Integrals and applications

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Outline

Currently we are here...

The Definite Integral

The Indefinite Integral

Techniques of Integration

Introduction to Integration

- We looked at differentiation: going from f to its derivative f'
  - associated notions: tangent line, monotonicity, extrema, ...
- Now we look at integration: going from f to F with $F' = f$.
  - What does such a primitive F of f tell us about f?
- Well, if $f(x) = F'(x) = \lim_{h \to 0} \frac{F(x+h)-F(x)}{h}$, then for small $h > 0$,

$$f(x) \cdot h \approx F(x+h) - F(x)$$

- So, the primitive F gives some information about the area under the graph of f
  - similarly, integration can also be used to calculate volumes, in more dimensions

The area problem and the (definite) integral

- Let f be a continuous function defined on the interval $[a, b]$. In order to estimate the area under f from a to b we divide $[a, b]$ into n subintervals: $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, $\ldots$, $[x_{n-1}, x_n]$, where $a = x_0, b = x_n$, each of length $\Delta x = \frac{b-a}{n}$.
(Hence we can write $x_i = a + i\Delta x, i = 0, \cdots, n$)
- The area S_i of the strip between x_{i-1} and x_i can be approximated as the area of the rectangle of width Δx and height $f({x_i}^*)$, for some ${x_i}^* \in [x_i, x_{i+1}]$. Hence $S_i \approx f(x_i^*) \cdot \Delta x$.
- So, the total area A under f is close to the sum of the S_i:

$$A \approx f({x_1}^*) \cdot \Delta x + f({x_2}^*) \cdot \Delta x + \ldots + f({x_n}^*) \cdot \Delta x = \sum_{i=1}^{n} f({x_i}^*) \cdot \Delta x$$

- The area A itself is then obtained as limit. This is the integral

$$\int_a^b f(x)\, dx = A = \lim_{n\to\infty} \sum_{i=1}^{n} f({x_i}^*)\Delta x$$

The fundamental theorem of calculus

Theorem

If f is a continuous function with primitive F, that is, with $F'(x) = f(x)$, then: $\int_a^b f(x)\,dx = F(b) - F(a)$.

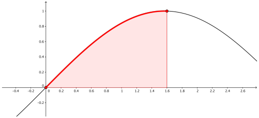
This difference $F(b) - F(a)$ is the area below f on $[a,b]$ (if $f(x) \geq 0$).

$F(b) - F(a)$ is abbreviated as $F(x)]_a^b$. So, $\int_a^b f(x)\,dx = F(x)]_a^b$.

Example

Compute the following integrals using the evaluation theorem:

- $\int_0^1 3\,dx$
- $\int_0^1 (-x^2)\,dx$
- $\int_2^4 x\,dx$
- $\int_0^{\frac{\pi}{2}} \sin(x)\,dx$



Linearity and interval properties of integrals

Lemma

(1) *integration of a constant function:* $\int_a^b c\, dx = c(b-a)$

(2) *addition:* $\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$

(3) *scalar multiplication:* $\int_a^b c \cdot f(x)\, dx = c \cdot \int_a^b f(x)\, dx$

Lemma

(1) $\int_a^a f(x)\, dx = 0$

(2) $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$, *where* $a < c < b$

(3) $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

Improper integral

Definition

An improper integral is the limit of a definite integral as one endpoint or both endpoints approach $\pm\infty$. Thus, we distinguish the following cases:

(1) If $\int_a^t f(x)\,dx$ exists for any $t > a$ and $\lim_{t\to\infty} \int_a^t f(x)\,dx$ exists, then
$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

(2) If $\int_t^b f(x)\,dx$ exists for any $t < b$ and $\lim_{t\to-\infty} \int_t^b f(x)\,dx$ exists, then
$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

(3) $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$; that is, it is defined as the sum of two improper integrals.

If the limit does not exist or infinite, the improper integral diverges.

Improper integral – Examples

Example

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left(-\frac{1}{x}\right)\Big]_1^b = 1$$

$$\int_{-\infty}^{-1} \frac{\ln(-x)}{x^2}\,dx = \frac{\ln(-x)+1}{x}\Big]_{-\infty}^{-1} = \lim_{x\to-\infty}\left(\frac{\ln(-x)+1}{x}\right) + 1 \overset{*}{=} 1$$

*Note: In the last step we applied L'Hôpital's rule.

$$\begin{aligned}
\int_{-\infty}^{\infty} \tfrac{dx}{1+x^2} &= \int_{-\infty}^{0} \tfrac{dx}{1+x^2} + \int_0^\infty \tfrac{dx}{1+x^2} = \arctan(x)]_{-\infty}^{0} + \arctan(x)]_0^\infty \\
&= \arctan(0) - \lim_{x\to-\infty}(\arctan(x)) + \lim_{x\to\infty}(\arctan(x)) - \arctan(0) \\
&= 0 - (-\frac{\pi}{2}) + \frac{\pi}{2} - 0 = \pi
\end{aligned}$$

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Order properties of integrals

Lemma

(1) *if* $f(x) \geq g(x)$ *for all* $x \in [a, b]$, *then* $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$
With two useful special cases:

(2) *if* $f(x) \geq 0$ *for all* $x \in [a, b]$, *then* $\int_a^b f(x)\,dx \geq 0$

(3) $m \leq f(x) \leq M$, *for* $x \in [a, b]$, *then*
$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

Currently we are here...

The Definite Integral

The Indefinite Integral

Techniques of Integration

Definite and indefinite integrals

There is a distinction between:

- The definite integral $\int_a^b f(x)\,dx$

  This is a number that represents the area under the curve $f(x)$ from $x = a$ to $x = b$.

- The indefinite integral $\int f(x)\,dx$

  This is notation for a function F with $F' = f$.

Indefinite integrals

Definition

(1) A function F such that $F'(x) = f(x)$ is called a primitive (or an antiderivative) function of f

- Note: $F + C$ is then also a primitive of f, for any constant C

(2) The indefinite integral $\int f(x)\,dx$ of f is used as notation for all these primitives. Thus: $\int f(x)\,dx = F + C$.

Table of indefinite integrals

- $\int 0 \; dx = C$
- $\int a \; dx = ax + C$, so $\int 1 \, dx = x + C$
- $\int x^n \, dx = \frac{1}{n+1} \cdot x^{n+1} + C$, for $n \neq -1$
- $\int \frac{1}{x} \, dx = \ln|x| + C$
- $\int e^x \, dx = e^x + C$
- $\int a^x \, dx = \frac{1}{\ln a} \cdot a^x + C$
- $\int \sin x \, dx = -\cos x + C$
- $\int \cos x \, dx = \sin x + C$
- $\int \frac{1}{cos^2 x} \, dx = \tan x + C$
- $\int \frac{1}{1+x^2} \, dx = \arctan x + C$
- $\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$

Examples

Example

- $$\begin{aligned}\int (3x^5 - 2x^2 + 1)\,dx &= \int 3x^5\,dx - \int 2x^2\,dx + \int 1\,dx\\ &= 3\int x^5\,dx - 2\int x^2\,dx + \int 1\,dx\\ &= \tfrac{1}{2}x^6 - \tfrac{2}{3}x^3 + x + C\end{aligned}$$
- $$\begin{aligned}\int (\sqrt[3]{x^2} - \tfrac{1}{x^2})\,dx &= \int x^{\frac{2}{3}}\,dx - \int x^{-2}\,dx\\ &= \frac{1}{\frac{5}{3}}x^{5/3} - \frac{1}{-1}x^{-1} + C\\ &= \tfrac{3}{5}x\sqrt[3]{x^2} + \tfrac{1}{x} + C\end{aligned}$$

Currently we are here...

Two useful techniques

There are no general rules for integration. We discuss the following two techniques.

(1) **Substitution**
- based on the chain rule $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

(2) **Integration by parts**
- based on the multiplication rule $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

They both require appropriate choices in individual cases. They are best learned by doing.

Ad 1. The substitution method

Lemma

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

where $g(x)$ is replaced by u.

Justification: Let $u = g(x)$ and $du/\,dx = g'(x)$. By the chain rule,

$$\left(\int f(u)du\right)'_x = \left(\int f(u)du\right)'_u \frac{du}{dx} = f(u)\cdot\frac{du}{dx} = f(g(x))\cdot g'(x).$$

Ad 1. The substitution method - Examples

Example

- $\int \cos \frac{x}{3}\, dx$ Let $u = \frac{x}{3}$. Then $du = \frac{1}{3}\, dx$. So, $dx = 3du$. By substitution, $\int \cos\left(\frac{x}{3}\right)\, dx = \int \cos(u) 3du = 3\sin(u) = 3\sin\left(\frac{x}{3}\right) + C$
- $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx$ Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}\, dx$. So, $2du = \frac{1}{\sqrt{x}}\, dx$. Thus, $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx = \int \cos u \cdot 2du = 2\sin(\sqrt{x}) + C$
- $\int x \sin(x^2)\, dx$ Let $u = x^2$.
- $\int \frac{x}{\cos^2(4x^2-5)}\, dx$ Let $u = 4x^2 - 5$. (Hint: $(\tan(x))' = \frac{1}{\cos^2(x)}$)

Ad 1. Adapting boundaries after substitution

When using substitution for definite integrals (still $u = g(x)$):

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example

Using subsitution $u = x^2 + 1$ we get $\frac{du}{dx} = 2x$ and so $x\,dx = \frac{1}{2}du$. Hence:

$$\begin{aligned}
\int_0^2 x\cos(x^2+1)\,dx &= \tfrac{1}{2}\int_0^2 2x\cos(x^2+1)\,dx \\
&= \tfrac{1}{2}\int_1^5 \cos(u)du \\
&= \tfrac{1}{2}\big[\sin(u)\big]_1^5 \\
&= \tfrac{1}{2}(\sin(5)-\sin(1)).
\end{aligned}$$

Ad 2. Integration by parts

- Recall the product rule for differentiation:
  $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.
- After integration we get:
  $f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$
  briefly,
  $\int f'g = fg - \int fg' \qquad (\text{or } \int fg' = fg - \int f'g)$

Example

- $\int x \ln x \, dx$
  $= \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx = \frac{1}{4}x^2(2\ln x - 1) + C$

	f	g
orig.	$\frac{x^2}{2}$	$\ln x$
der.	x	$\frac{1}{x}$

Ad 2. Examples, for integration by parts

Example

Compute the following indefinite integrals using the method of integration by parts:

- $\int xe^x \, dx$
- $\int x^2 \cdot \ln(x) \, dx$
- $\int x \sin(x) \, dx$
- $\int x^3 e^{x^2} \, dx$ (Hint: $u = x^2$)

Example

Compute the following definite integrals using integration by parts:

- $\int_0^1 xe^x \, dx$
- $\int_1^e x^2 \cdot \ln(x) \, dx$
- $\int_0^{\pi/2} x \sin(x) \, dx$
- $\int_0^1 x^3 e^{x^2} \, dx$

Learning by doing – Further examples

Example (substitutions)

- $\int \sin^5(x)\cos(x)\,dx = \frac{1}{6}\sin^6(x)$
- $\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\ln(1+x^2)$
- $\int \tan(x)\,dx = -\ln(\cos(x))$
- $\int \frac{1}{x\ln(x)}\,dx = \ln(\ln(x))$.

Example (integration by parts)

- $\int \ln(x)\,dx \quad$ (Hint: $\ln(x) = 1\cdot\ln(x)$)
- $\int e^x \sin(x)\,dx \quad$ (Hint: $\int A = \ldots - \int A \Longrightarrow 2\cdot\int A = \ldots$)
- $\int \sin^2(x)\,dx$

Areas and arc lengths

- Recall: the area below a function f on $[a, b]$ is
$$\int_a^b f(x)\, dx$$

- The area between f, g on $[a, b]$ is
$$\int_a^b (f(x) - g(x))\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$
Here we assume $f(x) \geq g(x)$, for $x \in [a, b]$.

Definition

Let f be a differentiable function on $[a, b]$. The arc length of f on this interval is
$$\int_a^b \sqrt{1 + f'(x)^2}\, dx$$

Area and arc length computations

- Compute the area below $f(x) = \sin^2(x)\cos(x)$ on $[0, \frac{\pi}{2}]$

  Substituting $u = \sin(x)$ yields:

$$\int_0^{\frac{\pi}{2}} \sin^2(x)\cos(x)\,dx = \int_0^1 u^2\,du = \frac{u^3}{3}\Big]_0^1 = \frac{1}{3}$$

- Compute the area bounded by $y^2 = x$ and $x - 4y = 0$. Solution: $\frac{32}{3}$.
- Find the length of the curve of $f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln(x)$ from $x = 1$ to $x = e$.
  - $f'(x) = \frac{1}{2}x - \frac{1}{2}\cdot\frac{1}{x} = \frac{x^2-1}{2x}$
  - $$\begin{aligned}\int_1^e \sqrt{1+\frac{(x^2-1)^2}{4x^2}}\,dx &= \int_1^e \frac{\sqrt{x^4+2x^2+1}}{2x}\,dx = \frac{1}{2}\int_1^e \frac{\sqrt{(x^2+1)^2}}{x}\,dx \\ &= \frac{1}{2}\int_1^e \frac{x^2+1}{x}\,dx = \frac{1}{2}\int_1^e (x+\frac{1}{x})\,dx \\ &= \frac{1}{2}(\frac{x^2}{2}+\ln x)\Big|_1^e = \frac{1}{4}(e^2+1) \approx 2.097\end{aligned}$$