Calculus and Probability

Assignment 3

Note:

- You can hand in your solutions as a single PDF via the assignment module in Blackboard. Note that the document should be in English and typeset with LaTeX, Word or a similar program. It should not be a scan or picture of your handwritten notes.
- Make sure that your name, student number and group number are on top of the first page!
- **Note that your submission should be an individual submission because it can influence your final grade for this course. If we detect that your work is not completely your own work, we will ask the exam committee to investigate whether it is plagiarism or not!**

Exercises to be presented during the exercise hours

Exercise 1
The function arcsin is the inverse function of sin.

a) What is the domain of the function $\arcsin(x)$? Why?

$[-1, 1]$, as this is the range of $\sin(x)$;

b) Compute the following values and explain how you got the result:

$$\arcsin(1) = ? \qquad \arcsin(0) = ? \qquad \arcsin\left(\frac{\sqrt{3}}{2}\right) = ?$$

$\sin(0) = 0$, $\sin(\pi/2) = 1$, $\sin(\pi/3) = \frac{\sqrt{3}}{2}$, so the results are respectively $\pi/2$, 0 and $\pi/3$

c) Find the derivative of f:

$$f(x) = \arcsin\left(\frac{2x}{1-x}\right).$$

Use the chain rule and the known derivative of arcsin. Alternatively, use the inverse rule. So we get $f'(x) = \frac{1}{\sqrt{1-(\frac{2x}{1-x})^2}} \frac{2}{(1-x)^2} = \frac{1-x}{\sqrt{1-2x-3x^2}} \frac{2}{(1-x)^2} = \frac{2}{(1-x)\sqrt{(x+1)(-3x+1)}}$.

Exercise 2
Find the limits of the following functions. (Note that before you can apply L'Hôpital's rule, you have to verify whether it is possible.)

a) $\lim\limits_{x\to\infty} \frac{e^{n-x}}{x^{-m}}$ with $m, n \in \mathbb{N}$; (Hint: if unclear first solve a particular case, e.g., $n = 0, m = 3$.)

$\lim\limits_{x\to\infty} \frac{e^{n-x}}{x^{-m}} = \lim\limits_{x\to\infty} \frac{e^{-(x+n)}}{x^{-m}} = \lim\limits_{x\to\infty} \frac{x^m}{e^{(x+n)}} \stackrel{!}{=} \lim\limits_{x\to\infty} \frac{m!}{e^{(x+n)}} = 0$
i.e., at some point the exponential grows quicker than any polynomial. Note that at the ! sign we have $\frac{\infty}{\infty}$ and applied L'Hôpital's rule m times.

b) If $\lim\limits_{x\to 0} \frac{\sqrt[3]{(a\cdot x+b)}-2}{x} = \frac{5}{12}$ with $a, b \in \mathbb{N}$ then $a \cdot b = ?$;

We have $\frac{\sqrt[3]{b}-2}{0}$ which is $\pm\infty$ unless $\sqrt[3]{b}-2 = 0$, hence $b = 8$. Then we have $\frac{0}{0}$ and can apply L'Hôpital's rule $\lim\limits_{x\to 0} \frac{\sqrt[3]{(a\cdot x+8)}-2}{x} = \lim\limits_{x\to 0} \frac{1}{3}(ax+b)^{-\frac{2}{3}} \cdot a$ which goes to $\frac{1}{3} \cdot 8^{-\frac{2}{3}} \cdot a = \frac{a}{12}$, hence $a = 5$. Thus $a \cdot b = 5 \cdot 8 = 40$.

Exercise 3
Given the function $f(x) = \log_3(2x)$: What is $f'''(x)$?

$f'''(x) = (\log_3(2x))''' = (\frac{1}{\ln 3}x^{-1})'' = (-\frac{1}{\ln 3}x^{-2})' = \frac{2}{\ln 3}x^{-3}$.

Exercise 4
We will investigate the function

$$f(x) = \frac{(x-2)^2}{x+2}.$$

Additionally, we prove that the line $y = x - 6$ is a slant asymptote on both sides.

a) Determine the domain of function f.

$\mathbb{R} \setminus \{-2\}$;

b) What are the roots of f? Where does the graph of f intersect the y axis?

Only zero is $x_0 = 2$. We have $f(0) = 2$.

c) Determine the limits at the edges of the domain.

We should consider limits to infinity, and limits from both sides to -2. This gives $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to-2^+} f(x) = \infty$, $\lim_{x\to-2^-} f(x) = -\infty$.

d) Find f' and f''.

Use the quotient rule. It gives $f'(x) = \frac{(2x-4)(x+2)-(x^2-4x+4)}{(x+2)^2} = \frac{x^2+4x-12}{x^2+4x+4}$. Using the quotient rule again we find $f''(x) = \frac{(2x+4)(x^2+4x+4)-(2x+4)(x^2+4x-12)}{(x+2)^4} = \frac{32}{(x+2)^3}$.

e) Find the zeros of f' and f''.

$f'(x) = 0 \iff x^2 + 4x - 12 = 0 \iff (x+6)(x-2) = 0$. So get two points $x = -6$ and $x = 2$. It is immediate that f'' has no zeros.

f) What are the critical points (determine their x and y coordinates)?

We have $f(2) = 0$ and $f(-6) = -16$, so the two points $(2, 0)$ and $(-6, -16)$.

g) Find the local minima and maxima.

As $f''(2) = \frac{32}{4^3} > 0$, this is a minimum, and as $f''(-6) = \frac{32}{(-4)^3} < 0$, this is a maximum.

h) Which parts of the function are convex and concave? Does function f have points of inflection? (Hint: Use the sign of the second derivative for answering both questions.)

We see that $f''(x) > 0 \iff \frac{32}{(x+2)^3} > 0 \iff x > -2$ and $f''(x) < 0 \iff \frac{32}{(x+2)^3} < 0 \iff x < -2$. So convex for $x > -2$ and concave for $x < -2$. There are no points such that $f''(x) = 0$, so no points of inflection.

i) Show that the line $y = x - 6$ is a slant asymptote of f. (Hint: Use the definition on slide 47 of the lecture and the following two limits.)

$$\lim_{x\to-\infty} (f(x) - (x-6)) = ? \qquad \text{and} \qquad \lim_{x\to+\infty} (f(x) - (x-6)) = ?$$

We have $(f(x) - (x-6)) = \frac{16}{x+2}$, so from this we see that $\lim_{x\to\pm\infty}(f(x) - (x-6)) = 0$. Hence it is a slant asymptote.

Exercise 5

a) Find the derivative of $f(x) = \ln(\cos(\ln(\cos(x))))$.

$f'(x) = (\ln(\cos(\ln(\cos(x)))))' = \frac{1}{\cos(\ln(\cos(x)))} \cdot (-\sin(\ln(\cos(x)))) \cdot \frac{1}{\cos(x)} \cdot (-\sin(x)) = \tan(\ln(\cos(x))) \cdot \tan(x)$.

b) Find a function $g(x)$ such that $g'(x) = \tan(2x)$.

From the previous problem we see that $(\ln(\cos(x)))' = -\tan(x)$. Thus, $(\ln(\cos(2x)))' = -2\tan(x)$ and $-\frac{1}{2}\ln(\cos(2x))' = \tan(x)$. So, $g(x) = -\frac{1}{2}\ln(\cos(2x)) + C$ where $C \in \mathbb{R}$. In grading the constant is not important here.

Exercises to be handed in

> You are expected to explain your answers, even if this is not explicitly stated in the exercises themselves.

Exercise 6
The function arccos is the inverse function of cos. Find the derivative of f (hint: for some ways to derive this, it can be useful to remember that $\sin^2 x + \cos^2 x = 1$):

$$f(x) = \arccos\left(\cos x^2\right).$$

1 pt

The direct solution is to recognize that within the domain of arccos, $\arccos\left(\cos x^2\right) = x^2$, and so $f'(x) = 2x$. Alternatively, using the chain rule and the known derivative of arccos, we get $f'(x) = \frac{-1}{\sqrt{1-(\cos x^2)^2}} * -2x \sin x^2 = \frac{2x \sin x^2}{\sqrt{\sin x^2}^2} = 2x$, by using the identity: $\sin^2\theta + \cos^2\theta = 1$.

Exercise 7
Find the limits of the following functions. (Note that before you can apply L'Hôpital's rule, you have to verify whether it is possible.)

a) $\lim\limits_{x\to\infty} \frac{x^2}{1+e^{-x}}$; 1 pt

We can not, nor need, to use L'Hôpital's rule in this case: the limit of the denominator goes to 1, while the limit of the numerator goes to ∞. So the limit of the function is ∞.

b) If $\lim\limits_{x\to 0} \frac{\sin(x)+Ax+Bx^3}{x^5} = \frac{1}{C}$ with $A, B, C \in \mathbb{Q}$, then $A \cdot B \cdot C = ?$. 1 pt

Since we have $\frac{0}{0}$, L'Hopital gives: $\lim\limits_{x\to 0} \frac{\cos(x)+A+3Bx^2}{5x^4} = \frac{1+A}{0}$ which is $\pm\infty$ unless $A = -1$. Then we have $\frac{0}{0}$ and can apply L'Hopital (repeatedly): $\lim\limits_{x\to 0} \frac{\cos(x)+A+3Bx^2}{5x^4} \stackrel{!}{=} \lim\limits_{x\to 0} \frac{-\sin(x)+6Bx}{20x^3} \stackrel{!}{=} \lim\limits_{x\to 0} \frac{-\cos(x)+6B}{60x^2} = \frac{-1+6B}{0}$ which is $\pm\infty$ unless $-1+6B = 0$, hence $B = \frac{1}{6}$. Continuing with L'Hopital gives $\lim\limits_{x\to 0} \frac{-\cos(x)+6B}{60x^2} \stackrel{!}{=} \lim\limits_{x\to 0} \frac{\sin(x)}{120x} \stackrel{!}{=} \lim\limits_{x\to 0} \frac{\cos(x)}{120}$, hence $C = 120$. Then $A \cdot B \cdot C = -1 \cdot \frac{1}{6} \cdot 120 = -20$.

Exercise 8
Given the function $g(x) = \cos(3x)$: What is $g^{(2015)}(x)$? (Hint: Start with finding the first few derivatives of g.) 1 pt

1. $f^{(1)}(x) = -3\sin(3x)$;
2. $f^{(2)}(x) = -3^2\cos(3x)$;
3. $f^{(3)}(x) = 3^3\sin(3x)$;
4. $f^{(4)}(x) = 3^4\cos(3x)$;

The remainder of 2015 divided by 4 is 3. From this we have $f^{(2015)}(x) = 3^{2015}\sin(3x)$.

Exercise 9
Investigate function $f = (x+1)^2(x-3)$ by following the steps below. (Do not start with drawing a graph. Of course, you may check your solution with GeoGebra or with some other tool.)

a) What are the roots of f? What is the y-intercept, that is, where is the intersection of the graph of f and the y-axis? 1 pt

Two roots $x_0 = -1$ and $x_1 = 3$. Have $f(0) = -3$.

b) Determine the limits at the edges of the domain. In this case, there are only two edges:

$$\lim_{x\to-\infty} f(x) \qquad \text{and} \qquad \lim_{x\to+\infty} f(x).$$

1 pt

Square goes to positive, so only depends on $x-3$. We get

$$\lim_{x\to-\infty} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to+\infty} f(x) = +\infty.$$

c) Find the local minima and maxima. 1 pt

Use product rule: $f'(x) = ((x^2+2x+1)(x-3))' = (2x+2)(x-3)+x^2+2x+1 = 3x^2-2x-5$. Now easily see $f''(x) = 6x-2$. $f'(x) = (x-1)(3x-5)$ so $x = 1$ and $x = \frac{5}{3}$ are the zeros of $f'(x)$. Easily see that $x = \frac{1}{3}$ is the only zero of $f''(x)$. We just found the x-coordinates of the critical points. We have $f(-1) = 0$ and $f(\frac{5}{3}) = -\frac{2^8}{3^3} \approx 9.5$. So get the points $(-1,0)$ and $(\frac{5}{3}, \frac{2^8}{3^3})$. We have $f''(-1) = -14 < 0$, so it is a maximum, and $f''(\frac{5}{3}) = 13 > 0$, so it is a minimum. These are the only critical points, so there are no more local minima and maxima.

d) Which parts of the function are convex and concave? Does function f have points of inflection? (Hint: Use the sign of the second derivative for answering both questions.) $\boxed{1 \text{ pt}}$

The function is concave $\iff f''(x) < 0 \iff 6x - 2 < 0 \iff x < \frac{1}{3}$. The function is convex $\iff f'' > 0 \iff 6x - 2 > 0 \iff x > \frac{1}{3}$. It has a point of inflection at $x = \frac{1}{3}$.

Exercise 10

a) Find a function $h(x)$ such that $h'(x) = \cos^3(x)$ (hint: recall that $\cos^2(x) + \sin^2(x) = 1$) $\boxed{1 \text{ pt}}$

Consider $h(x) = \sin(x) - \frac{1}{3}\sin^3(x)$, then by the product rule $h'(x) = \cos(x) - \sin^2(x)\cos(x)$, so using $\cos^2(x) + \sin^2(x) = 1$, $h'(x) = \cos^3(x)$.

b) Find three functions f_1, f_2, f_3 such that $f_1'(x) = f_2'(x) = f_3'(x) = \sin(x)\cos(x)$. $\boxed{1 \text{ pt}}$

$f_i(x) = \frac{1}{2}\sin^2(x) + C_i$. Pick 3 distinct values for C_i.

Your final grade is the sum of your scores divided by 1.0.