Calculus and Probability Theory

Assignment 4, February 23, 2017

Handing in your answers:

- submission via Blackboard (`http://blackboard.ru.nl`);
- one single pdf file (make sure that if you scan/photo your handwritten assignment, the result is clearly readable);
- all of your solutions are clearly and convincingly explained;
- make sure to write your name, your student number.

Deadline: Friday, March 3, 14:30 sharp!

Goals: After completing these exercises successfully you should be able to:

- analyse and sketch real functions;
- apply differentiation rules to determine higher-order partial derivatives;
- find primitives of well-known functions;
- compute definite integrals when the primitive function is known;
- compute improper integrals.

Marks: You can score a total of 100 points (and 16 bonus points) There are three bonus exercises.

1. **(20 points)** Investigate the function $f(x) = \frac{x}{\ln(x)}$ as follows. (Do not start with drawing a graph by means of a device or some web resource. Of course you may check your result when you're done.)

   (a) Determine the domain of the function f.

   (b) What are the roots of f?

   (c) Determine the limits at 1 and ∞. (Hint: there are 3 cases, use l'Hôpital!)

   (d) Find f' and f''.

   (e) Find the zeros of f' and f''.

   (f) What are the critical points (determine their x and y coordinates)?

   (g) Find the local minimums and maximums.

   (h) Which parts of the function are convex and concave? Does function f have points of inflection? (Hint: Use the sign of the second derivative for answering both questions.)

   (i) Draw the graph of function f. (If you collect all intervals and special points in a table, it helps a lot in drawing the graph. Moreover, you get some extra points!)

   Solution:

   (a) $(0, 1) \cup (1, \infty)$;

   (b) There are no zeros.

   (c) i. $\lim_{x\to\infty} \frac{x}{\ln(x)} = \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty$

   ii. $\lim_{x\to 1^+} \frac{x}{\ln(x)} = \infty$

   iii. $\lim_{x\to 1^-} \frac{x}{\ln(x)} = -\infty$

   For the first we used l'Hôpital (condition: $\frac{\infty}{\infty}$).

   (d) Use the quotient rule in both cases. We get $f'(x) = \frac{1\cdot\ln(x) - x\cdot 1/x}{(\ln(x))^2} = \frac{\ln(x)-1}{(\ln(x))^2} = \frac{1}{\ln(x)} - \frac{1}{(\ln(x))^2}$ and $f''(x) = \frac{\partial}{\partial x}\left[\frac{1}{\ln(x)} - \frac{1}{(\ln(x))^2}\right] = \frac{-1}{x(\ln(x))^2} - \frac{-2}{x(\ln(x))^3} = \frac{2-\ln(x)}{x(\ln(x))^3}$.

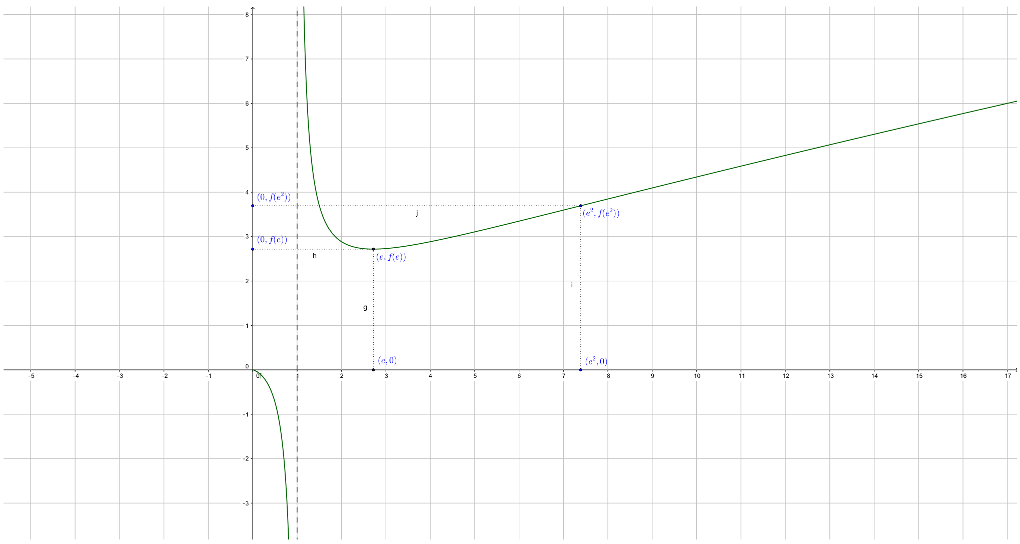
(e) From the above we can easily read that $f'(x) = 0 \iff \ln(x) = 1 \iff x = e$ and $f''(x) = 0 \iff \ln(x) = 2 \iff x = e^2$.

(f) The only critical point (i.e., with $f'(x) = 0$) has $x = e$, and for this we have $f(e) = \frac{e}{\ln(e)} = e$. So (e, e) is the only critical point.

(g) We have $f''(e) = \frac{1}{e} > 0$, so (e, e) is a local minimum. Another method is to look at $f'(x)$: since $f'(x) < 0$ if $x < e$ and $f'(x) > 0$ if $x > e$, (e, e) is a local minimum. The domain has no edge points or points in which f is not differentiable (since f is not defined in 0 and 1) so there are no other local extrema.

(h) i. on $(0, 1)$, $f''(x) < 0$ thus concave (since $f''(\frac{1}{e}) = -3e < 0$)
ii. on $(1, e^2)$, $f''(x) > 0$ thus convex (since $f''(e) = \frac{1}{e} > 0$)
iii. on (e^2, ∞), $f''(x) < 0$ thus concave (since $f''(e^3) = \frac{-1}{9e^3} < 0$)
iv. on e^2, $f''(e^2) = 0$ which is an inflection point as $f''(x)$ changes from convex to concave.

(i) Sketch:



[[**Grading Instruction:**

Grading (total 20):	
aspect:	points
(a)	1
(b)	1
(c) three simple limits	2
(c) limit with L'Hôpital and condition	2
(d) f'	2
(d) f''	2
(e) zero of f'	1
(e) no zero of f''	1
(f)	1
(g)	2
(h)	4
(i)	1
small mistake	-2

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2. **(6 points)** Show that the derivative of an odd function is even and that the derivative of an even function is odd.

Solution: Assume f is odd, i.e., $f(-x) = -f(x)$. Then because of the chain rule $\frac{\partial}{\partial x} f(-x) = f'(x) \cdot -1 = -f'(-x)$, but when f is odd we also have $\frac{\partial}{\partial x} f(-x) = \frac{\partial}{\partial x} - f(x) = -f'(x)$. Hence $f'(-x) = f'(x)$ and f' is even. Analogously, when f is even, i.e., $f(-x) = f(x)$ then (chain rule) $\frac{\partial}{\partial x} f(-x) = -f'(-x)$ and (f even) $\frac{\partial}{\partial x} f(-x) = f(x) = f'(x)$. Thus $f'(-x) = -f'(x)$ and f' is odd.

[[**Grading Instruction:**

Grading (total 6 points):	
aspect:	points
correct answer, per part	3

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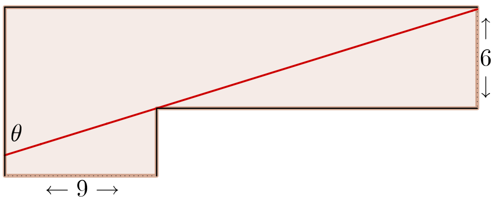
3. **(14 points)** Optimization problem

   (a) Find the point on the parabola $y^2 = 2x$ that is closest to $A = (1, 4)$.

   (b) A steel rod is carried down a hallway of 9 meter wide. At the end there is corner to the right into a narrower hallway of 6 meter wide. What is the maximum length of the steel rod that can be carried horizontally around the corner?



   (Hint: What happens at $\theta \to 0$ and $\theta \to \frac{1}{4}\pi$? Show that the angle at which the minimum is obtained is at $\theta = \arctan\left(\sqrt[3]{\frac{3}{2}}\right) \approx 0,853$.)

Solution:

(a) Let $B = (a, b)$ be this point. Then $b^2 = 2a \Leftrightarrow a = \frac{1}{2}b^2$. The squared distance $D = d(A, B)^2 = (1-a)^2 + (4-b)^2 = (1 - \frac{1}{2}b^2)^2 + (4-b)^2$. We find the minimum by differentiating D w.r.t. b, i.e., $\frac{\partial}{\partial b} D = 2 \cdot (1 - \frac{1}{2}b^2) \cdot -b + 2 \cdot (4-b) \cdot -1 = b^3 - 8$ which is 0 when $b = 2$. Then $a = \frac{1}{2}2^2 = 2$ and the point with minimum distance is $(2, 2)$. (Note: mimimizing the squared distance gives the same result as minimizing the distance but is easier to work with. An alternative solution can be found by substituting $b = \sqrt{2a}$ and taking the derivative w.r.t. a.)

(b) Paradoxically, we solve the maximization problem by solving a minimization problem. Let L be the lenght of the rod going from wall to wall. If $\theta \to 0$ or $\theta \to \frac{\pi}{2}$ then $L \to \infty$, but for some angle θ there will be a minimum length L such that the rod just fits around the corner. Let $L = L_1 + L_2$ with L_1 the lenght in the lower corridor, L_2 the length in the upper corridor. Then $\sin\theta = \frac{9}{L_1} \Leftrightarrow L_1 = \frac{9}{\sin\theta}$ and $\cos\theta = \frac{6}{L_2} \Leftrightarrow L_2 = \frac{6}{\cos\theta}$.The angle at which the minimum length is obtained is found by differentiating wrt θ and setting equal to 0. Thus $\frac{\partial}{\partial\theta} L = \frac{-9\cos\theta}{\sin^2\theta} + \frac{-6 - \sin\theta}{\cos^2\theta} = \frac{-9\cos^3\theta + 6\sin^3\theta}{\sin^2\theta\cos^2\theta}$ which is 0 when $6\sin^3\theta = 9\cos^3\theta \Leftrightarrow \tan^3\theta = \frac{9}{6} \Leftrightarrow \theta = \arctan\left(\sqrt[3]{\frac{3}{2}}\right) \approx 0,853$. Substituting this in the equation for L gives us $L \approx 21,07$ meter.

[[**Grading Instruction:**

Grading (total 6 points):	
aspect:	points
correct answer, per part	7

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4. **(20 points)** Given function f, find the partial derivatives. If it is necessary, simplify the result.

   (a) $f(x, y) = \cos(4y - xy)$; $\frac{\partial}{\partial x} f(x, y) = ?$ and $\frac{\partial}{\partial y} f(x, y) = ?$

(b) Show that $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}f(x,y)\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}f(x,y)\right)$.

Solution:

(a) First all partial derivatives for $f(x,y) = \cos(4y - xy)$.

i. $\frac{\partial f(x,y)}{\partial x} = -\sin(4y - xy) * (-y) = y\sin(4y - xy)$;

ii. $\frac{\partial f(x,y)}{\partial y} = -\sin(4y - xy)(4 - x)$;

(b) Second, the higher order partial derivatives

i. $\frac{\partial^2 f(x,y)}{\partial y \partial x} = \sin(4y - xy) + y\cos(4y - xy)(4 - x)$;

ii. $\frac{\partial^2 f(x,y)}{\partial x \partial y} = -\cos(4y - xy) * (-y) * (4 - x) + \sin(4y - xy) = \sin(4y - xy) + y\cos(4y - xy)(4 - x)$.

[[**Grading Instruction:**

Grading (total 20 points):	
aspect:	points
(a) per correct partial derivative	4
(b) per correct partial derivative	6

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5. **(20 points)** Evaluate the following definite integrals.

(a) $\int_{-1}^{1}(x^3 + x - 1)\ dx$;

(b) $\int_{1}^{2}(3\sqrt{x} + \frac{3}{x^2})\ dx$;

(c) $\int_{0}^{\pi}(\sin(x) + \cos(x))\ dx$;

(d) $\int_{1}^{e}\frac{1-\ln x}{x^2}\ dx$

Solution:

(a)

$$\begin{aligned}
\int_{-1}^{1}(x^3 + x - 1)\ dx &= \int_{-1}^{1} x^3\ dx + \int_{-1}^{1} x\ dx + \int_{-1}^{1} -1\ dx \\
&= \frac{1}{4}x^4|_{-1}^{1} + \frac{1}{2}x^2|_{-1}^{1} + (-x)|_{-1}^{1} \\
&= \frac{1}{4} - \frac{1}{4} + \frac{1}{2} - \frac{1}{2} - 1 - 1 \\
&= \underline{\underline{-2}}.
\end{aligned}$$

(b)

$$\begin{aligned}
\int_{1}^{2}(3\sqrt{x} + \frac{3}{x^2})\ dx &= 3\left(\int_{1}^{2}\sqrt{x}\ dx + \int_{1}^{2}\frac{1}{x^2}\ dx\right) \\
&= 3\left(\frac{2}{3}x^{\frac{3}{2}}|_{1}^{2} + \frac{-1}{x}|_{1}^{2}\right) \\
&= 3\left(\frac{2}{3}(2\sqrt{2} - 1) + \frac{1}{2}\right) \\
&= \underline{\underline{4\sqrt{2} - \frac{1}{2}}}.
\end{aligned}$$

(c)

$$\begin{aligned}
\int_{0}^{\pi}(\sin(x) + \cos(x))\ dx &= \int_{0}^{\pi}\sin(x)\ dx + \int_{0}^{\pi}\cos(x)\ dx \\
&= -\cos(x)|_{0}^{\pi} + \sin(x)|_{0}^{\pi} \\
&= 1 + 1 + 0 + 0 \\
&= \underline{\underline{2}}.
\end{aligned}$$

(d)

$$\begin{aligned}\int_1^e \frac{1-\ln x}{x^2}\, dx &= \frac{\ln x}{x}\Big|_1^e \\ &= \frac{1}{e} - 0 \\ &= \underline{\underline{\frac{1}{e}}}.\end{aligned}$$

[[**Grading Instruction:**

Grading (5 points per part; 20 points total):

aspect:	points
method	3
answer	2

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6. **(20 points)** Evaluate the following improper integrals.

(a) $\int_{-1}^{1} \frac{1}{x^n}\, dx$, n an integer such that $n \geq 2$; (Hint: distinguish two cases)

(b) $\int_{-\infty}^{-\pi/2} \frac{x\cos(x)-\sin(x)}{x^2}\, dx$; (Hint: use the quotient rule for derivation to find the primitive)

(c) $\int_2^\infty \frac{-1}{x\ln^2(x)}\, dx$. (Hint: use a fraction of well known functions to find the primitive)

Solution:

(a)

$$\begin{aligned}\int_{-1}^{1} \frac{1}{x^n}\, dx &= \lim_{\epsilon\to 0^+} \int_{-1}^{-\epsilon} \frac{1}{x^n}\, dx + \int_{\epsilon}^{1} \frac{1}{x^n}\, dx \\ &= \lim_{\epsilon\to 0^+} \left(\frac{-1}{n-1} x^{-n+1}\Big|_{-1}^{-\epsilon} + \frac{-1}{n-1} x^{-n+1}\Big|_{\epsilon}^{1} \right) \\ &= \frac{1}{n-1} \lim_{\epsilon\to 0^+} \left(-x^{-n+1}\Big|_{-1}^{-\epsilon} - x^{-n+1}\Big|_{\epsilon}^{1} \right) \\ \text{Case } n \text{ is odd}: &= \frac{1}{n-1} \lim_{\epsilon\to 0^+} \left(-\epsilon^{-n+1} + 1 - 1 + \epsilon^{-n+1} \right) = \underline{\underline{0}} \\ \text{Case } n \text{ is even}: &= \frac{1}{n-1} \lim_{\epsilon\to 0^+} \left(\epsilon^{-n+1} - 1 - 1 + \epsilon^{-n+1} \right) = \underline{\underline{\infty}}.\end{aligned}$$

(b)

$$\begin{aligned}\int_{-\infty}^{-\pi/2} \frac{x\cos(x)-\sin(x)}{x^2}\, dx &= \lim_{b\to\infty} \left(\frac{\sin(x)}{x}\Big|_{-b}^{-\pi/2} \right) \\ &= \lim_{b\to\infty} \left(\frac{-1}{-\pi/2} - \frac{\sin(b)}{b} \right) \\ &= \underline{\underline{\frac{2}{\pi}}}.\end{aligned}$$

(c)

$$\begin{aligned}\int_2^\infty \frac{-1}{x\ln^2(x)}\, dx &= \lim_{b\to\infty} \left(\frac{1}{\ln(x)}\Big|_2^b \right) \\ &= \lim_{b\to\infty} \left(\frac{1}{\ln(b)} - \frac{1}{\ln(2)} \right) \\ &= \underline{\underline{\frac{-1}{\ln(2)}}}.\end{aligned}$$

[[**Grading Instruction:**

Grading (total 20 points):	
aspect:	points
correct answer, per part	5
correct reasoning, total	5

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7. **(bonus, +6 points)** Find primitives of the following functions f. That is, find F such that $F'(x) = f(x)$.

   (a) $f(x) = \frac{1}{2\sqrt{x}} - \frac{1}{x^2}$;

   (b) $f(x) = 2\sin(x)\cos(x)$;

   (c) $f(x) = \frac{2}{1+4x^2}$;

   Solution:

   (a) $F(x) = \sqrt{x} + \frac{1}{x}$;

   (b) $F(x) = \sin^2(x)$;

   (c) $F(x) = \arctan(2x)$;

   [[**Grading Instruction:**

Grading (total 6 points):	
aspect:	points
2 point per part	

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8. **(bonus, 10 points)** If $f(x, y) = \frac{xy}{x+y}$, show that

$$x^2 \cdot \frac{\partial}{\partial x}\big(\frac{\partial}{\partial x} f(x,y)\big) \;+\; 2xy \cdot \frac{\partial}{\partial x}\big(\frac{\partial}{\partial y} f(x,y)\big) \;+\; y^2 \cdot \frac{\partial}{\partial y}\big(\frac{\partial}{\partial y} f(x,y)\big) \;=\; 0.$$

   (Hint: First compute all the second partial derivatives of f, then substitute the results in the expression on the left-hand side.)

   Solution: Compute the partial derivatives.

   (a) $\frac{\partial f}{\partial x} = \frac{y(x+y)-xy}{(x+y)^2} = \frac{y^2}{(x+y)^2}$;

   (b) $\frac{\partial^2 f}{\partial^2 x} = \frac{-2(x+y)y^2}{(x+y)^4} = \frac{-2y^2}{(x+y)^3}$;

   (c) $\frac{\partial f}{\partial y} = \frac{x(x+y)-xy}{(x+y)^2} = \frac{x^2}{(x+y)^2}$;

   (d) $\frac{\partial^2 f}{\partial^2 y} = \frac{-2(x+y)x^2}{(x+y)^4} = \frac{-2x^2}{(x+y)^3}$;

   (e) $\frac{\partial^2 f}{\partial y \partial x} = \frac{2y(x+y)^2-2(x+y)y^2}{(x+y)^4} = \frac{2y(x+y)-2y^2}{(x+y)^3} = \frac{2xy}{(x+y)^3}$.

   So the identity becomes

$$x^2\frac{-2y^2}{(x+y)^3} + 2xy\frac{2xy}{(x+y)^3} + y^2\frac{-2x^2}{(x+y)^3}$$

   which is clearly zero.

   [[**Grading Instruction:**

Grading (total 10 points):	
aspect:	points
for all (of at most the three) correct second order derivatives	5
concluding that identity holds	5
small mistakes	-2

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