Calculus and Probability

Assignment 2

Note:

- You can hand in your solutions as a single PDF via the assignment module in Brightspace. Note that the document should be in English and typeset with LaTeX, Word or a similar program. It should not be a scan or picture of your handwritten notes.
- Make sure that your name, student number and group number are on top of the first page!
- **Note that your submission should be an individual submission because it can influence your final grade for this course. If we detect that your work is not completely your own work, we will ask the exam committee to investigate whether it is plagiarism or not!**

Exercises to be presented during the exercise hours

Exercise 1
Find the limits. (Hint: try to simplify as much as possible before applying the limit!)

a) $\lim\limits_{x\to\infty} \frac{3x^2+8}{x+1}$;

Again assume $x \neq 0$ and divide by x^2. We find $\frac{3x^2+8}{x+1} = \frac{3+\frac{8}{x^2}}{\frac{1}{x}+\frac{1}{x^2}}$. So the numerator goes to 3, while the denominator goes to 0 as x goes to infinity. Hence $\underline{\underline{\lim\limits_{x\to\infty} \frac{3x^2+8}{x+1} = \infty}}$.

b) $\lim\limits_{x\to a} \frac{x^n-a^n}{x-a}$ for some parameter $a \in \mathbb{R}$. (*Hint*: Do you recognize this limit? If not, you can always simplify the fraction using long division)

We can recognize this as the derivative of x^n in the point a. So the limit becomes $\underline{\underline{\lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}}}$. We could also use long division to get $\frac{x^n-a^n}{x-a} = (x^{n-1}+ax^{n-2}+\ldots+a^{n-1})$ and $\lim_{x\to a}(x^{n-1}+ax^{n-2}+\ldots+a^{n-1}) = na^{n-1}$. You can also use L'Hopital to obtain this result.

Exercise 2

Recall that if a function f is differentiable at a, then the derivative at the point a is defined as

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

Use this definition to find the derivative of the following functions at the point a:

a) $f(x) = \pi$, $a = 2$;

We have $f(2+h) = f(2) = \pi$, so that $f(2+h) - f(2) = 0$. Hence

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

So $\underline{\underline{f'(2) = 0}}$.

b) $f(x) = x^2 + 2x + 1$, $a = 3$;

We have $f(3+h) = (3+h)^2 + 2(3+h) + 1 = h^2 + 8h + 16$ and $f(3) = 3^2 + 2*3 + 1 = 16$, so $f(3+h) - f(3) = h^2 + 8h$. So

$$\lim_{h\to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0} \frac{h^2+8h}{h} = \lim_{h\to 0}(h+8) = 8.$$

Hence $\underline{\underline{f'(3) = 8}}$.

c) $f(x) = \frac{5x-7}{4x+3}$, any $a \neq -\frac{3}{4}$.

We have

$$f(a+h) = \frac{5(a+h)-7}{4(a+h)+3} = \frac{5a-7+5h}{4a+3+4h}.$$

So

$$\begin{aligned} f(a+h)-f(a) &= \frac{5a-7+5h}{4a+3+4h}-\frac{5a-7}{4a+3} \\ &= \frac{(5a-7+5h)(4a+3)-(5a-7)(4a+3+4h)}{(4a+3+4h)(4a+3)} \\ &= \frac{43h}{(4a+3+4h)(4a+3)}. \end{aligned}$$

Hence

$$\begin{aligned} \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0}\frac{43h}{(4a+3+4h)(4a+3)}\frac{1}{h} \\ &= \lim_{h\to 0}\frac{43}{(4a+3+4h)(4a+3)} \\ &= \frac{43}{(4a+3)^2}. \end{aligned}$$

So $\underline{\underline{f'(a)=\frac{43}{(4a+3)^2}}}$ for every a in $\mathbb{R}\setminus\{-\frac{3}{4}\}$.

Exercise 3

Given the equation $4y-x+2(1-\ln 2)=0$ and the function $f(x)=a\ln x$, $a>0$.

a) Find the slope of the line having the equation above.

We rewrite $4y-x+2(1-\ln 2)=0$ as $4y=x-2(1-\ln 2)$ and finally $y=\frac{1}{4}x-\frac{1}{2}(1-\ln 2)$. We know that the slope of a line given by an equation of this form is the coefficient of x. Hence the slope is $\underline{\underline{\frac{1}{4}}}$.

b) There exists a value for a such that the line is the tangent line to $f(x)$ in the point $x=2$. Find a.

If the line is tangent to f at $x=2$ we know that the slope of f must also be $\frac{1}{4}$ at $x=2$, i.e. $f'(2)=\frac{1}{4}$. As $f'(x)=\frac{a}{x}$ this gives $\frac{a}{2}=f'(2)=\frac{1}{4}$, so $\underline{\underline{a=\frac{1}{2}}}$.

c) Find the tangent line to $f(x)$ in the point $x=2$ for every $a>0$.

From the above we know that $f'(2) = \frac{a}{2}$ so that the tangent line is $y = \frac{a}{2}x + b$ for some b. We know the point $(2, f(2)) = (2, a\ln 2)$ lies on the line. This gives $a\ln 2 = \frac{a}{2} * 2 + b$, i.e. $b = a\ln 2 - a = a(\ln 2 - 1)$. So the tangent line is $\underline{\underline{y = \frac{a}{2}x + a(\ln 2 - 1)}}$.

Note that we could also use the equivalent point-slope notation for a line here, which is $(y - a\ln 2) = \frac{a}{2}(x - 2)$.

Exercise 4

Find the derivative on the domain of the following functions. You can freely use all the differentiation rules that were discussed in the lecture. Simplify the result as much as you can.

a) $f(x) = x^4 - 2x^3 + 7$;

Use the fact that $(x^n)' = nx^{n-1}$, $(a)' = 0$ for $a \in \mathbb{R}$ and the sum rule. We find $\underline{\underline{f'(x) = 4x^3 - 6x^2}}$.

b) $f(x) = \frac{x^2+5}{x-7}$;

Use the rules above and the quotient rule. Write $f(x) = \frac{g(x)}{h(x)}$ for $g(x) = x^2 + 5$ and $h(x) = x - 7$. Then $g'(x) = 2x$ and $h'(x) = 1$, so $f'(x) = \frac{g'(x)h(x)-g(x)h'(x)}{(h(x))^2} = \frac{2x(x-7)-(x^2+5)}{(x-7)^2} = \underline{\underline{\frac{x^2-14x-5}{(x-7)^2}}} = 1 - \frac{54}{(x-7)^2}$.

c) $f(x) = \sin^2(\sqrt{x})$;

Use the chain rule and $(\sin x)' = \cos x$. Then $f'(x) = 2\sin(\sqrt{x})\cos(\sqrt{x})\frac{1}{2\sqrt{x}} = \underline{\underline{\frac{\sin(\sqrt{x})\cos(\sqrt{x})}{\sqrt{x}}}}$, which is equivalent to $\frac{\sin(2\sqrt{x})}{2\sqrt{x}}$.

d) $f(x) = 1 - \cos^2(\sqrt{x})$;

Note that this is the same function as in (c), so the same solution.

e) $f(x) = \arcsin(1 - 2x)$;

Use $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ and the chain rule. Then $f'(x) = \frac{1}{\sqrt{1-(1-2x)^2}} * (-2) = \frac{-2}{\sqrt{4x-4x^2}} = \underline{\underline{\frac{-1}{\sqrt{x-x^2}}}}$.

f) $f(x) = 10^{x^2}$.

Use $(a^x)' = a^x \ln a$ and the chain rule. Then $f'(x) = 10^{x^2} \ln 10 * (2x) = \underline{\underline{2(\ln 10)x10^{x^2}}}$.

Exercise 5

Apply any rules (including chain or inverse rules) and the logarithmic differentiation as appropriate to compute the result. Try to solve each question in two different ways.

a) $f(x) = e^{\sin x}$, compute $f'(x)$;

(i) Use chain rule to get $\underline{\underline{f'(x) = e^{\sin x} \cos x}}$.

(ii) Use logarithmic differentiation to get $\ln e^{\sin x} = \sin x$. So $\frac{f'(x)}{f(x)} = \cos x$ and therefore $f'(x) = e^{\sin x} \cos x$.

b) $f(x) = e^{2x}$, compute $(f^{-1})'(x)$;

(i) First compute the inverse and then differentiate. We have $f^{-1}(x) = \frac{\ln x}{2}$, so $\underline{\underline{(f^{-1})'(x) = \frac{1}{2x}}}$.

(ii) Use the inverse rule. We have again $f^{-1}(x) = \frac{\ln x}{2}$ and $f'(x) = 2e^{2x}$. Hence $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{2e^{\ln x}} = \frac{1}{2x}$.

Exercise 6

Consider a function $f(x)$ satisfying $f(x) = f(4x-2)$ for all real values x. If $f(x)$ is differentiable for all x and $f'(4) = 40$, then what is the value of $f'(54)$?

We have that

$$f'(x) = [f(x)]' = [f(4x-2)]' = f'(4x-2) \cdot 4$$

and that

$$f(54) = f(4 \cdot 14 - 2) = f(14) = f(4 \cdot 4 - 2) = f(4)$$

thus

$$40 = f'(4) = 4 \cdot f'(14) = 4 \cdot 4 \cdot f'(54)$$

hence

$$f'(54) = \frac{40}{4 \cdot 4} = \underline{\underline{2\frac{1}{2}}}$$

Exercises to be handed in

You are expected to explain your answers, even if this is not explicitly stated in the exercises themselves.

Exercise 7
Find the limits. (Hint: try to simplify as much as possible before applying the limit!)

a) $\lim\limits_{x \to -\infty} \frac{x^3+2x^2+2}{3x^3+x+4}$; 1 pt

As x is tending to (minus) infinity we can surely assume $x \neq 0$. So dividing the numerator and denominator by x^3 we get $\frac{x^3+2x^2+2}{3x^3+x+4} = \frac{1+\frac{2}{x}+\frac{2}{x^3}}{3+\frac{1}{x^2}+\frac{4}{x^3}}$. Hence the numerator goes to 1, while the denominator goes to 3 as x goes to infinity. Hence $\underline{\underline{\lim\limits_{x \to -\infty} \frac{x^3+2x^2+2}{3x^3+x+4} = \frac{1}{3}}}$

b) $\lim\limits_{x \to \infty} \frac{2x+1}{x^2+x}$; 1 pt

Divide by x^2 to get $\frac{2x+1}{x^2+x} = \frac{\frac{2}{x}+\frac{1}{x^2}}{1+\frac{1}{x}}$. So the numerator goes to 0, while the denominator goes to 1. Hence $\underline{\underline{\lim\limits_{x \to \infty} \frac{2x+1}{x^2+x} = 0.}}$

Exercise 8
Recall that if a function f is differentiable at a, then the derivative at the point a is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Use this definition to find the derivative of the function $f(x) = 2x + 3$ for any point a in $\mathbb{R}$ 1 pt

We have $f(a+h) = 2(a+h)+3 = 2a+2h+3$ and $f(a) = 2a+3$. So $f(a+h)-f(a) = 2a+2h+3-2a-3 = 2h$ and

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{2h}{h} = 2.$$

So $\underline{\underline{f'(a) = 2}}$ for every a in $\mathbb{R}$.

Exercise 9
Consider the function $f(x) = \frac{1}{(1+\frac{1}{x})}$.

a) Find the tangent line of $f(x)$ at $x = 2$. $\boxed{\text{1 pt}}$

$f(x) = \frac{1}{(1+\frac{1}{x})}$, so by using the chain rule we find $f'(x) = -\frac{1}{(1+\frac{1}{x})^2} * -\frac{1}{x^2} = \frac{1}{x^2(1+\frac{1}{x})^2}$. The tangent line at $x = a$ is $y = \frac{1}{a^2(1+\frac{1}{a})^2}x + b$ for some b. We know $f(a) = f'(a)x + b$ so, $\frac{1}{(1+\frac{1}{a})} = \frac{1}{a^2(1+\frac{1}{a})^2}a + b$, and $b = \frac{a^2(1+\frac{1}{a})-a}{a^2(1+\frac{1}{a})^2}$. Therefore $b = \frac{a^2}{a^2(1+\frac{1}{a})}$. So we have $y = \frac{1}{a^2(1+\frac{1}{a})^2}x + \frac{1}{(1+\frac{1}{a})^2}$. So for $a = 2$, $y = \frac{1}{4(1+\frac{1}{2})^2}x + \frac{1}{(1+\frac{1}{2})^2} = \frac{1}{9}x + \frac{4}{9}$.

Alternatively, we could first rewrite $f(x) = \frac{1}{(1+\frac{1}{x})} = \frac{x}{x+1} = 1 - \frac{1}{x+1}$. Then $f(2) = \frac{2}{3}$ and $f'(x) = \frac{1}{(x+1)^2}$, which will give the same line equation.

Instead of the slope-intercept line notation, we could also use the equivalent point-slope line notation. As the slope is $\frac{1}{9}$ through point $(2, \frac{2}{3})$ an equivalent line equation would be $(y - \frac{2}{3}) = \frac{1}{9}(x-2)$.

b) What is the tangent line of $f(x)$ as $x \to \infty$? $\boxed{\text{1 pt}}$

From the first part, we know we are interested in $\lim\limits_{a\to\infty} \frac{1}{a^2(1+\frac{1}{a})^2}x + \frac{1}{(1+\frac{1}{a})^2}$. The slope converges to 0 because $\lim\limits_{a\to\infty} a^2(1+\frac{1}{a})^2 = \infty$ as $a \to \infty$. The intercept converges to 1. So the tangent line converges to $y = 1$ as $x \to \infty$.

Exercise 10
Find the derivative on the domain of the following functions. You can freely use all the differentiation rules that were discussed in the lecture. Simplify the result as much as you can.

a) $f(x) = \exp(\tan(x))$; $\boxed{\text{1 pt}}$

Use $(e^x)' = e^x$, $(\tan x)' = \frac{1}{\cos^2 x}$ and the chain rule. Then $f'(x) = \exp(\tan(x))\frac{1}{\cos^2 x} = \underline{\underline{\frac{\exp(\tan(x))}{\cos^2 x}}}$.

b) $f(x) = -\ln(\cos(x))$; 1 pt

Use $(\ln x)' = \frac{1}{x}$, $(\cos x)' = -\sin x$ and the chain rule. Then $f'(x) = -\frac{1}{\cos x} * -\sin x = \underline{\underline{\frac{\sin x}{\cos x}}} = \tan(x)$.

Exercise 11

Apply any rules (including chain or inverse rules) and the logarithmic differentiation as appropriate to compute the result. Try to solve each question in two different ways.

a) $f(x) = (\exp x)^{\exp x}$, compute $f'(x)$; (*Hint:* use logarithmic differentiation or the chain rule) 1 pt

(i) Use logarithmic differentiation. We have $\ln((\exp x)^{\exp x}) = x\exp x$, so $\frac{f'(x)}{f(x)} = \exp x(x+1)$. We conclude $\underline{\underline{f'(x) = (\exp x)^{\exp x+1}(x+1)}} = e^{x(e^x+1)}(x+1)$.

(ii) We can also use the chain rule. Note that for $g(x) = x^x$ and $h(x) = \exp x$ we have $f(x) = (g \circ h)(x)$. So we use the chain rule. Compute $g'(x) = x^x(\log x + 1)$, $h'(x) = \exp x$ and thus $f'(x) = g'(h(x))h'(x) = (\exp x)^{\exp x}(x+1)\exp x = (\exp x)^{\exp x+1}(x+1) = e^{x(e^x+1)}(x+1)$.

b) $f(x) = \sqrt{x-2}$, compute $(f^{-1})'(x)$ (for $x > 2$). 1 pt

(i) First compute the inverse and then differentiate. We have $f^{-1}(x) = x^2 + 2$, so $\underline{\underline{(f^{-1})'(x) = 2x}}$.

(ii) Use the inverse rule. Again $f^{-1}(x) = x^2 + 2$ and $f'(x) = \frac{1}{2\sqrt{x-2}}$. So $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = 1/\frac{1}{2\sqrt{(x^2+2)-2}} = 1/\frac{1}{2x} = 2x$.

Exercise 12

The polynomial $P(x)$ satisfies the following identity 1 pt

$$P(P(x)+x) = 11 \cdot (P(x)+x)^2 - 4 \cdot (P(x)+x) + 5$$

What is the value of $P'(6)$?

We know from the chain rule that

$$[P(P(x)+x)]' \quad = \quad P'(P(x)+x)\cdot(P'(x)+1)$$

and that (by taking the derivative of the right hand side)

$$\begin{aligned}[P(P(x)+x)]' &= 22\cdot(P(x)+x)\cdot(P'(x)+1)-4\cdot(P'(x)+1)\\ &= (22\cdot(P(x)+x)-4)\cdot(P'(x)+1)\end{aligned}$$

This holds for all x thus dividing both these terms by $(P'(x)+1)$ gives

$$P'(P(x)+x) = 22\cdot(P(x)+x)-4$$

Setting $P(x)+x=6$ then gives

$$P'(6) = 22\cdot 6 - 4 = \underline{\underline{128}}$$

Your final grade is the sum of your scores divided by 1.0.