Calculus and Probability Theory
Assignment 7

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1. (**20 points**) An experiment consists of drawing 3 cards in succession from a well-shuffled ordinary deck of cards (standard 52-card deck). Let A_1 denote the event "Ace on first draw", A_2 the event "Ace on second draw" and A_3 the event "Ace on third draw". State in words the meaning of each of the following probabilities.

   (a) $P(A_1 \cap \neg A_2)$
   Solution:

   The probability to draw an Ace on the first draw and to draw no Ace on second draw.
   You can also say: The probability to draw an Ace on the first draw and to draw anything else except an Ace on the second draw.

   (b) $P(A_1 \cup A_2)$
   Solution:

   The probability to draw an Ace on the first draw or an Ace on second draw.

   (c) $P(\neg A_1 \cap \neg A_2 \cap \neg A_3)$
   Solution:

   The probability to draw no Ace on first draw and no Ace on second draw and no Ace on third draw.
   You can also say: The probability to draw anything in the first three draws except aces.

   (d) $P[(A_1 \cap \neg A_2) \cup (\neg A_2 \cap A_3)]$
   Solution:

The probability to draw an ace on first draw and no Ace on second draw or to draw no ace on second draw and an ace on third draw.

(e) $P(\neg A_1 \cup \neg A_2 | A_1)$
Solution:

The probability to draw no ace on first draw or no ace on second draw, given that an Ace has already been drawn on first draw.

2. **(25 points)** Compute the probabilities (a)-(e) in Exercise 1.

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ P(A|B) \cdot P(B) &= P(A \cap B) \\ P(A \cup B) &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

(a)

$$\begin{aligned} P(A_1 \cap \neg A_2) &= P(\neg A_2 | A_1) \cdot P(A_1) \\ &= \frac{48}{51} \cdot \frac{4}{52} \\ &= \frac{16}{221} \\ &\approx 0.0724 \end{aligned}$$

(b)

$$\begin{aligned} P(A_1 \cup A_2) &= P(A_1) + P(A_2) - P(A_1 \cap A_2) \\ &= \frac{4}{52} + \frac{4}{51} - (\frac{4}{52} * \frac{3}{51}) \\ &= \frac{100}{664} \\ &\approx 0.1508 \end{aligned}$$

(c)

$$\begin{aligned} P(\neg A_1 \cap \neg A_2 \cap \neg A_3) &= P(\neg A_2 \cap \neg A_3 | \neg A_1) \cdot P(\neg A_1) \\ &= (\frac{47}{51} \cdot \frac{46}{50}) \cdot \frac{48}{52} \\ &= \frac{1081}{1275} \cdot \frac{48}{52} \\ &= \frac{4324}{5525} \\ &\approx 0.7826 \end{aligned}$$

(d)

$$\begin{aligned}
P[(A_1 \cap \neg A_2) \cup (\neg A_2 \cap A_3)] = \\
P(A_1 \cap \neg A_2) + P(\neg A_2 \cap A_3) - P[(A_1 \cap \neg A_2) \cap (\neg A_2 \cap A_3)] = \\
(\frac{4}{52} \cdot \frac{48}{51}) + (\frac{48}{51} \cdot \frac{4}{50}) - (\frac{4}{52} \cdot \frac{48}{51} \cdot \frac{3}{50}) = \\
\frac{792}{5525} \approx \\
0.1433
\end{aligned}$$

(e)

$$\begin{aligned}
P(\neg A_1 \cup \neg A_2 | A_1) &= \frac{(\neg A_1 \cup \neg A_2) \cap A_1}{P(A_1)} \\
&= \frac{\frac{4}{52} \cdot \frac{48}{51}}{\frac{4}{52}} \\
&= \frac{16}{17} \\
&\approx 0.9411
\end{aligned}$$

3. **(20 points)** Assume that a pair of fair dice are to be tossed, and let the random variable X denote the sum of the points.

   (a) Make a table of the probability function containing the possible values of X and their probabilities.
   Solution:

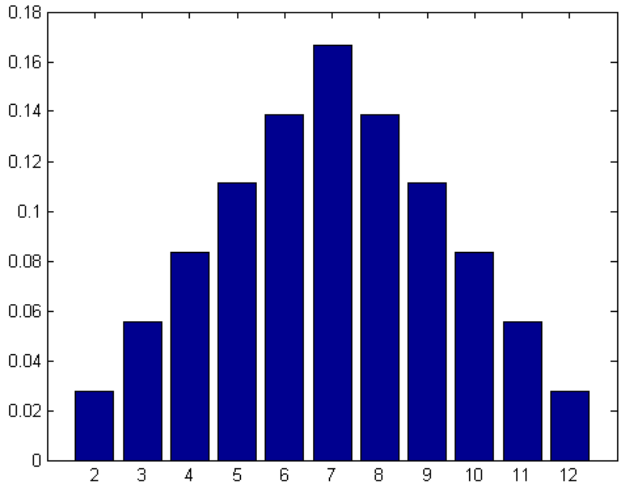
   Sum of dice:

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

   Probabilities of values:

2	3	4	5	6	7	8	9	10	11	12
$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

(b) Draw a histogram of this probability function.
Solution:



(c) What is $P(X\ is\ even)$?
Solution:

$$\begin{aligned}
P(X\ is\ even) &= P(X=2)+P(X=4)+P(X=6) \\
&+ P(X=8)+P(X=10)+P(X=12) \\
&= \frac{1}{36}+\frac{1}{12}+\frac{5}{36}+\frac{5}{36}+\frac{1}{12}+\frac{1}{36} \\
&= \frac{1}{2} \\
&= 0.5
\end{aligned}$$

(d) What is $P(X\ is\ even|X=2)$?
Solution:

$$\begin{aligned}
P(X\ is\ even|X=2) &= \frac{P(X\ is\ even \cap X=2)}{P(X=2)} \\
&= \frac{\frac{1}{36}}{\frac{1}{36}} \\
&= 1
\end{aligned}$$

(d) What is $P(X = 2|X \text{ } is \text{ } even)$?
Solution:

$$\begin{aligned} P(X = 2|X \; is \; even) &= \frac{P(X = 2 \cap X \; is \; even)}{P(X \; is \; even)} \\ &= \frac{\frac{1}{36}}{\frac{1}{2}} \\ &= \frac{1}{18} \\ &\approx 0.055 \end{aligned}$$

4. **(20 points)** If 10% of the bolts produced by a machine are defective, determine the following probabilities.

(a) Out of four bolts chosen at random 1 bolt will be defective.
Solution:

We have $n = 4$, defective probability $p = \frac{1}{10}$ and one bolt defective.

$$\begin{aligned} b(1) &= \binom{4}{1}(\frac{1}{10})^1(\frac{9}{10})^3 \\ &= \frac{4}{10} \cdot \frac{729}{1000} \\ &= \frac{729}{2500} \\ &\approx 0.2916 \end{aligned}$$

The probability that out of four bolts chosen at random, 1 bolt will be defective, is 29.16%.

(b) Out of four bolts chosen at random 0 bolt will be defective.
Solution:

We have $n = 4$, defective probability $p = \frac{1}{10}$ and 0 bolt defective.

$$\begin{aligned} b(0) &= \binom{4}{0}(\frac{1}{10})^0(\frac{9}{10})^4 \\ &= 1 \cdot \frac{6561}{10000} \\ &\approx 0.6561 \end{aligned}$$

The probability that out of four bolts chosen at random, 0 bolts will be defective, is 65.61%.

(c) Out of four bolts chosen at random less than 2 bolts will be defective.
Solution:

We can just add the probabilities from (a) and (b)here:

$$P = \frac{6561}{10000} + \frac{729}{2500} \approx 0.9477$$

(d) Out of four bolts chosen at random 2,3 or 4 bolts are defective.
Solution:

$$\begin{aligned} b(2)+b(3)+b(4) &= \binom{4}{2}(\frac{1}{10})^2(\frac{9}{10})^2 + \binom{4}{3}(\frac{1}{10})^3(\frac{9}{10})^1 + \binom{4}{4}(\frac{1}{10})^4(\frac{9}{10})^0 \\ &= \frac{243}{5000} + \frac{9}{2500} + \frac{1}{10000} \\ &= \frac{523}{10000} \\ &= 0.0523 \end{aligned}$$

5. **(15 points)** Urn A has 2 white and 3 red balls. Urn B has 4 white and 1 red ball. And Urn C has 3 white and 4 red balls. An urn is selected at random and a ball drawn at random is found to be white. Find the probability that Urn A was selected.
Solution:

There are 17 balls in total. 9 white balls and 8 red balls.

Urn A:

- 5 balls in total $\rightarrow \frac{5}{17}$ of all balls
- 2 white balls $\rightarrow \frac{2}{9}$ of all white balls $\rightarrow \frac{2}{5}$ of all balls in Urn A are white
- 3 red balls $\rightarrow \frac{3}{8}$ of all red balls $\rightarrow \frac{3}{5}$ of all balls in Urn A are red

Urn B:

- 5 balls in total $\rightarrow \frac{5}{17}$ of all balls
- 4 white balls $\rightarrow \frac{4}{9}$ of all white balls $\rightarrow \frac{4}{5}$ of all balls in Urn B are white
- 1 red balls $\rightarrow \frac{1}{8}$ of all red balls $\rightarrow \frac{1}{5}$ of all balls in Urn B are red

Urn C:

- 7 balls in total $\rightarrow \frac{7}{17}$ of all balls
- 3 white balls $\rightarrow \frac{3}{9}$ of all white balls $\rightarrow \frac{3}{7}$ of all balls in Urn C are white

- 4 red balls $\rightarrow \frac{4}{8}$ of all red balls $\rightarrow \frac{4}{7}$ of all balls in Urn C are red

If you draw a ball at random then the chance that it came from Urn A is $\frac{5}{17}$

Question reformulated:

$$\begin{aligned}
P(Urn\ A\ selected|Ball = white) &= \frac{P(Ball = white|Urn\ A\ selected) \cdot P(Urn\ A\ selected)}{P(Ball = white)} \\
&= \frac{\frac{2}{5} \cdot \frac{1}{3}}{\frac{9}{17}} \\
&= \frac{\frac{2}{15}}{\frac{9}{17}} \\
&= \frac{34}{135} \\
&\approx 0.2519
\end{aligned}$$