# Calculus and Probability

Assignment 4

Note:

- You can hand in your solutions as a single PDF via the assignment module in Blackboard. Note that the document should be in English and typeset with LaTeX, Word or a similar program. It should not be a scan or picture of your handwritten notes.
- Make sure that your name, student number and group number are on top of the first page!
- **Note that your submission should be an individual submission because it can influence your final grade for this course. If we detect that your work is not completely your own work, we will ask the exam committee to investigate whether it is plagiarism or not!**

## Exercises to be presented during the exercise hours

**Exercise 1**

Investigate the function $f(x) = \frac{e^x}{x+1}$ as follows. (Do not start with drawing a graph by means of a device or some web resource. Of course you may check your result when you're done.)

**a)** Determine the domain of the function $f$.

$\mathbb{R} \setminus \{-1\}$;

**b)** What are the roots of $f$?

There are no zeros

**c)** Determine the limits at the edges of the domain. (Hint: there are 4 cases, use l'Hôpital!)

Three out of the four are straightforward to see.

(i) $\lim_{x\to-\infty} \frac{e^x}{x+1} = 0$

(ii) $\lim_{x\to-1^+} \frac{e^x}{x+1} = \infty$

(iii) $\lim_{x\to-1^-} \frac{e^x}{x+1} = -\infty$

For the first one we use l'Hôpital (condition: $\frac{\infty}{\infty}$). We get $\lim_{x\to\infty} \frac{e^x}{x+1} = \lim_{x\to\infty} \frac{e^x}{1} = \infty$.

**d)** Find $f'$ and $f''$.

Use the quotient rule in both cases. We get $f'(x) = \frac{xe^x}{(x+1)^2}$ and $f''(x) = \frac{(x^2+1)e^x}{(x+1)^3}$.

**e)** Find the zeros of $f'$ and $f''$.

From the above we can easily read that $f'(x) = 0 \iff x = 0$ and $f''(x) = 0 \iff (x^2+1) = 0$. As $x^2+1 > 0$ there are no zeros for $f''$.

**f)** What are the critical points (determine their $x$ and $y$ coordinates)?

The only critical point has $x = 0$, and for this we have $f(0) = \frac{e^0}{0+1} = 1$. So $(0,1)$ is the only critical point.

**g)** Find the local minima and maxima.

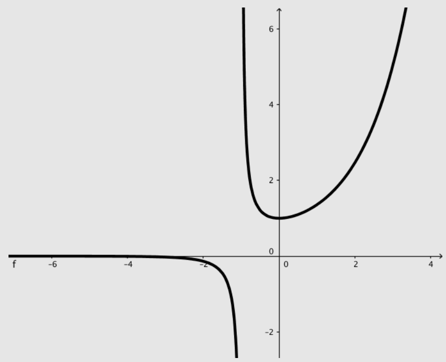
We have $f''(0) = 1 > 0$, so $(0,1)$ is a minimum.

**h)** Which parts of the function are convex and concave? Does function $f$ have points of inflection? (Hint: Use the sign of the second derivative for answering both questions.)

The numerator of $f''$ is always positive, so we find that $f''(x) > 0 \iff x > -1$ and $f''(x) < 0 \iff x < -1$. So convex for $x > -1$ and concave for $x < -1$. Also no inflection points.

**i)** Draw the graph of function $f$. (If you collect all intervals and special points in a table, it helps a lot in drawing the graph.)

Sketch:



**Exercise 2**
If $f(x,y) = \frac{xy}{x+y}$, show that

$$x^2 \cdot \tfrac{\partial}{\partial x}\big(\tfrac{\partial}{\partial x} f(x,y)\big) \ + \ 2xy \cdot \tfrac{\partial}{\partial x}\big(\tfrac{\partial}{\partial y} f(x,y)\big) \ + \ y^2 \cdot \tfrac{\partial}{\partial y}\big(\tfrac{\partial}{\partial y} f(x,y)\big) \ = \ 0.$$

(Hint: First compute all the second partial derivatives of $f$, then substitute the results in the expression on the left-hand side.)

Compute the partial derivatives.

1. $\frac{\partial f}{\partial x} = \frac{y(x+y)-xy}{(x+y)^2} = \frac{y^2}{(x+y)^2}$;
2. $\frac{\partial^2 f}{\partial^2 x} = \frac{-2(x+y)y^2}{(x+y)^4} = \frac{-2y^2}{(x+y)^3}$;
3. $\frac{\partial f}{\partial y} = \frac{x(x+y)-xy}{(x+y)^2} = \frac{x^2}{(x+y)^2}$;
4. $\frac{\partial^2 f}{\partial^2 y} = \frac{-2(x+y)x^2}{(x+y)^4} = \frac{-2x^2}{(x+y)^3}$;
5. $\frac{\partial^2 f}{\partial y \partial x} = \frac{2y(x+y)^2-2(x+y)y^2}{(x+y)^4} = \frac{2y(x+y)-2y^2}{(x+y)^3} = \frac{2xy}{(x+y)^3}$.

So the identity becomes

$$x^2 \frac{-2y^2}{(x+y)^3} + 2xy \frac{2xy}{(x+y)^3} + y^2 \frac{-2x^2}{(x+y)^3}$$

which is clearly zero.

**Exercise 3**
Evaluate the following definite integrals. (Hint: use slide 38 of the lectures about derivatives, and slide 13 of the lectures about primitives)

**a)** $\int_{-1}^{1}(x^3 + x - 1)\ dx$;

$$\begin{aligned} \int_{-1}^{1}(x^3 + x - 1)\ dx &= \int_{-1}^{1} x^3\ dx + \int_{-1}^{1} x\ dx + \int_{-1}^{1} -1\ dx \\ &= \frac{1}{4}x^4|_{-1}^{1} + \frac{1}{2}x^2|_{-1}^{1} + (-x)|_{-1}^{1} \\ &= \frac{1}{4} - \frac{1}{4} + \frac{1}{2} - \frac{1}{2} - 1 - 1 \\ &= \underline{\underline{-2}}. \end{aligned}$$

**b)** $\int_{0}^{\pi}(\sin(x) + \cos(x))\ dx$;

$$\begin{aligned} \int_{0}^{\pi}(\sin(x) + \cos(x))\ dx &= \int_{0}^{\pi} \sin(x)\ dx + \int_{0}^{\pi} \cos(x)\ dx \\ &= -\cos(x)|_{0}^{\pi} + \sin(x)|_{0}^{\pi} \\ &= 1 + 1 + 0 + 0 \\ &= \underline{\underline{2}}. \end{aligned}$$

**Exercise 4**
Evaluate the following improper integrals.

**a)** $\int_{1}^{\infty} \frac{1}{x^n}\ dx$, $n$ an integer such that $n \geq 2$; (Hint: this generalizes an example solved in the lecture on slide 8)

$$\begin{aligned} \int_{1}^{\infty} \frac{1}{x^n}\ dx &= \lim_{b\to\infty}\left(\frac{-1}{n-1}x^{-n+1}|_{1}^{b}\right) \\ &= \lim_{b\to\infty}\left(\frac{-1}{n-1}b^{-n+1} - \frac{-1}{n-1}\right) \\ &= \underline{\underline{\frac{1}{n-1}}}. \end{aligned}$$

**b)** $\int_2^\infty \frac{-1}{x \ln^2(x)}\, dx$. (Hint: use a fraction of well known functions to find the primitive)

$$\begin{aligned} \int_2^\infty \frac{-1}{x \ln^2(x)}\, dx &= \lim_{b \to \infty} \left( \frac{1}{\ln(x)} |_2^b \right) \\ &= \lim_{b \to \infty} \left( \frac{1}{\ln(b)} - \frac{1}{\ln(2)} \right) \\ &= \underline{\underline{\frac{-1}{\ln(2)}}}. \end{aligned}$$

**Exercise 5**

Compute the following limit

$$\lim_{x \to 1^+} x^{\frac{1}{1-x}}$$

Let $y = \ln\left(x^{\frac{1}{1-x}}\right) = \frac{1}{1-x} \ln(x) = \frac{\ln(x)}{1-x}$. Then

$$\lim_{x \to 1^+} \frac{\ln(x)}{1-x} \left(= \frac{0}{0}\right) \stackrel{!}{=} \lim_{x \to 1^+} \frac{1/x}{-1} = -1$$

Thus

$$\lim_{x \to 1^+} x^{\frac{1}{1-x}} = \lim_{x \to 1^+} = e^y = \underline{\underline{e^{-1}}}$$

where at the ! sign we applied L'Hôpital's rule.

## Exercises to be handed in

> You are expected to explain your answers, even if this is not explicitly stated in the exercises themselves.

**Exercise 6**
Consider the function $f(x) = e^x \sin(x)$.

**a)** What are the roots of $f$? Where does the graph of $f$ intersect the $y$ axis? 1 pt

As $e^x > 0$ for all $x$ we must solve $\sin(x) = 0$, which gives $x = n\pi$ for $n \in \mathbb{Z}$. Also $f(0) = e^0 \sin(0) = 0$.

**b)** Find all the zeros of $f'$ and $f''$. 1 pt

Use the chain rule to find $f'(x) = e^x(\cos(x) + \sin(x))$. Again using the chain rule twice results in $f''(x) = 2e^x \cos(x)$. Because $e^x > 0$ we have $f'(x) = 0 \iff \cos(x) + \sin(x) = 0$. Dividing by $\cos(x)$ this is $\tan(x) = -1$, which holds for $x = \frac{-\pi}{4} + n\pi$ for $n \in \mathbb{Z}$. So $\underline{\underline{f'(x) = 0 \iff x = \frac{-\pi}{4} + n\pi}}$. Clearly $\cos(x) = 0 \iff x = \frac{\pi}{2} + m\pi$ so $\underline{\underline{f''(x) = 0 \iff x = \frac{\pi}{2} + m\pi}}$.

**Exercise 7**
Given function $f$, find the partial derivatives. If it is necessary, simplify the result.

**a)** $f(x, y) = \cos(4y - xy)$; $\frac{\partial}{\partial x} f(x, y) =?$ and $\frac{\partial}{\partial y} f(x, y) =?$ 1 pt

First all partial derivatives for $f(x, y) = \cos(4y - xy)$.

(i) $\frac{\partial f(x,y)}{\partial x} = -\sin(4y - xy) * (-y) = y \sin(4y - xy)$;

(ii) $\frac{\partial f(x,y)}{\partial y} = -\sin(4y - xy)(4 - x)$;

(iii) $\frac{\partial^2 f(x,y)}{\partial y \partial x} = \sin(4y - xy) + y \cos(4y - xy)(4 - x)$;

(iv) $\frac{\partial^2 f(x,y)}{\partial x \partial y} = -\cos(4y - xy) * (-y) * (4 - x) + \sin(4y - xy) = \sin(4y - xy) + y \cos(4y - xy)(4 - x)$.

**b)** $f(x, y) = e^{\frac{x}{y}}$; $\frac{\partial}{\partial x} f(x, y) =?$ and $\frac{\partial}{\partial y} f(x, y) =?$ 1 pt

Second for $f(x,y) = e^{\frac{x}{y}}$.

(i) $\frac{\partial f(x,y)}{\partial x} = e^{\frac{x}{y}} \frac{1}{y}$;

(ii) $\frac{\partial f(x,y)}{\partial y} = e^{\frac{x}{y}} \frac{-x}{y^2}$

(iii) $\frac{\partial^2 f(x,y)}{\partial y \partial x} = e^{\frac{x}{y}} \frac{-x}{y^2} \frac{1}{y} - e^{\frac{x}{y}} \frac{1}{y^2} = -\frac{1}{y^2} e^{\frac{x}{y}} (\frac{x}{y} + 1)$;

(iv) $\frac{\partial^2 f(x,y)}{\partial x \partial y} = e^{\frac{x}{y}} \frac{1}{y} \frac{-x}{y^2} + e^{\frac{x}{y}} \frac{-1}{y^2} = -\frac{1}{y^2} e^{\frac{x}{y}} (\frac{x}{y} + 1)$.

**Exercise 8**

Evaluate the following definite integrals. (Hint: use slide 38 of the lectures about derivatives, and slide 13 of the lectures about primitives)

**a)** $\int_1^2 (3\sqrt{x} + \frac{3}{x^2})\, dx$; 1 pt

$$\begin{aligned}
\int_1^2 (3\sqrt{x} + \frac{3}{x^2})\, dx &= 3\left(\int_1^2 \sqrt{x}\, dx + \int_1^2 \frac{1}{x^2}\, dx\right) \\
&= 3\left(\frac{2}{3} x^{\frac{3}{2}}|_1^2 + \frac{-1}{x}|_1^2\right) \\
&= 3\left(\frac{2}{3}(2\sqrt{2} - 1) + \frac{1}{2}\right) \\
&= \underline{\underline{4\sqrt{2} - \frac{1}{2}}}.
\end{aligned}$$

**b)** $\int_{-1}^1 \frac{-5}{\sqrt{1-x^2}}\, dx$. 1 pt

$$\begin{aligned}
\int_{-1}^1 \frac{-5}{\sqrt{1-x^2}}\, dx &= 5 \arccos(x)|_{-1}^1 \\
&= 5\,(\arccos(1) - \arccos(-1)) \\
&= 5\,(0 - \pi) \\
&= \underline{\underline{-5\pi}}.
\end{aligned}$$

**Exercise 9**

Evaluate the following improper integrals.

**a)** $\int_{-\infty}^{-\pi/2} \frac{x\cos(x) - \sin(x)}{x^2}\, dx$; (Hint: use the quotient rule for derivation to find the primitive) 1 pt

$$\begin{aligned}\int_{-\infty}^{-\pi/2} \frac{x\cos(x)-\sin(x)}{x^2}\,dx &= \lim_{b\to\infty}\left(\frac{\sin(x)}{x}\Big|_{-b}^{-\pi/2}\right)\\ &= \lim_{b\to\infty}\left(\frac{-1}{-\pi/2}-\frac{\sin(b)}{b}\right)\\ &= \underline{\underline{\frac{2}{\pi}}}.\end{aligned}$$

**b)** $\int_1^\infty \frac{1-\ln(x)}{x^2}\,dx$. 1 pt

$$\begin{aligned}\int_{1}^{\infty} \frac{1-\ln(x)}{x^2}\,dx &= \lim_{b\to\infty}\left(\frac{\ln(x)}{x}\Big|_{1}^{b}\right)\\ &= \lim_{b\to\infty}\left(\frac{\ln(b)}{b}-\frac{\ln(1)}{1}\right)\\ &= \underline{\underline{0}}.\end{aligned}$$

**Exercise 10**

**a)** Let $m, n \in \mathbb{R}$. Compute the following limit 1 pt

$$\lim_{x\to 0}\frac{\cos(mx)-\cos(nx)}{x^2}$$

$$\begin{aligned}&\lim_{x\to 0}\tfrac{\cos(mx)-\cos(nx)}{x^2} \qquad \left(=\tfrac{0}{0}\right)\\ &\quad\overset{!}{=} \lim_{x\to 0}\tfrac{-m\sin(mx)+n\sin(nx)}{2x} \qquad \left(=\tfrac{0}{0}\right)\\ &\quad\overset{!}{=} \lim_{x\to 0}\tfrac{-m^2\cos(mx)+n^2\cos(nx)}{2}\\ &\quad= \underline{\underline{\tfrac{1}{2}(n^2-m^2)}}\end{aligned}$$

where at the ! sign we applied L'Hôpital's rule.

**b)** Compute the following limit 1 pt

$$\lim_{x\to\infty} x^{\frac{\ln 2}{1+\ln x}}$$

Let $y = \ln\left(x^{\frac{\ln 2}{1+\ln x}}\right) = \frac{\ln 2}{1+\ln x}\ln(x) = \frac{\ln(2)\ln(x)}{1+\ln x}$. Then

$$\lim_{x\to\infty} \frac{\ln(2)\ln(x)}{1+\ln x} \left(= \frac{\infty}{\infty}\right) \overset{!}{=} \lim_{x\to\infty} \frac{\ln(2)(1/x)}{(1/x)} = \ln 2$$

Thus

$$\lim_{x\to\infty} x^{\frac{\ln 2}{1+\ln x}} = \lim_{x\to\infty} = e^y = e^{\ln 2} = \underline{\underline{2}}$$

where at the ! sign we applied L'Hôpital's rule.

Your final grade is the sum of your scores divided by 1.0.