Calculus and Probability Theory
Assignment 5, March 2, 2017

Handing in your answers:

- submission via Blackboard (`http://blackboard.ru.nl`);
- one single pdf file (make sure that if you scan/photo your handwritten assignment, the result is clearly readable);
- all of your solutions are clearly and convincingly explained;
- make sure to write your name, your student number

Deadline: Friday, March 10, 14:30 sharp!

Goals: After completing these exercises successfully you should be:

- familiar with definite and indefinite integrals;
- able to apply the most important integration methods; more specifically, substitution and integration by parts;
- confident about switching between different representations of a function;
- able to compute area of a finite or infinite region;
- able to apply the formula for the arc length of a function over a finite interval.

Marks: You can score a total of 100 points. (Additionally, you can collect +10 bonus points.)

1. **(20 points)** Compute the following indefinite integrals. You can use *substitution* or *integration by parts*.

   In each problem *verify* your result, and don't forget about the constant term. You may need some of the following, well-known trigonometric identities:

   $$\sin(2x) = 2\sin(x)\cos(x), \quad \cos(2x) = \cos^2(x) - \sin^2(x), \quad \sin^2(x) + \cos^2(x) = 1.$$

   Also, it is highly recommended to consult with the lecture slides and solve the problems there before you start with these ones.

   (a) $\int \sin(x)\cos(x)\,dx$
   (b) $\int \ln(ax)\,dx$ where $a > 0$
   (c) $\int \cos^2(x)\,dx$
   (d) $\int \frac{1}{\sqrt{1-4x^2}}\,dx$
   (e) $\int e^{3x}\sin(x)\,dx$

2. **(20 points)** Compute the length of the curve $f(x) = \sqrt{1-x^2}$ where $x \in [-1, 1]$

   (a) using calculus, and
   (b) using a geometric argument.

   [Hint: (b) what is the shape of $\sqrt{1-x^2}$?]

3. **(20 points)** Compute the definite integral $\int_{-1}^{1} \sqrt{1-x^2}\,dx$

   (a) using calculus, and
   (b) using a geometric argument.

   [Hint: (a) instead of substituting a function of x by u, now substitute $x = \sin(u)$.]

4. **(15 points)** Compute the following improper integrals.

   (a) $\int_0^\infty re^{-r^2}\,dr$;

   (b) $\int_0^{2\pi}\left(\int_0^\infty re^{-r^2}\,dr\right)dt$;

   (c) (bonus, +3 points) Prove that $\int_{-\infty}^\infty e^{-z^2}\,dz=\sqrt{\pi}$.

   You may use the fact that $\int_{-\infty}^\infty\left(\int_{-\infty}^\infty e^{-(x^2+y^2)}\,dx\right)dy=\int_0^{2\pi}\left(\int_0^\infty re^{-r^2}\,dr\right)dt$.

   (d) $\int_0^\infty e^{-z^2}\,dz$.

5. **(15 points)** Compute the following improper integrals.

   (a) $\int_0^\infty e^{-x}\,dx$;

   (b) $\int_0^\infty xe^{-x}\,dx$ using integration by parts;

   (c) (bonus, +2 points) $\int_0^\infty x^n e^{-x}\,dx$ for all $n\in\{0,1,\dots\}$;

   (d) $\int_0^\infty x^{-\frac{1}{2}}e^{-x}\,dx$.

   [Hint: (c) Try first for $n=0,1,2,3$; (d) substitute $u=\sqrt{x}$ and, at the end, some information from a previous exercise turns out to be useful.]

6. **(10 points)**

   (a) Given three lines, $y=x+2$, $y=-x+6$ and $y=2x-3$, enclosing a triangle. Determine the *coordinates* of the three vertices and the *area* of the triangle.

   (b) Compute the area of the region bounded by $y=(x-1)^3$ and $y=(x-1)^2$.

7. **(bonus, 5 points)** The figure shows a horizontal line $y=c$ intersecting the curve $y=-(x-2)^2+4$. Find the number c such that the areas of the shaded regions are equal.