Integrals and applications

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Outline

Introduction to Integration

- We looked at differentiation: going from f to its derivative f'
  - associated notions: tangent line, monotonicity, extrema, ...
- Now we look at integration: going from f to F with $F' = f$.
  - What does such a primitive F of f tell us about f?
- Well, if $f(x) = F'(x) = \lim_{h \to 0} \frac{F(x+h)-F(x)}{h}$, then for small $h > 0$,
$$f(x) \cdot h \approx F(x+h) - F(x)$$
- So, the primitive F gives some information about the surface under the graph of f
  - integration can also be used to calculate volumes, in more dimensions

The area problem and the (definite) integral

- Let f be a continuous function defined on the interval $[a,b]$. In order to estimate the area under f from a to b we divide $[a,b]$ into n subintervals: $[x_0,x_1]$, $[x_1,x_2]$, $[x_2,x_3]$, ..., $[x_{n-1},x_n]$, where $a = x_0, b = x_n$, each of length $\Delta x = \frac{b-a}{n}$. (Hence we can write $x_i = a + i\Delta x, i = 0, \cdots, n$)
- The area S_i of the strip between x_{i-1} and x_i can be approximated as the area of the rectangle of width Δx and height $f(x_i{}^*)$, for some $x_i{}^* \in [x_i, x_{i+1}]$. Hence $S_i \approx f(x_i^*) \cdot \Delta x$.
- So, the total area A under f is close to the sum of the S_i:
$$A \approx f(x_1{}^*) \cdot \Delta x + f(x_2{}^*) \cdot \Delta x + \ldots + f(x_n{}^*) \cdot \Delta x$$
- The area A itself is then obtained as limit. This is the integral
$$\int_a^b f(x)dx = A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i{}^*)\Delta x$$

The evaluation theorem

Theorem

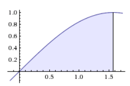
If f is a continuous function with primitive F, that is, with $F'(x) = f(x)$, then: $\int_a^b f(x)dx = F(b) - F(a)$.
This difference $F(b) - F(a)$ is the area below f on $[a,b]$.

$F(b) - F(a)$ is abbreviated as $F(x)]_a^b$. So, $\int_a^b f(x)dx = F(x)]_a^b$.

Example

Compute the following integrals using the evaluation theorem:

- $\int_0^1 3dx$
- $\int_0^1 x^2 dx$
- $\int_2^4 xdx$
- $\int_0^{\frac{\pi}{2}} \sin xdx$



Linearity and interval properties of integrals

Lemma

1. *integration of a constant function: $\int_a^b cdx = c(b-a)$*
2. *addition: $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$*
3. *scalar multiplication: $\int_a^b c \cdot f(x)dx = c \cdot \int_a^b f(x)dx$*

Lemma

1. $\int_a^a f(x)dx = 0$
2. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, *where $a < c < b$*
3. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Order properties of integrals

Lemma

1. *if $f(x) \geq g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$*
   With two useful special cases:
2. *if $f(x) \geq 0$ for all $x \in [a,b]$, then $\int_a^b f(x)dx \geq 0$*
3. *$m \leq f(x) \leq M$, for $x \in [a,b]$, then*
   $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

Definite and indefinite integrals

There is a distinction between:

- The definite integral $\displaystyle\int_a^b f(x)dx$
  This is a number that represents the area under the curve $f(x)$ from $x = a$ to $x = b$.
- The indefinite integral $\displaystyle\int f(x)dx$
  This is notation for a function F with $F' = f$.

Indefinite integrals

Definition

1. A function F such that $F'(x) = f(x)$ is called a primitive (or an antiderivative) function of f
   - Note: $F + C$ is then also a primitive of f, for any constant C
2. The indefinite integral $\int f(x)dx$ of f is used as notation for all these primitives. Thus: $\int f(x)dx = F + C$.

Table of indefinite integrals

- $\int 0\, dx = C$
- $\int a\, dx = ax + C$, so $\int 1 dx = x + C$
- $\int x^n dx = \frac{1}{n+1} \cdot x^{n+1} + C$, for $n \neq -1$
- $\int \frac{1}{x} dx = \ln|x| + C$
- $\int e^x dx = e^x + C$
- $\int a^x dx = \frac{1}{\ln a} \cdot a^x + C$
- $\int \sin x dx = -\cos x + C$
- $\int \cos x dx = \sin x + C$
- $\int \frac{1}{cos^2 x} dx = \tan x + C$
- $\int \frac{1}{1+x^2} dx = \arctan x + C$
- $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$

Examples

Example

- $$\begin{aligned} \int (3x^5 - 2x^2 + 1)dx &= \int 3x^5 dx - \int 2x^2 dx + \int 1 dx \\ &= 3\int x^5 dx - 2\int x^2 dx + \int 1 dx \\ &= \tfrac{1}{2}x^6 - \tfrac{2}{3}x^3 + x + C \end{aligned}$$
- $$\begin{aligned} \int (\sqrt[3]{x^2} - \tfrac{1}{x^2})dx &= \int x^{\frac{2}{3}} dx - \int x^{-2} dx \\ &= \frac{1}{\frac{5}{3}}x^{5/3} - \frac{1}{-1}x^{-1} + C \\ &= \tfrac{3}{5}x\sqrt[3]{x^2} + \tfrac{1}{x} + C \end{aligned}$$

Two useful techniques

There are no general rules for integration. We discuss the following two techniques.

1. Substitution
2. Integration by parts

They both require appropriate choices in individual cases. They are best learned by doing.

Ad 1. The substitution method

Lemma

$$\int f(g(x))g'(x)dx = \int f(u)du$$

where $g(x)$ is replaced by u.

Justification: Let $u = g(x)$ and $du/dx = g'(x)$. By the chain rule,

$$\left(\int f(u)du\right)'_x = \left(\int f(u)du\right)'_u \frac{du}{dx} = f(u) \cdot \frac{du}{dx} = f(g(x)) \cdot g'(x).$$

Ad 1. The substitution method - Examples

Example

- $\int \cos \frac{x}{3} dx$ Let $u = \frac{x}{3}$. Then $du = \frac{1}{3}dx$. So, $dx = 3du$. By substitution,
  $\int \cos\left(\frac{x}{3}\right) dx = \int \cos(u)3du = 3\sin(u) = 3\sin\left(\frac{x}{3}\right) + C$
- $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx$ Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}dx$. So, $2du = \frac{1}{\sqrt{x}}dx$. Thus,
  $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx = \int \cos u \cdot 2du = 2\sin(\sqrt{x}) + C$
- $\int x\sin(x^2)dx$ Let $u = x^2$.
- $\int \frac{x}{\cos^2(4x^2-5)} dx$ Let $u = 4x^2 - 5$. (Hint: $(\tan(x))' = \frac{1}{\cos^2(x)}$)

Ad 1. Adapting boundaries after substitution

When using substitution for definite integrals:

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Example

Using subsitution $u = x^2 + 1$ we get $\frac{du}{dx} = 2x$ and so $xdx = \frac{1}{2}du$. Hence:

$$\begin{aligned}
\int_0^2 x\cos(x^2+1)dx &= \frac{1}{2}\int_0^2 2x\cos(x^2+1)dx \\
&= \frac{1}{2}\int_1^5 \cos(u)du \\
&= \frac{1}{2}\left[\sin(u)\right]_1^5 \\
&= \frac{1}{2}(\sin(5) - \sin(1)).
\end{aligned}$$

Ad 2. Integration by parts

- Recall the product rule for differentiation:
  $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.
- After integration we get:
  $f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$
  or
  $\int f'g = fg - \int fg'$

Example

- $\int x \ln x dx$
  $= \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}dx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int xdx = \frac{1}{4}x^2(2\ln x - 1) + C$

	f	g
orig.	$\frac{x^2}{2}$	$\ln x$
der.	x	$\frac{1}{x}$

Ad 2. Examples, for integration by parts

Example

Compute the following indefinite integrals using the method of integration by parts:

- $\int xe^x dx$
- $\int x\sin(x)dx$
- $\int x^2 \cdot \ln(x)dx$
- $\int x^3 e^{x^2} dx$ (Hint: $x^2 \cdot xe^{x^2}$; $u = x^2$)

Example

Compute the following definite integrals using integration by parts:

- $\int_0^1 xe^x dx$
- $\int_0^{\pi/2} x\sin(x)dx$
- $\int_1^e x^2 \cdot \ln(x)dx$
- $\int_0^1 x^3 e^{x^2} dx$

Suppose you are working on self-driving cars . . .

Setting and question

- Assume the brake of a car is used with constant push/power
- Hence the change in velocity over time satisfies $v'(t) = -k$
- How far does the car go, starting from initial velocity v_0?

Answer

- We get $v(t) = \int v'dt = \int -kdt = -k \cdot t + C$
  - We have $v(0) = C = v_0$, so $v(t) = -k \cdot t + v_0$.
- Thus $v(t) = 0$, for $t_1 = \frac{v_0}{k}$
- The distance function $s(t)$ satisfies $s' = v$. Hence:
  - $s = \int s'dt = \int vdt = \int -k \cdot t + v_0 dt = -\frac{1}{2}kt^2 + v_0 t + C$
- At $t_1 = \frac{v_0}{k}$, when speed is 0, the distance is:
  $s_1 = \int_0^{\frac{v_0}{k}} s'dt = -\frac{1}{2}kt^2 + v_0 t\Big]_0^{\frac{v_0}{k}} = -\frac{1}{2}k\left(\frac{v_0}{k}\right)^2 + v_0\frac{v_0}{k} = \frac{v_0^2}{2k}$

Learning to find substitutions

Example (See LNBS for details)

- $\int \sin^5(x)\cos(x)dx = \frac{1}{6}\sin^6(x)$
- $\int \frac{x}{1+x^2}dx = \frac{1}{2}\ln(1+x^2)$
- $\int \tan(x)dx = -\ln(\cos(x))$
- $\int \frac{1}{x\ln(x)}dx = \ln(\ln(x))$.

Areas and arc lengths

- Recall: the area below a function f on $[a, b]$ is
$$\int_a^b f(x)dx$$
- The area between f, g on $[a, b]$ is
$$\int_a^b (f(x) - g(x))dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$
Here we assume $f(x) \geq g(x)$, for $x \in [a, b]$.

Definition

Let f be a differentiable function on $[a, b]$. The arc length of f on this interval is
$$\int_a^b \sqrt{1 + f'(x)^2}dx$$

Area and arc length computations

- Compute the area below $f(x) = \sin^2(x)\cos(x)$ on $[0, \frac{\pi}{2}]$
Substituting $u = \sin(x)$ yields:
$$\int_0^{\frac{\pi}{2}} \sin^2(x)\cos(x)dx = \int_0^1 u^2 du = \frac{u^3}{3}\Big]_0^1 = \frac{1}{3}$$
- Compute the area bounded by $y^2 = 4x$ and $4x - 5y + 4 = 0$.
Solution: $\frac{9}{8}$.
- Find the length of the curve of $f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln(x)$ from $x = 1$ to $x = e$.
  - $f'(x) = \frac{1}{2}x - \frac{1}{2} \cdot \frac{1}{x} = \frac{x^2-1}{2x}$
  - $$\begin{aligned} \int_1^e \sqrt{1 + \frac{(x^2-1)^2}{4x^2}} &= \int_1^e \frac{\sqrt{x^4+2x^2+1}}{2x} = \frac{1}{2}\int_1^e \frac{\sqrt{(x^2+1)^2}}{x} \\ &= \frac{1}{2}\int_1^e \frac{x^2+1}{x} = \frac{1}{2}\int_1^e (x + \frac{1}{x})dx \\ &= \frac{1}{2}(\frac{x^2}{2} + \ln x)\Big|_1^e = \frac{1}{4}(e^2+1) \end{aligned}$$