Calculus and Probability Theory
Assignment 5

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After completing these exercises successfully you should be confident with the following topics:

- familiar with definite and indefinite integrals
- able to apply the most important integration methods, more specifically, substitution and integration by parts
- confident about switching between different representations of a function
- able to compute area of a finite or infinite region
- able to apply the formula for the arc length of a function over a finite interval

1. (**20 points**) Compute the following indefinite integrals. You can use *substitution* or *integration by parts*. In each problem *verify* your result, and don't forget about the constant term. You may need some of the following, well-known trigonometric identities:

$$\sin(2x) = 2\sin(x)\cos(x), \quad \cos(2x) = \cos^2(x) - \sin^2(x), \quad \sin^2(x) + \cos^2(x) = 1$$

Also, it is highly recommended to consult with the lecture slides and solve the problems there before you start with these ones.

(a) $\int \sin(x)\cos(x)\,dx$
Solution:

Applying substitution seems to be the best approach here. Note: everytime a factor in the integrand is a derivative of the other factor in the integrand, it is a good idea to use substitution instead of integration by parts.

Substitution

$$\int_a^b f(u(x)) \cdot u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$$

Let $u = \sin(x)$. Then $du = \cos(x)dx$. So, $dx = \frac{du}{\cos(x)}$

$$\begin{aligned}\int \sin(x)\cos(x)\ dx &= \int u\cos(x)\frac{du}{\cos(x)} \\ &= \int u\ du \\ &= \frac{1}{2}u^2 \\ &= \frac{1}{2}\sin^2(x) + C\end{aligned}$$

(b) $\int \ln(ax)\ dx$ where $a > 0$
Solution:

Integration by parts

$$\int_a^b u(x) \cdot v'(x)dx = [u(x) \cdot v(x)] - \int_a^b u'(x) \cdot v(x)dx$$

$$\int \ln(ax)\ dx = \int ln(ax) \cdot 1\ dx$$

Let $v' = 1$ and $u = ln(ax) \rightarrow v = x$ and

$$u' = (ln(ax))' = \frac{1}{ax} \cdot a = \frac{1}{x}$$

$$\begin{aligned}\int ln(ax) \cdot 1\ dx &= [ln(ax) \cdot x] - \int \frac{1}{x} \cdot x\ dx \\ &= [ln(ax) \cdot x] - \int 1\ dx \\ &= [ln(ax) \cdot x] - x + C \\ &= x(ln(ax) - 1) + C\end{aligned}$$

(c) $\int \cos^2(x)\ dx$
Solution:

Using Integration by parts:

$$\int \cos^2(x)\ dx = \int cos(x)cos(x)\ dx$$

Let $u = cos(x), u' = -sin(x), v = sin(x), v' = cos(x)$

$$\begin{aligned}\int \cos^2(x) &= [cos(x)sin(x)] - \int -sin(x)sin(x)\ dx \\ &= [cos(x)sin(x)] + \int sin^2(x)\ dx\end{aligned}$$

Using trigonometric identity: $sin^2(x) + cos^2(x) = 1$

$$\begin{aligned}\int \cos^2(x) &= [cos(x)sin(x)] + \int 1 - cos^2(x)\ dx \\ \int \cos^2(x) &= [cos(x)sin(x)] + \int 1\ dx - \int cos^2(x)\ dx \qquad | + \int cos^2(x)\ dx \\ 2\int \cos^2(x) &= [cos(x)sin(x)] + \int 1\ dx \\ 2\int \cos^2(x) &= [cos(x)sin(x)] + x + C \qquad | \cdot \frac{1}{2} \\ \int \cos^2(x) &= \frac{1}{2}cos(x)sin(x) + x + C\end{aligned}$$

(d) $\int \frac{1}{\sqrt{1-4x^2}}\ dx$
Solution:

Using Substitution and the fact that $\int \frac{1}{\sqrt{1-x^2}}\ dx = arcsin(x) + C$ (It's given as a well know indefinite integral in the slides, so I assume that I can use it.)

Let $u^2 = 4x^2 \rightarrow u = 2x$. Then $du = 2\ dx$. So, $dx = \frac{1}{2}du$

$$\begin{aligned}\int \frac{1}{\sqrt{1-4x^2}}\ dx &= \int \frac{1}{\sqrt{1-u^2}}\ \frac{1}{2}\ du \\ &= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\ du \\ &= \frac{1}{2}arcsin(2x) + C\end{aligned}$$

(e) $\int e^{3x}\sin(x)\ dx$
Solution:

Using integration by parts.

Let $u = e^{3x}, u' = e^{3x}3, v = -cos(x), v' = sin(x)$

$$\begin{aligned}\int e^{3x}\sin(x)\,dx &= \left[-\cos(x)\cdot e^{3x}\right] - \int -\cos(x)\cdot 3e^{3x} \\ &= \left[-\cos(x)\cdot e^{3x}\right] - \left[-3\int cos(x)\cdot e^{3x}\right]\end{aligned}$$

Applying Integration by parts one more time.
Let $u = e^{3x}, u' = e^{3x}3, v = v = sin(x), v' = cos(x)$

$$\begin{aligned}\int e^{3x}\sin(x)\,dx &= \left[-\cos(x)\cdot e^{3x}\right] - \left[-3\left[e^{3x}sin(x) - \int e^{3x}3sin(x)dx\right]\right] \\ &= \left[-\cos(x)\cdot e^{3x}\right] - \left[-3\left[e^{3x}sin(x) - 3\int e^{3x}sin(x)dx\right]\right] \\ &= 3\left(e^{3x}sin(x) - 3\int sin(x)e^{3x}dx\right) - e^{3x}cos(x) \\ &= -\frac{1}{10}e^{3x}(cos(x) - 3sin(x)) + C\end{aligned}$$

2. (**20 points**) Compute the length of the curve $f(x) = \sqrt{1-x^2}$ where $x \in [-1, 1]$

   (a) using calculus, and
   Solution:

   We can use the definition of arc length to come to a solution:
   Let f be a differentiable function on $[a, b]$. The arc length of f on this interval is

$$\int_a^b \sqrt{1 + f'(x)^2}\,dx$$

$$\begin{aligned}f(x) &= \sqrt{1-x^2} = (1-x^2)^{\frac{1}{2}} \\ f'(x) &= \frac{1}{2}(1-x^2)^{-\frac{1}{2}}\cdot(-2x) \\ &= -x(1-x^2)^{-\frac{1}{2}} \\ &= -\frac{x}{\sqrt{1-x^2}}\end{aligned}$$

   Therefore:

$$\int_{-1}^{1}\sqrt{1 + \left(-\frac{x}{\sqrt{1-x^2}}\right)^2}\,dx = \int_{-1}^{1}\sqrt{1 + \frac{x^2}{1-x^2}}\,dx$$

Apply Integration by parts:

$$\int \sqrt{1+\frac{x^2}{1-x^2}}\, dx$$

Let $u = \sqrt{1+\frac{x^2}{1-x^2}}$, $u' = \frac{x}{(1-x^2)^{\frac{3}{2}}}$, $v' = 1$, $v = x$

$$\int \sqrt{1+\frac{x^2}{1-x^2}}\, dx = \sqrt{1+\frac{x^2}{1-x^2}}x - \int \frac{x}{(1-x^2)^{\frac{3}{2}}}x\, dx$$

$$\int \frac{x^2}{(1-x^2)^{\frac{3}{2}}}\, dx = \frac{x}{\sqrt{1-x^2}} - arcsin(x) + C$$

$$\int \sqrt{1+\frac{x^2}{1-x^2}}\, dx = x\sqrt{1+\frac{x^2}{1-x^2}} - \left(\frac{x}{\sqrt{1-x^2}} - arcsin(x)\right) + C$$

Computing the definite integrals:

$$\int_{-1}^{1} \sqrt{1+\left(-\frac{x}{\sqrt{1-x^2}}\right)^2}\, dx = \left[x\sqrt{1+\frac{x^2}{1-x^2}} - \left(\frac{x}{\sqrt{1-x^2}} - arcsin(x)\right)\right]_{-1}^{1}$$

Applying L'Hopital's rule gives us

$$\begin{aligned}\int_{-1}^{1} \sqrt{1+\left(-\frac{x}{\sqrt{1-x^2}}\right)^2}\, dx &= \frac{\pi}{2} - (-\frac{\pi}{2}) \\ &= \pi\end{aligned}$$

(b) using geometric argument. [Hint: what is the shape of $\sqrt{1-x^2}$]?
Solution:

When we look at the roots of the function $\sqrt{1-x^2}$, then we can clearly see that the roots are at -1 and 1. After plugging in all possible x- and y-intercepts, we discover that the curve is representing a circle. A unit circle to be precise.
We know that the unit circle has a perimeter of 2π. The given task ask for the interval from -1 to 1, which represents the half perimeter of the unit circle which is $\frac{2\pi}{2} = \pi$. Even without knowing the unit circle, we could compute the perimeter by the known formular of $2\pi r$ where r is the radius. In this case, the radius is 1 and therefore we come to the same solution of π.

3. (**20 points**) Compute the definite integral $\int_{-1}^{1} \sqrt{1-x^2}\, dx$

(a) using calculus [hint: instead of substituting a function of x by u, now substitute $x = \sin(u)$.]
Solution:

Let $x = sin(u)$. Then $du = \frac{dx}{cos(u)}$. So, $dx = cos(u)du$

$$\int \sqrt{1-x^2}\, dx = \int \sqrt{1-sin^2(u)}cos(u)\, du$$

Using trigonometric identity: $1 - sin^2(x) = cos^2(x)$

$$\begin{aligned}
\int \sqrt{1-sin^2(u)}cos(u)\, du &= \int \sqrt{cos^2(u)}cos(u)\, du \\
&= \int cos(u)cos(u)\, du \\
&= \int cos^2(u)\, du \\
&= \int \frac{1+cos(2u)}{2}\, du \\
&= \frac{1}{2}\int 1 + cos(2u)\, du \\
&= \frac{1}{2}(u + \frac{1}{2}sin(2u)) + C \\
&= \frac{1}{2}\left(arcsin(x) + \frac{1}{2}sin(2arcsin(x))\right) + C
\end{aligned}$$

$$\begin{aligned}
\int_{-1}^{1} \sqrt{1-x^2}\, dx &= \left[\frac{1}{2}\left(arcsin(x) + \frac{1}{2}sin(2arcsin(x))\right)\right]_{-1}^{1} \\
&= \frac{1}{2}\pi
\end{aligned}$$

(b) using geometric argument?
Solution:

The previous exercise asked for the length of the arc which was π. This exercise asks for half of the area enclosed by the unit circle which has a radius of 1. The area we wanted to compute is enclosed by the function which draws the unit circle and the x-axis. The center of this unit circle is at the origin. Therefore, the x-axis cuts the unit circle in half. A circle of radius r has the area πr^2. The unit's circle radius is 1 and therefore the enclosed area is equal to $\frac{1}{2}\pi 1^2 = \frac{1}{2}\pi$.

4. (**15 points**) Compute the following improper integrals.

(a) $\int_0^\infty re^{-r^2}\ dr$;
Solution:

Apply Substitution.
Let $u = -r^2$. $\frac{du}{dr} = -2r$, $du = -2r\ dr$, $dr = (-\frac{1}{2r})du$

$$
\begin{aligned}
\int re^{-r^2}\ dr &= \int re^u(-\frac{1}{2r})du \\
&= \int -\frac{e^u}{2}du \\
&= -\frac{1}{2}\int e^u\ du \\
&= -\frac{1}{2}e^u + C \\
&= -\frac{1}{2}e^{-r^2} + C \\
&= -\frac{e^{-r^2}}{2} + C
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\infty re^{-r^2}\ dr &= \lim_{b\to\infty}\left[-\frac{e^{-r^2}}{2}\right]_0^b \\
&= \lim_{b\to\infty}\left[(-\frac{e^{-b^2}}{2}) - (-\frac{e^{-0^2}}{2})\right] \\
&= \lim_{b\to\infty}\left[(-\frac{e^{-b^2}}{2}) + \frac{1}{2}\right] \\
&= 0 + \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}
$$

(b) $\int_0^{2\pi}(\int_0^\infty re^{-r^2}\ dr)\ dt$;
Solution:

$$\int_0^{2\pi}(\int_0^\infty re^{-r^2}\ dr)\ dt = \int_0^{2\pi}\frac{1}{2}\ dt$$

$$\int \frac{1}{2}\ dt = \frac{1}{2}t + C$$

$$\int_0^{2\pi} (\int_0^{\infty} re^{-r^2}\ dr)\ dt = \left[\frac{1}{2}t\right]_0^{2\pi}$$
$$= \left[\frac{1}{2}2\pi\right] - \left[\frac{1}{2}0\right]$$
$$= \pi$$

(c) (bonus, +3 points) Prove that $\int_{-\infty}^{\infty} e^{-z^2}\ dz = \sqrt{x}$.
You may use the fact that $\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx)\ dy = \int_0^{2\pi}(\int_0^{\infty} re^{-r^2}\ dr)\ dt$
Solution:

(d) $\int_0^{\infty} e^{-z^2}\ dz$
Solution:

5. (**15 points**) Compute the following improper integrals.

(a) $\int_0^{\infty} e^{-x}\ dx$
Solution:

Using Substitution
Let $u = -x$. $\frac{du}{dx} = -1$, $du = -1dx$, $dx = -1du$

$$\int e^{-x}\ dx = \int -e^u\ du$$
$$= -\int e^u\ du$$
$$= -e^u + C$$
$$= -e^{-x} + C$$

$$\int_0^{\infty} e^{-x}\ dx = \lim_{b\to\infty} \left[-e^{-x}\right]_0^b$$
$$= \lim_{b\to\infty} \left[(-e^{-b}) - (-e^{-0})\right]$$
$$= \lim_{b\to\infty} \left[(-e^{-b}) - (-1)\right]$$
$$= 0 + 1$$
$$= 1$$

(b) $\int_0^{\infty} xe^{-x}\ dx$ using integration by parts;
Solution:

Using Integration by parts.
Let $u = x, u' = 1, v' = e^{-x}, v = -e^{-x}$

$$\begin{aligned}
\int xe^{-x}\,dx &= x(-e^{-x}) - \int 1\cdot(-e^{x})\,dx \\
&= -e^{-x}x - \int -e^{-x}\,dx \\
&= -e^{-x}x - e^{-x} + C
\end{aligned}$$

$$\begin{aligned}
\int_0^\infty xe^{-x}\,dx &= \lim_{b\to\infty}\left[-e^{-x}x - e^{-x}\right]_0^b \\
&= \lim_{b\to\infty}\left[(-e^{-b}b - e^{-b}) - (-e^{-0}0 - e^{-0})\right] \\
&= \lim_{b\to\infty}\left[(-e^{-b}b - e^{-b}) - (-1)\right] \\
&= 0 + 1 \\
&= 1
\end{aligned}$$

(c) (bonus, +2 points) $\int_0^\infty x^n e^{-x}\,dx$ for all $n \in \{0, 1, ...\}$
[Hint: Try first for n = 0,1,2,3]
Solution:

(d) $\int_0^\infty x^{-\frac{1}{2}} e^{-x}\,dx$
[Hint: Substitute $u = \sqrt{x}$ and, at the end, some information from a previous exercise turns out to be useful.]
Solution:

Applying Substitution.
Let $u = \sqrt{x}$, $\frac{du}{dx} = \frac{1}{2u}$, $du = \frac{1}{2u}dx$, $dx = 2udu$

$$\begin{aligned}
\int x^{-\frac{1}{2}} e^{-x}\,dx &= \int \frac{e^{-x}}{\sqrt{x}}\,dx \\
&= \int \frac{e^{-x}}{u} 2udu \\
&= 2e^{-x}du
\end{aligned}$$

$$u = \sqrt{x} \Rightarrow x = u^2$$

$$
\begin{aligned}
\int x^{-\frac{1}{2}} e^{-x}\, dx &= \int 2e^{-u^2}\, du \\
&= 2\int e^{-u^2} du \\
&= 2\frac{\sqrt{\pi}}{2} erf(u) + C \\
&= 2\frac{\sqrt{\pi}}{2} erf(\sqrt{x}) + C
\end{aligned}
$$

erf = error function. I have no idea what this means, I had to use the wisdom of the internet for that.

$$
\begin{aligned}
\int_0^{\infty} x^{-\frac{1}{2}} e^{-x}\, dx &= \lim_{b\to\infty} \left[2\frac{\sqrt{\pi}}{2} erf(\sqrt{x})\right]_0^b \\
&= \lim_{b\to\infty} \left[(2\frac{\sqrt{\pi}}{2} erf(\sqrt{x})) - (2\frac{\sqrt{\pi}}{2} erf(\sqrt{x}))\right]
\end{aligned}
$$

6. (**10 points**) .

(a) Given three lines, $y = x + 2, y = -x + 6$ and $y = 2x - 3$ enclosing a triangle. Determine the *coordinates* of the three vertices and the *area* of the triangle.
Solution:

(b) Compute the area of the region bounded by $y = (x - 1)^3$ and $y = (x - 1)^2$
Solution:

7. (**bonus, 5 points**) The figure shows a horizontal line $y = c$ intersecting the curve $y = -(x-2)^2 + 4$. Find the number c such that the areas of the shaded regions are equal.