Calculus and Probability Theory

Assignment 4, February 23, 2017

Handing in your answers:

- submission via Blackboard (`http://blackboard.ru.nl`);
- one single pdf file (make sure that if you scan/photo your handwritten assignment, the result is clearly readable);
- all of your solutions are clearly and convincingly explained;
- make sure to write your name, your student number.

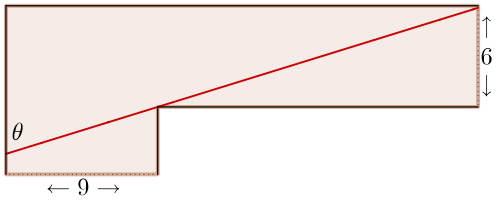
Deadline: Friday, March 3, 14:30 sharp!

Goals: After completing these exercises successfully you should be able to:

- analyse and sketch real functions;
- apply differentiation rules to determine higher-order partial derivatives;
- find primitives of well-known functions;
- compute definite integrals when the primitive function is known;
- compute improper integrals.

Marks: You can score a total of 100 points (and 16 bonus points) There are three bonus exercises.

1. **(20 points)** Investigate the function $f(x) = \frac{x}{\ln(x)}$ as follows. (Do not start with drawing a graph by means of a device or some web resource. Of course you may check your result when you're done.)
   (a) Determine the domain of the function f.
   (b) What are the roots of f?
   (c) Determine the limits at 1 and ∞. (Hint: there are 3 cases, use l'Hôpital!)
   (d) Find f' and f''.
   (e) Find the zeros of f' and f''.
   (f) What are the critical points (determine their x and y coordinates)?
   (g) Find the local minimums and maximums.
   (h) Which parts of the function are convex and concave? Does function f have points of inflection? (Hint: Use the sign of the second derivative for answering both questions.)
   (i) Draw the graph of function f. (If you collect all intervals and special points in a table, it helps a lot in drawing the graph. Moreover, you get some extra points!)
2. **(6 points)** Show that the derivative of an odd function is even and that the derivative of an even function is odd.
3. **(14 points)** Optimization problem
   (a) Find the point on the parabola $y^2 = 2x$ that is closest to $A = (1, 4)$.
   (b) A steel rod is carried down a hallway of 9 meter wide. At the end there is corner to the right into a narrower hallway of 6 meter wide. What is the maximum length of the steel rod that can be carried horizontally around the corner?



(Hint: What happens at $\theta \to 0$ and $\theta \to \frac{1}{4}\pi$? Show that the angle at which the minimum is obtained is at $\theta = \arctan\left(\sqrt[3]{\frac{3}{2}}\right) \approx 0,853$.)

4. **(20 points)** Given function f, find the partial derivatives. If it is necessary, simplify the result.
   (a) $f(x,y) = \cos(4y - xy)$; $\frac{\partial}{\partial x} f(x,y) =$? and $\frac{\partial}{\partial y} f(x,y) =$?
   (b) Show that $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x,y)\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x,y)\right)$.

5. **(20 points)** Evaluate the following definite integrals.
   (a) $\int_{-1}^{1} (x^3 + x - 1)\, dx$;
   (b) $\int_{1}^{2} (3\sqrt{x} + \frac{3}{x^2})\, dx$;
   (c) $\int_{0}^{\pi} (\sin(x) + \cos(x))\, dx$;
   (d) $\int_{1}^{e} \frac{1 - \ln x}{x^2}\, dx$

6. **(20 points)** Evaluate the following improper integrals.
   (a) $\int_{-1}^{1} \frac{1}{x^n}\, dx$, n an integer such that $n \geq 2$; (Hint: distinguish two cases)
   (b) $\int_{-\infty}^{-\pi/2} \frac{x\cos(x) - \sin(x)}{x^2}\, dx$; (Hint: use the quotient rule for derivation to find the primitive)
   (c) $\int_{2}^{\infty} \frac{-1}{x \ln^2(x)}\, dx$. (Hint: use a fraction of well known functions to find the primitive)

7. **(bonus, +6 points)** Find primitives of the following functions f. That is, find F such that $F'(x) = f(x)$.
   (a) $f(x) = \frac{1}{2\sqrt{x}} - \frac{1}{x^2}$;
   (b) $f(x) = 2\sin(x)\cos(x)$;
   (c) $f(x) = \frac{2}{1+4x^2}$;

8. **(bonus, 10 points)** If $f(x,y) = \frac{xy}{x+y}$, show that

$$x^2 \cdot \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f(x,y)\right) + 2xy \cdot \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x,y)\right) + y^2 \cdot \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} f(x,y)\right) = 0.$$

(Hint: First compute all the second partial derivatives of f, then substitute the results in the expression on the left-hand side.)