Calculus en Kansrekenen (NWI-IBC017) – EXAMINATION

October 26, 2015, 12:30–15:30

This exam consists of seven problems. You can score a maximum of 100 points. Each question indicates how many points it is worth. The exam is closed book. You are NOT allowed to use a programmable calculator, a computer or a mobile phone. The only device you are allowed to use is the calculator provided at the exam. **Explain your approach and answers** briefly. You may give the answers in Dutch or in English. Please write clearly! Do not forget to put **your name** and **your student number** on the top of **each page**.

1. **(16 points)** Consider the function given by

$$f(x) = \frac{e^x}{x-1}.$$

Investigate the function by following the steps below. (If you need to compute with it, you can use the approximation $e \approx 2.71$.)

(a) Determine f's domain.

(b) What are the x and y intercepts?

(c) What are the limits of f at the edges of the domain?

(d) What is $f'(x)$ and what are its zeros?

(e) Find the critical points. Where is f increasing? Where is f decreasing?

(f) Find the points of inflection. Where is f convex? Where is f concave?

(g) Sketch the graph of f.

2. **(15 points)** Compute the following indefinite integrals. (You can verify your results; you don't get any points for the verification.)

(a) $\int (1+x)^2 \, dx$

(b) $\int x \cdot \sin(x^2) \, dx$

(c) $\int \cos^2(x) \, dx$

3. **(12 points)** In this problem we will calculate $\int \arctan(x) \, dx$. Follow the steps below.

(a) Find the derivative of the function $\tan(x)$.

(b) Using this result, find the derivative of the function $\arctan(x)$. (*Hint:* derivative of inverse function)

(c) By definition $x^2 = \tan^2(\arctan(x))$. Show that the result in (b) is identical to $\frac{1}{1+x^2}$.

(d) So, we know that $(\arctan(x))' = \frac{1}{1+x^2}$. Calculate $\int \arctan(x) \, dx$ by applying integration by parts on $\int 1 \cdot \arctan(x) \, dx$.

4. **(12 points)** Consider the set

$$E = \left\{ (x, y) \in \mathbb{R}^2 \colon \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \right\},$$

where $a > 0$ and $b > 0$.

(a) Draw E with $a = 1$ and $b = 1$.

(b) Draw E with $a = 1$ and $b = 2$.

(c) Show in general that the area enclosed by E is πab.
(*Hint:* Compute $\int_{-a}^{a} b\sqrt{1 - (x/a)^2} \, dx$ by substituting x by $a \sin(u)$.)

.... **The exam continues on the next page!** ....

5. **(15 points)** Solve the following differentiation problems. Simplify the results as much as possible.

   (a) Find the derivative of the function $f(x) = e^{\ln(x^2)} + \frac{1}{3}x^4$.

   (b) Find the derivative of the function $f(x) = (\sin(x))^{\sin(x)}$ (where $x \in (0, \pi/2)$).

   (c) Demonstrate that the theorem of Schwarz holds in case of the following function: $f(x, y) = \frac{x^2+2}{y}$. (*Hint:* The theorem of Schwarz states that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.)

6. **(15 points)** Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ cxe^{-x^2} & \text{if } x \geq 0. \end{cases}$$

   (a) Find c.

   (b) Find the cumulative distribution function F.

   (c) Compute the probability $P(1 < X < \sqrt{2})$.

7. **(15 points)**

   (a) Draw the first six rows of Pascal's triangle.

   (b) Determine the expansion of $(x-y)^4$. (*Hint:* For instance, the expansion of $(x+y)^2$ is $x^2+2xy+y^2$.)

   (c) Determine the expansion of $\left(2a + \frac{b}{3}\right)^5$.

   (d) Cars stop at red lights but not at green or yellow lights. At the junction at the Mercator 1 building the traffic light on the Heyendaalseweg is red for 30 seconds, yellow for 10 seconds, and green for 20 seconds. Let us introduce a random variable X for the number of cars stopped by the red light out of five randomly arriving cars.

      i. What is the probability $P(X = 3)$?

      ii. What is $P(X = 0)$?

      iii. And what is $\sum_{i=1}^{5} P(X = i)$?