Calculus en Kansrekenen (NWI-IBC017) – EXAMINATION

April 5, 2017, 12:30–15:30

— S O L U T I O N —

This exam consists of eight problems. You can score a maximum of 100 points. Each question indicates how many points it is worth. All questions can be made independently of each other. The exam is closed book. You are NOT allowed to use a programmable calculator, a computer or a mobile phone. The only device you are allowed to use is the calculator provided at the exam. **Explain your approach and answers** briefly. You may give the answers in Dutch or in English. Please write clearly! Do not forget to put **your name** and **your student number** on the top of **each page**.

1. **(5+6+5+6=22 points)** [Tested skills: (in)definite integral, partial integration, substitution]
   Compute the following (in)definite integrals.

   (a) $\int (x^2+2x)e^x dx$

   (b) $\int_{-1}^{1}\left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 1dx\right)dy$

   (c) $\int_1^\infty \frac{1}{x^{\frac{3}{2}}}dx$

   (d) $\int \frac{3}{2}e^{\sqrt{3x}}dx$

   Hint: (b) First work out the inner integral, then use the substitution $y = \sin(u)$ for the outer integral. Make use of trignometric equalities like $\sin(x)^2 + \cos(x)^2 = 1$ and $\cos(x)^2 = \frac{1}{2} + \frac{1}{2}\cos(2x)$. (c) Note that the integral is *improper*. (d) Use substitution and use $u = f(x) \Rightarrow du = f'(x)dx \Rightarrow dx = \frac{1}{f'(x)}du$.

   Solution:

   (a) $\int (x^2+2x)e^x dx = \int x^2e^x dx + \int 2xe^x dx \overset{p.i.}{=} x^2e^x - \int 2xe^x dx + \int 2xe^x dx = \underline{\underline{x^2e^x + C}}$ where we used partial integration in the 2nd step.

   (b)

$$\begin{aligned}
\int_{-1}^{1}\left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 1dx\right)dy &= \int_{-1}^{1}\sqrt{1-y^2} - -\sqrt{1-y^2}dy \\
&= \int_{-1}^{1} 2\sqrt{1-y^2}dy \\
&= \int_{-\pi/2}^{\pi/2} 2\sqrt{1-(\sin u)^2}\cos u du \\
&= \int_{-\pi/2}^{\pi/2} 2(\cos u)^2 du \\
&= \int_{-\pi/2}^{\pi/2} 1+\cos(2u)du \\
&= u + \frac{1}{2}\sin(2u)|_{-\pi/2}^{\pi/2} \\
&= (\pi/2+0) - (-\pi/2+0) \\
&= \underline{\underline{\pi}}
\end{aligned}$$

(c)

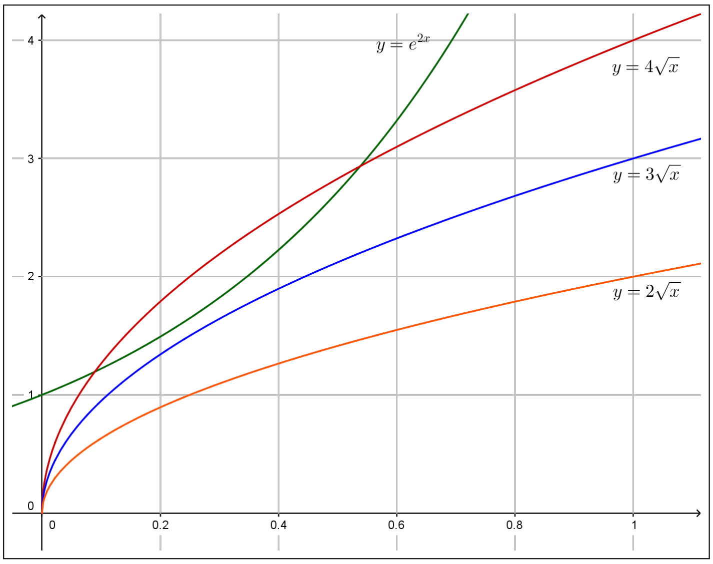
$$\begin{aligned}\int_1^\infty \frac{1}{x^{\frac{3}{2}}}dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x^{\frac{3}{2}}}dx\\ &= \lim_{t\to\infty} \frac{-2}{x^{\frac{1}{2}}}\Big|_1^t\\ &= \lim_{t\to\infty} \frac{-2}{t^{\frac{1}{2}}} - \frac{-2}{1^{\frac{1}{2}}}\\ &= 0 - \frac{-2}{1}\\ &= \underline{\underline{2}}\end{aligned}$$

(d)

$$\begin{aligned}\int \frac{3}{2}e^{\sqrt{3x}}dx &= \int ue^u du\\ &= ue^u - \int e^u du\\ &= (u-1)e^u + C\\ &= \underline{\underline{(\sqrt{3x}-1)e^{\sqrt{3x}} + C}}\end{aligned}$$

where we used the substitution $u = \sqrt{3x}$, $du = \sqrt{3}\frac{1}{2\sqrt{x}}dx \Rightarrow dx = \frac{2}{3}udu$ followed by partial integration.

2. **(6 points)** [Tested skills: derivative]



Let $f(x) = e^{2x}$ and $g(x) = k\sqrt{x}$. On the left graphs are shown for f and g for several values of k. Clearly there is a value of $k \in [3,4]$ such that the graphs of f and g intersect in exactly one point. Compute the value of k such that f and g intersect in exactly one point. (Hint: define a to the point of intersection. What do the two functions have in common in a?)

Solution: Let a be the point of intersection, i.e.,

$$e^{2a} = k\sqrt{a} \tag{1}$$

From the graphs we see that f and g intersect in exactly 1 point if they share the tangent line, i.e., taking the derivative of f and g in a gives

$$2e^{2a} = k\frac{1}{2\sqrt{a}} \tag{2}$$

Substituting Eq (1) into Eq (2) and solving for a gives $a = \frac{1}{4}$. Substituting this value back in Eq (1) and solving for k gives $\underline{\underline{k = 2\sqrt{e}}}$.

3. **(6 points)** [Tested skills: functions]
Let $f(x) = \frac{1}{(1-x)^3}$. Find all points on f that have a tangent $f'(x)$ that is parallel to the line $y - 3x = 0$.

Solution: The line is given by $y - 3x = 0 \Leftrightarrow y = 3x$, i.e., the slope is 3. A tangent line is parallel if it has the same slope, i.e., the derivative $f'(x) = \frac{3}{(1-x)^4} = 3$. Solving gives $x \in \{0, 2\}$ and the coordinates we are looking for are thus $\underline{\underline{(0,1)}}$ and $\underline{\underline{(2,-1)}}$. The tangent lines through these points are given by $y = 3x + 1$ and $y = 3x - 7$.

4. **(6+6+6=18 points)** [Tested skills: logarithmic differentiation, chain rule, partial derivatives]

   (a) $f(x) = 3 \cdot x^{(3x)}$, compute $\frac{d}{dx}f(x)$
   (b) $f(x) = \exp(\sin(\exp(\sin(x))))$, compute $\frac{d}{dx}f(x)$
   (c) $f(x, y, t) = e^{-t}(\sin(x) \cdot \cos(y))$. Demonstrate that the theorem of Schwarz holds for the cases $\frac{\partial^3}{\partial x \partial y \partial t}f(x, y, t)$ and $\frac{\partial^3}{\partial y \partial t \partial x}f(x, y, t)$

   Solution:

   (a) $\ln(3 \cdot x^{(3x)}) = \ln 3 + \ln(x^{(3x)}) = \ln 3 + 3x\ln(x)$
   $\frac{\partial}{\partial x}\ln(f(x)) = \frac{\partial}{\partial x}\ln 3 + 3x\ln(x) = 0 + 3\ln(x) + \frac{3x}{x} = 3(\ln(x) + 1)$
   $\frac{\partial}{\partial x}f(x) = f(x)\frac{\partial}{\partial x}\ln(f(x)) = 3 \cdot x^{(3x)} \cdot 3(\ln(x) + 1) = \underline{\underline{9 \cdot x^{(3x)} \cdot (\ln(x) + 1)}}$
   (b) $f'(x) = \underline{\underline{\exp(\sin(\exp(\sin(x)))) \cdot \cos(\exp(\sin(x))) \cdot \exp(\sin(x)) \cdot \cos(x)}}$
   (c)

$$\begin{aligned}
\frac{\partial^3}{\partial x \partial y \partial t}f(x, y, t) &= \frac{\partial^3}{\partial x \partial y \partial t}e^{-t}(\sin(x) \cdot \cos(y)) \\
&= \frac{\partial^2}{\partial y \partial t}e^{-t}(\cos(x) \cdot \cos(y)) \\
&= \frac{\partial}{\partial t} - e^{-t}(\cos(x) \cdot \sin(y)) \\
&= \underline{\underline{e^{-t}(\cos(x) \cdot \sin(y))}}
\end{aligned}$$

   and

$$\begin{aligned}
\frac{\partial^3}{\partial y \partial t \partial x}f(x, y, t) &= \frac{\partial^3}{\partial y \partial t \partial x}e^{-t}(\sin(x) \cdot \cos(y)) \\
&= \frac{\partial^2}{\partial t \partial x} - e^{-t}(\sin(x) \cdot \sin(y)) \\
&= \frac{\partial}{\partial x}e^{-t}(\sin(x) \cdot \sin(y)) \\
&= \underline{\underline{e^{-t}(\cos(x) \cdot \sin(y))}}
\end{aligned}$$

5. **(6+6=12 points)** [Tested skills: differentiation, L'Hôpital]
   Use L'Hôpital to compute the following limits. (Make sure to validate the requirements for L'Hôpital and to transform the limit if the conditions are not satisfied.)

   (a) $\lim_{x \to \pi} \frac{e^{\sin(x)} - 1}{x - \pi}$
   (b) $\lim_{t \to 0^+} t\ln(t)$

   Solution:

   (a) Since the limit is of the form $\frac{0}{0}$ we can use L'Hôpital:
   $\lim_{x \to \pi} \frac{e^{\sin(x)} - 1}{x - \pi} \overset{\text{LHR}}{=} \lim_{x \to \pi} \frac{\cos(x)e^{\sin(x)}}{1} = \cos(\pi)e^{\sin(\pi)} = -1$
   (b) As the limit is of the form $\frac{0}{-\infty}$ we cannot use L'Hôpital directly. First we transform the limit into the form $\frac{\pm\infty}{\pm\infty}$ and then apply L'Hôpital:

$$\lim_{t \to 0^+} t\ln(t) = \lim_{t \to 0} \frac{1/t}{1/t}t\ln(t) = \lim_{t \to 0^+} \frac{1}{1/t}\ln(t) = \lim_{t \to 0^+} \frac{\ln(t)}{1/t} \overset{\text{LHR}}{=} \lim_{t \to 0^+} \frac{1/t}{-1/t^2} = \lim_{t \to 0^+} \frac{1}{-1/t} = \underline{\underline{0}}$$

6. **(6+6+6=18 points)** [Tested skills: combinatorics]

(a) A library has 'a' copies of one book. 'b' copies each of two books, 'c' copies each of three books and single copy of 'd' books. What is 1) the total number of books, and 2) the total number of ways in which these books can be arranged in a shelf (i.e., in same row).

(b) If letters of the word 'BAYES' are written in all possible orders and arranged in a dictionary using the lexicographic ordering, then 1) how many words are in the dictionary, and 2) what is the rank of the word 'BAYES' in the dictionary?

(c) How many 3 letter words can be formed from the letters of the word 'PROBABILITY'?

Solution:

(a) Total number of books: $\underline{\underline{(a+2b+3c+d)}}$, which can be arranged in $\underline{\underline{\frac{(a+2b+3c+d)!}{a!(b!)^2(c!)^3}}}$ ways.

(b) 1) 5 different letters gives $5! = \underline{\underline{120}}$ possible words.
2) Alphabetical order is A, B, E, S, Y.

Total words starting with	A	$= 4! =$	24
Total words starting with	BAE	$= 2! =$	2
Total words starting with	BAS	$= 2! =$	2

Next word will be BAYES $= 24 + 2 + 2 + 1 = \underline{\underline{29}}$

(c) We have 1A, 2B, 2I, 1L, 1O, 1P, 1R, 1T, 1Y, thus

Total words with 3 different letters	$=$	$9 \cdot 8 \cdot 7$	$=$	504
Total words including BB	$=$	$8 \cdot 3$	$=$	24
Total words including II	$=$	$8 \cdot 3$	$=$	24

Total number of 3 letter words is $504 + 24 + 24 = \underline{\underline{552}}$

7. **(6+6=12 points)** [Tested skills: Probability, Bayes rule]
Recall that the Bayes' rule states that

$$P(H \mid E) = \frac{P(E \mid H) \cdot P(H)}{P(E)}.$$

Moreover, according the total probability lemma, for a partition $A_1, \ldots, A_n$ and an arbitrary event B,

$$P(B) = P(B \mid A_1) \cdot P(A_1) + \cdots + P(B \mid A_n) \cdot P(A_n).$$

(a) On any given day, the probability that stock A and stock B both go up is 0.4 and the probability that they both go down is 0.2. Otherwise one goes up and the other goes down. If you know that at least one went up, what is the probability that both went up?

(b) Suppose that you have a jar with 4 fair coins and 1 unfair coint (which has heads on both sides). You choose one of the 5 coins at random, flip it five times, and get all heads. What is the probability that you chose the unfair coin?

Solution:

(a) Given that at least one went up, they either both stocks went up (0.4) or only one stock went up $(1 - 0.4 - 0.2 = 0.4)$ Thus the probability of both going up is

$$\frac{0.4}{0.4 + 0.4} = 0.5$$

(b) Let U be the unfair coin, F a fair coin. Then $p(U) = \frac{1}{5}, p(F) = \frac{4}{5}$, thus

$$\begin{aligned}
p(U|\text{“5 heads”}) &= \frac{p(\text{“5 heads”|U)p(U)}}{p(\text{“5 heads”})} \\
&= \frac{p(\text{“5 heads”|U)p(U)}}{p(\text{“5 heads”|F)p(F)} + \text{p(“5 heads”|U)p(U)}} \\
&= \frac{1 \cdot \frac{1}{5}}{\left(\frac{1}{2}\right)^5 \cdot \left(\frac{4}{5}\right) + 1 \cdot \left(\frac{1}{5}\right)} \\
&= \frac{\left(\frac{1}{5}\right)}{\left(\frac{1}{40}\right) + \left(\frac{1}{5}\right)} \\
&= \frac{\left(\frac{8}{40}\right)}{\left(\frac{9}{40}\right)} \\
&= \underline{\underline{\frac{8}{9}}}
\end{aligned}$$

8. **(6 points)** [Tested skills: Area between curves]
Find the area of the region bounded by the curves $y = \sin(x)$, $y = \cos(x)$, $x = 0$, $x = \pi/2$. (Hint: use the fact that $\sin(\pi/4) = \cos(\pi/4) = \frac{1}{\sqrt{2}}$.)

Solution:

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} |\cos(x) - \sin(x)| dx &= \int_0^{\frac{\pi}{4}} \cos(x) - \sin(x) dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x) - \cos(x) dx \\
&= (\sin(x) + \cos(x))|_0^{\frac{\pi}{4}} + (-\cos(x) - \sin(x))_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right) + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) \\
&= \underline{\underline{2\sqrt{2} - 2}}
\end{aligned}$$