Calculus and Probability Theory
Assignment 3

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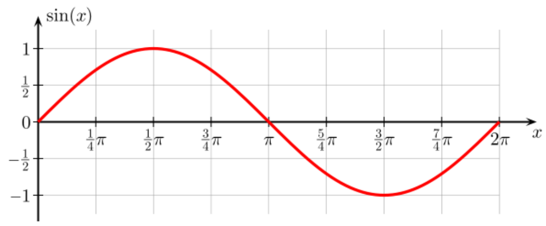
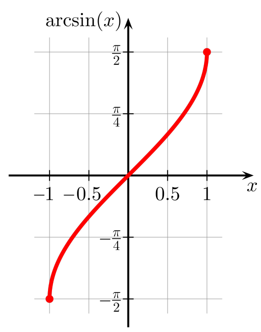
After completing these exercises successfully you should be confident with the following topics:

- Apply all differentiation rules on elementary and transcendental functions
- Solve problems including higher-order derivatives
- Apply l'Hopital's rules when applicable
- analyse graphs of a given real function

1. (**10 points**) The function arcsin is the inverse function of sin.

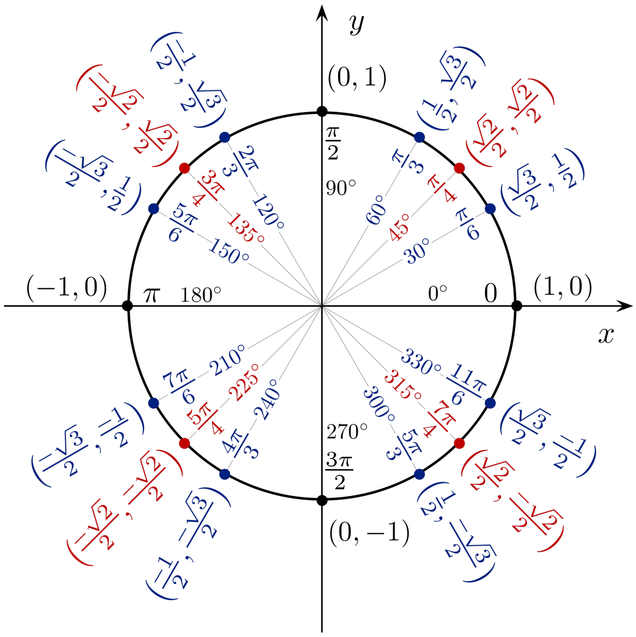
   (a) What is the domain of the function $\arcsin(x)$? Why?
   Solution:





As we already know, the domain of sin is $\mathbb{R}$ and its range is $[-1, 1]$. Because arcsin is the inverse function of sin and reverses the output of sin, the domain of arcsin is the range of sin. Therefore, the domain of arcsin is $[-1, 1]$.

(b) Compute the following values and explain how you got the result:



- $\arcsin(1) = ?$
  Solution:

  This can be rewritten as: What angle would I have to take the *sine* of in order to get 1? $\to \sin(?) = 1$
  The answer is $\arcsin(1) = \frac{\pi}{2}$, because I know (by remembering the Unit Circle and the Sine function) that $\sin(\frac{\pi}{2}) = 1$.

- $\arcsin(0) = ?$
  Solution:

  This can be rewritten as: What angle would I have to take the *sine* of in order to get 0? $\to \sin(?) = 0$

  The problem is that $\arcsin(x)$ is NOT the true inverse of $\sin(x)$. In fact, $\sin(x)$ doesn't have an inverse function since it fails the horizontal line test. $\arcsin(x)$ is actually just the inverse of $\sin(x)$ on the interval $[-\pi/2, \pi/2]$ (the only interval where $\sin(x)$ is strictly increasing that contains the origin).

  Thus, the value of $\arcsin(0)$ is the value of x on the interval $[\pi/2, \pi/2]$ that satisfies $\sin(x) = 0$. The only value of x on $[-\pi/2, \pi/2]$ that does this is $x = 0$, so: $\arcsin(0) = 0$.

- $\arcsin(\frac{\sqrt{3}}{2}) = ?$

Solution:

By remembering the Unit Circle, I know that $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$. Therefore, $\arcsin\frac{\sqrt{3}}{2} = \frac{\pi}{3}$.

This can be rewritten as: What angle would I have to take the *sine* of in order to get $\frac{\sqrt{3}}{2}$? $\rightarrow \sin(?) = \frac{\sqrt{3}}{2}$

(c) Find the derivative of f:

$$f(x) = \arcsin(\frac{2x}{1-x})$$

Solution:

Using chainrule with quotientrule.

$$\begin{aligned}
y &= \arcsin(x) \\
\sin(y) &= x \\
(\cos(y) \cdot y') &= 1 \\
y' &= \frac{1}{\cos(y)} \\
\cos^2(y) + \sin^2(y) &= 1 \\
\cos^2(y) &= 1 - \sin^2(y) \\
\cos(y) &= \sqrt{1 - \sin^2(y)} \\
y' &= \frac{1}{\cos(y)} \\
&= \frac{1}{\sqrt{1 - \sin^2(y)}} \\
&= \frac{1}{\sqrt{1 - x^2}}
\end{aligned}$$

Done with outher derivative.

$$\begin{aligned}
\frac{d}{dx}\left(\frac{2x}{1-x}\right) &= \frac{[2 \cdot (1-x)] - [2x \cdot (-1)]}{(1-x)^2} \\
&= \frac{2}{x^2 - 2x + 1}
\end{aligned}$$

Done with inner derivative. Applying chain rule.

$$f'(x) = \frac{1}{\sqrt{\frac{2x}{1-x} - 1}} \cdot \frac{2}{x^2 - 2x + 1}$$

2. (**15 points**) Find the limits of the following functions. (Note that before you can apply L'Hopital's rule, you have to verify whether it is possible.)

   (a) $\lim_{x\to\infty} \frac{e^{n-x}}{x^{-m}}$ with $m, n \in \mathbb{N}$(Hint: if unclear first solve a particular case, e.g., n = 0, m = 3.)
   Solution:

   (b) If $\lim_{x\to 0} \frac{\sqrt[3]{(a\cdot x+b)}-2}{x} = \frac{5}{12}$ with $a, b \in \mathbb{N}$ then $a \cdot b =?$
   Solution:

   (c) If $\lim_{x\to 0} \frac{\sin(x)+Ax+Bx^3}{x^5} = \frac{1}{C}$ with $A, B, C \in \mathbb{Q}$, then $A \cdot B \cdot C =?$
   Solution:

3. (**10 points**) Given the functions $f(x) = \log_3(2x)$ and $g(x) = \cos(3x)$.

(a) What is $f'''(x)$?
Solution:

$$\begin{aligned} f'(x) &= \frac{1}{2x \cdot \ln(3)} \cdot 2 \\ &= \frac{2}{2x \cdot \ln(3)} \\ &= \frac{1}{x \cdot \ln(3)} \end{aligned}$$

$$\begin{aligned} f''(x) &= \frac{0 - [1 \cdot (1 \cdot \ln(3)) + (x \cdot 0))]}{(x \cdot \ln(3))^2} \\ &= \frac{-\ln(3)}{x^2 \cdot \ln(3)^2} \\ &= -\frac{1}{x^2 \cdot \ln(3)} \end{aligned}$$

$$\begin{aligned} f''(x) &= -\frac{0 - \left[2x \cdot \ln(3) + x^2 \cdot 0\right]}{(x^2 \cdot \ln(3))^2} \\ &= \frac{2x \cdot \ln(3)}{x^4 \cdot \ln(3)^2} \\ &= \frac{2}{x^3 \cdot \ln(3)} \end{aligned}$$

(b) What is $g^{(2015)}(x)$? (Hint: Start with finding the first few derivatives of g.)
Solution:

$$\begin{aligned} g'(x) &= -\sin(3x) \cdot 3 \\ g''(x) &= [(-\cos(3x) \cdot 3) \cdot 3] + [(-\sin(3x)) \cdot 0] \\ &= -\cos(3x) \cdot 9 \\ g'''(x) &= [(\sin(3x) \cdot 3) \cdot 9] + [-\cos(3x) \cdot 0] \\ &= \sin(3x) \cdot 27 \\ g''''(x) &= [(\cos(3x) \cdot 3) \cdot 27] + [\sin(3x) \cdot 0] \\ &= \cos(3x) \cdot 81 \\ g^{(5)}(x) &= -\sin(3x) \cdot 3^5 \\ g^{(2015)}(x) &= -\sin(3x) \cdot 3^{2015} \end{aligned}$$

4. (**5 points**) For which values of c has the equation $\ln x = cx^2$ precisely one solution. (Hint: There is a value $0.1 < c < 0.2$ for which the curves just touch each other. What do these curves also have in common, besides the point of intersection?)
Solution:

5. (**25 points**) Investigate function $f = (x+1)^2(x-3)$ by following the steps below. (Do not start with drawing a graph. Of course you may check your solution with GeoGebra or with some other tool.)

   (a) Determine the domain of function f.
   Solution:

   $$D(f) = \mathbb{R}$$

   (b) What are the roots of f? What is the y-intercept, that is, where is the intersection of the graph of f and the y-axis?
   Solution:

   Rewriting the term gives us:

   $$f(x) = (x+1)^2(x-3) = (x+1)(x+1)(x-3) = x^3 - x^2 - 5x - 3$$

   The original form already shows use the roots (x-intercepts) of the function, namely $\{-1, 3\}$.

   To get the y-intercept, we just have to plug-in 0:

   $$\begin{aligned} f(x) &= x^3 - x^2 - 5x - 3 \\ f(0) &= 0^3 - 0^2 - 5 \cdot 0 - 3 \\ &= -3 \end{aligned}$$

   Y-intercept at -3.

   (c) Determine the limits at the edges of the domain. In this case, there are only two edges:

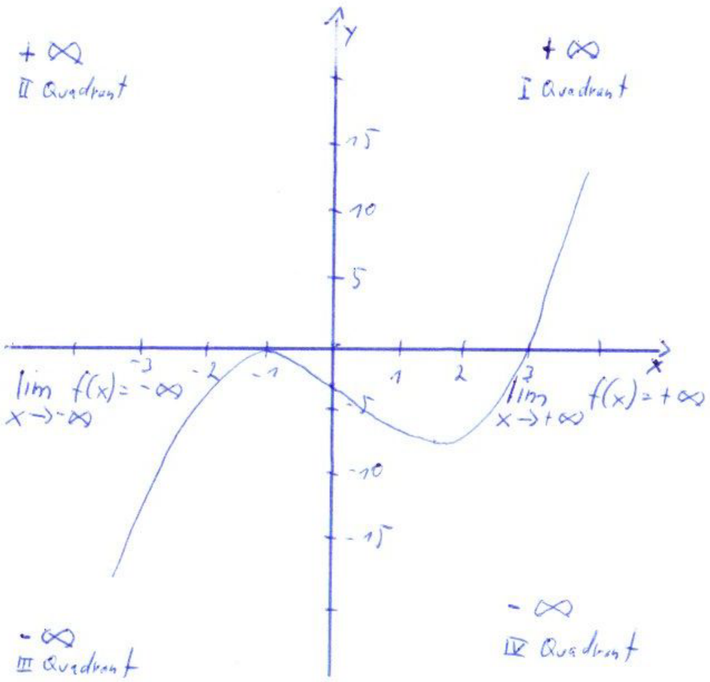
   $$\lim_{x \to -\infty} f(x) \qquad and \qquad \lim_{x \to +\infty} f(x)$$

   Solution:

   If we want to determine the limits of $f(x)$ in regards to $-\infty$ and ∞, we just have to take the highest exponent of the equation into account, because all others would be outperformed towards their values to $-\infty$ and ∞. We already computed the expanded form of f in 5b) which is $x^3 - x^2 - 5x - 3$ and will work with this form.

$$\lim_{x \to -\infty} x^3 - x^2 - 5x - 3 \approx \lim_{x \to -\infty} x^3$$
$$\lim_{x \to -\infty} x^3 = -\infty$$

$$\lim_{x \to \infty} x^3 - x^2 - 5x - 3 \approx \lim_{x \to \infty} x^3$$
$$\lim_{x \to \infty} x^3 = \infty$$



As we can see, the graph goes up towards ∞ and goes down towards $-\infty$.

(d) Find f' and f''.
Solutions:

$$f(x) = (x+1)^2(x-3) = (x+1)(x+1)(x-3) = x^3 - x^2 - 5x - 3$$

$$f'(x) = 3x^2 - 2x - 5$$
$$f''(x) = 6x - 2$$

(e) Find the zeros of f' and f''.
Solution:

$$3x^2 - 2x - 5 = 0$$

Apply abc-formula:

$$\begin{aligned}
x &= \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 3 \cdot (-5)}}{6} \\
&= \frac{2 \pm \sqrt{64}}{6} \\
&= \frac{2 \pm 8}{6} \\
x_1 &= -1 \\
x_2 &= \frac{5}{3}
\end{aligned}$$

Zeros of $f' = \{-1, \frac{5}{3}\}$

$$\begin{aligned}
6x - 2 &= 0 \\
6x &= 2 \\
x &= \frac{1}{3}
\end{aligned}$$

Zeros of $f''(x) = \frac{1}{3}$

(f) What are the critical points (determine their x and y coordinates)?
Solution:

A critical point of a function $f : D \to \mathbb{R}$, is a point $a \in D$ such that $f'(a) = 0$. The value $f(a)$ is called a critical value of f.

Determine critical points by plugging in the values where $f'(x) = 0$ into the original function

$$\begin{aligned}
(-1)^3 - (-1)^2 - 5(-1) - 3 &= -1 - 1 + 5 - 3 \\
&= 0
\end{aligned}$$

Critical point 1: $(-1, 0)$

$$
\begin{aligned}
(\frac{5}{3})^3 - (\frac{5}{3})^2 - 5(\frac{5}{3}) - 3 &= \frac{125}{27} - \frac{25}{9} - 5\frac{25}{3} - 3 \\
&= \frac{125}{27} - \frac{75}{27} - \frac{225}{27} - 3 \\
&= -\frac{256}{27}
\end{aligned}
$$

Critical point 2: $(\frac{5}{3}, -\frac{256}{27})$

(g) Find the local minima and maxima.
Solution:

When a function's slope is zero at x, and the second derivative at x is:

- less than 0, it is a local maximum
- greater than 0, it is a local minimum
- equal to 0, then the test failes

Plugging-in the zeros of f' into $f''(x)$:

$$6(-1) - 2 = -8$$

Therefore, $(-1, 0)$ is a local maximum.

$$6(\frac{5}{3}) - 2 = 8$$

Therefore, $(\frac{5}{3}, -\frac{256}{27})$ is a local minimum.

(h) Which parts of the function are convex and concave? Does function f have points of inflection? (Hint: Use the sign of the second derivative for anwering both questions.)
Solution:

A **point of inflection** on a curve $y = f(x)$ is a point at which f changes from concave to convex or vice versa.

- If $f''(x) > 0$, for all $x \in (a, b)$, then f is convex on (a, b)
- If $f''(x) < 0$, for all $x \in (a, b)$, then f is concave on (a, b)
- If f has an inflection point at x and f'' exists in $(x - \delta, x + \delta)$, for some $\delta > 0$, then $f''(x) = 0$

Calculating points of inflection:

$$\begin{aligned} f''(x) &= 0 \\ 6x - 2 &= 0 \\ x &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} f'''(x) &= 6 \\ f'''(\frac{1}{3}) &> 0 \rightarrow \text{change from concave to convex} \end{aligned}$$

Calculating point of inflection by plugging in $\frac{1}{3}$ into f:

$$\begin{aligned} f(\frac{1}{3}) &= \frac{1}{3}^3 - \frac{1}{3}^2 - \frac{5}{3} - 3 \\ &= \frac{1}{27} - \frac{1}{9} - \frac{5}{3} - 3 \\ &= -4\frac{20}{27} \end{aligned}$$

Point of inflection: $\left(\frac{1}{3}, -4\frac{20}{27}\right)$
We know the local minima/maxima, behaviour towards ∞ and $-\infty$ and the point of inflection by now. Therefore, we can divide the function into the parts we want to inspect with regards to their convexity and concavity:

- $(-\infty, -1)$, $f''(-2) = -14 \rightarrow$ concave
- $(-1, \frac{1}{3})$, $f''(0) = -2 \rightarrow$ concave
- $\frac{1}{3}, \frac{5}{3}$, $f''(\frac{2}{3}) = 2 \rightarrow$ convex
- $(\frac{5}{3}, \infty)$, $f''(2) = 10 \rightarrow$ convex

6. (**25 points**) We will investigate the function

$$f(x) = \frac{(x-2)^2}{x+2}$$

following similar steps at the ones in the previous problem. Additionally, we prove that the line $y = x - 6$ is a slant asymptote on both sides

(a) Determine the domain of function f.
Solution:

$$D(f) = \{x \in \mathbb{R} | x \neq -2\}$$

(b) What are the roots of f? Where does the graph of f intersect the y axsis?
Solution:

To get the roots of f (x-intercepts) which is a quotient in that case, we just have to set the numerator to zero.

Therefore, the root is 2.
We could also expand the numerator to $x^2 - 4x + 4$ and then apply the abc-formula, but that's kind of an overkill in this case.

To get the y-intercept, we just have to plug-in 0:

$$\begin{aligned} f(x) &= \frac{(x-2)^2}{x+2} \\ f(0) &= \frac{(0-2)^2}{0+2} \\ &= 2 \end{aligned}$$

Y-intercept at 2.

(c) Determine the limits at the edges of the domain. In this case, there are only two edges:

$$\lim_{x \to -\infty} f(x) \qquad \textit{and} \qquad \lim_{x \to +\infty} f(x)$$

Solution:

$$\begin{aligned} \lim_{x \to -\infty} \frac{x^2 - 4x + 4}{x+2} &= \frac{\lim_{x \to -\infty} x^2 - 4x + 4}{\lim_{x \to -\infty} x + 2} \\ &= \frac{\infty}{-\infty} \\ &= \text{undefined expression} \end{aligned}$$

Therefore, we have to apply L'Hopital's rule.

$$\begin{aligned} \frac{d}{dx} x^2 - 4x + 4 &= 2x - 4 \\ \frac{d}{dx} x + 2 &= 1 \end{aligned}$$

$$\lim_{x \to -\infty} \frac{2x - 4}{1} = -\infty$$

The same procedure can be applied for $\lim_{x\to\infty} \frac{x^2-4x+4}{x+2}$

$$\lim_{x\to\infty} \frac{2x-4}{1} = \infty$$

(d) Find f' and f''.
Solutions:

$$\begin{aligned} f(x) &= \frac{(x-2)^2}{x+2} \\ &= \frac{x^2-4x+4}{x+2} \end{aligned}$$

$$\begin{aligned} f'(x) &= \frac{[(2x-4)(x+2))] - \left[(x^2-4x+4)\cdot 1\right]}{(x+2)^2} \\ &= \frac{x^2-4x-12}{(x+2)^2} \\ &= \frac{(x-2)(x+6)}{(x+2)^2} \\ &= \frac{x^2+4x-12}{(x+2)^2} \end{aligned}$$

$$\begin{aligned} f''(x) &= \frac{\left[(2x+4)(x+2)^2\right] - \left[(x^2-4x-12)(2x+4)\right]}{(x+2)^4} \\ &= \frac{32}{(x+2)^3} \end{aligned}$$

(e) Find the zeros of f' and f''.
Solution:

Zeros of $f'(x) = \{2, -6\}$. Known by the products in the numerator.

Zeros of $f''(x)$ are undetermined.

(f) What are the critical points (determine their x and y coordinates)?
Solution:

A critical point of a function $f : D \to \mathbb{R}$, is a point $a \in D$ such that $f'(a) = 0$. The value $f(a)$ is called a critical value of f.

Determine critical points by plugging in the values where $f'(x) = 0$ into the original function

$$\frac{(2-2)^2}{2+2} = 0$$

Critical point 1: $(2, 0)$

$$\begin{aligned}\frac{(-6-2)^2}{-6+2} &= \frac{64}{-4} \\ &= -16\end{aligned}$$

Critical point 2: $(-6, -16)$

(g) Find the local minima and maxima.
Solution:

When a function's slope is zero at x, and the second derivative at x is:

- less than 0, it is a local maximum
- greater than 0, it is a local minimum
- equal to 0, then the test failes

Plugging-in the zeros of f' into $f''(x)$:

$$\frac{32}{(2+2)^3} = \frac{1}{2}$$

Therefore, $(2, 0)$ is a local minimum.

$$\frac{32}{(-6+2)^3} = -\frac{1}{2}$$

Therefore, $(-6, -16)$ is a local maximum.

(h) Which parts of the function are convex and concave? Does function f have points of inflection? (Hint: Use the sign of the second derivative for anwering both questions.)
Solution:

$f''(x) = 0$ is undetermined. Therefore, there is no point of inflection.

(i) Show that the line $y = x - 6$ is a slant asymptote of f. (Hint: use the definition on slide 47 of the lecture and the following two limits.)

$$\lim_{x\ to -\infty} (f(x) - (x-6)) =? \quad and \quad \lim_{x \to +\infty} (f(x) - (x-6)) =?$$

Solution:

7. (**10 points**) Misc

(a) Find the derivative of $f(x) = \ln(\cos(\ln(\cos(x))))$.
Solution:

$$\begin{aligned} f'(x) &= \frac{1}{\cos(\ln(\cos(x)))} \cdot (-\sin(\ln(\cos(x)))) \cdot \frac{1}{\cos(x)} \cdot (-\sin(x)) \\ &= \frac{-\sin(\ln(\cos(x)))}{\cos(\ln(\cos(x)))} \cdot \frac{-\sin(x)}{\cos(x)} \\ &= -\tan(\ln(\cos(x))) \cdot (-\tan(x)) \end{aligned}$$

(b) Find a function $g(x)$ such that $g'(x) = \tan(2x)$.
Solution:

$$g(x) = -\frac{1}{2}\ln(\cos(2x)) + c$$

(c) Find three functions f_1, f_2, f_3 such that $f_1'(x) = f_2'(x) = f_3'(x) = \sin(x)\cos(x)$.
Solution: