Calculus en Kansrekening

Mock exam (optional Assignment 8), October 13, 2015

— S O L U T I O N —

Marks: You can score a total of 100 points.

1. **(25 points)**

   Sketch the graph of $y = f(x) = \frac{2x-1}{(x-1)^2}$. Investigate first all the points required *i.e.* domain, parity, limits, extremes, monotonicity and asymptotes, points of inflection and convexity/concavity.

   Solution:

   The domain is $\mathbb{R} \setminus \{1\}$. We see that $f(x) = 0 \iff x = \frac{1}{2}$. Moreover $f(0) = -1$.

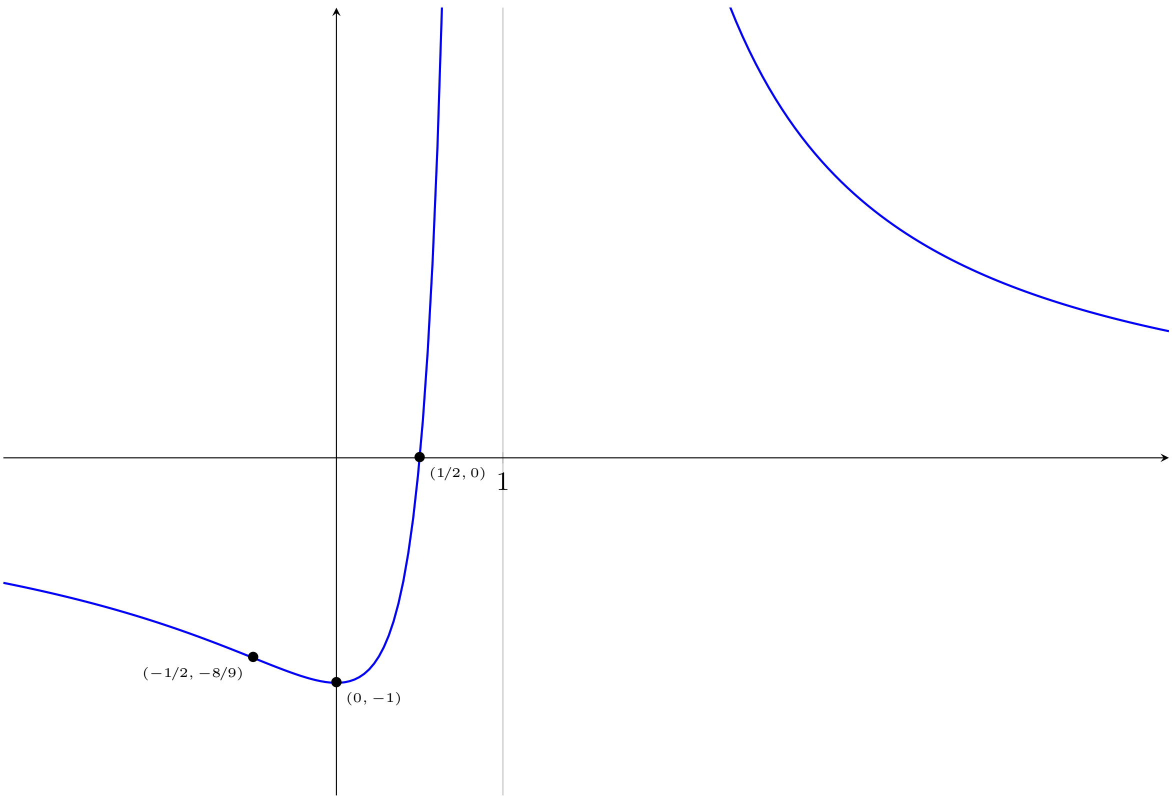
   The $\lim_{x\to 1} f(x) = \infty$, from either side. By writing out the denominator and dividing by x^2 we also see that

   $$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{\frac{2}{x} - \frac{1}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}} = 0.$$

   Use the quotient rule to compute $f'(x) = \frac{-2x}{(x-1)^3}$. From this we see that $f'(x) = 0 \iff x = 0$, so only a single extreme point. Also

   (a) if $x < 0$ then $f'(x) < 0$;
   (b) if $0 < x < 1$ then $f'(x) > 0$;
   (c) if $x > 1$ then $f'(x) < 0$.

   Now we compute $f''(x) = \frac{2(2x+1)}{(x-1)^4}$. We have $f''(0) = 2 > 0$, so this is a minimum. Also $f''(x) = 0 \iff x = -\frac{1}{2}$. As this is not an extreme point, this is an inflection point. The function f is convex if and only if $f''(x) > 0$ which happens if and only $x > -\frac{1}{2}$. Similarly the function is concave if and only if $x < -\frac{1}{2}$.

2. **(15 points)**

   The following function is given: $f(x) = e^{1-x^2}$. Find the tangent lines to the function f in the points where the graph of f intersects the line $y = 1$.

Solution:

First, find the points of intersection by solving $f(x) = 1$. Then, find the derivative of the function that will provide the slope of the tangent line in each point. Taking the log of both sides gives $1 - x^2 = 0$, so $x = \pm 1$. Use the chain rule to compute $f'(x) = e^{1-x^2} \cdot (-2x) = -2xe^{1-x^2}$. Finally, determine the equations of both tangent lines:

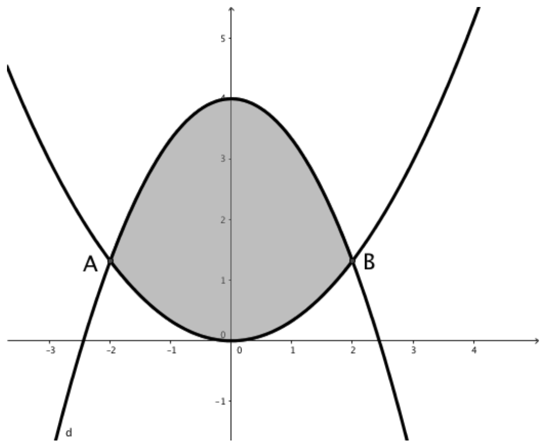
i. First consider the line at $x = -1$. We compute the slope $f'(-1) = 2$, so the line is of the form $y = 2x + b$. Since it must certainly intersect $(-1, 1)$ we know that $b = 1 + 2 = 3$. So the tangent line in $x = -1$ is $\underline{\underline{y = 2x + 3}}$.

ii. Now consider the line at $x = 1$. The slope is $f'(1) = -2$, so the line has the form $y = -2x + c$. Now it intersects the point $(1, 1)$ so we have $c = 1 + 2 = 3$. Hence the tangent line in $x = 1$ is $\underline{\underline{y = -2x + 3}}$.

3. **(20 points)**

Compute the area bounded by the two parabolas: $y = \frac{x^2}{3}$ and $y = 4 - \frac{2}{3}x^2$. Sketch the area first.

Solution:



First find the edges of our area of interest by solving $\frac{x^2}{3} = 4 - \frac{2}{3}x^2$. This gives $x = \pm 2$. Both functions are even, so it's sufficient to first compute the half of the area, then double it. Hence

$$\int_0^2 \left((4 - \frac{2x^2}{3}) - \frac{x^2}{3} \right) dx = \frac{16}{3}.$$

Thus, the area is $\underline{\underline{\frac{32}{3}}}$.

4. **(20 points)**

We are considering families with 4 children and we assume the probability of a male birth is 1/2. Answer the following questions:

- Find the probability that that there will be at least 1 boy in a family.
- Find the probability that that there will be at least 1 boy and at least 1 girl in a family.
- Out of 2000 families with 4 children each, how many would you expect to have exactly 2 boys?

Solution:

- The event that no boys are born is the event that only girls are born. This happens with probability $(\frac{1}{2})^4 = \frac{1}{16}$. So the event that at least 1 boy is born happens with probability $1 - \frac{1}{16} = \underline{\underline{\frac{15}{16}}}$.
- Let A be the event that at least 1 boy is born, and B be the event that at least 1 girl is born. Then $P(A \cap B) = 1 - P(\neg(A \cap B)) = 1 - P(\neg A \cup \neg B)$. But $\neg A$ is the event that no boys are born, i.e. only girls are born. Similarly $\neg B$ is the event that no girls are born, i.e. only boys are born. These are clearly mutually exclusive, so by Axiom 2, $P(\neg A \cup \neg B) = P(\neg A) + P(\neg B) = \frac{1}{16} + \frac{1}{16} = \frac{2}{16}$. Thus, $P(A \cap B) = 1 - \frac{2}{16} = \underline{\underline{\frac{7}{8}}}$.

- The probability of having exactly 2 boys is $\binom{4}{2} \cdot (\frac{1}{2})^4 = \frac{6}{16}$. So, we would expect $2000 \cdot \frac{6}{16} = \underline{\underline{750}}$ families with exactly 2 boys.

5. **(20 points)**

   A random variable X has density function:

   $$f(x) = \begin{cases} cx^2, & 1 \leq x \leq 2 \\ cx, & 2 < x < 3 \\ 0, & \text{otherwise} \end{cases}.$$

   Find: (a) the constant c, (b) $P(X > 2)$, (c) $P(2 < X < 3/2)$.

   Solution:

   (a) We solve $1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_1^2 cx^2\,dx + \int_2^3 cx\,dx = \left[\frac{1}{3}cx^3\right]_1^2 + \left[\frac{1}{2}cx^2\right]_2^3 = \frac{29}{6}c$. So $\underline{\underline{c = \frac{6}{29}}}$.

   (b) $P(X > 2) = \int_2^{\infty} f(x)\,dx = \frac{6}{29}\int_2^3 x\,dx = \frac{6}{29}\left[\frac{1}{2}x^2\right]_2^3 = \frac{6}{29}\frac{5}{2} = \underline{\underline{\frac{15}{29}}}$.

   (c) Note that $2 < X < \frac{3}{2}$ is *nonsense*, since $2 > \frac{3}{2}$. However, we can still use the definition of a probability for a continuous random variable, that is: $P(2 < X < \frac{3}{2}) = \int_2^{\frac{3}{2}} f(x)\,dx = -\int_{\frac{3}{2}}^2 f(x)\,dx = -\frac{6}{29}\int_{\frac{3}{2}}^2 x^2\,dx = -\frac{6}{29}\left[\frac{1}{3}x^3\right]_{\frac{3}{2}}^2 = -\frac{6}{29}\left(\frac{1}{3}\cdot 8 - \frac{1}{3}\frac{27}{8}\right) = -\frac{6}{29}\frac{37}{24} = -\frac{37}{116}$.

   It is important to realise that this is not a proper probability (see the definition of a *probability measure*), since $-\frac{37}{116} \notin [0,1]$!