Mathematical Methods in Finance

Lecture 10: Black-Scholes in Practice: Greeks and Volatility-Arbitrage

Fall 2013

Overview

- Black-Scholes Sensitivities (Greeks)
- Black-Scholes in practice: Delta-Neutral and Long Gamma Trading
- Implied Volatility vs. Realized Volatility: Volatility-Arbitrage

Recall: the Black-Scholes-Merton PDE

- Consider the Black-Scholes-Merton model:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t),$$

and suppose that the interest rate is r.

- Let $C(t) = c(t, S(t))$ be the value of a call option with maturity T with payoff $(s-K)^+$.
- $c(t, x)$ satisfies the Black-Scholes-Merton equation.

$$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x) \quad \text{for all } t \in [0,T), \tag{1}$$

with a terminal condition $c(T, x) = (x-K)^+$.

The Black-Scholes-Merton formula

The Black-Scholes-Merton formula: For any $t \in [0, T)$ and $x > 0$,

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)), \tag{2}$$

where $N(y)$ is the CDF of standard normal distribution and

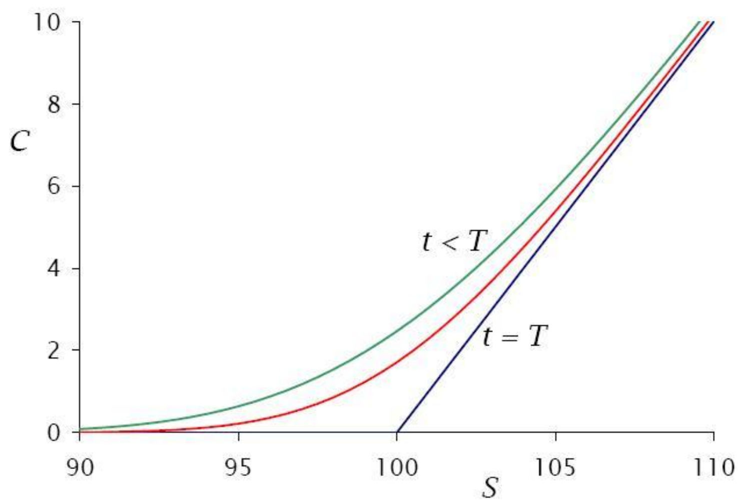
$$\begin{aligned} d_+(\tau,x) &= \frac{1}{\sigma\sqrt{\tau}}\left[\log\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau\right], \\ d_-(\tau,x) &= \frac{1}{\sigma\sqrt{\tau}}\left[\log\left(\frac{x}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau\right]. \end{aligned} \tag{3}$$

The Black-Scholes-Merton formula: Greeks

Greeks: The derivatives of $c(t,x)$ w.r.t. various variables are called Greeks.

- Delta: $\Delta = c_x(t,x) = N(d_+(T-t,x)) > 0$;
- Theta: $\Theta = c_t(t,x) =$
  $-rKe^{-r(T-t)}N(d_-(T-t,x)) - \dfrac{\sigma x}{2\sqrt{T-t}}N'(d_+(T-t,x)) < 0$;
- Gamma: $\Gamma = c_{xx}(t,x) = \dfrac{1}{\sigma x\sqrt{T-t}}N'(d_+(T-t,x)) > 0$;
- Vega: $\mathcal{V} = c_\sigma(t,x) = xN'(d_+(T-t,x))\sqrt{T-t} > 0$.

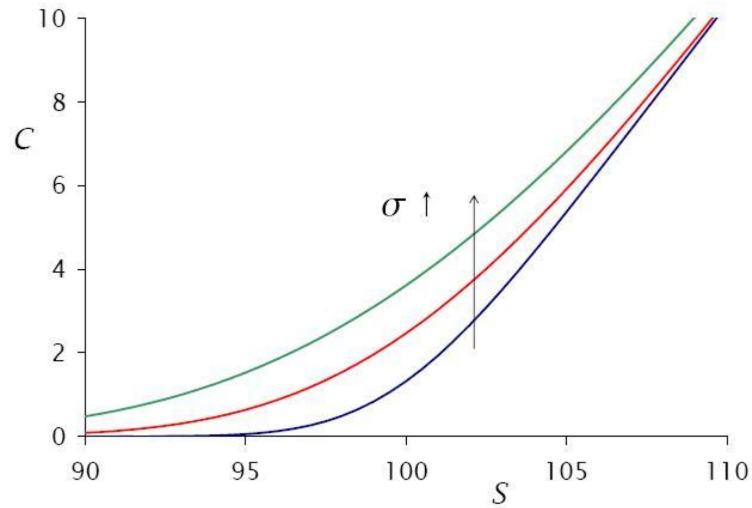
Black-Scholes-Merton Formula: Sensitivity to S and T



Call price:

- $C \uparrow$ as $S \uparrow$
- $C \downarrow$ as $t \to T$

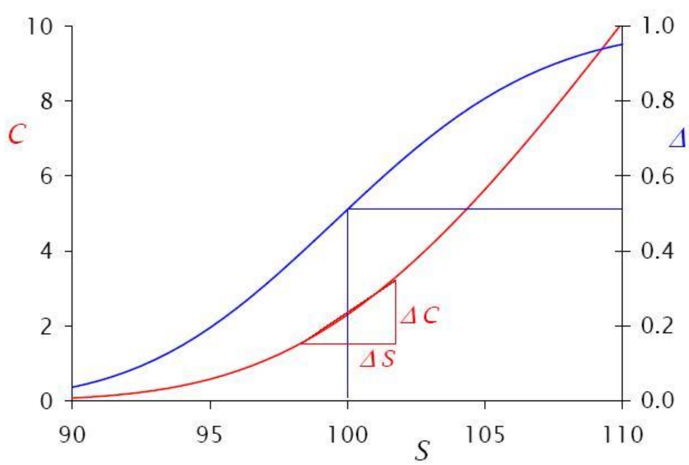
Black-Scholes-Merton Formula: Sensitivity to σ



Call price:

- $C \uparrow$ as $\sigma \uparrow$

Black-Scholes Delta



Delta:

- $\Delta \to 0$ as $S \downarrow 0$
- $\Delta \approx 1/2$ at $S = K$ (because $d_1 \approx 0$ at $S = K$ and so $N(d_1) \approx 1/2$)
- $\Delta \to 1$ as $S \uparrow \infty$

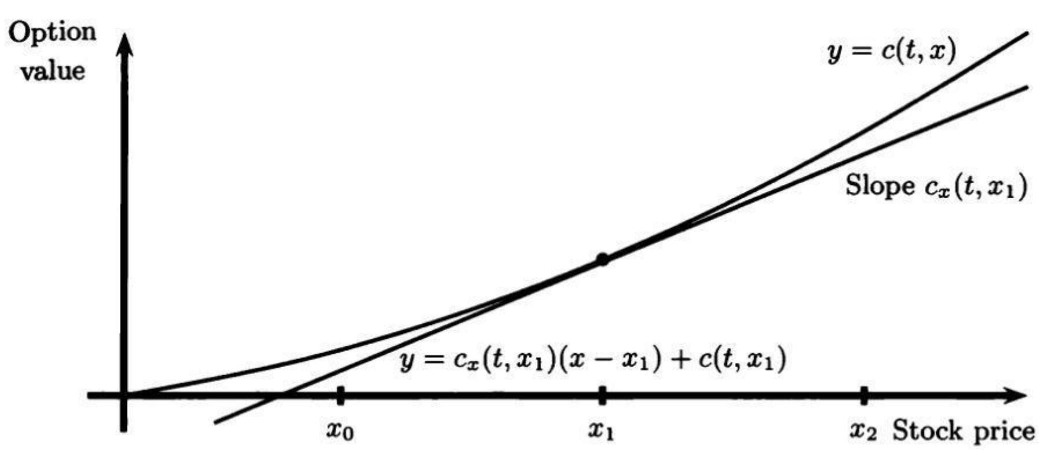
Understanding the Black-Scholes-Merton formula

- An explanation of the Black-Scholes-Merton formula (2), i.e., $c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)) = xc_x(t,x) - e^{-r(T-t)}[KN(d_-(T-t,x))]$.
- It means that to hedge a short position, at time t, we hold $c_x(t,x) \equiv N(d_+(T-t,x))$ shares of stocks and borrow $e^{-r(T-t)}KN(d_-(T-t,x))$ from the money market.
- Consider taking a long position on the option and then hedging it. Then at time t and given stock price x_1, the initial value of the portfolio is:

$$c(t,x_1) - x_1 c_x(t,x_1) + M, \tag{4}$$

where $M = x_1 c_x(t,x_1) - c(t,x_1)$ (short stock, buy the option, and lend the money to the bank).

Delta Neutral Position



Delta Neutral Position

- Delta neutral: Short $c_x(t, x_1)$ shares of stocks so that the change of portfolio values due to the change of the stock price, i.e. $\Delta c(t, x)$, is nearly offset by the change in the value of our short position in the stock , i.e.

$$\Delta x c_x(t, x) \approx \Delta x \frac{\Delta c(t, x)}{\Delta x} = \Delta c(t, x).$$

  - If we short more than $c_x(t, x_1)$ shares, the portfolio value would decrease (or increase) when stock price rises (or falls).
  - If we short less than $c_x(t, x_1)$ shares, the portfolio would decrease (or increase) when stock price falls (or rises).
  - If we have no speculation on the movement of the stock price, we would short exactly $c_x(t, x_1)$ shares.

Long Gamma Position

- Long gamma: See (4). If the stock price were to instantaneously fall or rise to x_0 and we do not change our position in the stock $c_x(t, x_1)$ or the money market account M, due to the convexity of $c(t, x)$ in x the total portfolio value would be

$$\begin{aligned} c(t, x_0) - x_0 c_x(t, x_1) + M =& c(t, x_0) - c_x(t, x_1)(x_0 - x_1) - c(t, x_1) \\ >& 0 \end{aligned}$$

- Therefore, a long gamma portfolio is profitable in times of high stock volatility.
- As time t moves forward a little bit, the stock price rises or falls.
  - Long gamma has a positive effect on the value of a long gamma portfolio;
  - However, the negative theta $c_t(t, x)$ has a negative effect on the value of a long gamma portfolio.
  - For the European options, the two effects above cancel.

How to specify the volatility? Two Notions: Implied vs. Realized

IMPLIED VOLATILITY: for option pricing

- The constant volatility parameter plugged into the Black-Scholes-Merton model for option pricing

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{5}$$

- The **implied volatility** σ^* is the volatility that equates model and market option prices, i.e.

$$C(\sigma^*) = C_{Market}. \tag{6}$$

- Different options imply different implied volatility!
- Different maturities and strikes $\Longrightarrow$ implied volatility surface

How to specify the volatility? Two Notions: Implied vs. Realized

REALIZED VOLATILITY: for risk management and forecasting etc.

- Statistical definition: capture the real fluctuation of the asset return!
- Independent of any model
- **Historical Observation**: $\{S(t_i)\}$,
- **Realized variance** for the period of $[0, T]$ is defined as:

$$RV_{0,T} := \frac{1}{(n-1)\Delta t} \sum_{i=0}^{n-1} \left(\log \frac{S_{t_{i+1}}}{S_{t_i}} \right)^2.$$

- **Realized volatility**: $\sqrt{RV_{0,T}}$

FTSE100 Index Option Prices, Spot$= 6045$

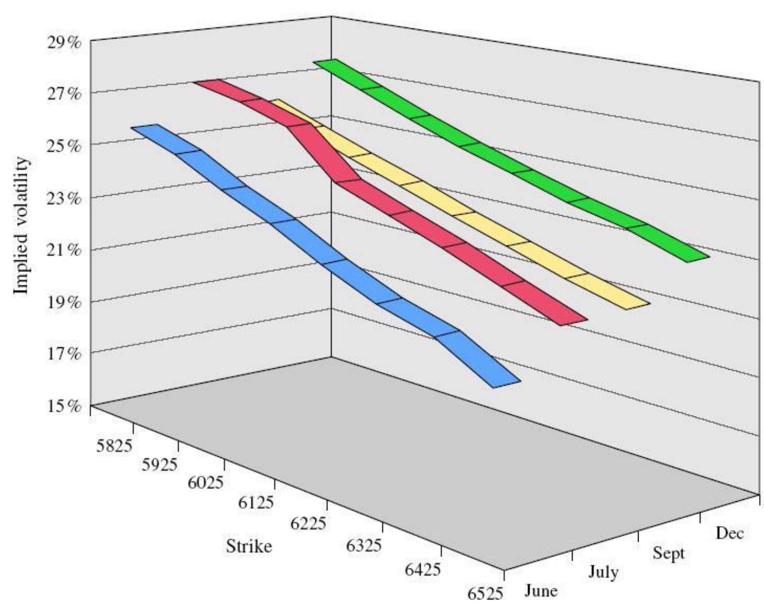
Source: Financial Times

EURO STYLE FTSE 100 INDEX OPTION (LIFFE) £10 per full index point 19 May

	5825		5925		6025		6125		6225		6325		6425		6525	
	C	P	C	P	C	P	C	P	C	P	C	P	C	P	C	P
May	**216½**	¼	**116½**	¼	**16½**	¼	¼	83½	¼	183½	¼	283½	¼	383½	¼	483½
Jun	**310½**	76½	**241½**	107	**179**	144	**127**	119½	**84**	248½	**52**	316	**30½**	393½	**15**	477½
Jul	**410**	144½	**347**	181	**288**	221	**224**	256	**175½**	306	**134**	363½	**98**	426½	**69**	496½
Sep	**506½**	216	**441½**	249	**380**	286	**323½**	327	**271**	373	**224**	424	**181½**	479½	**145½**	541½
Dec†	**663½**	301½	**597**	331	**533½**	364½	**474**	401½	**418½**	442½	**366½**	487½	**320**	537	**273½**	587½

Calls 15,531; Puts 32,579. * Underlying Index value. Premiums shown are based on settlement prices. † Long dated expiry months.

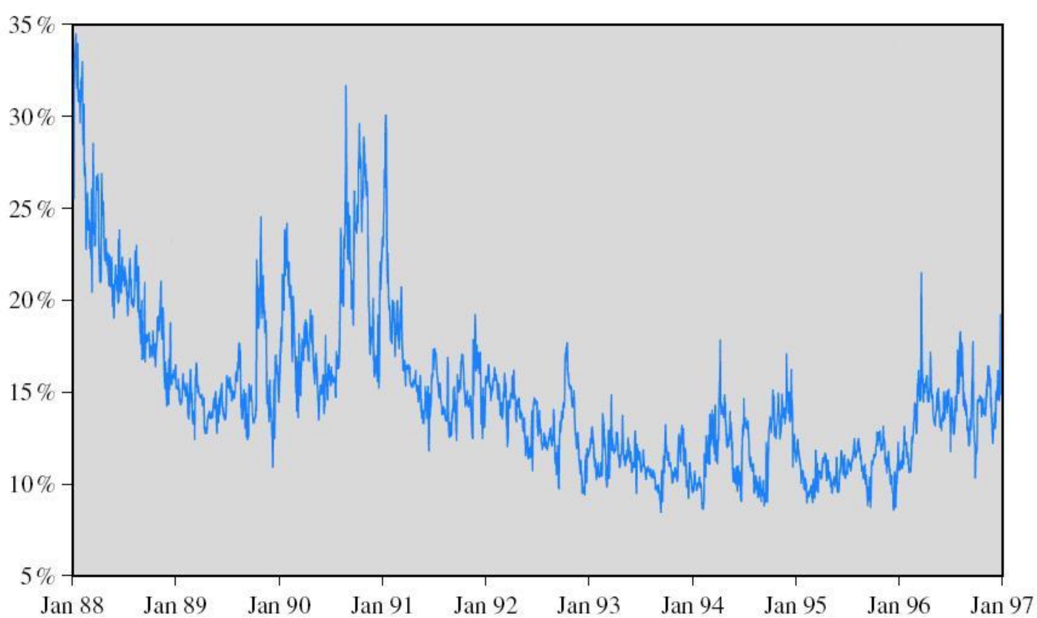
Implied Volatility Surface for FTSE100 Index Option



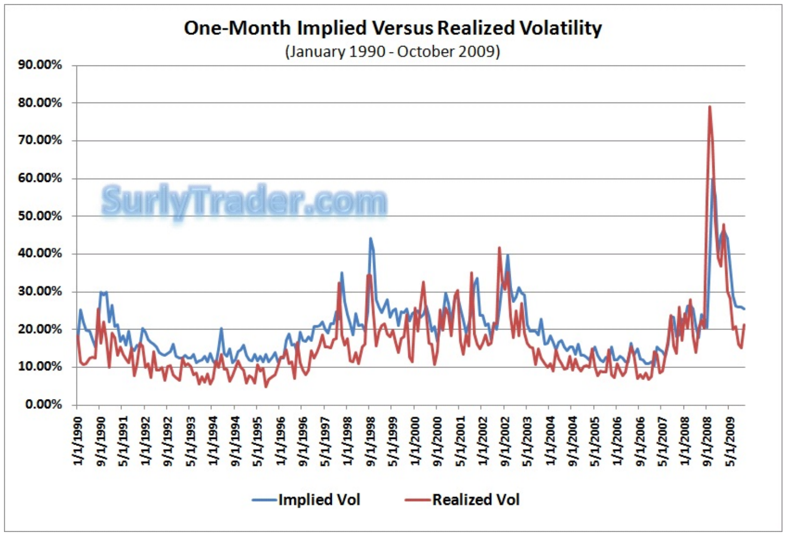
S&P500 Option Implied Volatility Skew

The Evolution of Implied Volatility Over Time

Implied Volatility for ATM Options on S&P500 Index, 1987-1997



Implied Volatility vs. Realized Volatility



	Implied Exceeds Realized	Realized Exceeds Implied	Total
Number of Observations	201	37	238
Percent of Observations	84.5%	15.5%	
Average Monthly Difference	5.90%	-5.03%	4.20%
Median Monthly Difference	5.36%	-1.68%	4.61%
Max/Min	17.43%	-39.85%	

Implied Volatility vs. Realized Volatility

Exhibit S&P 500 Implied Volatility vs. Subsequent Realized Volatility

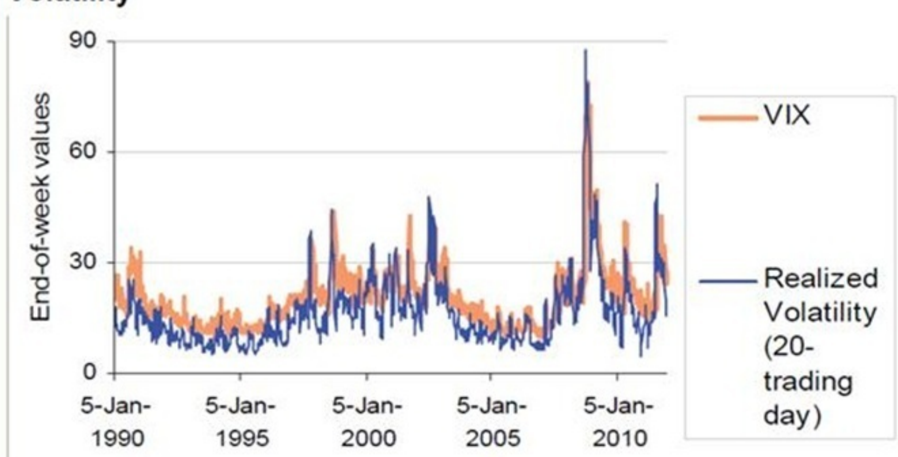


Exhibit In this Exhibit the average value for implied volatility (as represented by VIX) was 20.27 and the average value for realized volatility was 16.38. A number of studies have shown that the implied volatility inherent in index options prices generally has exceeded subsequent realized volatility over multi-year periods (see www.cboe.com/benchmarks). Richly priced index options could provide advantages to the option seller.

Black-Scholes in Practice: Volatility Arbitrage

- Suppose the true dynamics for the underlying asset follows

$$dS(t) = \alpha(t)S(t)dt + \beta(t)S(t)dW(t),$$

  where $\alpha(t)$ and $\beta(t)$ are two stochastic processes adapted to a prespecified filtration. Such a model is quite flexible and general.
- $\alpha(t)$ represents the return; $\beta(t)$ represents the volatility
- Taking a short position of an option with maturity T and payoff $p(S(T))$, a trader believes that $S(t)$ satisfied Black-Scholes with the implied volatility σ_{imp}. She/he prices the option and hedges accordingly.
- Suppose $V(t) = c(t, S(t))$ is the no-arbitrage price, where $c(t, s)$ is governed by the Black-Scholes-Merton equation.

Black-Scholes in Practice: Volatility Arbitrage

- A self-financing hedging portfolio has value $X(t)$ at time t by holding $\Delta(t) = \dfrac{\partial c(t, S(t))}{\partial s}$ shares of $S(t)$ and investing the rest $X(t) - \Delta(t)S(t)$ in the riskless account. Thus,

$$dX(t) = \Delta(t)dS(t) + r\left(X(t) - \Delta(t)S(t)\right)dt$$

  with $X(0) = c(0, S(0))$ if the option is written at the Black-Scholes price.
- Consider

$$Y(t) = X(t) - C(t, S(t))$$

  We have

$$dY(t) = rY(t)dt + \frac{1}{2}S^2(t)\frac{\partial^2 c}{\partial s^2}(\sigma_{imp}^2 - \beta(t)^2)dt$$

  with $Y(0) = 0$,

Black-Scholes in Practice: Volatility Arbitrage

- Equivalently, we have

$$Y(T) = \frac{1}{2}\int_0^T e^{r(T-t)} S^2(t) \frac{\partial^2 c}{\partial s^2} (\sigma_{imp}^2 - \beta(t)^2) dt.$$

- If the implied volatility σ_{imp} is higher than the realized volatility $\beta(t)$ (as mostly is the case), the trader makes a positive profit due to the positive Gamma for put and call options.
- Successful hedging is entirely a matter of good volatility estimation!

Further Topics

- More theory and mathematical tools on stochastic calculus
- More realistic models
- More financial products (futures, exotic options, American style derivatives, etc.)
- Term structure modeling
- Portfolio planning
- Numerical techniques
- Many other topics...

Supplementary Material

Suggested Reading Material (We only need to focus on the material parallel to our course slides):

- Selected material from Shreve Vol. II: 4.5.5

Suggested Exercises (Do Not Hand In; For Your Deeper Understanding Only)

- Shreve Vol. II: 4.9, 4.11, 4.12, 4.21 (some of these are challenging questions)

Thank You