Calculus and Probability Theory
Assignment 5

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After completing these exercises successfully you should be confident with the following topics:

- familiar with definite and indefinite integrals
- able to apply the most important integration methods, more specifically, substitution and integration
- confident about switching between different representations of a function
- able to compute area of a finite or infinite region
- able to apply the formula for the arc length of a function over a finite interval

1. (**20 points**) Compute the following indefinite integrals. You can use *substitution* or *integration by parts*. In each problem *verify* your result, and don't forget about the constant term. You may need some of the following, well-known trigonometric identities:

$$\sin(2x) = 2\sin(x)\cos(x), \quad \cos(2x) = \cos^2(x) - \sin^2(x), \quad \sin^2(x) + \cos^2(x) = 1$$

Also, it is highly recommended to consult with the lecture slides and solve the problems there before you start with these ones.

(a) $\int \sin(x)\cos(x)\, dx$
Solution:

$$\begin{aligned} u &= \sin(x) \\ du &= \cos(x)dx \\ dx &= \frac{du}{\cos(x)} \end{aligned}$$

$$
\begin{aligned}
\int \sin(x)\cos(x)\ dx &= \int u\cos(x)\frac{du}{\cos(x)} \\
&= \int u\ du \\
&= \frac{1}{2}u^2 \\
&= \frac{1}{2}\sin^2(x) + C
\end{aligned}
$$

(b) $\int \ln(ax)\ dx$ where $a > 0$
Solution:

Using integration by parts $= \int f' \cdot g = f \cdot g - \int f \cdot g'$

$$
\begin{aligned}
g &= \ln(ax) \\
g' &= \frac{1}{x} \\
f &= x \\
f' &= 1
\end{aligned}
$$

$$
\begin{aligned}
\int \ln(ax)\ dx &= x \cdot \ln(ax) - \int x \cdot \frac{1}{x} \\
&= x \cdot \ln(ax) - \int 1 \\
&= x \cdot \ln(ax) - x + C
\end{aligned}
$$

(c) $\int \cos^2(x)\ dx$
Solution:

$$
\begin{aligned}
\int \cos^2(x)\ dx &= \sin(x)\cos(x) - \int \sin(x) \cdot (-\sin(x))dx \\
&= \sin(x)\cos(x) + \int \sin^2(x)\ dx \\
&= \sin(x)\cos(x) + \int 1 - \cos^2(x)\ dx \\
\int \cos^2(x)\ dx &= \sin(x)\cos(x) + x + K - \int \cos^2(x)\ dx \\
2\int \cos^2(x)\ dx &= \sin(x)\cos(x) + x + K \\
\int \cos^2(x)\ dx &= \frac{1}{2}(x + \sin(x)\cos(x)) + c
\end{aligned}
$$

(d) $\int \frac{1}{\sqrt{1-4x^2}}\,dx$
Solution:

$$\int \frac{1}{\sqrt{1-4x^2}}\,dx = \frac{1}{2}\arcsin(2x) + c$$

(e) $\int e^{3x}\sin(x)\,dx$
Solution:

Using integration by parts $= \int f' \cdot g = f \cdot g - \int f \cdot g'$

$$\begin{aligned} g &= e^{3x} \\ g' &= 3e^{3x} \\ f &= -\cos(x) \\ f' &= sin(x) \end{aligned}$$

$$\begin{aligned} \int e^{3x}\sin(x)\,dx &= -\cos(x)\cdot e^{3x} - \int -\cos(x)\cdot 3e^{3x} \\ &= -\cos(x)\cdot e^{3x} + \int \cos(x)\cdot 3e^{3x} \end{aligned}$$

Using integration by parts again:

$$\begin{aligned} g &= 3e^{3x} \\ g' &= 9e^{3x} \\ f &= sin(x) \\ f' &= cos(x) \end{aligned}$$

$$\begin{aligned} \int e^{3x}\sin(x)\,dx &= -\cos(x)e^{3x} + \sin(x)3e^{3x} - \int \sin(x)9e^{3x} \\ &= e^{3x}(3\sin(x) - \cos(x)) - 9\int \sin(x)e^{3x} \\ 10\int e^{3x}\sin(x) &= e^{3x}(3\sin(x) - \cos(x)) \\ &= \frac{1}{10}(e^{3x}(3\sin(x) - \cos(x))) \end{aligned}$$

2. (**20 points**) Compute the length of the curve $f(x) = \sqrt{1-x^2}$ where $x \in [-1, 1]$

(a) using calculus, and
Solution:

Arc length of this invervall

$$\int_{-1}^{1} \sqrt{1+(f'(x))^2}\, dx$$

$$\begin{aligned} f'(x) &= \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot (-2x) \\ &= \frac{1}{2}\frac{1}{\sqrt{1-x^2}} \cdot (-2x) \\ &= -\frac{2x}{2} \cdot \frac{1}{\sqrt{1-x^2}} \\ &= -x \cdot \frac{1}{\sqrt{1-x^2}} \\ &= -\frac{x}{\sqrt{1-x^2}} \end{aligned}$$

$$(-\frac{x}{\sqrt{1-x^2}})^2 = \frac{x^2}{1-x^2}$$

(b) using geometric argument. [Hint: what is the shape of $\sqrt{1-x^2}$]?
Solution:

3. (**20 points**) Compute the definite integral $\int_{-1}^{1} \sqrt{1-x^2}\, dx$

(a) using calculus [hint: instead of substituting a function of x by u, now substitute $x = \sin(u)$.]
Solution:

(b) using geometric argument?
Solution:

4. (**15 points**) Compute the following improper integrals.

(a) $\int_0^\infty re^{-r^2}\, dr$;
Solution:

$$\begin{aligned} \int_0^\infty re^{-r^2}\, dr &= -\frac{e^{-r^2}}{2} \\ &= \frac{1}{2} \end{aligned}$$

(b) $\int_0^{2\pi}(\int_0^{\infty} re^{-r^2}\ dr)\ dt$;
Solution:

$$\int_0^{2\pi}(\int_0^{\infty} re^{-r^2}\ dr)\ dt = \pi$$

(c) (bonus, +3 points) Prove that $\int_{-\infty}^{\infty} e^{-z^2}\ dz = \sqrt{x}$.
You may use the fact that $\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx)\ dy = \int_0^{2\pi}(\int_0^{\infty} re^{-r^2}\ dr)\ dt$
Solution:

(d) $\int_0^{\infty} e^{-z^2}\ dz$
Solution:

$$\int_0^{\infty} e^{-z^2}\ dz = \frac{\sqrt{\pi}}{2}$$

5. (**15 points**) Compute the following improper integrals.

(a) $\int_0^{\infty} e^{-x}\ dx$
Solution:

$$\int_0^{\infty} e^{-x}\ dx = 1$$

(b) $\int_0^{\infty} xe^{-x}\ dx$ using integration by parts;
Solution:

$$\int_0^{\infty} xe^{-x}\ dx = 1$$

(c) (bonus, +2 points) $\int_0^{\infty} x^n e^{-x}\ dx$ for all $n \in \{0, 1, ...\}$
[Hint: Try first for n = 0,1,2,3]
Solution:

(d) $\int_0^{\infty} x^{-\frac{1}{2}} e^{-x}\ dx$
[Hint: Substitute $u = \sqrt{x}$ and, at the end, some information from a previous exercise turns out to be useful.]
Solution:

$$\int_0^{\infty} x^{-\frac{1}{2}} e^{-x}\ dx = \sqrt{\pi}$$

6. (**10 points**) .

(a) Given three lines, $y = x + 2, y = -x + 6$ and $y = 2x - 3$ enclosing a triangle. Determine the *coordinates* of the three vertices and the *area* of the triangle.
Solution:

(b) Compute the area of the region bounded by $y = (x-1)^3$ and $y = (x-1)^2$
Solution:

7. (**bonus, 5 points**) The figure shows a horizontal line $y = c$ intersecting the curve $y = -(x-2)^2 + 4$. Find the number c such that the areas of the shaded regions are equal.