Annu. Rev. Astron. Astrophys. 1992. 30: 499–542

THE COSMOLOGICAL CONSTANT

Sean M. Carroll and William H. Press

Harvard-Smithsonian Center for Astrophysics, Cambridge, Massachusetts 02138

Edwin L. Turner

Princeton University Observatory, Princeton, New Jersey 08544

KEY WORDS: general relativity, vacuum energy, age of universe, galaxy counts, gravitational lenses

1. INTRODUCTION

Astronomy and physics bring different perspectives to the "cosmological constant problem." Originally introduced by Einstein as a new term in his gravitational field equations [and later regretted by him as "the biggest blunder of my life" (quoted in Gamow 1970)], the cosmological constant, Λ, confronts observational astronomers as a possible additional term in the equation that, according to general relativity, governs the expansion factor of the universe $R(t)$,

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho_M + \frac{\Lambda}{3} - \frac{k}{R^2}. \qquad 1.$$

Here ρ_M is the mass density; $k = -1, 0, +1$ for a Universe that is respectively open, "flat," and closed; and H is the Hubble constant, whose observable value at the present epoch t_0 is denoted H_0.

Equation 1 says that three competing terms drive the universal expansion: a matter term, a cosmological constant term, and a curvature term. It is convenient to assign symbols to their respective fractional contributions at the present epoch. We define

$$\Omega_M \equiv \frac{8\pi G}{3H_0^2}\rho_{M0}, \quad \Omega_\Lambda \equiv \frac{\Lambda}{3H_0^2}, \quad \Omega_k \equiv -\frac{k}{R_0^2 H_0^2}, \qquad 2.$$

0066-4146/92/0915-0499$02.00

where zero subscripts refer to the present epoch. Equation 1 then implies that

$$1 = \Omega_M + \Omega_\Lambda + \Omega_k. \qquad 3.$$

It is also sometimes convenient to define $\Omega_{tot} \equiv \Omega_M + \Omega_\Lambda = 1 - \Omega_k$. It is an observational question whether a nonzero Ω_Λ is required to achieve consistency in Equation 3. This is the astronomer's cosmological constant problem.

The Heisenberg uncertainty principle allows particle-antiparticle pairs spontaneously to appear and disappear. The theoretical particle physicist thus sees the Ω_Λ term in Equation 3 as an inevitable concomitant to the Ω_M term. As Ω_M is associated with a density of real particles, so Ω_Λ is associated with virtual, "vacuum" states of those same particles' species—that is, with the energy-momentum density of their vacuum states. The gravitational effect of these virtual particles gives the vacuum an energy density ρ_{vac} (Zel'dovich 1967). Although particle physicists do not know how to compute ρ_{vac} exactly, theory allows one to estimate its value. Unfortunately, the estimates disagree with observational limits by a factor of 10^{120}. This is the physicist's cosmological constant problem.

In this review, we sample both the astronomer's and the physicist's viewpoints. The differing perspectives lead to different perceived goals. An epochal astronomical discovery would be to establish by convincing observation that Λ is nonzero. An important physics discovery, on the other hand, would be to adduce a convincing theoretical model that requires Λ to be exactly zero.

Attempts to measure fundamental cosmological parameters have consumed enormous observational (and intellectual) resources but have met with only limited success (see Sandage 1987 for a historical review). Hubble's constant H_0 is physically the most fundamental such parameter, yet independent determinations via classical techniques (see reviews by Tammann 1987 and de Vaucouleurs 1981) or the latest new methods (Tonry 1991, Roberts et al 1991, Press et al 1991, Birkinshaw et al 1991, Jacoby et al 1990, Aaronson et al 1989) give values that vary over a factor of two. Attempts to measure the value of Ω_M have been even less conclusive (see reviews by Peebles 1986, Trimble 1987, Fukugita 1991). New results supporting apparently inconsistent values continue to appear (e.g. Toth & Ostriker 1992, Richstone et al 1992).

In comparison with H_0 and Ω_M, attempts to measure Ω_Λ have been infrequent and modest in scope. Moreover, in many respects, the physical signatures of Ω_Λ are smaller and more subtle than those of the other two parameters, at least from an observational perspective. These considerations should severely limit our expectations for current observational information concerning Ω_Λ.

Table 1 lists five fiducial cosmological models, parametrized by Ω_M and Ω_Λ, which we will refer to in following sections as Models A through E. They represent extreme, though not impossible, limits on the present state of knowledge. In fact, we will see that distinguishing among these models, and thus among variations in Equation 3 having dominant versus negligible Ω_Λ terms, is quite challenging at present.

2. WHY A COSMOLOGICAL CONSTANT SEEMS INEVITABLE

In this section we discuss the form that the vacuum energy-momentum tensor must take, and why the predicted value of ρ_{vac} is unreasonably high (Weinberg 1989; for nontechnical introductions see Abbott 1988 and Freedman 1990).

To a particle physicist, the word "vacuum" has a different meaning than to an astronomer. Rather than denoting "empty space," vacuum is used to mean the ground state (state of lowest energy) of a theory. In general, this ground state must be Lorentz invariant, that is, must look the same to all observers. If this is the case, then the stress-energy-momentum tensor $\mathscr{T}_{\mu\nu}$ of vacuum must be proportional (in any locally inertial frame) simply to the diagonal Minkowski metric, diag$(-1, 1, 1, 1)$, because this is the *only* 4×4 matrix that is invariant under Lorentz boosts in special relativity (as can easily be checked). As is well known, a perfect fluid with density ρ and pressure P has the stress-energy-momentum tensor diag(ρ, P, P, P). (See, e.g. Misner et al 1973; in this section, we choose units with $c = 1$). Comparing to the Minkowski metric, it follows that (*a*) "vacuum" is a perfect fluid, and (*b*) it has the equation of state

$$P_{vac} = -\rho_{vac}. \qquad 4.$$

Not by coincidence, this equation of state is precisely the one that, under

Table 1 Five fiducial cosmological models

Model	Ω_{tot}	Ω_M	Ω_Λ	Description
A	1	1	0	flat, matter dominated, no Λ
B	0.1	0.1	0	open, plausible matter, no Λ
C	1	0.1	0.9	flat, Λ plus plausible matter
D	0.01	0.01	0	open, minimal matter, no Λ
E	1	0.01	0.99	flat, Λ plus minimal matter

application of the first law of thermodynamics, causes ρ_{vac} to remain constant if a volume of vacuum is adiabatically compressed or expanded: PdV work provides exactly the amount of mass-energy to fill the new volume dV to the same density ρ_{vac}. Thus ρ_{vac} remains truly a constant. Its relation to Λ is simply $\Lambda = 8\pi G\rho_{vac}$.

In nongravitational physics, the energy of the vacuum is irrelevant. In nongravitational classical mechanics, for example, we speak of particles with energy $E = T+V$, where T is the kinetic energy and V the potential energy. The force on a particle is given by the gradient of V; therefore, we may add an arbitrary constant to V without affecting its motion. Often we choose this arbitrary constant so that the minimum of V is zero, and we say that the particle has zero energy in its vacuum state.

In quantum mechanics the situation is more complicated. Consider, for example, a simple harmonic oscillator of frequency ω; that is, a particle of mass m moving in a one dimensional potential well $V(x) = \frac{1}{2}m\omega^2x^2$. We have chosen the potential such that it has a minimum $V(0) = 0$. However, the uncertainty principle forbids us from isolating the particle in a state with zero kinetic energy and zero potential energy (cf Cohen-Tannoudji et al 1977). In fact, the vacuum state has a zero-point energy $E_0 = \frac{1}{2}\hbar\omega$. Note that we could have set this energy to zero, simply by subtracting $\frac{1}{2}\hbar\omega$ from the definition of the potential; quantum mechanics does not restrict our freedom to pick the zero point of energy. However, it does imply that the energy of a vacuum state will differ from our classical expectation, and that the difference will depend on the physical system (in this case it is a function of ω).

The generalization of this phenomenon to quantum field theory is straightforward (Feynman & Hibbs 1965, Mandl & Shaw 1984). A relativistic field may be thought of as a collection of harmonic oscillators of all possible frequencies. A simple example is provided by a scalar field ϕ (i.e. a spinless boson) of mass m. For this system, the vacuum energy is simply a sum of contributions

$$E_0 = \sum_j \frac{1}{2}\hbar\omega_j, \tag{5.}$$

where the sum is over all possible modes of the field, i.e. over all wave-vectors $\mathbf{k}$. We can do the sum by putting the system in a box of volume L^3, and letting L go to infinity. If we impose periodic boundary conditions, forcing the wavelength (in, say, the ith direction) to be $\lambda_i = L/n_i$ for some integer n_i, then, since $k_i = 2\pi/\lambda_i$, there are $dk_iL/2\pi$ discrete values of k_i in the range (k_i, k_i+dk_i). Therefore expression 5 becomes

$$E_0 = \frac{1}{2}\hbar L^3 \int \frac{d^3\mathbf{k}}{(2\pi)^3}\omega_{\mathbf{k}}. \tag{6.}$$

The energy density ρ_{vac} is obtained by letting $L \to \infty$ while simultaneously dividing both sides by the volume L^3. To perform the integral, we must use $\omega_{\mathbf{k}}^2 = k^2 + m^2/\hbar^2$, and impose a cutoff at a maximum wavevector $k_{\max} \gg m/\hbar$. Then the integral gives

$$\rho_{\text{vac}} \equiv \lim_{L\to\infty} \frac{E_0}{L^3} = \hbar \frac{k_{\max}^4}{16\pi^2}. \qquad 7.$$

As we let the cutoff $k_{\max}$ approach infinity, ρ_{vac} becomes divergent. In the venerable rhetoric of quantum field theory, this is known as an "ultraviolet divergence," since it comes about due to the contribution from modes with very high k. Such divergences are only modestly worrisome. We know that no simple low-energy theory is likely to be exactly true at high energies, where other particles, and possibly new kinds of forces, become important. Therefore, we can estimate $k_{\max}$ as the energy scale at which our confidence in the formalism no longer holds. For example, it is widely believed that the Planck energy $E^* \approx 10^{19}$ GeV $\approx 10^{16}$ erg marks a point where conventional field theory breaks down due to quantum gravitational effects. Choosing $k_{\max} = E^*/\hbar$, we obtain

$$\rho_{\text{vac}} \approx 10^{74}\,\text{GeV}^4\hbar^{-3} \approx 10^{92}\,\text{g/cm}^3. \qquad 8.$$

This, as we will see later, is approximately 120 orders of magnitude larger than is allowed by observation.

We might boldly ignore a discrepancy this large, if it were not for gravity. As in classical mechanics, the absolute value of the vacuum energy has no measurable effect in (nongravitational) quantum field theory. However, one of the postulates of general relativity is that gravitation couples universally to all energy and momentum; this must include the energy of the vacuum. Since gravity is the only force for which this is true, the only manifestation of vacuum energy will be through its gravitational influence. For a density as high as given by Equation 8, this manifestation is dramatic: if $\rho_{\text{vac}} = 10^{92}$ g/cm^3, the cosmic microwave background would have cooled below 3 K in the first 10^{-41} s after the Big Bang.

One may object that we have simply chosen an unrealistically high value for $k_{\max}$. However, to satisfy cosmologically observed constraints, we would need $k_{\max} < 10^{-3}$ cm^{-1}; in other words, we must neglect effects at energies higher than 10^{-14} erg $\approx 10^{-2}$ eV. This is not very high at all; the binding energy of the electron in a hydrogen atom is much larger, and is experimentally tested to very high precision. Moreover, there is direct experimental evidence for the reality of a vacuum energy density in the Casimir (1948) effect: The vacuum energy between two parallel plate conductors depends on the separation between the plates. This leads to a force between the plates, experimentally measured by Sparnaay (1957; also Tabor & Winterton 1969), who found agreement with Casimir's prediction. Fulling (1989), in a lucid discussion of the Casimir effect, notes that "No worker in the field of overlap of quantum theory and general relativity can fail to point this fact out in tones of awe and reverence."

One can postulate an additional "bare" cosmological constant, opposite in sign and exactly equal in magnitude to $8\pi G\rho_{\mathrm{vac}}$, so that the "net" cosmological constant is exactly zero. However, the vacuum energy of quantum field theory does not simply result from the fluctuations of a single scalar field. In the real world there are many different particles, each with its own somewhat different contribution, and with additional contributions derived from their interactions. Given the large number of elementary fields in the standard model of particle physics, it is most unlikely that they conspire to produce a vanishing vacuum energy.

We should finally note that possible solutions to the cosmological constant problem are particularly constrained if they are to be compatible with the inflationary universe scenario (Guth 1981, Linde 1982, Albrecht & Steinhardt 1982; recent reviews are by Narlikar & Padmanabhan 1991, Linde 1990, and Kolb & Turner 1990). Inflationary cosmology postulates an early, exponential expansion driven by the vacuum energy density of a scalar field trapped in a "false vacuum," away from the true minimum of its potential. During the exponential phase, this vacuum energy density is, in fact, a nonzero (and quite large) cosmological constant. Thus, to be compatible with inflation, whatever physical process enforces $\Lambda = 0$ today must also allow it to have had a large value in the past.

For physicists, then, the cosmological constant problem is this: There are independent contributions to the vacuum energy density from the virtual fluctuations of each field, from the potential energy of each field, and possibly from a bare cosmological constant itself. Each of these contributions should be much larger than the observational bound; yet, in the real world, they seem to combine to be zero to an uncanny degree of accuracy. Most particle theorists take this situation as an indication that new, unknown physics must play a decisive role. The quest to solve this puzzle has led to a number of intriguing speculations, some of which we will review in Section 5.

3. EFFECTS OF A NONZERO COSMOLOGICAL CONSTANT

In this section we discuss the observable effects of a nonzero cosmological constant, deferring to Section 4 discussion of the present state of observational limits that derive from these effects. In some cases we are able to give formulas that are simpler than those found elsewhere in the literature. For other summaries of effects, see Kolb & Turner (1990), Charlton & Turner (1987), Felten & Isaacman (1986), and Weinberg (1972). Earlier compendiums include Sandage (1961a,b) and Refsdal et al (1967).

3.1 *Expansion Dynamics*

If $a \equiv 1/(1+z) \equiv R/R_0$ is the expansion factor relative to the present (z being the redshift), and if $\tau \equiv H_0 t$ is a dimensionless time variable (time in units of the measured Hubble time $1/H_0$), then Equation 1 can be rewritten in terms of measurable quantities as

$$\left(\frac{da}{d\tau}\right)^2 = 1+\Omega_{\mathrm{M}}\left(\frac{1}{a}-1\right)+\Omega_\Lambda(a^2-1). \qquad 9.$$

Note that Ω_{M} and Ω_Λ here serve as constants that parametrize the past (or future) evolution in terms of quantities at the present epoch. Equivalently, it was formerly common to parametrize the evolution by Ω_{M} (or $\sigma_0 \equiv \Omega_{\mathrm{M}}/2$) and the deceleration parameter $q_0 = -(R\ddot{R}/\dot{R}^2)_0$. Equation 9 then readily yields the relation

$$q_0 = \frac{1}{2}\Omega_{\mathrm{M}} - \Omega_\Lambda. \qquad 10.$$

We will often use the parametrization Ω_{M} and $\Omega_{\mathrm{tot}} \equiv \Omega_{\mathrm{M}}+\Omega_\Lambda = 1-\Omega_k$, since it is $\Omega_{\mathrm{tot}} < 1$ (>1) that makes the universe spatially open (closed)—a fundamental issue in cosmology. For different assumed values of Ω_{M} and Ω_{tot} (or any other parametrization) one gets qualitatively different expansion histories. Figure 1 displays the various regimes. Felten & Isaacman (1986) show graphs of $a(\tau)$ for various values of Ω_{M} and Ω_Λ.

Qualitatively, the effect of a nonzero Ω_Λ can be described as follows: Looking from now towards the future, a positive value of Λ (or Ω_Λ) tries to drive the universe towards unbounded exponential expansion—asymptotically becoming a DeSitter spacetime. It can fail at this only if the matter density Ω_{M} is so large as to cause the universe to recollapse before it reaches a sufficiently large size for the Λ-driven term (which scales asymptotically as a^2 in Equation 9) to become significant—the narrow wedge in the upper-right corner of Figure 1. A universe fated to recollapse has some value a greater than 1 (the present value), such that the right hand side of Equation 9 vanishes. Some manipulation of the resulting cubic equations (Glanfield 1966, Felten & Isaacman 1986) yields an analytic formula for the boundary between recollapsing and perpetually expanding universes in the $(\Omega_{\mathrm{M}}, \Omega_\Lambda)$ plane (see Figure 1): Unbounded expansion occurs when

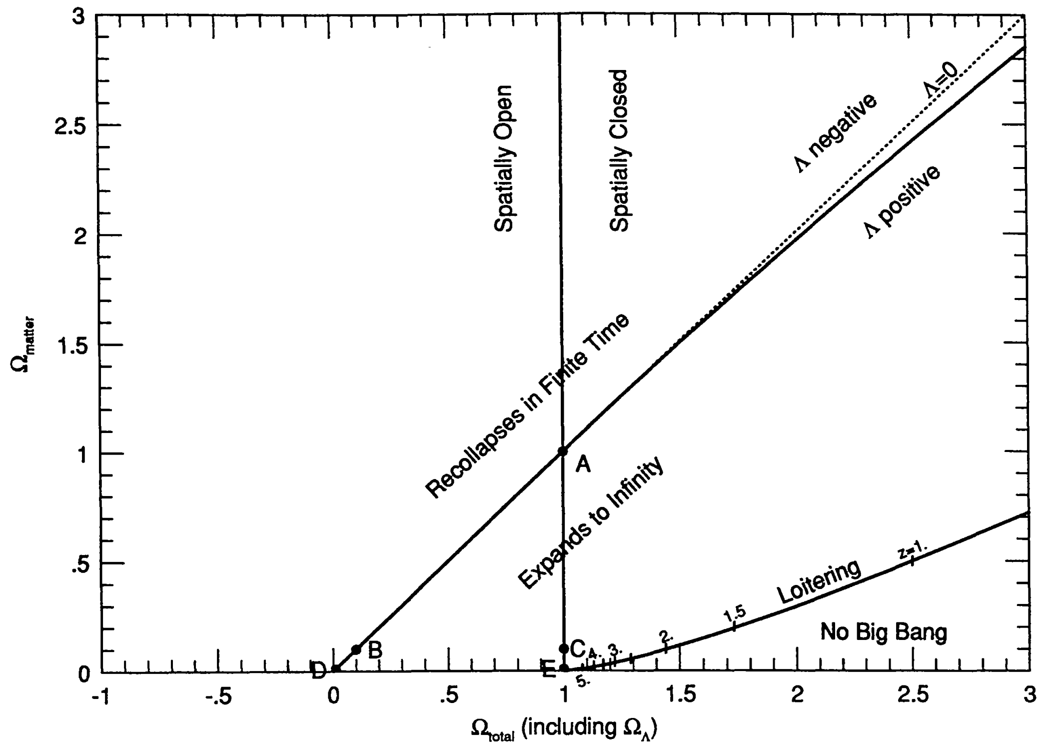

Figure 1 Qualitative behavior of cosmological models in the (Ω_M, Ω_{tot}) plane. Flat models, with $\Omega_{tot} = 1$ and nonzero Ω_Λ, are on the vertical line ACE. Models with $\Omega_\Lambda = 0$ lie on the diagonal line ABD. Use this figure as a "finding chart" for Figures 4, 7, and 9.

$$\Omega_\Lambda \geq \begin{cases} 0 & 0 \leq \Omega_M \leq 1 \\ 4\Omega_M \left\{ \cos\left[\frac{1}{3}\cos^{-1}\left(\frac{1-\Omega_M}{\Omega_M}\right) + \frac{4\pi}{3}\right]\right\}^3 & \Omega_M > 1. \end{cases} \qquad 11.$$

Otherwise the universe recollapses. In particular, negative Ω_Λ implies inevitable recollapse, even for spatially open universes, because the effect of Λ is in the same direction as gravity (attraction) rather than opposing it (repulsion).

For *large*, positive values of Ω_Λ, the universe has a turning point in its *past*, that is, it collapsed from infinite size to a finite radius and is now reexpanding. This occurs when

$$\Omega_\Lambda \geq 4\Omega_M \left\{ \mathrm{coss}\left[\frac{1}{3}\mathrm{coss}^{-1}\left(\frac{1-\Omega_M}{\Omega_M}\right)\right]\right\}^3, \qquad 12.$$

where "coss" is defined as being cosh when $\Omega_M < 1/2$ and cos when $\Omega_M > 1/2$. (The join at $\Omega_M = 1/2$ is perfectly analytic. The need for two formulas to represent a single function is an artifact of solving cubic equations. Here and below it is sometimes useful to use the identities $\sinh^{-1} x = \ln[x+(x^2+1)^{1/2}]$ and $\cosh^{-1} x = \ln[x+(x^2-1)^{1/2}]$.) The redshift z_c of the "bounce" [which is the maximum redshift of any object in the universe, since the universe never gets smaller than $a = (1+z_c)^{-1}$] satisfies

$$z_c^2(z_c+3) = (z_c+1)^3 - 3(z_c+1) + 2 \leq \frac{2}{\Omega_M} \qquad 13.$$

(see, e.g. Börner & Ehlers 1988). Inequality 13 can be solved for z_c, giving

$$z_c \leq 2\,\mathrm{coss}\left(\frac{1}{3}\mathrm{coss}^{-1}\left[\frac{1-\Omega_M}{\Omega_M}\right]\right) - 1, \qquad 14.$$

where "coss" is as defined above. In general, such "bounce" cosmologies are ruled out by the mere existence of high redshift quasars and (even more strongly) by the cosmic microwave background (see Section 4.1).

First noted by Lemaitre (1931), so-called "hesitating" or "loitering" universes occur when Ω_Λ is close to, but barely outside, the bounce region of Equation 12. These are big-bang universes that are now expanding, exponentially in fact, but formerly had an epoch of indecision about whether to recollapse (from their matter content) or to continue expanding (due to their large positive cosmological constant). They thus spent a period of proper time loitering at a nearly constant value of a. (The closer Equation 12 is to an equality, the longer they coast.) The redshift of the loiter satisfies Equations 13 and 14 as equalities. This redshift is plotted in Figure 1 as a parameter along the loitering boundary. One sees that, analogously with bouncing universes, a high redshift loiter requires unreasonably small Ω_M today. The present value Ω_Λ in a universe that had a loitering phase is related to z_c (or Ω_M) by

$$\Omega_\Lambda = \frac{1}{2}\Omega_M(1+z_c)^3 \qquad 15.$$

(cf Equations 12 and 14).

In view of the above arguments, and the observations described in Section 4.1, it should be no surprise that universes with large, positive values for Ω_Λ are presently out of fashion. We think they will remain so. Universes with $\Omega_M > 1$ are of course out of fashion, since all evidence is that there is a "missing mass problem," and not an "excess mass problem." Our attention henceforth will therefore focus on the "fashionable" region in Figure 1, bounded approximately by $0 < \Omega_M < 1$ and $0 < \Omega_{\rm tot} < 1$. In this region the big questions are: (*a*) Is $\Omega_{\rm tot}$ exactly equal to 1, as is required by inflationary scenarios? And, (*b*) if it is equal to 1, is $\Omega_{\rm tot}$ made of Ω_M

(cold matter), Ω_Λ (vacuum energy), or some more exotic form of matter (Peebles 1984, M. S. Turner 1991)?

Figure 2 shows the past and future expansion history of the models of Table 1, found by integrating Equation 9. Model A shows the familiar $t^{2/3}$ expansion law. Nearly empty (non-Λ) models B and D show nearly identical histories, except close to $a = 0$ where B's larger matter content has an effect. Models C and E—flat models with a cosmological constant—have nearly identical future histories, since both have already entered their exponential expansion phase. Model E, being emptier of matter, shows a longer exponential phase to the past, while C's matter content asserts itself more readily and drives the expansion to a more recent big bang ($a = 0$).

3.2 *The Age of the Universe*

By a trivial change of variables in Equation 9, from a to z and from τ to t, we obtain an integral that relates redshift z_1 to lookback time from the present,

$$t_0 - t_1 = H_0^{-1} \int_0^{z_1} (1+z)^{-1}[(1+z)^2(1+\Omega_M z) - z(2+z)\Omega_\Lambda]^{-1/2}\, dz. \qquad 16.$$

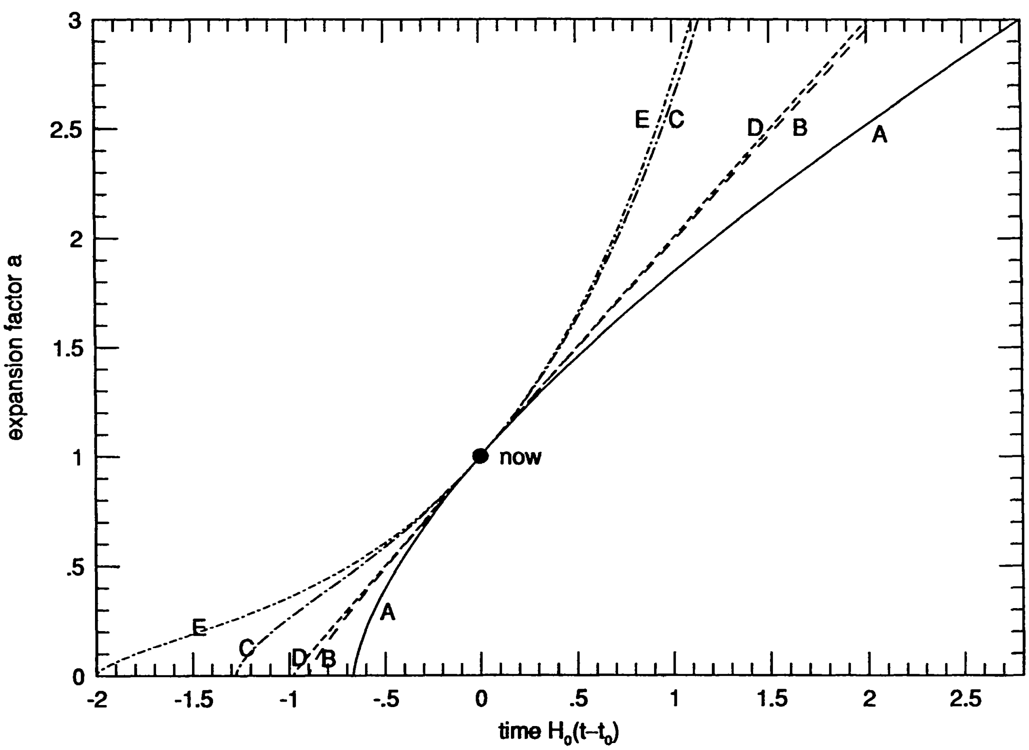

Figure 2 Expansion history of the five models A–E shown in Table 1 and Figure 1. Ω_Λ-dominated models like C and E have already entered an exponential expansion phase; also, they are older than the open $\Omega_\Lambda = 0$ models B and D.

This integral can be solved analytically in some special cases, e.g. when $\Omega_\Lambda = 0$ (Kolb & Turner 1990, equations 3.22–3.25; Sandage 1961a) or when $\Omega_{tot} = 1$ (Weinberg 1989). In general, it can be calculated numerically without difficulty. By inspection of the integrand, one sees that at fixed Ω_M, increasing Ω_Λ lengthens the lookback time to any redshift. Eliminating Ω_Λ in favor of Ω_{tot}, one likewise finds that at fixed Ω_{tot}, the lookback time to any z lengthens for decreasing Ω_M. Figure 3 shows the lookback time as a function of redshift for the five models A–E.

The integral in Equation 16 goes to a finite limit, the age of the universe, as $z_1 \to \infty$. Kolb & Turner (1990) give analytic formulas for the special cases $\Omega_\Lambda = 0$ and $\Omega_{tot} = 1$ (see also Sandage 1961a). Figure 4 shows the results of numerical integration for general cases with $\Omega_\Lambda, \Omega_{tot}$ in the same range that was shown in Figure 1. One sees that if Ω_M, Ω_Λ is bounded to a plausible range, then the age of the universe is between about 0.5 and 2 Hubble times. In Section 4.2 we will compare these ages to observational constraints.

Because the contours in Figure 4 are not too different from lines of constant slope, one can readily write a simple approximation that is valid to within a few percent in the range $0 < \Omega_M \leq 1, 0 < \Omega_{tot} \leq 1$, and serviceable anywhere away from the loitering line:

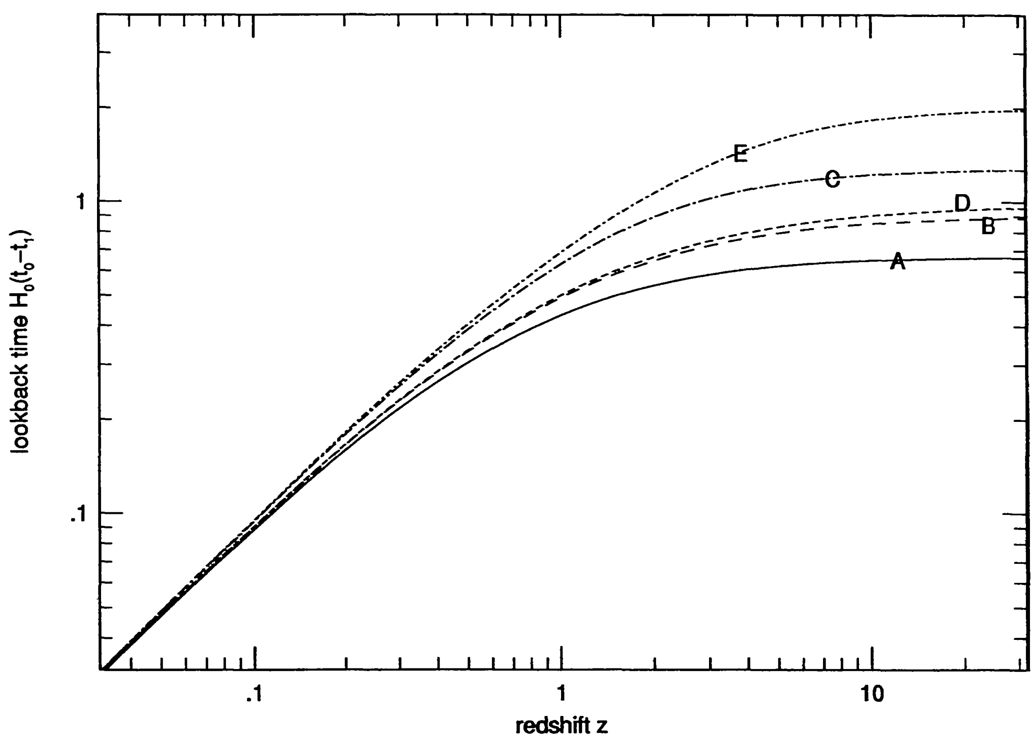

Figure 3 Lookback time as a function of redshift for the five models A–E. Even at moderate redshifts ($z \approx 1$), the Ω_Λ-dominated models separate cleanly from the open models. However, the absolute differences are small.

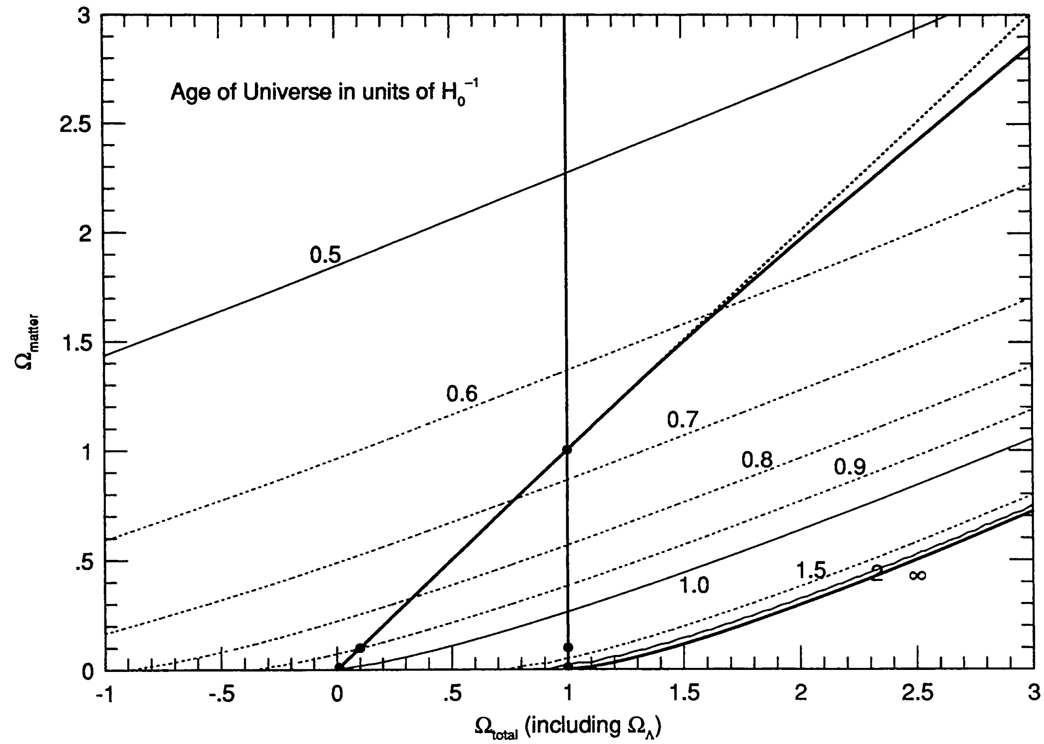

Figure 4 Contours of the age of the universe are shown in the (Ω_M, Ω_{tot}) plane. Away from the infinity line, the contours are close to straight lines, and Equation 17 is a good analytic approximation.

$$t_0 \approx \frac{2}{3} H_0^{-1} \frac{\operatorname{sinn}^{-1}[\sqrt{|1-\Omega_a|/\Omega_a}]}{\sqrt{|1-\Omega_a|}} \qquad 17.$$

where

$$\Omega_a \equiv \Omega_M - 0.3\Omega_{tot} + 0.3 \qquad 18.$$

and "sinn^{-1}" is defined as $\sinh^{-1}$ if $\Omega_a \leq 1$ (the usual case) and as $\sin^{-1}$ if $\Omega_a > 1$. (In fact, Equation 17 is the exact result when $\Omega_{tot} = 1$.)

3.3 *Distance Measures*

As we look out from our self-defined position at $r = 0$ to observe some object at a radial coordinate value r_1, we are also looking back in time to some time $t_1 < t_0$, and back to some expansion factor $R_1 = R(t_1)$ that is smaller than the current value R_0. Note, however, that neither r_1, t_1, nor R_1 are directly measurable quantities. Rather, the measurable quantities are things like the redshift z; the angular diameter distance

$$d_A = D/\theta, \qquad 19.$$

where D is a known (or assumed) proper size of an object and θ is its apparent angular size; the proper motion distance

$$d_{\rm M} = u/\dot{\theta}, \tag{20.}$$

where u is a known (or assumed) transverse proper velocity and $\dot{\theta}$ is an apparent angular motion; and the luminosity distance

$$d_{\rm L} = \left(\frac{\mathscr{L}}{4\pi\mathscr{F}}\right)^{1/2} \tag{21.}$$

where $\mathscr{L}$ is a known (or assumed) rest-frame luminosity and $\mathscr{F}$ is an apparent flux. The relation of the measurables to the unmeasurables turns out to be (Lightman et al 1975, Section 19.9)

$$(1+z) = R_0/R_1, \quad d_{\rm A} = R_1 r_1, \quad d_{\rm M} = R_0 r_1, \quad d_{\rm L} = R_0^2 r_1/R_1. \tag{22.}$$

One sees in particular that $d_{\rm A}$, $d_{\rm M}$, and $d_{\rm L}$ are not independent, but related by

$$d_{\rm L} = (1+z)d_{\rm M} = (1+z)^2 d_{\rm A} \tag{23.}$$

independent of the dynamics of $R(t)$. This is perhaps a disappointment, since it means that we cannot learn anything about $\Omega_{\rm M}, \Omega_\Lambda$ simply by comparing two distance indicators of a single object. Rather, the information about $\Omega_{\rm M}, \Omega_\Lambda$ is contained in the dependence of the distance indicators on redshift z, which we now calculate.

Looking back along a light ray, R, r, and t are related by the equation for a radial, null geodesic of the Friedmann-Robertson-Walker metric, namely

$$\frac{dr}{dt} = \frac{(1-kr^2)^{1/2}}{R}. \tag{24.}$$

Multiplying this equation by R_0, and using Equations 22 and 16, and the definitions of $\Omega_{\rm k}$ and z, one obtains the integral formula for the distance measure at redshift z_1,

$$H_0 d_{\rm M} = \frac{1}{|\Omega_{\rm k}|^{1/2}} \operatorname{sinn}\left\{|\Omega_{\rm k}|^{1/2} \int_0^{z_1} [(1+z)^2(1+\Omega_{\rm M} z) - z(2+z)\Omega_\Lambda]^{-1/2}\, dz\right\}, \tag{25.}$$

where "sinn" is now defined as sinh if $\Omega_{\rm k} > 0$ (open universe) and as sin if $\Omega_{\rm k} < 0$. Remember that $\Omega_{\rm k}$ is not independent, but given by Equation 3. (In the flat case of $\Omega_{\rm k} = 0$, i.e. $\Omega_{\rm tot} = 1$, the sinn and $\Omega_{\rm k}$s disappear from Equation 25, leaving only the integral.) The integral in Equation 25 can be done analytically in the usual special cases $\Omega_\Lambda = 0$ and $\Omega_{tot} = 1$ (see Weinberg 1972 and Kolb & Turner 1990), but in general is straightforward to evaluate numerically. Because of the dependence on $\Omega_k = 1 - \Omega_{tot}$ in the sinn function, the qualitative behavior of Equation 25 is not completely obvious by inspection. At fixed Ω_{tot}, or when $\Omega_\Lambda = 0$, the distance measures all increase with decreasing Ω_M, for all z. For fixed Ω_M, however, there is no monotonicity as Ω_Λ is increased: The distance measure will generally increase at small redshifts, but decrease at redshifts greater than some particular value. Figure 5 illustrates these effects for the specific models A–E. For clarity we plot d_A instead of d_M because the extra factor of $(1+z)^{-1}$ (Equation 23) spreads the curves apart.

3.4 *Comoving Density of Objects*

One is sometimes able to count objects (e.g. galaxies) in observable volume elements, that is, per solid angle $d\Omega$ and per redshift interval dz. If the objects counted can be identified with objects of a known comoving density (e.g. galaxies today), then one has in effect another distance measure in

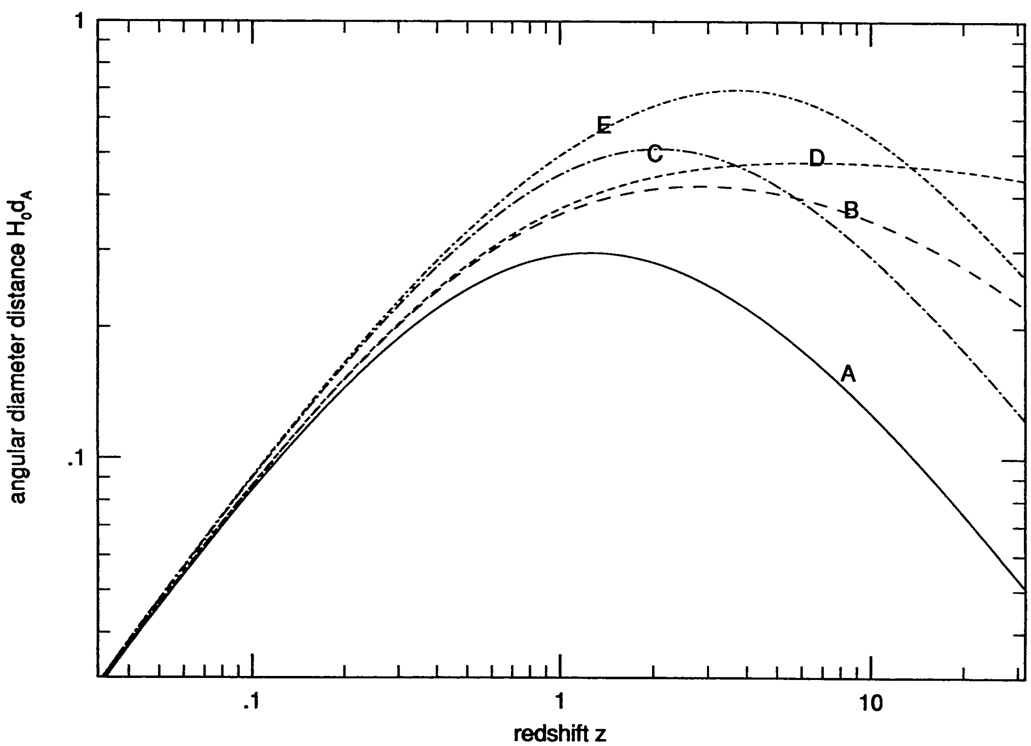

Figure 5 Angular diameter distance as a function of redshift for models A–E. Objects of a small fixed z look farther away (have smaller angular diameters) in Ω_Λ-dominated models. At larger redshifts the situation reverses. Correspondingly, a fixed angular beam subtends, at high redshift, a smaller scale for Ω_Λ-dominated models than for open models, giving smaller cosmic microwave background fluctuations for the Ω_Λ-dominated case.

the relation between comoving density and redshift, and another opportunity to learn about Ω_M, Ω_Λ. The comoving volume element of the Friedmann-Robertson-Walker metric is

$$dV = R_0^3 \frac{r^2}{(1-kr^2)^{1/2}} dr\, d\Omega = \frac{d_M^2}{(1+\Omega_k H_0^2 d_M^2)^{1/2}} d(d_M) d\Omega. \qquad 26.$$

Notice that the volume element is *not* simply a function of d_M, or of d_M and z, but has an additional dependence on Ω_k. This shows that number counts fundamentally probe a different aspect of the universe's geometry than do the distance measures of Equation 22.

Equation 26 has the consequence that, given a population of objects of constant (or calibratable) density and determinable distance measures, one can in principle directly measure Ω_k (or Ω_{tot}) and determine whether the universe is open or closed, in an almost model-independent fashion: One "simply" determines (e.g. along a pencil beam) whether the volume V scales as d_M^3, or whether it shows evidence of the denominator in Equation 26. If d_M, the proper motion distance, were directly accessible to measurement, this test could be performed without measuring any redshifts! Unfortunately, d_M is the least accessible of distance measures. Using d_L or d_A instead, the test requires that redshifts be known, or estimated from a model of the sources (as in Sandage 1988).

More model-dependently, one can calculate from Equations 26 and 25 the dependence of dV on z, and use observed number counts to constrain the values of Ω_M, Ω_{tot} (Loh 1986). Figure 6 shows how the comoving volume element $dV/dzd\Omega$ varies with z for the five models A–E. Notice that at modest redshifts (e.g. $z = 1/2$) the fractional variation among the models is significantly larger in Figure 6 than for the other distance measures in Figure 5. This is an attractive feature of number count tests, but (as we will see) it must be weighed against their susceptibility to evolutionary and selection effects. At redshifts $1+z \sim 2$, the models with significant Ω_Λ have volumes-per-redshift larger than the open models by a factor ~ 2, and larger than the flat $\Omega_M = 1$ model by a factor ~ 4. This results in Λ-models becoming fashionable whenever excess counts of high-redshift objects are claimed to exist (see Section 4.3 below).

Equation 26 can be integrated analytically to give the comoving volume out to a distance d_M,

$$V(d_M) = H_0^{-3}(2\Omega_k)^{-1}[H_0 d_M (1+\Omega_k H_0^2 d_M^2)^{1/2} - |\Omega_k|^{-1/2} \operatorname{sinn}^{-1}(H_0 d_M |\Omega_k|^{1/2})], \qquad 27.$$

where "sinn" is as defined after Equation 25.

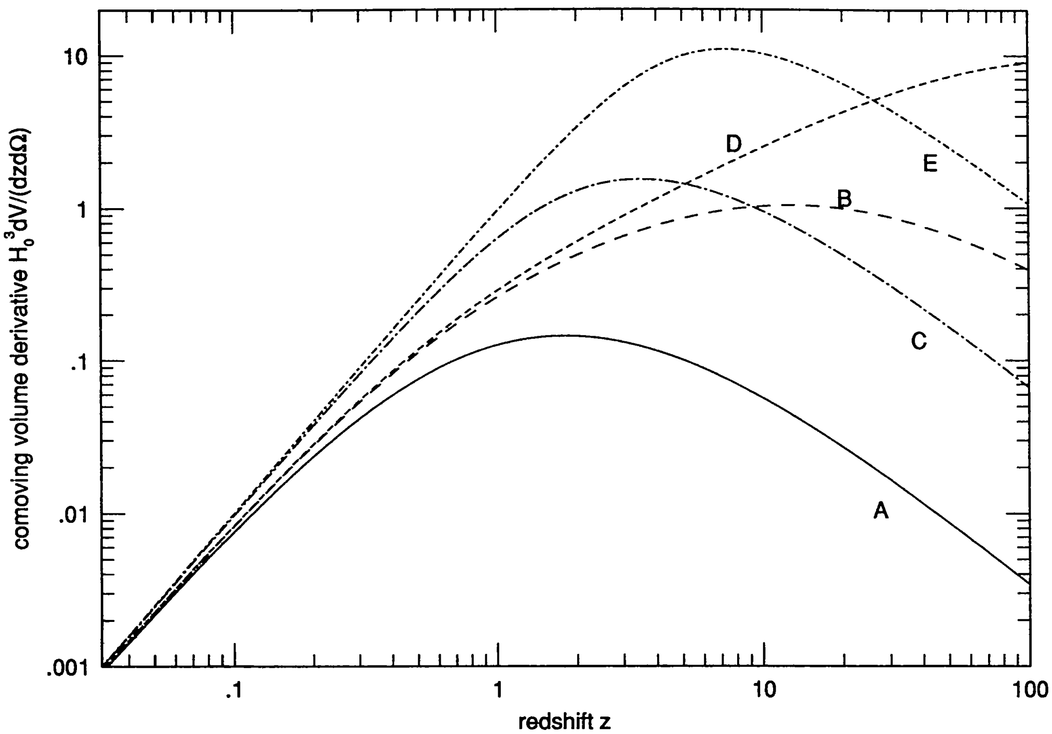

Figure 6 Volume derivative as a function of redshift for models A–E. At redshifts $z < 3$ there is a clean, and quite large, separation between the Ω_Λ-dominated and open models. Number counts would be a powerful test, were it not for a morass of evolutionary and selection effects.

3.5 *Growth of Linear Perturbations*

In all the homogeneous and isotropic cosmologies, linear cold matter perturbations $\delta \equiv \delta\rho/\rho$ grow at a rate that does not depend on their comoving spatial scale (e.g. Peebles 1980). An explicit expression for the amplitude of a growing perturbation (Heath 1977) is

$$\delta(a) = \frac{5\Omega_M}{2a}\frac{da}{d\tau}\int_0^a \left(\frac{da'}{d\tau}\right)^{-3} da', \qquad 28.$$

where a' is the dummy integration variable, and $da/d\tau$ is to be viewed as a known function of a or a', in our case given explicitly by Equation 9. Equation 28 is normalized so that the fiducial case of $\Omega_M = 1, \Omega_\Lambda = 0$ gives the familiar scaling $\delta(a) = a$, with coefficient unity.

Different values of Ω_M, Ω_Λ lead to different linear growth factors from early times ($a \approx 0$) to the present ($a = 1$, $da/d\tau = 1$). Denoting the ratio of the current linear amplitude to the fiducial case by $\delta_0(\Omega_M, \Omega_\Lambda)$, we have

$$\delta_0(\Omega_M, \Omega_\Lambda) = \frac{5}{2}\Omega_M \int_0^1 \left(\frac{da'}{d\tau}\right)^{-3} da'$$

$$\approx \frac{5}{2}\Omega_M \left[\Omega_M^{4/7} - \Omega_\Lambda + \left(1+\frac{1}{2}\Omega_M\right)\left(1+\frac{1}{70}\Omega_\Lambda\right)\right]^{-1}. \qquad 29.$$

(The remarkable approximation formula—good to a few percent in regions of plausible Ω_M, Ω_Λ—follows from Lahav et al 1991 and Lightman & Schechter 1990.) Figure 7 shows numerical values for $\delta_0(\Omega_M, \Omega_\Lambda)$ for the region in the $(\Omega_\Lambda, \Omega_{tot})$ plane previously seen in Figures 1 and 4. One sees that as Ω_M is reduced from unity, both along the line $\Omega_\Lambda = 0$ and along the line $\Omega_{tot} = 1$, the growth of perturbations is suppressed, but somewhat less suppressed in the $\Omega_{tot} = 1$ case. The reason is that, for fixed Ω_M, linear growth effectively stopped at a redshift $(1+z) = \Omega_M^{-1}$ in the open case (when the universe became curvature dominated), but, more recently, at $(1+z) = \Omega_M^{-1/3}$ in the flat case (when the universe became Λ dominated).

To the right of the line $\Omega_{tot} = 1$ in Figure 7, one sees values of $\delta_0(\Omega_M, \Omega_\Lambda)$

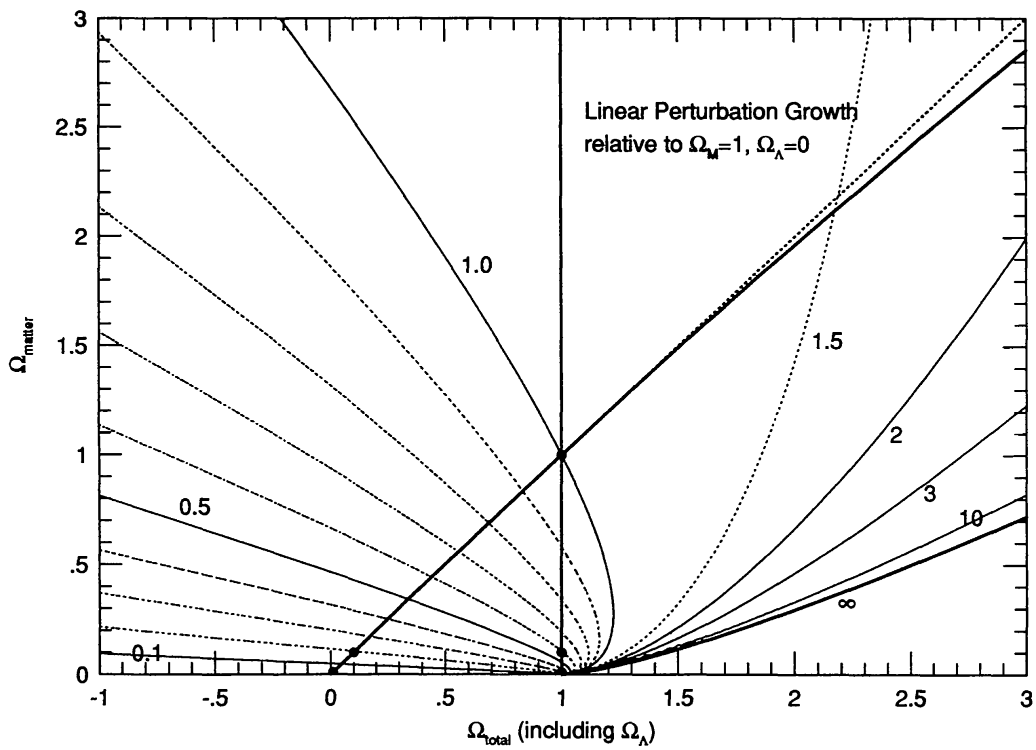

Figure 7 Growth factor for linear perturbations, as contours in the (Ω_M, Ω_{tot}) plane, normalized to unity for the case $\Omega_M = 1$, $\Omega_\Lambda = 0$. There is relatively less suppression of growth as Ω_M is decreased along the line $\Omega_{tot} = 1$ than along the line $\Omega_\Lambda = 0$; but for credible values of Ω_M the difference is not a large factor. Perturbation growth approaches ∞ at the "loiter line," but for credible Ω_M it occurs at too small a redshift to explain quasars (see Figure 1).

that are greater than 1, in fact approaching infinity at the loitering-cosmology line (cf Figure 1 and discussion above). Loitering cosmologies allow the arbitrarily large growth of linear perturbations, since the perturbations continue to grow during the (arbitrarily long) loiter time.

Related to the growth of linear perturbations is the relation between peculiar velocity **v** and peculiar acceleration **g**, or (as a special case) the radial infall velocity $v_{\rm rad}$ around a spherical perturbation of radius $\mathscr{R}$. These quantities depend not directly on Equation 28, but on its logarithmic derivative, the exponent in the momentary power law relating δ to a,

$$f \equiv (d\ln\delta)/(d\ln a), \qquad 30.$$

the relation being

$$\mathbf{v} = \frac{2f\mathbf{g}}{3H\Omega_{\rm M}} \quad \frac{v_{\rm rad}}{H\mathscr{R}} = \frac{1}{3}f\langle\delta\rangle \qquad 31.$$

where $\langle\delta\rangle$ is the overdensity averaged over the interior of the sphere of radius $\mathscr{R}$ (Peebles 1980, Section 14). One can calculate f accurately by taking the derivative of Equation 28, using Equation 9, and solving the resulting integral numerically. Lahav et al (1991), however, give an approximation formula valid for all redshifts z,

$$f = f(z) \approx \left[\frac{\Omega_{\rm M}(1+z)^3}{\Omega_{\rm M}(1+z)^3 - (\Omega_{\rm M}+\Omega_\Lambda - 1)(1+z)^2 + \Omega_\Lambda}\right]^{4/7}. \qquad 32.$$

Figure 8 plots $f(z)$ for our standard models A–E. One sees that, at small redshifts, peculiar velocities depend almost entirely on $\Omega_{\rm M}$ and are quite insensitive to Ω_Λ. This is because they are driven by the matter perturbations in primarily the most recent Hubble time. Looking back to redshifts $z \gtrsim 1$, however, the peculiar velocities do start depending on Ω_Λ, allowing in principle for observational tests (but see Lahav et al 1991 for caveats).

3.6 *Gravitational Lens Probabilities*

One effect of a nonzero cosmological constant is to change, in some cases drastically, the probability that quasars are gravitationally lensed by intervening galaxies (Fukugita et al 1990a, Turner 1990). While the absolute lens probability obviously depends on the absolute density and gravitational potential of the lensing galaxies, a useful statistic is the probability for lensing by a population of isothermal spheres of constant comoving

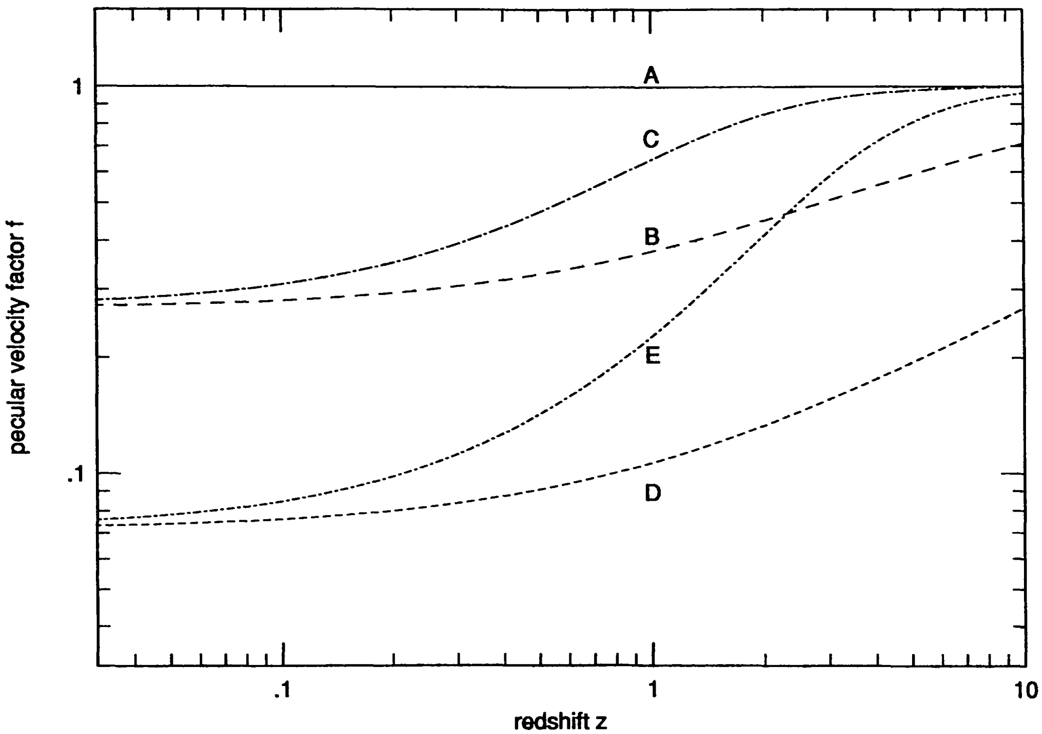

Figure 8 Peculiar velocities around fixed-density condensations as a function of redshift, for models B–E, relative to model A. For $z \lesssim 1$, peculiar velocities, and other related dynamical effects, are extremely insensitive to the value of Ω_Λ.

density *relative* to the fiducial case $\Omega_M = 1$, $\Omega_\Lambda = 0$, given by the integral

$$P_{\text{lens}} = \frac{15}{4}\left[1 - \frac{1}{(1+z_s)^{1/2}}\right]^{-3}$$

$$\times \int_0^{z_s} \frac{(1+z)^2}{[(1+z)^2(1+\Omega_M z) - z(z+2)\Omega_\Lambda]^{1/2}} \left[\frac{d(0,z)d(z,z_s)}{d(0,z_s)}\right]^2 dz \qquad 33.$$

(Fukugita et al 1992). Here z_s is the redshift of the source (quasar). The prefactor normalizes the fiducial value to unity. The function $d(z_1, z_2)$ is the angular diameter distance from redshift z_1 to redshift z_2, given by the generalization of Equation 25,

$$d(z_1, z_2) = \frac{1}{(1+z_2)|\Omega_k|^{1/2}}$$

$$\times \operatorname{sinn}\left\{|\Omega_k|^{1/2} \int_{z_1}^{z_2} [(1+z)^2(1+\Omega_M z) - z(2+z)\Omega_\Lambda]^{-1/2} dz\right\}. \qquad 34.$$

Equation 33 quantifies the geometrical differences affecting ray paths and volumetric factors among different Ω_M and Ω_Λ models. Figure 9 plots the value of $P_{\rm lens}$ in the $(\Omega_M, \Omega_{\rm tot})$ plane for the specific (but reasonable) choice $z_s = 2$. Along the diagonal line $\Omega_\Lambda = 0$, one sees that lens probabilities increase as the universe becomes emptier, but only by a modest factor ~ 2. By contrast, as the matter density is decreased along the line $\Omega_{\rm tot} = 1$ (that is, compensated by increasing Ω_Λ), the lens probability rises dramatically, by a factor ~ 10. We will see below that gravitational lensing, because it distinguishes so sharply between low Ω_M universes of differing Ω_Λ, holds great promise for putting firm limits on Ω_Λ.

4. OBSERVATIONAL STATUS OF THE COSMOLOGICAL CONSTANT

In the preceding section's catalog of effects, we avoided discussion of the present status of actual observations. Here, we redress the balance.

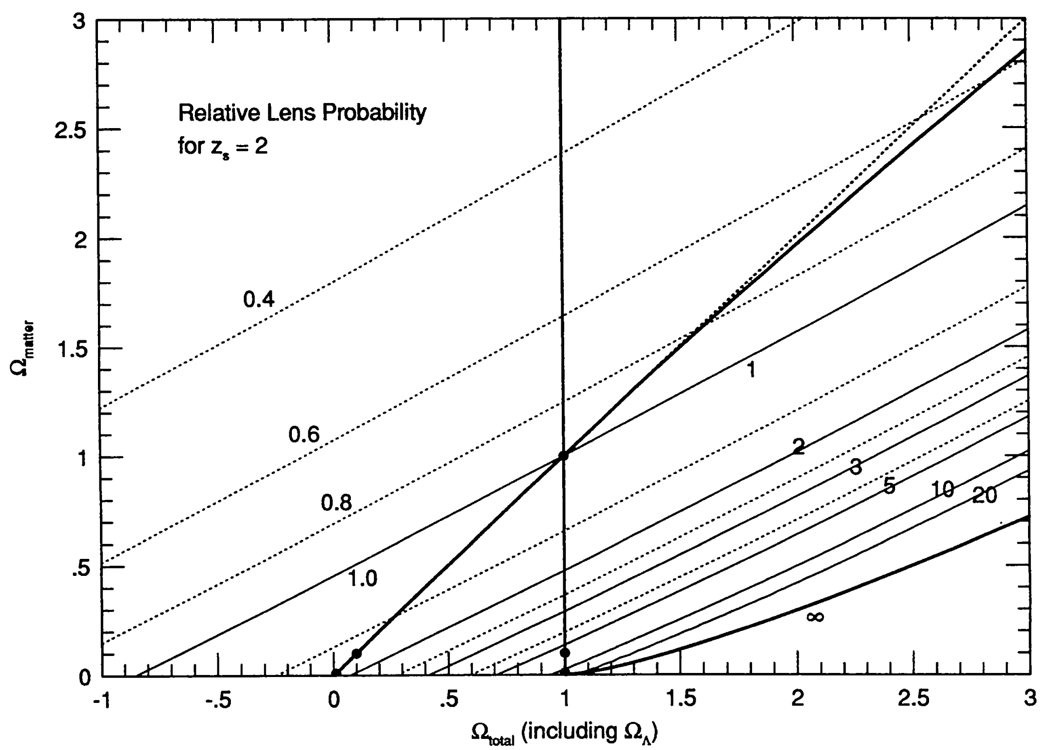

Figure 9 Probability for observing a gravitational lens, as contours in the $(\Omega_M, \Omega_{\rm tot})$ plane, normalized to unity for the case $\Omega_M = 1$, $\Omega_\Lambda = 0$. Gravitational lens statistics are the most promising method for ruling out Ω_Λ-dominated models along the $\Omega_{\rm tot} = 1$ line, and already give the best bounds on Ω_Λ to the right of that line.

4.1 *Existence of High-Redshift Objects*

Bounce cosmologies are ruled out by the mere existence of high-redshift phenomena. High-redshift quasars are known with $z = 4.89$ (Schneider et al 1991). The restriction on Ω_M implied by Equation 13 is thus $\Omega_M < 0.01$, which is quite unlikely on direct observational grounds, and also incompatible with the successful predictions of the theory of big bang nucleosynthesis (Olive et al 1990, Peebles et al 1991). Thermalization of the microwave background at $z > 10^3$ implies $\Omega_M < 2 \times 10^{-9}$, which is surely impossible (Trimble 1987).

The case against loitering cosmologies is strong, though not quite so airtight. At one time, there was some belief that an excess number of quasars with $z \approx 1.95$ pointed to the existence of a loitering model with $\Omega_M \approx 0.106$, $\Omega_\Lambda = 1.361$ (Burbidge 1967, Shklovsky 1967). However, if a loitering phase lasts long enough, then the point of the universe antipodal to us becomes visible, at high magnification and with various exotic effects. Petrosian et al (1967), Shklovsky (1967), and Rowan-Robinson (1968) investigated this for redshifts around 1.95.

More recently, Gott (1985), Gott & Rees (1987), and Gott et al (1989) have found constraints on the redshift of the antipodes (and therefore any loiter) arising from gravitational lensing. Under rather general assumptions, they find that no quasar at a redshift larger than that of the antipodes can be lensed to more than one image. The existence of the lensed quasar QSO 2016 at a redshift $z = 3.27$ then bounds possible values of Ω_Λ significantly away from the critical value of Equation 12 if $\Omega_M \gtrsim 0.03$—as seems quite likely on dynamical grounds (see below; Lahav et al 1991; note, however, Paczynski's caveat mentioned in Gott 1985). Durrer & Kovner (1990) have argued that the range $0.01 < \Omega_M < 0.03$ is conceivably viable and have investigated possible effects of antipodal focusing of cosmic microwave background fluctuations, while R. D. Blandford (unpublished) has directed attention to peculiarities in the observed lens. Such arguments seem forced, however. To circumvent these limits on Ω_M, one can postulate a time-variable cosmological constant (Sahni et al 1992), but such models are also artificial.

We have already noted (Section 3.5) that linear density perturbations can grow by a large factor during a loitering phase. However, the existence of high-redshift quasars (lensed or not) argues against the theoretical invocation of a loitering cosmology to magnify perturbations: Since the gravitational condensation that creates (or fuels) the quasar (E. L. Turner 1991) must come after the perturbations have grown, the implied redshift of loiter should be larger than the redshift of any observed quasar; this is inconsistent with the firm fact that $\Omega_M > 0.01$ (Equations 13 or 14).

4.2 *Age Concordance: Globular Clusters and Cosmic Nuclear Data*

Perhaps the most compelling, plausibly achievable demonstration of a nonzero value of Ω_Λ would be the identification of objects or material older than H_0^{-1}. Figure 4 shows that one would be forced to invoke models with Ω_Λ significantly greater than zero if $\Omega_M > 0.1$. Such an argument would be strong because it is difficult to imagine escaping it through the usual sort of loopholes of "astrophysical complications" which prevent definite conclusions in so many cosmological considerations. In other words, the universe ought to be at least as old as the objects and material it contains. Of course astrophysical complications are still able to enter the picture when we get down to the quantitative question of how old the oldest objects actually are!

Galactic globular clusters are the stellar systems with the most reliably determined extreme ages. The calibration of stellar ages is a complex and highly developed subject with many thorough reviews (e.g. Rood 1990, VandenBerg & Smith 1988, VandenBerg 1990) and even whole volumes concerning it (Philip 1988). Here we only comment briefly on the most salient issues.

There are basically two techniques for using the models of stellar evolutionary theory to derive ages from observed globular cluster H-R diagrams. One may fit the theoretical isochrones directly to the observed main sequence color-magnitude track and turn-off in order to determine the mass of stars which have just exhausted their central hydrogen fuel. Alternately, one may use the magnitude difference between the main sequence and the horizontal branch (HB) to find the turn-off luminosity (taking the HB to have a fixed luminosity). The primary advantage of the former method is that it relies on the most secure regime of stellar evolution theory. Its worst disadvantage is that very small errors in matching theoretical to observed colors (which could be due to inaccurate reddening corrections, stellar atmosphere models, photometric calibration, and so on) lead to 5–7 times larger fractional errors in the derived ages. In other words, a 0.04 magnitude systematic shift in the color match corresponds to a 20–30% error in the age. The main advantage of the latter method is that it avoids color fitting (and hence these problems) altogether. Its primary difficulty is that the HB absolute magnitude is poorly known (based on RR Lyrae star studies) with the uncertainty being at least 0.2 magnitudes (Sandage 1990) corresponding to a 20% age uncertainty. Both techniques are discussed in detail and applied to the best available data for a large sample of globular clusters by Sandage & Cacciari (1990).

Despite these formidable difficulties, observational and theoretical, the consensus of expert opinion concerning the ages of the oldest globular clusters is impressive. All seem to agree that the best-fit ages are 15–18 Gyr or more, perhaps considerably more. (It turns out to be easier to extend the main sequence lifetime of low mass stars by introducing theoretical complications, which typically provide additional nuclear fuel or added support against gravity, than to lower the ages.) Of more interest in the present context, one wishes to know the lower limit on these oldest stellar ages; unfortunately, it is not a matter of formal errors but rather of informed judgments of how far various effects and uncertainties can be pushed. The range of expert opinion clusters around 12–14 Gyr whether based on considerations of many clusters (Sandage & Cacciari 1990, Rood 1990) or the few best studied cases such as 47 Tuc and M92 (VandenBerg 1990, Pagel 1990).

It may provide a useful perspective to note that determination of an age with some fractional accuracy corresponds to determining the distance twice as accurately. Thus, a 20% age uncertainty (the difference between 15 and 12 Gyr) corresponds to claiming a 10% uncertainty in the distance! This holds true whichever of the two techniques is employed.

Nuclear chronometers also offer the possibility of obtaining a useful lower limit on the age of the universe. They give the age of the Solar System with great precision (Anders 1963), and a few chronometric pairs (notably ^{232}Th–^{238}U, ^{235}U–^{238}U, and ^{187}Re–^{187}Os) can, in principle, yield a mean heavy element age prior to the condensation of the Solar System (Schramm & Wasserburg 1970). Unfortunately, due to both observational uncertainties in their relative abundances and to the necessity of relying on highly speculative and poorly constrained models for the Galactic history of nucleosynthesis, the indicated age of the universe is extremely uncertain (Clayton 1988, Arnould & Takahashi 1990, Cowan et al 1991). Nevertheless, a conservative analysis (essentially assuming all of the heavy elements were synthesized promptly at the beginning of the universe) which allows for the abundance uncertainties indicates a somewhat interesting *lower* limit of 9.6 Gyr (Schramm 1990) for the age of the oldest heavy elements. Although this is less restrictive than the lower limits obtained from globular cluster studies discussed above, it may be more secure because the physics of nuclear decay is so much better understood than that of stellar evolution.

With these lower limits for the universe's age, we could obtain decisive information on Ω_Λ from Figure 4 if only we had an accurately measured value of H_0. Figure 10 illustrates the situation. The shaded boxes show a reasonable range of observational determinations of the dynamical component of Ω_M, and of H_0. Taking 0.1 as a lower limit of Ω_M from dynamical studies (Trimble 1987), the open ($k = -1$) models with $\Omega_\Lambda = 0$ require H_0

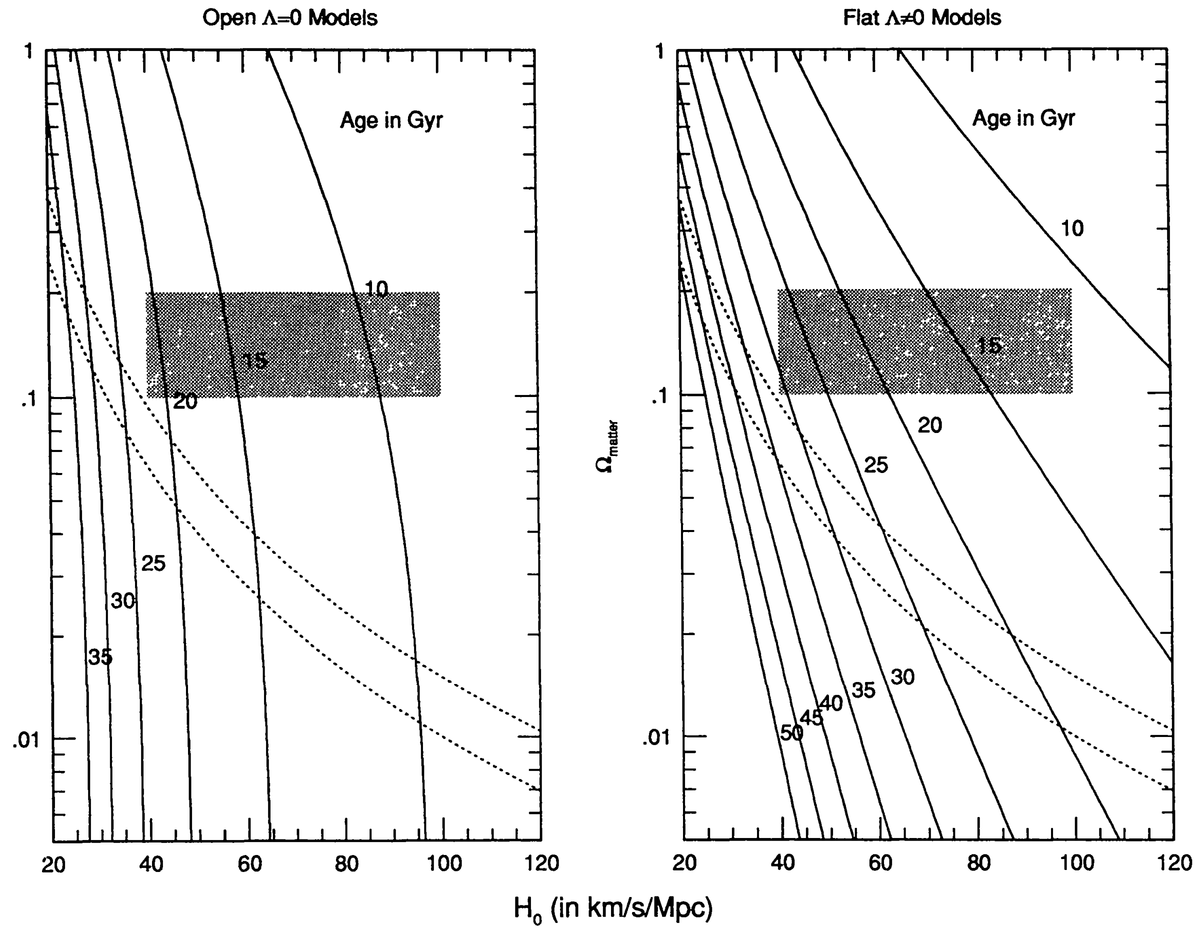

Figure 10 The age of the universe is shown as solid contours in the plane defined by H_0 and Ω_M, (*left*) for open models with $\Omega_\Lambda = 0$, and (*right*) for flat models with $\Omega_{tot} = 1$. Shaded boxes indicate likely observational ranges for these quantities. The contribution of baryons to Ω_M is bounded by nucleosynthesis to lie between the dotted curves. See text for discussion.

to be no larger than 64–76 km/s/Mpc, to be consistent with globular cluster ages. In the extreme case of $\Omega_M = 1, \Omega_\Lambda = 0$, the range is 48–57 km/s/Mpc. There would be an additional 25% increase to 71 or 95 km/s/Mpc (for the $k = 0$ and $k = -1$ cases, respectively) if one more cautiously used the nucleochronometer limit quoted above. If the true value of H_0 were shown to be larger than these limits (pick your favorite!), then a nonzero Ω_Λ would be required.

A more extreme version of this argument is to take best-fit globular cluster ages of 15–20 Gyr and note that this places upper limits of 33–43 km/s/Mpc if one insists on a model with $\Omega_M = 1, \Omega_\Lambda = 0$.

Clearly there is the possibility of a discovery here, but is there really any problem at the moment? Some optimistic commentators (Fukugita 1991, Peacock 1991, Fukugita & Hogan 1990) have been encouraged by the precision and consistency of several modern extragalactic distance indicators (Aaronson et al 1989, Jacoby et al 1990, Tonry 1991, Fukugita & Hogan 1991) to conclude that H_0 is quite likely to be within 10% of 80 km/s/Mpc. Such a value would require either invoking a nonzero cosmological constant or both abandoning a $k = 0$ cosmology and stretching the globular cluster ages to roughly the limit of their usually claimed uncertainties. This view has been a significant motivation for the recent renewed interest in nonzero Ω_Λ. Moreover, even if the various extragalactic indicators (cited above) that indicate a large H_0 value are accurate indicators of relative distance, the resulting H_0 value is still dependent on the distances of the same few local calibrators, which are not established beyond reasonable doubt either (though see Madore & Freedman 1991, Freedman 1990). On the other hand, there is also some substantial evidence for small H_0 values (Arnett 1982, Arnett et al 1985, Branch 1987, Tammann 1987, Sandage 1988b,c, Eastman & Kirshner 1989, Roberts et al 1991, Press et al 1992, Narayan 1991) which we have no reason to disregard. In summary, a value of H_0 small enough to avoid any age concordance problems, even in an $\Omega_M = 1, \Omega_\Lambda = 0$ model, is not yet excluded.

As a matter of related interest, Figure 10 also shows as dotted lines the upper and lower bounds on $\Omega_{\rm baryon} H_0^2$ that derive from cosmological light element abundances (Olive et al 1990, Walker et al 1991). At the lower-left corner of the shaded box in the left-hand figure, one notes that an open model consisting entirely of baryons with $\Lambda = 0$ and $\Omega_M = 0.1$ is by no means strongly excluded. That such a model is currently so unfashionable testifies to the strength of theoretical prejudice for one or more of (*a*) inflation, (*b*) CDM theory (see below), or (*c*) exotic dark matter. The corresponding flat $\Lambda \neq 0$ model (right-hand figure) has an age that exceeds 30 Gyr, and is thus much less plausible: Not one observed stellar system even approaches such an age. One sees that a flat Λ model thus effectively requires a nonbaryonic component of dynamical matter. The necessity of invoking two speculative elements should perhaps be counted as a strike against Λ models.

4.3 *Galaxy Counts as a Function of Redshift or Apparent Magnitude*

As discussed in Section 3.4 and illustrated in Figure 6, the variation of dV/dz with cosmological parameters is substantial and offers a promising effect on which Ω_Λ determinations might be based. This has been pursued primarily in terms of counting galaxies, either as a function of their estimated redshifts or their apparent magnitudes in some band. Neither technique has yet given a decisive result despite significant efforts reviewed below.

There are two major difficulties which make galaxy number–redshift counts problematic. First, redshifts for a very large unbiased sample of faint galaxies are required as input data. The sample must be unbiased with respect to redshift, despite the expected strong anti-correlation between redshift and surface brightness. Second, one must take careful account of any galaxy evolution with redshift to make sure the low-z counts refer to the same population of objects as the high-z counts.

The most notable and fearless use of this technique was reported by Loh & Spillar (1986, see also Loh 1986) who used broadband photometry in six optical–near IR bands to estimate redshifts for 1000 galaxies down to a brightness level of approximately 22 in I. They used a very simple model for galaxy evolution (essentially assuming that the luminosities of all galaxies at any past epoch to be a constant factor times their present luminosities) described by a single parameter which they simultaneously fit to the data. Their results were very well fit by a $k = 0$, $\Omega_M = 1$, $\Omega_\Lambda = 0$ model and were seen as strong support for that popular cosmological scenario; however, they were in fact consistent with a significant range of other possibilities, including nonzero Ω_Λ cases (Peebles 1988).

More seriously, consideration of even slightly more realistic models of galactic evolution (e.g. allowing for the possibly separate evolution of early and late type galaxy populations) greatly increases the error ranges for the cosmological parameters derived by Loh & Spillar (Bahcall & Tremaine 1988, Yoshii & Takahara 1989). Worse still, the peculiar behavior of faint galaxy counts and color-magnitude diagrams (see below), the surprisingly rapid evolution of the spectroscopic populations of rich galaxy clusters (Dressler 1984), and the redshifts distributions for small samples of faint galaxies studied with slit spectroscopy (Broadhurst et al 1988, Cowie 1991) all make it abundantly clear that low-redshift galaxy evolution is a complex process which probably cannot be described by any simple model with a small number of parameters. These same complications call into doubt the accuracy of photometric redshifts which must rely on the overall spectral shape of galaxy optical–IR emission changing in a predictable and simple way.

It thus appears unlikely that any compelling constraints on cosmological parameters can be derived from galaxy number–redshift counts until there are great improvements in both empirical information on faint galaxies and our theoretical understanding of their evolution.

An observationally less challenging approach to dV/dz tests is to abandon redshift determinations altogether and simply count galaxies as a function of flux (apparent magnitude). In fact, galaxy number–magnitude counts in the B (blue) band (Tyson 1988, Maddox et al 1990b) are substantially better fit by Ω_Λ-dominated cosmologies than by $\Omega_\Lambda = 0$ ones, even taking into account the uncertainties in relatively sophisticated modern galaxy evolution models (Fukugita et al 1990b). This realization also prompted a flurry of interest in the cosmological constant, but was soon undermined by the discovery that near-IR K band counts do not require (and, in fact, are poorly fit by) Ω_Λ-dominated models (Cowie 1991) and that selection effects, including seeing-dependent ones, must be carefully accounted for in the interpretation of such number–magnitude counts (Fukugita 1991).

Once again, it seems difficult not only to draw any firm conclusions but even to discern which cosmological models are marginally favored by the available data.

4.4 *Dynamical Tests of* Ω_Λ

Since a nonzero Ω_Λ might be thought of as producing significant non-gravitational long range forces in the evolution of the universe, it is natural to hope that the large-scale dynamics of the material in the universe (i.e. large-scale galaxy clustering) might be sensitive to its value and thus provide some useful tests. Unfortunately, as Martel & Wasserman (1990), Martel (1991), and Lahav et al (1991) have shown in detail, the properties of present day structures and galaxy clusters are remarkably insensitive to Ω_Λ; it is doubtful that anything significant can be learned about the cosmological constant from their study.

On the other hand, if one considers not merely the present day clustering but also some information on its derivatives (time evolution), there is hope of some purchase on the Λ issue. For example, Carlberg (1991) has shown that the expected rate of galaxy mergers increases much more rapidly with redshift (at $z \leq 1$) for zero Λ models than for Λ-dominated ones, at least for conventional models of structure formation, and has interpreted some evidence for a high rate of galaxy mergers at moderate redshifts as evidence against a significant value of Ω_Λ. However, since it may be reasonably doubted that galaxy mergers were ever a common process (Ostriker 1980) and since it is anything but clear how cosmic structure formed (Peebles & Silk 1990), it is probably more sensible to regard this test as an interesting idea for further investigation than as yet giving any clear result.

Similarly, Richstone et al (1992) have pointed out that the mean density (in absolute units or relative to the critical density) of just collapsing structures are expected to be somewhat lower in Λ-dominated cosmologies (assuming only gravity-driven structure formation) than in conventional ones, because the increased age of the universe allows time for their slower dynamical evolution. Thus, if one could use the presence of unrelaxed substructure, galaxy populations, or some other indicator to identify just post-collapse clusters and could measure their mean cluster densities accurately enough, a test might be feasible. Again, available data and our current understanding of cluster evolution are still far from up to the task.

Recently, a nonzero Ω_Λ term has been advocated (Efstathiou et al 1990, M. S. Turner 1991) as a means of saving the cold dark matter (CDM) model of structure formation (e.g. Davis et al 1985, Bardeen et al 1986) from the contrary discoveries of excess matter perturbations on large scales (e.g. Maddox et al 1990a, Geller & Huchra 1990). In CDM theory, there is a change of logarithmic slope in the perturbation spectrum, caused by suppression of the growth of perturbations that are smaller than the size of the horizon during the radiation-dominated era. The length scale of this break becomes larger if the epoch of matter dominance is made more recent, i.e. if Ω_M is decreased. Therefore, for fixed (observed) normalization of the perturbations at the small-scale end, the amplitudes of large-scale matter perturbations increases as Ω_M decreases. A value $\Omega_M \approx 0.2$ is found to give best agreement with observation.

If $\Omega_\Lambda = 0$, such a value is incompatible not only with theoretical prejudices in favor of inflationary models with $\Omega_{tot} = 1$, but also directly with anisotropy measurements of the cosmic microwave background (Bond et al 1990, Vittorio et al 1991). One can see the problem in Figure 5, whose ordinate is proportional (by Equation 19) to the proper size of a scale that subtends a fixed angle θ: Models A, B, and D, with progressively decreasing Ω_M and $\Omega_\Lambda = 0$, subtend respectively larger proper scales, which are therefore less correlated, implying increasing anisotropies. The sequence A, C, E, where $\Omega_{tot} = 1$ and decreasing Ω_M is compensated by increasing Ω_Λ, yields much smaller increases in scale, therefore smaller increases in anisotropy (Vittorio & Silk 1985, Kofman & Starobinskii 1985, Sugiyama et al 1990, Gorski et al 1991). A model with $\Omega_M = 0.2$, $\Omega_\Lambda = 0.8$ is claimed to be compatible with both observed large-scale structure and present microwave anisotropy limits. Whether this model can be confirmed or ruled out by other tests—e.g. gravitational lensing (see below) or the X-ray temperature distribution of clusters of galaxies (Lilje 1992)—is an important current question.

The patching up of CDM, by itself, can hardly be taken as firm evidence of a nonzero Ω_Λ. CDM theory has been perhaps unjustifiably wed to the assumption of a single, constant bias factor b relating mass to light. Large-scale structure is no more direct evidence of a nonzero Ω_Λ than it is evidence of a scale-dependent value of b. In fact, scale- and velocity-dependent biasing is seen in recent, as yet unpublished, numerical simulations by Carlberg (1991), and by Cen & Ostriker (1992); these simulations include hydrodynamical and radiative effects and attempt to calculate, rather than assume, biasing effects.

4.5 *Quasar Absorption Line Statistics*

Gas clouds, believed to be associated with the halos of galaxies, are distributed through intergalactic space and cause narrow absorption features in quasar spectra. These are the most numerous objects that can be counted to high redshift. Their statistical distribution with redshift, dN/dz, offers another possible Ω_Λ test somewhat akin to the dV/dz count tests. Potentially, the evolution of ionized gas clouds might be more easily understandable than that of physically much more complex whole galaxies; at least in the case of the clouds, one probably knows what the relevant fundamental equations and physical effects are. This gives some hope that evolution and cosmology might someday be clearly disentangled for quasar absorption line statistics.

In an early application of this idea, Tytler (1981) used the absence of strong features in the distribution of quasar absorption line redshifts to argue against loitering cosmologies. Turner & Ikeuchi (1992) have studied quasar absorption line statistics in cosmological models corresponding to our cases A and C, using an identical simple physical model for the clouds in each case to account for their evolution. They find that no clear choice between these two extreme possibilities can be made from the available data; however, the extrapolation to low redshift of high-redshift fits of various absorption lines predicts quite different frequencies for the two models. Typically 2–3 times more frequent absorptions are predicted for model C than for model A. When space far-UV spectroscopy of quasars provides a substantial body of data on low-redshift quasar absorption lines (not now available), this effect may well provide an interesting test. Also, flat Ω_Λ-dominated models predict an inflection in dN/dz versus z for all types of lines at a redshift of about $(\Omega_\Lambda/\Omega_M)^{1/3}$ due to the universe's transition into roughly exponential expansion. Consistent detection of such a feature in a wide variety of classes of absorbers might give some confidence that it was due to a cosmological, rather than an evolutionary, effect. This is clearly an area which deserves further work, especially as data on low-redshift absorption systems becomes available.

The first Hubble Space Telescope (HST) data on low-redshift Lyman-alpha clouds (Morris et al 1991, Bahcall et al 1991) have been tentatively interpreted on this basis as evidence against Ω_Λ-dominated models by Fukugita & Lahav (1991), with, however, a variety of caveats (Turner & Ikeuchi 1992, Ikeuchi & Turner 1991).

4.6 *Gravitational Lensing*

As described in Section 3.6 and illustrated in Figure 9, gravitational lensing frequencies are potentially sensitive indicators of a nonzero Ω_Λ, especially along the fashionable $\Omega_k = 0$ line. This fact, implicit in the lensing statistics analysis of Gott et al (1989), was pointed out explicitly by Fukugita et al (1990a) and by Turner (1990). It is an effect that has the potential for making a decisive test of the possibility of an Ω_Λ-dominated universe. Earlier work on lensing with nonzero Λ values (Paczynski & Gorski 1981, Alcock & Anderson 1986) concentrated on quantities such as image angular separations which are quite insensitive indicators (Fukugita et al 1992) and thus gave little hope for a useful test.

Whether or not currently available data on, and understanding of, gravitational lens statistics yet allows any clear conclusion is a somewhat controversial question. Turner (1990) found that a naive calculation of the expected lensing rates in flat Ω_Λ-dominated models predicted far more lens systems in known quasar samples than have been observed and concluded that the data excluded large Ω_Λ values (with various caveats). Fukugita & Turner (1991) reexamined the issue attempting to take into account more carefully both observational and theoretical uncertainties and concluded that although the strength of the conclusion was weakened, models as Ω_Λ-dominated as model C in Table 1 could only be accommodated by stretching both sorts of uncertainties to their plausible limits (i.e. that it was only marginally allowed). A yet more elaborate treatment by Fukugita et al (1992) reached a similar conclusion.

The principal difficulties in calculating lensing frequencies and comparing the results to observational determinations include: (*a*) characterizing the mass distributions of the low-redshift galaxy population accurately enough to allow a determination of its lensing effectiveness (the critical issues being the space density of galaxies, the distribution of their potential well depths, their mass core radii, and their ellipticities); (*b*) accounting for possible evolution of the galaxy population (note that here one need only consider evolution of the galaxies' mass distributions without regard to any possible luminosity evolution); (*c*) determining the selection biases in specific quasar and lens surveys, particularly those which might cause lens systems to be entirely omitted from the sample (e.g. by the rejection of objects with nonstellar images) or to go unrecognized (e.g. by lack of sufficient resolution to detect the multiple images); and (*d*) adjusting the predictions for the effect of amplification biases—the sometimes strong tendency of lens systems to be preferentially included in flux limited samples due to the boosting of their brightnesses (Turner 1980, Turner et al 1984).

These are a formidable set of complications, which cannot yet be dealt with precisely; however, the uncertainties in accounting for them amount to a factor of 1.5 or perhaps 2, while the differences associated with substantial variations of Ω_Λ are substantially larger, typically an order of magnitude (see Figure 9). Furthermore, the effect of increasing Ω_Λ is to make some of these uncertainties smaller; for example, lensing cross sections become less sensitive to galaxy core radii, and significant galaxy evolution at redshifts that dominate the total lensing integrated probabilities become less astrophysically plausible (because the universe is vacuum rather than mass dominated). It is also important that large Ω_Λ values tend to predict too many lensing events; a prediction of too few events would be far easier to explain away by invoking an otherwise unknown population of lenses or by supposing that physical multiples were being mistaken for lens systems. It is these considerations which give some reason for confidence in the upper limits on Ω_Λ (now typically about 0.9) in $\Omega_k = 0$ cosmologies that have been adduced from available calculations and observations.

On the other hand, Kochanek (1991) and Mao (1991) have emphasized these possible sources of systematic error, and believe that firm conclusions are premature. Since both improved theoretical (numerical) predictions are possible (Kochanek 1991) and since a variety of carefully controlled quasar surveys (in which lensing events may be found with predictable efficiencies) are becoming available or are in progress (Crampton 1991, Hartwick & Schade 1990), rapid progress should be possible for this test. In the end, its value may be limited by our understanding of galaxy properties (i.e. the lens population) and their evolution (Mao 1991), just as for several of the other Λ tests already discussed.

Recently, Kochanek (1992) has suggested a new test of Ω_Λ. He considers the expected lens redshift distribution for systems with given source redshift and image separations (i.e. angular diameters of the lens Einstein ring) and shows that flat, zero Ω_Λ models predict much lower typical lens redshifts than do Ω_Λ-dominated flat models (like model C). Comparing this to the data for the small number of known lens systems for which all of the required data is available, he concludes that the results significantly favor the $\Omega_\Lambda = 0$ model. This technique is extremely promising, although it too needs to be examined for possible systematic problems (e.g. closer lenses are easier to detect and have their redshifts measured more readily) and for possible worries about its sensitivity to details of the lens (galaxy) properties and their evolution.

On balance, it is probably fair to conclude that gravitational lens statistics (of both sorts discussed above) currently offer the biggest empirical challenge to cosmological models with significant Ω_Λ terms, and that they are perhaps the most immediately promising area for further study, both observational and theoretical. However, no conclusions strong enough to deter either theoretical Λ enthusiasts nor the pursuit of other observational tests are yet in hand.

4.7 *Astrophysics of Distant Objects*

One possible test for the cosmological constant has been explored very little. A zero value of Ω_Λ is almost invariably assumed by investigators whose interest is focused not on cosmology per se but on attempts to build detailed physical models of distant cosmic objects. Since the various cosmic distance measures depend on Ω_Λ significantly (see Section 3.3), the physical properties (sizes, velocities, luminosities, etc) of distant objects are influenced by these choices. At least in principle, it is possible that physical models of some such object or class of objects might work (or at least be plausible) for certain values of Ω_Λ and Ω_M and not for others. Possible candidate types of objects include high-redshift radio source lobes, quasars, the gas clouds that produce quasar absorption lines, superluminal motion VLBI sources, and thermal x-ray sources in high-redshift galaxy clusters. One such possibility which has been explored slightly (Malhotra & Turner 1992) is the population properties of quasars which differ significantly in flat Ω_Λ-dominated models from those normally considered (based on zero cosmological constant cosmologies). Of course, our astrophysical understanding of extragalactic objects is not generally so firm (nor scale dependent) that this approach offers hope of easy progress, but it may deserve at least selective exploration.

5. POSSIBLE SOLUTIONS TO THE PHYSICIST'S COSMOLOGICAL CONSTANT PROBLEM

5.1 *Wormholes and the Cosmological Constant*

One of the most provocative explanations for the small value of the cosmological constant invokes quantum cosmology and fluctuations in the topology of spacetime known as "wormholes." Although we will not give a technical description of the relevant arguments, we will try to give a pedagogical introduction to the essential ideas and the troubles with their realization.

Quantum cosmology is the study of the universe as a quantum gravitational system (Wheeler 1968, DeWitt 1967). Since one does not have a consistent quantum theory of gravity, it is common to use approximation schemes based on Feynman's path integral formulation of quantum mechanics (Feynman & Hibbs 1965). In this picture, we compute the wavefunction for a particle with initial state ϕ_0 to be in state ϕ by integrating over all paths that connect the two states:

$$\Psi(\phi) \sim \int [d\rho] e^{iS[\rho]/\hbar}, \qquad 35.$$

where ρ is a path from ϕ_0 to ϕ, and $S[\rho]$ is the action for the path. In quantum cosmology, a "state" is a three-dimensional slice Σ of a four-dimensional spacetime, and the wave function of a particle is replaced by the "wave function of the universe" $\Psi(\Sigma)$, which is the probability amplitude that the universe contains Σ.

Since an oscillating integral such as that of Equation 35 will generally not converge, it is common to analytically continue the time parameter to imaginary values: $t \to i\tau$. This transformation changes the signature of the metric from $(-+++)$ to $(++++)$, so the resulting paths are in a Euclidean space rather than a Lorentzian one. At the same time, the action becomes imaginary, so that we may write $S \to iS_E$, where S_E is called the Euclidean action. The path integral is then damped by a decaying exponential, and will converge if S_E is bounded below. In quantum cosmology, this transformation implies that we should integrate over manifolds of Euclidean signature rather than Lorentzian spacetimes. We therefore compute the wave function of the universe via

$$\Psi(\Sigma) \sim \int [dM] e^{-S_E[M]/\hbar}, \tag{36.}$$

where M is a four-dimensional Euclidean space containing a three-dimensional slice Σ. (We will not discuss the contentious issue of boundary conditions; see Hartle & Hawking 1983; Vilenkin 1982, 1988; Linde 1984).

Observation tells us that our universe is large and smooth on a global scale; therefore, our next step is to estimate the integral in Equation 36 for three-surfaces Σ which are large and smooth. Although performing the integral is well beyond our capabilities, it is possible to estimate it as the exponential of an "effective action" $\Gamma[M]$. The effective action may be thought of as an action with all quantum fluctuations integrated out: $\int [dM] \exp(-S_E[M]/\hbar) = \exp(-\Gamma[M_c]/\hbar)$, where M_c is the "classical" space, for which Γ is stationary. As Coleman (1988b) points out, it may not seem very useful to define a function (Γ) in terms of a path integral over another function (S_E) which we do not know; however, an approximate expression for Γ for large spaces is known. The leading terms are simply those of the (Euclidean) action for general relativity:

$$\Gamma = \frac{1}{16\pi G} \int d^4x \sqrt{g}(2\Lambda - R) + \ldots \tag{37.}$$

where g is the determinant of the metric $g_{\mu\nu}$ and R is the Ricci scalar ($R \equiv g^{\mu\nu} R_{\mu\nu}$). This expression may be roughly thought of as a power series expansion in the inverse size of the space; for large manifolds, gravitation is always dominant, and we may neglect terms representing other fields.

Since Equation 37 is simply the action for general relativity, its stationary point is the solution to Einstein's equations with cosmological constant; in Euclidean space, this is a four-dimensional sphere. For such spaces, $R = 4\Lambda$ and $\int d^4x\sqrt{g} = 24\pi^2/\Lambda^2$. Inserting these into Equation 37 yields $\Gamma = -3\pi/G\Lambda$. Since the path integral in Equation 36 is the exponential of $-\Gamma/\hbar$, we have

$$\Psi \sim e^{3\pi/\hbar G\Lambda}. \tag{38}$$

If we consider Λ as an independent parameter, this expression is infinitely peaked at $\Lambda = 0$; the cosmological constant problem is solved! The answer is simply that universes in which $\Lambda = 0$ dominate the path integral, making it overwhelmingly probable that the cosmological constant vanishes.

However, the cosmological constant may not normally be thought of as a free parameter. Hawking (1984; see also Baum 1984) proposed a field which would contribute to the action in such a way as to mimic a cosmological constant; this field would be varied in the path integral, turning Λ into a free parameter and making Equation 38 the wave function of the universe. However, there is no compelling reason to believe in the existence of such a field (except for solving the cosmological constant problem).

A more natural mechanism for making Λ a free parameter is provided by wormholes—topologically nontrivial spacetime geometries. Roughly, a wormhole may be thought of as a thin tube which connects two separated regions of a Euclidean space (see Figure 11). (The Euclidean wormholes we consider are distinct from wormholes which connect spatial regions in

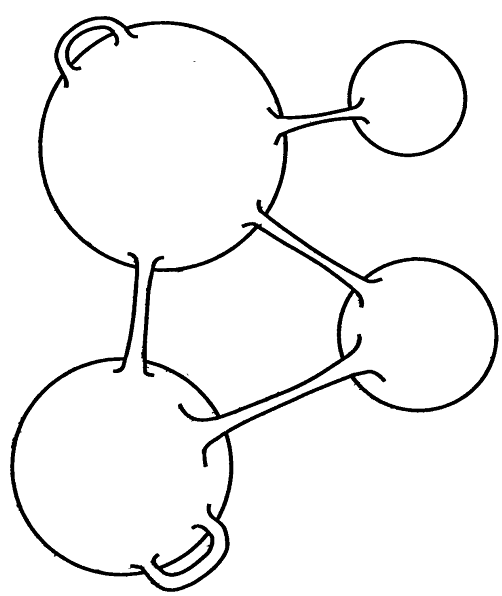

Figure 11 Example of a Euclidean space that contributes to the Feynman path integral for quantum cosmology. This manifold consists of large spheres connected by wormholes.

a Lorentzian geometry; see Wheeler 1964 and Morris et al 1988.) Since the action for an infinitesimally small wormhole is negligible, manifolds consisting of large spheres connected by wormholes are approximate stationary points of the effective action (Equation 37), and therefore contribute to the path integral; the effect of these configurations has been the object of some debate (Hawking 1979, 1982, 1988; Teitelboim 1982; Strominger 1984; Gross 1984; Lavrelashvili et al 1987). A resolution was provided by Coleman (1988a) and Giddings & Strominger (1988), who found that wormholes induced a distribution of values for all the constants of nature—precisely what is necessary to solve the cosmological constant problem. In other words, the interaction of our "universe" with other universes through wormholes allows the cosmological constant to attain a range of values; since the effective action is stationary at $\Lambda = 0$, this value is singled out. (One must not take this concept too literally—the universes being spoken of are fictional Euclidean spaces used to calculate a path integral, not alternate worlds that coexist with our own.)

This is essentially the argument assembled by Coleman in his celebrated paper (Coleman 1988b). (See also Banks 1988; for later variations, see Accetta et al 1989, Adler 1989, Elizalde & Gaztañaga 1990, Hosoya 1989, Kosower 1989, Rubakov 1988, Unruh 1989b, and Veneziano 1989.) One subtlety arises because there are many connected spheres contributing to the path integral; the associated combinatorics makes the wave function of the universe a double exponential, $\Psi \sim \exp(e^{3\pi/\hbar G\Lambda})$; this displays the infinite peak at $\Lambda = 0$ in an even more impressive way. As a solution to the cosmological constant problem, this proposal has at least two very favorable features. First, although highly speculative physics is essential to the argument, there was no need to introduce any new laws or invent new phenomena; all that was necessary was to include in the path integral wormhole configurations which should be there anyway. Second, the communication with other large universes explains how our universe "knows ahead of time" to set $\Lambda = 0$ at low temperature, rather than at early times. In Coleman's phrase, "prearrangement is replaced by precognition" (Coleman 1988b).

At the same time, there are many unanswered questions relating to Coleman's proposal; we will mention just a few. Before looking at wormholes specifically, it is worth noting a long-standing problem of quantum cosmology: The Euclidean action for gravitation is not bounded below, and therefore the path integral of Equation 36 does not converge. Many remedies to this problem have been proposed, including allowing t to vary along a complex contour (Gibbons et al 1978), adding additional terms to the action such that it becomes bounded below (Horowitz 1985), or staying in Lorentzian-signature space all along (Farhi 1989, Strominger 1989).

Unfortunately, the solution to the cosmological constant problem seems to depend intimately on the "wrong" sign for the action (Giddings & Strominger 1989), and attempts to base analogous calculations in Lorentzian space do not find a peak at $\Lambda = 0$ (Fischler et al 1989, Cline 1989).

If we accept for the moment the viability of Euclidean quantum gravity, there is still some question of the reliability of approximating the path integral by large spheres connected by small wormholes. V. Kaplunovsky (unpublished) and Fischler & Susskind (1989) have suggested that large wormholes may dominate small ones; since then a debate has raged back and forth with no clear winner (Preskill 1989; Coleman & Lee 1989, 1990; Polchinski 1989a; Iwazaki 1989). Equally troubling is an argument by Polchinski (1989b) that the integration over spheres connected by wormholes induces a phase $(-i)^{d+2}$ in the wave function, where d is the dimension of spacetime. Thus, in a four-dimensional universe $\Psi \sim \exp(-e^{3\pi/\hbar G\Lambda})$, which exhibits no peak at $\Lambda = 0$. Lastly, several authors have explored a suggestion by Coleman (1988b) that wormholes may determine all of the constants of nature. To date, attempts to implement this plan have not met with great success (Preskill 1989, Hawking 1990, Klebanov et al 1989, Preskill et al 1989).

These results serve to emphasize that quantum cosmology is an ambitious but unsettled subject, insufficiently developed for crucial questions to be definitively answered. The solution to the cosmological constant problem offered by wormholes is certainly elegant as well as provocative; only further work will allow us to judge its physical relevance.

5.2 *Other Explanations*

Although quantum cosmology has attracted significant attention recently, there are many other proposed alternative solutions to the cosmological constant problem. We briefly review several here, noting in advance that while many are provocative, none could be described as compelling. More details and references to many of these proposals may be found in Zee (1985) and Weinberg (1989).

A popular explanation for various unusual coincidences in physics is the anthropic principle, which holds that life (and scientists) will exist only if the laws of physics so allow; therefore, constants of nature must have friendly values. This argument has been applied to the cosmological constant by Banks (1985), Abbott (1985), Brown & Teitelboim (1987), and Linde (1989). An interesting consequence of this argument is that Λ should not be zero, but only small enough for life to exist. Weinberg (1987, 1989) argues that this bound is very close to the observational limits. The possibility that Λ is small for anthropic reasons is therefore of interest to astronomers, since they should then be able to detect a nonvanishing value.

Another suggestion which allows for a nonzero cosmological constant today is to let Λ vary smoothly with time (Freese et al 1987, Özer & Taha 1987, Peebles & Ratra 1988, Ratra & Peebles 1988, Chen & Wu 1990, Abdel-Rahman 1990, Berman 1991, Fujii & Nishioka 1991). (Even in conventional theories, Λ varies rapidly with time during cosmological phase transitions.) The extra degree of freedom introduced allows models to be constructed in which Λ is appreciable, either today or in the early universe. Unfortunately, attempts at constructing a realistic field theory incorporating such features run into difficulty with cosmological nucleosynthesis and observations of cosmic background radiation (Freese et al 1987, Weinberg 1989).

Many authors have proposed the existence of a scalar field which serves to cancel out the cosmological constant (Dolgov 1982, Zee 1985, Ford 1987, Peccei et al 1987, Barr & Hochberg 1988, Solá 1989, Tomboulis 1990). A similar procedure has found theoretical, if not observational, success with the CP-violating parameter of QCD (Peccei & Quinn 1977, Weinberg 1978, Wilczek 1978). Unfortunately, none of these models has proven to be workable. Weinberg (1989) argues that there is a good reason for this failure: The condition that a scalar field relax the cosmological constant to zero will generally overdetermine the field equations, such that no solution can be found without fine-tuning. On the other hand, he notes that this argument relies on technical assumptions which may be in error.

It is natural to wonder, given that the cosmological constant problem involves the overlap of quantum theory with general relativity, whether a solution will eventually be provided by a true quantum theory of gravity. Although such a theory is not available at present, progress has been made in understanding the cosmological constant problem in the context of supersymmetry and superstring theory. In supersymmetry, every boson is associated with a fermion of equal mass. Both bosons and fermions contribute identically to the energy of the vacuum, as given in Equation 6; however, they contribute with opposite signs! Therefore, in the presence of supersymmetry the net contribution of quantum fluctuations to the vacuum energy is zero (Zumino 1975). Unfortunately, we do not observe supersymmetry in the real world—if it exists, it must be spontaneously broken, in which case the vacuum energies of the bosons and fermions will no longer cancel. Nevertheless, the startling cancellation has led many workers to search for supergravity or superstring theories in which the cosmological constant remains zero even after supersymmetry breaking (Christensen et al 1980, Cremmer et al 1983, Witten 1985, Dine et al 1985, Moore 1987, Siopsis 1989). It is probably safe to say that no firm conclusions can be drawn until the theories themselves are better understood.

Other proposals have been made. Linde (1988), in a precursor to the wormhole proposal, suggested a model in which two interacting universes contain particles with energies of opposite sign, in which the effective cosmological constant in each universe vanished. Pagels (1984) proposed a theory of gravitation in which the metric did not enter into the action. Many groups (Zee 1985, Buchmüller & Dragon 1989, Henneaux & Teitelboim 1989, Unruh 1989a) have noted that Λ multiplies $(-g)^{1/2}$ in the action for general relativity; they therefore suggest changing gravity in a way which demands $\sqrt{-g}$ be fixed, so that Λ becomes a Langrangian multiplier. Weinberg (1989) notes that this "does not solve the cosmological constant problem, but it does change it in a suggestive way," while Ng & Dam (1990) maintain that, in the context of quantum cosmology, it does provide a solution. La (1991) has proposed an "elastic vacuum theory," in which the vacuum energy oscillates rapidly but averages to zero. Taylor & Veneziano (1989) propose that quantum gravity corrections (not involving wormholes) can serve to rearrange the vacuum energy to produce a vanishing cosmological constant today. Finally, it has been argued that quantum fluctuations could destabilize a universe dominated by a cosmological constant, although there are many issues still to be resolved (Mottola 1986, Traschen & Hill 1986, Ford 1985, Isaacson & Rogers 1991).

The multiplicity of proposed solutions to the cosmological constant problem is telling—one correct solution would be enough. However, the search for a solution has led in some instances to increased understanding of the relationship between gravitation, field theory, and cosmology. While it is difficult to judge the relative likelihood that any of the above proposals will ultimately succeed, one can predict with confidence that the cosmological constant problem will continue to produce creative, and sometimes interesting, speculations.

6. CONCLUSIONS

The cosmological constant Λ is an idea whose time has come . . . and gone . . . and come . . . and so on. The most recent cycle of interest derives from a mutually supportive combination of aggressive theoretical prejudice and new, suggestive, observations.

Theorists, in aggregate, strongly believe (on the basis of little or no observational evidence) that $\Omega_{\rm tot} = 1$. This belief is not only supported by the Copernican view that the present cosmological epoch should not be special, but is also the firm prediction of inflationary models (which also explain several, other otherwise, mystifying cosmological puzzles). Nucleosynthetic evidence against baryons providing more than $\Omega_M = 0.1$ does not sway this conviction, but only fuels equally fervent belief in nonbaryonic dark matter.

However, the preponderance of evidence against *any* form of dynamical matter able to provide $\Omega_M > 0.2$ or so is a definite embarrassment. Even the tentative evidence of large-scale velocity flows, which may allow $\Omega_M \approx 1$, cannot be warmly embraced by the many theorists who favor the Cold Dark Matter theory of structure formation in its canonical form: CDM with $\Omega_M = 1$ and a constant bias factor does not provide sufficient power on large spatial scales.

Postulating an Ω_Λ-dominated model seems to solve a lot of problems at once. The cosmological constant supplies the "missing matter" to make $\Omega_{tot} = 1$. It modifies CDM to put more (perhaps sufficient) power on large scales, *and* it does so in a way compatible with anisotropy limits on the cosmic microwave background. Simultaneously, it cleans up that old embarrassment: the apparent discrepancy, for larger values of H_0 in its observationally viable range, between the age of the universe and the age of globular clusters.

On the observational side, the new cycle of interest in Λ was for a time supported by evidence of an excess of faint galaxies in B band number vs magnitude counts, and by the realization that previous number vs redshift evidence *against* a significant Ω_Λ (Loh-Spillar) was flawed in its reliance on an overly simple model for galaxy evolution.

Unfortunately, this new evidence has been undermined by near-IR K band counts that show an opposite trend, and by new appreciation of the importance of selection effects.

Furthermore, while arguably convenient, a nonzero Ω_Λ is not really necessary for solving the theorists' problems: $\Omega_M = 1$ in the form of dynamical baryonic plus nonbaryonic dark matter (of unknown character!) is not ruled out, and is perhaps supported by large-scale streaming velocities. Closing the universe with Ω_Λ in fact does not remove the need to postulate nonbaryonic matter, unless one is willing to have the universe be *older* than 30 Gyr *and* have a very low value for H_0 (Figure 10). CDM theory can be fixed by abandoning the assumption (made originally as a matter of convenient simplification, not physical necessity) of a constant bias factor; indeed this may be forced on the theory by new numerical simulations, and by the *COBE* microwave anisotropy measurements.

In terms of ruling *in* a nonzero cosmological constant, the situation now is not too different than it has been in the past. A high value of H_0 (>80 km/s/Mpc, say), combined with no loss of confidence in a value 12–14 Gyr as a *minimum* age for some globular clusters, would effectively prove the existence of a significant Ω_Λ term. Given such observational results, we would know of no convincing alternative hypotheses.

What is most different now from in the past, and what provides hope for breaking the seemingly endless alternation between Λ-fashionability and Λ-rejection, is the existence of a new set of tests—gravitational lens statistics—that have the ability to rule *out* a dominant Ω_Λ contribution. Both the raw number of expected lenses, and also the statistics of their redshifts, are highly sensitive to Ω_Λ as it approaches 1 along the $\Omega_{tot} = 1$ line (Figure 9). While there are formidable complications to be dealt with, there is a good case that, along the $\Omega_{tot} = 1$ line, gravitational lens tests already bound Ω_Λ to be less than 0.9, about the same as the bound from the existence of dynamical matter in amounts $\Omega_M \approx 0.1$. It is possible that bounds to less than 0.5 can be achieved, by which point Λ is rendered uninteresting as a solution for theoretical ills—its "constituency" ought to evaporate.

It will never be possible to rule out a sufficiently small fractional value for Ω_Λ, particularly since the effects of Ω_Λ are smaller in the higher-redshift past than they are today.

The particle theorist who has no prejudice for $\Omega_{tot} = 1$ might want to know current, observationally secure, bounds on Λ. For negative Λ, a bound derives from the minimum age of the universe (Figure 4). Taking $\Omega_M < 1$, $t_0 > 10$ Gyr, and $H_0 > 40$ km/s/Mpc, one gets $H_0 t > 0.40$, $\Omega_\Lambda > -7$, and $\Lambda > -2 \times 10^{-29}$ g/cm^3. For positive Λ, the best bound derives from gravitational lens statistics (Figure 9), although a bound from the simple existence of high-redshift objects would be not much less stringent. Taking $\Omega_M < 1$, one gets $\Omega_\Lambda < 2$; with $H_0 < 100$ km/s/Mpc, one obtains $\Lambda < 4 \times 10^{-29}$ g/cm^3. If these bounds seem broad in cosmological terms, astronomers can nevertheless take satisfaction in bounding Λ to a fractional range of one part in 10^{120} of that allowed by contemporary particle theory, thus making it the most precisely measured constant in all of physics. That same precision convinces most theoretical physicists that Λ must be precisely zero.

Acknowledgments

We thank many colleagues for responding to our requests for information about their work. For helpful conversations and comments on the manuscript, we thank Robert Brandenberger, Sidney Coleman, George Field, Chris Kochanek, Ramesh Narayan, Jerry Ostriker, Ted Pyne, Martin Rees, George Rybicki, and Helmut Zaglauer. This work was supported in part by NSF grant PHY-91-06678 and by NASA contracts NAGW-931 and NAGW-2448. SMC acknowledges the support of an NSF predoctoral fellowship.

Literature Cited

Aaronson, M., Bothun, G. D., Cornell, M. E., Dawe, J. A., Dickens, R.J., et al. 1989. *Ap. J.* 338: 654
Abbott, L. 1985. *Phys. Lett.* B150: 427
Abbott, L. 1988. *Sci. Am.* 258(5): 106
Abdel-Rahman, A.-M. M. 1990. *Gen. Relativ. Gravit.* 22: 655
Accetta, F. S., Chodos, A., Cooper, F., Shao, B. 1989. *Phys. Rev. D* 39: 452
Adler, S. L. 1989. *Phys. Rev. Lett.* 62: 373
Albrecht, A., Steinhardt, P. J. 1982. *Phys. Rev. Lett.* 48: 1220
Alcock, C., Anderson, N. 1986. *Ap. J.* 302: 43
Anders, E. 1963. In *The Moon, Meteorites, and Comets—The Solar System IV*, ed. B. M. Middlehurst, G. P. Kuiper, p. 277. Chicago: Univ. Chicago Press
Arnett, W. D. 1982. *Ap. J.* 254: 1
Arnett, W. D., Branch, D., Wheeler, J. C. 1985. *Nature* 314: 337
Arnould, M., Takahashi, K. 1990. See Vangioni-Flam et al 1990, p. 325
Bahcall, J., Jannuzi, B. T., Schneider, D. P., Hartig, G. F., Bohlin, R., Junkkarinen, V. 1991. *Ap. J. Lett.* 377: L5
Bahcall, S. R., Tremaine, S. 1988. *Ap. J. Lett.* 326: L1
Banks, T. 1985. *Nucl. Phys.* B249: 332
Banks, T. 1988. *Nucl. Phys.* B309: 493
Bardeen, J. M., Bond, J. R., Kaiser, N., Szalay, A. S. 1986. *Ap. J.* 304: 15
Barr, S. M., Hochberg, D. 1988. *Phys. Lett.* B211: 49
Baum, E. 1984. *Phys. Lett.* B133: 185
Berman, M. S. 1991. *Phys. Rev. D* 43: 1075
Birkinshaw, M., Hughes, J. P., Arnaud, K. A. 1991. *Ap. J.* 379: 466
Bond, J. R., Efstathiou, G., Lubin, P. M., Meinhold, P. R. 1991. *Phys. Rev. Lett.* 17: 2179
Börner, G. , Ehlers, J. 1988. *Astron. Astrophys.* 204: 1
Branch, D. 1987. *Ap. J. Lett.* 320: L23
Broadhurst, T. J., Ellis, R. S., Shanks, T. 1988. *MNRAS* 235: 827
Brown, J. D., Teitelboim, C. 1987. *Nucl. Phys.* B297: 787
Buchmüller, W., Dragon, N. 1989. *Phys. Lett.* B223: 313
Burbidge, G. R. 1967. *Ap. J.* 147: 851
Carlberg, R. G. 1991. *Ap. J.* 375: 429
Casimir, H. B. G. 1948. *K. Ned. Akad. Wet., Proc. Sec. Sci.* 51: 793
Cen, R., Ostriker, J. P. 1992. Preprint
Charlton, J. C., Turner, M. S. 1987. *Ap. J.* 313: 495
Chen, W., Wu, Y.-S. 1990. *Phys. Rev. D* 41: 695
Christensen, S. M., Duff, M. J., Gibbons, G. W., Rocek, M. 1980. *Phys. Rev. Lett.* 45: 161
Clayton, D. D. 1988. *MNRAS* 234: 1
Cline, J. M. 1989. *Phys. Lett.* B224: 53
Cohen-Tannoudji, C., Diu, B., Laloë, F. 1977. *Quantum Mechanics.* New York: Wiley
Coleman, S. 1988a. *Nucl. Phys.* B307: 867
Coleman, S. 1988b. *Nucl. Phys.* B310: 643
Coleman, S., Lee, K. 1989. *Phys. Lett.* B221: 242
Coleman, S., Lee, K. 1990. *Nucl. Phys.* B341: 101
Cowan, J. J., Thielemann, F.-K., Truran, J. W. 1991. *Annu. Rev. Astron. Astrophys.* 29: 447
Cowie, L. L. 1991. In *Primordial Nucleosynthesis and Evolution of Early Universe*, ed. K. Sato, J. Audouze, p. 425. Dordrecht: Kluwer
Crampton, D., ed. 1991. *Workshop on the Space Distribution of Quasars.* San Francisco: Astron. Soc. Pac. In press
Cremmer, E., Ferrara, S., Kounnas, C., Nanopoulos, D. V. 1983. *Phys. Lett.* B133: 61
Davis, M., Efstathiou, G., Frenk, C. S., White, S. D. M. 1985. *Ap. J* 292: 371
de Vaucouleurs, G. 1981. *Ann. NY Acad. Sci.* 375: 90
DeWitt, B. S. 1967. *Phys. Rev.* 160: 1113
Dine, M., Rohm, R., Seiberg, N., Witten, E. 1985. *Phys. Lett* B156: 55
Dolgov, A. D. 1982. In *The Very Early Universe: Proceedings of the* 1982 *Nuffield Workshop*, ed. G. W. Gibbons, S. W. Hawking, S. T. C. Siklos. Cambridge: Cambridge Univ. Press
Dressler, A. 1984. *Annu. Rev. Astron. Astrophys.* 22: 185
Durrer, R., Kovner, I. 1990. *Ap. J.* 356: 49
Eastman, R. G., Kirshner, R. P. 1989. *Ap. J.* 347: 771
Efstathiou, G., Sutherland, W. J., Maddox, S. J. 1990. *Nature* 348: 705
Elizalde, E., Gaztañaga, E. 1990. *Phys. Lett.* B234: 265
Farhi, E. 1989. *Phys. Lett.* B219: 403
Felten, J. E., Isaacman, R. 1986. *Rev. Mod. Phys.* 58: 689
Feynman, R. P., Hibbs, A. H. 1965. *Quantum Mechanics and Path Integrals.* New York: McGraw-Hill
Fischler, W., Klebanov, I., Polchinski, J., Susskind, L. 1989. *Nucl. Phys.* B327: 157
Fischler, W., Susskind, L. 1989. *Phys. Lett.* B217: 48
Ford, L. H. 1985. *Phys. Rev. D* 31: 709
Ford, L. H. 1987. *Phys. Rev. D* 35: 2339
Freedman, D. H. 1990. *Discover* 11(7): 46
Freedman, W. L. 1990. *Ap. J. Lett.* 355: L35

Freese, K., Adams, F. C., Frieman, J. A., Mottola, E. 1987. *Nucl. Phys.* B287: 797
Fujii, Y., Nishioka, T. 1991. *Phys. Lett.* B254: 347
Fukugita, M. 1991. In *Primordial Nucleosynthesis and Evolution of Early Universe*, ed. K. Sato, J. Audouze, p. 599. Dordrecht: Kluwer
Fukugita, M., Hogan, C. J. 1990. *Nature* 347: 120
Fukugita, M., Hogan, C. J. 1991. *Ap. J. Lett.* 368: L11
Fukugita, M., Lahav, O. 1991. *MNRAS* 253: 17p
Fukugita, M., Turner, E. L. 1991. *MNRAS* 253: 99
Fukugita, M., Futamase, T., Kasai, M., Turner, E. L. 1992. *Ap. J.* In press
Fukugita, M., Futamase, T., Kasai, M. 1990a. *MNRAS* 246: 24
Fukugita, M., Takahara, F., Yamashita, K., Yoshii, Y. 1990b. *Ap. J. Lett.* 361: L1
Fulling, S. A. 1989. *Aspects of Quantum Field Theory in Curved Space-Time*. Cambridge: Cambridge Univ. Press
Gamow, G. 1970. *My World Line*. New York: Viking. p. 44
Geller, M. J., Huchra, J. P. 1990. *Science* 246: 897
Gibbons, G., Hawking, S. W., Perry, M. 1978. *Nucl. Phys.* B138: 141
Giddings, S. B., Strominger, A. 1988. *Nucl. Phys.* B307: 854
Giddings, S. B., Strominger, A. 1989. *Nucl. Phys.* B321: 481
Glanfield, J. R. 1966. *MNRAS* 131: 271
Gorski, K. M., Silk, J., Vittorio, N. 1992. *Phys. Rev. Lett.* 68: 733
Gott, J. R. 1985. In *IAU Symposium* 117, *Dark Matter in the Universe*, ed. G. Knapp, J. Kormendy. Dordrecht: Reidel
Gott, J. R., Park, M. G., Lee, H. M. 1989. *Ap. J.* 338: 1
Gott, J. R., Rees, M. J. 1987. *MNRAS* 227: 453
Gross, D. 1984. *Nucl. Phys.* B236: 349
Guth, A. H. 1981. *Phys. Rev. D* 23: 347
Hartle, J., Hawking, S. W. 1983. *Phys. Rev. D* 28: 2960
Hartwick, F. D. A., Schade, D. 1990. *Annu. Rev. Astron. Astrophys.* 28: 437
Hawking, S. W. 1979. In *General Relativity: An Einstein Centenary Survey*, ed. S. W. Hawking, W. Israel. Cambridge: Cambridge Univ. Press
Hawking, S. W. 1982. *Commun. Math. Phys.* 87: 395
Hawking, S. W. 1984. *Phys. Lett.* B134: 403
Hawking, S. W. 1988. *Phys. Rev. D* 37: 904
Hawking, S. W. 1990. *Nucl. Phys.* B335: 155
Heath, D. J. 1977. *MNRAS* 179: 351
Henneaux, M., Teitelboim, C. 1989. *Phys. Lett.* B222: 195
Horowitz, G. T. 1985. *Phys. Rev. D* 31: 1169
Hosoya, A. 1989. *Prog. Theor. Phys.* 81: 1248
Ikeuchi, S., Turner, E. L. 1991. *Ap. J. Lett.* 375: 499
Isaacson, J. A., Rogers, B. 1991. *Nucl. Phys.* B364: 381
Iwazaki, A. 1989. *Phys. Lett.* B229: 211
Jacoby, G. H., Ciardullo, R., Ford, H. 1990. *Ap. J.* 356: 332
Klebanov, I., Susskind, L., Banks, T. 1989. *Nucl. Phys.* B317: 665
Kochanek, C. S. 1991. *Ap. J.* 379: 517
Kochanek, C. S. 1992. *Ap. J.* 384: 1
Kofman, L. A., Starobinskii, A. A. 1985. *Sov. Astron. Lett.* 11: 271
Kolb, E. W., Turner, M. S. 1990. *The Early Universe*. Redwood City, CA: Addison-Wesley
Kosower, D. A. 1989. *Mod. Phys. Lett. A* 24: 2323
La, D. 1991. Preprint
Lahav, O., Lilje, P. B., Primack, J. R., Rees, M. J. 1991. *MNRAS* 251: 128
Lavrelashvili, G. V., Rubakov, V. A., Tinyakov, P. G. 1987. *JETP Lett.* 46: 167
Lemaitre, A. 1931. *MNRAS* 91: 483
Lightman, A. P., Press, W. H., Price, R. H., Teukolsky, S. A. 1975. *Problem Book in Relativity and Gravitation*. Princeton: Princeton Univ. Press
Lightman, A. P., Schechter, P. L 1990. *Ap. J. Suppl. Ser.* 74: 831
Lilje, P. B. 1992. *Ap. J. Lett.* 386: L33
Linde, A. D. 1982. *Phys. Lett.* B108: 389
Linde, A. D. 1984. *Sov. Phys. JETP* 60: 211
Linde, A. D. 1988. *Phys. Lett.* B200: 272
Linde, A. D. 1989. *Phys. Lett.* B227: 352
Linde, A. D. 1990. *Particle Physics and Inflationary Cosmology*. New York: Gordon & Breach
Loh, E. D. 1986. *Phys. Rev. Lett.* 57: 2865
Loh, E. D., Spillar, E. J. 1986. *Ap. J. Lett.* 307: L1
Maddox, S. J., Sutherland, W. J., Efstathiou, G., Loveday, J., Peterson, B. A. 1990b. *MNRAS* 247: 1p
Maddox, S. J., Efstathiou, G., Sutherland, W. J., Loveday, J. 1990a. *MNRAS* 242: 43p
Madore, B. F., Freedman, W. L. 1991. *Publ. Astron. Soc. Pac.* 103: 933
Malhotra, S., Turner, E. L. 1992. Preprint
Mandl, F., Shaw, G. 1984. *Quantum Field Theory*. New York: Wiley
Mao, S. 1991. *Ap. J.* 380: 9
Martel, H. 1991. *Ap. J.* 366: 353
Martel, H., Wasserman, I. 1990. *Ap. J.* 348: 1
Misner, C., Thorne, K. S., Wheeler, J. A. 1973. *Gravitation*. New York: Freeman
Moore, G. 1987. *Nucl. Phys.* B293: 139

Morris, M. S., Thorne, K. S., Yurtsever, U. 1988. *Phys. Rev. Lett.* 61: 1446
Morris, S. L., Weymann, R. J., Savage, B. D., Gilliland, R. L. 1991. *Ap. J. Lett.* 377: L21
Mottola, E. 1986. *Phys. Rev. D* 33: 2136
Narayan, R. 1991. *Ap. J. Lett.* 378: L5
Narlikar, J. V., Padmanabhan, T. 1991. *Annu. Rev. Astron. Astrophys.* 29: 325
Ng, Y. J., van Dam, H. 1990. *Phys. Rev. Lett.* 65: 1972
Olive, K. A., Schramm, D. N., Steigman, G., Walker, T. P. 1990. *Phys. Lett. B* 236: 454
Ostriker, J. P. 1980. *Comments Astrophys.* VIII: 177
Ozer, M., Taha, M. O. 1987. *Nucl. Phys.* B287: 776
Paczynski, B., Gorski, K. 1981. *Ap. J. Lett.* 248: L101
Pagel, B. 1990. See Vangioni-Flam et al. 1990, p. 493
Pagels, H. 1984. *Phys. Rev. D* 29: 1960
Peacock, J. 1991. *Nature* 352: 378
Peccei, R. D., Quinn, H. 1977. *Phys. Rev. D* 16: 1791
Peccei, R. D., Solá, J., Wetterlich, C. 1987. *Phys. Lett.* B195: 183
Peebles, P. J. E. 1980. *The Large Scale Structure of the Universe*. Princeton: Princeton Univ. Press
Peebles, P. J. E. 1984. *Ap. J.* 284: 439
Peebles, P. J. E. 1986. *Nature* 321: 27
Peebles, P. J. E. 1988. *Publ. Astron. Soc. Pac.* 100: 670
Peebles, P. J. E., Ratra, B. 1988. *Ap. J.* 325: L17
Peebles, P. J. E., Schramm, D. N., Turner, E. L., Kron, R. G. 1991. *Nature* 352: 769
Peebles, P. J. E., Silk, J. 1990. *Nature* 346: 233
Petrosian, V., Salpeter, E., Szekeres, P. 1967. *Ap. J.* 147: 1222
Philip, A. G. D., ed. 1988. *Calibration of Stellar Ages*. Schenectady: L. Davis
Polchinski, J. 1989a. *Nucl. Phys.* B325: 619
Polchinski, J. 1989b. *Phys. Lett.* B219: 251
Preskill, J. 1989. *Nucl. Phys.* B323: 141
Preskill, J., Trivedi, S. P., Wise, M. B. 1989. *Phys. Lett.* B223: 26
Press, W. H., Rybicki, G. B., Hewitt, J. N. 1992. *Ap. J.* 385: 416
Ratra, B., Peebles, P. J. E. 1988. *Phys. Rev. D* 37: 3406
Refsdal, S., Stabell, R., deLange, F. G. 1967. *MNRAS* 71: 143
Richstone, D. O., Loeb, A., Turner, E. L. 1992. *Ap. J.* In press
Roberts, D. H., Lehar, J., Hewitt, J. N., Burke, B. F. 1991. *Nature* 352: 43
Rood, R. T. 1990. See Vangioni-Flam et al 1990, p. 313
Rowan-Robinson, M. 1968. *MNRAS* 141: 455
Rubakov, V. A. 1988. *Phys. Lett.* B214: 503
Sahni, V., Feldman, H., Stebbins, A. 1992. *Ap. J.* In press
Sandage, A. 1961a. *Ap. J.* 133: 355
Sandage, A. 1961b. *Ap. J.* 134: 916
Sandage, A. 1987. In *Observational Cosmology*, ed. A. Hewitt, G. Burbidge, L. Z. Fang, p. 1. Dordrecht: Reidel
Sandage, A. 1988a. *Annu. Rev. Astron. Astrophys.* 26: 561
Sandage, A. 1988b. *Ap. J.* 331: 583
Sandage, A. 1988c. *Ap. J.* 331: 605
Sandage, A. 1990. *Ap. J.* 350: 631
Sandage, A., Cacciari, C. 1990. *Ap. J.* 350: 645
Schneider, D. P., Schmidt, M., Gunn, J. E. 1991. *Astron. J.* 102: 837
Schramm, D. N. 1990. See Vangioni-Flam et al. 1990. p. 365
Schramm, D. N., Wasserburg, G. J. 1970. *Ap. J.* 162: 57
Shkolvsky, J. 1967. *Ap. J. Lett.* 150: L1
Siopsis, G. 1989. *Phys. Lett.* 233: 301
Solá, J. 1989. *Phys. Lett.* B228: 317
Sparnaay, M. J. 1957. *Nature* 180: 334
Strominger, A. 1984. *Phys. Rev. Lett.* 52: 1733
Strominger, A. 1989. *Nucl. Phys.* B319: 722
Sugiyama, N., Gouda, N., Sasaki, M. 1990. *Ap. J.* 365: 432
Tabor, D., Winterton, R. H. S. 1969. *Proc. R. Soc. London Ser. A* 312: 435
Tammann, G. A. 1987. In *Observational Cosmology*, ed. A. Hewitt, G. Burbidge, L. Z. Fang, p. 151. Dordrecht: Reidel
Taylor, T. R., Veneziano, G. 1989. *Phys. Lett.* B228: 311
Teitelboim, C. 1982. *Phys. Rev. D* 25: 3159
Tomboulis, E. T. 1990. *Nucl. Phys.* B329: 410
Tonry, J. L. 1991. *Ap. J. Lett.* 373: L1
Toth, G., Ostriker, J. P. 1992. *Ap. J.* In press
Traschen, J., Hill, C. 1986. *Phys. Rev. D* 33: 3519
Trimble, V. 1987. *Annu. Rev. Astron. Astrophys.* 25: 425
Turner, E. L. 1980. *Ap. J. Lett.* 242: L13
Turner, E. L. 1990. *Ap. J. Lett.* 365: L43
Turner, E. L. 1991. *Astron. J.* 101: 5
Turner, E. L., Ikeuchi, S. 1992. *Ap. J.* In press
Turner, E. L., Ostriker, J. P., Gott, J. R. 1984. *Ap. J.* 284: 1
Turner, M. S. 1991. In *Proc. Nobel Symposium No. 79: The Birth and Early Evolution of Our Universe*, ed. B. Gustafsson, Y. Nilsson, B. Skagerstam. Singapore: World Scientific
Tyson, J. A. 1988. *Astron. J.* 96: 1
Tytler, D. 1981. *Nature* 291: 289
Unruh, W. G. 1989a. *Phys. Rev. D* 40: 1048
Unruh, W. G. 1989b. *Phys. Rev. D* 40: 1053

VandenBerg, D. A. 1990. See Vangioni-Flam et al. 1990, p. 241
VandenBerg, D. A., Smith, G. H. 1988. *Publ. Astron. Soc. Pac.* 100: 314
Vangioni-Flam, E. , Casse, M., Audouze, J., Tran Thanh Van, T., eds. 1990. *Astrophysical Ages and Dating Methods*. Gif sur Yvette: Editions Frontieres. 507 pp.
Veneziano, G. 1989. *Mod. Phys. Lett. A* 4: 695
Vilenkin, A. 1982. *Phys. Lett.* B117: 25
Vilenkin, A. 1988. *Phys. Rev. D* 37: 888
Vittorio, N., Silk, J. 1985. *Ap. J. Lett.* 297: L1
Vittorio, N., Meinhold, P., Muciaccia, P. F., Lubin, P., Silk, J. 1991. *Ap. J. Lett.* 372: L1
Walker, T. P., Steigman, G., Schramm, D. N., Olive, K. A., Kang, H.-S. 1991. *Ap. J.* 376: 51
Weinberg, S. 1972. *Gravitation and Cosmology*. New York: Wiley
Weinberg, S. 1978. *Phys. Rev. Lett.* 40: 223
Weinberg, S. 1987. *Phys. Rev. Lett.* 59: 2607
Weinberg, S. 1989. *Rev. Mod. Phys.* 61: 1
Wheeler, J. A. 1964. In *Relativity, Groups and Topology*, ed. B. S. DeWitt, C. M. DeWitt, p. 317. New York: Gordon & Breach
Wheeler, J. A. 1968. In *Batelle Rencontres*, ed. C. M. DeWitt, J. A. Wheeler, p. 242. New York: Benjamin
Wilczek, F. 1978. *Phys. Rev. Lett.* 40: 279
Witten, E. 1985. *Phys. Lett.* B155: 151
Yoshii, Y., Takahara, F. 1989. *Ap. J.* 346: 28
Zee, A. 1985. In *High Energy Physics: Proceedings of the 20th Annual Orbis Scientae*, 1983, ed. S. L. Mintz, A. Perlmutter, p. 211. New York: Plenum
Zeldovich, Ya. B. 1967. *JETP Lett.* 6: 316
Zumino, B. 1975. *Nucl. Phys.* B89: 535