Y_0 —— Y_1

C_0 C_1

Time Horizon
(1m / 1yr / 5-10yr)

金融经济学

第二讲(B)
Consumer Choice in the Risk Dimension

张宇

Finance: transfer risk

Consumer Optimization: The Risk Dimension

In the 1950s and 1960s, Kenneth Arrow (US, 1921-2017, Nobel Prize 1972) and Gerard Debreu (France, 1921-2004, Nobel Prize 1983) extended consumer theory to accommodate risk and uncertainty.

To do so, they drew on earlier ideas developed by others, but added important insights of their own.

Building Blocks of Arrow-Debreu Theory

binary pricing

1. Fisher's (1930) intertemporal model of consumer decision-making.
2. From probability theory: uncertainty described with reference to "states of the world." (Andrey Kolmogorov, 1930s).
   概率论的语言（测度理论）
3. Expected utility theory (John von Neumann and Oskar Morgenstern, 1947).
4. Contingent claims – stylized financial assets – a powerful analytic device of their own invention.

Consumer Optimization: The Risk Dimension

To be more specific about the source of risk, let's suppose that there are two possible outcomes for income next year, good and bad:

$$Y_0 = \text{income today}$$
$$Y_1^G = \text{income next year in the "good" state}$$
$$Y_1^B = \text{income next year in the "bad" state}$$

where the assumption $Y_1^G > Y_1^B$ makes the "good" state good and where

$$\pi = \text{probability of the good state}$$
$$1 - \pi = \text{probability of the bad state}$$

Consumer Optimization: The Risk Dimension

把世界分为三个点：

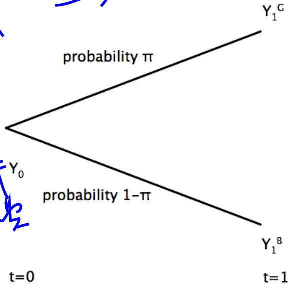
Y_0, Y_1^G, Y_1^B

Good　Bad

⇒ 用于具体的定价

选择 c_0, c_1^G, c_1^B

可以根据未来的确定情况



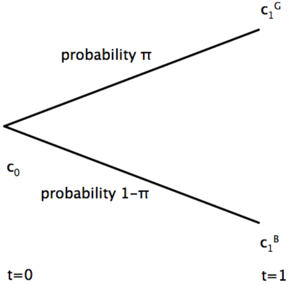
An event tree highlights randomness in income as the source of risk.

Consumer Optimization: The Risk Dimension

Arrow and Debreu used the probabilistic idea of states of the world to extend Irving Fisher's work, recognizing that under these circumstances, the consumer chooses between three goods:

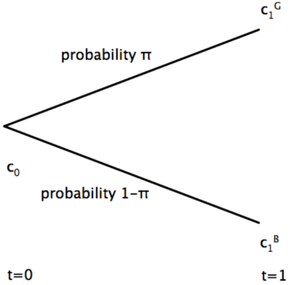
$$
\begin{aligned}
c_0 &= \text{consumption today} \\
c_1^G &= \text{consumption next year in the good state} \\
c_1^B &= \text{consumption next year in the bad state}
\end{aligned}
$$

Consumer Optimization: The Risk Dimension



Under uncertainty, the consumer chooses consumption today and consumption in both states next year.

Consumer Optimization: The Risk Dimension



Uncertainty about future income "induces" randomness in future consumption as well.

Consumer Optimization: The Risk Dimension

Suppose that the consumer's utility function is

$$u(c_0) + \beta\pi u(c_1^G) + \beta(1-\pi)u(c_1^B),$$

so that the terms involving next year's consumption are weighted by the probability that each state will occur as well as by the discount factor β.

Consumer Optimization: The Risk Dimension

In probability theory, if a random variable X can take on n possible values, $X_1, X_2, \ldots, X_n$, with probabilities $\pi_1, \pi_2, \ldots, \pi_n$, then the expected value of X is

$$E(X) = \pi_1 X_1 + \pi_2 X_2 + \ldots + \pi_n X_n.$$

Consumer Optimization: The Risk Dimension

Hence, by assuming that the consumer's utility function is

$$u(c_0) + \beta\pi u(c_1^G) + \beta(1-\pi)u(c_1^B),$$

we are assuming that the consumer's seeks to maximize expected utility

$$u(c_0) + \beta E[u(c_1)].$$

期望效用

Consumer Optimization: The Risk Dimension

But by writing out all three terms,

$$u(c_0) + \beta\pi u(c_1^G) + \beta(1-\pi)u(c_1^B),$$

we can see that concavity of the function u, which in the standard microeconomic case represents a preference for diversity, represents here a preference for smoothness in consumption over time and across states in the future – the consumer is risk averse in the sense that he or she does not want consumption in the bad state to be too much different from consumption in the good state.

Contingent Claims

Consumer Optimization: The Risk Dimension

To implement these state-contingent consumption plans, Arrow and Debreu imagined that the consumer would trade contingent claims for both future states.

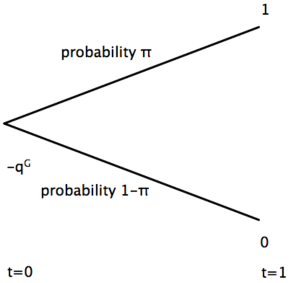
只有一种情况下的钱

A contingent claim for the good state costs q^G today, and delivers one unit of consumption next year in the good state and zero units of consumption next year in the bad state.

可以 buy/sell

A contingent claim for the bad state costs q^B today, and delivers one unit of consumption next year in the bad state and zero units of consumption next year in the good state.

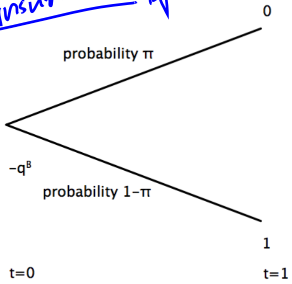
Consumer Optimization: The Risk Dimension



Payoffs for the contingent claim for the good state (a long position).

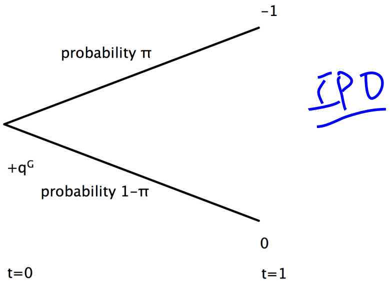
Consumer Optimization: The Risk Dimension

insurance 保险



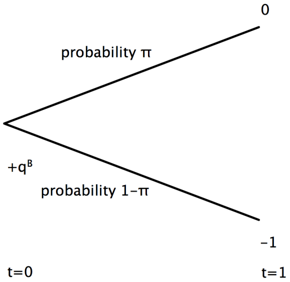
Payoffs for the contingent claim for the bad state (a long position).

Consumer Optimization: The Risk Dimension



Payoffs for a short position in the contingent claim for the good state.

Consumer Optimization: The Risk Dimension



卖保险的

Payoffs for a short position in the contingent claim for the bad state.

Consumer Optimization: The Risk Dimension

一个证券：
和固定时间，资产状态的

Trading Strategy	Claim	Cash Flow at $t = 0$	Cash Flow in Good State at $t = 1$	Cash Flow in Bad State at $t = 1$
Long	Good	$-q^G$	$+1$	0
Long	Bad	$-q^B$	0	$+1$
Short	Good	$+q^G$	-1	0
Short	Bad	$+q^B$	0	-1

Like a sophisticated form of saving and borrowing, where the investor can "fine-tune" the future state in which payments are received or made.

使得未来的可以决定

Consumer Optimization with Contingent Claims

Consumer Optimization: The Risk Dimension

Today, the consumer divides his or her income up into an amount to be consumed and amounts used to purchase the two contingent claims:

$$Y_0 \geq c_0 + q^G s^G + q^B s^B,$$

where s^G and s^B denote the number of each contingent claim purchased or sold short.

S: savings
(1기 저축 | 2기 state)

If either s^G or s^B is negative, the consumer is taking a short position in that claim.

Consumer Optimization: The Risk Dimension

Next year, the consumer simply spends his or her income, including payoffs on contingent claims:

$$Y_1^G + s^G \geq c_1^G$$

in the good state and

$$Y_1^B + s^B \geq c_1^B$$

in the bad state.

Consumer Optimization: The Risk Dimension

$$Y_0 \geq c_0 + q^G s^G + q^B s^B$$

$$Y_1^G + s^G \geq c_1^G$$

$$Y_1^B + s^B \geq c_1^B$$

Multiply both sides of the second equation by q^G and both sides of the third equation by q^B, Then add them all up to get the lifetime budget constraint

$$\underline{Y_0 + q^G Y_1^G + q^B Y_1^B \geq c_0 + q^G c_1^G + q^B c_1^B.}$$

手段：打包

Consumer Optimization: The Risk Dimension

The problem is to choose c_0, c_1^G, and c_1^B to maximize expected utility

$$u(c_0) + \beta\pi u(c_1^G) + \beta(1-\pi)u(c_1^B),$$

subject to the budget constraint

$$Y_0 + q^G Y_1^G + q^B Y_1^B \geq c_0 + q^G c_1^G + q^B c_1^B.$$

This was Arrow and Debreu's key insight: that finance is like grocery shopping. Mathematically, making decisions over time and under uncertainty is no different from choosing apples, bananas, and pears!

contigent claim

①：确定未来收入

②掌握预算约束

Consumer Optimization: The Risk Dimension

The Lagrangian is

$$\begin{aligned} L &= u(c_0) + \beta\pi u(c_1^G) + \beta(1-\pi)u(c_1^B) \\ &\quad + \lambda\left(Y_0 + q^G Y_1^G + q^B Y_1^B - c_0 - q^G c_1^G - q^B c_1^B\right), \end{aligned}$$

and the first-order conditions are

$$u'(c_0^*) - \lambda^* = 0$$

$$\beta\pi u'(c_1^{G*}) - \lambda^* q^G = 0$$

$$\beta(1-\pi)u'(c_1^{B*}) - \lambda^* q^B = 0$$

Consumer Optimization: The Risk Dimension

The first-order conditions

$$u'(c_0^*) - \lambda^* = 0$$

$$\beta\pi u'(c_1^{G*}) - \lambda^* q^G = 0$$

$$\beta(1-\pi)u'(c_1^{B*}) - \lambda^* q^B = 0$$

imply that marginal rates of substitution equal relative prices:

$$\frac{u'(c_0^*)}{\beta\pi u'(c_1^{G*})} = \frac{1}{q^G} \text{ and } \frac{u'(c_0^*)}{\beta(1-\pi)u'(c_1^{B*})} = \frac{1}{q^B}$$

$$\text{and } \frac{\pi u'(c_1^{G*})}{(1-\pi)u'(c_1^{B*})} = \frac{q^G}{q^B}.$$

Pricing Stocks and Bonds with Contingent Claims

Consumer Optimization: The Risk Dimension

Do we really observe consumers trading in contingent claims?

Yes, if we think of financial assets as “bundles” of contingent claims.

This insight is also Arrow and Debreu’s.

Consumer Optimization: The Risk Dimension

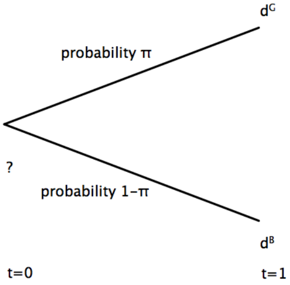
A "stock" is a risky asset that pays dividend d^G next year in the good state and d^B next year in the bad state.

These payoffs can be replicated by buying d^G contingent claims for the good state and d^B contingent claims for the bad state.

用 contingent claim

建构 股票/债券

Consumer Optimization: The Risk Dimension



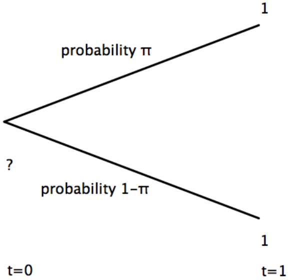
Payoffs for the stock.

Consumer Optimization: The Risk Dimension

A "bond" is a safe asset that pays off one next year in the good state and one next year in the bad state.

These payoffs can be replicated by buying one contingent claim for the good state and one contingent claim for the bad state.

Consumer Optimization: The Risk Dimension



Payoffs for the bond.

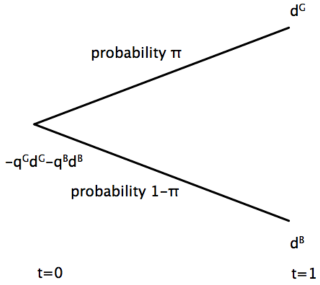
Consumer Optimization: The Risk Dimension

If we start with knowledge of the contingent claims prices q^G and q^B, then we can infer that the stock must sell today for

$$q^{stock} = q^G d^G + q^B d^B.$$

Since if the stock cost more than the equivalent bundle of contingent claims, traders could make profits for sure by short selling the stock and buying the contingent claims; and if the stock cost less than the equivalent bundle of contingent claims, traders could make profits for sure by buying the stock and selling the contingent claims.

Consumer Optimization: The Risk Dimension



"Pricing" the stock.

Consumer Optimization: The Risk Dimension

Likewise, if we start with knowledge of the contingent claims prices q^G and q^B, then we can infer that the bond must sell today for

$$q^{bond} = q^G + q^B.$$

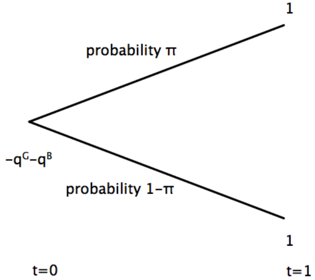
Since the bond pays off one for sure next year, the interest rate, defined as the return on the risk-free bond, is

$$1 + r = \frac{1}{q^{bond}} = \frac{1}{q^G + q^B}.$$

The bond price relates to the interest rate via

$$q^{bond} = \frac{1}{1 + r}.$$

Consumer Optimization: The Risk Dimension



Pricing the bond.

Replicating Contingent Claims with Stocks and Bonds

determine q^B & q^G

Consumer Optimization: The Risk Dimension

We've already seen how contingent claims can be used to replicate the stock and the bond.

Now let's see how the stock and the bond can be used to replicate the contingent claims.

Consumer Optimization: The Risk Dimension

Consider buying s shares of stock and b bonds, in order to replicate the contingent claim for the good state.

In the good state, the payoffs should be

$$sd^G + b = 1$$

and in the bad state, the payoffs should be

$$sd^B + b = 0$$

since the contingent claim pays off one in the good state and zero in the bad state.

Consumer Optimization: The Risk Dimension

To replicate the contingent claim for the good state:

考试

会有这种题

$$sd^G + b = 1$$

$$sd^B + b = 0 \Rightarrow b = -sd^B$$

Substitute the second equation into the first to solve for

$$s = \frac{1}{d^G - d^B} \text{ and } b = \frac{-d^B}{d^G - d^B}$$

Since s and b are of opposite sign, this requires going "long" one asset and "short" the other.

Consumer Optimization: The Risk Dimension

To replicate the contingent claim for the good state:

$$s = \frac{1}{d^G - d^B} \text{ and } b = \frac{-d^B}{d^G - d^B}$$

If we know the prices q^{stock} and q^{bond} of the stock and bond, we can infer that in the absence of arbitrage, the claim for the good state would have price

$$q^G = q^{stock}s + q^{bond}b = \frac{q^{stock} - d^B q^{bond}}{d^G - d^B}.$$

计算 q^G，代入 s, b

Consumer Optimization: The Risk Dimension

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Consumer Optimization: The Risk Dimension

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Consumer Optimization: The Risk Dimension

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If we know the prices q^{stock} and q^{bond} of the stock and bond, we can infer that in the absence of arbitrage, the claim for the good state would have price

$$q^G = q^{stock}s + q^{bond}b = \frac{q^{stock} - d^B q^{bond}}{d^G - d^B}.$$

Consumer Optimization: The Risk Dimension

Consider buying s shares of stock and b bonds, in order to replicate the contingent claim for the bad state.

In the good state, the payoffs should be

$$sd^G + b = 0$$

and in the bad state, the payoffs should be

$$sd^B + b = 1$$

since the contingent claim pays off one in the bad state and zero in the good state.

Consumer Optimization: The Risk Dimension

To replicate the contingent claim for the bad state:

$$sd^G + b = 0 \Rightarrow b = -sd^G$$

$$sd^B + b = 1$$

Substitute the first equation into the second to solve for

$$s = \frac{-1}{d^G - d^B} \text{ and } b = \frac{d^G}{d^G - d^B}$$

Once again, this requires going long one asset and short the other.

Consumer Optimization: The Risk Dimension

To replicate the contingent claim for the bad state:

$$s = \frac{-1}{d^G - d^B} \text{ and } b = \frac{d^G}{d^G - d^B}$$

Once again, if we know the prices q^{stock} and q^{bond} of the stock and bond, we can infer that in the absence of arbitrage, the claim for the bad state would have price

$$q^B = q^{stock}s + q^{bond}b = \frac{d^G q^{bond} - q^{stock}}{d^G - d^B}.$$

financial engineering → 制造 contingency claim 的一个方法

Pricing Options with Contingent Claims

在 $t=1$ 时定价

Black-Scholes Option Pricing

A call option is a contract that gives the buyer the right, but not the obligation, to purchase a share of stock at the strike price K at $t = 1$.

At $t = 1$, the call is said to be in the money if the actual share price is above the strike price and out of the money if the actual share price is below the strike price.

At $t = 1$, the option will have value only if it is in the money. But at $t = 0$, the option will have value even if there is only a probability of it being in the money at $t = 1$.

Black-Scholes Option Pricing

Fischer Black (US, 1938-1995) and Myron Scholes (Canada/US, b.1941, Nobel Prize 1997) were the first to derive a formula for the price of an option.

Robert Merton (US, b.1944, Nobel Prize 1997) arrived at the same formula in a simpler way, by showing how options prices could be inferred from assumptions about and observations on the underlying stock price.

Black-Scholes Option Pricing

The arguments used by Merton were not exactly those from Arrow-Debreu no-arbitrage theory that would use the price of the stock and bond to infer contingent claims prices, then use contingent claims prices to compute the price of the option.

But his analysis followed along similar lines, and today it is recognized that one could use the Arrow-Debreu approach to obtain the same results.

Black-Scholes Option Pricing

Their papers were both published in 1973.

Fischer Black and Myron Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* Vol.81 (May-June 1973): pp.637-654.

Robert Merton, "Theory of Rational Option Pricing," *The Bell Journal of Economics and Management Science* Vol.4 (Spring 1973): pp.141-183.

Black-Scholes Option Pricing

To see how the theory works, assume a simple two-period structure, with $t = 0$ and $t = 1$, and assume as well, that there are only two states, $i = G$ and $i = B$, at $t = 1$. Let

$$q^s = \text{ price of the stock at } t = 0$$

$$P^G = \text{ price of the stock in state } i = G \text{ at } t = 1$$

$$P^B = \text{ price of the stock in state } i = B \text{ at } t = 1$$

Black-Scholes Option Pricing

Likewise, let

$$q^b = \text{ price of the bond at } t = 0$$

$$1 = \text{ payoff from bond at } i = G \text{ at } t = 1$$

$$1 = \text{ payoff from bond at } i = B \text{ at } t = 1$$

Black-Scholes Option Pricing

Now consider a call option on the stock with strike price K. Let

$$q^o = \text{ price of the call at } t = 0$$

$$C^G = \text{ payoff generated by the call in state } i = G \text{ at } t = 1$$

$$C^B = \text{ payoff generated by the call in state } i = B \text{ at } t = 1$$

Assume, for now, that the call is in the money in both states at $t = 1$. Then:

$$C^G = P^G - K \text{ and } C^B = P^B - K$$

Black-Scholes Option Pricing

One of the key insights that underlies the Black-Scholes formula is that we don't need to make any specific assumptions about risk or risk aversion to price the option.

Instead, we can use a no-arbitrage argument that:

1. Replicates the option's payoffs using a portfolio of the stock and the risk-free bond.
2. Values the option based on the cost of assembling the portfolio.

Black-Scholes Option Pricing

也可以用
price of stock
↓
解出来 $q^B \cdot q^s$

State	Stock's Payoff	Bond's Payoff	Option's Payoff
G	P^G	1	$P^G - K$
B	P^B	1	$P^B - K$

We want to construct a portfolio consisting of s shares of the stock and b bonds that replicates the payoffs from the option in both states at $t = 1$:

$$sP^G + b = P^G - K$$

$$sP^B + b = P^B - K$$

Black-Scholes Option Pricing

$$sP^G + b = P^G - K$$
$$sP^B + b = P^B - K$$

This is a set of two linear equations in the two unknowns: s and b. The solution is

$$s = 1 \text{ and } b = -K$$

Since the stock costs q^s and the bond costs q^b, the cost of this portfolio at $t = 0$ is

$$q^s - q^b K$$

Black-Scholes Option Pricing

The option's payoffs are replicated by a portfolio with

$$s = 1 \text{ and } b = -K$$

and since the stock costs q^s and the bond costs 1, the cost of this portfolio at $t = 0$ is

$$q^s - q^b K$$

But this means that the price of the option must also be

$$q^o = q^s - q^b K$$

Black-Scholes Option Pricing

Next, let's consider the case in which the call is in the money in the good state and out of the money in the bad state at $t = 1$.

Then

$$C^G = P^G - K \text{ and } C^B = 0$$

Black-Scholes Option Pricing

State	Stock's Payoff	Bond's Payoff	Option's Payoff
G	P^G	1	$P^G - K$
B	P^B	1	0

Again we want to construct a portfolio consisting of s shares of the stock and b bonds that replicates the payoffs from the option in both states at $t = 1$:

$$sP^G + b = P^G - K$$

$$sP^B + b = 0$$

Black-Scholes Option Pricing

$$sP^G + b = P^G - K$$

$$sP^B + b = 0$$

Again this is a set of two linear equations in the two unknowns: s and b. The solution is

$$s = \frac{P^G - K}{P^G - P^B} \text{ and } b = -\frac{P^B(P^G - K)}{P^G - P^B}$$

Since the stock costs q^s and the bond costs q^b, the cost of this portfolio at $t = 0$ is

$$\left(\frac{P^G - K}{P^G - P^B}\right) q^s + \left[-\frac{P^B(P^G - K)}{P^G - P^B}\right] q^b$$

Black-Scholes Option Pricing

But since the portfolio of the stock and bond again replicates the payoffs from the option, this implies that the option's price must be

$$
\begin{aligned}
q^o &= \left(\frac{P^G - K}{P^G - P^B}\right) q^s + \left[-\frac{P^B(P^G - K)}{P^G - P^B}\right] q^b \\
&= \frac{(q^s - q^b P_B)(P^G - K)}{P^G - P^B}
\end{aligned}
$$

Black-Scholes Option Pricing

Finally, there is the easy case in which the call is out of the money in both states at $t = 1$.

Then

$$C^G = 0 \text{ and } C^B = 0$$

The option's payoffs can be replicated by a portfolio consisting of zero shares of the stock and zero bonds, which costs zero at $t = 0$. Equivalently, an asset that pays off nothing should cost nothing.

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