Calculus en Kansrekening

Assignment 7, October 13, 2015

— S O L U T I O N —

Goals: After completing these exercises successfully you should be able to:

- recognize and use common distributions of discrete and continuous random variables;
- compute the expectation and variance of discrete and continuous random variables.

Marks: You can score a total of 100 points.

1. **(20 points)** A shooter has exactly 6 bullets and shoots on a target. A random variable X is the number of bullets used *until he/she hits it for the first time.* The probability of a bullet hitting the target is 0.4 for every attempt.

   (a) Find the probability distribution of X; that is, give the probabilities for all possible values.
   (b) What is the expected value for X?
   (c) What is the the variance?
   (d) What is the the standard deviation?

   Solution: Although explicitly not mentioned in the lectures, this is a Geometric distribution.

   (a) The shooter uses 1, 2, 3, 4, 5 or, at most, 6 bullets. Clearly, $P(X = 0) = 0$ (that is, with 0 bullet the shooter can't hit the target).
      i. $P(X = 1) = 0.4$;
      ii. $P(X = 2) = 0.6 \cdot 0.4 = 0.24$;
      iii. $P(X = 3) = (0.6)^2 \cdot 0.4 = 0.144$;
      iv. $P(X = 4) = (0.6)^3 \cdot 0.4 = 0.0864$;
      v. $P(X = 5) = (0.6)^4 \cdot 0.4 = 0.05184$;
      vi. $P(X = 6) = 1 - \sum_{i=1}^{5} P(X = i) = 0.07776$.

   (b) $E(X) = \sum_{i=1}^{6} i \cdot P(X = i) = \underline{\underline{2.38336}}$.
   (c) $Var(X) = \sum_{i=1}^{6} P(X = i) \cdot (i - E[X])^2 \approx \underline{\underline{2.45336}}$.
   (d) $\sigma(X) = \sqrt{Var(X)} \approx \sqrt{2.45336} \approx \underline{\underline{1.56632}}$.

2. **(20 points)** Consider a class where students have to hand in exercises every week. They have to hand in eight assignments in total and have to pass at least five to be able to attend the exam. Student A does not study very hard, so for each assignment he/she has a probability of 0.5 to pass. Student B studies very hard, so for each assignment he/she has a probability of 0.8 to pass. The random variable X_A is the number of passes of student A and the random variable X_B is the number of passes of student B.

   (a) Find $P(X_A = 5)$;
   (b) Find $P(X_A \geq 5)$;
   (c) Find $P(X_B \geq 5)$.

   Solution:

   Both X_A and X_B have binomial distributions with parameters $(8, 0.5)$ and $(8, 0.8)$, respectively.

   (a) $P(X_A = 5) = \binom{8}{5} \cdot (0.5)^8 = \underline{\underline{0.21875}}$;
   (b) i. $P(X_A = 6) = \binom{8}{6} \cdot (0.5)^8 = 0.109375$;
      ii. $P(X_A = 7) = \binom{8}{7} \cdot (0.5)^8 = 0.03125$;
      iii. $P(X_A = 8) = \binom{8}{8} \cdot$ '$(0.5)^8 = 0.00390625$;
      So $P(X_A \geq 5) = \sum_{i=5}^{8} P(X_A = i) = \underline{\underline{0.36328125}}$.

(c) i. $P(X_B = 5) = \binom{8}{5} \cdot (0.8)^5 \cdot (0.2)^3 \approx 0.1468$;

ii. $P(X_B = 6) = \binom{8}{6} \cdot (0.8)^6 \cdot (0.2)^2 \approx 0.2936$;

iii. $P(X_B = 7) = \binom{8}{7} \cdot (0.8)^7 \cdot (0.2)^1 \approx 0.3355$;

iv. $P(X_B = 8) = \binom{8}{8} \cdot (0.8)^8 \cdot (0.2)^0 \approx 0.1678$;

So $P(X_B \geq 5) = \sum_{i=5}^{8} P(X_B = i) \approx \underline{\underline{0.9437}}$.

3. **(20 points)** A continuous random variable X has the following probability density function:

$$f(x) = \begin{cases} a \cdot (1 - 4x^2) & \text{if } -\frac{1}{2} < x < \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant a.

(b) Find the cumulative distribution function $F(x)$.

(c) Compute the probability $P(X = \frac{1}{4})$.

(d) Compute the probability $P(0 < X < \frac{1}{4})$.

Solution:

(a) As f is a pdf, we solve $1 = \int_{-\infty}^{\infty} f(x)\, dx = a \cdot \int_{-1/2}^{1/2} (1 - 4x^2)\, dx = a \cdot [x - \frac{4}{3}x^3]_{-1/2}^{1/2} = a \cdot \frac{2}{3}$. So $\underline{\underline{a = \frac{3}{2}}}$.

(b) According to (a), on the interval $(-\frac{1}{2}, \frac{1}{2})$, $f(x) = \frac{3}{2} \cdot (1 - 4x^2)$. The indefinite integral of f is $F(x) = \frac{3}{2}(x - \frac{4}{3}x^3) + C$ (on this interval). Thus, the cumulative distribution function F is given by

$$F(x) = \begin{cases} 0 & \text{if } x \leq -\frac{1}{2}, \\ \frac{3}{2}(x - \frac{4}{3}x^3) + C & \text{if } -\frac{1}{2} < x < \frac{1}{2}, \\ 1 & \text{if } x \geq \frac{1}{2}. \end{cases}$$

From $F(-1/2) = 0$ we have $C = \frac{1}{2}$, so $\underline{\underline{F(x) = \frac{3}{2}(x - \frac{4}{3}x^3) + \frac{1}{2}}}$. Note that the restriction $F(1/2) = 1$ gives the same result.

(c) This is $\underline{\underline{\text{zero}}}$. (As is for any point in a continuous distribution!)

(d) We compute $\int_0^{1/4} f(x)\, dx = F(1/4) - F(0) = \frac{3}{2}(\frac{1}{4} - \frac{4}{3}(\frac{1}{4})^3) = \underline{\underline{\frac{11}{32}}}$.

4. **(20 points)** TV sets with various defects are brought to the service for reparation. The time of reparation is a continuous random variable T. The cumulative distribution function of T is given as:

$$F(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 - e^{-kt} & \text{if } t \geq 0, \end{cases}$$

where $k > 0$.

(a) Find the probability density function f of the random variable.

(b) Find the expectation and variance.

Solution:

(a) $f(t) = \frac{d}{dt}F(t) = \begin{cases} 0 & \text{if } t < 0, \\ ke^{-kt} & \text{if } t \geq 0. \end{cases}$

(b) i. $E[T] = \int_{-\infty}^{\infty} tf(t)\, dt = \int_0^{\infty} tke^{-kt}\, dt = \lim_{a \to \infty} \int_0^a tke^{-kt}\, dt$. Use integration by parts once to find $\int_0^a tke^{-kt}\, dt = \left[-te^{-kt}\right]_0^a + \int_0^a e^{-kt}\, dt = ae^{-ka} + \left[-\frac{1}{k}e^{-kt}\right]_0^a = ae^{-ka} - \frac{1}{k}e^{-ka} + \frac{1}{k}$. From this we see that

$$E[T] = \lim_{a \to \infty} \int_0^a tke^{-kt}\, dt = \lim_{a \to \infty} \left(ae^{-ka} - \frac{1}{k}e^{-ka} + \frac{1}{k} \right) = \underline{\underline{\frac{1}{k}}}.$$

ii. To compute the variance ($Var[T] = E[T^2] - E[T]^2$), we first need to compute $E[T^2]$. $E[T^2] = \int_{-\infty}^{\infty} t^2 f(t)\, dt = \lim_{a\to\infty} \int_0^a t^2 k e^{-kt}\, dt$. Again use integration by parts to get $\int_0^a t^2 k e^{-kt}\, dt = \left[-t^2 e^{-kt}\right]_0^a + 2\int_0^a t e^{-kt}\, dt = -a^2 e^{-ka} + \frac{2}{k}\int_0^a t k e^{-kt}\, dt = -a^2 e^{-ka} + \frac{2}{k}\left(a e^{-ka} - \frac{1}{k} e^{-ka} + \frac{1}{k}\right)$. Note that in the last equality we used the integral from i.. We see that

$$E[T^2] = \lim_{a\to\infty}\left(-a^2 e^{-ka} + \frac{2}{k}\left(a e^{-ka} - \frac{1}{k} e^{-ka} + \frac{1}{k}\right)\right) = \frac{2}{k^2}.$$

Finally, we compute

$$Var[T] = E[T^2] - E[T]^2 = \frac{2}{k^2} - \frac{1}{k^2} = \underline{\underline{\frac{1}{k^2}}}.$$

5. **(20 points)** A normal random variable X has probability density function

$$f(x) = \frac{1}{3}\exp\left(-\frac{\pi}{9}(x^2 - 4x + 4)\right).$$

(a) Find the mean μ and the variance σ.

(b) Let Y be the random variable defined by $Y = \frac{X-\mu}{\sigma}$. For a real number a show that

$$P(Y \le -a) = 1 - P(Y \le a).$$

(Hint: use that $\int_\alpha^\beta \phi(x)\, dx = -\int_\beta^\alpha \phi(x)\, dx$.)

Solution:

(a) Note that the exercise was supposed to say "standard deviation" σ, which is the standard notation for the normal distribution. In this case, the solution is as follows: from $\frac{1}{\sigma\sqrt{2\pi}} = \frac{1}{3}$ we get $\sigma = \frac{3}{\sqrt{2\pi}}$. Also

$$-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 = \frac{-1}{2\sigma^2}(x-\mu)^2 = \frac{-\pi}{9}\left(x^2 - 2\mu x + \mu^2\right),$$

so we see that $\mu = 2$.

(When σ is the variance instead of the standard deviation, we have $\sigma = \left(\frac{3}{\sqrt{2\pi}}\right)^2 = \frac{9}{2\pi}$ instead.)

(b) Write $\phi(x)$ for the probability density function of Y. All we need is that the mean of Y is zero, from which by symmetry follows that $\phi(-x) = \phi(x)$. Then

$$\begin{aligned}
P(Y \le -a) &= \int_{-\infty}^{-a} \phi(x)\, dx \\
&= \lim_{b\to\infty} \int_{-b}^{-a} \phi(x)\, dx \\
&= \lim_{b\to\infty} \int_{b}^{a} -\phi(y)\, dy \quad \text{[substitute } y = -x \text{, use } \phi(y) = \phi(x)\text{]} \\
&= \lim_{b\to\infty} \int_{a}^{b} \phi(y)\, dy \\
&= P(Y \ge a) \\
&= 1 - P(Y \le a).
\end{aligned}$$