Calculus and Probability Theory

Assignment 6, March 9, 2017

Handing in your answers:

- submission via Blackboard (`http://blackboard.ru.nl`);
- one single pdf file (make sure that if you scan/photo your handwritten assignment, the result is clearly readable);
- all of your solutions are clearly and convincingly explained;
- make sure to write your name, your student number

Deadline: Friday, March 17, 14:30 sharp!

Goals: After completing these exercises successfully you should be able to:

- compute combinatorial problems;
- recognize the birthday paradox;
- work with the basic definitions of probability theory.

Marks: You can score a total of 100 points.

1. **(10 points)** Vanessa wants to give her friend potted plants as present. At the local florist, the flowers come in five colours, and there are four types of flower pots.
   (a) If Vanessa buys one potted flower (any combination of a flower and a pot), how many different options can Vanessa choose from?
   (b) If Vanessa buys two potted flowers and she wants two different combinations, how many options does she have? (Two combinations are different if at least the colours of the flowers or the types of the pots are different.)
2. **(5 points)** In how many ways can five out of eight people be seated on a sofa if there are only five seats available.
3. **(5 points)** Three cards are drawn at random (without replacement) from an ordinary deck of 52 cards.[1] Find the number of ways in which one can draw
   (a) a diamond and a club and a heart in this order;
   (b) one hearts and then two clubs or two spades.
4. **(10 points)** How many numbers, consisting of five different digits each, can be made from the digits $1, 2, 3, \ldots, 9$ if
   (a) the numbers must be even?
   (b) exactly two of the digits are odd?
   (c) How many numbers are in (a) and (b) if repetitions of the digits are allowed?
5. **(10 points)** According to Wikipedia[2] a new scheme was introduced for license plates in the Netherlands. The short summary of the scheme is shown below. In practice, some letters are excluded to avoid confusion and obscene language; however, *we ignore this* in the exercise!

   Answer the following questions and give brief explanations.

   (a) Consider only the image in the second entry of the table above. How many possible license plates can be made with this setup. (We assume that 26 letters can be used where X is displayed, and 10 digits can be used where 9 is displayed.)

[1] The four suits are as follows. spades ♠, hearts ♥, diamonds ♦ and clubs ♣.

[2] `https://en.wikipedia.org/w/index.php?title=Vehicle_registration_plates_of_the_Netherlands&oldid=682267623`

2015–present	XXX-99-X	Side Code 11	This series is currently issued for mopeds (DBB-01-B till FZZ-99-Z)

Table 1: New license plates scheme in the Netherlands

(b) In the fourth column of Table 1 a range is provided for mopeds. (Note that it is a range, so for example `EAA-01-A` is in the range!) Using this, count how many license plates can be issued for mopeds?

(c) Assuming that there are only cars and mopeds in this scheme, can you tell what kind of vehicle has the license plate `DFA-78-V`? And `DAF-78-A`? And `DCF-78-V`?

(d) Assuming that there are only cars and mopeds in this scheme, how many cars can be with this type of a license plate?

6. **(10 points)** Expand the following expressions, either directly or via binomial coefficients. Make it clear how you proceed.

   (a) $(x-7)^3$;

   (b) $(x+2y)^4$;

   (c) $(x^3-3)^4$;

7. **(5 points)** Consider a team of 11 players wishing each other luck before a match. If everyone shakes everyone else's hand exactly once, how many handshakes occur?

8. **(5 points)** From four consonants and five vowels, how many six-letter words can be formed consisting of three different consonants and three different vowels? The words need not have a meaning.

9. **(5 points)** A school has four maths teachers, three English teachers and three IT teachers. From this whole group, a five teacher committee has to be established. Calculate the number of ways that this committee can be formed if at least one IT teacher must be on the committee.

10. **(10 points)** There are twelve provinces in the Netherlands. Suppose that the birth of Dutch people is uniformly distributed over all twelve provinces, *i.e.* for every province $\frac{1}{12}$ of the population is born there. What is the probability that at least two of r randomly selected Dutch-born people were born in the same province, where $r=1$, $r=2$, $r=3$, $r=4$, $r=5$ or $r=6$?

11. **(10 points)** We toss a coin twice.

    (a) Give a corresponding sample space S.

    (b) Give the set of outcomes corresponding to each of the following events:

       i. A: "we throw heads exactly once";

       ii. B: "we throw heads at least once";

       iii. C: "tails did not appear before a head appeared".

    (c) Give a probability measure P_1 for the sample space S.

12. **(5 points)** Let P be a probability measure on space S. Prove that $P(\emptyset)=0$.

13. **(bonus, +5 points)** Let P be a probability measure on space S. Prove that for pairwise mutually exclusive events $A_1, A_2, \ldots A_n$ (*i.e.* $A_i \cap A_j = \emptyset$ for all $i \neq j$ and $n \geq 2$) one has $P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$. (Hint: use induction.)

14. **(10 points)** An experiment consists of drawing 3 cards in succession from a well-shuffled ordinary deck of cards.[3] Let A_1 denote the event "Ace on first draw", A_2 the event "Ace on second draw" and A_3 the event "Ace on third draw". State in words the meaning of each of the following probabilities.

    (a) $P(A_1 \cap \neg A_2)$;

    (b) $P(A_1 \cup A_2)$;

    (c) $P(\neg A_1 \cap \neg A_2 \cap \neg A_3)$;

    (d) $P[(A_1 \cap \neg A_2) \cup (\neg A_2 \cap A_3)]$.

[3] If you are not familiar with the standard 52-card deck, visit Wikipedia: `http://en.wikipedia.org/wiki/Standard_52-card_deck`.