Calculus and Probability Theory
Assignment 4

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After completing these exercises successfully you should be confident with the following topics:

- Analyse and sketch real functions
- Apply differentiation rules to determine higher-order partial derivatives
- Find primitives of well-known functions
- Compute definite integrals when the primitive function is known
- computer improper integrals

1. (**20 points**) Investigate the function $f(x) = \frac{x}{\ln(x)}$ as follows. (Do not start with drawing a graph by means of a device or some web resource. Of couse you may check your result when you're done.)

   (a) Determine the domain of the function f.
   Solutions:

   In this case, we just have to look out for the values for x when the denominator could become 0 or undefined. First of all, ln(0) is undefined. Second, ln(1) = 0. Note for myself: ln(1) = 0 answers the question, what is the value for x so $e^x = 1$?. Therefore:

   $$D(f) = \{x \in \mathbb{R} \mid 0 < x < 1 \wedge x > 1\}$$

   (b) What are the roots of f?
   Solutions:

   To determine the roots of f, we just have to determine the x value when the whole quotient equals 0. As we take the denominator as the first step, then we plug in x = 0 and get ln(0) = 1 and in the numerator 0. Therefore $\frac{0}{\ln(0)} = \frac{0}{1} = 0$.

The roots of f: $x_1 = 0$

The y-intercept would therefore also be 0.

(c) Determine the limits at 1 and ∞. (Hint: there are 3 cases, use L'Hopital!)
Solutions:

$$\begin{aligned}
\lim_{x\to 1^-} \frac{x}{\ln(x)} &= \lim_{x\to 1^-} \frac{1}{\frac{1}{x}} \\
&= \lim_{x\to 1^-} \frac{1}{\frac{1}{x}} \\
&= \lim_{x\to 1^-} x \\
&= -\infty
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to 1^+} \frac{x}{\ln(x)} &= \lim_{x\to 1^+} \frac{1}{\frac{1}{x}} \\
&= \lim_{x\to 1^+} \frac{1}{\frac{1}{x}} \\
&= \lim_{x\to 1^+} x \\
&= \infty
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to \infty} \frac{x}{\ln(x)} &= \lim_{x\to \infty} \frac{1}{\frac{1}{x}} \\
&= \lim_{x\to \infty} \frac{1}{\frac{1}{x}} \\
&= \lim_{x\to \infty} x \\
&= \infty
\end{aligned}$$

(d) Find f' and f''.
Solutions:

Use Quotient rule.

$$\begin{aligned}
f(x)' &= \frac{1 \cdot ln(x) - x \cdot \frac{1}{x}}{ln^2(x)} \\
&= \frac{\ln(x) - 1}{\ln^2(x)}
\end{aligned}$$

$$
\begin{aligned}
f(x)'' &= \frac{\frac{1}{x}\ln^2(x) - ((\ln(x)-1)\frac{2\ln(x)}{x})}{ln^4(x)} \\
&= \frac{\frac{\ln^2(x)-((\ln(x)-1)2\ln(x))}{x}}{\ln^4(x)} \\
&= \frac{\frac{\ln(x)-(2\ln(x)-2)}{x}}{\ln^3(x)} \\
&= \frac{\frac{-\ln(x)+2}{x}}{\ln^3(x)} \\
&= \frac{2-\ln(x)}{x\ln^3(x)}
\end{aligned}
$$

(e) Find the zeros of f' and f''.
Solutions:

$f'(x) = \frac{ln(x)-1}{ln^2(x)}$ so $x = e$ is the only zero of $f'(x)$.
$f''(x) = \frac{2-\ln(x)}{x\ln^3(x)}$ so $x = e^2$ is the only zero of $f''(x)$.

(f) What are the critical points (determine their x and y coordinates)?
Solutions:

A critical point of a function $f : D \to \mathbb{R}$, is a point $a \in D$ such that $f'(a) = 0$. The value $f(a)$ is called a critical value of f.

We just found the x-coordinates of the critical point. We have $f(e) = \frac{e}{ln(e)} = \frac{e}{1} = 1$. So we get the point (e,e).

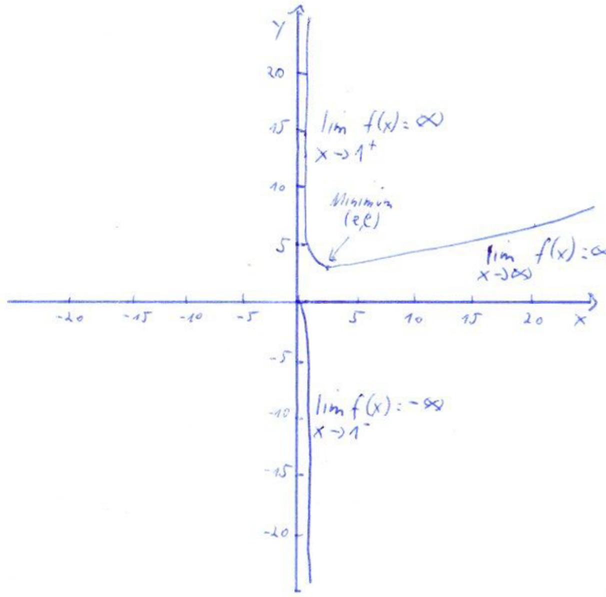
(g) Find the local minimums and maximums.
Solutions:

We have $f''(e) = \frac{2-ln(e)}{x\ln^3(e)} = \frac{2-1}{e\cdot 1} = \frac{1}{e} > 0$, so it is a minimum. This is the only critical point, so there are no local maxima and only one minimum.

(h) Which parts of the function are convex and concave? Does function f have points of inflection? (Hint: Use the sign of the second derivative for answering both questions.)
Solutions:

The function is concave $\iff f''(x) < 0 \iff \frac{2-\ln(x)}{x\ln^3(x)} < 0 \iff x < e^2$.
The function is convex $\iff f''(x) > 0 \iff \frac{2-\ln(x)}{x\ln^3(x)} > 0 \iff x > e^2$.
It has a point of inflection at $x = e^2$

(i) Draw the graph of function f. (If you collect all intervals and special points in a table, it helps a lot in drawing the graph. Moreover, you get some extra points!)
Solutions:



2. (**6 points**) Show that the derivative of an odd function is even and that the derivative of an even function is odd.
Solutions:

- A function $f : (-a, a) \to \mathbb{R}$ is even if $f(-x) = f(x)$, for all $x \in (a-, a)$ and odd if $f(-x) = -f(x)$, for all $x \in (-a, a)$
- $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

If f is odd then

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} = -\lim_{h\to 0} \frac{f(x-h) - f(x)}{h} = -f'(x)$$

3. (**14 points**) Optimization problem

(a) Find the point on the parabola $y^2 = 2x$ that is closest to $A = (1, 4)$
Solutions:

$y^2 = 2x$ is a sideways parabola with the equation $x = \frac{y^2}{2}$.
The distance formula from unknown point (x,y) to known point A = (1,4) is

$$d(x,y) = \sqrt{(x-1)^2 + (y-4)^2}$$

Substitute $\frac{y^2}{2}$ for x:

$$d(\frac{y^2}{2}, y) = \sqrt{(\frac{y^2}{2} - 1) + (y-4)^2}$$
$$d(y) = \sqrt{\frac{y^4}{4} - y^2 + 1 + 2 - 8y + 16}$$
$$= \sqrt{\frac{y^4}{4} - 8y + 17}$$

Take the derivative of the distance with respect to y

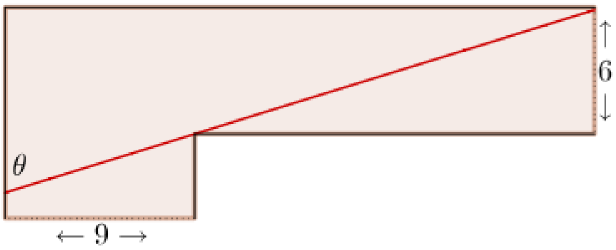
$$d'(y) = \frac{y^3 - 8}{\sqrt{y^4 - 32y + 68}}$$

The minimum will occur when the derivative is zero:

$$y^3 - 8 = 0$$
$$y^3 = 8$$
$$y = 2$$
$$x = \frac{2^2}{2}$$
$$x = 2$$

The point (2,2) is the closest point on the parabola to A = (1,4).

(b) A steel rod is carried down a hallway of 9 meter wide. At the end there is a corner to the right into a narrower hallway of 6 meter wide. What is the maximum length of the steel rod that can be carried horizontally around the corner?



(Hint: What happens at $\theta \to 0$ and $\theta \to \frac{1}{4}\pi$? Show that the angle at which the minimum is obtained is at $\theta = arctan(\sqrt[3]{\frac{3}{2}}) \approx 0,853$.)

Solutions:

4. (**20 points**) Given function f, find the partial derivatives. If it is necessary, simplify the result.

a.i $f(x,y) = \cos(4y - xy)$; $\frac{\partial}{\partial x} f(x,y) =?$
Solutions:

$$\begin{aligned}\frac{\partial}{\partial x} f(x,y) &= -sin(4y - xy)(-y)\\ &= y\sin(4y - xy)\end{aligned}$$

a.ii $f(x,y) = \cos(4y - xy)$; $\frac{\partial}{\partial y} f(x,y) =?$
Solutions:

$$\begin{aligned}\frac{\partial}{\partial y} f(x,y) &= -sin(4y - xy)(4 - x)\\ &= (x-4)sin(4y - xy)\end{aligned}$$

b Show that $\frac{\partial}{\partial x}(\frac{\partial}{\partial y} f(x,y)) = \frac{\partial}{\partial y}(\frac{\partial}{\partial x} f(x,y))$
Solutions:
The **Theorem of Schwarz** says that it does not matter in which order you take two partial derivatives:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

If we differentiate f first with respect to x and then with respect to y we get the derivative $\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})$ (if it exists). It is more usually denoted by $\frac{\partial^2 f}{\partial y \partial x}$ or $(f'_x)'_y$ or f''_{xy}

Alternatively, if we differentiate first with respect to y and then x we get $\frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial x \partial y}$ (if it exists). Or $(f'_y)'_x$ or f''_{yx}
Let's rewrite the left-hand side of the formula $\frac{\partial}{\partial x}(\frac{\partial}{\partial y} f(x,y)) = \frac{\partial}{\partial y}(\frac{\partial}{\partial x} f(x,y))$ like this:

$$\frac{\partial}{\partial x}(\frac{\partial}{\partial y} f(x,y)) = \frac{\partial}{\partial x}(\lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h})$$

5. (**20 points**) Evaluate the following definite integrals. (Hint: use slide 38 of the lectures about derivatives, and slide 13 of the lectures about primitives)

$$\int_a^b f(x)dx = F(b) - F(a)$$

(a) $\int_{-1}^{1}(x^3+x-1)dx$

Solutions:

$$\begin{aligned}
\int(x^3+x-1)dx &= \frac{1}{4}x^4+\frac{1}{2}x^2-x+C \\
\int_{-1}^{1}(x^3+x-1)dx &= \left[\frac{1}{4}1^4+\frac{1}{2}1^2-1\right]-\left[\frac{1}{4}(-1)^4+\frac{1}{2}(-1)^2-(-1)\right] \\
&= \frac{1}{4}-\frac{1}{2}-\frac{1}{4}-\frac{1}{2}-1 \\
&= -2
\end{aligned}$$

(b) $\int_{1}^{2}(3\sqrt{x}+\frac{3}{x^2})dx$

Solutions:

$$\begin{aligned}
\int(3\sqrt{x}+\frac{3}{x^2})dx &= 3\left[\frac{2x^{\frac{3}{2}}}{3}-\frac{1}{x}\right]+C \\
\int_{1}^{2}(3\sqrt{x}+\frac{3}{x^2})dx &= \left[3(\frac{2(2)^{\frac{3}{2}}}{3}-\frac{1}{2})\right]-\left[3(\frac{2(1)^{\frac{3}{2}}}{3}-1)\right] \\
&= 4\sqrt{2}-\frac{1}{2} \\
&\approx 5.1569
\end{aligned}$$

(c) $\int_{0}^{\pi}(\sin(x)+\cos(x))dx$

Solutions:

$$\begin{aligned}
\int(\sin(x)+\cos(x))dx &= -\cos(x)+\sin(x)+C \\
\int_{0}^{\pi}(\sin(x)+\cos(x))dx &= [-\cos(\pi)+\sin(\pi))]-[-\cos(0)+\sin(0)] \\
&= 1+0+1+0 \\
&= 2
\end{aligned}$$

(d) $\int_{1}^{e}(\frac{1-\ln x}{x^2})dx$

Solutions:
Integration by parts:

$$\begin{aligned}
\int (\frac{1-\ln x}{x^2})dx &= \int ((1-\ln x)\frac{1}{x^2})dx = \int ((1-\ln x)x^{-2})dx \\
&= \left[(1-lnx)(-\frac{1}{x})\right] - \int -\frac{1}{x}(-\frac{1}{x}) \\
&= \left[(1-lnx)(-\frac{1}{x})\right] - \int \frac{1}{x^2} \\
&= \left[(1-lnx)(-\frac{1}{x})\right] + \frac{1}{x} \\
&= \frac{lnx}{x} \\
\int_1^e (\frac{1-\ln x}{x^2})dx &= \frac{ln(e)}{e} - \frac{ln(1)}{1} \\
&= \frac{1}{e} - \frac{0}{1} \\
&= \frac{1}{e}
\end{aligned}$$

6. (**20 points**) Evaluate the following improper integrals.

   (a) $\int_{-1}^{1}(\frac{1}{x^n})dx$, n an integer such that $n \geq 2$; (Hint: distinguish two cases).
   Solutions:

   (b) $\int_{-\infty}^{-\pi/2} \frac{x\cos(x)-\sin(x)}{x^2}dx$; (Hint: use the quotient rule for derivation to find the primitive)
   Solutions:

$$\begin{aligned}
\int (\frac{x\cos(x)-\sin(x)}{x^2})dx &= \frac{\sin(x)}{x} + C \\
\int_{-\infty}^{-\pi/2} (\frac{x\cos(x)-\sin(x)}{x^2})dx &= \frac{2}{\pi}
\end{aligned}$$

   (c) $\int_2^{\infty} \frac{-1}{x\ln^2(x)}dx$; (Hint: use a fraction of well known functions to find the primitive)
   Solutions:

$$\begin{aligned}
\int (\frac{-1}{x\ln^2(x)})dx &= \frac{1}{\ln(x)} + C \\
\int_2^{\infty} (\frac{-1}{x\ln^2(x)})dx &= \frac{1}{\ln(2)}
\end{aligned}$$

7. (**bonus, +6 points**) Find primitves of the following functions f. That is, find F such that $F'(x) = f(x)$.

   (a) $f(x) = \frac{1}{2\sqrt{x}} - \frac{1}{x^2}$
   Solutions:

$$\int (\frac{1}{2\sqrt{x}} - \frac{1}{x^2})dx = \sqrt{x} + \frac{1}{x} + C$$

(b) $f(x) = 2\sin(x)\cos(x)$
Solutions:
Integration by parts:

$$\int (2\sin(x)\cos(x))dx = -\frac{1}{2}\cos(2x) + C$$

(c) $f(x) = \frac{2}{1+4x^2}$
Solutions:
Integration by parts:

$$\int (\frac{2}{1+4x^2})dx = \arctan(2x) + C$$

8. (**bonus, 10 points**) If $f(x,y) = \frac{xy}{x+y}$, show that

$$x^2 \cdot \frac{\partial}{\partial x}(\frac{\partial}{\partial x} f(x,y)) + 2xy \cdot \frac{\partial}{\partial x}(\frac{\partial}{\partial y} f(x,y)) + y^2 \cdot \frac{\partial}{\partial y}(\frac{\partial}{\partial y} f(x,y)) = 0$$

(Hint: First compute all the second partial derivatives of f, then substitute the results in the expression on the left-hand side.)
Solutions: