

# Linear Systems

## 1 Basics

Consider the system of,

$$y_1 = 2x_1 + x_2 \quad (1)$$

$$y_2 = 2x_2 \quad (2)$$

This can be written as,

$$y = Ax \quad (3)$$

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad (4)$$

## 2 Differential Equations

Consider the system of,

$$\dot{x} = y \quad (5)$$

$$\dot{y} = -x \quad (6)$$

which as above is (writing  $x$  as a vector now, where  $x_1 = x$  and  $x_2 = y$  above),

$$\dot{x} = Ax \quad (7)$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (8)$$

The solution to differential equations of the form  $\dot{x} = Ax$  is  $x = Ce^{At}$  where  $C$  is some constant. This is true even when  $A$  is a matrix, though in this case  $C$  becomes a vector. The value of this exponent is found from the sum of an infinite power series.

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + At + \frac{(At)^2}{2} + \dots \quad (9)$$

This is not a special case for the exponentiation of a matrix, but just an extension of the definition of the exponential function.

$$e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!} = 1 + a + \frac{a^2}{2} + \frac{a^3}{6} + \dots \quad (10)$$

for the case of  $A$  above,  $e^{At}$  evaluates to,

$$\begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \quad (11)$$

and so we have

$$x = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \quad (12)$$

How do we choose  $C$ ? For that we need the initial conditions. Let's say that,

$$x(0) \equiv x_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad (13)$$

This gives us the set of equations at  $t = 0$ ,

$$x_1 = C_1 \cos(0) + C_2 \sin(0) = 1 \quad (14)$$

$$x_2 = C_2 \cos(0) - C_1 \sin(0) = 2 \quad (15)$$

which we can solve for  $C_1 = 1$ ,  $C_2 = 2$ . So,

$$x = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad (16)$$

This was a bit easier than the totally general case as the eigenvectors of  $E^{At}$  at  $t = 0$  are along the axes.