Wiskunde 1b: Continuity, Limits and Differentiation

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Outline

Continuous functions

Definition

A function $f : D \to \mathbb{R}$ is continuous in point $a \in D$ if $f(x)$ is close to $f(a)$ for each x that is close to a.

More formally: $f : D \to \mathbb{R}$ is continuous in point $a \in D$ if:
$\forall \epsilon > 0\, \exists \delta > 0\, \forall x \in D\ |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$.

Example

The function $f(x) = sign(x)$ is not continuous in 0
Indeed, $\exists \epsilon > 0\, \forall \delta > 0\, \exists x$ with $|x - 0| = |x| < \delta$ but $|f(x) - f(0)| = |f(x)| \geq \epsilon$
Choose $\epsilon = \frac{1}{2}$, then any $x \neq 0$ with $|x| < \delta$ has $|sign(x)| = 1$. The function values around 0 do not fall into the ϵ-interval.

Definition

A function $f : D \to \mathbb{R}$ is continuous if it is continuous in all $a \in D$.

Limits

Definition

A function $f : D \to \mathbb{R}$ has limit b for $x \to a$ if:
$\forall \epsilon > 0\, \exists \delta > 0\, \forall x \in D\ |x - a| < \delta \Rightarrow |f(x) - b| < \epsilon$.

In that case we write $\lim_{x \to a} f(x) = b$.

(Note: a does not have to be in D.)

Example

- $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x-2} = 4$.
- $\lim_{x \to 1} \frac{-1}{x-1}$ is undefined

Limits involving infinity

We also need $\lim_{x \to \infty} f(x)$ or $\lim_{x \to -\infty} f(x)$. What does this mean?

Definition

A function $f : D \to \mathbb{R}$ has limit b for $x \to \infty$ if:
$\forall \epsilon > 0\, \exists n \in \mathbb{N}\, \forall x \in D\ x > n \Rightarrow |f(x) - b| < \epsilon$.

In that case we write $\lim_{x \to \infty} f(x) = b$. Formulate yourself what $\lim_{x \to -\infty} f(x) = b$ means.

Example

- $\lim_{x \to \pm\infty} \frac{1}{x} = 0$
- $\lim_{x \to \pm\infty} \frac{6x^2 + 2x + 1}{5x^2 - 3x + 4}$
- $\lim_{x \to +\infty} \frac{2x^3}{x^2 + 1}$

Derivatives

Definition

A function $f : D \to \mathbb{R}$ is differentiable at a if $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists. In this case the limit is denoted $f'(a)$ and is called the derivative of f at a.

f is differentiable if f is differentiable at a for every $a \in D$.

If $f(x) = \cdots$ then f' ("Lagrange notation") is sometimes written as $\frac{df}{dx}$ ("Leibniz notation")

We also define the tangent line to f at a to be the line through $(a, f(a)) \in G(f)$ with slope $f'(a)$.

More examples

Example (Geometric interpretation)

Find a tangent line of a curve $f(x) = \frac{1}{x}$ in $x = 2$.

Example

Investigate if $f(x) = |x|$ is differentiable in 0.

Differentiation rules

Let $f, g : D \to \mathbb{R}$ be differentiable functions in $a \in D$

- For a constant function $f(x) = c, c \in \mathbb{R}$, we have $f'(x) = 0$
- $f(x) = x$, then $f'(x) = 1$.
- plus/minus rule $(f \pm g)'(a) = f'(a) \pm g'(a)$.
- scalar rule: $(c \cdot f)'(a) = c \cdot f'(a)$.
- multiplication rule $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$.
- division rule $(\frac{f}{g})'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}$.
- chain/composition rule $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$
- if f has an inverse f^{-1}, then $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$

Derivatives of powers

Lemma

1. *For $n \in \mathbb{N}$ and $f(x) = x^n$ we have $f'(x) = nx^{n-1}$*
   - *This can be shown by induction on n*
2. *In fact, for $n \in \mathbb{Z}$ and $f(x) = x^n$ we have $f'(x) = nx^{n-1}$*
   - *This follows from the previous point, using the division rule.*
3. *It can be shown that $\frac{dx^a}{dx} = ax^{a-1}$, for each $a \in \mathbb{R}$.*

Monotonicity and the derivative

Definition

Let $f : D \to \mathbb{R}$ be a function.

- f is increasing if $x_1 < x_2 \implies f(x_1) \leq f(x_2)$, for all $x_1, x_2 \in D$
- f is strictly increasing if $x_1 < x_2 \implies f(x_1) < f(x_2)$, for all $x_1, x_2 \in D$
- f is decreasing if $x_1 < x_2 \implies f(x_1) \geq f(x_2)$, for all $x_1, x_2 \in D$
- f is strictly decreasing if $x_1 < x_2 \implies f(x_1) > f(x_2)$, for all $x_1, x_2 \in D$

How related to derivatives

- If $f'(x) \geq 0$, $\forall x \in [a, b] \Rightarrow f$ is increasing on $[a, b]$.
- If $f'(x) \leq 0$, $\forall x \in [a, b] \Rightarrow f$ is decreasing on $[a, b]$.

Derivation exercises

Example

- $f(x) = (2 + 3x)(2 - 3x)$. Find f'.
- $y = x^5 - 3x^3 + 4x^2 - 3$. Find f'.
- $y = \sqrt{2 - 5x}$. Find f'.
- $f(x) = x^2$. Find $(f^{-1})'$.