# Calculus and Probability Theory

**Assignment 3, February 16, 2017**

**Handing in your answers:**

- submission via Blackboard (`http://blackboard.ru.nl`);
- one single pdf file (make sure that if you scan/photo your handwritten assignment, the result is clearly readable);
- all of your solutions are clearly and convincingly explained;
- make sure to write your name, your student number

**Deadline: Friday, February 24, 14:30 sharp!**

**Goals:** After completing these exercises successfully you should be able to:

- apply all differentiation rules on elementary and transcendental functions;
- solve problems including higher-order derivatives;
- apply l'Hôpital's rules when applicable;
- analyse graphs of a given real function.

**Marks:** You can score a total of 100 points.

1. **(10 points)** The function arcsin is the inverse function of sin.

   (a) What is the domain of the function $\arcsin(x)$? Why?

   (b) Compute the following values and explain how you got the result:

   $$\arcsin(1) = ? \qquad \arcsin(0) = ? \qquad \arcsin\left(\frac{\sqrt{3}}{2}\right) = ?$$

   (c) Find the derivative of $f$:

   $$f(x) = \arcsin\left(\frac{2x}{1-x}\right).$$

   **Solution:**

   (a) $[-1, 1]$, as this is the range of $\sin(x)$;

   (b) $\sin(0) = 0$, $\sin(\pi/2) = 1$, $\sin(\pi/3) = \frac{\sqrt{3}}{2}$, so the results are respectively $\pi/2$, 0 and $\pi/3$.

   (c) Use the chain rule and the known derivative of arcsin. Alternatively, use the inverse rule. So we get $f'(x) = \frac{1}{\sqrt{1-(\frac{2x}{1-x})^2}} \frac{2}{(1-x)^2} = \frac{1-x}{\sqrt{1-2x-3x^2}} \frac{2}{(1-x)^2} = \frac{2}{(1-x)\sqrt{(x+1)(-3x+1)}}$.

   **[[Grading Instruction:**

   | **Grading (total 10):** | |
   |---|---|
   | aspect: | points |
   | (a) (with explanation) | 2 |
   | (b) explanation | 2 |
   | (b) 3 correct results | 2 |
   | (c) | 4 |
   | small mistake | -2 |

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2. **(15 points)** Find the limits of the following functions. (Note that before you can apply L'Hôpital's rule, you have to verify whether it is possible.)

   (a) $\lim_{x\to\infty} \frac{e^{n-x}}{x^{-m}}$ with $m, n \in \mathbb{N}$; (Hint: if unclear first solve a particular case, e.g., $n = 0, m = 3$.)

   (b) If $\lim_{x\to 0} \frac{\sqrt[3]{(a\cdot x+b)}-2}{x} = \frac{5}{12}$ with $a, b \in \mathbb{N}$ then $a \cdot b = ?$;

   (c) If $\lim_{x\to 0} \frac{\sin(x)+Ax+Bx^3}{x^5} = \frac{1}{C}$ with $A, B, C \in \mathbb{Q}$, then $A \cdot B \cdot C = ?$.

   **Solution:**

   (a) $\lim_{x\to\infty} \frac{e^{n-x}}{x^{-m}} = \lim_{x\to\infty} \frac{e^{-(x+n)}}{x^{-m}} = \lim_{x\to\infty} \frac{x^m}{e^{(x+n)}} \stackrel{!}{=} \lim_{x\to\infty} \frac{m!}{e^{(x+n)}} = 0$
   i.e., at some point the exponential grows quicker than any polynomial. Note that at the ! sign we have $\frac{\infty}{\infty}$ and applied L'Hôpital's rule $m$ times.

   (b) We have $\frac{\sqrt[3]{b}-2}{0}$ which is $\pm\infty$ unless $\sqrt[3]{b} - 2 = 0$, hence $b = 8$. Then we have $\frac{0}{0}$ and can apply L'Hôpital's rule $\lim_{x\to 0} \frac{\sqrt[3]{(a\cdot x+8)}-2}{x} = \lim_{x\to 0} \frac{1}{3}(ax + b)^{-\frac{2}{3}} \cdot a$ which goes to $\frac{1}{3} \cdot 8^{-\frac{2}{3}} \cdot a = \frac{a}{12}$, hence $a = 5$. Thus $a \cdot b = 5 \cdot 8 = 40$.

   (c) Since we have $\frac{0}{0}$, L'Hopital gives: $\lim_{x\to 0} \frac{\cos(x)+A+3Bx^2}{5x^4} = \frac{1+A}{0}$ which is $\pm\infty$ unless $A = -1$. Then we have $\frac{0}{0}$ and can apply L'Hopital (repeatedly): $\lim_{x\to 0} \frac{\cos(x)+A+3Bx^2}{5x^4} \stackrel{!}{=} \lim_{x\to 0} \frac{-\sin(x)+6Bx}{20x^3} \stackrel{!}{=} \lim_{x\to 0} \frac{-\cos(x)+6B}{60x^2} = \frac{-1+6B}{0}$ which is $\pm\infty$ unless $-1+6B = 0$, hence $B = \frac{1}{6}$. Continuing with L'Hopital gives $\lim_{x\to 0} \frac{-\cos(x)+6B}{60x^2} \stackrel{!}{=} \lim_{x\to 0} \frac{\sin(x)}{120x} \stackrel{!}{=} \lim_{x\to 0} \frac{\cos(x)}{120}$, hence $C = 120$. Then $A \cdot B \cdot C = -1 \cdot \frac{1}{6} \cdot 120 = -20$.

   [[**Grading Instruction:**

   | **Grading (total 15):** | |
   |---|---|
   | aspect: | points |
   | (a) (with check the conditions) | 5 |
   | (b) (with check the conditions) | 5 |
   | (c) (with check the conditions) | 5 |
   | small mistake | -2 |

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3. **(10 points)** Given the functions $f(x) = \log_3(2x)$ and $g(x) = \cos(3x)$.

   (a) What is $f'''(x)$?

   (b) What is $g^{(2015)}(x)$? (Hint: Start with finding the first few derivatives of $g$.)

   **Solution:**

   (a) $f'''(x) = (\log_3(2x))''' = (\frac{1}{\ln 3}x^{-1})'' = (-\frac{1}{\ln 3}x^{-2})' = \frac{2}{\ln 3}x^{-3}$.

   (b) i. $f^{(1)}(x) = -3\sin(3x)$;
   ii. $f^{(2)}(x) = -3^2\cos(3x)$;
   iii. $f^{(3)}(x) = 3^3\sin(3x)$;
   iv. $f^{(4)}(x) = 3^4\cos(3x)$;
   The remainder of 2015 divided by 4 is 3. From this we have $f^{(2015)}(x) = 3^{2015}\sin(3x)$.

   [[**Grading Instruction:**

   | **Grading (total 10):** | |
   |---|---|
   | aspect: | points |
   | (a) first derivative | 2 |
   | (a) second derivative | 2 |
   | (a) final result | 1 |
   | (b) idea | 3 |
   | (b) final computation | 2 |

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4. **(5 points)** For which values of $c$ has the equation $\ln x = cx^2$ precisely one solution. (Hint: There is a value $0.1 < c < 0.2$ for which the curves just touch each other. What do these curves also have in common, besides the point of intersection?)

   **Solution:** Let $a$ be the point at which the curves just touch each other: $\ln a = ca^2$. In this point $a$ **the gradient of both curves must be similar**, or else they would intersect or not meet. Hence, $\frac{1}{a} = 2ca$ or equivalently $\frac{1}{2} = ca^2$. Using this in the first equation gives $\ln a = \frac{1}{2}$ or $a = e^{\frac{1}{2}}$. Using this in the second equation and solving for $c$ gives $c = \frac{1}{2e}$. Clearly, the equations also intersect in one point when $c$ is negative or zero, thus the full answer is $c \in (-\infty, 0] \cup \{\frac{1}{2e}\}$.

   [[**Grading Instruction:**

| **Grading (total 5):** | |
|---|---|
| aspect: | points |
| (a) $c \in (-\infty, 0]$ | 1 |
| (b) $= \frac{1}{2e}$ | 4 |

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5. **(25 points)** Investigate function $f = (x+1)^2(x-3)$ by following the steps below. (Do not start with drawing a graph. Of course, you may check your solution with GeoGebra or with some other tool.)

   (a) Determine the domain of function $f$.

   (b) What are the roots of $f$? What is the $y$-intercept, that is, where is the intersection of the graph of $f$ and the $y$-axis?

   (c) Determine the limits at the edges of the domain. In this case, there are only two edges:

   $$\lim_{x \to -\infty} f(x) \qquad \text{and} \qquad \lim_{x \to +\infty} f(x).$$

   (d) Find $f'$ and $f''$.

   (e) Find the zeros of $f'$ and $f''$.

   (f) What are the critical points (determine their $x$ and $y$ coordinates)?

   (g) Find the local minima and maxima.

   (h) Which parts of the function are convex and concave? Does function $f$ have points of inflection? (Hint: Use the sign of the second derivative for answering both questions.)

   **Solution:**

   (a) $\mathbb{R}$;

   (b) Two roots $x_0 = -1$ and $x_1 = 3$. Have $f(0) = -3$.

   (c) Square goes to positive, so only depends on $x - 3$. We get

   $$\lim_{x \to -\infty} f(x) = -\infty \qquad \text{and} \qquad \lim_{x \to +\infty} f(x) = +\infty.$$

   (d) Use product rule. $f'(x) = ((x^2+2x+1)(x-3))' = (2x+2)(x-3) + x^2 + 2x + 1 = 3x^2 - 2x - 5$. Now easily see $f''(x) = 6x - 2$.

   (e) $f'(x) = (x+1)(3x-5)$ so $x = -1$ and $x = \frac{5}{3}$ are the zeros of $f'(x)$. Easily see that $x = \frac{1}{3}$ is the only zero of $f''(x)$.

   (f) We just found the $x$-coordinates of the critical points. We have $f(-1) = 0$ and $f(\frac{5}{3}) = -\frac{2^8}{3^3} \approx 9.5$. So get the points $(-1, 0)$ and $(\frac{5}{3}, \frac{2^8}{3^3})$.

   (g) We have $f''(-1) = -14 < 0$, so it is a maximum, and $f''(\frac{5}{3}) = 13 > 0$, so it is a minimum. These are the only critical points, so there are no more local minima and maxima.

   (h) The function is concave $\iff f''(x) < 0 \iff 6x - 2 < 0 \iff x < \frac{1}{3}$. The function is convex $\iff f'' > 0 \iff 6x - 2 > 0 \iff x > \frac{1}{3}$. It has a point of inflection at $x = \frac{1}{3}$.

[[**Grading Instruction:**

| **Grading (total 25):** | |
|---|---|
| aspect: | points |
| (a) | 2 |
| (b) | 3 |
| (c) | 4 |
| (d) $f'$ | 3 |
| (d) $f''$ | 2 |
| (e) | 2 |
| (f) | 2 |
| (g) | 2 |
| (h) | 5 |
| small mistake | -2 |

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6. **(25 points)** We will investigate the function

$$f(x) = \frac{(x-2)^2}{x+2}.$$

following similar steps as the ones in the previous problem. Additionally, we prove that the line $y = x - 6$ is a slant asymptote on both sides.

(a) Determine the domain of function $f$.

(b) What are the roots of $f$? Where does the graph of $f$ intersect the $y$ axis?

(c) Determine the limits at the edges of the domain.

(d) Find $f'$ and $f''$.

(e) Find the zeros of $f'$ and $f''$.

(f) What are the critical points (determine their $x$ and $y$ coordinates)?

(g) Find the local minima and maxima.

(h) Which parts of the function are convex and concave? Does function $f$ have points of inflection? (Hint: Use the sign of the second derivative for answering both questions.)

(i) Show that the line $y = x - 6$ is a slant asymptote of $f$. (Hint: Use the definition on slide 47 of the lecture and the following two limits.)

$$\lim_{x\to-\infty} (f(x) - (x-6)) = ? \qquad \text{and} \qquad \lim_{x\to+\infty} (f(x) - (x-6)) = ?$$

**Solution:**

(a) $\mathbb{R} \setminus \{-2\}$;

(b) Only zero is $x_0 = 2$. We have $f(0) = 2$.

(c) We should consider limits to infinity, and limits from both sides to -2. This gives $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to-2^+} f(x) = \infty$, $\lim_{x\to-2^-} f(x) = -\infty$.

(d) Use the quotient rule. It gives $f'(x) = \frac{(2x-4)(x+2)-(x^2-4x+4)}{(x+2)^2} = \frac{x^2+4x-12}{x^2+4x+4}$. Using the quotient rule again we find $f''(x) = \frac{(2x+4)(x^2+4x+4)-(2x+4)(x^2+4x-12)}{(x+2)^4} = \frac{32}{(x+2)^3}$.

(e) $f'(x) = 0 \iff x^2 + 4x - 12 = 0 \iff (x+6)(x-2) = 0$. So get two points $x = -6$ and $x = 2$. It is immediate that $f''$ has no zeros.

(f) We have $f(2) = 0$ and $f(-6) = -16$, so the two points $(2, 0)$ and $(-6, -16)$.

(g) As $f''(2) = \frac{32}{4^3} > 0$, this is a minimum, and as $f''(-6) = \frac{32}{(-4)^3} < 0$, this is a maximum.

(h) We see that $f''(x) > 0 \iff \frac{32}{(x+2)^3} > 0 \iff x > -2$ and $f''(x) < 0 \iff \frac{32}{(x+2)^3} < 0 \iff x < -2$. So convex for $x > -2$ and concave for $x < -2$. There are no points such that $f''(x) = 0$, so no points of inflection.

(i) We have $(f(x)-(x-6))=\frac{16}{x+2}$, so from this we see that $\lim\limits_{x\to\pm\infty}(f(x)-(x-6))=0$. Hence it is a slant asymptote.

[[**Grading Instruction:**

| **Grading (total 25):** | |
|---|---|
| aspect: | points |
| (a) | 2 |
| (b) | 2 |
| (c) | 4 |
| (d) $f'$ | 3 |
| (d) $f''$ | 3 |
| (e) | 1 |
| (f) | 1 |
| (g) | 3 |
| (h) | 2 |
| (i) | 4 |
| small mistake | -2 |

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7. **(10 points)**

(a) Find the derivative of $f(x)=\ln(\cos(\ln(\cos(x))))$.

(b) Find a function $g(x)$ such that $g'(x)=\tan(2x)$.

(c) Find three functions $f_1, f_2, f_3$ such that $f_1'(x)=f_2'(x)=f_3'(x)=\sin(x)\cos(x)$.

**Solution:**

(a) $f'(x)=(\ln(\cos(\ln(\cos(x)))))'=\frac{1}{\cos(\ln(\cos(x)))}\cdot(-\sin(\ln(\cos(x))))\cdot\frac{1}{\cos(x)}\cdot(-\sin(x))=\tan(\ln(\cos(x)))\cdot\tan(x)$.

(b) From the previous problem we see that $(\ln(\cos(x)))'=-\tan(x)$. Thus, $(\ln(\cos(2x)))'=-2\tan(x)$ and $-\frac{1}{2}\ln(\cos(2x))'=\tan(x)$. So, $g(x)=-\frac{1}{2}\ln(\cos(2x))+C$ where $C\in\mathbb{R}$. In grading the constant is not important here.

(c) $f_i(x)=\frac{1}{2}\sin^2(x)+C_i$. Pick 3 distinct values for $C_i$. ($\frac{-1}{4}\cos(2x)+C_i$ also works.)

[[**Grading Instruction:**

| **Grading (total 10):** | |
|---|---|
| aspect: | points |
| (a) | 4 |
| (b) | 3 |
| (c) | 3 |
| small mistake | -2 |

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