Homework Assignment #3
Due: In class, two weeks after distribution

§1 Random Walk and Martingales

Suppose X_i are i.i.d. random variables with mean μ and variance σ^2, i.e.

$$EX_i = \mu, \quad \mathrm{Var}X_i = \sigma^2.$$

Let $M_n = \sum_{k=1}^n X_k,\ M_0 = 0$ be a random walk. Show that:

1. $\{M_n - n\mu\}$ is a martingale adapted to the filtration $\{\mathcal{F}_n\}$;
2. if $\mu = 0$, $\{M_n^2 - n\sigma^2\}$ is a martingale adapted to the filtration $\{\mathcal{F}_n\}$.

Here $\mathcal{F}_n$ is the "information" generated by $(X_1, X_2, ..., X_n)$.

§2 Wald Martingale

Define the process W by

$$W_0 = 1,\ W_n = \frac{e^{\theta \sum_{j=1}^n X_j}}{(\phi(\theta))^n},$$

where X_j are I.I.D random variables and $\phi(\theta) = \mathbb{E}e^{\theta X_i}$ is the moment generating function of X_i.

1. Verify that W is a martingale;
2. Denote by τ the first time $\sum_{j=1}^n X_j$ attains some level a, i.e.,

$$\tau = \min\left\{ n \in \mathbb{N}; \sum_{j=1}^n X_j = a \right\}.$$

   What can you say about the distribution of τ?

§3 Poisson Martingales

Verify that, for a Poisson process $\{N(t); t \geq 0\}$ with intensity λ,

1. the following compensated Poisson process is martingale:

$$M(t) = N(t) - \lambda t;$$

2. the following geometric Poisson process is a martingale:

$$S(t) = \exp\{N(t)\log(\sigma+1) - \lambda\sigma t\} \equiv (\sigma+1)^{N(t)} \exp(-\lambda\sigma t),$$

   where $\sigma > -1$.

§4 Asymmetric Random Walk and Gambler's Problem

Consider an asymmetric random walk on the integers with probability $p < 1/2$ of moving to the right and probability of $1-p$ of moving to the left. Let S_n be the value at time n and assume that $S_0 = a$, where $0 < a < N$.

1. Prove that
$$\left(\frac{1-p}{p}\right)^{S_n}$$
is a martingale;

2. Define the first hitting time
$$\tau = \min\{n \in \mathbb{N}, S_n = 0 \text{ or } N\}.$$
Prove that
$$\mathbb{P}(S_\tau = N) = \frac{1 - \left(\frac{1-p}{p}\right)^{-a}}{\left(\frac{1-p}{p}\right)^{N-a} - \left(\frac{1-p}{p}\right)^{-a}}.$$