Calculus en Kansrekenen (NWI-IBC017) – EXAMINATION

May 24, 2017, 18:00–21:00

— S O L U T I O N —

This exam consists of nine problems. You can score a maximum of 100 points. Each question indicates how many points it is worth. All questions and almost all subquestions can be made independently of each other. The exam is closed book. You are NOT allowed to use a programmable calculator, a computer, or a mobile phone. The only device you are allowed to use is the calculator provided at the exam. **Explain your approach and answers** briefly. You may give the answers in Dutch or in English. Please write clearly! Do not forget to put **your name** and **your student number** on the top of **each page**. At the end is a table you can use to keep track of some of your numerical answers. They follow a simple pattern that, if done correctly, you can use to validate your computations.

1. **(7+7=14 points)** [Tested skills: limits, L'Hôpital, differentiation]

   (a) $\lim_{x\to 0} \frac{2(e^x-1-x)}{x^2}$ (Hint: Check if you can use L'Hôpital.)

   (b) Let $a \in \mathbb{R}$ be given. Find $b \in \mathbb{R}$ such that the limit $\lim_{x\to a} \frac{x^2-b}{x-a}$ is defined. What is this limit? What is the limit in the specific case of $a = \frac{1}{2}, b = \frac{1}{4}$? (Hint: Either use L'Hôpital or factorization.)

Solution:

(a)

$$\begin{aligned}\lim_{x\to 0} \frac{2(e^x-1-x)}{x^2} &\overset{\text{LHR}}{=} \lim_{x\to 0} \frac{2(e^x-1)}{2x}\\ &\overset{\text{LHR}}{=} \lim_{x\to 0} \frac{2e^x}{2}\\ &= \underline{\underline{1}}\end{aligned}$$

In both steps we applied L'Hôpital (LHR) since in both we have an indeterminate form $\frac{0}{0}$.

(b) Since $\lim_{x\to a} x - a = 0$, it must hold that $\lim_{x\to a} x^2 - b = a^2 - b = 0$ or else the limit is not defined, hence $\underline{\underline{b = a^2}}$. Then

$$\begin{aligned}\lim_{x\to a} \frac{x^2-b}{x-a} &= \lim_{x\to a} \frac{x^2-a^2}{x-a}\\ &= \lim_{x\to a} \frac{(x-a)(x+a)}{x-a}\\ &= \lim_{x\to a} x+a\\ &= \underline{\underline{2a}}\end{aligned}$$

For $a = \frac{1}{2}, b = \frac{1}{4}$ the limit satisfies the conditions above and is thus $\underline{\underline{1}}$.

2. **(7+7=14 points)** [Tested skills: (In)definite Integral, Substitution, Partial integration, Improper integral]

   (a) If $\int (\frac{1}{2}\sqrt{\sin(x)} + 1) \cdot e^{\sqrt{\sin(x)}} \cdot \cos(x)dx = \int (u^2 + au)e^u du$ then $a = ?$. (Hint: Use substitution $u = \sqrt{\sin(x)}$.)

   (b) $\int_0^2 - \ln\left(\frac{1}{2}x\right) dx$ (Hint: Use partial integration. Note that the integral is improper. You can use the fact that $\lim_{t\to 0} t \cdot \ln\left(\frac{1}{2}t\right) = 0$.)

Solution:

(a) If $u = \sqrt{\sin(x)}$, then $du = \frac{1}{2\sqrt{\sin(x)}} \cdot \cos(x)dx = \frac{1}{2u} \cdot \cos(x)dx$. Hence $\cos(x)dx = 2udu$. Thus

$$\begin{aligned}\int \left(\frac{1}{2}\sqrt{\sin(x)}+1\right) \cdot e^{\sqrt{\sin(x)}} \cdot \cos(x)dx &= \int \left(\frac{1}{2}u+1\right) \cdot e^{u} \cdot 2udu \\ &= \int (u^2+2u)e^u\end{aligned}$$

and thus $a = \underline{\underline{2}}$.

(b) First the indefinite integral

$$\begin{aligned}\int -\ln\left(\frac{1}{2}x\right)dx &= \int -1 \cdot \ln\left(\frac{1}{2}x\right)dx \\ &= -x \cdot \ln\left(\frac{1}{2}x\right) - \int -x \cdot \frac{1}{\frac{1}{2}x} \cdot \frac{1}{2}dx \\ &= -x \cdot \ln\left(\frac{1}{2}x\right) + \int 1dx \\ &= -x \cdot \ln\left(\frac{1}{2}x\right) + x\end{aligned}$$

Thus

$$\begin{aligned}\int_0^2 -\ln\left(\frac{1}{2}x\right)dx &= \lim_{t\to 0}\int_t^2 -\ln\left(\frac{1}{2}x\right)dx \\ &= \lim_{t\to 0} -x \cdot \ln\left(\frac{1}{2}x\right) + x\Big|_t^2 \\ &= \lim_{t\to 0}(-2 \cdot \ln(1)+2) - \left(-t \cdot \ln\left(\frac{1}{2}t\right)+t\right) \\ &= (0+2) - (0+0) = \underline{\underline{2}}\end{aligned}$$

3. **(7+7+7=21 points)** [Tested skills: Logarithmic Differentiation, Anti-derivative, Partial derivatives]

(a) $\int_1^2 x^x(\ln(x)+1)dx$ (Hint: If you are not sure how to tackle the integral, consider the behaviour of $\frac{d}{dx}x^x dx$ using Logarithmic Differentiation.)[1]

(b) Let $f(x,y) = \sin(xy)$. Show that $\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial^2 f(x,y)}{\partial y \partial x}$.

(c) Find the derivative of $\exp(\ln(\tan(x)))$.

Solution:

(a) Logarithmic differentiation shows that x^x is the anti-derivative of $x^x(\ln(x)+1)$ since

$$\begin{aligned}\frac{d}{dx}x^x dx &= x^x \cdot \left(\frac{d}{dx}\ln(x^x)\right) \\ &= x^x \cdot \left(\frac{d}{dx}x\ln(x)\right) \\ &= x^x \cdot \left(1 \cdot \ln(x) + x \cdot \frac{1}{x}\right) \\ &= x^x \cdot (\ln(x)+1)\end{aligned}$$

We have $\int_1^2 x^x(\ln(x)+1)dx = x^x|_1^2 = 2^2 - 1^1 = 4 - 1 = \underline{\underline{3}}$.

[1] The 'NK integreren' archive `https://www.a-eskwadraat.nl/Vereniging/Commissies/cieinfinity/archiefnki.html` has many more interesting integrals like these.

(b)

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\sin(xy)\right) = \frac{\partial}{\partial x} x\cos(xy)) = \cos(xy) - xy\sin(xy).$$
$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\sin(xy)\right) = \frac{\partial}{\partial y} y\cos(xy)) = \cos(xy) - xy\sin(xy).$$

(c) $\frac{\partial}{\partial x}\exp(\ln(\tan(x))) = \frac{\partial}{\partial x}\tan(x) = \frac{\partial}{\partial x}\frac{\sin(x)}{\cos(x)} = \frac{\sin(x)^2+\cos(x)^2}{\cos(x)^2} = \frac{1}{\cos(x)^2}.$

4. **(7 points)** [Tested skills: Area between curves]
Compute the area of the region bounded by (i.e., in between) $f(x) = x^2$, $g(x) = 2 - x^2$, $x = 0$, $x = 2$. Besides giving the numeric answer also clearly provide i) the graph of the functions and integration area, ii) the intersection point, iii) the integral expression for the area (i.e., sum of 2 regions).

Solution: Clearly, in $[0, 2]$ f and g intersect in the point $x = 1$ thus the area A is given by

$$\begin{aligned}
A &= \int_0^1 (2-x^2) - x^2 dx + \int_1^2 x^2 - (2-x^2) dx \\
&= \int_0^1 2 - 2x^2 dx + \int_1^2 2x^2 - 2dx \\
&= 2x - \frac{2}{3}x^3|_0^1 + \frac{2}{3}x^3 - 2x|_1^2 \\
&= \left(2 - \frac{2}{3}\right) - (0-0) + \left(\frac{2}{3}\cdot 8 - 4\right) - \left(\frac{2}{3} - 2\right) \\
&= \frac{4}{3} - 0 + \frac{4}{3} + \frac{4}{3} = 3\cdot\frac{4}{3} = \underline{\underline{4}}
\end{aligned}$$

5. **(7+7+7=21 points)** [Tested skills: Integration, PDF, CDF, Derivative, Graph sketching]
A continuous random variable X has the following probability density function

$$f(x) = \begin{cases} \frac{3}{16}\cdot(1+ax^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value for a.
(b) Find the cumulative distribution function $F(x)$.
(c) Give (short) answers to the following subquestions:
   i. What is $f'(x)$ and what are its zeros? Is it a minimum or maximum?
   ii. Is the function even, odd, or neither?
   iii. Sketch the graph of f
   iv. What is the expected value $\mathrm{E}[X]$?

Solution:

(a) As f is a pdf it must integrate to 1, thus

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \frac{3}{16}\cdot(1+ax^2) \\
&= \int_{-1}^{1} \frac{3}{16}\cdot(1+ax^2) \\
&= \frac{3}{16}\cdot\left(x + \frac{a}{3}x^3\right)|_{-1}^{1} \\
&= \frac{3}{16}\cdot\left(\left(1+\frac{a}{3}\right) - \left(-1-\frac{a}{3}\right)\right) \\
&= \frac{3}{16}\cdot\left(2\left(1+\frac{a}{3}\right)\right)
\end{aligned}$$

Hence $a = \left(\frac{16}{3} \cdot \frac{1}{2} - 1\right) \cdot 3 = \underline{\underline{5}}$.

(b) From (a) follows that the indefinite integral on $(-1, 1)$ is given by $\frac{3}{16}\left(x + \frac{5}{3}x^3\right) + C$, thus the cumulative distribution function $F(x)$ is given by

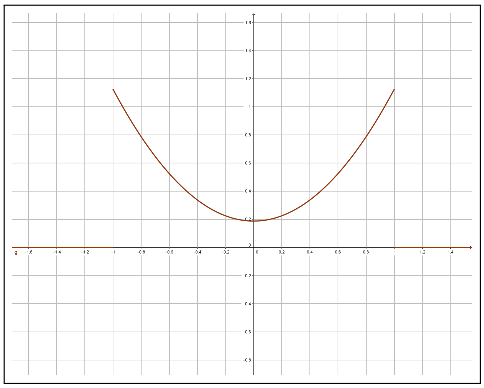
$$F(x) = \begin{cases} 0 & x \leq -1 \\ \frac{3}{16}\left(x + \frac{5}{3}x^3\right) + C & -1 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

C follows by looking the value at the endpoints of the domain:

$$1 = F(1) = \frac{3}{16}(1 + \frac{5}{3}) + C = \frac{3}{16} \cdot \frac{8}{3} + C = \frac{1}{2} + C$$
$$0 = F(-1) = \frac{3}{16}(-1 - \frac{5}{3}) + C = \frac{3}{16} \cdot -\frac{8}{3} + C = -\frac{1}{2} + C$$

either one gives $C = \frac{1}{2}$.

(c)



(a) $f'(x) = 2 \cdot a \cdot \frac{3}{16}x = \frac{3 \cdot a}{8}x = \underline{\underline{\frac{15}{8}}}x$ which is 0 for $\underline{\underline{x = 0}}$, which is a $\underline{\underline{\textit{minimum}}}$.

(b) The function is $\underline{\underline{\textit{even}}}$, since $f(x) = f(-x)$.

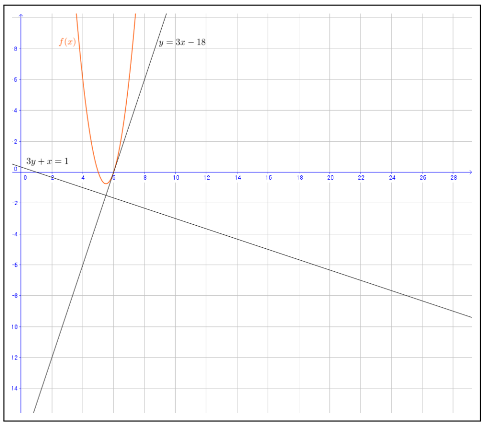
(c) See graph, where

$$f(-1) = f(1) = \frac{3(1 + a)}{16} = \frac{18}{16} = \frac{9}{8} = 1.125,$$
$$f(0) = \frac{3}{16} = 0.1875$$

(d) The expectation of an even function is $\underline{\underline{0}}$.

6. **(7 points)** [Tested skills: Derivative, Lines]
Let $f(x) = 3(x - 5)^2 - 3(x - 5)$. Find one point on f that has a tangent that is perpendicular to the line $3y + x = 1$. (Two lines are perpendicular if the product of their slopes is -1.) In addition give the equation of the line going through this point.

Solution:



We have $3y + x = 1 \Leftrightarrow y = -\frac{1}{3}x + \frac{1}{3}$, thus the slope of the line is $-\frac{1}{3}$ and since $-\frac{1}{3} \cdot 3 = -1$, the line perpendicular to it has slope 3. Furthermore, $f'(x) = 6(x - 5) - 3x = 6x - 33$. Solving $6x - 33 = 3$ gives the x-coordinate 6. Hence, the point on the curve f we are looking for is $(6, f(6)) = \underline{\underline{(6, 0)}}$. The line going through this point with slope 3 is given by $\underline{\underline{y = 3x - 18}}$.

7. **(2 points)** [Tested skills: Combinatorics]
How many ways can eight people sit around a circular table, i.e., cyclic patterns such as 12345678 and 23456781 are considered the same. You don't need to evaluate the ! operator into a number. Motivate your answer.

Solution: The first person has 8 choices, the second 7, etc. thus their are 8! permutations, however, each permutation is counted 8 times (there are 8 choices for the first person and then the rest is fixed), thus $\frac{8!}{8} = \underline{\underline{7!}}$.

8. **(7 points)** Suppose that you like to throw a party for surviving your last Calculus and Probability exam at the beach but that it is very windy. Luckily you brought your flexible windscreen with a length of 8m. Maximize the rectangular area by shielding the area on 3 sides with your windscreen. What is the size of this area? What are the lengths of the sides? For example, setting up your windscreen with sides $1m - 6m - 1m$ would give a rectangular area of $1m \times 6m = 6m^2$.

   (Hint: The area has 3 sides, expressible with 2 variables. Express one of the variables in terms of the other to obtain an expression for the total area in terms of 1 variable. Then find the optimum using differentiation.)

   Solution: Let the sides 3 sides of rectangular area be given by x, x, y with a total length of 8m. Thus $2x + y = 8$ or $y = 8 - 2x$. The area is given by $A = xy = x(8 - 2x) = 8x - 2x^2$. Maximizing is done by taking the derivative of A with respect to x and finding the optimum. $A' = 8 - 4x$ which is 0 when $x = 2$. Thus $y = 8 - 2 \cdot 2 = 4$. The total area is thus $A = 2 \cdot 4 = \underline{\underline{8m^2}}$ with with <u>two sides of 2m</u> and <u>one side of 4m</u>.

9. **(7 points)** [Tested skills: Probability, Bayes rule]
   Recall that the Bayes' rule states that

$$P(H \mid E) = \frac{P(E \mid H) \cdot P(H)}{P(E)}.$$

   Moreover, according the total probability lemma, for a partition $A_1, \ldots, A_n$ and an arbitrary event B,

$$P(B) = P(B \mid A_1) \cdot P(A_1) + \cdots + P(B \mid A_n) \cdot P(A_n).$$

   Suppose that there are two boxes. In Box 1 there are 10 apples and 90 oranges. In Box 2 there are 80 apples and 20 oranges. Take one of the two boxes at random, and pick a fruit at random. Suppose that you drew an apple, then what is the probablity that you chose from Box 1? (Use B_i for Box i, A for apple, O for orange.)

   Solution:

$$\begin{aligned} P(B_1|A) &= \frac{P(A|B_1)P(B_1)}{P(A)} \\ &= \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2)} \\ &= \frac{\frac{10}{100} \cdot \frac{1}{2}}{\frac{10}{100} \cdot \frac{1}{2} + \frac{80}{100} \cdot \frac{1}{2}} \\ &= \frac{10}{10 + 80} = \underline{\underline{\frac{1}{9}}} \end{aligned}$$

Nr.	**(a)**	**(b)**	**(c)**
1	Limit: 1	Limit ($a = \frac{1}{2}, b = \frac{1}{4}$): 1	
2	a: 2	Integral: 2	
3	Integral: 3	NaN	NaN
4	Area: 4		
5	a: 5	NaN	NaN
6	x-coordinate: 6		
7	Nr. Combinations (with !): 7!		
8	Area: $8m^2$		
9	Probability: $\frac{1}{9}$		

Figure 1: Keep track of your progress for parts of the exercises. NaN = Not a number. The numerical value corresponds to the exercise number.