Julia Set, Fatou Set, and Mandelbrot Set for Complex Polynomial Functions

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1 Introduction and History[1]

The study of iterating complex analytic functions began with Ernst Schroder in the 1870s when he studied the convergence of Newton's method for finding roots. This corresponds to the iteration of the function $z - \frac{f(z)}{f'(z)}$ in $\mathbb{C}$. Independently, Arthuer Cayley and Schroder obtained the results of sensitive dependence in initial conditions of quadratic polynomials. But they failed to understand higher degree polynomials.

In World War I, Prerre Fatou and Gaston Julia studied the theory of iterated rational functions. Using Montel's theory of normal families, they independently showed the fractal nature of the Julia set together with other properties.

Not until the 1980s did the field of iteration of one complex variable thrive again, probabily due to the help of the computers, which can visualize the extremely beautiful and complex patterns that were understood by Julia and Fatou.

The project is based on the following articles and books, [2], [3], [4]. In this project, we investigate the simpler model of rational functions, namely, the complex polynomials. We start with introducing some useful properties of the Julia set and the Fatou set for polynomial functions. Then we look at a toy model, the quadratic family $\{z^2 + c,\ c \in \mathbb{C}\}$ and introduce the Mandelbrot set. We discuss the algorithm for visualizations and provide more insights into the structure of these sets. For the last part, we give a glance on the theory for rational functions and discuss Newton's iteration.

2 General Theory for Julia set

For convenience of our discussion, we also give the general definition of Julia set for any holomorphic function f.

Definition 2.1. We define the filled-in Julia set of f as

$$K(f) = \{z \in \mathbb{C},\ f^k(z) \nrightarrow \infty\}, \tag{1}$$

and the Julia set as the boundary of $K(f)$, i.e., $J(f) = \partial K(f)$.

It is noticeable that if $z \in J(f)$, then for any neighborhood U of z, we can find $u, v \in U$ such that $f^k(u) \nrightarrow \infty$ and $f^k(v) \to \infty$.

[1]Reader can refer to [1]

We also define the Fatou set of f as the complement of the Julia set, that is, $F(f) = J(f)^C$. Then the Fatou set contains the points whose orbits are tame under iterates of f.

Two important theorems from Complex Analysis would help us understand the dynamics of complex polynomial.

Theorem 2.2. *Given $f(z) = \sum_{j=0}^{n} a_j z^j$, $a_n \neq 0$ and $n \geq 2$, then there is a positive real number r such that if $|z| \geq r$, then $|f(z)| \geq 2|z|$. In particular, if for some $m \in \mathbb{N}$, $|f^m(z)| \geq r$, then $f^k(z) \to \infty$ as $k \to \infty$.*

Proof. We pick r such that if $|z| \geq r$, then $\frac{1}{2}|a_n||z^n| \geq 2|z|$ and $\sum_{j=0}^{n-1} |a_j||z^j| \leq \frac{1}{2}|a_n||z^n|$. We can always choose such r since $|z^j| = |z|^j$ and power of the highest degree dominates the growth of the polynomial. Then we have

$$\begin{aligned} |f(z)| &\geq |a_n||z^n| - \sum_{j=0}^{n-1} |a_j||z^j| \\ &\geq \frac{1}{2}|a_n||z^n| \geq 2|z|. \end{aligned}$$

If $|f^m(z) \geq r|$, inductively we have that $|f^{m+k}(z)| \geq 2^k r$, then $f^k(z) \to \infty$. $\square$

Corollary 2.3 (Dichotomy for Polynomial Function)**.** *For any complex polynomial f and initial point $z \in \mathbb{C}$, we either have $f^k(z) \to \infty$ or $\{f^k(z)\}$ is bounded.*

These immediately give some properties of the Julia set of a polynomial f.

- The filled-in Julia set $K(f)$ and the Julia set $J(f)$ are non-empty compact subsets of $\mathbb{C}$.
- $J(f) \subset K(f)$.
- $J(f)$ has empty interior.
- $J = J(f)$ is forward and backward invariant, i.e., $f(J) = f^{-1}(J)$.

The last one uses the fact that any polynomial maps surjectively onto $\mathbb{C}$.

The second theorem is the famous Montel's theorem. We first give the definition of a normal family: we say that $\{g_\alpha,\ \alpha \in \mathcal{A}\}$ is a normal class on an open domain U if any sequence taken from the family have a subsequence converges uniformly on any compact subset of U. We say that $\{g_\alpha\}$ is a normal class at a point z, if there is an open neighborhood U of z such that the family is normal on U.

Theorem 2.4 (Montel)**.** *If $\{g_\alpha\}$ is a family of holomorphic functions on an open domain U but not a normal family, then for all $w \in \mathbb{C}$ with at most one exception, we have $g_\alpha(z) = w$ for some $z \in U$ and some $\alpha \in \mathcal{A}$.*

We first observe that the Julia set can be rewritten as

$$J(f) = \{z \in \mathbb{C},\ \textit{the family } \{f^k\} \textit{ is not normal at } z\}. \tag{2}$$

It is clearly that $J(f)$ belongs to the right hand side. If we recall the definition of the Julia set, for any neighborhood U of $z \in J(f)$, there is points $u, v \in U$ such that $f^k(u) \to \infty$ but $f^k(v)$ remains bounded. Thus, no subsequence converges uniformly on U and the family is not normal at z. Conversely, if $z \in F(f)$, we divide into two cases. Firstly, if the orbit of z

is bounded, i.e., it's in $int(K(f))$, then by Montel's theorem, $\{f^k\}$ is normal at z. Secondly, if the orbit of z tends to infinity, then exists $r > 0$ such that $|f^k(z)| \geq r$ for some k. By continuity, we have an open neighborhood such that all points v in this neighborhood satisfy $|f^k(z)| \geq r$. If we recall *Theorem 2.2* and if f is polynomial, we pick r as in *Theorem 2.2*. Then f^k converge uniformly to infinity on this neighborhood.

This characterization extends the definition of the Julia and Fatou sets for the general complex functions. It is also useful since we can derive the mixing property of f near the Julia set.

Arguing with Montel's theorem, we obtain the following important properties of the Julia set, for general complex functions.

- For all $z \in \mathbb{C}$ except for at most one point, if U is an open set intersecting $J(f)$, then $f^{-k}(z)$ intersects U for infinitely many values of k.
- If $z \in J(f)$, then $J(f) = \overline{\bigcup_{k=1}^{\infty} f^{-k}(z)}$.
- $J(f)$ is closed and has no isolated points.

If f is a polynomial, we additionally have that $J(f) = \overline{\{repelling\ periodic\ points\ of\ f\}}$. This characterization tells that the dynamics near Julia set is chaotic.

3 The Mandelbrot Set and the Julia Set of the Quadratic Polynomials

3.1 The choice of quadratic family $\{z^2 + c\}$

One should notice that any quadratic polynomial is topologically conjugate to $P_c(z) = z^2 + c$. Consider $h(z) = az + b$ $(a \neq 0)$, then we have

$$h^{-1} \circ P_c \circ h(z) = (a^2 z^2 + 2abz + b^2 + c - b)/a. \tag{3}$$

Choosing appropriate a, b and c, we can make the above expression into any quadratic polynomial.

Thus, all these quadratic polynomials would have the similar topological properties. Moreover, h is a linear transform, therefore the Julia sets of these quadratic polynomials are geometrically similar. However, this is not the case for higher degree polynomials.

3.2 The Mandelbrot Set

The Mandelbrot set is defined in the parameter plane for quadratic family $\{z^2 + c\}$ with parameter c. The definition of Mandelbrot set is the following:

Definition 3.1. Consider quadratic polynomials $P_c(z) = z^2 + c$, the Mandelbrot set is defined as

$$\mathcal{M} = \{c \in \mathbb{C},\ \textit{the orbit of } 0 \textit{ under } P_c \textit{ is bounded}\}. \tag{4}$$

There are deep connections between the Mandelbrot set and the Julia sets of P_c. We shall classify the Julia sets of P_c using the Mandelbrot set.

3.3 Escape Time Algorithm

To visualize the Mandelbrot set and the filled-in Julia sets, we use the *Escape Time Algorithm* which depends on the definitions (4) and (1). In the spirit of *Theorem 2.2*, we see that if we want $|P_c(z)| > |z|$, it is sufficient to let $|P_c(z)| \geq |z|^2 - |c| > |z|$, which leads to

$$|z| > \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}. \tag{5}$$

And if this is the case, $P_c^k(z) \to \infty$. By recording the time of iterations that $|P_c^k(z)| > \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$ or not, we can assign heatmap to the complex plane to visualize the filled-in Julia set of P_c for particular c.

For the Mandelbrot set, if we repeat the idea for *Theorem 2.2*, we want that $\frac{1}{2}|z^2| > |z|$, then we have $|z| > 2$. In this case, $P_c^k(z) \to \infty$. Note that $P_c^k(z) = P_c^{k-1}(c)$, then we take the threshold to be $|c| = 2$. The output is in *Figure 1*.

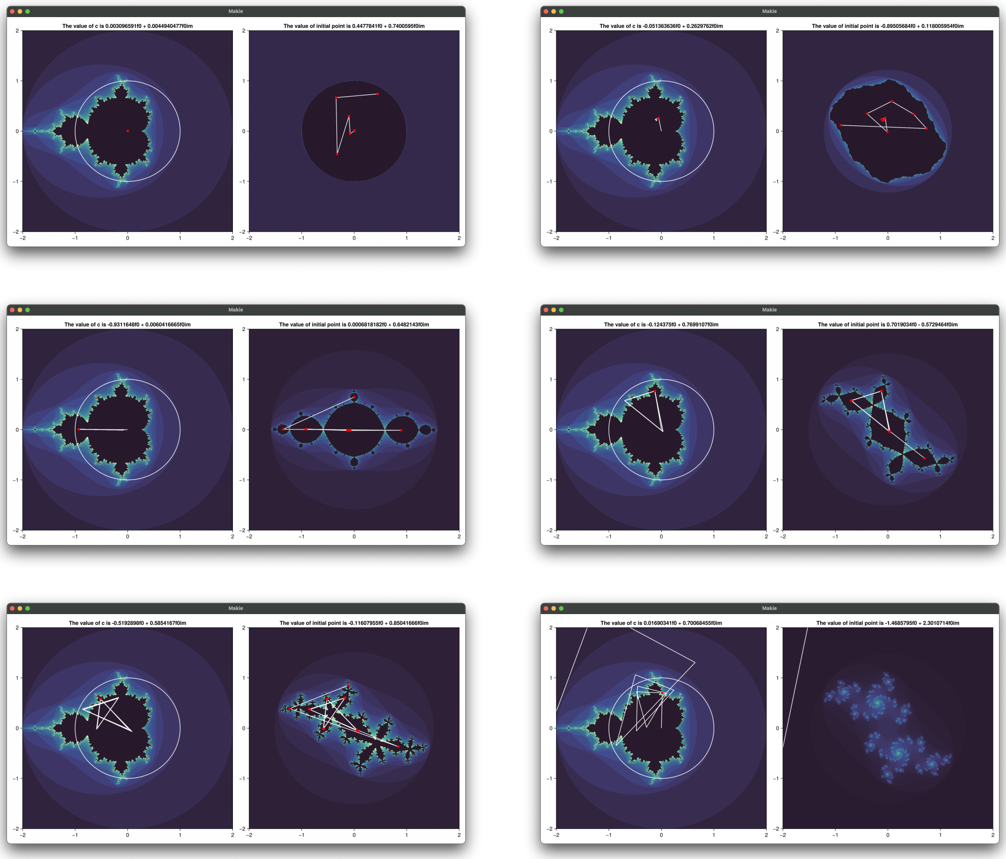


Figure 1: Visualizations of the Mandelbrot set (left) and the Julia sets (right) with corresponding parameter c chosen in the Mandelbrot set (red dot in left pictures). The white lines connects the first 30 iterations, where in the Mandelbrot set, the initial point is 0; while in the filled-Julia sets, the initial point is arbitrarily chosen. 1st: $c = 0$, 2nd: quadratic polynomial with attracting fixed point, 3rd: with attracting 2-cycle, 4th: with attracting 3-cycle, 5th: with attracting 5-cycle, 6th: "The dust", where the Julia set is homeomorphic to a Cantor set. One can access codes at my GitHub page `https://github.com/ChristopheOshino/Mandelbrot_Julia_Netwon/blob/main`.

In the visualization, we see that the Mandelbrot set consists of a main cardioid and many bulbs. One can perform a direct calculation to show that the area contained in the main cardioid corresponds to the parameter c such that P_c has an attracting fixed point. Note that if $P_c(z) = z$, then $z_{1,2} = \frac{1}{2}(1 \pm \sqrt{1-4c})$. And at least one of them are attracting means that $|P_c'(z_1)| < 1$ or $|P_c'(z_2)| < 1$. This leads to the parametric expression $c = \frac{1}{2}e^{2\pi i\theta} - \frac{1}{4}e^{4\pi i\theta}$, where $2\pi\theta$ is the clockwise rotation of the circle.

Notice that $\mathcal{M}$ contains all attracting cycles, which . From the visualization it's not hard to see that every bulb in the Mandelbrot set corresponds to an attracting n-cycle. In fact, $J(P_c)$ would have only one finite attractive cycle. This is due to a wonderful theorem by Fatou

Theorem 3.2 (Fatou)**.** Every attracting cycle for a polynomial or rational function attracts at least one critical point.

A critical point is a point at which the derivative of the function is 0. For quadratic family $\{z^2 + c\}$, the only finite critical point is 0. If there were two attractive cycles, both would attract 0, which is impossible.

One way to identify n is to look at the corresponding Julia set, if P_c has an attracting n-cycle, its Julia set will have $n-1$ "ears" connecting to the central area. See *Figure 1* for details.

In fact, we can locate exactly where the bulb is. Recall the parametrization of the main cardioid, $c = \frac{1}{2}e^{2\pi i\theta} - \frac{1}{4}e^{4\pi i\theta}$, where θ, if $\theta = \frac{p}{q}$ is a rational in irreducible form, then c is at the connecting point (also called a root point) where a period q-bulb is tangent to the main cardioid. However, we cannot tell immediately how many such bulbs there are.

Another interesting observation is that when we moving c on the parameter plane where the Mandelbrot set lies in, we notice that in the beginning the Julia set is homeomorphic to the circle, but later it becomes less connected and more intricate. In fact, we have the following dichotomy:

Theorem 3.3 (Dichotomy for Julia Sets of Quadratic Family)**.** Let J be the Julia set of $P_c(z) = z^2 + c$, then

- if the orbit of c is unbounded, then J is homeomorphic to a Cantor set;
- if the orbit of c is bounded, then J is connected.

Roughly speaking, the above theorem comes from the study of the *critical points* of P_c, which is just the zeros of the derivative. But for quadratic polynomials, there is only one critical point, which is 0. Using the idea of loops in complex plane[2], i.e., smooth, closed, simple curves, we can easily show that if $P_c^k(0)$ is bounded, then $J(P_c)$ is connected. We take a large circle C in the complex plane that all iterates $P_c^k(0)$ are inside C and take the pre-images $P_c^{-k}(C)$. Note that only near the critical point 0, P_c has no local inverse, one can show that the each set $P_c^{-k}(C)$ is i) a loop, ii) $P_c^{-(k+1)}(C)$ is contained in the interior of $P_c^{-k}(C)$, iii) 0 and $c = P_c(0)$ are inside any $P_c^{-k}(C)$. By taking all the points inside and on such loops, we get the filled-in Julia set for P_c. Then the filled-in Julia set is an intersection of decreasing simply connected sets, it is simply connected. Therefore, the boundary $J(P_c)$ is connected. By similar argument, we can show that if $J(P_c)$ is not connected, then the orbit of c is unbounded. For the homeomorphism to Cantor set, refer to the blog of W.R. Lim at `https://willierushrush.github.io/posts/2020/07/mandelbrot-set/`.

[2] One can refer to [3], Chapter 14.2

In particular, if $|c| < \frac{1}{4}$, then the Julia set J is a simple closed curve. If $|c| > \frac{1}{4}(5+2\sqrt{6})$, then J is totally disconnected. One can find proof in [3], Chapter 14.

3.4 Inverse Iteration Algorithm

There is another way to coding Julia set according to the properties comes from the Motel's theorem, which is that if $z \in J(f)$, then Jf is the closure of the union of all k-pre-images of z for all k. For the quadratic family P_c, in particular, one of the fixed point $z_2 = \frac{1}{2}(1+\sqrt{1-4c})$ is always repelling and thus is in the Julia set. We inductively collect points

$$P_c^{-(n+1)}(z_2) = \{\pm\sqrt{w-c},\ w \in P_c^{-n}(z_2)\}, \tag{6}$$

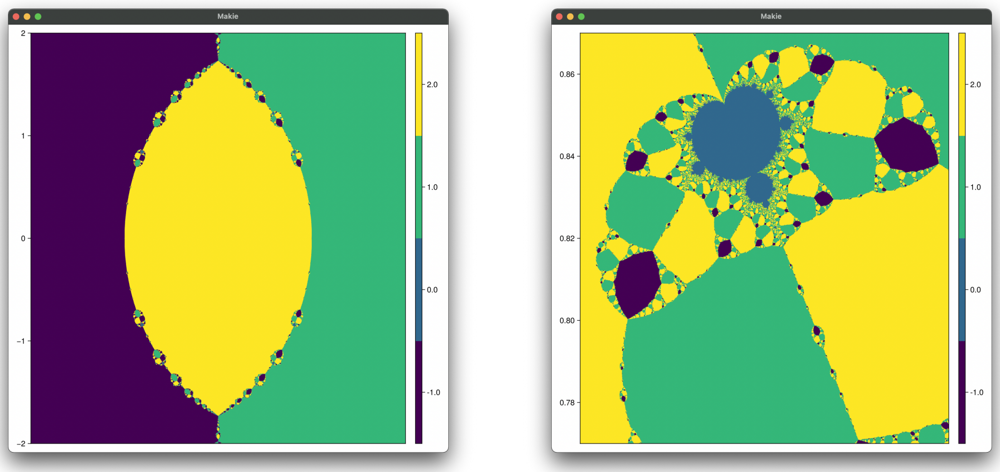
then we would have 2^{n+1} distinct elements from $P_c^{-(n+1)}(z_2)$.

4 Newton Iteration and Rational Functions

The theory based on Montel's theorem can be easily extended to rational functions, which is important in the study of the Newton iteration for finding roots of complex polynomials. If we have a polynomial p, we want to design an algorithm that if we start with any point on the complex plane, an iteration map would quickly take the point to a root. Newton's method uses iteration map

$$N_p(z) = z - \frac{p(z)}{p'(z)}. \tag{7}$$

It is fine for real functions, but the dynamics of N_p can fall into attracting cycles on the complex plane, leading to the failure of convergence to a root.



(a) (b)

Figure 2: Parameter plane for iterations of p. If orbit of $\frac{\lambda}{3}$ converges to -1, we colored region purple; 1 for green; λ for yellow; and not converging to roots for blue. Codes on my GitHub page `https://github.com/ChristopheOshino/Mandelbrot_Julia_Netwon/blob/main`.

A simple case is $p(z) = (z-1)(z+1)(z-\lambda)$. We wonder if there is any point will never converge to any of the root. Again, by the useful theorem from Fatou, *Theorem 3.2*, we look at the critical points of N_p. Let $N_p'(z) = 0$, we have

$$\frac{p(z)p''(z)}{p'(z)^2} = 0, \tag{8}$$

which leads to $z = \pm 1, \lambda, \frac{\lambda}{3}$. Then the only possible point to fall into an attracting n-cycle for some $n \geq 2$ is $\frac{\lambda}{3}$.

Like what we did for the quadratic family, we look at the parameter space, i.e., the space of λ where the orbit of $\frac{\lambda}{3}$ never converges to any of the roots $\pm 1, \lambda$. We certainly observe some small part that does not converge to any roots. If you zoom in, there will be Mandelbrot-like shape of these points. (See *Figure 2.*) Any interested reader should refer to [4], [5]

References

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