Calculus and Probability Theory

Assignment 6, March 9, 2017

Handing in your answers:

- submission via Blackboard (`http://blackboard.ru.nl`);
- one single pdf file (make sure that if you scan/photo your handwritten assignment, the result is clearly readable);
- all of your solutions are clearly and convincingly explained;
- make sure to write your name, your student number

Deadline: Friday, March 17, 14:30 sharp!

Goals: After completing these exercises successfully you should be able to:

- compute combinatorial problems;
- recognize the birthday paradox;
- work with the basic definitions of probability theory.

Marks: You can score a total of 100 points.

1. **(10 points)** Vanessa wants to give her friend potted plants as present. At the local florist, the flowers come in five colours, and there are four types of flower pots.

   (a) If Vanessa buys one potted flower (any combination of a flower and a pot), how many different options can Vanessa choose from?

   (b) If Vanessa buys two potted flowers and she wants two different combinations, how many options does she have? (Two combinations are different if at least the colours of the flowers or the types of the pots are different.)

   Solution:

   (a) She has 5 options for the colours, and 4 for the pot. So $5 \cdot 4 = 20$ options.

   (b) She has again 20 options for the first, and then $20 - 1 = 19$ options for the second. So $20 \cdot 19 = 380$ options.

   [[**Grading Instruction:**

Grading (total 10):	
aspect:	points
(a)	5
(b)	5
small mistake	-2

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2. **(5 points)** In how many ways can five out of eight people be seated on a sofa if there are only five seats available.

   Solution:

   There are 8 people for the first seat, then 7 for the second, etc. So $8 * 7 * 6 * 5 * 4 = 6720 = \frac{8!}{3!}$.

   [[**Grading Instruction:**

Grading (total 5):	
aspect:	points
correct	5
small mistake	-2

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3. **(5 points)** Three cards are drawn at random (without replacement) from an ordinary deck of 52 cards.[1] Find the number of ways in which one can draw

   (a) a diamond and a club and a heart in this order;

   (b) one hearts and then two clubs or two spades.

   Solution:

   (a) There are 13 diamonds, 13 clubs and 13 hearts. So $13^3 = 2197$ ways.

   (b) There are 13 hearts, and then either 13 clubs, then 12 clubs left or 13 spades and then 12 spades left. So $13 * 13 * 12 + 13 * 13 * 12 = 4056$.

   [[**Grading Instruction:**

Grading (total 5):	
aspect:	points
(a)	2
(b)	3
small mistake	-1

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4. **(10 points)** How many numbers, consisting of five different digits each, can be made from the digits $1, 2, 3, \ldots, 9$ if

   (a) the numbers must be even?

   (b) exactly two of the digits are odd?

   (c) How many numbers are in (a) and (b) if repetitions of the digits are allowed?

   Solution:

   (a) A number is even if and only if the last digit is even. The even digits are $2, 4, 6, 8$, and then there are 8 digits left. So $4 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 6720$ numbers.

   (b) The numbers $1, 3, 5, 7, 9$ are odd. So $\binom{5}{2}$ options for the positions of the two odd digits. We then have $5 \cdot 4$ ways to put odd digits in those positions. For the remaining three positions there are $4 \cdot 3 \cdot 2$ options (they must be even), so $\binom{5}{2} \cdot 5 \cdot 4 \cdot 4 \cdot 3 \cdot 2 = 4800$ numbers.

   (c) i. We now have 4 options for the last digits, and then 9 for all others. So $4 \cdot 9^4 = 26244$.

   ii. Again *binom*52 ways to choose the positions of the odd digits, and $5 \cdot 5$ ways to put odd digits. The remaining must be even, so $4 \cdot 4 \cdot 4$ ways to do this. We get $\binom{5}{2} \cdot 5^2 \cdot 4^3 = 16000$.

   [[**Grading Instruction:**

Grading (total 10):	
aspect:	points
(a)	2
(b)	3
(c)→(a)	2
(c)→(b)	3
small mistake	-1

[1] The four suits are as follows. spades ♠, hearts ♥, diamonds ♦ and clubs ♣.

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5. **(10 points)** According to Wikipedia[2] a new scheme was introduced for license plates in the Netherlands. The short summary of the scheme is shown below. In practice, some letters are excluded to avoid confusion and obscene language; however, *we ignore this* in the exercise!

2015–present		Side Code 11	This series is currently issued for mopeds (DBB-01-B till FZZ-99-Z)

Table 1: New license plates scheme in the Netherlands

Answer the following questions and give brief explanations.

(a) Consider only the image in the second entry of the table above. How many possible license plates can be made with this setup. (We assume that 26 letters can be used where X is displayed, and 10 digits can be used where 9 is displayed.)

(b) In the fourth column of Table 1 a range is provided for mopeds. (Note that it is a range, so for example `EAA-01-A` is in the range!) Using this, count how many license plates can be issued for mopeds?

(c) Assuming that there are only cars and mopeds in this scheme, can you tell what kind of vehicle has the license plate `DFA-78-V`? And `DAF-78-A`? And `DCF-78-V`?

(d) Assuming that there are only cars and mopeds in this scheme, how many cars can be with this type of a license plate?

Solution:

(a) $26^4 \cdot 10^2 = 45,697,600$

(b) First we look at all cases in the lexicographic ordering from DAA-00-A till DBB-01-A

from	**till**	**count formula**	**nr combinations**
DAA-00-A	DAZ-99-Z	$26^2 \cdot 10^2$	67,600
DBA-00-A	DBA-99-Z	$26 \cdot 10^2$	2,600
DBB-00-A	DBB-00-Z	26	26
DBB-01-A	DBB-01-A	1	1
DAA-00-A	DBB-01-A		70,227

Then we subtract this result from DAA-00-A till FZZ-99-Z to obtain all combinations in DBB-01-B till FZZ-99-Z

from	**till**	**count formula**	**nr combinations**
DAA-00-A	FZZ-99-Z	$3 \cdot 26^3 \cdot 10^2$	5,272,800
DAA-00-A	DBB-01-A		70,227
DBB-01-B	FZZ-99-Z		5,202,573

(c) `DFA-78-V`: moped. `DAF-78-A`: car; `DCF-78-V`: moped.

(d) It can be computed by subtracting the result of (b) from that of (a): $45,697,600 - 5,202,573 = 40,495,027$.

[[**Grading Instruction:**

Grading (total 10):	
aspect:	points
(a)	2
(b)	3
(c) 1 point each	3
(d)	2
small mistake	-1

[2] `https://en.wikipedia.org/w/index.php?title=Vehicle_registration_plates_of_the_Netherlands&oldid=682267623`

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6. **(10 points)** Expand the following expressions, either directly or via binomial coefficients. Make it clear how you proceed.

   (a) $(x-7)^3$;

   (b) $(x+2y)^4$;

   (c) $(x^3-3)^4$;

   Solution:

   (a) $(x-7)^3 = \sum_{i=0}^{3} \binom{3}{i} x^i (-7)^{3-i} = \binom{3}{0}(-7)^3 + \binom{3}{1}(-7)^2 x + \binom{3}{2}(-7)x^2 + \binom{3}{3}x^3 = \underline{\underline{x^3 - 21x^2 + 147x - 343}}$;

   (b) $(x+2y)^4 = \binom{4}{0}(2y)^4 + \binom{4}{1}x(2y)^3 + \binom{4}{2}x^2(2y)^2 + \binom{4}{3}x^3(2y) + \binom{4}{4}x^4 = \underline{\underline{16y^4 + 32xy^3 + 24x^2y^2 + 8x^3y + x^4}}$;

   (c) $(x^3-3)^4 = \binom{4}{0}(-3)^4 + \binom{4}{1}(x^3)\cdot(-3)^3 + \binom{4}{2}(x^3)^2\cdot(-3)^2 + \binom{4}{3}(x^3)^3\cdot(-3)^1 + \binom{4}{4}(x^3)^4 = \underline{\underline{x^{12} - 12x^9 + 54x^6 - 108x^3 + 8}}$

   [[**Grading Instruction:**

Grading (total 10):	
aspect:	points
(a)	4
(b)	3
(c)	3
small mistake	-1

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7. **(5 points)** Consider a team of 11 players wishing each other luck before a match. If everyone shakes everyone else's hand exactly once, how many handshakes occur?

   Solution:

   (a) Solution 1: There are 11 players, and each handshake involves two players. So $\binom{11}{2} = 55$ handshakes.

   (b) Solution 2: The first player shakes 10 hands, the seconds 9, etc. So $10 + 9 + 8 + \ldots + 1 = 55$.

   [[**Grading Instruction:**

Grading (total 5):	
aspect:	points
correct	5
small mistake	-2

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8. **(5 points)** From four consonants and five vowels, how many six-letter words can be formed consisting of three different consonants and three different vowels? The words need not have a meaning.

   Solution:

   The three consonants can be chosen in $\binom{4}{3}$ ways, the three vowels in $\binom{5}{3}$ ways. The letters can then be ordered in 6! different ways. So there are $\binom{4}{3} \cdot \binom{5}{3} \cdot 6! = 4 \cdot 10 \cdot 720 = 28,800$ words.

   [[**Grading Instruction:**

Grading (total 5):	
aspect:	points
correct	5
small mistake	-2

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9. **(5 points)** A school has four maths teachers, three English teachers and three IT teachers. From this whole group, a five teacher committee has to be established. Calculate the number of ways that this committee can be formed if at least one IT teacher must be on the committee.

   Solution:

   (a) Solution 1: There are either one, two, or three teachers in the committee. There are then four, three respectively two other teachers left from the seven maths/English teachers. So we compute $\binom{3}{1} \cdot \binom{7}{4} + \binom{3}{2} \cdot \binom{7}{3} + \binom{3}{3} \cdot \binom{7}{2} = 231$.

   (b) Solution 2: There are $\binom{10}{5}$ ways to form a five teacher committee, of which $\binom{7}{5}$ contain no IT teacher. So we compute $\binom{10}{5} - \binom{7}{5} = 231$.

   [[**Grading Instruction:**

Grading (total 5):	
aspect:	points
correct	5
small mistake	-2

   ]]

10. **(10 points)** There are twelve provinces in the Netherlands. Suppose that the birth of Dutch people is uniformly distributed over all twelve provinces, *i.e.* for every province $\frac{1}{12}$ of the population is born there. What is the probability that at least two of r randomly selected Dutch-born people were born in the same province, where $r = 1$, $r = 2$, $r = 3$, $r = 4$, $r = 5$ or $r = 6$?

    Solution:

    This problem is a toy example for the Birthday paradox. Thus, we can apply the same reasoning here:

    $$p(r) = 1 - \frac{12!}{(12-r)! \cdot 12^r},$$

    and the particular probabilities are the following:

r	$p(r)$	$p(r)$
1	0	0
2	0.0833	${}^{1}/_{12}$
3	0.2361	${}^{17}/_{72}$
4	0.4271	${}^{41}/_{96}$
5	0.6181	${}^{89}/_{144}$
6	0.7772	${}^{1343}/_{1728}$

    [[**Grading Instruction:**

Grading (total 10):	
aspect:	points
correct formula	4
correct probility for r ($\times 6$)	1
small mistake	-1

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11. **(10 points)** We toss a coin twice.

(a) Give a corresponding sample space S.
(b) Give the set of outcomes corresponding to each of the following events:
   i. A: "we throw heads exactly once";
   ii. B: "we throw heads at least once";
   iii. C: "tails did not appear before a head appeared".
(c) Give a probability measure P_1 for the sample space S.

Solution:

(a) For example $S = \{HH, HT, TH, TT\}$. But there are more options for valid sample spaces!
(b) i. $A = \{HT, TH\}$;
   ii. $B = \{HH, HT, TH\}$;
   iii. $C = \{HH, HT, TT\}$;
(c) Most obvious is $P_1(HH) = P_1(HT) = P_1(TH) = P_1(TT) = \frac{1}{4}$. Note that we have to define the probability measure on all $16 = 2^4$ subsets of S, but all others can be constructed from these 4. Make sure the students do this, or mention it!

[[**Grading Instruction:**

Grading (total 10):	
aspect:	points
(a)	3
(b) (1,1,2)	4
(c)	3
small mistake	-2

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12. **(5 points)** Let P be a probability measure on space S. Prove that $P(\emptyset) = 0$.

Solution:

We have that for any event A of the sample space $P(A) = P(A \cup \emptyset) = P(A) + P(\emptyset)$. As the left and right side must be equal, we have $P(\emptyset) = 0$.

[[**Grading Instruction:**

Grading (total 5):	
aspect:	points
correct	5
small mistake	-2

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13. **(bonus, +5 points)** Let P be a probability measure on space S. Prove that for pairwise mutually exclusive events $A_1, A_2, \dots A_n$ (*i.e.* $A_i \cap A_j = \emptyset$ for all $i \neq j$ and $n \geq 2$) one has $P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$. (Hint: use induction.)

Solution:

For $n = 2$ this follows by definition. Suppose that $A_1, A_2, \dots A_{k+1}$ are mutually disjoint. Then $A_1 \cup A_2 \cup \ldots \cup A_k$ and A_{k+1} are disjoint, so $P((A_1 \cup A_2 \cup \ldots \cup A_k) \cup A_{k+1}) = P(A_1 \cup A_2 \cup \ldots \cup A_k) + P(A_{k+1})$. But by induction $P(A_1 \cup A_2 \cup \ldots \cup A_k) = P(A_1) + P(A_2) + \ldots P(A_k)$, so the result follows.

[[**Grading Instruction:**

Grading (total 5):	
aspect:	points
correct	5
small mistake	-2

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14. **(10 points)** An experiment consists of drawing 3 cards in succession from a well-shuffled ordinary deck of cards.[3] Let A_1 denote the event "Ace on first draw", A_2 the event "Ace on second draw" and A_3 the event "Ace on third draw". State in words the meaning of each of the following probabilities.

    (a) $P(A_1 \cap \neg A_2)$;

    (b) $P(A_1 \cup A_2)$;

    (c) $P(\neg A_1 \cap \neg A_2 \cap \neg A_3)$;

    (d) $P[(A_1 \cap \neg A_2) \cup (\neg A_2 \cap A_3)]$.

    Solution:

    (a) The probability that the first draw is an ace, and the second draw is not an ace;

    (b) The probability that the first draw or the second draw is an ace, or both;

    (c) The probability that in the first, second and third draw, no ace is drawn;

    (d) Notice that this is $P[\neg A_2 \cap (A_1 \cup A_3)]$. So the probability that the second draw is not an ace, and the first, third or both are an ace.

    [[**Grading Instruction:**

Grading (total 10):	
aspect:	points
(a)	3
(b)	3
(c)	2
(d)	2
small mistake	-1

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[3] If you are not familiar with the standard 52-card deck, visit Wikipedia: `http://en.wikipedia.org/wiki/Standard_52-card_deck`.