Calculus and Probability

Assignment 1

Note:

- You can hand in your solutions as a single PDF via the assignment module in Brightspace. Note that the document should be in English and typeset with LaTeX, Word or a similar program. It should not be a scan or picture of your handwritten notes.
- Make sure that your name, student number and group number are on top of the first page!
- **Note that your submission should be an individual submission because it can influence your final grade for this course. If we detect that your work is not completely your own work, we will ask the exam committee to investigate whether it is plagiarism or not!**

Exercises to be presented during the exercise hours

Exercise 1
Let $f(x) = \frac{1}{2} - \frac{1}{1+(x-1)^2}$. Determine the values x for which

a) $f(x) = 0$

For $f(x) = 0$ we have $\frac{1}{2} = \frac{1}{1+(x-1)^2}$ so $2 = 1+(x-1)^2$ and $2 = 2+x(x-2)$ so $x \in \{0, 2\}$.

b) $f(x) > 0$

If $x < 0$, $x(x-2) > 0$ so $f(x) > 0$. When $0 < x < 2$, $x(x-2) < 0$, so $f(x) < 0$. When $x > 2$, $x(x-2) > 0$ so $f(x) > 0$.

Exercise 2
Let's assume that x runs through the interval $(\frac{1}{2}, 1)$. What values does y run through for $y = \frac{a}{x}$, where $a \in \mathbb{R}$?

We distinguish 3 cases.

1. When $a = 0$, we have $y = \{0\}$
2. $a > 0$. Using the fact that $x < 1$ we see that $y = \frac{a}{x} > a$ and because $x > \frac{1}{2}$, $y = \frac{a}{x} < 2a$. So y takes values in the interval $(a, 2a)$. Does y reach every value in this interval? Let c be in $(a, 2a)$. Then
$$c = \frac{a}{x_c} \iff x_c = \frac{a}{y}.$$
Because c lies in $(a, 2a)$, we see that x_c lies in $(\frac{1}{2}, 1)$. So for every element c in $(a, 2a)$, there exists an element x_c in $(\frac{1}{2}, 1)$ such that $c = \frac{a}{x_c}$. Hence y runs through all values in $(a, 2a)$.
3. $a < 0$. Again using $x \in (\frac{1}{2}, 1)$ we find $y = \frac{a}{x} < a$ and $y = \frac{a}{x} > 2a$, so y takes values in $(2a, a)$. Using the same approach as above we find that y runs through all values in $(2a, a)$.

Exercise 3
Are the following functions even or odd? In your explanation use the definition.

a) $\sqrt{x^2}$

$f(-x) = \sqrt{(-x)^2} = \sqrt{x^2} = f(x)$, so even.

b) $f(x) = x^3 + 5x^2 + 2x + 10 + (5 - x)(x^2 + 2)$

Note by rewriting: $f(x) = (5+x)(x^2+2)+(5-x)(x^2+2)$, so $f(-x) = (5-x)((-x)^2+2)+(5+x)((-x)^2+2) = (5+x)(x^2+2)+(5-x)(x^2+2) = f(x)$, so even.

Exercise 4
Determine the domains and ranges of the following functions. In your answer use $D(f)$ and $R(f)$ to be the domain respectively the range of the function f.

a) $f(x) = \frac{1}{1+\sqrt{x}}$;

Because of the square root we need $x \geq 0$, so $\underline{\underline{D(f) = [0, \infty)}}$. This implies $1 \leq 1 + \sqrt{x}$ and so $\underline{\underline{R(f) = (0, 1]}}$.

b) $f(x) = \frac{x-5}{x^2-3x-10}$;

The only elements which are not in the domain are the points where the denominator evaluates to 0. We note that $x^2 - 3x - 10 = (x-5)(x+2)$ so that $\underline{\underline{D(f) = \mathbb{R} \setminus \{-2, 5\}}}$ (all points in $\mathbb{R}$ except -2 and 5). This also means that $f(x) = \frac{1}{x+2}$ if $x \neq -2, 5$. The range of $\frac{1}{x+2}$ is $\mathbb{R} \setminus \{0\}$, but we cannot include the value at $x = 5$. So $\underline{\underline{R(f) = \mathbb{R} \setminus \{0, \frac{1}{7}\}}}$.

Exercise 5
Find the limits. (Hint: simplify as much as possible before applying the limit!)

a) $\lim\limits_{x \to 0} \frac{3(x-1)+3}{x}$;

We note that for all $x \neq 0$ we have $\frac{3(x-1)+3}{x} = \frac{3x-3+3}{x} = \frac{3x}{x} = 3$. So we must have $\underline{\underline{\lim\limits_{x \to 0} \frac{3(x-1)+3}{x} = 3}}$.

b) $\lim\limits_{x \to -1} \frac{5x^2-5}{x^3+2x^2-x-2}$;

For $x \notin \{-2, -1, 1\}$, we have $\frac{5x^2-5}{x^3+2x^2-x-2} = \frac{5(x^2-1)}{(x^2-1)(x+2)}$. So $\lim\limits_{x \to -1} \frac{5x^2-5}{x^3+2x^2-x-2} = \lim\limits_{x \to -1} \frac{5}{(x+2)} = \underline{\underline{5}}$.

Exercises to be handed in

You are expected to explain your answers, even if this is not explicitly stated in the exercises themselves.

Exercise 6
Let $f(x) = x - x^3$. Determine the values x for which

a) $f(x) = 0$ $\boxed{\text{1 pt}}$

We note that $x - x^3 = x(1-x^2) = x(1-x)(1+x)$ so $f(x) = 0$ for $\underline{\underline{x \in \{-1, 0, 1\}}}$.

b) $f(x) > 0$ 1 pt

Note that $(1 - x^2) > 0 \iff |x| < 1 \iff x \in (-1, 1)$. So we get four cases:

1. $x < -1$. Then $x < 0$ and $(1 - x^2) < 0$, so $f(x) > 0$.
2. $x \in (-1, 0)$. Then $x < 0$ and $(1 - x^2) > 0$, so $f(x) < 0$.
3. $x \in (0, 1)$. Then $x > 0$ and $(1 - x^2) > 0$, so $f(x) > 0$.
4. $x > 1$. Then $x > 0$ and $(1 - x^2) < 0$, so $f(x) < 0$.

Thus $f(x) > 0$ for $\underline{\underline{x \in (-\infty, -1) \cup (0, 1)}}$.

Exercise 7

Let's assume that x runs through the interval $(0, 1)$. What values does y run through for $y = a + (b - a)x$, where $a, b \in \mathbb{R}$? 1 pt

We distinguish 3 cases. Clearly, when $a = b$, we have $y = \{a\}$. Furthermore,

1. $b > a$. Then $(b - a) \geq 0$. Using the fact that $x \in (0, 1)$ we see that $y = a + (b - a)x > a$ and $y = a + (b - a)x < a + (b - a) = b$. So y takes values in the interval (a, b). Does y reach every value in this interval? Let c be in (a, b). Then

$$c = a + (b - a)x_c \iff x_c = \frac{c - a}{b - a}.$$

   Because c lies in (a, b), we see that x_c lies in $(0, 1)$. So for every element c in (a, b), there exists an element x_c in $(0, 1)$ such that $c = a + (b - a)x_c$. Hence y runs through all values in (a, b).

2. $b < a$. Then $(b - a) < 0$. Again using $x \in (0, 1)$ we find $y = a + (b - a)x < a$ and $y = a + (b - a)x > a + (b - a) = b$, so y takes values in (b, a). Using the same approach as above we find that y runs through all values in (b, a).

Exercise 8

Are the following functions even or odd? In your explanation use the definition. 1 pt

a) $f(x) = 3x - x^3$;

$f(-x) = 3(-x) - (-x)^3 = -3x + x^3 = -(3x - x^3) = -f(x)$, so odd.

b) $f(x) = \sqrt[3]{(1-x)^2} + \sqrt[3]{(1+x)^2}$;

$f(-x) = \sqrt[3]{(1-(-x))^2} + \sqrt[3]{(1+(-x))^2} = \sqrt[3]{(1+x)^2} + \sqrt[3]{(1-x)^2} = f(x)$, so even.

Exercise 9

Determine the domains and ranges of the following functions. In your answer use $D(f)$ and $R(f)$ to be the domain respectively the range of the function f. 2 pt

a) $f(x) = \sqrt{7 - x^2} + 1$

To correctly compute the square root we need $7 - x^2 \geq 0$, i.e. $-\sqrt{7} \leq x \leq \sqrt{7}$. So $\underline{\underline{D(f) = [-\sqrt{7}, \sqrt{7}]}}$. As $0 \leq 7 - x^2 \leq 7$ we have $0 \leq \sqrt{7 - x^2} \leq \sqrt{7}$, so $1 \leq f(x) \leq \sqrt{7} + 1$. Hence $\underline{\underline{R(f) = [1, \sqrt{7} + 1]}}$.

b) $f(x) = \frac{1}{|x|}$

Clearly the domain only excludes $x = 0$, i.e. $\underline{\underline{D(f) = \mathbb{R} \setminus \{0\}}}$. The range is $\underline{\underline{R(f) = (0, \infty)}}$.

Exercise 10

a) What is the inverse of 1 pt

$$y = \frac{ax + b}{cx + d} \qquad (ad - bc \neq 0)$$

We find $y(cx + d) = ax + b$ which results in $x(cy - a) = -dy + b$ so that $x = \frac{-dy+b}{cy-a}$. Hence the inverse function g is given by $g(x) = \frac{-dx+b}{cx-a}$.

b) When is it equal to the original function? 1 pt

It is equal to f when $d = -a$. Note that g is defined for all x except $x = \frac{a}{c}$. But $f(x) = \frac{a}{c} \iff ad - bc = 0$, so g is still defined on the whole range of f.

Exercise 11

Find the limits. (Hint: simplify as much as possible before applying the limit!) $\boxed{\text{2 pt}}$

a) $\lim\limits_{x\to 2} \frac{x-2}{x^2+x-6}$;

We note that for all $x \neq 2$ we have $\frac{x-2}{x^2+x-6} = \frac{x-2}{(x-2)(x+3)} = \frac{1}{x+3}$. So $\lim\limits_{x\to 2} \frac{x-2}{x^2+x-6} = \lim\limits_{x\to 2} \frac{1}{x+3} = \underline{\underline{\frac{1}{5}}}$.

b) $\lim\limits_{x\to 1} \frac{x^2-4x+3}{x^2+x-2}$.

For all $x \neq 1$ we have $\frac{x^2-4x+3}{x^2+x-2} = \frac{(x-1)(x-3)}{(x-1)(x+2)} = \frac{x-3}{x+2}$. So $\lim\limits_{x\to 1} \frac{x^2-4x+3}{x^2+x-2} = \lim\limits_{x\to 1} \frac{x-3}{x+2} = \underline{\underline{-\frac{2}{3}}}$.

Exercise 12 (Old Exam Question (Bonus, 2pt))

This exercise combines several aspects of the theory of this week.

Let $f(x) = \frac{x}{2x+3}$ and let $g(x)$ be defined as follows

$$g(x) = \begin{cases} f^{-1}(x) & \text{if } x < -2 \text{ or } x > 3 \\ ax + b & \text{if } -2 \leq x \leq 3 \end{cases}$$

If $g(x)$ is continuous, then what is $\frac{b}{a}$?

First we find the inverse of f. Start with $y = \frac{x}{2x+3}$. Interchanging x and y gives $x = \frac{y}{2y+3}$. Then

$$\begin{aligned} x = \frac{y}{2y+3} &\Rightarrow (2y+3)x = y \\ &\Rightarrow 2yx + 3x = y \\ &\Rightarrow 2yx - y = -3x \\ &\Rightarrow y(2x-1) = -3x \\ &\Rightarrow y = \frac{-3x}{2x-1} \\ &\Rightarrow y = \frac{3x}{1-2x} \end{aligned}$$

Hence, $f^{-1}(x) = \frac{3x}{1-2x}$. Since $f^{-1}(-2) = -\frac{6}{5}$ and $f^{-1}(3) = -\frac{9}{5}$ we are looking for a, b such that the line goes through the points $(-2, -\frac{6}{5})$ and $(3, -\frac{9}{5})$. The slope a is given by

$$a = \frac{\Delta y}{\Delta x} = \frac{-\frac{9}{5} - (-\frac{6}{5})}{3 - (-2)} = \frac{-\frac{3}{5}}{5} = -\frac{3}{25}$$

Since $-\frac{6}{5} = -\frac{3}{25} \cdot -2 + b$ it follows that $b = -\frac{6}{5} - \frac{6}{25} = -\frac{36}{25}$. Hence $\frac{b}{a} = \frac{-36}{-3} = \underline{\underline{12}}$.

Your final grade is the sum of your scores divided by 1.0.