Calculus and Probability Theory
Lecture Notes

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General Information

- Teachers
  - Perry Groot
- Grading
  - Exam: 5. April, 12:30 - 15:30, HG00.304, HG00.062
  - Condition: at least five sufficient homework assignments
  - Final mark: the result of the written exam

Lecture I

- Foundations
- Limits and continuous functions
- Derivatives

Foundations

Definition: Functions (domain, codomain, range)

A **real function** $f : D \to \mathbb{R}$, for $D \subseteq \mathbb{R}$, is a rule which assignes to each $x \in D$ precisely one $f(x) \in \mathbb{R}$.

- In this situation the subset $D \subseteq \mathbb{R}$ is called the **domain** of f. Sometimes we write D(f) for D.
- $\mathbb{R}$ is the **codomain** of f, and the subset $R(f) = \{f(x)|x \in D\} \subseteq \mathbb{R}$ is called the **range** of f.

Definition: injective, surjective, bijective, isomorphism

A function $f : D \to \mathbb{R}$ is **injective** or **one-to-one** if $f(x) = f(y)$ implies $x = y$, for all $x, y \in D$
A function $f : D \to R(\subseteq \mathbb{R})$ is **surjective** or **onto** if its range is equal to its codomain

- This means: for any $y \in R$ there is an $x \in D$ such that $f(x) = y$. Symbolically: $\forall y \in R\ \exists x \in D\ f(x) = y$.

A function $f : D \to R$ is **bijective** if it is both injective and surjective. Then it is an **isomorphism** $f : D \xrightarrow{\cong} R$

Definition: Graph of a real function

For a function $f : D \to \mathbb{R}$, the **graph** $G(f) \subseteq D \times \mathbb{R}$ of f contains all pairs $(x, f(x))$. So, we write: $G(f) = \{(x, f(x)) \mid x \in D\}$.

Definition: Inverse and composition

If a function $f : D \to \mathbb{R}$, is **injective**, we can define an **inverse** function $f^{-1} : R(f) \to D \subseteq \mathbb{R}$, namely:

- for $y \in R(f)$, say $y = f(x)$, define $f^{-1}(y) = x$
- this x is uniquely determined: if $f(x) = y = f(x')$, then $x = x'$, since f is **injective**
- by construction: $f(f^{-1}(y)) = y$ and also $f^{-1}(f(x)) = x$.

The **composition** of $f : X \to Y$ and $g : Y \to Z$ is $h = g \circ f : X \to Z$, for which $h(x) = g(f(x))$, for each $x \in X$.

Definition: Parity of a function

A function $f : (-a, a) \to \mathbb{R}$ is **even** if $f(-x) = f(x)$, for all $x \in (-a, a)$, and **odd** if $f(-x) = -f(x)$, for all $x \in (-a, a)$.

Limits and continuous functions

Definition: Approach to limit

A function f approaches the limit b at an input a, if we can make $f(x)$ as close as we like to b by requiring that x be sufficiently close, but not equal to a.

Definition: Limit

A function $f : D \to \mathbb{R}$ has **limit** b for $x \to a$ if:

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in D \qquad 0 < |x - a| < \delta \Rightarrow |f(x) - b| < \epsilon$$

In that case we write $\lim_{x \to a} f(x) = b$. *the limit of f of x as x approaches a equals b.* Note. a does not have to be in D.

Example:

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{x - 3} = \lim_{x \to 3} (x + 3) = 6$$

$$\lim_{x \to 1} \frac{-1}{x - 1} \text{is undefined}$$

Computation of Limits

Let c be a constant, a a real number, and assume that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist and are equal to L_1 and L_2

- $\lim_{x \to a} c = c$
- $\lim_{x \to a} x = a$
- $\lim_{x \to a} [c \cdot f(x)] = c \cdot \lim_{x \to a} f(x) = c \cdot L_1$
- $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L_1 \pm L_2$
- $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L_1 \cdot L_2$
- $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L_1}{L_2}$ (if $L_2 \neq 0$)

Computation of Limits

Let $p(x) = c_n x^n + c_{n-1}x^{n-1} + ... + c_1 x + c_0$ be a polynomial, a a real number and $\lim x \to a g(x)$ both exist and are equal to L_1 and L_2

- $\lim_{x\to a} p(x) = p(a)$
- $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L_1}$
- $\lim_{x\to a} \sin(x) = \sin(a)$
- $\lim_{x\to a} \cos(x) = \cos(a)$

Let $f(x) \leq g(x) \leq h(x)$ for all x on some interval (c,d), except maybe for $a \in (c,d)$. If

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

Then

$$\lim_{x\to a} g(x) = L$$

Example:
$\lim_{x\to 0}[x^2 \cos(\frac{1}{x})] = 0$, Since $-1 \leq \cos(\frac{1}{x}) \leq 1$ we have $-x^2 \leq x^2 \cos(\frac{1}{x}) \leq x^2$ and since $\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$. Note that the rule for products fails.

Definition: Limits involving infinity

A function $f : D \to \mathbb{R}$ has **limit** b for $x \to \infty$ if:

$$\forall \epsilon > 0 \exists n \in \mathbb{N} \forall x \in D\ x > n \Rightarrow |f(x) - b| < \epsilon$$

In that case we write $\lim_{x\to\infty} f(x) = b$. Formulate yourself what $\lim_{x\to-\infty} f(x) = b$ means.

Definition: Continuous functions

A function $f : D \to \mathbb{R}$ is continuous in a point $a \in D$ if $f(x)$ is close to $f(a)$ for each x that is close to a. More formally: $f : D \to \mathbb{R}$ is **continuous in point** $a \in D$ if:

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in D\ |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$$

A function $f : D \to \mathbb{R}$ is **continuous** if it is continuous in all $a \in D$.

Derivatives

Recall: Points and lines

- Equation of a line
  - y-intercept (b), **slope** (m)
  - most convenient: $y = mx + b$
  - can be determined from e.g., two points, or a point and the slope
- Distance of two points (x_1, y_1) and (x_2, y_2)
  - by Pythagorean theorem
  - $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
- Sample Questions
  - What is the intercept and the slope: $3x - 4y = -8$?
  - Determine the distance of points $(2, -3)$ and $(-3, 9)$
  - Equation of the line? Slope: $m = \frac{2}{5}$, a point on it: $(-1, 2)$

Definition: Derivatives (differentiable, tangent line)

A function $f : D \to \mathbb{R}$ is **differentiable at a** if $\lim_{h\to 0}(\frac{f(a+h)-f(a)}{h})$ exists. In this case the limit is denoted $f'(a)$ and is called the **derivative of f at a**.
f is **differentiable** if f is differentiable at a for every $a \in D$.
We also define the **tangent line** to f at a to be the line through $(a, f(a)) \in G(f)$ with slope $f'(a)$.
If f is a differentiable function then f' ("Lagrange notation") is sometimes written as $\frac{df}{dx}$ ("Leibniz notation").

Example:

- (Geometric interpretation) Find a tangent line of a curve $f(x) = \frac{1}{x}$ in $a = 2$
- Check that $f(x) = |x|$ is not differentiable in 0.

Differentiation rules

Let $f, g : D \to \mathbb{R}$ be differentiable functions in $a \in D$

- For a **constant function** $f(x) = c, c \in \mathbb{R}$, we have $f'(x) = 0$
- $f(x) = x$, then $f'(x) = 1$
- **sum/subtraction rule** $(f \pm g)'(a) = f'(a) \pm g'(a)$
- **scalar rule** $(c \cdot f)'(a) = c \cdot f'(a)$
- **product rule** $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$
- **division rule** $(\frac{f}{g})'(a) = \frac{f'(a)g(a)-f(a)g'(a)}{g^2(a)}$
- **chain/composition rule** $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$
- if f has an inverse f^{-1}, then $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$

Lemma: Derivatives of powers

- For $n \in \mathbb{N}$ and $f(x) = x^n$ we have $f'(x) = nx^{n-1}$. This can be shown by induction on n
- In fact, for $n \in \mathbb{Z}$ and $f(x) = x^n$ we have $f'(x) = nx^{n-1}$. This follows from the previous point, using the division rule.
- It can be shown that $(x^a)' = ax^{a-1}$, for any $a \in \mathbb{R}$

Recall: exponential and logarithm

Exponential, for $a \geq 0$

- $a^0 = 1, a^{x+y} = a^x \cdot a^y$
- $a^1 = a, a^{x \cdot y} = (a^x)^y$
- $a^{-x} = \frac{1}{a^x}$, and thus $a^{x-y} = \frac{a^x}{a^y}$

The logarithm is defined as inverse of power: $x = \log_a(y) \Longleftrightarrow a^x = y$, for $y > 0$

Logarithm

- $\log_a(a^x) = x$ and $a^{\log_a x} = x$
- $log_a(x \cdot y) = \log_a(x) + \log_a(y)$, and $log_a(x^y) = y \cdot \log_a(x)$
- $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$
- $\frac{\log_a x}{\log_a b} = \log_b x$

Euler's number e

Consider $f_a(x) = a^x$. Then:

$$\begin{aligned} f_a'(x) = (a^x)' &= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x(a^h - 1)}{h} \\ &= a^x \cdot \lim_{h \to 0} \frac{a^{0+h} - a^0}{h} = a^x \cdot \lim_{x \to 0} \frac{f_a(0+h) - f_a(0)}{h} \\ &= a^x \cdot f_a'(0) \end{aligned}$$

- We have $f_a'(0) = 1$ for $a = e = 2.71828...$
- and thus $(e^x)' = e^x$
- The **natural logarithm** ln uses base e; notation: $\ln \equiv \log_e$

Important derivatives with logarithms

- $(a^x)' = a^x \cdot \ln(a)$
- $(\ln(y))' = \frac{1}{y}$
- We have $(e^{f(x)})' = e^{f(x)} \cdot f'(x)$ by the chain rule
- Thus: $(a^x)' = a^x \cdot \ln(a)$, since
  $(a^x)' = ((e^{\ln(a)})^x)' = (e^{\ln(a)\cdot x})' = e^{\ln(a)\cdot x} \cdot \ln(a) = a^x \cdot \ln(a)$
- For $f(x) = e^x$ we have $f'(x) = e^x$ and $f^{-1}(y) = \ln y$
- We use the inverse function law $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$
- Thus $\ln'(y) = \frac{1}{f'(\ln y)} = \frac{1}{e^{\ln y}} = \frac{1}{y}$

Definition: Logarithmic differentiation

According to the chain rule:

$$(\ln f(x))' = \ln'(f(x)) \cdot f'(x) = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}$$

Briefly: $(\ln f)' = \frac{f'}{f}$. This is called the **logarithmic derivative of f** and this law is called **logarithmic differentiation**.

Example:

For $f(x) = \frac{6x}{\sqrt{x-1}}$ we can compute $f'(x)$ via the fraction rule, but also first taking logarithms on both sides:

$$\ln(f(x)) = \ln(\frac{6x}{\sqrt{x-1}}) = \ln(6x) - \ln((x-1)^{\frac{1}{2}}) = \ln(6x) - \frac{1}{2}\ln(x-1)$$

Differentiating on both sides gives:

$$\frac{f'(x)}{f(x)} = \frac{6}{6x} - \frac{1}{2} \cdot \frac{1}{x-1} = \frac{1}{x} - \frac{1}{2(x-1)} = \frac{2(x-1)-x}{2x(x-1)} = \frac{x-2}{2x(x-1)}$$

Hence:

$$f'(x) = f(x) \cdot \frac{x-2}{2x(x-1)} = \frac{6x}{\sqrt{x-1}} \cdot \frac{x-2}{2x(x-1)} = \frac{3(x-2)}{(x-1)^{\frac{3}{2}}}$$

Recall: sine, cosine and tangent

- Geometric interpretation with $\sin(90°) = \sin(\frac{\pi}{2}) = 1$
- $\sin^2(x) + \cos^2(x) = 1$
- Sum rules:
  * $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
  * $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
  * $\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
  * $\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
- $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$
- $\tan(x) = \frac{\sin(x)}{\cos(x)}$, with $\tan'(x) = \frac{1}{\cos^2(x)}$

Logarithmic differentation Example

Logarithmic differentation is useful for reducing products to sum, fractions to differences, and powers to products.
Take $f(x) = (\sin(x))^x$

$$\ln f(x) = \ln((\sin x)^x) = x \cdot \ln(\sin(x))$$

Thus:

$$\frac{f'(x)}{f(x)} = \ln(\sin x) + x \cdot \frac{1}{\sin x} \cdot \cos x$$

And:

$$f'(x) = f(x) \cdot (\ln(\sin x) + \frac{x \cos x}{\sin x}) = (\sin x)^x (\ln(\sin x) + \frac{x \cos x}{\sin x}).$$

Derivatives of special functions

- $f(x) = a^x$ then $f'(x) = a^x \cdot \ln a$. Special case $(e^x)' = e^x$
- $(\log_a x)' = \frac{1}{x \cdot \ln a}$ with special case $(\ln x)' = \frac{1}{x}$
- $(\sin x)' = \cos x$
- $(\cos x)' = -\sin x$
- $(\tan x)' = \frac{1}{\cos^2 x}$ where $\tan x = \frac{\sin x}{\cos x}$
- $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ where $\arcsin = \sin^{-1}$
- $(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}$ where $\arccos = \cos^{-1}$
- $(\arctan x)' = \frac{1}{1+x^2}$ where $\arctan = \tan^{-1}$

L'Hopitals rule

Let $f, g : D \to \mathbb{R}$ be functions that are differentiable and

- $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$ or $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, moreover
- $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists with
- $g'(x) \neq 0$ (except for perhaps at a), then
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Example:

- $\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\cos(x)}{1} = \cos(0) = 1$
- $\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$
- $\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \lim_{x\to 0} \frac{\sin(x)}{2x} = \lim_{x\to 0} \frac{\cos(x)}{2} = \frac{1}{2}$

Higher order derivatives

Let $f(x)$ be a real function.

- One writes $f' = \frac{df}{dx}$
- The second derivative is written as: $f'' = \frac{d}{dx} f' = \frac{d^2 f}{dx^2}$
- The n-th derivative is: $f^{(n)} = \frac{d}{dx} f^{n-1}$ with $f^{(0)} = f$

Example: Let $f(x) = x^n$, find $f^{(n)}(x)$