Calculus and Probability Theory

Assignment 3, February 16, 2017

Handing in your answers:

- submission via Blackboard (`http://blackboard.ru.nl`);
- one single pdf file (make sure that if you scan/photo your handwritten assignment, the result is clearly readable);
- all of your solutions are clearly and convincingly explained;
- make sure to write your name, your student number

Deadline: Friday, February 24, 14:30 sharp!

Goals: After completing these exercises successfully you should be able to:

- apply all differentiation rules on elementary and transcendental functions;
- solve problems including higher-order derivatives;
- apply l'Hôpital's rules when applicable;
- analyse graphs of a given real function.

Marks: You can score a total of 100 points.

1. **(10 points)** The function arcsin is the inverse function of sin.
   (a) What is the domain of the function $\arcsin(x)$? Why?
   (b) Compute the following values and explain how you got the result:
   $$\arcsin(1) =? \qquad \arcsin(0) =? \qquad \arcsin\left(\frac{\sqrt{3}}{2}\right) =?$$
   (c) Find the derivative of f:
   $$f(x) = \arcsin\left(\frac{2x}{1-x}\right).$$
2. **(15 points)** Find the limits of the following functions. (Note that before you can apply L'Hôpital's rule, you have to verify whether it is possible.)
   (a) $\lim\limits_{x\to\infty} \frac{e^{n-x}}{x^{-m}}$ with $m, n \in \mathbb{N}$; (Hint: if unclear first solve a particular case, e.g., $n = 0, m = 3$.)
   (b) If $\lim\limits_{x\to 0} \frac{\sqrt[3]{(a\cdot x+b)}-2}{x} = \frac{5}{12}$ with $a, b \in \mathbb{N}$ then $a \cdot b = ?$;
   (c) If $\lim\limits_{x\to 0} \frac{\sin(x)+Ax+Bx^3}{x^5} = \frac{1}{C}$ with $A, B, C \in \mathbb{Q}$, then $A \cdot B \cdot C = ?$.
3. **(10 points)** Given the functions $f(x) = \log_3(2x)$ and $g(x) = \cos(3x)$.
   (a) What is $f'''(x)$?
   (b) What is $g^{(2015)}(x)$? (Hint: Start with finding the first few derivatives of g.)
4. **(5 points)** For which values of c has the equation $\ln x = cx^2$ precisely one solution. (Hint: There is a value $0.1 < c < 0.2$ for which the curves just touch each other. What do these curves also have in common, besides the point of intersection?)
5. **(25 points)** Investigate function $f = (x+1)^2(x-3)$ by following the steps below. (Do not start with drawing a graph. Of course, you may check your solution with GeoGebra or with some other tool.)
   (a) Determine the domain of function f.

(b) What are the roots of f? What is the y-intercept, that is, where is the intersection of the graph of f and the y-axis?

(c) Determine the limits at the edges of the domain. In this case, there are only two edges:

$$\lim_{x\to-\infty} f(x) \qquad \text{and} \qquad \lim_{x\to+\infty} f(x).$$

(d) Find f' and f''.

(e) Find the zeros of f' and f''.

(f) What are the critical points (determine their x and y coordinates)?

(g) Find the local minima and maxima.

(h) Which parts of the function are convex and concave? Does function f have points of inflection? (Hint: Use the sign of the second derivative for answering both questions.)

6. **(25 points)** We will investigate the function

$$f(x) = \frac{(x-2)^2}{x+2}.$$

following similar steps as the ones in the previous problem. Additionally, we prove that the line $y = x-6$ is a slant asymptote on both sides.

(a) Determine the domain of function f.

(b) What are the roots of f? Where does the graph of f intersect the y axis?

(c) Determine the limits at the edges of the domain.

(d) Find f' and f''.

(e) Find the zeros of f' and f''.

(f) What are the critical points (determine their x and y coordinates)?

(g) Find the local minima and maxima.

(h) Which parts of the function are convex and concave? Does function f have points of inflection? (Hint: Use the sign of the second derivative for answering both questions.)

(i) Show that the line $y = x - 6$ is a slant asymptote of f. (Hint: Use the definition on slide 47 of the lecture and the following two limits.)

$$\lim_{x\to-\infty} (f(x) - (x-6)) = ? \qquad \text{and} \qquad \lim_{x\to+\infty} (f(x) - (x-6)) = ?$$

7. **(10 points)**

(a) Find the derivative of $f(x) = \ln(\cos(\ln(\cos(x))))$.

(b) Find a function $g(x)$ such that $g'(x) = \tan(2x)$.

(c) Find three functions f_1, f_2, f_3 such that $f_1'(x) = f_2'(x) = f_3'(x) = \sin(x)\cos(x)$.