First notes: WTA Networks with Feedback

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1 Simple contextual modulation

We consider a simple generative model of the form $Z \rightarrow Y \rightarrow X$. Here, Z and Y are multinomial variables whereas X is a binary random vector. Note that this is an extension of the model considered in [1] where only two levels, in our notation with variables Y and X, were considered.

In this model, we have a prior distribution $P(Z = j)$ over Z. Z then defines the prior over Y, i.e., $P(Y = i|Z = j) = p_{ij}$. Finally, the distribution over $P(X|Y)$ is defined as in [1]. We are interested in the distribution over Y for given X and Z. Here, X is interpreted as the sensory input and Z as a context variable which biases the inference over Y. For example, when you are on the street in Austria it is improbable that you see a lion, which changes in the context of a zoo-visit.

1.1 Inference of hidden state for given context

We assume the context is given and we want to infer the hidden variable Y, i.e., we want to compute $P(Y = i|X = \mathbf{x}, Z = j)$. This probability is given by

$$
\begin{aligned}
P(Y = i|X = \mathbf{x}, Z = j) &= \frac{P(Y = i, X = \mathbf{x}, Z = j)}{P(X = \mathbf{x}, Z = j)} && (1)\\
&= \frac{P(Z = j)P(Y = i|Z = j)P(X = \mathbf{x}|Y = i)}{\sum_k P(Z = j)P(Y = k|Z = j)P(X = \mathbf{x}|Y = k)} && (2)\\
&= \frac{P(Y = i|Z = j)P(X = \mathbf{x}|Y = i)}{\sum_k P(Y = k|Z = j)P(X = \mathbf{x}|Y = k)}. && (3)
\end{aligned}
$$

Assume that a context vector $\mathbf{z} \in \{0, 1\}^{n_z}$ indicates the current context in a one-hot encoded manner, where n_z is the number of possible contexts. Note that $\ln(P(Y =$

$i|Z = j)P(X = \mathbf{x}|Y = i)) = \ln P(Y = i|Z = j) + \ln p_{ij}$. We define

$$u_i(\mathbf{x}, \mathbf{z}) = \ln P(X = \mathbf{x}|Y = i) + \sum_j z_j w_{ij}^{\mathrm{fb}}, \tag{4}$$

with $w_{ij}^{\mathrm{fb}} = \ln p_{ij}$. Then, the probability we aimed for is given by

$$P(Y = i|X = \mathbf{x}, Z = \mathbf{z}) = \frac{\exp u_i}{\sum_k \exp u_k}, \tag{5}$$

which is again a WTA formulation as in [1]. Note that there is just one subtle difference in Eq.(4) as compared to the standard formulation. The bias term disappeared and is replaced by $\sum_j z_j \ln p_{ij}$, since the prior probability for Y depends on Z.

The feed-forward weights can be learned as in [1]. Similarly, the feedback-weights have to converge to $\ln P(Y = i|Z = j)$. Hence, the learning rule should be the same as for the feed-forward weights.

1.2 Inference of hidden state for a context-belief

Now assume that we have only a belief about the context, i.e., for each j we have $P(Z = j)$.

$$\begin{aligned}
P(Y = i|X = \mathbf{x}) &= \frac{P(Y = i, X = \mathbf{x})}{P(X = \mathbf{x})} && (6)\\
&= \frac{\sum_l P(Z = l)P(Y = i|Z = l)P(X = \mathbf{x}|Y = i)}{\sum_k \sum_l P(Z = l)P(Y = k|Z = l)P(X = \mathbf{x}|Y = k)} && (7)\\
&= \frac{P(X = \mathbf{x}|Y = i) \sum_l P(Z = l)p_{il}}{\sum_k P(X = \mathbf{x}|Y = k) \sum_l P(Z = l)p_{kl}} && (8)\\
&= \frac{\exp(\ln(P(X = \mathbf{x}|Y = i))) \sum_l P(Z = l)p_{il}}{\sum_k \exp(\ln(P(X = \mathbf{x}|Y = k))) \sum_l P(Z = l)p_{kl}}. && (9)
\end{aligned}$$

This could in principle also be computed, e.g. if the Z-neurons output $P(Z = l)$, but it is biologically less plausible.

References

[1] Bernhard Nessler, Michael Pfeiffer, Lars Buesing, and Wolfgang Maass. Bayesian computation emerges in generic cortical microcircuits through spike-timing-dependent plasticity. *PLoS computational biology*, 9(4):e1003037, 2013.