(\rho)\|\mathcal{M}(\sigma)) \leq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) + \epsilon \quad \forall \rho, \sigma \tag{A.4}$$

and simply $\mathcal{N} \succeq_{\mathrm{rel}} \mathcal{M}$ if $\epsilon = 0$. An important technical result in [35] was the following observation,

$$\mathcal{N} \succeq_{\mathrm{rel}} \mathcal{M} \quad \Leftrightarrow \quad \mathcal{N} \succeq_{\mathrm{l.n.}} \mathcal{M}. \tag{A.5}$$

Following the proof in [35] it can be easily seen that also in the approximate case we still have

$$\mathcal{N} \succeq_{\mathrm{rel}}^{\epsilon} \mathcal{M} \quad \Rightarrow \quad \mathcal{N} \succeq_{\mathrm{l.n.}}^{\epsilon} \mathcal{M}. \tag{A.6}$$

However, we will now see that the opposite direction is generally not true. For this, consider two erasure channels $\mathcal{E}_1$ and $\mathcal{E}_2$ with erasure probabilities ϵ_1 and ϵ_2, respectively. We know that for an erasure channel one has $I(A:B) = (1-\epsilon)I(A:A')$ and $D(\mathcal{E}(\rho)\|\mathcal{E}(\sigma)) = (1-\epsilon)D(\rho\|\sigma)$. First, consider the approximate less noisy condition that for our example evaluates to

$$(1-\epsilon_2)I(U:A) \leq (1-\epsilon_1)I(U:A) + \epsilon. \tag{A.7}$$

It can now easily be seen that the two channels are always ϵ-approximately less noisy if $\epsilon = \max\{0, 2(\epsilon_1 - \epsilon_2)\log|A|\}$, because $I(U:A) \leq 2\log|A|$. Next, note that the condition for the partial order based on relative entropy defined in (A.4) can be written as

$$(1-\epsilon_2)D(\rho\|\sigma) \leq (1-\epsilon_1)D(\rho\|\sigma) + \epsilon. \tag{A.8}$$

Since the relative entropy can be arbitrarily large for suitably chosen ρ and σ, there is in general no ϵ such that the above inequality always holds provided that $\epsilon_1 > \epsilon_2$. This proves that the reverse implication of Equation (A.6) cannot hold. Finally, the example can easily be specialized to the case where $\mathcal{M} = \mathcal{N}^c$ because the complementary channel of an erasure channel with erasure probability ϵ_1 is an erasure channel with erasure probability $\epsilon_2 = 1 - \epsilon_1$.

B Energy-constrained partial orders on quantum channels

In this section we return to the theme of quantum channels but we add a twist by considering energy-constrained settings. We will again base our partial orders on the quantum and private capacities of a quantum channel, but this time we focus on their energy-constrained variants. Based on [38] we consider the following quantities:[2] The energy-constrained quantum capacity

$$Q_{H_A,E}(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} Q^{(1)}_{H_{A^n},nE}(\mathcal{N}^{\otimes n}) \tag{B.1}$$

$$\text{with} \quad Q^{(1)}_{H_A,E}(\mathcal{N}) = \sup_{\substack{\rho_A \\ \operatorname{Tr} H_A\rho_A \leq E}} H(\mathcal{N}(\rho_A)) - H(\mathcal{N}^c(\rho_A)), \tag{B.2}$$

and the energy-constrained private capacity

$$P_{H_A,E}(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} P^{(1)}_{H_{A^n},nE}(\mathcal{N}^{\otimes n}) \tag{B.3}$$

$$\text{with} \quad P^{(1)}_{H_A,E}(\mathcal{N}) = \sup_{\substack{\rho_{UA} \\ \operatorname{Tr} H_A\rho_A \leq E}} I(U:B) - I(U:E). \tag{B.4}$$

For the energy constraint we define the Hamiltonian H_{A^n} on n copies of the input quantum system as the extension of the single system Hamiltonian H_A as

$$H_{A^n} = H_A \otimes \mathbb{I}_A \otimes \cdots \otimes \mathbb{I}_A + \cdots + \mathbb{I}_A \otimes \cdots \otimes \mathbb{I}_A \otimes H_A. \tag{B.5}$$

Throughout this section we will assume that the finite output entropy condition holds, that is

$$\sup_{\substack{\rho_A \\ \operatorname{Tr} H_A\rho_A \leq E}} H(\mathcal{N}(\rho_A)) < \infty. \tag{B.6}$$

It was shown in [38] that if this condition holds for a channel $\mathcal{N}$ it also holds for the complementary channel $\mathcal{N}^c$. We can now define the following energy-constrained partial orders.

Definition B.1. *A channel $\mathcal{N}$ is called:*

- *(ϵ, H_A, E)-regularized less noisy if $P_{H_A,E}(\mathcal{N}^c) \leq \epsilon$;*

[2] My understanding is that these quantities are not fully proven to be the actual capacities, but [38] gives strong evidence and we stick with the name for now.

- (ϵ, H_A, E)*-less noisy if* $P^{(1)}_{H_A,E}(\mathcal{N}^c) \leq \epsilon$;
- (ϵ, H_A, E)*-regularized more capable if* $Q_{H_A,E}(\mathcal{N}^c) \leq \epsilon$;
- (ϵ, H_A, E)*-more capable if* $Q^{(1)}_{H_A,E}(\mathcal{N}^c) \leq \epsilon$.

Note that each of these partial orders has an equivalent definition via a mutual information-based condition similar to the unconstrained case, with the difference that the condition only needs to be checked on states satisfying the energy constraint. For the less noisy orderings this is fairly obvious from the definition. For the more capable orderings consider a state $\rho_A = \sum_i \lambda_i |\Psi_i\rangle\langle\Psi_i|$ and its extension $\rho_{UA} = \sum_i \lambda_i |i\rangle\langle i| \otimes |\Psi_i\rangle\langle\Psi_i|$. The energy constraint $\operatorname{Tr} H_A \rho_A \leq E$ remains the same and can, similar to the less noisy setting, be interpreted as an average energy constraint of the ensemble $\{\lambda_i, |\Psi_i\rangle\}$ as $\sum_i \lambda_i \operatorname{Tr} H_A |\Psi_i\rangle\langle\Psi_i| \leq E$.

In the unconstrained case we saw that the less noisy order is closely related to the concavity of a channel's coherent information. A careful check reveals that the same holds true if an energy constraint is to be obeyed.

Lemma B.2. *A channel* $\mathcal{N}$ *is* (ϵ, H_A, E)*-approximate less noisy if and only if its channel coherent information is approximately concave for all quantum states* ρ^i_A *and probability distributions* $p(i)$ *satisfying* $\operatorname{Tr} H_A \rho_A \leq E$ *with* $\rho_A = \sum_i p(i)\rho^i_A$*, i.e.,*

$$\sum_i p(i) I(A\rangle B)_{\rho_i} \leq I(A\rangle B)_\rho + \epsilon, \tag{B.7}$$

where $I(A\rangle B)_\rho$ *is evaluated on the state* $\mathcal{N}(\Psi_{AA'})$ *with* $\Psi_{AA'}$ *a purification of* ρ_A*. Similarly, a channel* $\mathcal{N}$ *being* (ϵ, H_A, E)*-approximate anti-less noisy is equivalent to*

$$\sum_i p(i) I(A\rangle B)_{\rho_i} \geq I(A\rangle B)_\rho - \epsilon. \tag{B.8}$$

For an arbitrary quantum channel $\mathcal{N}$ *and states obeying* $\operatorname{Tr} H_A \rho_A \leq E$ *we have*

$$I(A\rangle B)_\rho - P^{(1)}_{H_A,E}(\mathcal{N}) \leq \sum_i p(i) I(A\rangle B)_{\rho_i} \leq I(A\rangle B)_\rho + P^{(1)}_{H_A,E}(\mathcal{N}^c). \tag{B.9}$$

Now we would like to briefly discuss to what extent the bounds and results on channel capacities can be extended to the energy-constrained setting. To this end, we briefly revisit the approach in [35]. Take quantum states ρ^u_A and a probability distribution $p(u)$, define $\rho_{UA} = \sum_u p(u)|u\rangle\langle u| \otimes \rho^u_A$ and $\rho_A = \operatorname{Tr}_U \rho_{UA}$. A central observation to the proofs in [35] is that the following equality holds,

$$I(U:B) - I(U:E) = I(A\rangle B)_\rho - \sum_i p(i) I(A\rangle B)_{\rho_i} \tag{B.10}$$

$$= I(A\rangle B)_\rho + \sum_i p(i) I(A\rangle E)_{\rho_i}. \tag{B.11}$$

If we fix ρ_{UA} to be the optimizing state in $P^{(1)}_{H_A,E}(\mathcal{N}^c)$, we easily get

$$P^{(1)}_{H_A,E}(\mathcal{N}) \leq Q^{(1)}_{H_A,E}(\mathcal{N}) + \sum_i p(i) I(A\rangle E)_{\rho_i}. \tag{B.12}$$

In the unconstrained case it is now easy to bound each of the remaining coherent informations with $Q^{(1)}(\mathcal{N}^c)$, making the average irrelevant, which leads to our previously stated inequality. However, in the constrained case, we can not simply do the same using $Q^{(1)}_{H_A,E}(\mathcal{N}^c)$ because the individual ρ_i might not fulfill the energy constraint $\operatorname{Tr} H_A \rho^i_A \leq E$. One might be tempted to remedy this problem by using concavity, but from the previous lemma it is clear that this doesn't seem to help for general channels. In [38] it was shown that $Q^{(1)}_{H_A,E}(\mathcal{N})$ equals both the energy-constrained quantum and private capacity of a degradable channel. How is this compatible with the above observations? To us, the most likely explanation seems to be the following.

Note that degradability is usually defined via the diamond norm and therefore considering all possible input states without constraint. Equivalently, we can prove

$$P^{(1)}_{H_A,E}(\mathcal{N}) \leq Q^{(1)}_{H_A,E}(\mathcal{N}) + Q^{(1)}(\mathcal{N}^c), \tag{B.13}$$

showing that $P^{(1)}_{H_A,E}(\mathcal{N}) = Q^{(1)}_{H_A,E}(\mathcal{N})$ for all ϵ-less noisy channels $\mathcal{N}$, which is a significantly weaker requirement than degradability.

One could define similarly a weaker form of degradability that obeys an energy constraint. It is an interesting question whether results like those in [35] hold under this requirement.

Definition B.3. *A channel $\mathcal{N}$ is called (ϵ, H_A, E)-degradable if there exists a channel $\mathcal{D}$ such that*

$$\|\mathcal{N}^c - \mathcal{D} \circ \mathcal{N}\|_\diamond^{H_A,E} \leq \epsilon, \tag{B.14}$$

where $\|\Delta\|_\diamond^{H_A,E}$ is the energy-constrained diamond norm defined in [25].

The energy-constrained diamond norm has already found several applications, in particular for infinite dimensional systems, see e.g. [25, 40].

Finally, we comment on single-letter upper bounds on regularized capacities. Note that following the proof in [4] one gets

$$I(A\rangle B^n) \leq \sum_i I(A\rangle B_i) + \sum_i [I(V|B_i) - I(V|E_i)]. \tag{B.15}$$

However, while the state ρ_{A^n} obeys the constraint $\operatorname{Tr} H_{A^n}\rho_{A^n} \leq nE$, the best we can say about the individual ρ_{A_i} is that they also obey $\operatorname{Tr} H_A\rho_{A_i} \leq nE$, leading us to the somewhat unsatisfying result

$$Q^{(1)}_{H_A,E}(\mathcal{N}^{\otimes n}) \leq nQ^{(1)}_{H_A,nE}(\mathcal{N}) + nP_E^{H_A,nE}(\mathcal{N}^c), \tag{B.16}$$

where $P_E^{H_A,E}$ is the energy-constrained entanglement-assisted private information defined as

$$P_E^{H_A,E}(\mathcal{N}) = \sup_{\substack{\rho_{AA'} \\ \operatorname{Tr} H_A\rho_{A'} \leq E}} I(A:B) - I(A:E). \tag{B.17}$$

If one wishes to regularize, the energy constraints on the right hand side would become meaningless, resulting in the inequality

$$Q_{H_A,E}(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + P_E(\mathcal{N}^c), \tag{B.18}$$

which implies once more that the desired simplifications only seem to hold if a requirement without energy-restriction holds. Note that the above behavior is certainly intuitive as the way energy-constrained capacities are regularized allows for strategies where a single input uses an arbitrarily high amount of energy as long as it is compensated by the other channels uses. The problem would be resolved if one considers a more restricted way of regularizing the quantities where each channel input is subject to a fixed energy constraint instead of an average energy constraint on the overall state. This might also sometimes be the practically more relevant scenario.