Calculus and Probability Theory

Assignment 1, February 2, 2017

Handing in your answers:

- submission via Blackboard (`http://blackboard.ru.nl`);
- one single pdf file (make sure that if you scan/photo your handwritten assignment, the result is clearly readable);
- all of your solutions are clearly and convincingly explained;
- please make sure that you write your name and your student number on the assignment.

Deadline: Friday, February 10, 14:30 sharp!

Goals: After completing these exercises successfully you should be confident with the following topics:

- Even and odd functions
- The domain and range of a function
- The limit of a function

Marks: You can score a total of 100 points.

1. **(10 points)** Let $f(x) = x - x^3$. Determine the values x for which

$$1.\ f(x) = 0; \qquad 2.\ f(x) > 0.$$

Solution:

We note that $x - x^3 = x(1-x^2) = x(1-x)(1+x)$ so $f(x) = 0$ for $x \in \{-1, 0, 1\}$. Now note that $(1-x^2) > 0 \iff |x| < 1 \iff x \in (-1, 1)$. So we get four cases:

(a) $x < -1$. Then $x < 0$ and $(1-x^2) < 0$, so $f(x) > 0$.

(b) $x \in (-1, 0)$. Then $x < 0$ and $(1-x^2) > 0$, so $f(x) < 0$.

(c) $x \in (0, 1)$. Then $x > 0$ and $(1-x^2) > 0$, so $f(x) > 0$.

(d) $x > 1$. Then $x > 0$ and $(1-x^2) < 0$, so $f(x) < 0$.

[[**Grading Instruction:**

Grading (total 10):	
aspect:	points
correct results for $f(x) = 0$	5
correct results for $f(x) > 0$	5
small mistakes	-2

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2. **(10 points)** Let's assume that x runs through the interval $(0, 1)$. What values does y run through for $y = a + (b-a)x$, where $a, b \in \mathbb{R}$?

Solution:

We distinguish 3 cases. Clearly, when $a = b$, we have $y = \{a\}$. Furthermore,

(a) $b > a$. Then $(b-a) \geq 0$. Using the fact that $x \in (0,1)$ we see that $y = a + (b-a)x > a$ and $y = a + (b-a)x < a + (b-a) = b$. So y takes values in the interval (a,b). Does y reach every value in this interval? Let c be in (a,b). Then

$$c = a + (b-a)x_c \iff x_c = \frac{c-a}{b-a}.$$

Because c lies in (a,b), we see that x_c lies in $(0,1)$. So for every element c in (a,b), there exists an element x_c in $(0,1)$ such that $c = a + (b-a)x_c$. Hence y runs through all values in (a,b).

(b) $b < a$. Then $(b-a) < 0$. Again using $x \in (0,1)$ we find $y = a + (b-a)x < a$ and $y = a + (b-a)x > a + (b-a) = b$, so y takes values in (b,a). Using the same approach as above we find that y runs through all values in (b,a).

[[**Grading Instruction:**

Grading (total 10):	
aspect:	points
distinguish $b > a$ and $b < a$	2
recognize that y is in (a,b) or (b,a)	5
every value in (a,b) or (b,a) is reached	3

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3. **(10 points)** Are the following functions even or odd? In your explanation use the definition.

(a) $f(x) = 3x - x^3$;

(b) $f(x) = \sqrt[3]{(1-x)^2} + \sqrt[3]{(1+x)^2}$;

Solution:

(a) $f(-x) = 3(-x) - (-x)^3 = -3x + x^3 = -(3x - x^3) = -f(x)$, so odd.

(b) $f(-x) = \sqrt[3]{(1-(-x))^2} + \sqrt[3]{(1+(-x))^2} = \sqrt[3]{(1+x)^2} + \sqrt[3]{(1-x)^2} = f(x)$, so even.

[[**Grading Instruction:**

Grading (total 10):	
aspect:	points
(a)	5
(b)	5

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4. **(10 points)** What is the inverse of

$$y = \frac{ax+b}{cx+d} \qquad (ad - bc \neq 0)?$$

When is it equal to the original function?

Solution:

We find $y(cx+d) = ax + b$ which results in $x(cy-a) = -dy + b$ so that $x = \frac{-dy+b}{cy-a}$. Hence the inverse function g is given by $g(x) = \frac{-dx+b}{cx-a}$. It is equal to f when $d = -a$. Note that g is defined for all x except $x = \frac{a}{c}$. But $f(x) = \frac{a}{c} \iff ad - bd = 0$, so g is still defined on the whole range of f.

[[**Grading Instruction:**

Grading (total 10):	
aspect:	points
g	5
$d = -a$	5
small mistakes	-3

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5. **(30 points)** Determine the domains and ranges of the following functions.
   (a) $f(x) = \sqrt{7 - x^2} + 1$;
   (b) $f(x) = \frac{x-5}{x^2-3x-10}$;
   (c) $f(x) = \frac{1}{|x|}$.

Solution:

Let $D(f)$ and $R(f)$ be the domain respectively the range of the function f.

(a) To correctly compute the square root we need $7-x^2 \geq 0$, i.e. $-\sqrt{7} \leq x \leq \sqrt{7}$. So $\underline{\underline{D(f) = [-\sqrt{7}, \sqrt{7}]}}$. As $0 \leq 7 - x^2 \leq 7$ we have $0 \leq \sqrt{7-x^2} \leq \sqrt{7}$, so $1 \leq f(x) \leq \sqrt{7}+1$. Hence $\underline{\underline{R(f) = [1, \sqrt{7}+1]}}$.

(b) The only elements which are not in the domain are the points where the denominator evaluates to 0. We note that $x^2 - 3x - 10 = (x-5)(x+2)$ so that $\underline{\underline{D(f) = \mathbb{R} \setminus \{-2, 5\}}}$ (all points in $\mathbb{R}$ except -2 and 5). This also means that $f(x) = \frac{1}{x+2}$ if $x \neq -2, 5$. The range of $\frac{1}{x+2}$ is $\mathbb{R} \setminus \{0\}$, but we cannot include the value at $x = 5$. So $\underline{\underline{R(f) = \mathbb{R} \setminus \{0, \frac{1}{7}\}}}$.

(c) Clearly the domain only excludes $x = 0$, i.e. $\underline{\underline{D(f) = \mathbb{R} \setminus \{0\}}}$. The range is $\underline{\underline{R(f) = (0, \infty)}}$.

[[**Grading Instruction:**

Grading (total 30):	
aspect:	points
correct domain	6
correct range	4
small mistakes	-2

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6. **(30 points)** Find the limits. (Hint: try to simplify as much as possible before applying the limit!)
   (a) $\lim\limits_{x\to 0} \frac{3(x-1)+3}{x}$;
   (b) $\lim\limits_{x\to 2} \frac{x-2}{x^2+x-6}$;
   (c) $\lim\limits_{x\to 1} \frac{x^2-4x+3}{x^2+x-2}$.

Solution:

(a) We note that for all $x \neq 0$ we have $\frac{3(x-1)+3}{x} = \frac{3x-3+3}{x} = \frac{3x}{x} = 3$. So we must have $\underline{\underline{\lim\limits_{x\to 0} \frac{3(x-1)+3}{x} = 3}}$.

(b) We note that for all $x \neq 2$ we have $\frac{x-2}{x^2+x-6} = \frac{x-2}{(x-2)(x+3)} = \frac{1}{x+3}$. So $\lim\limits_{x\to 2} \frac{x-2}{x^2+x-6} = \lim\limits_{x\to 2} \frac{1}{x+3} = \underline{\underline{\frac{1}{5}}}$.

(c) For all $x \neq 1$ we have $\frac{x^2-4x+3}{x^2+x-2} = \frac{(x-1)(x-3)}{(x-1)(x+2)} = \frac{x-3}{x+2}$. So $\lim\limits_{x\to 1} \frac{x^2-4x+3}{x^2+x-2} = \lim\limits_{x\to 1} \frac{x-3}{x+2} = \underline{\underline{-\frac{2}{3}}}$.

[[**Grading Instruction:**

Grading (total 30):

aspect:	points
correct limit	10
small mistakes	-3

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