Astro 212: Dynamical Astronomy
Optional Problem Set

Numerical Integration Methods.[1] You will implement three variants of a simple numerical integrator for the N-body problem and test your implementations using the masses, positions, and velocities of the Sun and planets in our solar system. A file containing the masses, position vectors, and velocity vectors of the Sun and planets at a specified time is on the class website.

1. The Lagrangian of a dynamical system is given by

$$\mathcal{L} = T - V \ , \tag{1}$$

where T is the kinetic energy of the system and V is the potential energy of the system. Write down the Lagrangian for the N-body problem. Let m_i be the mass of the i-th body, and let $\vec{r}_i = (x_i, y_i, z_i)$ and $\dot{\vec{r}}_i = (\dot{x}_i, \dot{y}_i, \dot{z}_i)$ be the position and velocity vectors of that body in an inertial frame with the origin located at some arbitrary position. Remember, there is one Lagrangian for the system of masses, rather than n different Lagrangians. Please use the notation $r_{ij} = |\vec{r}_i - \vec{r}_j| = [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2]^{1/2}$. Use the Lagrangian to derive the second-order equations of motion, and show that the equation of motion of the i-th body is:

$$\ddot{\vec{r}}_i = -\sum_{j \neq i} \frac{G m_j}{r_{ij}^3} (\vec{r}_i - \vec{r}_j) \ , \tag{2}$$

2. The equations of motion, as written above, are a second-order system of ordinary differential equations. Rewrite them as a first-order system of equations, using the velocity vectors, $\vec{v}_i = \dot{\vec{r}}_i$, as intermediate variables:

$$\dot{\vec{r}}_i = \vec{v}_i \tag{3}$$

$$\dot{\vec{v}}_i = -\sum_{j \neq i} \frac{G m_j}{r_{ij}^3} (\vec{r}_i - \vec{r}_j) \ , \tag{4}$$

for a total of six equations for each body. Then implement a first-order numerical integrator for the resulting system of equations.

$$\vec{r}_i' = \vec{r}_i + \dot{\vec{r}}_i \Delta t \tag{5}$$

$$\vec{v}_i' = \vec{v}_i + \dot{\vec{v}}_i \Delta t \ , \tag{6}$$

where the step size Δt and the $\mathcal{O}(\Delta t^2)$ terms have been ignored. There is a set of these equations for each body. The primed variables represent the dynamical state at the next time step. This integration technique is called Eulers method.

3. Explore the conservation of the center of mass and the total energy as a function of time, for different values of the step size (try step sizes of 0.1 day, 0.2 day, 0.4 day, 0.8 day, ...). Plot the relative energy error, $|(E - E_0)/E_0|$, as a function of time. If you include only the Sun and the Earth, does the orbit close? What is the period? How small does the time step need to be for the orbit to close?

[1]This problem set comes from Matt Holman, one of the inventors of the widely-used Wisdom-Holman symplectic integration algorithm.

4. Returning to the Lagrangian, determine the components of canonical momentum for the i-th mass: $p_{x_i} = \partial\mathcal{L}/\partial\dot{x}_i$, $p_{y_i} = \partial\mathcal{L}/\partial\dot{y}_i$, $p_{z_i} = \partial\mathcal{L}/\partial\dot{z}_i$. Then, using the Lagrangian and the canonical momenta, derive the Hamiltonian for the N-body problem,

$$H = \sum_i (\dot{x}_i p_{x_i} + \dot{y}_i p_{y_i} + \dot{z}_i p_{z_i}) - \mathcal{L} \tag{7}$$

rewritten in terms of coordinates and canonical momenta only. Use the Hamiltonian and Hamiltons equations to derive the first-order equations of motion.

5. Notice that if the Hamiltonian above is written

$$H = T + V, \tag{8}$$

where T contains the kinetic energy and V contains the potential energy terms, T de- pends only upon the momenta and V depends only on the coordinates. Lets introduce a "mapping Hamiltonian"

$$H_{map} = T + 2\pi\delta_{2\pi}(nt)V, \tag{9}$$

where $2\pi\delta_{2\pi}(nt)$ is a periodic sequence of delta functions with period $\Delta t = 2\pi/n$, such that

$$\int_{-\epsilon/2}^{\epsilon/2} 2\pi\delta_{2\pi}(nt)dt = \frac{2\pi}{n} = \Delta t, \tag{10}$$

for arbitrarily small ϵ. The time average of $2\pi\delta_{2\pi}(nt)$ is unity. Use the Fourier expansion of the delta function to show that

$$\delta_{2\pi}(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \cos(nt). \tag{11}$$

H_{map} differs from H only by a series of n-dependent terms, that are presumably of high frequency. Derive the equations of motion for H_{map}. Note that between the delta functions, only the terms in T are non-zero. When crossing a delta function, the V terms dominate. Suppose you start with the coordinates and momenta at a time $t = \epsilon/2$, just after a delta function. Show that the time evolution of the Hamiltonian equations of motion up to and across the next delta function yields

$$x_i' = x_i + \left.\frac{\partial T}{\partial p_{x_i}}\right|_{p_{x_i}} \Delta t \tag{12}$$

$$p_{x_i}' = p_{x_i} - \left.\frac{\partial V}{\partial x_i}\right|_{x_i'} \Delta t. \tag{13}$$

This is the essence of a first-order "symplectic T+V" integrator for the N-body problem. Implement such an integrator. Unlike Eulers method, this symplectic integrator advances the dynamical state in two steps. First the positions are updated. Then the velocities are updated, based on the revised positions. Notice that this integrator is an exact solution to a somewhat different Hamiltonian, rather than an approximate solution to the actual Hamiltonian.

6. Evaluate the first-order symplectic integrator using the same tests as before. Do you see any long-term trend in the energy or center of mass?

7. A second-order "symplectic $T{+}V$" integrator for the N-body problem can be derived by a simple change
$$H_{map} = T + 2\pi\delta_{2\pi}(nt - \pi)V, \tag{14}$$
which shifts the phase of the delta functions. Starting at time $t = 0$, the dynamical state is now advanced in three steps: a half step of T, a full step of V, and another half step of T. Write down and implement the equations for the time evolution of the coordinates and momenta for one cycle of the mapping Hamiltonian. These are the equations for the "leap frog" integrator. Notice that if you were outputting the dynamical state infrequently, the third step can be combined with the next first step.

8. Evaluate the second-order symplectic integrator using the same tests as before. Do you see any long-term trend in the energy or center of mass? Compare the step sizes required by this leap frog integrator and Eulers method from part 1 to achieve similar accuracy.