Deterministic Chaos in Lotka-Volterra Equations

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1. What is deterministic chaos?

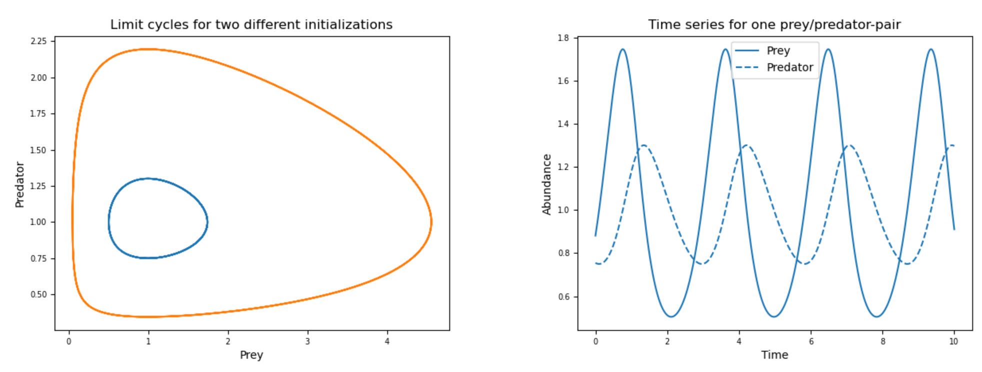
- Chaos is a behaviour shown by dynamical systems, that on first look **may resemble randomness**.
- Exact same initial values for a chaotic system will always give the same behaviour, but **small differences** between initial values will **lead to exponentially growing differences**.
- Chaotic systems are **deterministic but not predictable** in the long term.

2. What are the Lotka-Volterra equations?

In a predator-prey system the number of preys (X) is moderated by the number of predators (Y) and vice versa. (There is also another set of equations for modeling competition).

$$\frac{dX}{dt} = X(a - bY) \quad \text{and} \quad \frac{dY}{dt} = Y(-c + dX)$$

3. No chaos with two species



With parameters a,b = 5 and c,d = 1 and random initialization values. Prey and predator abundance vary periodically.

4. Going to 3D

Chaos can only be found in continuous nonlinear systems with at least three dimensions. As this requires more parameters, the generalized Lotka-Volterra equation collects the growing rates in a vector and the interactions in a matrix. The diagonal fields cover intraspecific competition.

$$\frac{dX_i}{dt} = X_i(r_i + \sum_{j=1}^{n} A_{ij} X_j)$$

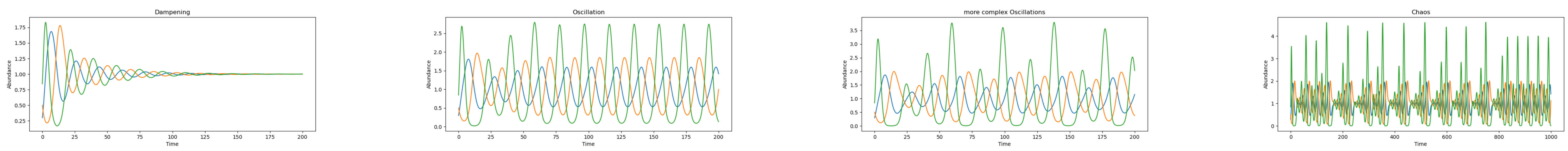
5. Towards chaos

Finding parameters that induce chaotic behaviour proves to be difficult, at least for the simple model. The following variant is proposed by [1] and adds nonlinear saturation. It also absorbs the growing rates into the matrix.

$$\frac{dX_i}{dt} = X_i(\sum_{j=1}^{3} A_{ij}(1 - X_j))$$

See [2] for a simplified presentation of this, see [3] for another approach.

6. Chaos with three species



model from 5. with matrix $\begin{pmatrix} 0.5 & 0.5 & 0.1 \\ -0.5 & -0.1 & 0.1 \\ \alpha & 0.1 & 0.1 \end{pmatrix}$

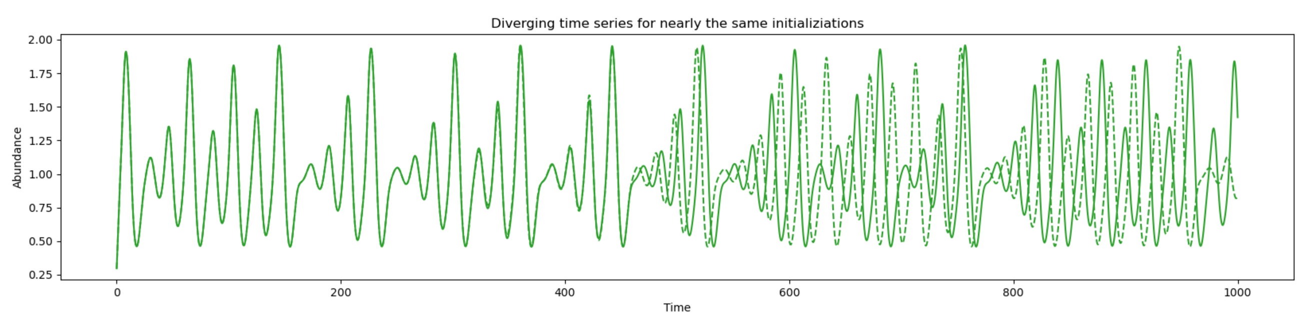
$\alpha = 0.75$: all populations go extinct

$\alpha = 1.2$: 3D limit cycle

$\alpha = 1.387$: different limit cycles

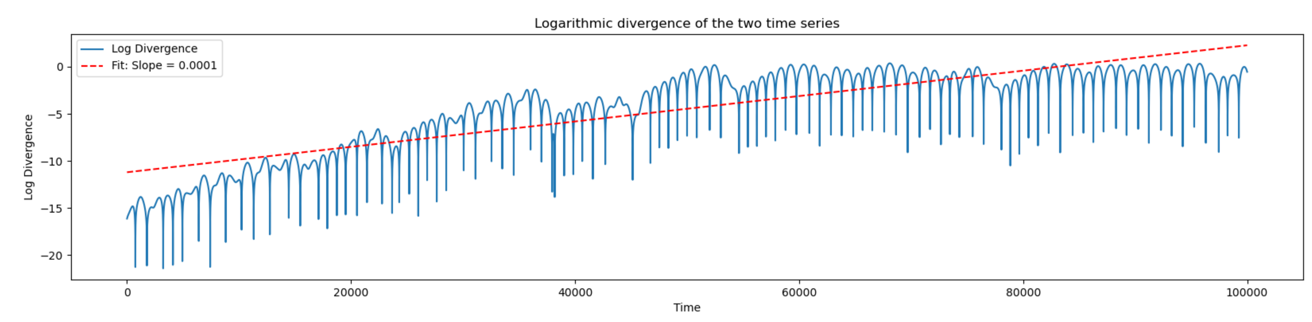
$\alpha = 1.5$: the system behaves chaotically

7. Is this really chaos?



With the initializations for the first species differing by only 1e-6 (or 0.000001) the populations will develop identically for some time before they diverge by large amounts. This is an indicator of chaotic behaviour.

8. How to prove



The absolute differences of the two time series transformed to log scale show a tendency, which is a prove for exponential growing. But the trend saturates and there are aperiodic changes which should be explained.

9. Does it occur in nature?

At least two approaches have been tried: Showing that found parameters are likely to occur in nature [4] and proving that measured time series can be modelled with chaos [5]. Although the probability of chaos in nature in general and in population ecology in particular has been a topic of extensive discussion, the evidence remains surprisingly sparse.

References

- [1] Arneodo, A., Coullet, P., Tresser, C. (1980). Occurrence of strange attractors in three-dimensional Volterra equations. *Phys. Lett. A*, 79A(4): 259-263.
- [2] Flake, G. W. (1998). *The computational beauty of nature: Computer explorations of fractals, chaos, complex systems, and adaptation.* MIT Press.
- [3] Hastings, A. and Powell, T. (1991). Chaos in a Three-Species Food Chain. *Ecology*, 72: 896-903.
- [4] McCann, K., Yodzis, P. (1994). Biological Conditions for Chaos in a Three-Species Food Chain, *Ecology*, 75(2): 561–564.
- [5] Benincà, E., Huisman, J., Heerkloss, R. et al. (2008). Chaos in a long-term experiment with a plankton community. *Nature* 451: 822–825.

Code for the graphics