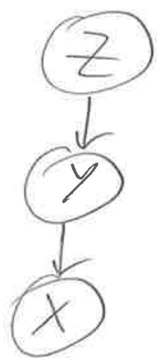
Consider a 3-node chain of discrete RVs X, Y, Z.

X is the input, Y is the intermed. repr. and Z is the high-level repr.

The generative model is



(this is the generative model of Sec. 3.2.2 in [1], but with variables Z and X renamed to X and Z respectively).

We assume $X=x$ is given and we want to infer $P(Z \mid X=x)$.

According to [1] this is

$$Bel(Z=z) = P(Z=z \mid X=x)$$

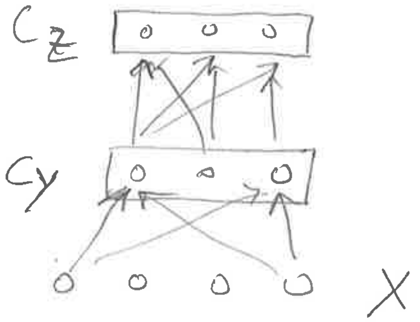
$$= \frac{\sum_{Y=y} P(X=x \mid Y=y) P(Y=y \mid Z=z) P(Z=z)}{P(X=x)}$$

Let the RV Y take on values from $1 \dots n_Y$ and RV Z take on values from $1 \dots n_Z$

Then

$$Bel(Z=j) = P(Z=j \mid X=x) = \frac{\sum_{i} P(X=x \mid Y=i) P(Y=i \mid Z=j) P(Z=j)}{P(X=x)} \quad (1)$$

Now consider two stacked WTA circuits C_y and C_z



~~Cz computes~~

Neuron i in C_y outputs

$$y_i = \frac{\lambda_i^y}{\sum_j \lambda_j^y} \quad \text{with} \quad \lambda_i^y = P(X=x \mid Y=i) \qquad (2)$$

Hence $$y_i = \frac{P(X=x \mid Y=i)}{\sum_j P(X=x \mid Y=j)}$$

The weight w_{ji} from neuron i in C_y to neuron j in C_z is given by

$$w_{ji} = P(Y=i \mid Z=j) \qquad (3)$$

Furthermore, neuron j in C_z computes

$$z_j = \frac{\lambda_j^z}{\sum_k \lambda_k^z} \quad \text{with} \quad \lambda_j^z = P(Z=j) \cdot \sum_i w_{ji}\, y_i \qquad (4)$$

Hence we have

$$\lambda_j^z = P(Z=j) \cdot \sum_i \frac{P(X=x \mid Y=i)}{\sum_k P(X=x \mid Y=k)} \cdot P(Y=i \mid Z=j)$$

$$= \frac{P(X=x)}{\sum_k P(X=x \mid Y=k)} \frac{\sum_i P(X=x \mid Y=i) P(Y=i \mid Z=j) P(Z=j)}{P(X=x)}$$

$$= \frac{P(X=x)}{\sum_k P(X=x \mid Y=k)} P(Z=j \mid X=x)$$

Now normalizing

$$z_j = \frac{\lambda_j^z}{\sum_e \lambda_e^z} = \frac{P(Z=j \mid X=x)}{\sum_e P(Z=e \mid X=x)} = \underline{\underline{P(Z=j \mid X=x)}}$$

Thus, the WTA network C_Z computes the ~~posterior~~ belief over the high-level latent variable Z.

[1] Bayesian Artificial Intelligence
chap. 3 Inference in Bayesian Networks

Neurons in C_Y

$$u_i = \sum_j W_{ij} X_j + \sum_j w'_{ij} (1 - x_j)$$

x_j is binary $\quad w_{ij} = \log P(x_j = 1 / Y = i)$

$$w'_{ij} = \log P(x_j = \emptyset / Y = i)$$

$$\Rightarrow u_i = \sum_j x_j \log P(x_j = 1 / Y = i)$$

$$+ \sum_j (1 - x_j) \log (x_j = \emptyset / Y = i)$$

$$= \log \prod_j P(x_j / Y = i)$$

$$= \log P(X = \underline{x} / Y = i)$$

Now let $\lambda_i^Y = \exp(u_i) = P(X = \underline{x} / Y = i)$

as desired by eq. (2).