Calculus and Probability Theory

Assignment 5, March 2, 2017

Handing in your answers:

- submission via Blackboard (`http://blackboard.ru.nl`);
- one single pdf file (make sure that if you scan/photo your handwritten assignment, the result is clearly readable);
- all of your solutions are clearly and convincingly explained;
- make sure to write your name, your student number

Deadline: Friday, March 10, 14:30 sharp!

Goals: After completing these exercises successfully you should be:

- familiar with definite and indefinite integrals;
- able to apply the most important integration methods; more specifically, substitution and integration by parts;
- confident about switching between different representations of a function;
- able to compute area of a finite or infinite region;
- able to apply the formula for the arc length of a function over a finite interval.

Marks: You can score a total of 100 points. (Additionally, you can collect +10 bonus points.)

1. **(20 points)** Compute the following indefinite integrals. You can use *substitution* or *integration by parts*.

   In each problem *verify* your result, and don't forget about the constant term. You may need some of the following, well-known trigonometric identities:

   $$\sin(2x) = 2\sin(x)\cos(x), \quad \cos(2x) = \cos^2(x) - \sin^2(x), \quad \sin^2(x) + \cos^2(x) = 1.$$

   Also, it is highly recommended to consult with the lecture slides and solve the problems there before you start with these ones.

   (a) $\int \sin(x)\cos(x)\,dx$

   (b) $\int \ln(ax)\,dx$ where $a > 0$

   (c) $\int \cos^2(x)\,dx$

   (d) $\int \frac{1}{\sqrt{1-4x^2}}\,dx$

   (e) $\int e^{3x}\sin(x)\,dx$

   Solution:

   (a) $\int \sin(x)\cos(x) = \frac{1}{2}\sin^2(x) + C$.

   (b) Substituting x by $\frac{y}{a}$ we see that $\int \ln(ax)\,dx = \int \ln(y)\frac{d(a^{-1}y)}{dy}\,dy = \int a^{-1}\ln(y)\,dy = a^{-1}y(\ln(y) - 1) + C = a^{-1}ax(\ln(ax) - 1) + C = x(\ln(ax) - 1) + C$.

   (c) $\int \cos^2(x)\,dx = \int \frac{1}{2} + \frac{1}{2}\cos(2x)\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$. (Another format: $\frac{1}{2}(\sin x\cos x + x) + C$.)

   (d) $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x) + C$.

(e) We compute

$$\begin{aligned}
\int e^{3x}\sin(x)\,dx &= -\int e^{3x}\frac{d\cos(x)}{dx}\,dx \\
&= \left(-e^{3x}\cos(x)\right) - \left(-\int \frac{de^{3x}}{dx}\cos(x)\,dx\right) \\
&= -e^{3x}\cos(x) + 3\int e^{3x}\cos(x)\,dx \\
&= -e^{3x}\cos(x) + 3\int e^{3x}\frac{d\sin(x)}{dx}\,dx \\
&= -e^{3x}\cos(x) + 3e^{3x}\sin(x) - 9\int e^{3x}\sin(x)\,dx.
\end{aligned}$$

Hence we see that

$$\int e^{3x}\sin(x)\,dx = \frac{1}{10}\,e^{3x}\,(3\sin(x) - \cos(x)) + C.$$

[[**Grading Instruction:**

Grading (total 20 points):	
aspect:	points
each part	4 points
missing $+C$	-1 point

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2. **(20 points)** Compute the length of the curve $f(x) = \sqrt{1-x^2}$ where $x \in [-1,1]$

(a) using calculus, and

(b) using a geometric argument.

[Hint: (b) what is the shape of $\sqrt{1-x^2}$?]

Solution:

(a) First note that $f'(x) = -\frac{x}{\sqrt{1-x^2}}$. Thus the length of the curve is

$$\begin{aligned}
\int_{-1}^{1}\sqrt{1+(f'(x))^2}\,dx &= \int_{-1}^{1}\sqrt{1+\frac{x^2}{1-x^2}}\,dx \\
&= \int_{-1}^{1}\sqrt{\frac{1-x^2+x^2}{1-x^2}}\,dx \\
&= \int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,dx \\
&= [\arcsin(x)]_{-1}^{1} \\
&= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.
\end{aligned}$$

(b) The length of the curve is half of the unit circle, the answer is π.

[[**Grading Instruction:**

Grading (total 20 points):	
aspect:	points
correct answer (π)	4 points
(a)	8 points
(b)	8 points

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3. **(20 points)** Compute the definite integral $\int_{-1}^{1} \sqrt{1-x^2}\,dx$

(a) using calculus, and

(b) using a geometric argument.

[Hint: (a) instead of substituting a function of x by u, now substitute $x = \sin(u)$.]

Solution:

(a) We compute:

$$\begin{aligned}
\int_{-1}^{1} \sqrt{1-x^2}\,dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2\vartheta}\,\frac{d\sin\vartheta}{d\vartheta}\,d\vartheta && \text{substitute } x \text{ by } \sin\vartheta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\vartheta \cdot \cos\vartheta\,d\vartheta && \int \cos^2(\vartheta) \text{ from 1(c)} \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} + \frac{1}{2}\cos(2\vartheta)\ d\vartheta \\
&= \left[\frac{1}{2}\vartheta + \frac{1}{4}\sin(2\vartheta)\right]_{\vartheta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \left(\frac{\pi}{4} + \frac{1}{4}\sin(\pi)\right) - \left(-\frac{\pi}{4} + \frac{1}{4}\sin(-\pi)\right) \\
&= \frac{\pi}{2} && \text{since } \sin(\pi) = \sin(-\pi) = 0
\end{aligned}$$

Another form of the primitive (with which some students may work): $\int \cos^2\vartheta\,d\vartheta = \frac{1}{2}(\sin\vartheta\cos\vartheta + \vartheta)$.

(b) Since the area under the function $\sqrt{1-x^2}$ on $[-1,1]$ is half of the unit disk, the answer is $\frac{\pi}{2}$.

[[**Grading Instruction:**

Grading (total 20 points):	
aspect:	points
correct answer ($\frac{\pi}{2}$)	4 points
(a)	8 points
(b)	8 points

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4. **(15 points)** Compute the following improper integrals.

   (a) $\int_0^\infty re^{-r^2}\,dr$;

   (b) $\int_0^{2\pi}\left(\int_0^\infty re^{-r^2}\,dr\right)dt$;

   (c) (bonus, +3 points) Prove that $\int_{-\infty}^\infty e^{-z^2}\,dz = \sqrt{\pi}$.

   You may use the fact that $\int_{-\infty}^\infty\left(\int_{-\infty}^\infty e^{-(x^2+y^2)}\,dx\right)dy = \int_0^{2\pi}\left(\int_0^\infty re^{-r^2}\,dr\right)dt$.

   (d) $\int_0^\infty e^{-z^2}\,dz$.

   Solution:

   (a) $\int_0^\infty re^{-r^2}\,dr \;=\; \left[-\frac{1}{2}e^{-r^2}\right]_{r=0}^\infty \;=\; 0-(-\frac{1}{2}) \;=\; \frac{1}{2}$.

   (b) $\int_0^{2\pi}\int_0^\infty re^{-r^2}\,drd\vartheta \;=\; \int_0^{2\pi}d\vartheta\cdot\int_0^\infty re^{-r^2}\,dr \;=\; 2\pi\cdot\frac{1}{2} \;=\; \pi$.

   (c) We have:

$$\begin{aligned}\int_{-\infty}^\infty e^{-z^2}\,dz \;&=\; \left(\int_{-\infty}^\infty e^{-x^2}\,dx\;\cdot\;\int_{-\infty}^\infty e^{-y^2}\,dy\right)^{\frac{1}{2}}\\ &=\; \left(\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-(x^2+y^2)}\,dxdy\right)^{\frac{1}{2}}\\ &=\; \left(\int_0^{2\pi}\int_0^\infty re^{-r^2}\,drd\vartheta\right)^{\frac{1}{2}} \;=\; \sqrt{\pi}\end{aligned}$$

   (d) We have $\int_{-\infty}^\infty e^{-z^2}\,dz = \sqrt{\pi}$. Solution 1: Since e^{-z^2} is even (*i.e.* $e^{-z^2} = e^{-(-z)^2}$), $\int_0^\infty e^{-z^2}\,dz = \frac{1}{2}\sqrt{\pi}$. Solution 2: $\int_{-\infty}^\infty e^{-z^2}\,dz \;=\; \int_0^\infty e^{-z^2}\,dz \;+\; \int_{-\infty}^0 e^{-z^2}\,dz$. By substituting z by $-w$, we see that $\int_{-\infty}^0 e^{-z^2}\,dz \;=\; \int_\infty^0 e^{-(-w)^2}\frac{d(-w)}{dw}\,dw \;=\; -\int_\infty^0 e^{-w^2}\,dw \;=\; \int_0^\infty e^{-w^2}\,dw$. Thus $\int_{-\infty}^\infty e^{-z^2}\,dz = 2\int_0^\infty e^{-z^2}\,dz$. Hence $\int_0^\infty e^{-z^2}\,dz \;=\; \frac{1}{2}\sqrt{\pi}$.

   [[**Grading Instruction:**

Grading (total 15 points):	
aspect:	points
(a)	5 points
(b)	5 points
(c)	bonus, +3 points
(d)	5 points

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5. **(15 points)** Compute the following improper integrals.

   (a) $\int_0^\infty e^{-x}\,dx$;

   (b) $\int_0^\infty xe^{-x}\,dx$ using integration by parts;

   (c) (bonus, +2 points) $\int_0^\infty x^ne^{-x}\,dx$ for all $n\in\{0,1,\ldots\}$;

   (d) $\int_0^\infty x^{-\frac{1}{2}}e^{-x}\,dx$.

   [Hint: (c) Try first for $n = 0,1,2,3$; (d) substitute $u = \sqrt{x}$ and, at the end, some information from a previous exercise turns out to be useful.]

   Solution:

   (a) $\int_0^\infty e^{-x}\,dx \;=\; [-e^{-x}]_0^\infty \;=\; 0-(-1) \;=\; 1$.

   (b) $\int_0^\infty e^{-x}\,dx \;=\; \int_0^\infty \frac{dx}{dx}e^{-x}\,dx \;=\; [xe^{-x}]_0^\infty \;-\; \int_0^\infty x\frac{de^{-x}}{dx}\,dx \;=\; (0-0)-(-\int_0^\infty xe^{-x}\,dx) \;=\; \int_0^\infty xe^{-x}\,dx$. Thus $\int_0^\infty xe^{-x}\,dx = 1$.

(c) Writing $\gamma_n = \int_0^\infty x^n e^x \, dx$, we have, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\gamma_n &= \int_0^\infty x^n e^{-x} \, dx \\
&= \frac{1}{n+1} \int_0^\infty \frac{dx^{n+1}}{dx} e^{-x} \, dx \\
&= \frac{1}{n+1} \left[x^{n+1} e^{-x} \right]_0^\infty \; - \; \frac{1}{n+1} \int_0^\infty x^{n+1} \frac{de^{-x}}{dx} \, dx \\
&= \frac{0-0}{n+1} \; + \; \frac{1}{n+1} \int_0^\infty x^{n+1} e^{-x} \, dx \\
&= \frac{\gamma_{n+1}}{n+1}.
\end{aligned}$$

Thus $\gamma_0 = 1$ by (a), and $\gamma_{n+1} = (n+1)\gamma_n$ for all $n \in \mathbb{N}$. Hence $\gamma_n = n!$.

(d) If we substitute x by y^2, we get $\int_0^\infty x^{-\frac{1}{2}} e^{-x} \, dx = \int_0^\infty y^{-1} e^{-y^2} \frac{dy^2}{dy} \, dy = 2 \int_0^\infty e^{-y^2} \, dy = \sqrt{\pi}$.

[[**Grading Instruction:**

Grading (total 15 points):	
aspect:	points
(a)	5 points
(b)	5 points
(c)	bonus, +2 points
(d)	5 points

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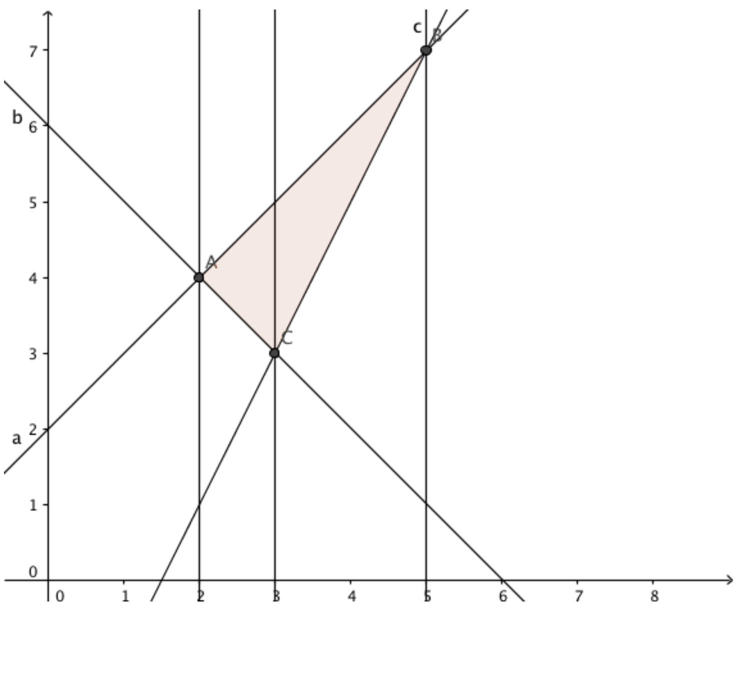
6. **(10 points)**

(a) Given three lines, $y = x + 2$, $y = -x + 6$ and $y = 2x - 3$, enclosing a triangle. Determine the *coordinates* of the three vertices and the *area* of the triangle.

(b) Compute the area of the region bounded by $y = (x-1)^3$ and $y = (x-1)^2$.

Solution:

(a) To get the vertices, we have to solve three equations: $x + 2 = -x + 6$, $2x - 3 = -x + 6$ and $x + 2 = 2x - 3$. We find that the three vertices are $(2, 4), (3, 3)$ and $(5, 7)$. To compute the area of the triangle there are multiple possibilities. We focus here on the one with definite integrals.



After sketching the lines we find that the area can be computed as

$$\int_2^5 (x+2)\,dx - \int_2^3 (-x+6)\,dx - \int_3^5 (2x-3)\,dx = [x^2+2x]_2^5 - [-x^2+6x]_2^3 - [x^2-3x]_3^5 = 3.$$

Solution 2: to compute the length of the edges and to apply some formula, such as Heron's, for the area; Solution 3: recognising that the first two lines orthogonal, one can compute the area easier; namely, $\sqrt{2}\cdot 3\sqrt{2}/2 = 3$.

(b) The intersections are $1, 2$ because $(x-1)^3 = (x-1)^2$ iff $(x-1)^3-(x-1)^2 = 0$ iff $(x-1)^2(x-1-1) = (x-1)^2(x-2) = 0$. (In fact, at $x = 1$ the two curves tangent to each other.)
If $x \in [1,2]$ $x-1 \in [0,1]$ and $(x-1)^3 \leq (x-1)^2$. Therefore, the area can be computed as

$$\int_1^2 ((x-1)^2 - (x-1)^3)\,dx = -\frac{x^4}{4} + \frac{4x^3}{3} - \frac{5x^2}{2} + 2x]_1^2 = \frac{1}{12}.$$

[[**Grading Instruction:**

Grading (total 10):

aspect:	points
(a) vertices	2
(a) proper definite integrals (or other solution)	2
(a) correct result (3)	1
(b) correct interval edges: 1, 2	2
(b) proper definite integral and computation	2
(b) correct result ($\frac{1}{12}$)	1

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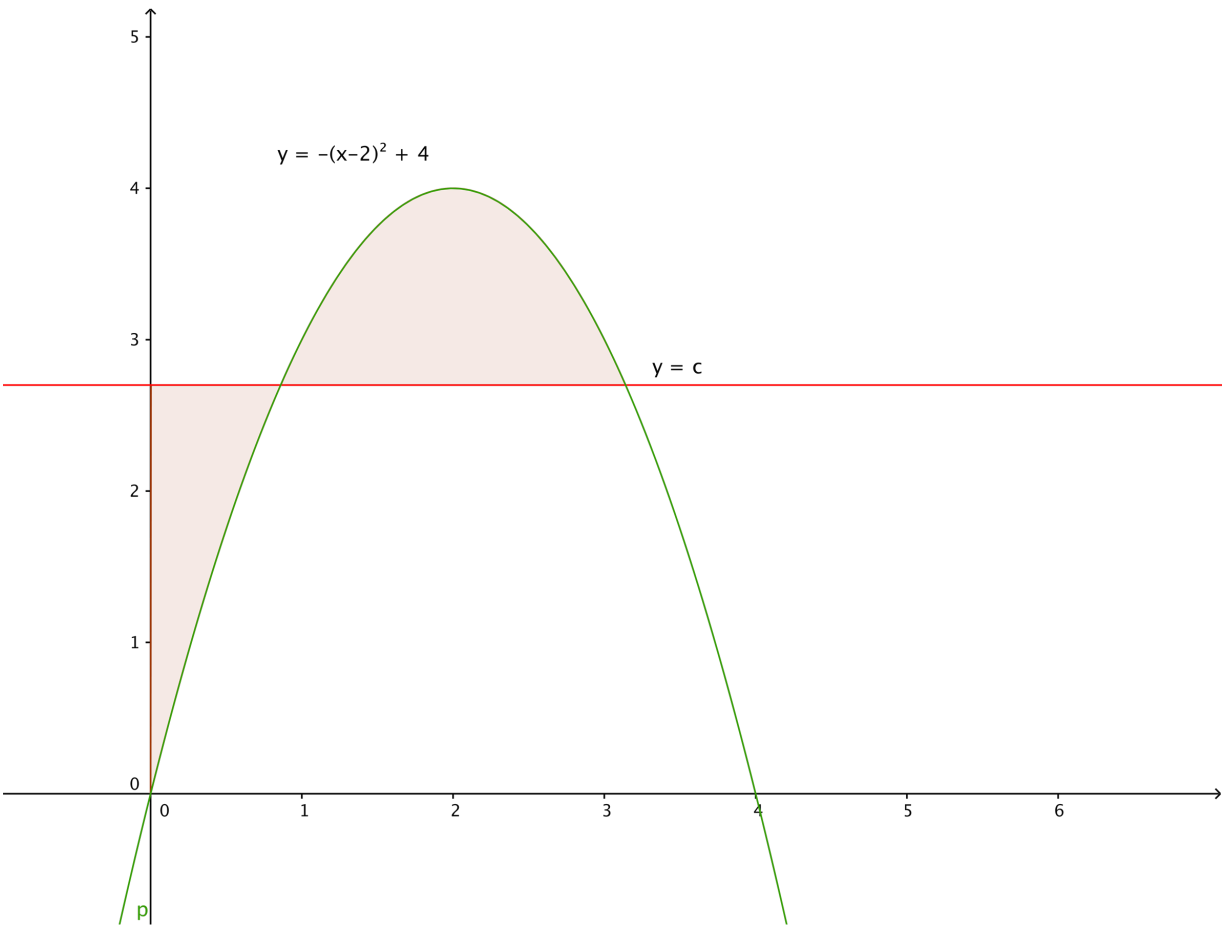
7. **(bonus, 5 points)** The figure shows a horizontal line $y = c$ intersecting the curve $y = -(x-2)^2 + 4$. Find the number c such that the areas of the shaded regions are equal.



Solution: $f(x) = -(x-2)^2 + 4 = -x^2 + 4x$ and $g(x) = c$. Let the x-coordinates of the two intersection points are a, b where $0 < a < b < 4$. Then the two areas are $\int_0^a (g - f) = \int_a^b (f - g)$. Moreover, $b = 4 - a$ and $f(a) = c$, that is, $-b^2 + 4b = c$.

$\int_0^a (g - f) = \int_0^a (c - (-x^2 + 4x))dx = \int_0^a (c + x^2 - 4x)dx = cx + \frac{1}{3}x^3 - 2x^2]_0^a = ca + \frac{1}{3}a^3 - 2a^2$

$\int_a^b (f - g) = \int_a^b ((-x^2 + 4x) - c)dx = -\frac{1}{3}x^3 + 2x^2 - cx]_a^b = -\frac{1}{3}b^3 + 2b^2 - cb + \frac{1}{3}a^3 - 2a^2 + ca.$

Therefore, we have three unknowns (a, b, c) and three equations:

$$-b^2 + 4b = c \tag{1}$$
$$b = 4 - a$$
$$-\frac{1}{3}b^3 + 2b^2 - cb + \frac{1}{3}a^3 - 2a^2 + ca = ca + \frac{1}{3}a^3 - 2a^2 \tag{2}$$

Equation (2) becomes $-\frac{1}{3}b^3 + 2b^2 - cb = 0$. Since $b \neq 0$ (as by the initial assumption $b > a > 0$) $-\frac{1}{3}b^2 + 2b - c = 0$ or $b^2 - 6b + 3c = 0$. Applying Equation (1), we get that $-c - 2b + 3c = 0$ or $b = c$. Using Equation (1) again, we get that $-c^2 + 3c = 0$, that is, $(c \neq 0)$ $c = 3$.

Indeed, if $c = 3$, $a = 1, b = 3$ and both shaded areas are $1 + 1/3$.

[[**Grading Instruction:**

Grading (total 5):	
aspect:	points
proper reasoning	2
proper equations and definite integrals	2
correct result ($c = 3$)	1

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