Calculus and Probability Theory
Assignment 3

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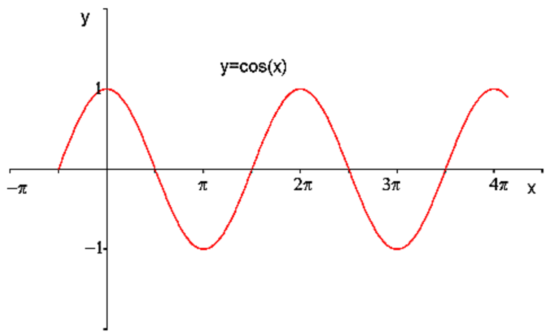
1. (**15 points**) As we learnt in the lecture (slide 33), the function arccos is the inverse function of cos.

   (a) What it the domain of the function arccos(x)? Why?
   Solution:

   As we already know from the cosine function, the domain of the cosine function is $\mathbb{R}$ and its range is $[-1, 1]$

   An Inverse function is a function that "reverses" another function: if the function f applied to an input x gives a result of y, then applying its inverse function g to y gives the reult x and vice versa.

   Let f be a function whose domain is the set X and whose range is the set Y. Then f is invertible if there exists a function g with domain Y and *range* X, with property

   $$f(x) = y \leftrightarrow g(y) = x$$

   Because we already know, that arccos is the inverse function of cosine, its domain has to be the range of cosine, which is $[-1, 1]$.

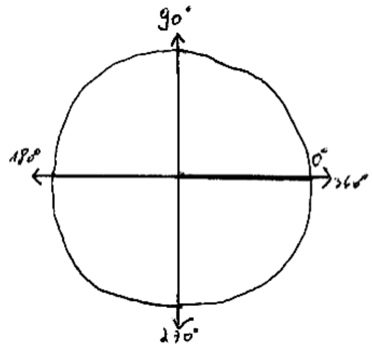
(b) Compute the following values and explain how you got the result:

- *arccos*(1) =?
  Solution:

$$\arccos x = \theta$$
$$\cos\theta = x$$

Question: What angle θ can i take the cosine of to get x?

$$\arccos(1) = \theta$$
$$\cos\theta = 1$$



From the picture of the cosine function in 1a) it is easy to see that the *arccos*(1) is 0. We can also take a look at the unit circle and read it that way.

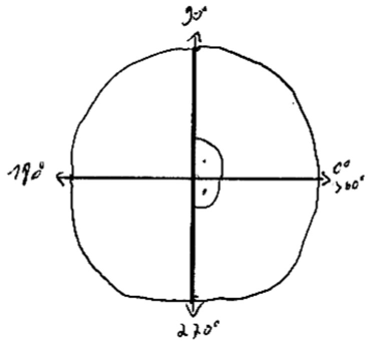
- *arccos*(0) =?
  Solution:

$$\arccos x = \theta$$
$$\cos\theta = x$$

Question: What angle θ can i take the cosine of to get x?

$$\arccos(0) = \theta$$
$$\cos\theta = 0$$

Again, we can just read the answer from the cosine function in 1a) and ask: where does the cosine function hit the x-axis? And



the answer is $\frac{\pi}{2}$.

Another explanation can be taken from the unit circle. If we let the cosine be 0, then there is an angle of 90. Therefore:

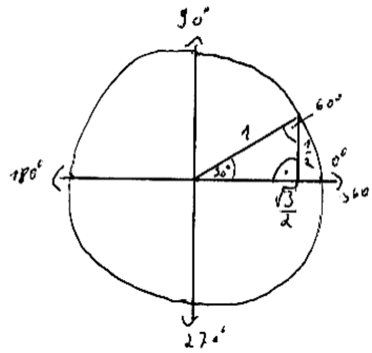
$90 \cdot \frac{\pi}{180} = \frac{\pi}{2}$

- $arccos(\frac{\sqrt{3}}{2}) =?$
  Solution:

$$\arccos x = \theta$$
$$\cos\theta = x$$

Question: What angle θ can i take the cosine of to get x?

$$\arccos(\frac{\sqrt{3}}{2}) = \theta$$
$$\cos\theta = \frac{\sqrt{3}}{2}$$



In this scenario, we have to work with the unit circle. We know, we have to draw the length of the arccos to be $\frac{\sqrt{3}}{2}$ as we see in the

picture.
After that, we can use pythagoras' theorem to compute the opposite/adjacent, which givues us $\frac{1}{2}$. We already know that the hypothenuse is 1, because it's a unit circle.

After that, we get an angle of 30 and therefore:

$30 \cdot \frac{\pi}{180} = \frac{\pi}{6}$

(c) Find the derivative of f:

$$f(x) = arccos(\frac{2x}{1-x})$$

Solution:

$$(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}$$

Using the chain rule $\frac{d}{dx}(cos^{-1}(\frac{2x}{1-x})) = \frac{d\cos^{-1}(u)}{du}\frac{du}{dx}$, where $u = \frac{2x}{1-x}$ and $\frac{d}{du}(\cos^{-1}(u)) = -\frac{1}{\sqrt{1-u^2}}$

$= -\frac{2\frac{d}{dx}(\frac{2x}{1-x})}{\sqrt{1-\frac{4x^2}{(1-x)^2}}}$

Use the quotient rule, $\frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$ where $u = x$ and $v = 1-x$

$= -\frac{2((1-x)(\frac{d}{dx}(x))-\frac{d}{dx}(1)-\frac{d}{dx}(x))}{(1-x)^2\sqrt{1-\frac{4x^2}{(1-x)^2}}}$

$= -\frac{2(x+1(1-x))}{(1-x)^2\sqrt{1-\frac{4x^2}{(1-x)^2}}}$

$= -\frac{2}{(1-x)^2\sqrt{1-\frac{4x^2}{(1-x)^2}}}$

2. (**15 points**) Given the functions $f(x) = log_2(4x)$ and $g(x) = sin(2x)$.

(a) What is $f'''(x)$?
Solution:

$$\frac{d}{dx}(\log_2(4x)) = \log_2(4x)' = \frac{\frac{d}{dx}(ln(4x))}{ln(2)}$$
$$= \frac{\frac{\frac{d}{dx}(4x)}{4x}}{\ln(2)}$$
$$= \frac{4\frac{d}{dx}(x)}{4x\ln(2)}$$
$$= \frac{\frac{d}{dx}(x)}{x\ln(2)}$$
$$= \frac{1}{x\ln(2)}$$

$$\frac{d}{dx}(\frac{1}{x\ln(2)}) = \log_2(4x)'' = \frac{\frac{d}{dx}(\frac{1}{x})}{\ln(2)}$$
$$= \frac{\frac{-1}{x^2}}{\ln(2)}$$
$$= -\frac{1}{x^2\ln(2)}$$

$$\frac{d}{dx}(-\frac{1}{x^2\ln(2)}) = \log_2(4x)''' = -\frac{\frac{d}{dx}(\frac{1}{x})}{\ln(2)}$$
$$= -\frac{\frac{-1}{x^2}}{\ln(2)}$$
$$= \frac{1}{x^2\ln(2)}$$

(b) What is $g^{(2014)}(x)$
Solution:

$\sin(2x)' = \frac{d}{dx}(\sin(2x)) = \cos(2x)(\frac{d}{dx}(2x)) = 2\cos(2x)$

$\sin(2x)'' = \frac{d}{dx}(2\cos(2x)) = -4\sin(2x)$

$\sin(2x)''' = \frac{d}{dx}(-4\sin(2x)) = -8\cos(2x)$

$\sin(2x)'''' = \frac{d}{dx}(-8\cos(2x)) = 16\sin(2x)$

$g^{2014}(x) = 2^{2014}\cdot(-\sin(2x))$

3. (**25 points**) Investigate function $f = (x-1)^2(x+2)$ by following the steps below. (Do not start with drawing a graph by means of a device or some web resource. Of course you may check your result when you are done.)

   (a) Determine the domain of function f.

   $D(f(x)) = \{x \in \mathbb{R}\}$

   (b) What are the roots of f? Where does the graph of f intersect the y axis?

   $f = (x-1)^2(x+2) = x^3 - 3x + 2$
   $x^3 - 3x + 2 = 0$

   First root by guessing: $1^3 - 3 + 2 = 1 - 3 + 2 = 0$
   $x = 1$

   Polynomdivision: $(x^3 - 3x + 2) : (x - 1) = x^2 + x - 2$

   ABC-Formula: $ax^2 + bx + c = 0 \rightarrow x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ or $\frac{-b-\sqrt{b^2-4ac}}{2a}$

   $x^2 + x - 2 = 0 \rightarrow x = \frac{-1-\sqrt{1^2+8}}{2} = \frac{-1-3}{2} = -2$

   Roots:

   $$x = -2$$
   $$x = 1$$

   Intersection with y-axsis: $f(0) = 0^3 - 3 \cdot 0 + 2 = 2$

   (c) Determine the limits at the edges of the domain. In this case, there are only two edges:

   $$\lim_{x\to-\infty} f(x) \qquad and \qquad \lim_{x\to+\infty} f(x)$$

   (d) Find f' and f''.

   $f = (x-1)^2x + 2 = x^3 - 3x + 2$
   $f' = \frac{d}{dx}(x^3 - 3x + 2) = 3x^2 - 3 = x(x^2 - 1)$

   $f'' = \frac{d}{dx}(3x^2 - 3) = 6x$

   (e) Find the zeros of f' and f''.

   $$f'' = x(x^2 - 1) \rightarrow x(x^2 - 1) = 0$$

Roots: $x = -1, x = 0, x = 1$

$$f'' = 6x \to 6x = 0$$

Roots: $x = 0$

(f) What are the critical points (determine their x and y coordinates)?

(g) Find the local minimums and maximums.

(h) Which parts of the function are convex and concave? Does the function f have points of inflection? (Hint: Use the sign of the second derivative for answering both questions.)

(i) Draw the graph of function f. (If you collect all intervals and special points in a table, it helps a lot in drawing the graph. Moreover, you get some extra points!)

4. (**25 points**) We will sketch the function

$$f(x) = \frac{x^2}{x-2}$$

following similar steps as the ones in the previous problem. Additionaly, we prove that the line $y = x+2$ is an asymptote on both sides. (Again, do not start with drawing a graph by means of a device or some web resource. Of course you may check your result when you are done.)

(a) Determine the domain of function f.

$D(f(x)) = \{x \in \mathbb{R} | x \neq 2\}$

(b) What are the roots of f? Where does the graph of f intersect the y axis?

Roots:

$$\begin{aligned} \frac{x^2}{x-2} &= 0 \\ \frac{x^2}{x-2} \cdot (x-2) &= 0 \\ x^2 &= 0 \\ x &= 0 \end{aligned}$$

The only root is $x = 0$

Intersection with y-axsis: $f(0) = \frac{0^2}{0-2} = 0$

(c) Determine the limits at the edges of the domain.

(d) Find f' and f''.

$$
\begin{aligned}
f' &= \frac{d}{dx}(\frac{x^2}{x-2}) \\
&= \frac{-x^2(\frac{d}{dx}(-2+x)) + (-2+x)(\frac{d}{dx}(x^2)}{(-2+x)^2} \\
&= \frac{-(x^2(\frac{d}{dx}(x))) + (-2+x)(\frac{d}{dx}(x^2))}{(-2+x)^2} \\
&= \frac{(-2+x)(\frac{d}{dx}(x^2)) - x^2}{(-2+x)^2} \\
&= \frac{-x^2 + (-2+x)2x}{(-2+x)^2} \\
&= \frac{(x-4)x}{(x-2)^2}
\end{aligned}
$$

$$
\begin{aligned}
f'' &= \frac{d}{dx}(\frac{(x-4)x}{(x-2)^2}) \\
&= \frac{x(\frac{d}{dx}(-4+x))}{(-2+x)^2} + (-4+x)(\frac{d}{dx}(\frac{x}{(-2+x)^2})) \\
&= \frac{x}{(-2+x)^2} + (-4+x)(\frac{d}{dx}(\frac{x}{(-2+x)^2})) \\
&= \frac{x}{(-2+x)^2} + (-4+x)(\frac{\frac{d}{dx}(x)}{(-2+x)^2} + \frac{-2\frac{d}{dx}(-2+x)}{(x-2)^3}x) \\
&= \frac{x}{(-2+x)^2} + (-4+x)(\frac{1}{(-2+x)^2} - \frac{2x}{(-2+x)^3}) \\
&= \frac{8}{(x-2)^3}
\end{aligned}
$$

(e) Find the zero of f' and f''.

$$f' = \frac{(x-4)x}{(x-2)^2} \rightarrow \frac{(x-4)x}{(x-2)^2} = 0$$

Roots for f':

$$x = 0$$
$$x = 4$$

$$f'' = \frac{8}{(x-2)^3} \rightarrow \frac{8}{(x-2)^3} = 0$$

Roots for f'': No roots exist.

(f) What are the critical points (determine their x and y coordinates)?

(g) Find the local minimums and maximums.

(h) Which parts of the function are convex and concave? Does function f have points of inflection? (Hint: Use the sign of the second derivative for answering both questions.)

(i) Show that the line $y = x + 2$ is a slant asymptote of f. (Hint: Use the definition on slide 41 of the lecture and the following two limits.)

$$\lim_{x \to -\infty} (f(x) - (x+2)) =? \qquad and \qquad \lim_{x \to +\infty} (f(x) - (x+2)) =?$$

(j) Draw the graph of function f.

5. (**12 points**) Given function f, find the partial derivatives. If it is necessary, simplify the result.

   (a) $f(x,y) = \sin(3x + 3xy)$; $\frac{\partial f(x,y)}{\partial x}$ =? and $\frac{\partial f(x,y)}{\partial y}$ =?

   (b) $f(x,y) = \ln(\frac{y}{x})$; $\frac{\partial f(x,y)}{\partial x}$ =? and $\frac{\partial f(x,y)}{\partial y}$ =?

6. (**8 points**) If $f(x,y) = \frac{xy}{x-y}$, show that

$$x^2 \frac{\partial^2 f(x,y)}{\partial x^2} + 2xy \frac{\partial^2 f(x,y)}{\partial x \partial y} + y^2 \frac{\partial^2 f(x,y)}{\partial y^2}$$

(Hint: First compute all the second partial derivatives of f, then substitute the results in the expression on the left-hand side.)