Calculus and Probability Theory

Organisation and derivatives

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Outline

About this course I – Lectures

- Weekly, 2 hours, on Mondays 10:45 (HG00.303)
- Presence not compulsory . . .
  - but active attitude expected, when present
- Covering the same material as in:
  - *Calculus* lecture notes by Bernd Souvignier
  - *Kansrekening* lecture notes by Bernd Souvignier
- Main material is on the lecture slides
- Communication via Blackboard

About this course II – Exercise sessions

- Also weekly meetings
  - Wednesdays from 8:45 to 10:30
  - Two locations: E 2.18 / HG00.633
  - Presence not compulsory
- Teaching assistants give the exercise classes
  - Joost Renes
  - Bram Westerbaan
- Activities
  - Questions about lectures/homework
  - Practicing methods
  - Solving problems

About this course III – Homework

- Handing in homework assignments is compulsory (at least five)
  - Homework exercises have to be done individually
- Schedule:
  - New assignment on the web on Tuesday (Blackboard), say in week n
  - You can try them yourself immediately and ask advice on Wednesday morning in week n
  - You can ask final questions, again on Wednesday in week $n + 1$
  - You have to hand-in, via Blackboard (Bb), before Wednesday 14:30 sharp, in week $n + 1$; late submissions will not be accepted.
- Student assistants correct the assignments
  - Arjen Zijlstra
  - Wouter van der Linde

About this course III – Exercise groups

- There will be two groups for the exercise classes, based on the levels of mathematical skills
- Rate your own skill honestly, according to
  - strong, *e.g.* $\geq 7\frac{1}{2}$ at secondary school $\longrightarrow$ HG00.633
  - "not so strong" $\longrightarrow$ E 2.18
  - If you are "uncertain",
    (1) do the self-assessment test (in Bb)
    (2) check it, results published: Tuesday, 16:00
    (3) register Tuesday afternoon: `J.Renes@cs.ru.nl`
- The classifications of the groups will not be used explicitly

About this course IV – Grading

- Exam: 26 Oct., 12:30-15:30, LIN 3 / LIN 8
- Condition: at least five sufficient homework assignments
- Final mark: the result of the written exam

About this course V – How to pass this course . . .

- Practice, practice, practice, think, practice . . .
- You don't learn it it by just staring at the slides!
  - Study the lecture slides (and your notes): read, write, compute
  - Look for similar problems, make your own ones (!) and solve them
  - Attend the exercise classes for further practice
  - Solve the homework first on paper, then make a clear assignment
- Exam questions will be in line with the exercises

About this course VI – Some special points

- You can succeed in this course!
- 3ec means $3 \times 28 = 84$ hours in total
  - Let's say 20 hours for exam
  - 64 hours for 8 weeks means: 8 hours per week!
  - on average 4 hours for studying & making exercises
- Why computer scientists need maths?
  - problem solving
  - thinking in a structured and accurate way
  - programming, esp. for embedded/hybrid systems, machine learning
  - computer hardware and architecture: computer networks, data encryption and compression, ...

About this course VII – And finally...

- Coming up-to-speed is your own responsibility
- If you lack background knowledge, or have forgotten basic mathematics:
  - Blackboard / Course Content / Voorkennis by Wim Gielen
  - Wikipedia
  - Khan Academy, ...
- Further help:
  - Computer algebra system: GeoGebra
  - Wolfram Alpha
- Office hours:
  - G. Alpár: Monday, 16:00-17:00 (Mercator 1, 3.03)
  - J. Reenes: Wednesday, 16:00-17:00 (Mercator 1, 2.16 $\xrightarrow{?}$ 3.17)

Different numbers

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

- In the natural numbers $\mathbb{N}$ one can add and multiply: $x + y$ with 0, $x \cdot y$ with 1.
- In the integers $\mathbb{Z}$ one can also subtract: $x - y$
- In the rationals $\mathbb{Q}$ one can divide: $\frac{x}{y}$, for $y \neq 0$
- In the reals $\mathbb{R}$, with the number line being complete,
  - one can take limits: $\lim_{n \to \infty} r_n$, and
  - thus, one can take roots $\sqrt{x}$ for $x \geq 0$.
- In the complex numbers $\mathbb{C}$ one can take all roots, in particular $\sqrt{-1} = i$.

Numbers: some basic properties

- associative laws, for addition and multiplication

$$a + (b + c) = (a + b) + c \qquad a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

- commutative laws, for addition and multiplication

$$a + b = b + a \qquad a \cdot b = b \cdot a$$

- distributive law

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

- existence of an additive and multiplicative identities:

$$a + 0 = a = 0 + a \qquad a \cdot 1 = a = 1 \cdot a$$

- existence of additive and multiplicative inverses

$$a + (-a) = 0 = (-a) + a \qquad a \cdot \frac{1}{a} = 1 = \frac{1}{a} \cdot a, \text{ for } a \neq 0$$

Function – Basic definitions

Definition (Functions)
A real function $f\colon D \to \mathbb{R}$, for $D \subseteq \mathbb{R}$, is a rule which assigns to each $x \in D$ precisely one $f(x) \in \mathbb{R}$.

- In this situation the subset $D \subseteq \mathbb{R}$ is called the domain of f. Sometimes we write $D(f)$ for D.
- $\mathbb{R}$ is the codomain of f, and the subset $R(f) = \{f(x) | x \in D\} \subseteq \mathbb{R}$ is called the range of f.

Example

- $f(x) = |x|$, "absolute value", with $D(f) = \mathbb{R}$, $R(f) = [0, \infty)$
- $f(x) = \sqrt{25 - 4x^2}$
- $f(x) = \mathrm{sign}(x)$

More on functions I

Definition

A function $f : D \to \mathbb{R}$ is injective or one-to-one if $f(x) = f(y)$ implies $x = y$, for all $x, y \in D$.

A function $f : D \to R \, (\subseteq \mathbb{R})$ is surjective or onto if its range is equal to its codomain

- This means: for any $y \in R$ there is an $x \in D$ such that $f(x) = y$. Symbolically: $\forall y \in R \, \exists x \in D \, f(x) = y$.

A function $f : D \to R$ is bijective if it is both injective and surjective.

Then it is an isomorphism $f : D \xrightarrow{\cong} R$.

Definition (Graph of a real function)

For a function $f : D \to \mathbb{R}$, the graph $G(f) \subseteq D \times \mathbb{R}$ of f contains all pairs $(x, f(x))$. So, we write: $G(f) = \{(x, f(x)) | x \in D\}$.

More on functions II

Definition (Inverse and composition)

If a function $f : D \to \mathbb{R}$, is injective, we can define an inverse function $f^{-1}: R(f) \to D \subseteq \mathbb{R}$, namely:

- for $y \in R(f)$, say $y = f(x)$, define $f^{-1}(y) = x$
- this x is uniquely determined: if $f(x) = y = f(x')$, then $x = x'$, since f is injective
- by construction: $f(f^{-1}(y)) = y$ and also $f^{-1}(f(x)) = x$.

The composition of functions $f : X \to Y$ and $g : Y \to Z$ is the function $h = g \circ f : X \to Z$, for which $h(x) = g(f(x))$, for each $x \in X$.

Definition (Parity of function)

A function $f : (-a, a) \to \mathbb{R}$ is even if $f(-x) = f(x)$, for all $x \in (-a, a)$, and odd if $f(-x) = -f(x)$, for all $x \in (-a, a)$.

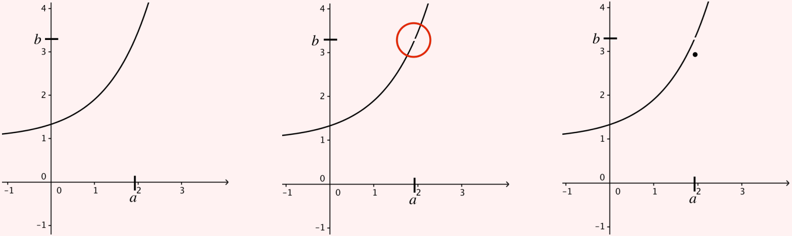
Example

$f(x) = x^3$ $\qquad g(x) = |x|$ $\qquad g \circ f =?, f \circ g =?,$ even?, odd?

Intuition fo limit

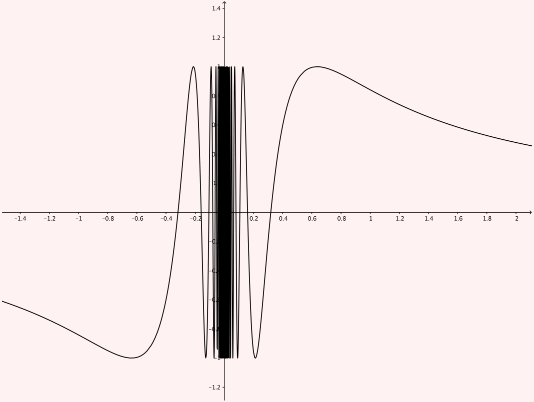
Definition ("Approach to limit")
A function f approaches the limit b at an input a, if we can make $f(x)$ as close as we like to b by requiring that x be sufficiently close, *but not equal*, to a.

Example



Example: No limit of $\sin\left(\frac{1}{x}\right)$ at 0

Example



Limits

Definition

A function $f : D \to \mathbb{R}$ has limit b for $x \to a$ if:

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in D \quad 0 < |x - a| < \delta \Rightarrow |f(x) - b| < \varepsilon.$$

In that case we write $\lim_{x \to a} f(x) = b$. "the limit of f of x as x approaches a equals b"

(Note: a does not have to be in D.)

Example

- $\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x \to 3} (x + 3) = 6.$
- $\lim_{x \to 1} \frac{-1}{x-1}$ is undefined

Limits involving infinity

We also need $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$. What does this mean?

Definition

A function $f : D \to \mathbb{R}$ has limit b for $x \to \infty$ if:

$$\forall \varepsilon > 0\, \exists n \in \mathbb{N}\, \forall x \in D\ x > n \Rightarrow |f(x) - b| < \varepsilon.$$

In that case we write $\lim_{x\to\infty} f(x) = b$. Formulate yourself what $\lim_{x\to-\infty} f(x) = b$ means.

Example

$\lim_{x\to\infty} \frac{1}{x} = 0$ $\quad$ $\lim_{x\to-\infty} \frac{1}{x} = 0$ $\quad$ $\lim_{x\to\pm\infty} \frac{11x^2-2x+3}{5x^2+3x-1}$ $\quad$ $\lim_{x\to\infty} \frac{-2.5x^5}{100x^4+1}$

Continuous functions

Definition

A function $f : D \to \mathbb{R}$ is continuous in point $a \in D$ if $f(x)$ is close to $f(a)$ for each x that is close to a.

More formally: $f : D \to \mathbb{R}$ is continuous in point $a \in D$ if:
$\forall \varepsilon > 0\, \exists \delta > 0\, \forall x \in D\; |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

Note: this is the same as $a \in D$, $\exists \lim_{x \to a} f(x)$ and $\lim_{x \to a} f(x) = f(a)$.

A function $f : D \to \mathbb{R}$ is continuous if it is continuous in all $a \in D$.

Example

The function $f(x) = \text{sign}(x)$ is not continuous in 0.
Indeed, $\exists \varepsilon > 0\, \forall \delta > 0\, \exists x$ with $|x - 0| = |x| < \delta$ but
$|f(x) - f(0)| = |f(x)| \geq \varepsilon$
Choose $\varepsilon = \frac{1}{2}$, then any $x \neq 0$ with $|x| < \delta$ has $|\text{sign}(x)| = 1$. The function values around 0 do not fall into the ε-interval.

Recall: Points and lines

- Equation of a line
  - y-intercept (b), slope (m)
  - most convenient: $y = mx + b$
  - can be determined from *e.g.*, two points, or a point and the slope
- Distance of two points (x_1, y_1) and (x_2, y_2)
  - by Pythagorean theorem
  - $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Example

- What is the intercept and the slope: $3x - 4y = -8$?
- Determine the distance of points $(2, -3)$ and $(-3, 9)$.
- Equation of the line? Slope: $m = 2/5$, a point on it: $(-1, 2)$.

Derivatives

Definition

A function $f : D \to \mathbb{R}$ is differentiable at a if $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ exists. In this case the limit is denoted $f'(a)$ and is called the derivative of f at a.

f is differentiable if f is differentiable at a for every $a \in D$.

We also define the tangent line to f at a to be the line through $(a, f(a)) \in G(f)$ with slope $f'(a)$.

If f is a differentiable function then f' ("Lagrange notation") is sometimes written as $\frac{df}{dx}$ ("Leibniz notation").

More examples

Example (Geometric interpretation)

Find a tangent line of a curve $f(x) = \frac{1}{x}$ in $a = 2$.

Example

Check that $f(x) = |x|$ is *not* differentiable in 0.

(Differentiable implies continuous, but not the other way around, as this example shows.)

Differentiation rules

Let $f, g : D \to \mathbb{R}$ be differentiable functions in $a \in D$

- For a constant function $f(x) = c, c \in \mathbb{R}$, we have $f'(x) = 0$
- $f(x) = x$, then $f'(x) = 1$.
- sum/subtraction rule $(f \pm g)'(a) = f'(a) \pm g'(a)$.
- scalar rule: $(c \cdot f)'(a) = c \cdot f'(a)$.
- product rule $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$.
- division rule $(\frac{f}{g})'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}$.
- chain/composition rule $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$
- if f has an inverse f^{-1}, then $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$

Derivatives of powers

Lemma

(1) *For $n \in \mathbb{N}$ and $f(x) = x^n$ we have $f'(x) = nx^{n-1}$*
- *This can be shown by induction on n*

(2) *In fact, for $n \in \mathbb{Z}$ and $f(x) = x^n$ we have $f'(x) = nx^{n-1}$*
- *This follows from the previous point, using the division rule.*

(3) *It can be shown that $(x^a)' = ax^{a-1}$, for any $a \in \mathbb{R}$.*

Derivation exercises

Example

- $f(x) = (-1 + 7x)(3 - 4x)$. Find f'.
- $f(x) = \dfrac{-1 + 7x}{3 - 4x}$. Find f'.
- $y = x^6 - 3x^4 + 4x - 3$. Find f'.
- $f(x) = x^2$. Find $(f^{-1})'$.
- $f(x) = \sqrt{2 - 5x}$. Find f'.

Recall exponential and logarithm

Exponential, for $a \geq 0$

- $a^0 = 1$, $a^{x+y} = a^x \cdot a^y$
- $a^1 = a$, $a^{x \cdot y} = (a^x)^y$
- $a^{-x} = \frac{1}{a^x}$, and thus $a^{x-y} = \frac{a^x}{a^y}$

The logarithm is defined as inverse of power: $x = \log_a(y) \iff a^x = y$, for $y > 0$.

Logarithm

- $\log_a(a^x) = x$ and $a^{\log_a x} = x$
- $\log_a(x \cdot y) = \log_a(x) + \log_a(y)$, and $\log_a(x^y) = y \cdot \log_a(x)$
- $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$
- $\dfrac{\log_a x}{\log_a b} = \log_b x$

Introducing Euler's number e

Consider $f_a(x) = a^x$. Then:

$$\begin{aligned}
f_a'(x) = (a^x)' &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h\to 0} \frac{a^x(a^h - 1)}{h} \\
&= a^x \cdot \lim_{h\to 0} \frac{a^{0+h} - a^0}{h} = a^x \cdot \lim_{h\to 0} \frac{f_a(0+h) - f_a(0)}{h} \\
&= a^x \cdot f_a'(0).
\end{aligned}$$

- We have: $f_a'(0) = 1$ for $a = e = 2.71828\ldots$
- and thus $\boxed{(e^x)' = e^x}$
- The natural logarithm ln uses base e; notation: $\ln \equiv \log_e$

Important derivatives with logarithms

$$(a^x)' = a^x \cdot \ln(a) \qquad \text{and} \qquad (\ln(y))' = \frac{1}{y}$$

- We have $(e^{f(x)})' = e^{f(x)} \cdot f'(x)$ by the chain rule
- Thus: $(a^x)' = a^x \cdot \ln(a)$, since:

$$(a^x)' = ((e^{\ln(a)})^x)' = (e^{\ln(a)\cdot x})' = e^{\ln(a)\cdot x} \cdot \ln(a) = a^x \cdot \ln(a).$$

- For $f(x) = e^x$ we have $f'(x) = e^x$ and $f^{-1}(y) = \ln y$
- We use the inverse function law $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$
- Thus $\ln'(y) = \frac{1}{f'(\ln y)} = \frac{1}{e^{\ln y}} = \frac{1}{y}$.

Logarithmic differentiation

Definition

According to the chain rule:

$$(\ln f(x))' = \ln'(f(x)) \cdot f'(x) = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}$$

Briefly: $(\ln f)' = \frac{f'}{f}$. This is called the logarithmic derivative of f and this law is called logarithmic differentiation.

Logarithmic differentiation: Example

For $f(x) = \frac{6x}{\sqrt{x-1}}$ we can compute $f'(x)$ via the fraction rule, but also by first taking logarithms on both sides:

$$\ln(f(x)) = \ln\left(\frac{6x}{\sqrt{x-1}}\right) = \ln(6x) - \ln((x-1)^{\frac{1}{2}}) = \ln(6x) - \frac{1}{2}\ln(x-1)$$

Differentiating on both sides gives:

$$\frac{f'(x)}{f(x)} = \frac{6}{6x} - \frac{1}{2}\cdot\frac{1}{x-1} = \frac{1}{x} - \frac{1}{2(x-1)} = \frac{2(x-1)-x}{2x(x-1)} = \frac{x-2}{2x(x-1)}$$

Hence:

$$f'(x) = f(x)\cdot\frac{x-2}{2x(x-1)} = \frac{6x}{\sqrt{x-1}}\cdot\frac{x-2}{2x(x-1)} = \frac{3(x-2)}{(x-1)^{\frac{3}{2}}}$$

Recall sine, cosine and tangent

- Geometric interpretation with $\sin(90^\circ) = \sin(\frac{\pi}{2}) = 1$ *etc.*
- $\sin^2(x) + \cos^2(x) = 1$
- Sum rules:
  - $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
  - $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
  - $\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
  - $\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
- $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$
- $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, with $\tan'(x) = \dfrac{1}{\cos^2(x)}$.

Another example

Logarithmic differentation is useful for reducing products to sum, fractions to differences, and powers to products.

Take $f(x) = (\sin x)^x$.

$$\ln f(x) = \ln\left((\sin x)^x\right) = x \cdot \ln(\sin x)$$

Thus:

$$\frac{f'(x)}{f(x)} = \ln(\sin x) + x \cdot \frac{1}{\sin x} \cdot \cos x$$

And:

$$f'(x) = f(x) \cdot \left(\ln(\sin x) + \frac{x \cos x}{\sin x}\right) = (\sin x)^x \left(\ln(\sin x) + \frac{x \cos x}{\sin x}\right).$$

Overview: derivatives of special functions

- $f(x) = a^x$ then $f'(x) = a^x \cdot \ln a$. Special case $(e^x)' = e^x$
- $(\log_a x)' = \frac{1}{x \cdot \ln a}$, with special case $(\ln x)' = \frac{1}{x}$
- $(\sin x)' = \cos x$
- $(\cos x)' = -\sin x$
- $(\tan x)' = \frac{1}{\cos^2 x}$, where $\tan x = \frac{\sin x}{\cos x}$
- $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ where $\arcsin = \sin^{-1}$
- $(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}$ where $\arccos = \cos^{-1}$
- $(\arctan x)' = \frac{1}{1+x^2}$ where $\arctan = \tan^{-1}$

L'Hôpital's rule

Let $f, g : D \to \mathbb{R}$ be functions that are differentiable and

- $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \pm\infty$ or $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, moreover
- $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists with
- $g'(x) \neq 0$ (except for perhaps at a), then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Example

- $\lim_{x \to 0} \frac{\sin(x)}{x} \stackrel{*}{=} \lim_{x \to 0} \frac{\cos(x)}{1} = \cos(0) = 1;$
- $\lim_{x \to \infty} \frac{\ln(x)}{\sqrt{x}} \stackrel{*}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \to \infty} \frac{2\sqrt{x}}{x} = \lim_{x \to \infty} \frac{2}{\sqrt{x}} = 0;$
- $\lim_{x \to 0} \frac{1-\cos(x)}{x^2} \stackrel{*}{=} \lim_{x \to 0} \frac{\sin(x)}{2x} \stackrel{*}{=} \lim_{x \to 0} \frac{\cos(x)}{2} = \frac{1}{2}.$

Higher order derivatives

Let $f(x)$ be a real function.

- One writes $f' = \frac{df}{dx}$
- The second derivative is written as: $f'' = \frac{d}{dx} f' = \frac{d^2 f}{dx^2}$
- The n-the derivative is: $f^{(n)} = \frac{d}{dx} f^{(n-1)}$ with $f^{(0)} = f$

Example

Let $f(x) = x^n$, find $f^{(n)}(x)$.

Monotonicity and the derivative

Definition

Let $f : D \to \mathbb{R}$ be a function.

- f is increasing if $x_1 < x_2 \Longrightarrow f(x_1) \leq f(x_2)$, for all $x_1, x_2 \in D$
- f is strictly increasing if $x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$, for all $x_1, x_2 \in D$
- f is decreasing if $x_1 < x_2 \Longrightarrow f(x_1) \geq f(x_2)$, for all $x_1, x_2 \in D$
- f is strictly decreasing if $x_1 < x_2 \Longrightarrow f(x_1) > f(x_2)$, for all $x_1, x_2 \in D$

Proposition

- If $f'(x) \geq 0$, $\forall x \in [a, b] \Rightarrow f$ is increasing on $[a, b]$.
- If $f'(x) \leq 0$, $\forall x \in [a, b] \Rightarrow f$ is decreasing on $[a, b]$.

Absolute vs local, for extreme (= minimum or maximum)

Definition

A real function $f : D \to \mathbb{R}$ has in $a \in D$ absolute minimum (or absolute maximum) if $f(a) \leq f(x)$ (or $f(a) \geq f(x)$), for all $x \in D$.

This f has in $a \in D$ a local minimum (or maximum) if $\exists\, \delta > 0$ such that $f(a) \leq f(x)$ (or $f(a) \geq f(x)$), for all $x \in (a - \delta, a + \delta)$.

Lemma

Let $f : D \to \mathbb{R}$ be differentiable in a. If f has a local minimum or a local maximum in a then $f'(a) = 0$.

Note: the first derivative need not exist in a local extreme. Example: $f(x) = |x|$ has a minimum in 0.

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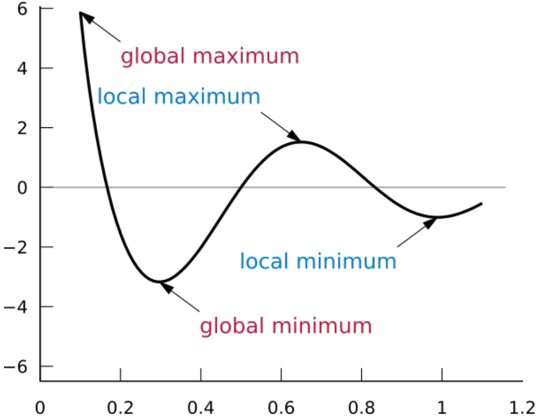
Absolute vs local, for extreme. Example



Source: Wikipedia

Extremes and critical points

Definition
A critical point of a function $f : D \to \mathbb{R}$, is a point $a \in D$ such that $f'(a) = 0$. The value $f(a)$ is called a critical value of f.

Fact

- *We saw: extremes are critical* *if the function is differentiable*
- *The reverse fails, see $f(x) = x^3$ in 0*

In order to find the maximum and minimum of $f : D \to \mathbb{R}$ three kinds of points must be considered:

- the critical points of f in D,
- points x in D such that f is not differentiable at x,
- points on the edge of D, that is, points $x \in D$ with $[x - \delta, x) \bigcap D = \emptyset$ or $(x, x + \delta] \bigcap D = \emptyset)$ for all $\delta \geq 0$.

Sufficient conditions for extremes

Theorem

A differentiable function $f(x)$ has a local minimum in a if $\exists\,\delta > 0$ such that $f'(a) = 0$, $f'(x) \leq 0$ for $x \in (a-\delta, a)$ and $f'(x) \geq 0$ on $(a, a+\delta)$. Especially, if $f'(a) = 0$ and $f''(a) > 0$, so that f' is increasing.

A differentiable function $f(x)$ has a local maximum in a if $\exists\,\delta > 0$ such that $f'(a) = 0$, $f'(x) \geq 0$ for $x \in (a-\delta, a)$ and $f'(x) \leq 0$ on $(a, a+\delta)$. Especially, if $f'(a) = 0$ and $f''(a) < 0$.

Example

Consider the function $f(x) = x^4 - 2x^2$. Critical points are $-1, 0, 1$. Which of them are min/max?

Convexity and Concavity

Definition

A function f is convex (or concave) on an interval if for all a and b in the interval, the line segment joining $(a, f(a))$ and $(b, f(b))$ lies above (or below) the graph of f.

Simply: convex = ☺ concave = ☹

(Convex and concave are sometimes called *concave up* and *concave down*, respectively.)

A point of inflection on a curve $y = f(x)$ is a point at which f changes from concave to convex or vice versa.

Theorem

- *If $f''(x) > 0$, for all $x \in (a, b)$, then f is convex on (a, b).*
- *If $f''(x) < 0$, for all $x \in (a, b)$, then f is concave on (a, b).*
- *If f has an inflection point at x and f'' exists in $(x - \delta, x + \delta)$, for some $\delta > 0$, then $f''(x) = 0$.*

Asymptotes

Definition

- A vertical line $x = a$ is called a vertical asymptote of f if both limits from below $\lim_{x \to a^-} f(x)$ or from above $\lim_{x \to a^+} f(x)$ are infinite.
- A horizontal line $y = b$ is called a horizontal asymptote of f if $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$.
- A line $y = ax + b$ is called a slant asymptote of f if $\lim_{x \to \pm\infty} f(x) - (ax + b) = 0$.
  - find a, b as: $a = \lim_{x \to \infty} \frac{f(x)}{x}$ and $b = \lim_{x \to \infty} f(x) - ax$

Example

$f(x) = \frac{x^2+3x+2}{x-2}$ has slant asymptote $y = x + 5$, since

$a = \lim_{x \to \infty} \frac{x+3+\frac{2}{x}}{x-2} = 1, \quad b = \lim_{x \to \infty} \frac{x^2+3x+2}{x-2} - \frac{x^2-2x}{x-2} = \lim_{x \to \infty} \frac{5x+4}{x-2} = 5.$

Curve Sketching

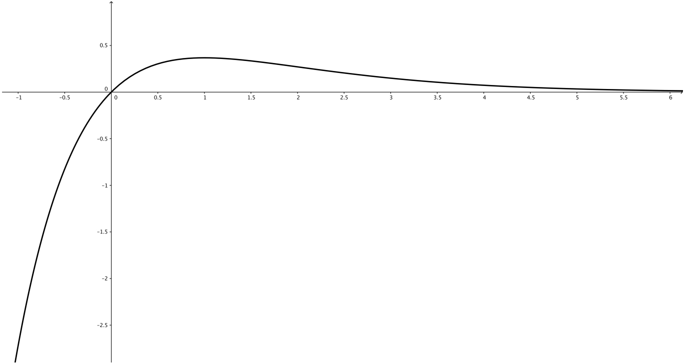
These steps should be followed to investigate function $f(x)$:

(1) domain of f
(2) parity *i.e.* is f even or odd
(3) points of intersection with x-axis and y-axis
(4) behaviour of f on the edges of the domain
(5) asymptotes
(6) monotonicity and min/max
(7) concavity/convexity and points of inflection
(8) table and sketch

Example

Example

Sketch the graph of $f(x) = xe^{-x}$.



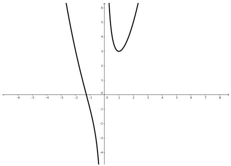
Example 2.

Example

Sketch the graph of $f(x) = x^2 + \frac{2}{x}$.

$f'(x) = 2x - \frac{2}{x^2}$ $\qquad$ $f''(x) = 2 + \frac{4}{x^3}$

	$(-\infty, -\sqrt[3]{2})$	$-\sqrt[3]{2}$	$(-\sqrt[3]{2}, 0)$	0	$(0, 1)$	1	$(1, +\infty)$
f	$\lim\limits_{-\infty} f = +\infty$	0	$\lim\limits_{0^-} f = -\infty$	$\notin D(f)$	$\lim\limits_{0^+} f = +\infty$	3	$\lim\limits_{+\infty} f = +\infty$
f'		$-$		$\vert$	$-$	0	$+$
f		$\searrow$		$\vert$	$\searrow$	loc.min.	$\nearrow$
f''	$+$	0	$-$	$\vert$		$+$	
f	$\smile$	infl	$\frown$	$\vert$		$\smile$	



Functions in several variables

- So far we have seen functions $f(x) = \cdots$ in one variable x.
- One can also have functions in several variables: $f(x, y) = \cdots$ or $g(x_1, \ldots, x_n) = \cdots$
  - These are functions $D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^n$, for some $n \in \mathbb{N}$

Example

- Distance from origin in $\mathbb{R}^2$, given by $f(x, y) = \sqrt{x^2 + y^2}$
- In physics: $f(x, y, t) = e^{-t} \cdot (\sin x + \cos y)$ describes heat distribution in a plane, as a function of position and time.
- Arbitrarily looking functions: $f(x_1, x_2, x_3) = \sin(x_3) \cdot \frac{x_2^{10}}{\ln(x_1)}$.

Differentiation in several variables: partial derivatives

- Given a function in several variables, say $f(x, y)$, one can take the derivative in each variable separately. These are called partial derivatives.
- Now the 'Leibniz' notation $\frac{df}{dx}$ and $\frac{df}{dy}$ is convenient. For partial derivatives they are written as curly d, as in: $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
- Thus:

$$\frac{\partial f}{\partial x}(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} \qquad \frac{\partial f}{\partial y}(a, b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h}$$

- The standard rules for derivatives apply, where the 'other' variables are treated as constants.

Second partial derivatives

- Consider $f(x, y, t) = e^{-t} \cdot (\sin x + \cos y)$
- Then:
  - $\frac{\partial f}{\partial x} = e^{-t} \cdot \cos x$
  - $\frac{\partial f}{\partial y} = -e^{-t} \cdot \sin y$
  - $\frac{\partial f}{\partial t} = -e^{-t} \cdot (\sin x + \cos x)$
- We can continue with successive partial derivatives:

$$\frac{\partial^2 f}{\partial x \partial t} = -e^{-t} \cdot \cos x$$

The Theorem of Schwarz says that it does not matter in which order you take the two partial derivatives:

$$\frac{\partial^2 f}{\partial x \partial t} = \frac{\partial^2 f}{\partial t \partial x}$$

(assuming all these derivatives exist and are continuous).