Econ 139: Lecture 10

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CONTENT

1.Investment in risky assets

$$\max_a E[u(w_0(1+r_f)+a(\tilde{r}-r_f))$$
$$F.O.C \quad E[u'(w_0(1+r_f)+a^*(\tilde{r}-r_f))(\tilde{r}-r_f)]=0$$
$$S.O.C \quad E[u''(w_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)^2]<0$$

$$Let \quad g(a)=E[u(w_0(1+r_f)+a(\tilde{r}-r_f))]$$
$$g'(a)=E[u'(w_0(1+r_f)+a^*(\tilde{r}-r_f))(\tilde{r}-r_f)]$$
$$g''(a)=E[u''(w_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)^2]<0$$
$$\Rightarrow$$
$$g'(a^*)=0 \, and \, g'(a) \, strictly \, decrease$$
$$g'(0)=E[u'(w_0(1+r_f))(\tilde{r}-r_f)]=u'(w_0(1+r_f))(E[\tilde{r}]-r_f)$$
$$g'(a) \, has \, the \, same \, sign \, as \, E[\tilde{r}]-r_f$$

Theorem:

$$Let \, a^*=\arg\max_a E[u(\tilde{w}_1)] \, and \, assume \, u'>0, \; u''<0$$
$$(1) \; a^*>0 \, iff \, E[\tilde{r}]>r_f$$
$$(2) \; a^*=0 \, iff \, E[\tilde{r}]=r_f$$
$$(3) \; a^*<0 \, iff \, E[\tilde{r}]<r_f$$

(2) WTS: $a^*=0 \;\, iff \;\, E[\tilde{r}]=r_f$

$(\Rightarrow)$ Suppose $\mathrm{a}^*=0$

know $g'(a^*)=0$ and $a^*=0$ (by supposition)

which implies $g'(0)=0$, further implies $E[\tilde{r}]-r_f=0$

$(\Leftarrow) Suppose E[\tilde{r}] = r_f$

E[]-r$_f = 0$ implies $g'(0) = 0$

since $g'(a)$ *strictly decrease and* $g'(a^*) = 0$, this implies $a^* = 0$

(1) WTS: $a^* > 0$ iff $E[\tilde{r}] > r_f$

$(\Rightarrow)$ *Suppose* a$^* > 0$

know $g'(a^*) = 0$ and $a^* > 0$ (by supposition)

which implies $g'(0) > 0$, further implies $E[\tilde{r}] - r_f > 0$

$(\Leftarrow) Suppose E[\tilde{r}] > r_f$

E[]-r$_f > 0$ implies $g'(0) > 0$

since $g'(a)$ strictly decrease and $g'(a^*) = 0$, this implies $a^* > 0$

(3) Likewise, $a^* < 0\, iff\; E[\tilde{r}] < r_f$

2.Example of logarithm utility function

Example:

Suppose u(w)=ln(w), so that u'(w)=$\frac{1}{w}$,

Stock with two possible returns:

$$\tilde{r} = \begin{cases} r_G & w.p. & \pi \\ r_B & w.p. & 1-\pi \end{cases} \qquad (r_G > r_f > r_B)$$

$E[\tilde{r}] = \pi r_G + (1-\pi) r_B > r_f$

Then we know:

$$W_1^G(a) = w_0(1 + r_f) + a(r_G - r_f)$$
$$W_1^B(a) = w_0(1 + r_f) + a(r_B - r_f)$$

Now we need to solve:

$$\max_a \; \pi \ln(W_1^G(a)) + (1-\pi)\ln(W_1^B(a))$$

FOC:

$$\frac{\pi(r_G - r_f)}{W_1^G(a^*)} + \frac{(1-\pi)(r_B - r_f)}{W_1^B(a^*)} = 0$$

After some simplifications, we get:

$$a^* = W_0[\frac{(1+r_f)(E[\tilde{r}] - r_f)}{(r_G - r_f)(r_f - r_B)}] > 0$$

Observations:

(i) a^* increases proportionally with w_0.

(ii) $\frac{a^*}{W_0}$ increases as $E[\tilde{r} - r_f]$ increases.

(iii) $\frac{a^*}{W_0}$ decreaces as r_f and r_B move away from r_f while maintaining $E[\tilde{r}]$.

r_f	r_G	r_B	π	$E[\tilde{r}]$	$\frac{a^*}{W_0}$
0.05	0.4	-0.2	0.5	0.1	0.6
0.05	0.3	-0.1	0.5	0.1	1.4
0.05	0.4	-0.2	0.z	0.2	2.4

3.Risk Aversion and Savings Behavior

Consider two investors, 1 and 2, both looking to invest to optimize their utilities. Assume the following situational constraints:

(i) Suppose for all wealth levels, $R_A^1(W) > R_A^2(W)$. Investor 1 is more risk averse always. Following from this assumption we know that $\alpha_1^*(W) < \alpha_2^*(W)$

(ii) Suppose for all wealth levels that $R_R^1(W) > R_R^2(W)$. Following from this assumption we know that $\alpha_1^*(W) < \alpha_2^*(W)$

We can Test this by generalizing u(W) to

$$\frac{W^{1-\gamma}}{1\gamma}$$

$$\gamma > 0, \gamma \neq 1$$

In this case we can rewrite

$$R_A(W) = -\frac{-\gamma W^{-\gamma-1}}{W^{-\gamma}} = \gamma W^{-1} = \frac{\gamma}{W} = R_A$$

$$R_R(W) = \frac{\gamma}{W} * W = \gamma$$

This means that γ is the constant relevant risk aversion. Now let's examine the implications on the safe and risky assets as the previous example.

$$\max_{\alpha} \pi \frac{[W_1^g(\alpha)]^{1-\gamma}}{1-\gamma} + (1-\pi)\frac{[W_1^b(\alpha)]^{1-\gamma}}{1-\gamma}$$

This has the First Order Condition:

$$\frac{\pi(r_g - r_f)}{W_1^g(\alpha^*)^\gamma} + \frac{(1-\pi)(r_b - r_f)}{W_1^g(\alpha^*)^\gamma} = 0$$

We can solve for α^* by plugging in, but the end solution is:

$$\frac{\alpha^*}{W_0} = \frac{(1+r_f)[(\pi(r_g - r_f))^{\frac{1}{\gamma}} - ((1-\pi)(r_b - r_f))^{\frac{1}{\gamma}}]}{(r_g - r_f)[(1-\pi)(r_f - r_b)]^{\frac{1}{\gamma}} + (r_f - r_b) - [\pi(r_g - r_f)]^{\frac{1}{\gamma}}}$$

We will look at examples setting $r_f = 0.05$, $r_g = 0.4$, $r_b = -0.1$, $\pi = 0.5$ and $\mathbb{E}[\tilde{r}] = 0.1$. As γ increases, there is more risk aversion and thus $\frac{\alpha^*}{W_0}$ decreases. See the following table of example results for varying γ:

γ	0.5	1	2	3	5	10
$\frac{\alpha^*}{W_0}$	1.2	0.6	0.3	0.2	0.1	0.06

Theorem

Let a*(ω) = arg max E[u(ω_1)]

(i) If $R^1{}_A(\omega) < 0$, then $\frac{d}{d\omega 0}\left(a^*(\omega_0)\right) > 0$

(ii) If $R^1{}_A(\omega) = 0$, then $\frac{d}{d\omega 0}\left(a^*(\omega_0)\right) = 0$

(iii) If $R^1{}_A(\omega) > 0$, then $\frac{d}{d\omega 0}\left(a^*(\omega_0)\right) < 0$

Case I: declining absolute risk aversion (DARA)

- Utility functions in this case:
  - $U(\omega) = \frac{\omega^{1-\gamma}}{1-\gamma}, \gamma > 0, \gamma \neq 0$
  - $U(\omega) = Ln(\omega)$
- We think this is the "normal" case.

Case II: constant absolute risk aversion (CARA)

- Amount invested in risky asset is unaffected by wealth.
- Utility function in this case:
  - $U(\omega) = -\mathrm{e}^{-\gamma*\omega}$

    Therefore, $R_A(\omega) = \gamma$

$\max\limits_{a} E[-\mathrm{e}^{-\gamma*\tilde{\omega}_1(a^*)}]$ where $\tilde{\omega}_1(a^*) = \omega_0\left(1+r_f\right) + a(\tilde{r} - r_f)$

FOC: $E\left[\gamma\mathrm{e}^{-\gamma*\tilde{\omega}_1(a^*)} * \left(\tilde{r} - r_f\right)\right] = 0$

Implicit Function Theorem

$If\ f(x,y) = 0$

$then, \frac{dx}{dy} = -\frac{f_y}{f_x}$

$$\frac{\partial a^*}{\partial \omega_0} = -\frac{E\left[-\gamma^2\left(\tilde{r} - r_f\right)\left(1+r_f\right)\mathrm{e}^{-\gamma*\tilde{\omega}_1(a^*)}\right]}{E\left[-\gamma^2\left(\tilde{r} - r_f\right)^2\mathrm{e}^{-\gamma*\tilde{\omega}_1(a^*)}\right]}$$

$$\frac{\partial a^*}{\partial \omega_0} = -\frac{\gamma\left(1+r_f\right) E\left[-\gamma\left(\tilde{r} - r_f\right)\mathrm{e}^{-\gamma*\tilde{\omega}_1(a^*)}\right]}{E\left[-\gamma^2\left(\tilde{r} - r_f\right)^2\mathrm{e}^{-\gamma*\tilde{\omega}_1(a^*)}\right]}$$

$$\frac{\partial a^*}{\partial \omega_0} = -\frac{\gamma\left(1+r_f\right) E\left[-\gamma\mathrm{e}^{-\gamma*\tilde{\omega}_1(a^*)} * \left(\tilde{r} - r_f\right)\right]}{E\left[-\gamma^2\left(\tilde{r} - r_f\right)^2\mathrm{e}^{-\gamma*\tilde{\omega}_1(a^*)}\right]}$$

$$\frac{\partial a^*}{\partial \omega_0} = -\frac{\gamma\left(1+r_f\right) * \ FOC}{E\left[-\gamma^2\left(\tilde{r} - r_f\right)^2\mathrm{e}^{-\gamma*\tilde{\omega}_1(a^*)}\right]}$$

$$\frac{\partial a^*}{\partial \omega_0} = -\frac{\gamma\left(1+r_f\right) * 0}{E\left[-\gamma^2\left(\tilde{r} - r_f\right)^2\mathrm{e}^{-\gamma*\tilde{\omega}_1(a^*)}\right]}$$

$$\frac{\partial a^*}{\partial \omega_0} = 0$$