Calculus and Probability

Assignment 7

Note:

- You can hand in your solutions as a single PDF via the assignment module in Blackboard. Note that the document should be in English and typeset with LaTeX, Word or a similar program. It should not be a scan or picture of your handwritten notes.
- Make sure that your name, student number and group number are on top of the first page!
- **Note that your submission should be an individual submission because it can influence your final grade for this course. If we detect that your work is not completely your own work, we will ask the exam committee to investigate whether it is plagiarism or not!**

Exercises to be presented during the exercise hours

Exercise 1
An experiment consists of drawing 3 cards in succession from a well-shuffled ordinary deck of cards.[1] Let A_1 denote the event "Ace on first draw", A_2 the event "Ace on second draw" and A_3 the event "Ace on third draw". State in words the meaning of each of the following probabilities.

a) $P(A_1 \cap \neg A_2)$;

> The probability that the first draw is an ace, and the second draw is not an ace;

b) $P(A_1 \cup A_2)$;

[1] If you are not familiar with the standard 52-card deck, visit Wikipedia: `http://en.wikipedia.org/wiki/Standard_52-card_deck`.

The probability that the first draw or the second draw is an ace, or both;

c) $P(\neg A_1 \cap \neg A_2 \cap \neg A_3)$;

The probability that in the first, second and third draw, no ace is drawn;

d) $P[(A_1 \cap \neg A_2) \cup (\neg A_2 \cap A_3)]$.

Notice that this is $P[\neg A_2 \cap (A_1 \cup A_3)]$. So the probability that the second draw is not an ace, and the first, third or both are an ace.

Exercise 2
Let P be a probability measure on space S. Prove that $P(\emptyset) = 0$.

We have that for any event A of the sample space $P(A) = P(A \cup \emptyset) = P(A) + P(\emptyset)$. As the left and right side must be equal, we have $P(\emptyset) = 0$.

Exercise 3
There are twelve provinces in the Netherlands. Suppose that the birth of Dutch people is uniformly distributed over all twelve provinces, *i.e.* for every province $\frac{1}{12}$ of the population is born there. What is the probability that at least two of r randomly selected Dutch-born people were born in the same province, where $r = 1$, $r = 2$, $r = 3$, $r = 4$, $r = 5$ or $r = 6$?

This problem is a toy example for the Birthday paradox. Thus, we can apply the same reasoning here:

$$p(r) = 1 - \frac{12!}{(12 - r)! \cdot 12^r},$$

and the particular probabilities are the following:

r	$p(r)$
1	0
2	0.0833
3	0.2361
4	0.4271
5	0.6181
6	0.7772

Exercise 4
A shooter has exactly 6 bullets and shoots on a target. A random variable X is the number of bullets used *until he/she hits it for the first time, or has no more bullets*. The probability of a bullet hitting the target is 0.4 for every attempt.

a) Find the probability distribution of X; that is, give the probabilities for all possible values.

The shooter uses 1, 2, 3, 4, 5 or, at most, 6 bullets. Clearly, $P(X = 0) = 0$ (that is, with 0 bullets the shooter can't hit the target). Furthermore, $P(X = k)$ for $0 < k < 6$ means there were $k - 1$ misses, and the kth shot is a hit. Therefore

(i) $P(X = 1) = 0.4$;

(ii) $P(X = 2) = 0.6 \cdot 0.4 = 0.24$;

(iii) $P(X = 3) = (0.6)^2 \cdot 0.4 = 0.144$;

(iv) $P(X = 4) = (0.6)^3 \cdot 0.4 = 0.0864$;

(v) $P(X = 5) = (0.6)^4 \cdot 0.4 = 0.05184$;

(vi) $P(X = 6) = 1 - \sum_{i=1}^{5} P(X = i) = 0.07776$.

Clearly, $P(X = k) = 0$ for $k > 6$ as there is a maximum of 6 bullets.

b) What is the expected value for X?

$E(X) = \sum_{i=1}^{6} i \cdot P(X = i) = \underline{\underline{2.38336}}$.

c) What is the variance?

$Var(X) = \sum_{i=1}^{6} P(X = i) \cdot (i - E[X])^2 \approx \underline{\underline{2.45336}}$.

d) What is the standard deviation?

$\sigma(X) = \sqrt{Var(X)} \approx \sqrt{2.45336} \approx \underline{\underline{1.56632}}$.

Exercise 5
TV sets with various defects are brought to the service for reparation. The time of reparation is a continuous random variable T. The cumulative distribution function of T is given as:

$$F(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 - e^{-kt} & \text{if } t \geq 0, \end{cases}$$

where $k > 0$.

a) Find the probability density function f of the random variable.

$$f(t) = \frac{d}{dt}F(t) = \begin{cases} 0 & \text{if } t < 0, \\ ke^{-kt} & \text{if } t \geq 0. \end{cases}$$

b) Find the expectation and variance.

(i) $E[T] = \int_{-\infty}^{\infty} tf(t)\ dt = \int_0^{\infty} tke^{-kt}\ dt = \lim\limits_{a\to\infty} \int_0^a tke^{-kt}\ dt$. Use integration by parts once to find $\int_0^a tke^{-kt}\ dt = \left[-te^{-kt}\right]_0^a + \int_0^a e^{-kt}\ dt = ae^{-ka} + \left[-\frac{1}{k}e^{-kt}\right]_0^a = ae^{-ka} - \frac{1}{k}e^{-ka} + \frac{1}{k}$. From this we see that

$$E[T] = \lim_{a\to\infty} \int_0^a tke^{-kt}\ dt = \lim_{a\to\infty} \left(ae^{-ka} - \frac{1}{k}e^{-ka} + \frac{1}{k}\right) = \underline{\underline{\frac{1}{k}}}.$$

(ii) To compute the variance ($Var[T] = E[T^2] - E[T]^2$), we first need to compute $E[T^2]$.
$E[T^2] = \int_{-\infty}^{\infty} t^2 f(t)\ dt = \lim\limits_{a\to\infty} \int_0^a t^2ke^{-kt}\ dt$. Again use integration by parts to get $\int_0^a t^2ke^{-kt}\ dt = \left[-t^2e^{-kt}\right]_0^a + 2\int_0^a te^{-kt}\ dt = -a^2e^{-ka} + \frac{2}{k}\int_0^a tke^{-kt}\ dt = -a^2e^{-ka} + \frac{2}{k}\left(ae^{-ka} - \frac{1}{k}e^{-ka} + \frac{1}{k}\right)$. Note that in the last equality we used the integral from i.. We see that

$$E[T^2] = \lim_{a\to\infty} \left(-a^2e^{-ka} + \frac{2}{k}\left(ae^{-ka} - \frac{1}{k}e^{-ka} + \frac{1}{k}\right)\right) = \frac{2}{k^2}.$$

Finally, we compute

$$Var[T] = E[T^2] - E[T]^2 = \frac{2}{k^2} - \frac{1}{k^2} = \underline{\underline{\frac{1}{k^2}}}.$$

Exercises to be handed in

You are expected to explain your answers, even if this is not explicitly stated in the exercises themselves.

Exercise 6
We toss a coin twice.

a) Give a corresponding sample space S. 1 pt

For example $S = \{HH, HT, TH, TT\}$. But there are more options for valid sample spaces!

b) Give the set of outcomes corresponding to each of the following events: 1 pt

(i) A: “we throw heads exactly once“;
(ii) B: “we throw heads at least once“;
(iii) C: “tails did not appear before a head appeared“.

(i) $A = \{HT, TH\}$;
(ii) $B = \{HH, HT, TH\}$;
(iii) $C = \{HH, HT, TT\}$;

c) Give a probability measure P_1 for the sample space S. 1 pt

Most obvious is $P_1(HH) = P_1(HT) = P_1(TH) = P_1(TT) = \frac{1}{4}$. Note that we have to define the probability measure on all $16 = 2^4$ subsets of S, but all others can be constructed from these 4. Make sure the students do this, or mention it!

Exercise 7
Let P be a probability measure on space S. Prove that for pairwise mutually exclusive events $A_1, A_2, \ldots A_n$ (*i.e.* $A_i \cap A_j = \emptyset$ for all $i \neq j$ and $n \geq 2$) one has $P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$. (Hint: use induction.) 1 pt

For $n = 2$ this follows by definition. Suppose that $A_1, A_2, \ldots A_{k+1}$ are mutually disjoint. Then $A_1 \cup A_2 \cup \ldots \cup A_k$ and A_{k+1} are disjoint, so $P((A_1 \cup A_2 \cup \ldots \cup A_k) \cup A_{k+1}) = P(A_1 \cup A_2 \cup \ldots \cup A_k) + P(A_{k+1})$. But by induction $P(A_1 \cup A_2 \cup \ldots \cup A_k) = P(A_1) + P(A_2) + \ldots P(A_k)$, so the result follows.

Exercise 8
Consider a class where students have to hand in exercises every week. They have to hand in eight assignments in total and have to pass at least five to be able to attend the exam. Student A does not study very hard, so for each assignment he/she has a probability of 0.5 to pass. Student B studies very hard, so for each assignment he/she has a probability of 0.8 to pass. The random variable X_A is the number of passes of student A and the random variable X_B is the number of passes of student B.

a) Find $P(X_A = 5)$; 1 pt

Both X_A and X_B have binomial distributions with parameters $(8, 0.5)$ and $(8, 0.8)$, respectively. $P(X_A = 5) = \binom{8}{5} \cdot (0.5)^8 = \underline{\underline{0.21875}}$;

b) Find $P(X_A \geq 5)$; 1 pt

(i) $P(X_A = 6) = \binom{8}{6} \cdot (0.5)^8 = 0.109375$;

(ii) $P(X_A = 7) = \binom{8}{7} \cdot (0.5)^8 = 0.03125$;

(iii) $P(X_A = 8) = \binom{8}{8} \cdot `(0.5)^8 = 0.00390625$;

So $P(X_A \geq 5) = \sum_{i=5}^{8} P(X_A = i) = \underline{\underline{0.36328125}}$.

c) Find $P(X_B \geq 5)$. 1 pt

(i) $P(X_B = 5) = \binom{8}{5} \cdot (0.8)^5 \cdot (0.2)^3 \approx 0.1468$;

(ii) $P(X_B = 6) = \binom{8}{6} \cdot (0.8)^6 \cdot (0.2)^2 \approx 0.2936$;

(iii) $P(X_B = 7) = \binom{8}{7} \cdot (0.8)^7 \cdot (0.2)^1 \approx 0.3355$;

(iv) $P(X_B = 8) = \binom{8}{8} \cdot (0.8)^8 \cdot (0.2)^0 \approx 0.1678$;

So $P(X_B \geq 5) = \sum_{i=5}^{8} P(X_B = i) \approx \underline{\underline{0.9437}}$.

Exercise 9
A continuous random variable X has the following probability density function:

$$f(x) = \begin{cases} a \cdot (1 - 4x^2) & \text{if } -\frac{1}{2} < x < \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

a) Find the constant a. 1 pt

As f is a pdf, we solve $1 = \int_{-\infty}^{\infty} f(x)\ dx = a \cdot \int_{-1/2}^{1/2}(1-4x^2)\ dx = a \cdot [x - \frac{4}{3}x^3]_{-1/2}^{1/2} = a \cdot \frac{2}{3}$. So $\underline{\underline{a = \frac{3}{2}}}$.

b) Find the cumulative distribution function $F(x)$. $\boxed{2 \text{ pt}}$

According to (a), on the interval $(-\frac{1}{2}, \frac{1}{2})$, $f(x) = \frac{3}{2} \cdot (1-4x^2)$. The indefinite integral of f is $F(x) = \frac{3}{2}(x - \frac{4}{3}x^3) + C$ (on this interval). Thus, the cumulative distribution function F is given by

$$F(x) = \begin{cases} 0 & \text{if } x \leq -\frac{1}{2}, \\ \frac{3}{2}(x - \frac{4}{3}x^3) + C & \text{if } -\frac{1}{2} < x < \frac{1}{2}, \\ 1 & \text{if } x \geq \frac{1}{2}. \end{cases}$$

From $F(-1/2) = 0$ we have $C = \frac{1}{2}$, so $\underline{\underline{F(x) = \frac{3}{2}(x - \frac{4}{3}x^3) + \frac{1}{2}}}$. Note that the restriction $F(1/2) = 1$ gives the same result.

c) Compute the probability $P(X = \frac{1}{4})$. $\boxed{1 \text{ pt}}$

This is $\underline{\underline{\text{zero}}}$. (As is for any point in a continuous distribution!)

d) Compute the probability $P(0 < X < \frac{1}{4})$. $\boxed{1 \text{ pt}}$

We compute $\int_0^{1/4} f(x)\ dx = F(1/4) - F(0) = \frac{3}{2}(\frac{1}{4} - \frac{4}{3}(\frac{1}{4})^3) = \underline{\underline{\frac{11}{32}}}$.

Exercise 10

A normal random variable X has probability density function

$$f(x) = \frac{1}{3} \exp\left(-\frac{\pi}{9}(x^2 - 4x + 4)\right).$$

a) Find the mean μ and the standard deviation σ. $\boxed{1 \text{ pt}}$

Note that the exercise says “standard deviation” σ, which is the standard notation for the normal distribution. In this case, the solution is as follows: from $\frac{1}{\sigma\sqrt{2\pi}} = \frac{1}{3}$ we get $\sigma = \frac{3}{\sqrt{2\pi}}$. Also

$$-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 = \frac{-1}{2\sigma^2}(x-\mu)^2 = \frac{-\pi}{9}\left(x^2 - 2\mu x + \mu^2\right),$$

so we see that $\mu = 2$.

(When σ is the variance instead of the standard deviation, we have $\sigma = \left(\frac{3}{\sqrt{2\pi}}\right)^2 = \frac{9}{2\pi}$ instead.)

b) Let Y be the random variable defined by $Y = \frac{X-\mu}{\sigma}$. For a real number a show that

$$P(Y \leq -a) = 1 - P(Y \leq a).$$

(Hint: use that $\int_\alpha^\beta \phi(x)\, dx = -\int_\beta^\alpha \phi(x)\, dx$.) 1 pt

Write $\phi(x)$ for the probability density function of Y. All we need is that the mean of Y is zero, from which by symmetry follows that $\phi(-x) = \phi(x)$. Then

$$\begin{aligned}
P(Y \leq -a) &= \int_{-\infty}^{-a} \phi(x)\, dx \\
&= \lim_{b\to\infty} \int_{-b}^{-a} \phi(x)\, dx \\
&= \lim_{b\to\infty} \int_{b}^{a} -\phi(y)\, dy \quad \text{[substitute } y = -x \text{, use } \phi(y) = \phi(x)\text{]} \\
&= \lim_{b\to\infty} \int_{a}^{b} \phi(y)\, dy \\
&= P(Y \geq a) \\
&= 1 - P(Y \leq a).
\end{aligned}$$

Your final grade is the sum of your scores divided by 1.4.