Calculus and Probability Theory
Assignment 4

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After completing these exercises successfully you should be confident with the following topics:

- Analyse and sketch real functions
- Apply differentiation rules to determine higher-order partial derivatives
- find primitives of well-known functions
- Compute definite integrals when the primitive function is known
- computer impproper integrals

1. (**20 points**) Investigate the function $f(x) = \frac{e^x}{x+1}$ as follows. (Do not start with drawing a graph by means of a device or some web resource. Of couse you may check your result when you're done.)

   (a) Determine the domain of the function f.
   Solution:

   $D(f) = \{x \in \mathbb{R} | c \neq -1\}$

   (b) What are the roots of f?
   Solution:

   There are no roots (x-interceptions) because e^x is never 0.

   (c) Determine the limits at the edges of the domain. (Hint: there are 4 cases, use L'Hoputial!)
   Solution:

   Vertical Asymptote:

For which value is $f(x)$ undefined?
We just have to come up with a value for the denominator which produces a zero. So:

$$\begin{aligned} x+1 &= 0 \\ x &= -1 \end{aligned}$$

Because $f(x)$ is undefined for -1, the vertical asymptote can be found at $x=-1$ and we can inspect 4 limits, namely

- $\lim_{x\to-\infty} \frac{e^x}{x+1}$
- $\lim_{x\to\infty} \frac{e^x}{x+1}$
- $\lim_{x\to-1^-} \frac{e^x}{x+1}$
- $\lim_{x\to-1^+} \frac{e^x}{x+1}$

$$\begin{aligned} \lim_{x\to-\infty} \frac{e^x}{x+1} &= \lim_{x\to-\infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x+1)} \\ &= \lim_{x\to-\infty} \frac{e^x}{1} \\ &= \lim_{x\to-\infty} e^x \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{x\to\infty} \frac{e^x}{x+1} &= \lim_{x\to\infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x+1)} \\ &= \lim_{x\to\infty} \frac{e^x}{1} \\ &= \lim_{x\to\infty} e^x \\ &= \infty \end{aligned}$$

$$\lim_{x\to-1^-} \frac{e^x}{x+1} = -\infty$$

$$\lim_{x\to-1^+} \frac{e^x}{x+1} = \infty$$

(d) Find f' and f''.
Solution:

$$\begin{aligned}
f(x) &= \frac{e^x}{x+1} \\
f'(x) &= \frac{[(e^x)(x+1)] - [e^x \cdot 1]}{(x+1)^2} \\
&= \frac{xe^x}{(x+1)^2} \\
f''(x) &= \frac{\left[e^x(x+1) \cdot (x+1)^2\right] - [xe^x \cdot (2x+2)]}{(x+1)^4} \\
&= \frac{\left[e^x(x+1)^3\right] - 2e^x x^2 - 2e^x x}{(x+1)^4} \\
&= \frac{e^x x^3 + 3e^x x^2 + 3e^x x + e^x - 2e^x x^2 - 2e^x x}{(x+1)^4} \\
&= \frac{e^x x^3 + e^x x^2 + e^x x + e^x}{(x+1)^4} \\
&= \frac{e^x(x^3 + x^2 + x + 1)}{(x+1)^4} \\
&= \frac{e^x(x+1)(x^2+1)}{(x+1)^4} \\
&= \frac{e^x(x^2+1)}{(x+1)^3}
\end{aligned}$$

(e) Find the zeros of f' and f''.
Solution:

Zeros of $f'(x)$:

Because we are dealing with a quotient, we are just interested in the the case when the numerator becomes zero. When the denominator becomes zero, the function is undefined and is not a x-intercept. Therefore:

$$\begin{aligned}
xe^x &= 0 \\
x &= \frac{0}{e^x} \\
x &= 0
\end{aligned}$$

Zeros of $f''(x)$:

$$
\begin{aligned}
e^x(x^2+1) &= 0 \\
e^x &= \frac{0}{(x^2+1)} \\
e^x &= 0
\end{aligned}
$$

There are no solutions to this equation and therefore no zeros for $f''(x)$.

(f) What are the critical points (determine their x and y coordinates)?
Solution:

A critical point of a function $f : D \to \mathbb{R}$, is a point $a \in D$ such that $f'(a) = 0$. The value $f(a)$ is called a critical value of f.

Determine critical points by plugging in the values where $f'(x) = 0$ into the original function

$$
\begin{aligned}
\frac{e^0}{0+1} &= \frac{1}{1} \\
&= 1
\end{aligned}
$$

Critical point 1: $(0, 1)$

(g) Find the local minimums and maximums.
Solution:

When a function's slope is zero at x, and the second derivative at x is:

- less than 0, it is a local maximum
- greater than 0, it is a local minimum
- equal to 0, then the test failes

Plugging-in the zeros of f' into $f''(x)$:

$$
\frac{e^0(0^2+1)}{(0+1)^3} = 1
$$

Therefore, $(0, 1)$ is a local minimum.

(h) Which parts of the function are convex and concave? Does function f have points of inflection? (Hint: Use the sign of the second derivative for answering both questions.)
Solution:

(i) Draw the graph of function f. (If you collect all intervals and special points in a table, it helps a low in drawing the graph. Moreover, you get some extra points.) **Solution:**

2. (**bonus, +1 points**) Write your name, student number, and the name of your TA on the first page.

3. (**bonus, +4 points**) Consider the function $f(x) = e^x \sin(x)$.

(a) Determine the domain of the function f.
Solution:

$$D(f) = \mathbb{R}$$

(b) What are the roots of f? Where does the graph of f intersect the y axis?
Solution:

We just have to come up with a value for x, where $\sin(x) = 0$. This value is π. And because sin is a periodic function, which repeats its x-intercept every n steps, we can write:

$$\begin{aligned} e^x \sin(x) &= 0 \\ x &= n \cdot \pi, n \in \mathbb{Z} \end{aligned}$$

(c) Find f' and f''.
Solution:

$$\begin{aligned} f'(x) &= e^x \cdot \sin(x) + e^x \cdot \cos(x) \\ &= e^x(\sin(x) + \cos(x)) \\ f''(x) &= [e^x \sin(x) + e^x \cos(x)] + [e^x \cos(x) - e^x \sin(x)] \\ &= 2e^x \cos(x) \end{aligned}$$

(d) Find all the zeros of f' and f''.
Solution:

$f'(x) = 0$

$$e^x(\sin(x) + \cos(x)) = 0$$
$$x = \pi \cdot n - \frac{\pi}{4}, n \in \mathbb{Z}$$

$f''(x) = 0$

$$2e^x \cos(x) = 0$$

$$x = \pi \cdot n - \frac{\pi}{2}, n \in \mathbb{Z}$$

4. (**20 points**) Given function f, find the partial derivatives. If it is necessary, simplify the result.

   a.i $f(x, y) = \cos(4y - xy)$
   Solutions:

$$\frac{\partial}{\partial x} f(x, y) = ((x-4)y'(x) + y)(-\sin((x-4)y))$$

$$\frac{\partial}{\partial y} f(x, y) = -(x-4)\sin((x-4)y)$$

   a.ii $f(x, y) = e^{\frac{x}{y}}$
   Solutions:

$$\frac{\partial}{\partial x} f(x, y) = \frac{e^{\frac{x}{y}}(y - xy'(x))}{y^2}$$

$$\frac{\partial}{\partial y} f(x, y) = -\frac{xe^{\frac{x}{y}}}{y^2}$$

   b For the two functions above, show that $\frac{\partial}{\partial x}(\frac{\partial}{\partial y} f(x, y)) = \frac{\partial}{\partial y}(\frac{\partial}{\partial x} f(x, y))$
   Solutions:

5. (**20 points**) If $f(x, y) = \frac{xy}{x+y}$, show that

$$x^2 \cdot \frac{\partial}{\partial x}(\frac{\partial}{\partial x} f(x, y)) + 2xy \cdot \frac{\partial}{\partial x}(\frac{\partial}{\partial y} f(x, y)) + y^2 \cdot \frac{\partial}{\partial y}(\frac{\partial}{\partial y} f(x, y)) = 0$$

   (Hint: First compute all the second partial derivatives of f, then substitute the results in the expression on the left-hand side.) **Solutions:**

6. (**20 points**) Evaluate the following definite integrals. (Hint: use slide 38 of the lectures about derivatives, and slide 13 of the lectures about primitives)

$$\int_a^b f(x)dx = F(b) - F(a)$$

(a) $\int_{-1}^{1}(x^3 + x - 1)dx$

Solutions:

$$\begin{aligned}
\int (x^3 + x - 1)dx &= \frac{1}{4}x^4 + \frac{1}{2}x^2 - x + C \\
\int_{-1}^{1}(x^3 + x - 1)dx &= \left[\frac{1}{4}1^4 + \frac{1}{2}1^2 - 1\right] - \left[\frac{1}{4}(-1)^4 + \frac{1}{2}(-1)^2 - (-1)\right] \\
&= \frac{1}{4} - \frac{1}{2} - \frac{1}{4} - \frac{1}{2} - 1 \\
&= -2
\end{aligned}$$

(b) $\int_{1}^{2}(3\sqrt{x} + \frac{3}{x^2})dx$

Solutions:

$$\begin{aligned}
\int (3\sqrt{x} + \frac{3}{x^2})dx &= 3\left[\frac{2x^{\frac{3}{2}}}{3} - \frac{1}{x}\right] + C \\
\int_{1}^{2}(3\sqrt{x} + \frac{3}{x^2})dx &= \left[3(\frac{2(2)^{\frac{3}{2}}}{3} - \frac{1}{2})\right] - \left[3(\frac{2(1)^{\frac{3}{2}}}{3} - 1)\right] \\
&= 4\sqrt{2} - \frac{1}{2} \\
&\approx 5.1569
\end{aligned}$$

(c) $\int_{0}^{\pi}(\sin(x) + \cos(x))dx$

Solutions:

$$\begin{aligned}
\int (\sin(x) + \cos(x))dx &= -\cos(x) + \sin(x) + C \\
\int_{0}^{\pi}(\sin(x) + \cos(x))dx &= [-\cos(\pi) + \sin(\pi))] - [-\cos(0) + \sin(0)] \\
&= 1 + 0 + 1 + 0 \\
&= 2
\end{aligned}$$

(d) $\int_{-1}^{1}(\frac{-5}{\sqrt{1-x^2}})dx$

Solutions:

$$\int (\frac{-5}{\sqrt{1-x^2}})dx = -5(\arcsin(x)) + C$$

$$\begin{aligned}\int_{-1}^{1} (\frac{-5}{\sqrt{1-x^2}}) &= [-5(\arcsin(1))] - [-5(\arcsin(-1))] \\ &= -5\frac{\pi}{2} - 5\frac{\pi}{2} \\ &= -5\pi\end{aligned}$$

7. (**20 points**) Evaluate the following improper integrals.

   (a) $\int_1^\infty (\frac{1}{x^n})dx$, n an integer such that $n \geq 2$; (Hint: this generalizes an example solved in the lecture on slide 8).
   Solutions:

   (b) $\int_{-\infty}^{-\pi/2} \frac{x\cos(x)-\sin(x)}{x^2}dx$; (Hint: use the quotient rule for derivation to find the primitive)
   Solutions:

$$\int (\frac{x\cos(x)-\sin(x)}{x^2})dx = \frac{\sin(x)}{x} + C$$

$$\int_{-\infty}^{-\pi/2} (\frac{x\cos(x)-\sin(x)}{x^2})dx = \frac{2}{\pi}$$

   (c) $\int_2^\infty \frac{-1}{x\ln^2(x)}dx$; (Hint: use a fraction of well known functions to find the primitive)
   Solutions:

$$\int (\frac{-1}{x\ln^2(x)})dx = \frac{1}{\ln(x)} + C$$

$$\int_2^\infty (\frac{-1}{x\ln^2(x)})dx = \frac{1}{\ln(2)}$$

8. (**bonus, +4 points**) Find primitves of the following functions f. That is, find F such that $F'(x) = f(x)$.

   (a) $f(x) = \frac{1}{2\sqrt{x}} - \frac{1}{x^2}$
   Solutions:

$$\int (\frac{1}{2\sqrt{x}} - \frac{1}{x^2})dx = \sqrt{x} + \frac{1}{x} + C$$

   (b) $f(x) = 2\sin(x)\cos(x)$
   Solutions:

$$\int (2\sin(x)\cos(x))dx = -\frac{1}{2}\cos(2x) + C$$

(c) $f(x) = \frac{2}{1+4x^2}$
Solutions:

$$\int (\frac{2}{1+4x^2})dx = \arctan(2x) + C$$

(d) $f(x) = \frac{1-\ln(x)}{x^2}$
Solutions:

$$\int (\frac{1-\ln(x)}{x^2})dx = \frac{\ln(x)}{x} + C$$