Calculus and Probability Theory

Assignment 2, February 9, 2017

Handing in your answers:

- submission via Blackboard (`http://blackboard.ru.nl`);
- one single pdf file (make sure that if you scan/photo your handwritten assignment, the result is clearly readable);
- all of your solutions are clearly and convincingly explained;
- make sure to write your name, your student number

Deadline: Friday, February 17, 14:30 sharp!

Goals: After completing these exercises successfully you should be confident with the following topics:

- Limits, possibly involving infinity
- The definition of the derivative
- The tangent line of a function
- Differentiation rules of (special) functions

Marks: You can score a total of 100 points (and 8 possible bonus points).

1. **(15 points)** Find the limits. (Hint: try to simplify as much as possible before applying the limit!)

   (a) $\lim\limits_{x\to-\infty} \frac{x^3+2x^2+2}{3x^3+x+4}$;

   (b) $\lim\limits_{x\to\infty} \frac{3x^2+8}{x+1}$;

   (c) $\lim\limits_{x\to\infty} \frac{2x+1}{x^2+x}$;

   (d) $\lim\limits_{x\to a} \frac{x^n-a^n}{x-a}$ for some parameter $a \in \mathbb{R}$. (*Hint*: Do you recognize this limit? If not, you can always simplify the fraction using long division)

   Solution:

   (a) As x is tending to (minus) infinity we can surely assume $x \neq 0$. So dividing the numerator and denominator by x^3 we get $\frac{x^3+2x^2+2}{3x^3+x+4} = \frac{1+\frac{2}{x}+\frac{2}{x^3}}{3+\frac{1}{x^2}+\frac{4}{x^3}}$. Hence the numerator goes to 1, while the denominator goes to 3 as x goes to infinity. Hence $\underline{\underline{\lim\limits_{x\to-\infty} \frac{x^3+2x^2+2}{3x^3+x+4} = \frac{1}{3}}}$

   (b) Again assume $x \neq 0$ and divide by x^2. We find $\frac{3x^2+8}{x+1} = \frac{3+\frac{8}{x^2}}{\frac{1}{x}+\frac{1}{x^2}}$. So the numerator goes to 3, while the denominator goes to 0 as x goes to infinity. Hence $\underline{\underline{\lim\limits_{x\to\infty} \frac{3x^2+8}{x+1} = \infty}}$.

   (c) Divide by x^2 to get $\frac{2x+1}{x^2+x} = \frac{\frac{2}{x}+\frac{1}{x^2}}{1+\frac{1}{x}}$. So the numerator goes to 0, while the denominator goes to 1. Hence $\underline{\underline{\lim\limits_{x\to\infty} \frac{2x+1}{x^2+x} = 0}}$.

   (d) We can recognize this as the derivative of x^n in the point a. So the limit becomes $\underline{\underline{\lim\limits_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}}}$. We could also use long division to get $\frac{x^n-a^n}{x-a} = (x^{n-1}+ax^{n-2}+\ldots+a^{n-1})$ and $\lim\limits_{x\to a}(x^{n-1}+ax^{n-2}+\ldots+a^{n-1}) = na^{n-1}$. You can also use L'Hoptial to obtain this result.

   [[Grading Instruction:

Grading (total 20):	
aspect:	points
(a),(b),(c) correct limit	4
(d) correct limit	3
small mistake	-2

]]

2. **(20 points)** Recall that if a function f is differentiable at a, then the derivative at the point a is defined as
$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$
Use this definition to find the derivative of the following functions at the point a:

(a) $f(x) = \pi$, $a = 2$;

(b) $f(x) = 2x + 3$, any a;

(c) $f(x) = x^2 + 2x + 1$, $a = 3$;

(d) $f(x) = \frac{5x-7}{4x+3}$, any $a \neq -\frac{3}{4}$.

Solution:

(a) We have $f(2+h) = f(2) = \pi$, so that $f(2+h) - f(2) = 0$. Hence
$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$
So $\underline{\underline{f'(2) = 0}}$.

(b) We have $f(a+h) = 2(a+h)+3 = 2a+2h+3$ and $f(a) = 2a+3$. So $f(a+h)-f(a) = 2a+2h+3-2a-3 = 2h$ and
$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{2h}{h} = 2.$$
So $\underline{\underline{f'(a) = 2}}$ for every a in $\mathbb{R}$.

(c) We have $f(3+h) = (3+h)^2 + 2(3+h) + 1 = h^2 + 8h + 16$ and $f(3) = 3^2 + 2*3 + 1 = 16$, so $f(3+h) - f(3) = h^2 + 8h$. So
$$\lim_{h\to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0} \frac{h^2+8h}{h} = \lim_{h\to 0}(h+8) = 8.$$
Hence $\underline{\underline{f'(3) = 8}}$.

(d) We have
$$f(a+h) = \frac{5(a+h)-7}{4(a+h)+3} = \frac{5a-7+5h}{4a+3+4h}.$$
So
$$\begin{aligned} f(a+h)-f(a) &= \frac{5a-7+5h}{4a+3+4h} - \frac{5a-7}{4a+3} \\ &= \frac{(5a-7+5h)(4a+3)-(5a-7)(4a+3+4h)}{(4a+3+4h)(4a+3)} \\ &= \frac{43h}{(4a+3+4h)(4a+3)}. \end{aligned}$$
Hence
$$\begin{aligned} \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0} \frac{43h}{(4a+3+4h)(4a+3)} \frac{1}{h} \\ &= \lim_{h\to 0} \frac{43}{(4a+3+4h)(4a+3)} \\ &= \frac{43}{(4a+3)^2}. \end{aligned}$$
So $\underline{\underline{f'(a) = \frac{43}{(4a+3)^2}}}$ for every a in $\mathbb{R} \setminus \{-\frac{3}{4}\}$.

[[**Grading Instruction:**

Grading (total 20):	
aspect:	points
computation of $f(a+h)-f(a)$	3
correct limit	2
small mistake	-2

]]

3. **(15 points)** Given the equation $4y - x + 2(1 - \ln 2) = 0$ and the function $f(x) = a \ln x$, $a > 0$.
   (a) Find the slope of the line having the equation above.
   (b) There exists a value for a such that the line is the tangent line to $f(x)$ in the point $x = 2$. Find a.
   (c) Find the tangent line to $f(x)$ in the point $x = 2$ for every $a > 0$.

   Solution:

   (a) We rewrite $4y - x + 2(1 - \ln 2) = 0$ as $4y = x - 2(1 - \ln 2)$ and finally $y = \frac{1}{4}x - \frac{1}{2}(1 - \ln 2)$. We know that the slope of a line given by an equation of this form is the coefficient of x. Hence the slope is $\underline{\underline{\frac{1}{4}}}$.
   (b) If the line is tangent to f at $x = 2$ we know that the slope of f must also be $\frac{1}{4}$ at $x = 2$, i.e. $f'(2) = \frac{1}{4}$. As $f'(x) = \frac{a}{x}$ this gives $\frac{a}{2} = f'(2) = \frac{1}{4}$, so $\underline{\underline{a = \frac{1}{2}}}$.
   (c) From the above we know that $f'(2) = \frac{a}{2}$ so that the tangent line is $y = \frac{a}{2}x + b$ for some b. We know the point $(2, f(2)) = (2, a \ln 2)$ lies on the line. This gives $a \ln 2 = \frac{a}{2} * 2 + b$, i.e. $b = a \ln 2 - a = a(\ln 2 - 1)$. So the tangent line is $\underline{\underline{y = \frac{a}{2}x + a(\ln 2 - 1)}}$.

[[**Grading Instruction:**

Grading (total 15):	
aspect:	points
(a) correct eqn $y = ..$	3
(a) correct slope	2
(b) recognize that $f'(2)$ is slope	3
(b) correct a	2
(c) $y = \frac{a}{2}x + b$	3
(c) $b = a(\ln 2 - 1)$	2

]]

4. **(30 points)** Find the derivative on the domain of the following functions. You can freely use all the differentiation rules that were discussed in the lecture. Simplify the result as much as you can.
   (a) $f(x) = x^4 - 2x^3 + 7$;
   (b) $f(x) = \frac{x^2+5}{x-7}$;
   (c) $f(x) = \sin^2(\sqrt{x})$;
   (d) $f(x) = 1 - \cos^2(\sqrt{x})$;
   (e) $f(x) = \exp(\tan(x))$;
   (f) $f(x) = -\ln(\cos(x))$;
   (g) $f(x) = \arcsin(1 - 2x)$;
   (h) $f(x) = 10^{x^2}$.

   Solution:

   (a) Use the fact that $(x^n)' = nx^{n-1}$, $(a)' = 0$ for $a \in \mathbb{R}$ and the sum rule. We find $\underline{\underline{f'(x) = 4x^3 - 6x^2}}$.

(b) Use the rules above and the quotient rule. Write $f(x) = \frac{g(x)}{h(x)}$ for $g(x) = x^2+5$ and $h(x) = x-7$. Then $g'(x) = 2x$ and $h'(x) = 1$, so $f'(x) = \frac{g'(x)h(x)-g(x)h'(x)}{(h(x))^2} = \frac{2x(x-7)-(x^2+5)}{(x-7)^2} = \underline{\underline{\frac{x^2-14x-5}{(x-7)^2}}} = 1 - \frac{54}{(x-7)^2}$.

(c) Use the chain rule and $(\sin x)' = \cos x$. Then $f'(x) = 2\sin(\sqrt{x})\cos(\sqrt{x})\frac{1}{2\sqrt{x}} = \underline{\underline{\frac{\sin(\sqrt{x})\cos(\sqrt{x})}{\sqrt{x}}}}$, which is equivalent to $\frac{\sin(2\sqrt{x})}{2\sqrt{x}}$.

(d) Note that this is the same function as in (c), so the same solution.

(e) Use $(e^x)' = e^x$, $(\tan x)' = \frac{1}{\cos^2 x}$ and the chain rule. Then $f'(x) = \exp(\tan(x))\frac{1}{\cos^2 x} = \underline{\underline{\frac{\exp(\tan(x))}{\cos^2 x}}}$.

(f) Use $(\ln x)' = \frac{1}{x}$, $(\cos x)' = -\sin x$ and the chain rule. Then $f'(x) = -\frac{1}{\cos x} * -\sin x = \underline{\underline{\frac{\sin x}{\cos x}}} = \tan(x)$.

(g) Use $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ and the chain rule. Then $f'(x) = \frac{1}{\sqrt{1-(1-2x)^2}} * (-2) = \frac{-2}{\sqrt{4x-4x^2}} = \underline{\underline{\frac{-1}{\sqrt{x-x^2}}}}$.

(h) Use $(a^x)' = a^x \ln a$ and the chain rule. Then $f'(x) = 10^{x^2}\ln 10 * (2x) = \underline{\underline{2(\ln 10)x10^{x^2}}}$.

[[**Grading Instruction:**

Grading (total 30):	
aspect:	points
(a-f) correct result	4
(g,h) correct result	3
small mistake	-2

]]

5. **(20 points)** Apply any rules (including chain or inverse rules) and the logarithmic differentiation as appropriate to compute the result. If you can solve the problem in two different ways, you get *two extra points.*

(a) $f(x) = e^{\sin x}$, compute $f'(x)$;

(b) $f(x) = (\exp x)^{\exp x}$, compute $f'(x)$; (*Hint:* use logarithmic differentiation or the chain rule)

(c) $f(x) = e^{2x}$, compute $(f^{-1})'(x)$;

(d) $f(x) = \sqrt{x-2}$, compute $(f^{-1})'(x)$ (for $x > 2$).

Solution:

(a) i. Use chain rule to get $\underline{\underline{f'(x) = e^{\sin x}\cos x}}$.

ii. Use logarithmic differentiation to get $\ln e^{\sin x} = \sin x$. So $\frac{f'(x)}{f(x)} = \cos x$ and therefore $f'(x) = e^{\sin x}\cos x$.

(b) i. Use logarithmic differentiation. We have $\ln((\exp x)^{\exp x}) = x\exp x$, so $\frac{f'(x)}{f(x)} = \exp x(x+1)$. We conclude $\underline{\underline{f'(x) = (\exp x)^{\exp x+1}(x+1)}} = e^{x(e^x+1)}(x+1)$.

ii. We can also use the chain rule. Note that for $g(x) = x^x$ and $h(x) = \exp x$ we have $f(x) = (g \circ h)(x)$. So we use the chain rule. Compute $g'(x) = x^x(\log x + 1)$, $h'(x) = \exp x$ and thus $f'(x) = g'(h(x))h'(x) = (\exp x)^{\exp x}(x+1)\exp x = (\exp x)^{\exp x+1}(x+1) = e^{x(e^x+1)}(x+1)$.

(c) i. First compute the inverse and then differentiate. We have $f^{-1}(x) = \frac{\ln x}{2}$, so $\underline{\underline{(f^{-1})'(x) = \frac{1}{2x}}}$.

ii. Use the inverse rule. We have again $f^{-1}(x) = \frac{\ln x}{2}$ and $f'(x) = 2e^{2x}$. Hence $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{2e^{\ln x}} = \frac{1}{2x}$.

(d) i. First compute the inverse and then differentiate. We have $f^{-1}(x) = x^2+2$, so $\underline{\underline{(f^{-1})'(x) = 2x}}$.

ii. Use the inverse rule. Again $f^{-1}(x) = x^2+2$ and $f'(x) = \frac{1}{2\sqrt{x-2}}$. So $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = 1/\frac{1}{2\sqrt{(x^2+2)-2}} = 1/\frac{1}{2x} = 2x$.

[[**Grading Instruction:**

Grading (total 20):

aspect:	points
correct result	5
two solutions	+2
small mistake	-2

]]