Calculus and Probability

Assignment 7

Note:

- You can hand in your solutions as a single PDF via the assignment module in Blackboard. Note that the document should be in English and typeset with LaTeX, Word or a similar program. It should not be a scan or picture of your handwritten notes.
- Make sure that your name, student number and group number are on top of the first page!
- **Note that your submission should be an individual submission because it can influence your final grade for this course. If we detect that your work is not completely your own work, we will ask the exam committee to investigate whether it is plagiarism or not!**

Exercises to be presented during the exercise hours

Exercise 1
An experiment consists of drawing 3 cards in succession from a well-shuffled ordinary deck of cards.[1] Let A_1 denote the event "Ace on first draw", A_2 the event "Ace on second draw" and A_3 the event "Ace on third draw". State in words the meaning of each of the following probabilities.

a) $P(A_1 \cap \neg A_2)$;

b) $P(A_1 \cup A_2)$;

c) $P(\neg A_1 \cap \neg A_2 \cap \neg A_3)$;

d) $P[(A_1 \cap \neg A_2) \cup (\neg A_2 \cap A_3)]$.

Exercise 2
Let P be a probability measure on space S. Prove that $P(\emptyset) = 0$.

[1] If you are not familiar with the standard 52-card deck, visit Wikipedia: `http://en.wikipedia.org/wiki/Standard_52-card_deck`.

Exercise 3
There are twelve provinces in the Netherlands. Suppose that the birth of Dutch people is uniformly distributed over all twelve provinces, *i.e.* for every province $\frac{1}{12}$ of the population is born there. What is the probability that at least two of r randomly selected Dutch-born people were born in the same province, where $r = 1$, $r = 2$, $r = 3$, $r = 4$, $r = 5$ or $r = 6$?

Exercise 4
A shooter has exactly 6 bullets and shoots on a target. A random variable X is the number of bullets used *until he/she hits it for the first time, or has no more bullets.* The probability of a bullet hitting the target is 0.4 for every attempt.

a) Find the probability distribution of X; that is, give the probabilities for all possible values.

b) What is the expected value for X?

c) What is the variance?

d) What is the standard deviation?

Exercise 5
TV sets with various defects are brought to the service for reparation. The time of reparation is a continuous random variable T. The cumulative distribution function of T is given as:

$$F(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 - e^{-kt} & \text{if } t \geq 0, \end{cases}$$

where $k > 0$.

a) Find the probability density function f of the random variable.

b) Find the expectation and variance.

Exercises to be handed in

You are expected to explain your answers, even if this is not explicitly stated in the exercises themselves.

Exercise 6
We toss a coin twice.

a) Give a corresponding sample space S. 1 pt

b) Give the set of outcomes corresponding to each of the following events: 1 pt

(i) A: "we throw heads exactly once";

(ii) B: "we throw heads at least once";

(iii) C: "tails did not appear before a head appeared".

c) Give a probability measure P_1 for the sample space S. 1 pt

Exercise 7
Let P be a probability measure on space S. Prove that for pairwise mutually exclusive events $A_1, A_2, \ldots A_n$ (*i.e.* $A_i \cap A_j = \emptyset$ for all $i \neq j$ and $n \geq 2$) one has $P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$. (Hint: use induction.) 1 pt

Exercise 8
Consider a class where students have to hand in exercises every week. They have to hand in eight assignments in total and have to pass at least five to be able to attend the exam. Student A does not study very hard, so for each assignment he/she has a probability of 0.5 to pass. Student B studies very hard, so for each assignment he/she has a probability of 0.8 to pass. The random variable X_A is the number of passes of student A and the random variable X_B is the number of passes of student B.

a) Find $P(X_A = 5)$; 1 pt

b) Find $P(X_A \geq 5)$; 1 pt

c) Find $P(X_B \geq 5)$. 1 pt

Exercise 9
A continuous random variable X has the following probability density function:

$$f(x) = \begin{cases} a \cdot (1 - 4x^2) & \text{if } -\frac{1}{2} < x < \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

a) Find the constant a. 1 pt

b) Find the cumulative distribution function $F(x)$. 2 pt

c) Compute the probability $P(X = \frac{1}{4})$. 1 pt

d) Compute the probability $P(0 < X < \frac{1}{4})$. $\boxed{\text{1 pt}}$

Exercise 10

A normal random variable X has probability density function

$$f(x) = \frac{1}{3} \exp\left(-\frac{\pi}{9}(x^2 - 4x + 4)\right).$$

a) Find the mean μ and the standard deviation σ. $\boxed{\text{1 pt}}$

b) Let Y be the random variable defined by $Y = \frac{X-\mu}{\sigma}$. For a real number a show that

$$P(Y \leq -a) = 1 - P(Y \leq a).$$

(Hint: use that $\int_\alpha^\beta \phi(x)\, dx = -\int_\beta^\alpha \phi(x)\, dx$.) $\boxed{\text{1 pt}}$

Your final grade is the sum of your scores divided by 1.4.