Calculus and Probability Theory
Assignment 4

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October 3, 2014

1. (**26 points**) Evaluate the following definite integrals. (If neccessary, round to two decimal places.).

   (a) $\int_1^4 (x + 3\sqrt{x} + 2)\ dx$;
   Solution:

$$\begin{aligned}
\int_1^4 (x + 3\sqrt{x} + 2) dx &= \int x\ dx + 3 \int x^{\frac{1}{2}}\ dx + \int 2\ dx \\
&= \frac{1}{2}x^2 + 3 \cdot \frac{2}{3}x^{\frac{3}{2}} + 2x \Big]_1^4 \\
&= \frac{1}{2}x^2 + 2x^{\frac{3}{2}} + 2x \Big]_1^4 \\
&= \frac{1}{2}x^2 + 2\sqrt{x^3} + 2x \Big]_1^4 \\
&= (\frac{1}{2}4^2 + 2\sqrt{4^3} + 2 \cdot 4) - (\frac{1}{2}1^2 + 2\sqrt{1^3} + 2 \cdot 1) \\
&= 32 - 4\frac{1}{2} \\
&= 27.5
\end{aligned}$$

(b) $\int_1^{16}(4x^{\frac{7}{3}} - \sqrt[4]{x} + \frac{\pi}{x})\ dx$;
Solution:

$$\begin{aligned}
\int_1^{16}(4x^{\frac{7}{3}} - \sqrt[4]{x} + \frac{\pi}{x})\ dx &= \int 4x^{\frac{7}{3}}\ dx - \int \sqrt[4]{x}\ dx + \int \frac{\pi}{x}\ dx \\
&= 4\int x^{\frac{7}{3}}\ dx - \int x^{\frac{1}{4}}\ dx + \pi \int \frac{1}{x}\ dx \\
&= 4 \cdot \frac{3}{10} x^{\frac{10}{3}} - \frac{4}{5}x^{\frac{5}{4}} + \pi \cdot \ln(x) \Big]_1^{16} \\
&= \frac{12}{10} x^{\frac{10}{3}} - \frac{4}{5}x^{\frac{5}{4}} + \pi \cdot \ln(x) \Big]_1^{16} \\
&= \frac{12}{10} \sqrt[3]{x^{10}} - \frac{4}{5}\sqrt[4]{x^5} + \pi \cdot \ln(x) \Big]_1^{16}
\end{aligned}$$

$$\begin{aligned}
&= (\frac{12}{10}\sqrt[3]{16^{10}} - \frac{4}{5}\sqrt[4]{16^5} + \pi \cdot \ln(16)) - (\frac{12}{10}\sqrt[3]{1^{10}} - \frac{4}{5}\sqrt[4]{1^5} + \pi \cdot \ln(1)) \\
&= 12368.63823 - 0.4 \\
&\approx 12368.24
\end{aligned}$$

(c) $\int_{-2}^{2}(e^{5x-1})\ dx$. (Hint: Apply substitution).
Solution:

$$\int_{-2}^{2}(e^{5x-1})\ dx$$

$Let\ u = 5x - 1$

$\frac{du}{dx} = 5 \rightarrow 5dx = du$

$$\int_{-2}^{2}(e^{u})\ \frac{du}{5} = \frac{1}{5}\int e^u\ du = \frac{1}{5}e^u\Big]_{-2}^{2} = \frac{1}{5}e^{5x-1}\Big]_{-2}^{2}$$

$$\begin{aligned}
\frac{1}{5}e^{5x-1}\Big]_{-2}^{2} &= (\frac{1}{5}e^{5(2)-1}) - (\frac{1}{5}e^{5(-2)-1}) \\
&\approx 1620.62
\end{aligned}$$

2. (**40 points**) Determine the indefinite integrals by applying the substitution method.

   (a) $\int \sqrt{x+1}\,dx$. Verify the result.
   Solution:

$$\int \sqrt{x+1}\,dx$$

Let $u = x + 1$

$\frac{du}{dx} = 1 \rightarrow 1dx = du$

$$\begin{aligned}\int \sqrt{u}\,du &= \int u^{\frac{1}{2}}\,du \\ &= \frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{2}{3}(x+1)^{\frac{3}{2}} + C\end{aligned}$$

Control:

$$\begin{aligned}\frac{d}{dx}(\frac{2}{3}(x+1)^{\frac{3}{2}} + c) &= \frac{d}{dx}(\frac{2}{3}(x+1)^{\frac{3}{2}}) \\ &= \frac{2}{3}\frac{d}{dx}(x+1)^{\frac{3}{2}}\end{aligned}$$

Let $u = x + 1$

$\frac{d}{du}u^{\frac{3}{2}} = \frac{3}{2}u^{\frac{1}{2}} = \frac{3\sqrt{u}}{2}$

$$\frac{2}{3}\frac{3\sqrt{x+1}\frac{d}{dx}(1+x)}{2} = \frac{2}{3}\frac{3\sqrt{x+1}}{2} = \frac{\frac{6}{3}\sqrt{x+1}}{2} = \sqrt{x+1}$$

   (b) $\int \sin(2x-3)\,dx$
   Solution:

$$\int \sin(2x-3)\,dx$$

Let $u = 2x - 3$

$\frac{du}{dx} = 2 \rightarrow 2dx = du \rightarrow dx = \frac{1}{2}du$

$$\begin{aligned}\int \frac{1}{2}\sin(u)\ du &= \frac{1}{2}\int \sin(u)\ du \\ &= \frac{1}{2}\cdot(-\cos(u))+C \qquad = \frac{1}{2}\cdot(-\cos(2x-1))+C \\ &= -\frac{1}{2}\cos(2x-3)+C\end{aligned}$$

(c) $\int 5x \cdot \cos(3x^2+5)\ dx$
Solution:

$$\int 5x \cdot \cos(3x^2+5)\ dx$$

Let $u = 3x^2+5$

$\frac{du}{dx} = 6x \rightarrow 6xdx = du \rightarrow dx = \frac{1}{6}xdu$

$$\begin{aligned}5\int x\cdot\cos(u)\ du &= \frac{5}{6}\int \cos(u)\ du \\ &= \frac{5}{6}\sin(u)+C \\ &= \frac{5}{6}\sin(3x^2+5)+C\end{aligned}$$

(d) $\frac{1}{3}\int 3^{x^4}\cdot x^3\ dx$. Verify the result.
Solution:

$$\frac{1}{3}\int 3^{x^4}\cdot x^3\ dx$$

Let $u = x^4$

$\frac{du}{dx} = 4x^3 \rightarrow 4x^3dx = du$

$$\begin{aligned}\frac{1}{3}(\frac{1}{4}\int 3^u\cdot 1\ du) &= \frac{1}{3}(\frac{1}{4}(\frac{1}{\ln(3)}3^u)+C) \\ &= \frac{1}{3}(\frac{1}{4}\frac{3^u}{\ln(3)}) \\ &= \frac{1}{3}(\frac{1}{4}\frac{3^{x^4}}{\ln(3)}) \\ &= \frac{1}{3}(\frac{3^{x^4}}{\ln(81)})\end{aligned}$$

Control:

$$\frac{d}{dx}(\frac{3^{x^4}}{\ln(81)})$$

Let $u = x^4$

$\frac{du}{dx} = 4x^3$

$\frac{d}{du}3^u = 3^u \ln(3)$

$$\frac{4x^3 3^{x^4} \ln(3)}{\ln(81)} = \frac{1}{4} 4x^3 3^{x^4} = 3^{x^4} \cdot x^3$$

Putting back the leading $\frac{1}{3}$ before the integral and we get back to the original.

(e) $\int \frac{2}{\sqrt{1-(2x+5)^2}} dx$. (Hint: arcsin).
Solution:

$$\int \frac{2}{\sqrt{1-(2x+5)^2}}\ dx = 2 \int \frac{1}{\sqrt{1-(2x+5)^2}}\ dx$$

Let $u = 2x + 5$

$\frac{du}{dx} = 2 \rightarrow 2dx = du \rightarrow du = \frac{1}{2}du$

$$\begin{aligned} \int \frac{1}{\sqrt{1-u}^2}\ du &= \arcsin(u) + C \\ &= \arcsin(2x+5) + C \end{aligned}$$

(f) $\int \frac{4x-10}{\sqrt{1-(4x^2-20x+25)^2}} dx$ (Hint: again?)
Solution:

$$\int \frac{4x-10}{\sqrt{1-(4x^2-20x+25)^2}} dx = \int 4x - 10 \frac{1}{\sqrt{1-(4x^2-20x+25)^2}}$$

Let $u = 4x^2 - 20x + 25$

$\frac{du}{dx} = 8x - 20 \rightarrow 8x - 20dx = du \rightarrow du = \frac{1}{8x-20dx}$

$$
\begin{aligned}
\int \frac{4x-10}{8x-20} \frac{1}{\sqrt{(1-u)^2}}\, du &= \int \frac{1}{2} \frac{1}{\sqrt{(1-u)^2}}\, du \\
&= \frac{1}{2} \int \frac{1}{(1-u)^2}\, du \\
&= \frac{1}{2} \arcsin(u) + C \\
&= \frac{1}{2} \arcsin(4x^2 - 20x + 25) + C
\end{aligned}
$$

3. (**10 points**) Given the function $f(x) = -x^2 + 8x - 7$. What is the area under the curve of f between the zeros of the function?
Solution:

1. Determine the roots of the function

Using the ABC-formula:

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

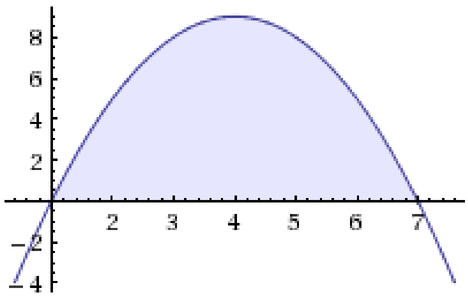
$x = \frac{-8 \pm \sqrt{8^2 - 4 \cdot (-1) \cdot (-7)}}{-2} = \frac{-8 \pm \sqrt{64-28}}{-2} = \frac{-8 \pm \sqrt{36}}{-2} = \frac{-8 \pm 6}{-2}$

$x_1 = 1$
$x_2 = 7$

2. Take the roots for the definite integral

$$
\begin{aligned}
\int_1^7 (-x^2 + 8x - 7)\, dx &= -\frac{1}{3}x^3 + 4x^2 - 7x \Big]_1^7 \\
&= (-\frac{1}{3}7^3 + 4 \cdot 7^2 - 7 \cdot 7) - (-\frac{1}{3}1^3 + 4 \cdot 1^2 - 7 \cdot 1) \\
&= (-\frac{1}{3} \cdot 343 + 4 \cdot 49 - 49) - (-\frac{1}{3} + 4 - 7) \\
&= -\frac{343}{3} + 147 + \frac{1}{3} + 3 \\
&= -114 + 147 + 3 \\
&= 36
\end{aligned}
$$

The area under the curve f between the zeros of the function (namely 1 and 7) is 36.



4. (**24 points**) Let $f(x) = \ln x$. Solve the problems below in order.

(a) Find $b \in \mathbb{R}$ such that $f(b) = 1$
Solution:

I already know by heart that $ln(e) = 1$.
So, $b = e$

(b) Determine the equation of the tangent line at the point $(b, 1)$.
Solution:

1. Find the derivative of this function to find the equation for the slope of the curve.

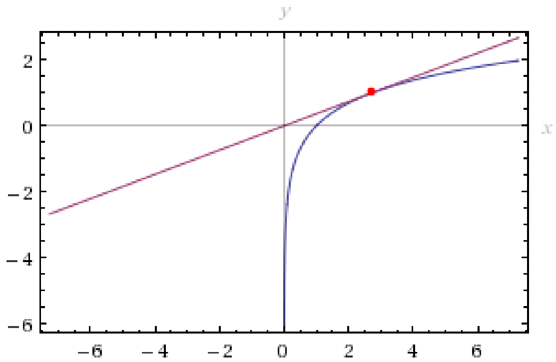
$f'(x) = \frac{1}{x}$

2. Plug the x-value of this point into the derived function to find the slope of the curve at that point.

$y = \frac{1}{e}$

3. Equation of the tangent line at the point (e,1)

$y - 1 = \frac{1}{e}(x - e)$

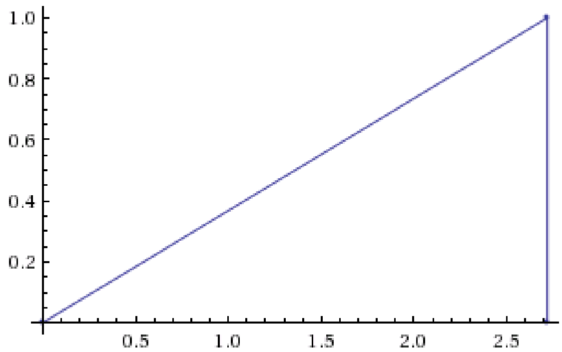


(c) Let this line intersect the x-axis at a. What is the area of the triangle with the following vertices:
$(a, 0), (b, 1), (b, 0)$
Solution:

Determine line intersect with x-axis

$$\begin{aligned} y - 1 &= \frac{1}{e}(x - e) \\ y &= \frac{1}{e}(x - e) + 1 \\ 0 &= \frac{1}{e}(x - e) + 1 \\ x &= 0 \rightarrow a = 0 \end{aligned}$$

We get a triangle with the following vertices: (0,0), (e,1), (e,0)



Because this gives us a right-angled triangle, we can use the following formula:

$$A = \frac{e \cdot 1}{2} = \frac{e}{2}$$

The area of the triangle is $\frac{e}{2}$

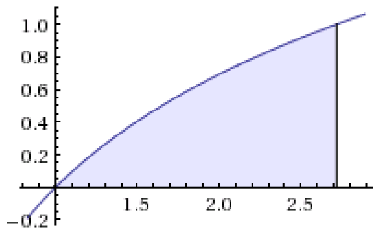
(d) Someone tells you that $(x \ln x - x)' = \ln x$. Verify whether it is true.
Solution:

$$\begin{aligned}
\frac{d}{dx}(x \ln x - x) &= -1 + \frac{d}{dx}(x \ln x) \\
&= -1 + \ln x \cdot 1 + x \frac{d}{dx}(\ln x) \\
&= -1 + \ln x + \frac{1}{x} x \\
&= -1 + \ln x + \frac{x}{x} \\
&= \ln x
\end{aligned}$$

(e) Determine $\int_1^b \ln x \, dx$.
Solution:

We already know from (d) that $(x \ln x - x)' = \ln x$, so we just have to plug in the values.

$$\begin{aligned}
\int_1^e \ln x \, dx &= x \ln x - x \Big]_1^e \\
&= (e \ln(e) - e) - (1 \cdot \ln(1) - 1) \\
&= (e \cdot 1 - e) - (1 \cdot 0 - 1) \\
&= 1
\end{aligned}$$



The area is 1.