Time-based Fixed Income Performance Attribution: I. Cash-Flow Profit and Loss Attribution

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1 Introduction

The British statistician George Box is often cited for his aphorism "all models are wrong, but some are useful"[1]. Humans have used models to understand complex behaviors, natural or artificial. Models are an economical means to make this understanding possible and manageable. Mathematical models are wrong in the sense that they are simplifications, or approximations, of reality. The fact that they are "only" simplifications, depending on what aspect of reality has been simplified, can be a potential source of frustration for anyone searching for a satisfactory explanation of those behaviors. Satisfaction in itself is a subjective feeling highly dependent on the context. Fortunately, this frustration can lead to creativity, and potentially be the source of new models. Those new models remain, however, simplifications and are consequently as wrong as their predecessors. Hopefully, they provide their creator with a more satisfying explanation of reality.

The approach presented here emerged out of a sense of frustration with the traditional approach to decomposing the profit generated by fixed income financial investments. This paper suggests a new approach for the attribution of the financial performance of single cash-flows. From a single cash-flow, it is relatively easy to generalise the methodology to an entire fixed income portfolio. This will be covered in a subsequent paper.

Once an investment decision has been implemented, market forces will modify the value of the financial transactions that makes up the investment. Shorty after inception, the investor will be interested in the current value of those transactions. This leads to the estimation of the investment balance sheet. In this paper, it is assumed that the estimation of the balance sheet can be accurately performed. At such point in time, the value of the investment will have changed and the difference with its initial value is called the profit or loss, optimistically shortened to profit for the sake of convenience. Part of this profit might have been realised, in the sense that it is the consequence of cash-flows that already have settled. Or, the profit is sometimes referred to as unrealised, ie the investment does not need to have been sold, or cash to have been received, to determine the profit, only, the new balance sheet value has changed and the impact of these changes could in theory be materialised by exiting the transactions. The challenge of the profit attribution is to provide a sensible explanation of the unrealised part of the profit.

To minimise potential confusion we assume a female investor with a male counterparty. A legitimate desire on the part of the investor is to understand the reasons underlying this profit. If it is a profit, is it commensurate with her expectations? Should she materialise the unrealised part by selling her investment? If it is a loss, should she materialise it so as to not lose even more? Or should she hold on to the position in expectation of profit to come that will compensate and ideally exceed the current level of losses? Those decisions must be informed and substantiated by a step called, in the classical investment process, the "profit and loss attribution", where "attribution" is sometimes replaced by "decomposition" or "explain".

What are the causes driving the change in the value of financial transactions? Ideally, one wishes to relate the events, be they economical, political or natural in nature, to the return generated by the investment. This is an extremely difficult task, and traditionally the decomposition of an investment return is performed at the level of so-called risk factors. Risk factors are financial quantities that contribute to the valuation functions of financial contracts. The number of risk factors can vary considerably, depending on the financial instruments contained in the portfolio, from a few to a few thousand. A typical explanation will state that a portfolio has a certain exposure to a particular risk factor and that in the analysed period the risk factor moved by a certain amount, and therefore contributed to a gain or a loss.

In the context of fixed income investment, the lowest extreme in the explanatory spectrum is to select a set of risk factors consisting of the market prices of the financial products. Although this approach has the highest regard of the international financial reporting standards, and therefore of the auditing profession, it contains close to no explanatory power. Instead of using the prices of the financial products, selecting interest rates as risk factors is a considerable step forward in the level of explanation of the profit.

The traditional approach for attributing performance within a fixed income portfolio is based on Taylor's expansion to this set of risk factors[2]. The set of risk factors is augmented by a fascinating new factor that is commonly called *time* but that should more precisely be called the passing of time. Indeed, if one imagines a world where nothing of a financial nature happens during the period under review, even indirectly, by virtue of the time value of money, the value of the transactions would have evolved and therefore profit will nevertheless have been generated. The pedagogical explanation for this change in value is that all cash-flows constituting the investment are closer to the present at the end of the analysed period compared with the start of the period and that "a dollar today is worth more than a dollar tomorrow", at least in an economic environment with positive interest rates. Although this method is the current market standard, it has many known drawback, for instance: the set of risk factors is arbitrary; and the attribution is never exact, as Taylor's expansion is an infinite series. Unfortunately, the unexplained term tends to become important when the market is volatile, at which point it attracts the greatest attention; the well-know roll-down effect does not naturally exist in this framework; and finally, the daily aggregation of the explanatory terms over a longer period of time is difficult to interpret.

In this short article, a new approach to performing the profit or loss attribution is put forward. This approach is based on cash-flows rather than sensitivities. It has the advantage of being exact in the sense that there are no unexplained terms. It is time-additive, meaning that the sum of partial contributions is identical, for the main attribution measures, to the contribution calculated directly over the entire reporting period. The roll-down effect emerges naturally from the theory. Finally, this approach is well suited to forecasting profit, assuming that the reinvestment strategy is provided. It should be added that the decomposition to the traditional set of risk factors can trivially be deduced from this approach.

However, this approach has costs. The first cost is that one needs to know all the cash-flows of the investment and perform the calculations at the cash-flow level. Second, it requires that the market information prevailing on the trade date be stored for each transaction. Those costs should be considered fully acceptable as, they represent only computational power and storage capacity, both of which have significantly decreased thanks to the Moore law. Furthermore, detailed knowledge of the cash-flows is nowadays required for liquidity disclosures. Therefore, this information is expected to be readily available.

2 Scope

The title of this article implies that the scope will be limited and a more practical subsequent article might be published to complement this more fundamental first step. In this paper, I expose the key decomposition for the simplest possible financial instrument, that is to say, a cash-flow of a fixed known amount to settle at a fixed known date. The author believes that applying this approach to more sophisticated fixed income financial instruments is almost trivial and will only lengthen and confuse the key message of this paper.

To further simplify the text, it is assumed that the investor is the beneficiary of this cash-flow. However, if the investor was to be the payer, all the mathematics can be reused, by simply inverting the sign of the amount.

A cash-flow can be described, for the purpose of performance attribution, by four features: the amount to be paid; the date at which the payment needs to be made; the currency of the payment; and the discount curve used to compute the time value of the cash-flow. This last feature is driven by two main factors: first, the so-called time value of money; and the second is the credit quality of the person who is supposed to make the payment. Indeed, one might want to factor in the present value of the probability that the debtor will not honor his debt. This paragraph assumes that the value is based on a discount curve, a concept that will be made explicit in the next section.

3 Curve-based valuation

This paper works on the assumption that the cash-flow valuation is based on a discount curve. A discount curve is a function that acquires as an input a date and provides as an output a discount factor D. The discount factor is the real number by which to multiply the amount to determine its present value. The reason for extracting discount factors instead of rates is to avoid the tedious discussion on day count conventions. The construction of discount curves is not the topic of this article, and excellent books, such as [3], detail how to properly build discount curves. In this paper, the discount curve is assumed to be given.

To establish some notations, let us call t_p the date of the payment, ie the date at which the cash-flow amount is in the investor's hands or in her bank account and that it is at her disposal. The payment currency is obvious from the context. Although this is a trivial extension, multi-currency transactions are not discussed in this first article. The amount to be received on the payment date is shown as N. At the risk of repetition, t_p and N are known and fixed.

The present value V of a cash-flow on a given date t_0, the valuation date, is given by the following mathematical expression:

$$V\left(t_0\right) = ND\left(t_0; t_p\right)$$

Because the value N is fixed and known, with the constant aim of keeping the equations as simple as possible, it will be assumed for the rest of the paper that N has the value 1.

4 Determining the discount factor

A priori, only two dates are necessary for determining the discount factor: the valuation date and the payment date. However, I argue that four dates are necessary to fully determine the value of the discount factor. Most of the time, it is assumed that three of them are identical.

What are those four dates? The first two have already been defined: the payment date and the valuation date. Often they are combined into a so-called tenor, ie the time between the valuation date and the payment date. The third date, t_c, is when the market data have been captured to construct the discount curve. Although it seems at first sight unnatural to use a date for t_c other than the valuation date, there are circumstances where it is useful to do so. As an example, one might be interested by the impact of moving the calendar one day forward, assuming the market data are as of today. It is a known technique for numerically computing the carry of a portfolio. Finally, based on the observation that all mathematical expressions of the discount factor are a composition of the tenor and an interest rate, the rate selected on the discount curve does not have to correspond to the rate of the payment date. I call t_g the date corresponding to the selected rate. Again, it does not seem intuitive that one might consider a rate other than the one corresponding to the payment date. But if she were interested in the impact of the shape of the discount curve without being influenced by the fact that the present value increases only because each time lapse brings the payment date closer (commonly known as the roll-down effect) she might want to do so.

On this basis, the discount factors in the rest of the paper will be written with one parameter and three arguments:

$$D_{t_c}\left(t_g; t_p - t_0\right)$$

To recap: t_0 is the valuation date; t_c is the date of the capture of the market information necessary for building the discount curve; t_g is the grid point selected on the discount curve to obtain the interest rate; and $t_p - t_0$ is the time between the valuation date and the payment date, or tenor.

The fact that four different dates are needed to determine the discount factor is the key observation supporting the suggested approach. This has led me to call this new attribution method the time-based performance attribution method.

5 Performance attribution

For cultural reasons, which give preference to addition against other mathematical operators, the objective of performance attribution is to decompose the change in the present value of a financial instrument into a sum of factors, each having an intuitive meaning. This objective is implemented in the current standard approach as a sum of sensitivities time changes in the associated risk factors. The change in the present value of fixed known cash-flows is given by the difference between the present value at time t_2 and that at time t_1, assuming t_2 is after t_1.

$$\Delta V\left(t_1, t_2\right) = D_{t_2}\left(t_2; t_p - t_2\right) - D_{t_1}\left(t_1; t_p - t_1\right)$$

A natural decomposition of the change in present value, $\Delta V\left(t_1, t_2\right)$, is to change, successively, t_1 into t_2 by adding and subtracting two identical terms and to pair terms that differ by only one parameter. Unfortunately, the order of the permutation will change the value of the pairs, and therefore their meaning, if any. There are six possibilities: the first change can be applied to any of the three t_1, so that there are only two t_1 available, with the third one a one for one consequence of the first two choices. Out of these six options, it might seem difficult to choose the right decomposition. To achieve time additivity, one is forced to disregard any of the six options. This will be explained in section 7. First, a mathematical definition of time additivity.

6 Time additivity

One of the weakness of the decomposition by sensitivities is that the resulting contributions are not time-additive. Time-additive should be understood to mean that a measure M determined between two dates, t_1 and t_3, can also be calculated as the sum of the same measure calculated between dates t_1 and t_2, and t_2 and t_3, such that:

$$M(t_1, t_3) = M(t_1, t_2) + M(t_2, t_3)$$

As an illustration, the profit is a time-additive measure. Indeed, the weekly profit can be calculated as the sum of the daily profits.

7 Interest income

7.1 Definition

To start the decomposition of the profit ΔV, let us first consider what is still, to this today, a mystery for the scientific community, the passing of time [4]. As mentioned above, the passing of time has an impact on the profit despite its predictability even if nothing of a financial nature were to happen in the analysed period. This last assumption the fact that the discount curve is assumed to be identical at the start and the end of the analysed period is what defines the first contribution to the profit. I decided to call it interest income, abbreviated to II in the equations. In short, interest income is a measure that reflects the gain acquired through patience.

First, we want to impose that the interest income is time-additive. Mathematically, this is expressed as:

$$II(t_1, t_3) = II(t_1, t_2) + II(t_2, t_3)$$

One can easily express this self-imposed constraint in terms of discount factors. Given that the payment date is known and fixed, the passage of time will impact the tenor, via the valuation date, and the rate selected on the discount curve. One should also impose that the curve is identical throughout the period under review. Given that there are three periods, we label t_c, $t_{c'}$ and $t_{c''}$ the three construction dates.

$$\begin{aligned} D_{t_c}(t_3; t_p - t_3) - D_{t_c}(t_1; t_p - t_1) &= D_{t_{c'}}(t_2; t_p - t_2) - D_{t_{c'}}(t_1; t_p - t_1) \\ &+ D_{t_{c''}}(t_3; t_p - t_3) - D_{t_{c''}}(t_2; t_p - t_2) \end{aligned}$$

The technical challenge is to select t_c, $t_{c'}$ and $t_{c''}$ such that this equality holds true in all circumstances. The only general solution is for $t_c = t'_c = t''_c$. The next step is to decide on t_c. The only value for t_c that will be available for the entire lifetime of the cash-flow is the date on which it was traded, and we therefore suggest $t_c = t_d$.

Consequently, the definition of the interest income is expressed as:

$$II\left(t_1, t_2\right) = D_{t_d}\left(t_2; t_p - t_2\right) - D_{t_d}\left(t_1; t_p - t_1\right) \tag{1}$$

with t_d as the trade date, ie the date when the investor has agreed with the counterparty that he will have to pay her the cash-flow amount.

7.2 Comparison

At this stage, it is interesting to compare the definition of the interest income suggested in this paper with other, more conventional approaches. I will limit myself to two of them: the carry as defined in Taylor's expansion of the profit; and the effective interest rate as suggested in the International Financial Reporting Standards (IFRS).

Let us start with the carry in Taylor's expansion of the profit. In this particular case, the main difference is the dependence of the yield on the evolution of the market. The definition given in equation (1) is indeed independent of the evolution of the risk factors after the trade date. It therefore can be computed for the entire lifetime of the cash-flow, ie between the trade date and the payment date.

The effective interest rate, also called the constant yield method, is also independent of the evolution of the discount curve after the trade date. The main difference between the two methods is that the effective interest rate method is based on an average of the rates composing the curve and does not benefit fully from the information content of the discount curve on the trade date. This averaging has the drawback of generating non-interest income profit, although the risk factors have not changed.

8 Valuation movement

The contribution of the profit that does not come from the interest income is called the valuation movement, abbreviated to VM in the equations. Mathematically, this can be easily expressed as:

$$VM\left(t_1, t_2\right) = \Delta V\left(t_1, t_2\right) - II\left(t_1, t_2\right)$$

To give meaning to this mathematical expression, it is useful to look at the discount factors that it is made up of:

$$\begin{aligned} VM\left(t_1, t_2\right) &= D_{t_2}\left(t_2; t_p - t_2\right) - D_{t_d}\left(t_2; t_p - t_2\right) \\ &+ D_{t_d}\left(t_1; t_p - t_1\right) - D_{t_1}\left(t_1; t_p - t_1\right) \end{aligned} \tag{2}$$

Although the equation might seem less intuitive than the interest income one, it simply states that the valuation movement is the impact of the change in the discount factor at the end of the analysed period compared with the discount factor as it was on the trade date, from which is removed the impact of the change of the discount factor at the beginning of the analysed period compared with the discount factor on the trade date. For accountants, this has a rather familiar ressemblance to the posting of reversal journals.

Let us now discuss the dynamic of the valuation movement from the trade date to the end of the first reporting period. We substitute t_d for t_1 and t_1 for t_2 in equation (2). This leads to:

$$\begin{aligned} VM\left(t_d, t_1\right) &= D_{t_1}\left(t_1; t_p - t_1\right) - D_{t_d}\left(t_1; t_p - t_1\right) \\ &+ D_{t_d}\left(t_d; t_p - t_d\right) - D_{t_d}\left(t_d; t_p - t_d\right) \\ &= D_{t_1}\left(t_1; t_p - t_1\right) - D_{t_d}\left(t_1; t_p - t_1\right) \end{aligned} \tag{3}$$

The interpretation of this is rather natural. For the tenor $t_p - t_1$, the valuation movement is the difference in the discount factors between the curve constructed on t_1 and the curve constructed on the trade date.

At the end of the second period, we use formula (2). Note that we reverse the valuation movement $VM\left(t_d, t_1\right)$ and report the valuation movement $VM\left(t_d, t_2\right)$.

This definition of the valuation movement has several advantages. Its meaning is clear. It is time-additive, as it is the difference between two time-additive measures. Furthermore, it is null if it is computed across the entire lifetime of the cash-flow, ie between trade date and payment date. This can be easily proved by using equation (3) with $t_1 = t_p$, and remembering that the discount factor for a null tenor is unity.

9 Further decomposition

Interest income and valuation movement are typically the measures expected for financial reporting. However, they are clearly not enough to provide a satisfactory explanation of the profit.

It is possible to decompose the interest income and the valuation movement further. Unfortunately, the decomposition will be path-dependent, ie it will depend on the evolution of the discount curve and the resulting measures will not be time-additive.

The interest income can be decomposed as a carry and a roll-down effect. The valuation movement can be decomposed as a pull-to-par effect and an impact of change in rates.

10 Decomposition of the interest income

As alluded to above, the effect of the passage of time has two qualitatively different consequences. First, the waiting time before the cash-flow is settled continuously decreases with the passage of time. In an economy with positive interest rates, it implies that the present value of the cash-flow continuously increases. There is a second effect well known to portfolio managers but rarely properly defined: the roll-down effect. The roll-down effect tries to capture the unavoidable fact that rates depend on the tenor. Because of this, the passage of time changes the selected rates even without changes to the discount curve. Most of the time, rates increase with tenor, and consequently the selected rate after the passing of a certain amount of time is usually smaller. This is the origin of the term "roll-down".

There are two mathematical ways to decompose the interest income of our approach. Either

$$\begin{aligned} II\left(t_1, t_2\right) &= D_{t_d}\left(t_2; t_p - t_2\right) - D_{t_d}\left(t_1; t_p - t_2\right) \\ &+ D_{t_d}\left(t_1; t_p - t_2\right) - D_{t_d}\left(t_1; t_p - t_1\right) \end{aligned}$$

where the first line on the right-hand side constitues a roll-down effect. Indeed, we keep the curve and the tenor unchanged. The only difference is the point selected on the curve. The second line constitutes a carry effect, where the curve and the selected rate are kept unchanged. The only difference is the tenor.

A second option could be:

$$\begin{aligned} II\left(t_1, t_2\right) &= D_{t_d}\left(t_2; t_p - t_2\right) - D_{t_d}\left(t_2; t_p - t_1\right) \\ &+ D_{t_d}\left(t_2; t_p - t_1\right) - D_{t_d}\left(t_1; t_p - t_1\right) \end{aligned}$$

where the first line on the right-hand side constitues a carry effect and the second line a roll-down effect using the same observations as for the first option.

It is not necessarily obvious how to decide between the two options. To justify the author's preference, the most extreme case where t_2 is the payment date t_p is analysed. In that particular situation, the following simplifications are observed. For the first case:

$$\begin{aligned} Roll-down &: D_{t_d}\left(t_p; t_p - t_p\right) - D_{t_d}\left(t_1; t_p - t_p\right) = 1 - 1 = 0 \\ Carry &: D_{t_d}\left(t_1; t_p - t_p\right) - D_{t_d}\left(t_1; t_p - t_1\right) = 1 - D_{t_d}\left(t_1; t_p - t_1\right) \end{aligned}$$

One therefore sees that this implies that there is no roll-down but only carry. However, if the discount curve has the most common shape, it would be desirable to report the impact of rolling down the curve. The second decomposition would provide it. Indeed, for the second option, in the extreme scenario, the carry becomes:

$$D_{t_d}(t_p; t_p - t_p) - D_{t_d}(t_p; t_p - t_1) \quad = \quad 1 - D_{t_d}(t_p; t_p - t_1)$$

and the roll-down is therefore non-null.

11 Decomposition of the valuation movement

There is an impact that is well-defined, and this is the impact of change in rate. By definition, the impact of change in rate is the impact of the value of the cash-flow because of the change in the discount curve, assuming that all else, the tenor and the selected rate, are equal.

As for the decomposition of the interest income, we are still confronted with multiple options for the tenor and the selected rate. For the tenor, $t_p - t_2$ is chosen because there should not be any impact on the change in rate for a cash-flow during a period that ends at the payment date.

For the selected rate, the choice is not obvious: it could be the rate at the beginning or at the end of the analysed period, or even a combination of the two. For ease of interpretability, the author has selected the grid point at the end of the analysed period.

$$IS(t_1, t_2) \quad = \quad D_{t_2}(t_2; t_p - t_2) - D_{t_1}(t_2; t_p - t_2)$$

Indeed, if we define the pull-to-par effect as the difference between the valuation movement and the impact of change in the discount curve, we obtain the following analytical formula:

$$\begin{aligned} PP(t_1, t_2) \quad &= \quad D_{t_1}(t_2; t_p - t_2) - D_{t_1}(t_1; t_p - t_1) \\ &+ \quad D_{t_d}(t_1; t_p - t_1) - D_{t_d}(t_2; t_p - t_2) \end{aligned}$$

These two couples of terms should be rather familiar to the reader by now. Indeed, they are the interest income of the starting curve minus the interest income on the original discount curve. One also observes that the pull-to-par does not require knowledge of the curve on t_2 which allows for an easy projection of the pull-to-par effect. As interest incomes, they can be decomposed into carry and roll-down.

12 Conclusions

In this short paper, we have laid the foundations of a new profit and loss attribution based solely on the decomposition of time variables (dates) of the curve-based valuation method of the fixed income financial product. The decomposition has four components:

$$\text{Profit} = II + VM = (CA + RD) + (IS + PP)$$

which can be made explicit as:

$$\begin{aligned}
& \Delta V\left(t_1, t_2\right) = \\
\text{Carry} \quad : \quad & D_{t_d}\left(t_2; t_p - t_2\right) - D_{t_d}\left(t_2; t_p - t_1\right) && (4) \\
\text{Roll-down} \quad : \quad & +D_{t_d}\left(t_2; t_p - t_1\right) - D_{t_d}\left(t_1; t_p - t_1\right) && (5) \\
\text{Change in rate} \quad : \quad & +D_{t_2}\left(t_2; t_p - t_2\right) - D_{t_1}\left(t_2; t_p - t_2\right) && (6) \\
\text{Reverse carry} \quad : \quad & +D_{t_d}\left(t_2; t_p - t_1\right) - D_{t_d}\left(t_2; t_p - t_2\right) && (7) \\
\text{New carry} \quad : \quad & +D_{t_1}\left(t_2; t_p - t_2\right) - D_{t_1}\left(t_2; t_p - t_1\right) && (8) \\
\text{Reverse roll-down} \quad : \quad & +D_{t_d}\left(t_1; t_p - t_1\right) - D_{t_d}\left(t_2; t_p - t_1\right) && (9) \\
\text{New roll-down} \quad : \quad & +D_{t_1}\left(t_2; t_p - t_1\right) - D_{t_1}\left(t_1; t_p - t_1\right) && (10)
\end{aligned}$$

It requires the calculation of seven discount factors that are then grouped into pairs and can be reported in numerous ways. The pairs are (4) for the carry; (7)+(8) for the change in carry; (5) for the roll-down; (9)+(10) for the change in roll-down; (6) for the impact of change in spread; (4)+(5) for the interest income; (7)+(8)+(9)+(10) for the pull-to-par; and the valuation movement is (6)+(7)+(8)+(9)+(10).

13 Perspective

This is obviously only the first step in the construction of a new performance attribution framework for fixed income securities. This work needs to be extended to traded financial instruments, with all their idiosyncratic challenges.

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