Linear Systems

1 Basics

Consider the system of,

$$y_1 = 2x_1 + x_2 \tag{1}$$
$$y_2 = 2x_2 \tag{2}$$

This can be written as,

$$y = Ax \tag{3}$$
$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \tag{4}$$

2 Differential Equations

Consider the system of,

$$\dot{x} = y \tag{5}$$
$$\dot{y} = -x \tag{6}$$

which as above is (writing x as a vector now, where $x_1 = x$ and $x_2 = y$ above),

$$\dot{x} = Ax \tag{7}$$
$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{8}$$

The solution to differential equations of the form $\dot{x} = Ax$ is $x = Ce^{At}$ where C is some constant. This is true even when A is a matrix, though in this case C becomes a vector. The value of this exponent is found from the sum of an infinite power series.

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + At + \frac{(At)^2}{2} + \dots \tag{9}$$

This is not a special case for the exponentiation of a matrix, but just an extension of the definition of the exponential function.

$$e^{a} = \sum_{k=0}^{\infty} \frac{a^k}{k!} = 1 + a + \frac{a^2}{2} + \frac{a^3}{6} + \dots \tag{10}$$

for the case of A above, e^{At} evaluates to,

$$\begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \tag{11}$$

and so we have

$$x = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \tag{12}$$

How do we choose C? For that we need the initial conditions. Let's say that,

$$x(0) \equiv x_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{13}$$

This gives us the set of equations,

$$x_1 = C_1 \cos(t) + C_2 \sin(t) = 1 \tag{14}$$

$$x_2 = C_2 \cos(t) - C_1 \sin(t) = 2 \tag{15}$$

which at $t = 0$ gives us, $C_1 = 1$, $C_2 = 2$. So,

$$x = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{16}$$