Calculus and Probability Theory
Assignment 2

Christoph Schmidl
s4226887
Informatica
c.schmidl@student.ru.nl
Exercise Teacher: Gergely Alpár

September 19, 2014

1. (**18 points**) Given a function f and $a \in D(f)$. Recall that the definition of the *derivative* of a function f at a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This is the slope of function f at point $(a, f(a))$. Compute the derivative of the function at the given a using the definition.

(a) $3x^2$ at point $a = -\frac{1}{2}$
Solution:

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \to 0} \frac{3(a+h)^2 - 3a^2}{h} \\ &= \lim_{h \to 0} \frac{3(-\frac{1}{2}+h)^2 - 3(-\frac{1}{2})^2}{h} \\ &= \lim_{h \to 0} \frac{3(-\frac{1}{2}+h)^2 - \frac{3}{4}}{h} \\ &= \lim_{h \to 0} 3h - 3 \\ &= -3 \end{aligned}$$

(b) $\frac{1}{x+2}$ at point $a = 1$
Solution:

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{(a+h)+2} - \frac{1}{a+2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{(1+h)+2} - \frac{1}{1+2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{h+3} - \frac{1}{3}}{h} \\
&= \lim_{h\to 0} -\frac{1}{3h+9} \\
&= -\frac{1}{9}
\end{aligned}
$$

(c) $sin\, 2x$ at point $a = 0$ (Hint: Use the fact the $\lim_{x\to 0} \frac{sin\, x}{x} = 1$)
Solution:

2. (**50 points**) Find the derivative of the following functions. You can freely use all the differentiation rules that were discussed on the lecture. If it is possible, simplify the result.

(a) $f(x) = x^5 + 5x^4 - 10x^2 + 6$
Solution:

Plus/Minus rule: $(f \pm g)'(a) = f'(a) \pm g'(a)$
Derivative of power rule: $f(x) = x^n \to f'(x) = nx^{n-1}$

$$
\begin{aligned}
\frac{d}{dx}(x^5 + 5x^4 - 10x^2 + 6) &= \frac{d}{dx}(x^5) + \frac{d}{dx}(5x^4) - \frac{d}{dx}(10x^2) + \frac{d}{dx}(6) \\
&= 5x^4 + 20x^3 - 20x
\end{aligned}
$$

(b) $f(x) = 5x^{\frac{1}{2}} - x^{\frac{3}{2}} + 2x^{-\frac{1}{2}}$
Solution:

Plus/Minus rule: $(f \pm g)'(a) = f'(a) \pm g'(a)$
Derivative of power rule: $f(x) = x^n \to f'(x) = nx^{n-1}$

$$
\begin{aligned}
\frac{d}{dx}(5x^{\frac{1}{2}} - x^{\frac{3}{2}} + 2x^{-\frac{1}{2}}) &= \frac{d}{dx}(5x^{\frac{1}{2}}) - \frac{d}{dx}(x^{\frac{3}{2}}) + \frac{d}{dx}(2x^{-\frac{1}{2}}) \\
&= \frac{5}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}} - x^{-\frac{3}{2}} \\
&= \frac{5}{2}\frac{1}{\sqrt{x}} - \frac{3}{2}\sqrt{x} - x^{-\frac{3}{2}} \\
&= \frac{5}{2\sqrt{x}} - \frac{3\sqrt{x}}{2} - \frac{1}{x^{\frac{3}{2}}}
\end{aligned}
$$

(c) $f(t) = \frac{1}{2t^2} + \frac{4}{\sqrt{t}}$
Solution:

Plus/Minus rule: $(f \pm g)'(a) = f'(a) \pm g'(a)$
Derivative of power rule: $f(x) = x^n \rightarrow f'(x) = nx^{n-1}$

$$
\begin{aligned}
\frac{d}{dx}(\frac{1}{2t^2} + \frac{4}{\sqrt{t}}) &= \frac{d}{dx}(\frac{1}{2t^2}) + \frac{d}{dx}(\frac{4}{\sqrt{t}}) \\
&= \frac{1}{2}(\frac{d}{dx}(\frac{1}{t^2})) + 4(\frac{d}{dx}(\frac{1}{\sqrt{t}})) \\
&= \frac{1}{2}(\frac{d}{dx}(t^{-2})) + 4(\frac{d}{dx}(t^{-\frac{1}{2}})) \\
&= \frac{1}{2} \cdot (-2t^{-3}) + 4 \cdot (-\frac{1}{2}t^{-\frac{3}{2}}) \\
&= -1t^{-3} - 2t^{-\frac{3}{2}} \\
&= -\frac{1}{t^3} - \frac{2}{t^{\frac{3}{2}}}
\end{aligned}
$$

(d) $y = (1 - 4x)^5$
Solution:

Chain/Composition rule: $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$

$$
\begin{aligned}
\frac{d}{dx}((1-4x)^5) &= 5(1-4x)^4 \cdot \frac{d}{dx}(1-4x) \\
&= 5(1-4x)^4 \cdot (-4) \\
&= -20(1-4x)^4
\end{aligned}
$$

(e) $f(x) = (x + 1)(x + 2)$
Solution:

Multiplication rule: $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$

$$\begin{aligned}\frac{d}{dx}((x+1)(x+2)) &= \frac{d}{dx}(x+1)\cdot(x+2)+(x+1)\cdot\frac{d}{dx}(x+2)\\ &= 1\cdot(x+2)+(x+1)\cdot 1\\ &= 2x+3\end{aligned}$$

(f) $f(x) = \frac{3x+1}{2x+4}$
Solution:

Division rule: $(\frac{f}{g}'(a)) = \frac{f'(a)g(a)-f(a)g'(a)}{g^2(a)}$

$$\begin{aligned}\frac{d}{dx}(\frac{3x+1}{2x+4}) &= \frac{\frac{d}{dx}(3x+1)\cdot(2x+4)-(3x+1)\cdot\frac{d}{dx}(2x+4)}{(2x+4)^2}\\ &= \frac{3(2x+4)-2(3x+1)}{(2x+4)^2}\\ &= \frac{6x+12-6x-2}{(2x+4)^2}\\ &= \frac{10}{4x^2+16x+16}\\ &= \frac{5}{2x^2+8x+8}\end{aligned}$$

(g) $f(x) = (\frac{x^2-1}{2x^3+1})^4$
Solution:

Chain/Composition rule: $(g\circ f)'(a) = g'(f(a))\cdot f'(a)$
Division rule: $(\frac{f}{g}'(a)) = \frac{f'(a)g(a)-f(a)g'(a)}{g^2(a)}$

$$\begin{aligned}\frac{d}{dx}((\frac{x^2-1}{2x^3+1})^4) &= 4(\frac{x^2-1}{2x^3+1})^3\cdot\frac{d}{dx}(\frac{x^2-1}{2x^3+1})\\ &= 4(\frac{x^2-1}{2x^3+1})^3\cdot\frac{\frac{d}{dx}(x^2-1)\cdot(2x^3+1)-(x^2-1)\cdot\frac{d}{dx}(2x^3+1)}{(2x^3+1)^2}\\ &= 4(\frac{x^2-1}{2x^3+1})^3\cdot\frac{2x(2x^3+1)-6x^2(x^2-1)}{(2x^3+1)^2}\\ &= \frac{4(x-1)^3(x+1)^3}{(2x^3+1)^3}\cdot\frac{2x(2x^3+1)-6x^2(x^2-1)}{(2x^3+1)^2}\\ &= \frac{8x(x^2-1)^3\cdot(x^3-3x-1)}{(2x^3+1)^2}\end{aligned}$$

3. (**32 points**) Apply any rules (including chain or inverse rules) and the logarithmic differentiation as appropriate to compute the result. If you can solve a problem in two different ways, you get two extra points.

(a) $f(x) = sin^2x + cos^2x; f'(x) = ?$
Solution:

Chain/Composition rule: $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$

$$\begin{aligned}
\frac{d}{dx}(sin^2x + cos^2x) &= \frac{d}{dx}(sin^2x) + \frac{d}{dx}(cos^2x) \\
&= \frac{d}{dx}(sin^2x) + 2cosx \cdot \frac{d}{dx}(cos) \\
&= \frac{d}{dx}(sin^2x) + (-sinx) \cdot 2cosx \\
&= \frac{d}{dx}(sin^2x) - 2cosx \cdot sinx \\
&= -2cosx \cdot sinx + 2\frac{d}{dx}(sinx) \cdot sinx \\
&= -2cosx \cdot sinx + cosx \cdot 2sinx \\
&= 0
\end{aligned}$$

(b) $f(x) = 3^{2x^2-1}; f'(x) = ?$
Solution:

Chain/Composition rule: $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$

$$\begin{aligned}
\frac{d}{dx}(3^{2x^2-1}) &= 3^{2x^2-1} \cdot log(3)(\frac{d}{dx}(-1 + 2x^2)) \\
&= \frac{d}{dx}(-1) + 2\frac{d}{dx}(x^2) \cdot 3^{2x^2-1} \cdot log(3) \\
&= 2x \cdot 2 \cdot 3^{2x^2-1} \cdot log(3) \\
&= 4 \cdot 3^{2x^2-1}xlog(3)
\end{aligned}$$

(c) $f(x) = 2^{ln(tan\,x)}; f'(x) = ?$
Solution:

Chain/Composition rule: $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$

$$\begin{aligned}
\frac{d}{dx}(2^{ln(tan\,x)}) &= 2^{ln(tan\,x)} \cdot ln(2)(\frac{d}{dx}(2^{ln(tan\,x)})) \\
&= cot(x)\frac{d}{dx}(tan(x))2^{ln(tan\,x)}ln(2) \\
&= sec(x)^2 \cdot 2^{ln(tan\,x)} \cdot cot(x)ln(2)
\end{aligned}$$

(d) $f(x) = \frac{5x^2-3}{\sqrt{x+1}}; f'(x) = ?$
Solution:

Multiplication rule: $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$
Chain/Composition rule: $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$

$$\begin{aligned}
\frac{d}{dx}(\frac{5x^2-3}{\sqrt{x+1}}) &= (-3+5x^2)(\frac{d}{dx}(\frac{1}{\sqrt{1+x}})) + \frac{\frac{d}{dx}(-3+5x^2)}{\sqrt{1+x}} \\
&= \frac{\frac{d}{dx}(-3+5x^2)}{\sqrt{1+x}} + (-3+5x^2) \cdot \frac{-\frac{d}{dx}(1+x)}{2(x+1)^{\frac{3}{2}}} \\
&= \frac{\frac{d}{dx}(-3+5x^2)}{\sqrt{1+x}} - \frac{(-3+5x^2)\frac{d}{dx}(1) + \frac{d}{dx}(x)}{2(1+x)^{\frac{3}{2}}} \\
&= \frac{10x}{\sqrt{1+x}} - \frac{-3+5x^2}{2(1+x)^{\frac{3}{2}}}
\end{aligned}$$

(e) $f(x) = x^3 - 2; (f^{-1})'(x) = ?$
Solution:

Inverse Rule: If f has an inverse f^{-1}, then $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$
Inverse: $\sqrt[3]{x+2}$

$$\begin{aligned}
\frac{d}{dx}(\sqrt[3]{x+2}) &= \frac{1}{\frac{d}{dx}(x^3-2) \cdot (\sqrt[3]{x+2})} \\
&= \frac{1}{3x^2 \cdot \sqrt[3]{x+2}}
\end{aligned}$$