Calculus and Probability

Assignment 5

Note:

- You can hand in your solutions as a single PDF via the assignment module in Blackboard. Note that the document should be in English and typeset with LaTeX, Word or a similar program. It should not be a scan or picture of your handwritten notes.
- Make sure that your name, student number and group number are on top of the first page!
- **Note that your submission should be an individual submission because it can influence your final grade for this course. If we detect that your work is not completely your own work, we will ask the exam committee to investigate whether it is plagiarism or not!**

Exercises to be presented during the exercise hours

Exercise 1
Consider the function $f(x) = (x^2 - 2)^3$.

a) What is the domain and range of $f(x)$?

$D(f) = \mathbb{R}$ and $R(f) = [-8, \infty)$

b) What are the roots of $f(x)$

For the roots we have that $(x^2 - 2)^3 = 0$, so $(x^2 - 2) = 0$. The roots therefore occur at $x = \sqrt{2}$ and $x = -\sqrt{2}$.

c) Find the local minima and maxima of $f(x)$

Taking the derivative, we have $f'(x) = 6x(x^2-2)^2$. For local optima, we have $f'(x) = 0$, which holds if $x = \sqrt{2}$, $x = -\sqrt{2}$ and $x = 0$. Considering the second derivative, we have: $f''(x) = 6(x^2-2)^2 + 24x^2(x^2-2)$. So in $x = \sqrt{2}$ and $x = -\sqrt{2}$, $f''(x) = 0$, meaning these are possible inflection points. For $x = 0$, $f''(x) > 0$, so this is a local minimum.

d) Which parts of the function are convex and concave? Does function f have points of inflection?

$f''(x) = 6(x^2-2)((5x^2-2)) = 0$, so $f''(x) = 0$ for $x = \sqrt{2}$, $x = -\sqrt{2}$, $x = \sqrt{\frac{2}{5}}$ and $x = -\sqrt{\frac{2}{5}}$. Using $f''(x)$, we can check that f(x) is convex for $x < -\sqrt{2}$, concave for $-\sqrt{2} < x < -\sqrt{\frac{2}{5}}$, convex for $-\sqrt{\frac{2}{5}} < x < \sqrt{\frac{2}{5}}$, concave for $\sqrt{\frac{2}{5}} < x < \sqrt{2}$, and again convex for $\sqrt{2} < x$. So it has 4 inflection points.

Exercise 2

For which values of c has the equation $\ln x = cx^2$ precisely one solution? (Hint: There is a value $0.1 < c < 0.2$ for which the curves just touch each other. What do these curves also have in common, besides the point of intersection?)

Let a be the point at which the curves just touch each other: $\ln a = ca^2$. In this point a **the gradient of both curves must be similar**, or else they would intersect or not meet. Hence, $\frac{1}{a} = 2ca$ or equivalently $\frac{1}{2} = ca^2$. Using this in the first equation gives $\ln a = \frac{1}{2}$ or $a = e^{\frac{1}{2}}$. Using this in the second equation and solving for c gives $c = \frac{1}{2e}$. Clearly, the equations also intersect in one point when c is negative or zero, thus the full answer is $c \in (-\infty, 0] \cup \{\frac{1}{2e}\}$.

Exercise 3

Find the limits of the following functions.

a) $\lim\limits_{x\to 1} \frac{x^a - ax + a - 1}{(x-1)^2}$;

Since we have $\frac{0}{0}$, we can apply L'Hôpital's rule: $\lim\limits_{x\to 1} \frac{x^a - ax + a - 1}{(x-1)^2} = \lim\limits_{x\to 1} \frac{ax^{(a-1)} - a}{2(x-1)}$. Using L'Hôpital's rule again, we have $\lim\limits_{x\to 1} \frac{(a-1)ax^{(a-2)}}{2} = \frac{1}{2}(a-1)a$.

b) $\lim\limits_{x\to\infty} \frac{\ln(2015x)}{x^3}$;

Since it's $\frac{\infty}{\infty}$, we can apply L'Hôpital's rule: $\lim\limits_{x\to\infty} \frac{\ln(2015x)}{x^3} = \lim\limits_{x\to\infty} \frac{\frac{1}{x}}{3x^2} = \lim\limits_{x\to\infty} \frac{1}{3x^3} = 0$.

Exercise 4
Find primitives of the following functions f. That is, find F such that $F'(x) = f(x)$.

a) $f(x) = \frac{1}{2\sqrt{x}} - \frac{1}{x^2}$;

$F(x) = \sqrt{x} + \frac{1}{x} + \mathrm{C}$;

b) $f(x) = 2x + 5(1 - x^2)^{-\frac{1}{2}}$

$F(x) = x^2 + 5\arcsin(x) + C$

Exercises to be handed in

You are expected to explain your answers, even if this is not explicitly stated in the exercises themselves.

Exercise 5
Consider the function: $f(x) = e^{x^3-x} - 1$.

a) What is the domain and range of $f(x)$? 1 pt

$D(f) = \mathbb{R}$ and $R(f) = (-1, \infty)$.

b) What are the roots of $f(x)$? 1 pt

For the roots we have that $e^{x^3-x} = 1$. Taking the logarithm on both sides, we get $x^3 - x = 0$ and $x(x^2-1) = 0$, so $f(x) = 0$ for $x = 0, -1, +1$.

c) Find the local minima and maxima of $f(x)$. 1 pt

Taking the derivative, we have $f'(x) = (3x^2-1)e^{(x^3-x)}$. For local optima, we have $f'(x) = 0$, which holds if $(3x^2-1) = 0$, so we have optima at $x = \frac{1}{\sqrt{3}}$ and $x = \frac{-1}{\sqrt{3}}$. By considering the second derivative $f''(x) = (6x + (3x^2-1)^2)e^{x^3-x}$ at these locations, we find that $x = \frac{1}{\sqrt{3}}$ is a local minimum and $x = \frac{-1}{\sqrt{3}}$ is a local maximum.

d) Find the limits of the function for the boundaries of the domain of the function. 1 pt

As $x \to -\infty$, $e^{x^3-x} \to 0$ so $\lim_{x\to-\infty} f(x) = -1$. For $\lim_{x\to\infty} e^{x^3-x} - 1 = \infty$ because x^3 grows faster than x.

Exercise 6
a) $f(x) = x^x$, find $f'(x)$; 2 pt

Using logarithmic differentiation, we have $\frac{f'(x)}{f(x)} = 1 + \ln x$, so $f'(x) = x^x(1 + \ln x)$.

b) $f^{-1}(x) = \sin(x^2)$, find $f'(x)$; $\boxed{2\text{ pt}}$

First note that $f(y) = \sqrt{\arcsin(y)}$. Using the inverse rule: $(f^{-1}))'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{2f^{-1}(x)\cos(f^{-1}(x)^2)} = \frac{1}{2\sqrt{\arcsin(y)}\cos(\arcsin(y))} = \frac{1}{2\sqrt{\arcsin(y)}\sqrt{1-y^2}}$. The last inequality is because of the identity: $\cos(\arcsin(y)) = \sqrt{1-y^2}$.

Alternatively, we could take the derivative of the inverse, which by the chainrule gives: $f(y) = \frac{1}{2\sqrt{\arcsin(y)}\sqrt{1-y^2}}$.

Exercise 7

Find the limits of the following functions.

a) $\lim\limits_{a\to-3} \frac{\sin(a\cdot\pi)}{a^2-9}$; $\boxed{2\text{ pt}}$

Since it's $\frac{0}{0}$, we can apply L'Hôpital's rule: $\lim\limits_{a\to-3} \frac{\sin(a\cdot\pi)}{a^2-9} = \lim\limits_{a\to-3} \frac{\pi\cos(a\cdot\pi)}{2a} = \frac{\pi\cos(-3\pi)}{2\cdot(-3)} = \frac{\pi\cdot(-1)}{-6} = \frac{\pi}{6}$.

b) $\lim\limits_{x\to-\infty} \frac{e^{3-x}}{7x^2}$. $\boxed{2\text{ pt}}$

Since it's $\frac{\infty}{\infty}$, we can apply L'Hôpital's rule: $\lim\limits_{x\to-\infty} \frac{e^{3-x}}{7x^2} = \lim\limits_{x\to-\infty} \frac{-e^{3-x}}{14x}$. This is still of type $\frac{-\infty}{-\infty}$, so we can't compute the limit yet, but L'Hôpital's rule can be applied again, which will give the solution: $\lim\limits_{x\to-\infty} \frac{-e^{3-x}}{14x} = \lim\limits_{x\to-\infty} \frac{e^{3-x}}{14} = \infty$.

Exercise 8

Find primitives of the following functions f. That is, find F such that $F'(x) = f(x)$.

a) $f(x) = 2\sin(x)\cos(x)$; $\boxed{2\text{ pt}}$

$F(x) = \sin^2(x) + C$;

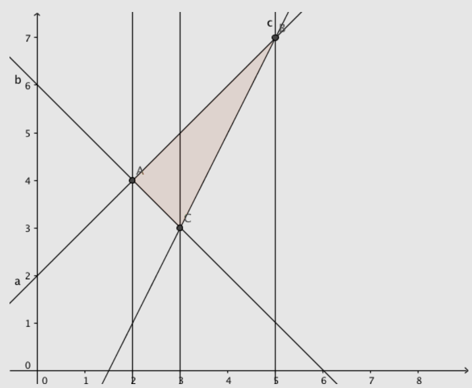
b) $f(x) = \frac{2}{1+4x^2}$; $\boxed{2\text{ pt}}$

$F(x) = \arctan(2x) + C;$

Exercise 9

a) Given three lines, $y = x + 2$, $y = -x + 6$ and $y = 2x - 3$, enclosing a triangle. Determine the *coordinates* of the three vertices and the *area* of the triangle. $\boxed{2\text{ pt}}$

To get the vertices, we have to solve three equations: $x + 2 = -x + 6$, $2x - 3 = -x + 6$ and $x + 2 = 2x - 3$. We find that the three vertices are $(2, 4), (3, 3)$ and $(5, 7)$. To compute the area of the triangle there are multiple possibilities. We focus here on the one with definite integrals.



After sketching the lines we find that the area can be computed as

$$\begin{aligned} \int_2^5 (x+2)\,dx - \int_2^3 (-x+6)\,dx - \int_3^5 (2x-3)\,dx \\ = [x^2 + 2x]_2^5 - [-x^2 + 6x]_2^3 - [x^2 - 3x]_3^5 \\ = 3 \end{aligned} \tag{1}$$

Solution 2: to compute the length of the edges and to apply some formula, such as Heron's, for the area; Solution 3: recognising that the first two lines orthogonal, one can compute the area easier; namely, $\sqrt{2} \cdot 3\sqrt{2}/2 = 3$.

b) Suppose the triangular area calculated in (a) represents a piece of land and that we want to enlarge the area by placing a fence at *exactly a* meters from the boundary of the triangular area. Compute both the area A and length of the boundary B of this new area. $\boxed{2\text{ pt}}$

First we compute the length of the triangular boundary edges:

$$\begin{aligned} d((2,4),(3,3)) &= \sqrt{(2-3)^2+(4-3)^2} = \sqrt{2} \\ d((2,4),(5,7)) &= \sqrt{(2-5)^2+(4-7)^2} = \sqrt{18} \\ d((3,3),(5,7)) &= \sqrt{(3-5)^2+(3-7)^2} = \sqrt{20} \end{aligned}$$

Each of the edges can be extended with a rectangular area of size a times the length of the boundary edge, i.e., $a \cdot (\sqrt{2}+\sqrt{18}+\sqrt{20})$. The remaining parts of the enlarged area around the corners of the triangle can be combined into a *circle*, which has an area of $\pi \cdot a^2$ and circumference of $2\pi \cdot a$. Thus we have:

$$\begin{aligned} \text{(area)} : A &= 3 + a \cdot (\sqrt{2}+\sqrt{18}+\sqrt{20}) + (\pi \cdot a^2) \\ \text{(boundary)} : B &= (\sqrt{2}+\sqrt{18}+\sqrt{20}) + (2\pi \cdot a) \end{aligned}$$

c) (Bonus 2 pt) If we want the values A and B from (b) to be equal, i.e., $A = B$, does the distance a has to be smaller or larger than 1?

Define $c = \sqrt{2}+\sqrt{18}+\sqrt{20}$ then we have

$$A - B = \pi \cdot a^2 + (c - 2\pi)a + (3 - c)$$

which is zero when

$$a = \frac{-(c-2\pi) \pm \sqrt{(c-2\pi)^2 - 4\pi(3-c)}}{2\pi} = \{-2.24, 1.01\}$$

Hence, since a must be positive, $a = 1.01$ which is larger than 1. (In this case we have $A = B = 16.5$.)

Your final grade is the sum of your scores divided by 2.0.