# 金融经济学

## 第一讲

张宇

# 《金融经济学》与《证券投资学》课程区别

# 培养方案的区别

- 《金融经济学》是大二分方向后，金经方向唯一的金融学必修课

<table>
<tr><th></th><th>金融方向</th><th>金经方向</th></tr>
<tr><td>一上，一下，二上</td><td colspan="2">高数，线代，概统<br>经原，微观，宏观<br>财务会计，公司财务</td></tr>
<tr><td>分方向后：专业必修</td><td>证券投资学<br>金融市场与金融机构<br>…</td><td>金融经济学</td></tr>
<tr><td>分方向后：专业选修</td><td>…</td><td>证券投资学</td></tr>
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# 培养目标的区别

- **金融经济学方向**
  - 注重训练学生的经济学理论思维和分析能力，强调经济学在金融领域的运用，
  - 培养具有扎实的理论基础、国际化的学术视野、较强的中英文沟通能力和善于知行结合的专业性人才。
  - 学生毕业后适于在经济和金融相关的政府部门、企业和研究机构工作。
- **金融学方向**
  - 致力于培养具有坚实的金融学理论基础和创新潜力、具有国际视野与较高应用技能的金融专业人才，
  - 同时具备较强的中英文沟通能力以及缜密的分析能力。
  - 学生毕业后适于在高端金融管理部门、各类金融机构和研究机构工作。

# 《金融经济学》与《证券投资学》课程区别

- 《证券投资学》
  - 国际通用的商学院金融专业投资学教材：
    Bodie, Kane, Marcus – Investments, 9ed/10ed/11ed
  - 侧重市场实务（培养目标）
- 《金融经济学》
  - 侧重构建理解金融市场的经济学框架和抽象思维能力（培养目标）
    该部分以P. Ireland等《金融经济学》课程内容为基础
  - 在框架基础上，提供中国金融市场基本背景知识（培养方案）
  - 无教材，考试内容以课件、作业为准

# Administrative issues

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# Requirements

- Evaluation:
  - Quiz: 10%
  - Homework 15%
  - Midterm: 35%
  - Final Exam: 40% - 部分前半学期

# 拓展阅读

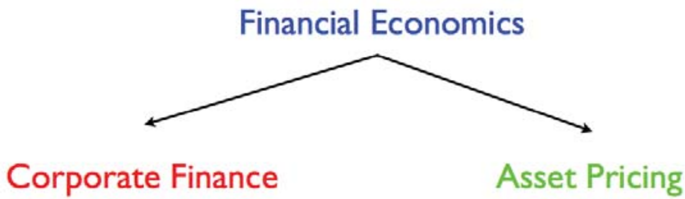
- Bodie, Kane, Marcus – Investments 9/10/11ed
  实务细节/CFA
- 徐高（光大证券/国发院）– 金融经济学二十五讲
  理论由来/现实意义/一些更深的内容
- 如何读
  - 理解本课程内容后，应当可以快速自学以上两本
  - 学习本课程过程中，如有时间，同时阅读这两本（尤其是徐高讲义）可能可带来帮助

这要看课件

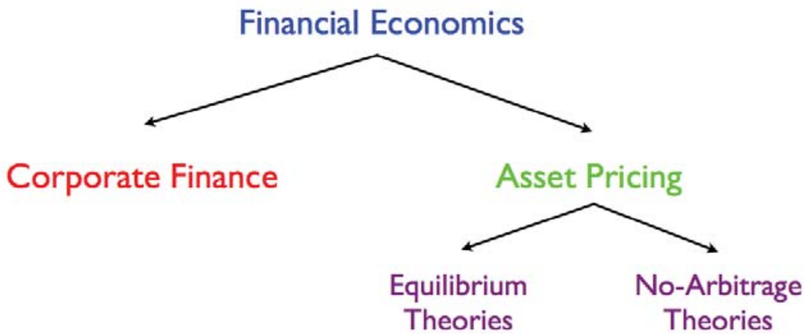
# What you will learn

1. 跨期消费投资决策理论
   Intertemporal consumption and saving choice
2. 风险下的投资决策 Decisions under risks
3. 现代资产组合理论 Modern Portfolio Theory
4. 资产定价模型 Capital Asset Pricing Model
5. 无套利定价模型 Arbitrage Pricing Theory
6. 阿罗德布鲁定价模型 Arrow-Debreu Pricing
7. 风险中性概率定价模型 Martingale Pricing
8. 宏观金融和金融摩擦 Macro-finance & Financial Frictions
9. 中国金融市场

# The Structure of Financial Economics



# The Structure of Financial Economics

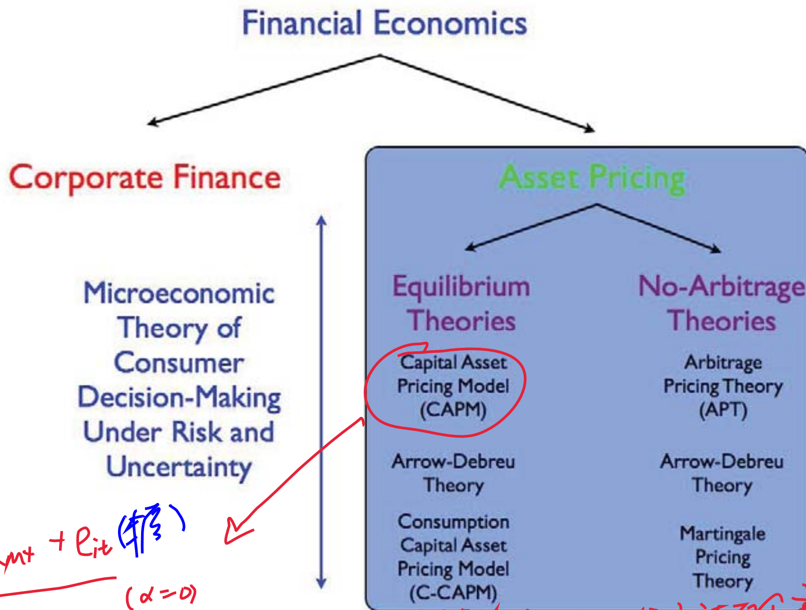


# The Structure of Financial Economics

- Financial Economics
  - Corporate Finance
  - Asset Pricing
    - Equilibrium Theories
      - Capital Asset Pricing Model (CAPM)
      - Arrow-Debreu Theory
      - Consumption Capital Asset Pricing Model (C-CAPM)
    - No-Arbitrage Theories
      - Arbitrage Pricing Theory (APT)
      - Arrow-Debreu Theory
      - Martingale Pricing Theory

Project financing ( stock / bond / loan )

# The Structure of Financial Economics




PE managers

$R_{it} = \alpha + \beta \cdot R_{mt} + e_{it}$ (样)

($\alpha = 0$)

无套利 (APT): $R_{it} = 0 + \underbrace{\Sigma \beta_f \cdot R_{ft}}_{\text{factors include } R_{mt}} + e_{it}$ (平均为0的一个无法解释的)

Arrow Debreu：（后半学期的内容）

$$P = E[\vec{m} \cdot \vec{x}]$$（与市场状态相关）

stochastic discount factor（在某个状态下钱的意义）

在我好的时候给我1000万 不如在遇难时给我1000万

# 导论 - 金融的经济学含义

# What is finance?

- Economics:
  - Mankiw: How Society Manages Scarce Resources
- Finance:
  - (For each investor) how to allocate scarce resources across time and states of the world
    - Saving versus consuming
    - Portfolio choice; Insurance decisions
  - (For society) financial markets work to channel funds from savers (money > current uses)
    to borrowers (money < current uses)
    - Prices in the financial markets:
      Interest rates, Risk premia (beta), Insurance premia, etc.
    - Savers and borrowers optimizes take as given such prices

# What is finance? More on the borrower/saver interpretation (1)

- Think of Finance as a game:
  - 2 Players: "Lender/Saver" & "Borrower/spender"
  - Borrower: has been exposed to some opportunities but lack the necessary funds to finance such opportunities.
  - Lender: has excess funds available
  - Finance happens when the borrower borrows money from the lender and be able to take the opportunities he thinks it worthwhile to take.
  - When this happens, the borrower acquires capital needed to increase his "real assets" through selling "financial assets" to the lender.

Source: Travis Ng, Lecture Notes on Financial Economics, Univ of Toronto (2007)

# What is finance? More on the borrower/saver interpretation (2)

- Real assets:
  - Any tangible assets that generates a stream of goods or services in the future. For example, machinery, land, building, vehicle, robots, antiques, etc.
- Financial assets:
  - Intangible, usually paper claims by one person against another.
  - Seller of financial assets (i.e., the borrower) establishes liabilities which is an obligation.
  - Buyer of financial assets (i.e., the lender) buys some rights to entitle some claims of the seller’s assets in the future.

Source: Travis Ng, Ibid.

# Example of financial assets

- Writer: Harry the Borrower
- Terms: I, Harry the Borrower, hereby swear to God that, on Feb 26, 2019, I will pay 10 RMB to whoever holds this paper contract with my big name signed on it. Sue me if I don’t fill my obligation.

- Question: Would you buy this IOU? At what price? And what would this price depend on?

Source: Travis Ng, Ibid.

# Costs and benefits of finance

Cost of Financing

金融机构专业完成这些任务

- Search costs – finding Harry
- Transaction costs – writing and enforcing the contract with Harry
- Negotiation costs – building trust with Harry?
- RISK!!! Cost of assessing risk and monitoring risk – keeping an eye on Harry

- If the presence of financial market can overcomes these costs, the benefits of finance would be:
  - Facilitate the channeling funds from saver to borrower in more efficient ways.
  - Savers are better off by lending and earn higher interest. Borrower can invest in productive opportunities. Overall the society is better off.
  - Theoretically, if costs of financing is low, more of such financing activities will boost the economy.

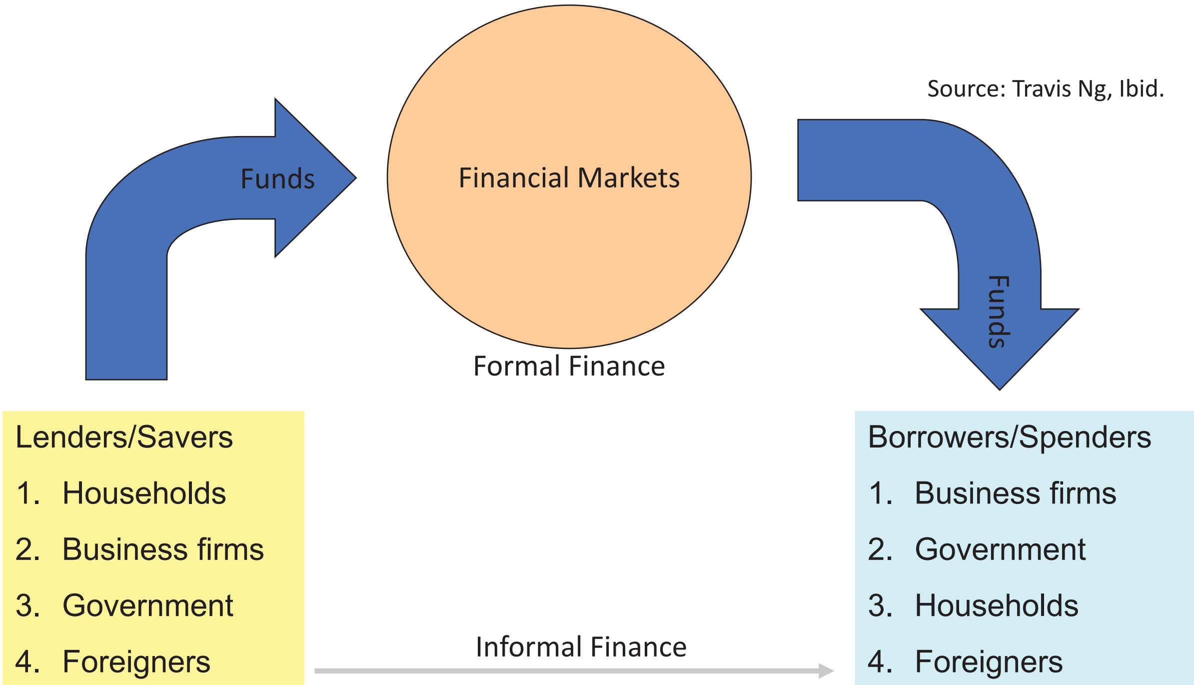
Source: Travis Ng, Ibid.

# Financial intermediaries

- Financial market has players (usually big ones) like the banks, insurance companies, etc, that act as financial intermediaries, and also primary and secondary markets.
- Primary market – trading of new issues of securities such as bond or stock.
- Secondary market – trading of securities that have been previously issued. This injects "liquidity" to the primary market, leading to lower risk for both lenders and borrowers.
- It is set up to overcome/lower the costs as mentioned.
- Lower transaction costs, equilibrium market risk-premium dictates the price (save bargaining costs), lower search costs, established trust, provide risk monitoring more effectively.

Source: Travis Ng, Ibid.

# Flows of funds



# Summary of finance

- In short, finance is an activity that involves borrowers and lenders. They can finance directly, or they can finance through intermediaries.
- It is easy if you lend money to your roommate (informal finance), but if you want to lend money to entrepreneurs say 马化腾 you can also do so through participating in the formal financial market (formal finance). No matter how much you admire him and how confident you are to his firm, you may never run into him throughout your entire life. So essentially, financial market provides a more efficient way of channeling funds.
- In this process, finance (both informal and formal) create ways for investors to allocate scarce resources across time and states of the world, as well as ways for the society to put scarce resources into more productive uses.
- If the financial market functions well, it should be transparent. People would feel comfortable dealing with it. Default rates should be low. Important banks should have low bad debt. And transaction costs would be low, and most important of all, funds will be channeled more efficiently. The society as a whole will be better off with a healthy financial market. [KING and LEVINE (1993)]

  Source: Travis Ng, Ibid.

# Why financial economics?

- With such a basic knowledge in mind, we'll go deeper into understanding why individuals lend and borrow.

- We study finance behaviors by employing the approach economists use. Economics provide an excellent set of tools for us to understand financial market.

Source: Travis Ng, Ibid.

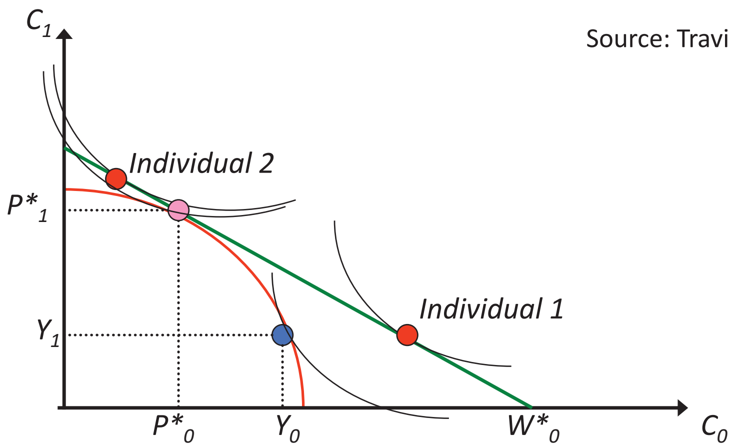
# Why financial economics?

A simple answer to why individuals lend and borrow:

- Our happiness depends not only on current consumption but also on our future consumptions.
- Thus, we want to maximize our happiness through optimizing the allocation of funds available to us across different time periods.
- And such actions involve finance.

# Roadmap for today and next week

- What we said on the last slide is called inter-temporal consumption choice. We study it next to have a concrete understanding of how finance allocates resources across people and time periods.
- (We will learn about another related framework to understand the role of finance in dealing with risks. )




- But first we review some basic math and economic tools.

# Course Roadmap

0. 最优化与微观经济学复习 Math and Economic Foundations
1. 跨期消费投资决策理论
   Intertemporal consumption and saving choice
2. 风险下的投资决策 Decisions under risks
3. 现代资产组合理论 Modern Portfolio Theory
4. 资产定价模型 Capital Asset Pricing Model
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9. 中国金融市场

# 0 Mathematical and Economic Foundations

A Mathematical Preliminaries

1 Unconstrained Optimization
2 Constrained Optimization

B Consumer Optimization

1 Graphical Analysis
2 Algebraic Analysis
3 The Time Dimension

# Mathematical Preliminaries

## Unconstrained Optimization

$$\max_x F(x)$$

## Constrained Optimization

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

# Unconstrained Optimization

To find the value of $x$ that solves

$$\max_x F(x)$$

you can:

1. Try out every possible value of $x$.
2. Use calculus.

Since search could take forever, let's use calculus instead.

# Unconstrained Optimization

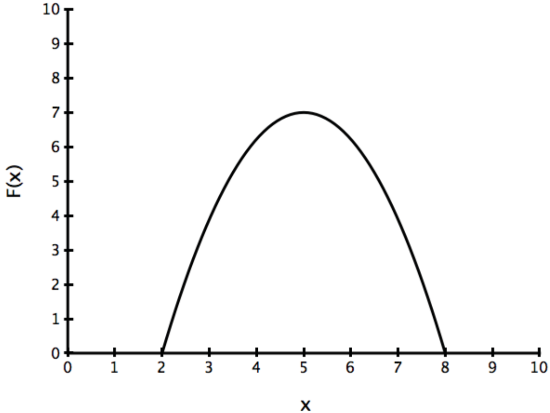
Theorem If $x^*$ solves

$$\max_x F(x),$$

then $x^*$ is a critical point of $F$, that is,

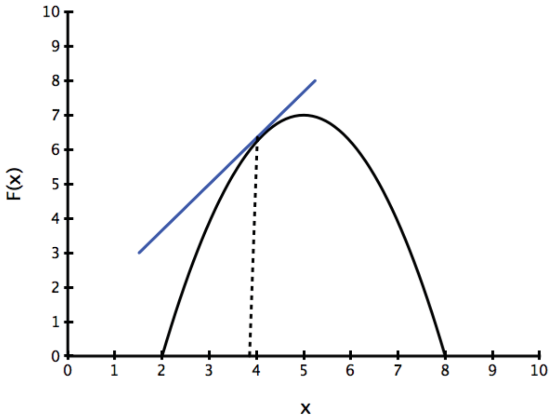
$$F'(x^*) = 0.$$

# Unconstrained Optimization



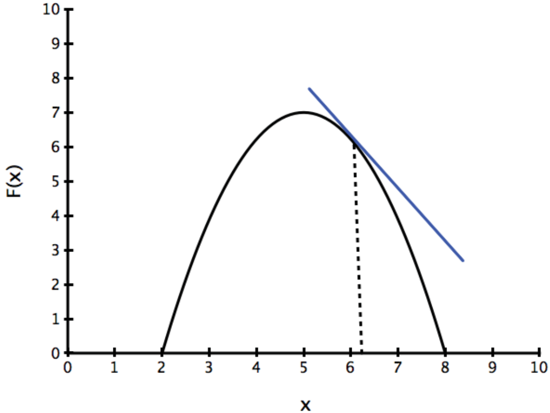

$F(x)$ maximized at $x^* = 5$

# Unconstrained Optimization



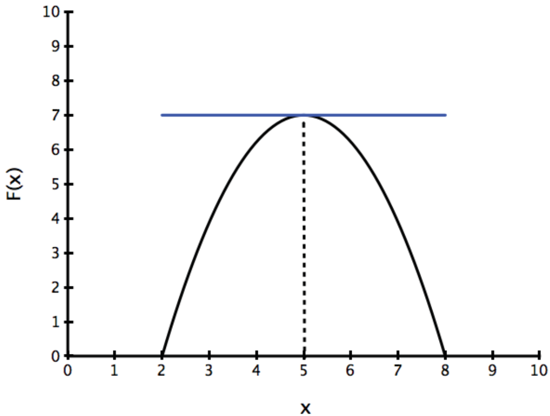

$F'(x) > 0$ when $x < 5$. $F(x)$ can be increased by increasing $x$.

# Unconstrained Optimization




$F'(x) < 0$ when $x > 5$. $F(x)$ can be increased by decreasing $x$.

# Unconstrained Optimization




$F'(x) = 0$ when $x = 5$. $F(x)$ is maximized.

# Unconstrained Optimization

Theorem If $x^*$ solves

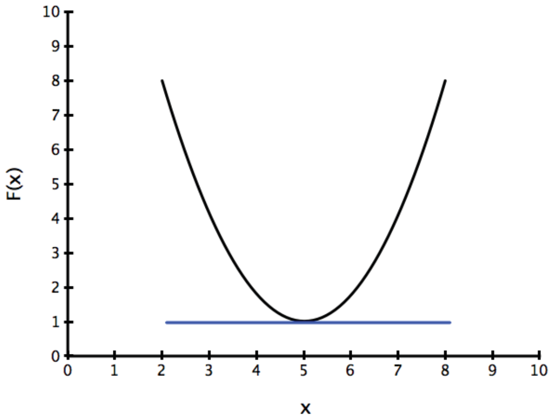
$$\max_x F(x),$$

then $x^*$ is a critical point of $F$, that is,

$$F'(x^*) = 0.$$

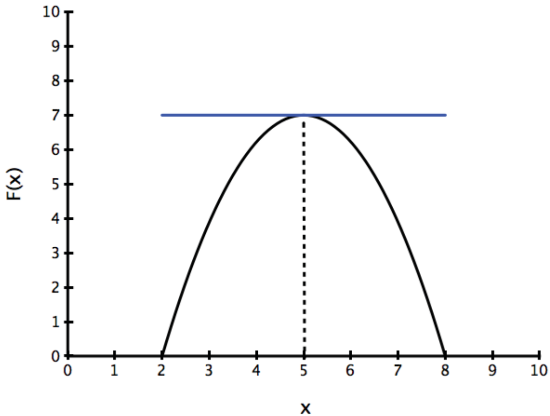
Note that the same first-order necessary condition $F'(x^*) = 0$ also characterizes a value of $x^*$ that minimizes $F(x)$.

# Unconstrained Optimization



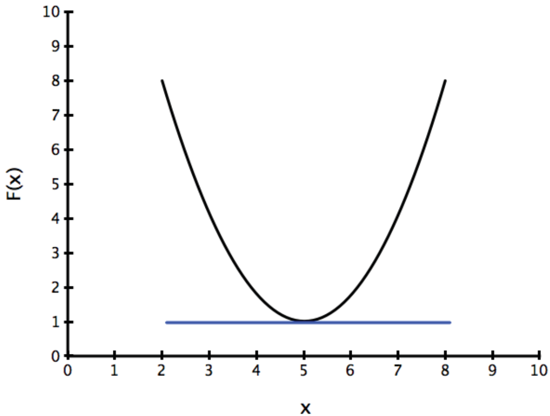

$F'(x) = 0$ when $x = 5$. $F(x)$ is minimized.

# Unconstrained Optimization




$F'(x) = 0$ and $F''(x) < 0$ when $x = 5$. $F(x)$ is maximized.

# Unconstrained Optimization




$F'(x) = 0$ and $F''(x) > 0$ when $x = 5$. $F(x)$ is minimized.

# Unconstrained Optimization

Theorem If $x^*$ solves

$$\max_x F(x),$$

then $x^*$ is a critical point of $F$, that is,

$$F'(x^*) = 0.$$

# Unconstrained Optimization

Theorem If

$$F'(x^*) = 0 \text{ and } F''(x^*) < 0,$$

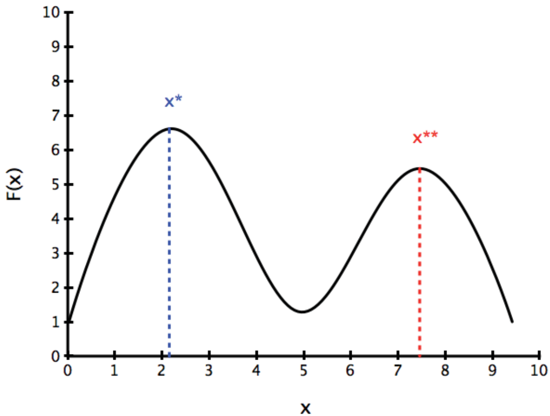
then $x^*$ solves

$$\max_x F(x)$$

(at least locally).

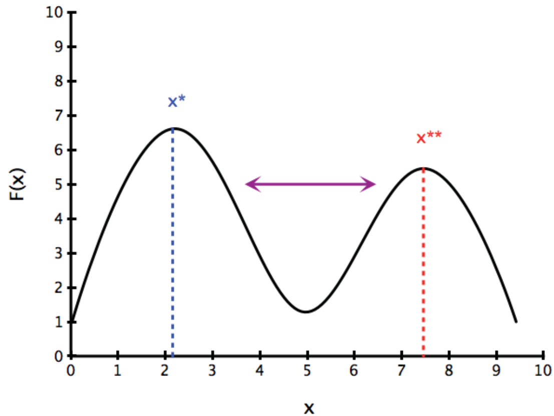
The first-order condition $F'(x^*) = 0$ and the second-order condition $F''(x^*) < 0$ are sufficient conditions for the value of $x$ that (locally) maximizes $F(x)$.

# Unconstrained Optimization




$F'(x^{**}) = 0$ and $F''(x^{**}) < 0$ at the local maximizer $x^{**}$ and
$F'(x^{*}) = 0$ and $F''(x^{*}) < 0$ at the global maximizer $x^{*}$.

# Unconstrained Optimization




$F'(x^{**}) = 0$ and $F''(x^{**}) < 0$ at the local maximizer $x^{**}$ and $F'(x^{*}) = 0$ and $F''(x^{*}) < 0$ at the global maximizer $x^{*}$, but $F''(x) > 0$ in between $x^{*}$ and $x^{**}$.

# Unconstrained Optimization

Theorem If

$$F'(x^*) = 0$$

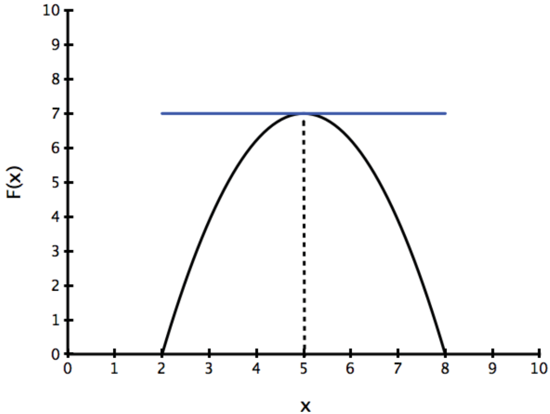
and

$$F''(x) < 0 \text{ for all } x \in \mathbb{R},$$

then $x^*$ solves

$$\max_x F(x).$$

# Unconstrained Optimization




$F''(x) < 0$ for all $x \in \mathbb{R}$ and $F'(5) = 0$. $F(x)$ is maximized when $x = 5$.

# Unconstrained Optimization

If $F''(x) < 0$ for all $x \in \mathbb{R}$, then the function $F$ is concave.

When $F$ is concave, the first-order condition $F'(x^*) = 0$ is both necessary and sufficient for the value of $x$ that maximizes $F(x)$.

And, as we are about to see, concave functions arise frequently and naturally in economics and finance.

# Unconstrained Optimization: Example 1

Consider the problem

$$\max_{x} \left(-\frac{1}{2}\right)(x-\tau)^2,$$

where $\tau$ is a number ($\tau \in \mathbb{R}$) that we might call the "target."

The first-order condition

$$-(x^* - \tau) = 0$$

leads us immediately to the solution: $x^* = \tau$.

# Unconstrained Optimization: Example 2

Consider maximizing a function of three variables:

$$\max_{x_1, x_2, x_3} F(x_1, x_2, x_3)$$

Even if each variable can take on only 1,000 values, there are one billion possible combinations of $(x_1, x_2, x_3)$ to search over!

This is an example of what Richard Bellman (US, 1920-1984) called the "curse of dimensionality."

# Unconstrained Optimization: Example 2

Consider the problem:

$$\max_{x_1,x_2,x_3} \left(-\frac{1}{2}\right)(x_1-\tau)^2 + \left(-\frac{1}{2}\right)(x_2-x_1)^2 + \left(-\frac{1}{2}\right)(x_3-x_2)^2.$$

Now the three first-order conditions

$$-(x_1^*-\tau)+(x_2^*-x_1^*)=0$$
$$-(x_2^*-x_1^*)+(x_3^*-x_2^*)=0$$
$$-(x_3^*-x_2^*)=0$$

lead us to the solution: $x_1^*=x_2^*=x_3^*=\tau$.

## Constrained Optimization

To find the value of $x$ that solves

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

you can:

1. Try out every possible value of $x$.
2. Use calculus.

Since search could take forever, let's use calculus instead.

# Constrained Optimization

A method for solving constrained optimization problems like

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

was developed by Joseph-Louis Lagrange (France/Italy, 1736-1813) and extended by Harold Kuhn (US, b.1925) and Albert Tucker (US, 1905-1995).

# Constrained Optimization

Associated with the problem:

$$\max_{x} F(x) \text{ subject to } c \geq G(x)$$

Define the Lagrangian

例如：不能低约定

$$L(x, \lambda) = F(x) + \lambda[c - G(x)],$$

where $\lambda$ is the Lagrange multiplier.

有多不喜欢这个约束

# Constrained Optimization

$$L(x, \lambda) = F(x) + \lambda[c - G(x)],$$

Theorem (Kuhn-Tucker) If $x^*$ maximizes $F(x)$ subject to $c \geq G(x)$, then there exists a value $\lambda^* \geq 0$ such that, together, $x^*$ and $\lambda^*$ satisfy the first-order condition

$$F'(x^*) - \lambda^* G'(x^*) = 0$$

and the complementary slackness condition

$$\lambda^*[c - G(x^*)] = 0.$$

# Constrained Optimization

In the case where $c > G(x^*)$, the constraint is non-binding. The complementary slackness condition

$$\lambda^*[c - G(x^*)] = 0$$

requires that $\lambda^* = 0$.

And the first-order condition

$$F'(x^*) - \lambda^* G'(x^*) = 0$$

requires that $F'(x^*) = 0$.

# Constrained Optimization

In the case where $c = G(x^*)$, the constraint is binding. The complementary slackness condition

$$\lambda^*[c - G(x^*)] = 0$$

puts no further restriction on $\lambda^* \geq 0$.

Now the first-order condition

$$F'(x^*) - \lambda^* G'(x^*) = 0$$

requires that $F'(x^*) = \lambda^* G'(x^*)$.

# Constrained Optimization: Example 1

For the problem

$$\max_x \left(-\frac{1}{2}\right)(x-5)^2 \text{ subject to } 7 \geq x,$$

$F(x) = (-1/2)(x-5)^2$, $c = 7$, and $G(x) = x$. The Lagrangian is

$$L(x, \lambda) = \left(-\frac{1}{2}\right)(x-5)^2 + \lambda(7-x).$$

# Constrained Optimization: Example 1

With

$$L(x, \lambda) = \left(-\frac{1}{2}\right)(x-5)^2 + \lambda(7-x),$$

the first-order condition

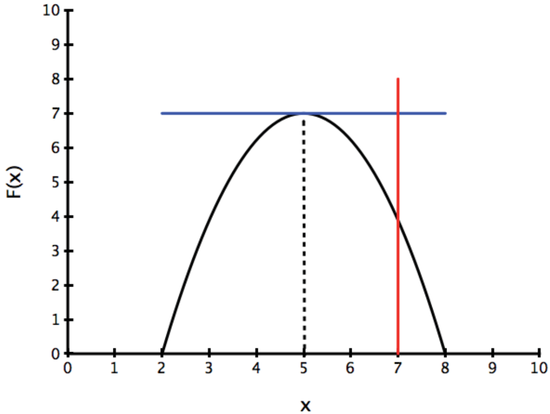
$$-(x^* - 5) - \lambda^* = 0$$

and the complementary slackness condition

$$\lambda^*(7 - x^*) = 0$$

are satisfied with $x^* = 5$, $F'(x^*) = 0$, $\lambda^* = 0$, and $7 > x^*$.

# Constrained Optimization: Example 1




Here, the solution has $F'(x^*) = 0$ since the constraint is nonbinding.

# Constrained Optimization: Example 2

For the problem

$$\max_x \left(-\frac{1}{2}\right)(x-5)^2 \text{ subject to } 4 \geq x,$$

$F(x) = (-1/2)(x-5)^2$, $c = 4$, and $G(x) = x$. The Lagrangian is

$$L(x, \lambda) = \left(-\frac{1}{2}\right)(x-5)^2 + \lambda(4-x).$$

# Constrained Optimization: Example 2

With

$$L(x, \lambda) = \left(-\frac{1}{2}\right)(x-5)^2 + \lambda(4-x),$$

the first-order condition

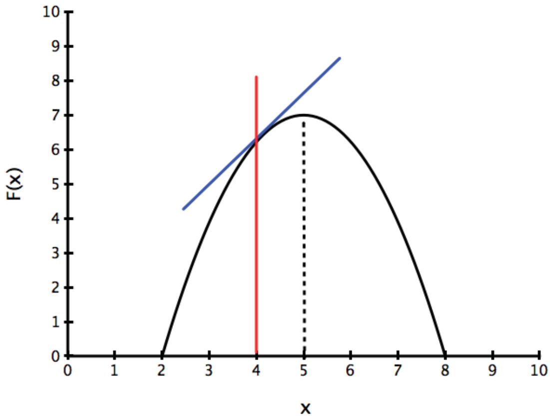
$$-(x^* - 5) - \lambda^* = 0$$

and the complementary slackness condition

$$\lambda^*(4 - x^*) = 0$$

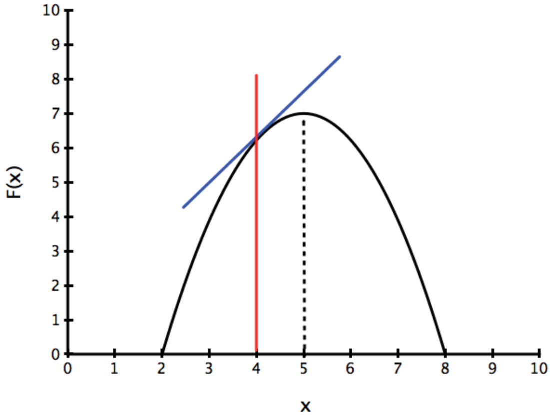
are satisfied with $x^* = 4$ and $F'(x^*) = \lambda^* = 1 > 0$.

# Constrained Optimization: Example 2




Here, the solution has $F'(x^*) = \lambda^* G'(x^*) > 0$ since the constraint is binding. $F'(x^*) > 0$ indicates that we'd like to increase the value of $x$, but the constraint won't let us.

# Constrained Optimization: Example 2




With a binding constraint, $F'(x^*) \neq 0$ but $F'(x^*) - \lambda^* G'(x^*) = 0$. The value $x^*$ that solves the problem is a critical point, not of the objective function $F(x)$, but instead of the entire Lagrangian $F(x) + \lambda[c - G(x)]$.

# Consumer Optimization

1. Graphical Analysis
2. Algebraic Analysis
3. Time Dimension
4. Risk Dimension

# Consumer Optimization

Alfred Marshall, *Principles of Economics*, 1890. – supply and demand

Francis Edgeworth, *Mathematical Psychics*, 1881.

Vilfredo Pareto, *Manual of Political Economy*, 1906. – indifference curves

# Consumer Optimization

John Hicks, *Value and Capital*, 1939. – wealth and substitution effects

Paul Samuelson, *Foundations of Economic Analysis*, 1947. – mathematical reformulation

Irving Fisher, *The Theory of Interest*, 1930. – intertemporal extension.

# Consumer Optimization

Gerard Debreu, *Theory of Value*, 1959.

Kenneth Arrow, "The Role of Securities in the Optimal Allocation of Risk Bearing," *Review of Economic Studies*, 1964.

Extensions to include risk and uncertainty.

# Consumer Optimization: Graphical Analysis

Consider a consumer who likes two goods: apples and bananas.

$$
\begin{aligned}
Y &= \text{income} \\
c_a &= \text{consumption of apples} \\
c_b &= \text{consumption of bananas} \\
p_a &= \text{price of an apple} \\
p_b &= \text{price of a banana}
\end{aligned}
$$

The consumer's budget constraint is

$$Y \geq p_a c_a + p_b c_b$$

# Consumer Optimization: Graphical Analysis

So long as the consumer always prefers more to less, the budget constraint will always bind:

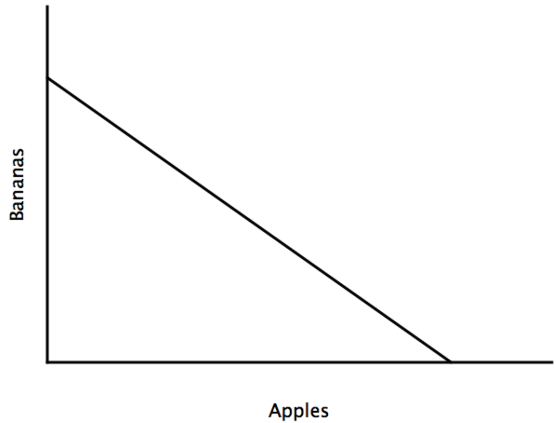
$$Y = p_a c_a + p_b c_b$$

or

$$c_b = \frac{Y}{p_b} - \left(\frac{p_a}{p_b}\right) c_a$$

Which shows that the graph of the budget constraint will be a straight line with slope $-(p_a/p_b)$ and intercept $Y/p_b$.

# Consumer Optimization: Graphical Analysis




The budget constraint is a straight line with slope $-(p_a/p_b)$ and intercept $Y/p_b$.

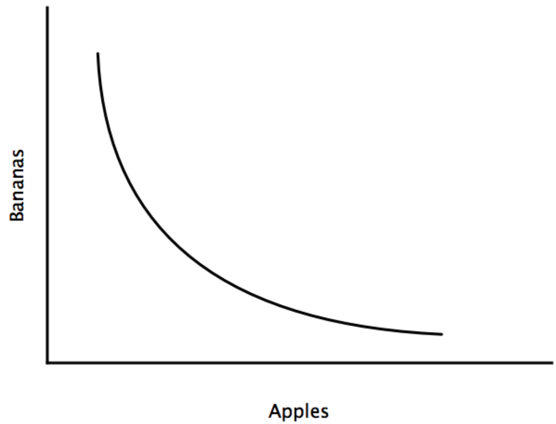
# Consumer Optimization: Graphical Analysis

The budget constraint describes the consumer's market opportunities.

Francis Edgeworth (Ireland, 1845-1926) and Vilfredo Pareto (Italy, 1848-1923) were the first to use indifference curves to describe the consumer's preferences.

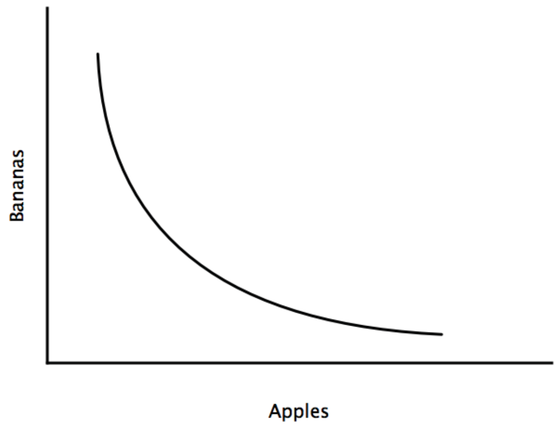
Each indifference curve traces out a set of combinations of apples and bananas that give the consumer a given level of utility or satisfaction.

# Consumer Optimization: Graphical Analysis



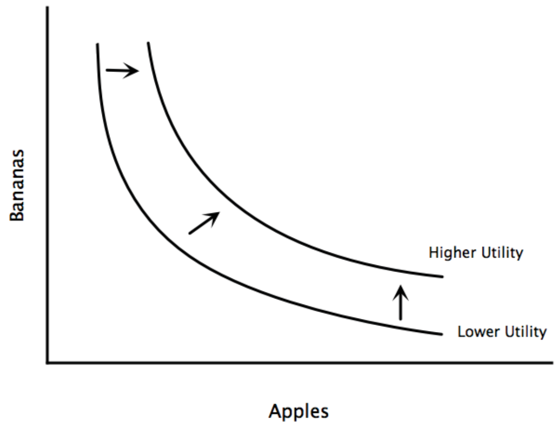

Each indifference curve traces out a set of combinations of apples and bananas that give the consumer a given level of utility.

# Consumer Optimization: Graphical Analysis



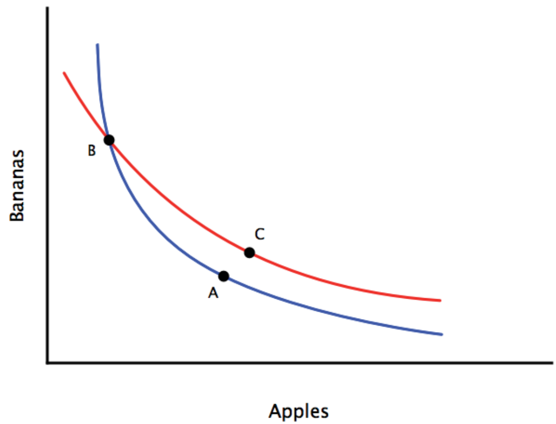

Each indifference curve slopes down, since the consumer requires more apples to compensate for a loss of bananas and more bananas to compensate for a loss of apples, if more is preferred to less.

# Consumer Optimization: Graphical Analysis



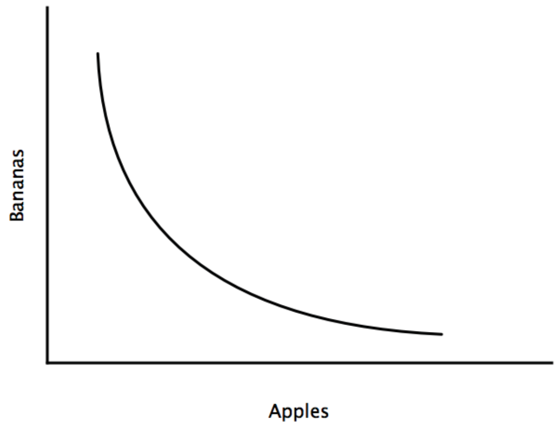

Indifference curves farther away from the origin represent higher levels of utility, if more is preferred to less.

# Consumer Optimization: Graphical Analysis



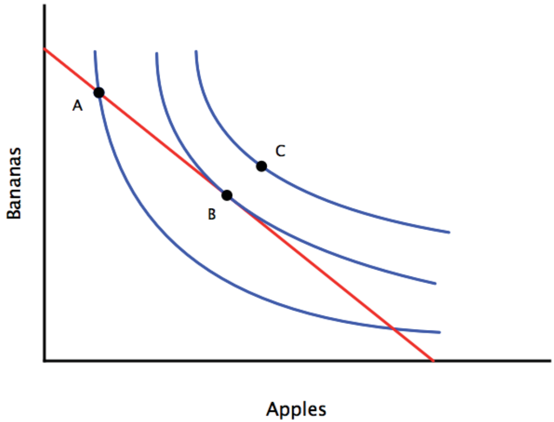

A and B yield the same level of utility, and B and C yield the same level of utility, but C is preferred to A if more is preferred to less. Indifference curves cannot intersect.

# Consumer Optimization: Graphical Analysis



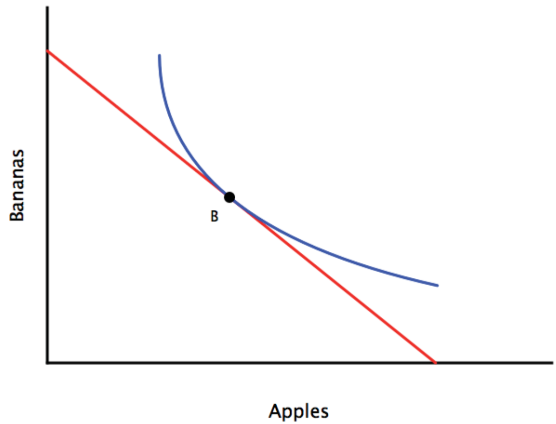

Indifference curves are convex to the origin if consumers have a preference for diversity.

# Consumer Optimization: Graphical Analysis




A is suboptimal and C is infeasible. B is optimal.

# Consumer Optimization: Graphical Analysis




At B, the optimal choice, the indifference curve is tangent to the budget constraint.

# Consumer Optimization: Graphical Analysis

Recall that the budget constraint

$$Y = p_a c_a + p_b c_b$$

or

$$c_b = \frac{Y}{p_b} - \left(\frac{p_a}{p_b}\right) c_a$$

has slope $-(p_a/p_b)$.

# Consumer Optimization: Graphical Analysis

Suppose that the consumer's preferences are also described by the utility function

$$u(c_a) + \beta u(c_b).$$

The function $u$ is increasing, with $u'(c) > 0$, so that more is preferred to less, and concave, with $u''(c) < 0$, so that marginal utility falls as consumption rises.

The parameter $\beta$ measures how much more (if $\beta > 1$) or less (if $\beta < 1$) the consumer likes bananas compared to apples.

# Consumer Optimization: Graphical Analysis

Since an indifference curve traces out the set of $(c_a, c_b)$ combinations that yield a given level of utility $\bar{U}$, the equation for an indifference curve is

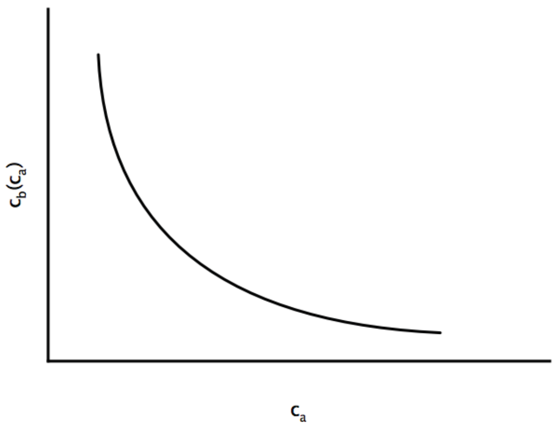
$$\bar{U} = u(c_a) + \beta u(c_b).$$

Use this equation to define a new function, $c_b(c_a)$, describing the number of bananas needed, for each number of apples, to keep the consumer on this indifference curve:

$$\bar{U} = u(c_a) + \beta u[c_b(c_a)].$$

无差别曲线的推导

# Consumer Optimization: Graphical Analysis




The function $c_b(c_a)$ satisfies $\bar{U} = u(c_a) + \beta u[c_b(c_a)]$.

# Consumer Optimization: Graphical Analysis

Differentiate both sides of

$$\bar{U} = u(c_a) + \beta u[c_b(c_a)]$$

to obtain

$$0 = u'(c_a) + \beta u'[c_b(c_a)]c_b'(c_a)$$

or

$$c_b'(c_a) = -\frac{u'(c_a)}{\beta u'[c_b(c_a)]}.$$

# Consumer Optimization: Graphical Analysis

This last equation,

$$c_b'(c_a) = -\frac{u'(c_a)}{\beta u'[c_b(c_a)]},$$

written more simply as

$$c_b'(c_a) = -\frac{u'(c_a)}{\beta u'(c_b)},$$

measures the slope of the indifference curve: the consumer's marginal rate of substitution.

# Consumer Optimization: Graphical Analysis

Thus, the tangency of the budget constraint and indifference curve can be expressed mathematically as

$$\frac{p_a}{p_b} = \frac{u'(c_a)}{\beta u'(c_b)}.$$

The marginal rate of substitution equals the relative prices.

# Consumer Optimization: Graphical Analysis

Returning to the more general expression

$$c_b'(c_a) = -\frac{u'(c_a)}{\beta u'[c_b(c_a)]},$$

we can see that $c_b'(c_a) < 0$, so that the indifference curve is downward-sloping, so long as the utility function $u$ is strictly increasing, that is, if more is preferred to less.

# Consumer Optimization: Graphical Analysis

$$c_b'(c_a) = -\frac{u'(c_a)}{\beta u'[c_b(c_a)]}$$

Differentiating again yields

$$c_b''(c_a) = -\frac{\beta u'[c_b(c_a)]u''(c_a) - u'(c_a)\beta u''[c_b(c_a)]c_b'(c_a)}{\{\beta u'[c_b(c_a)]\}^2},$$

which is positive if $u$ is strictly increasing (more is preferred to less) and concave (diminishing marginal utility). In this case, the indifference curve will be convex. Again, we see how concave functions have mathematical properties and economic implications that we like.

# Consumer Optimization: Algebraic Analysis

Graphical analysis works fine with two goods.

But what about three goods? That depends on how good an artist you are!

And what about four or more goods? Our universe won't accommodate a graph like that!

But once again, calculus makes it easier!

# Consumer Optimization: Algebraic Analysis

Consider a consumer who likes three goods:

$$
\begin{aligned}
Y &= \text{income} \\
c_i &= \text{consumption of goods } i = 0, 1, 2 \\
p_i &= \text{price of goods } i = 0, 1, 2
\end{aligned}
$$

Suppose the consumer's utility function is

$$u(c_0) + \alpha u(c_1) + \beta u(c_2),$$

where $\alpha$ and $\beta$ are weights on goods 1 and 2 relative to good 0.

# Consumer Optimization: Algebraic Analysis

The consumer chooses $c_0$, $c_1$, and $c_2$ to maximize the utility function

$$u(c_0) + \alpha u(c_1) + \beta u(c_2),$$

subject to the budget constraint

$$Y \geq p_0 c_0 + p_1 c_1 + p_2 c_2.$$

The Lagrangian for this problem is

$$L = u(c_0) + \alpha u(c_1) + \beta u(c_2) + \lambda(Y - p_0 c_0 - p_1 c_1 - p_2 c_2).$$

# Consumer Optimization: Algebraic Analysis

$$L = u(c_0) + \alpha u(c_1) + \beta u(c_2) + \lambda(Y - p_0 c_0 - p_1 c_1 - p_2 c_2).$$

First-order conditions:

$$u'(c_0^*) - \lambda^* p_0 = 0$$

$$\alpha u'(c_1^*) - \lambda^* p_1 = 0$$

$$\beta u'(c_2^*) - \lambda^* p_2 = 0$$

# Consumer Optimization: Algebraic Analysis

The first-order conditions

Quant

$$u'(c_0^*) - \lambda^* p_0 = 0$$

$$\alpha u'(c_1^*) - \lambda^* p_1 = 0$$

$$\beta u'(c_2^*) - \lambda^* p_2 = 0$$

imply

$$\frac{u'(c_0^*)}{\alpha u'(c_1^*)} = \frac{p_0}{p_1} \text{ and } \frac{u'(c_0^*)}{\beta u'(c_2^*)} = \frac{p_0}{p_2} \text{ and } \frac{\alpha u'(c_1^*)}{\beta u'(c_2^*)} = \frac{p_1}{p_2}.$$

The marginal rate of substitution equals the relative prices.

# Consumer Optimization: The Time Dimension

Irving Fisher (US, 1867-1947) was the first to recognize that the basic theory of consumer decision-making could be used to understand how to optimally allocate spending intertemporally, that is, over time, as well as how to optimally allocate spending across different goods in a static, or point-in-time, analysis.

# Consumer Optimization: The Time Dimension

Following Fisher, return to the case of two goods, but reinterpret:

$$c_0 = \text{consumption today}$$

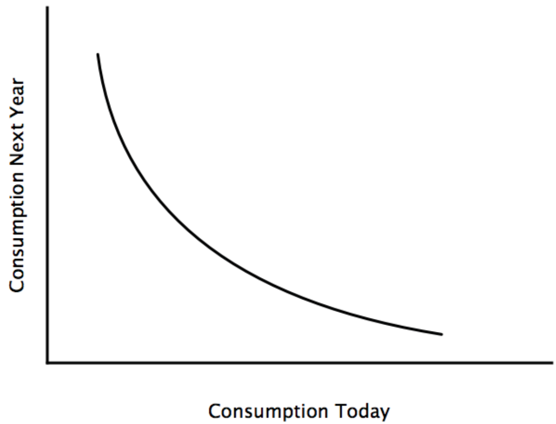
$$c_1 = \text{consumption next year}$$

Suppose that the consumer's utility function is

$$u(c_0) + \beta u(c_1),$$

where $\beta$ now has a more specific interpretation, as the discount factor, a measure of patience.

# Consumer Optimization: The Time Dimension




A concave utility function implies that indifference curves are convex, so that the consumer has a preference for a smoothness in consumption.

# Consumer Optimization: The Time Dimension

Next, let

$$Y_0 = \text{income today}$$
$$Y_1 = \text{income next year}$$
$$s = \text{amount saved (or borrowed if negative) today}$$
$$r = \text{interest rate}$$

# Consumer Optimization: The Time Dimension

Today, the consumer divides his or her income up into an amount to be consumed and an amount to be saved:

$$Y_0 \geq c_0 + s.$$

Next year, the consumer simply spends his or her income, including interest earnings if $s$ is positive or net of interest expenses if $s$ is negative:

$$Y_1 + (1 + r)s \geq c_1.$$

# Consumer Optimization: The Time Dimension

Divide both sides of next year's budget constraint by $1 + r$ to get

$$\frac{Y_1}{1+r} + s \geq \frac{c_1}{1+r}.$$

Now combine this inequality with this year's budget constraint

$$Y_0 \geq c_0 + s.$$

to get

$$Y_0 + \frac{Y_1}{1+r} \geq c_0 + \frac{c_1}{1+r}.$$

# Consumer Optimization: The Time Dimension

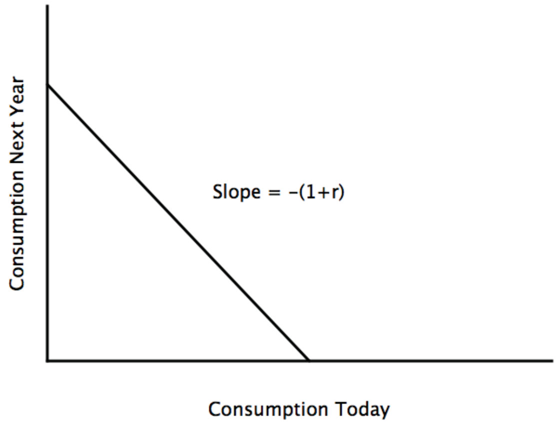
The "lifetime" budget constraint

$$Y_0 + \frac{Y_1}{1+r} \geq c_0 + \frac{c_1}{1+r}$$

says that the present value of income must be sufficient to cover the present value of consumption over the two periods. It also shows that the "price" of consumption today relative to the "price" of consumption next year is related to the interest rate via

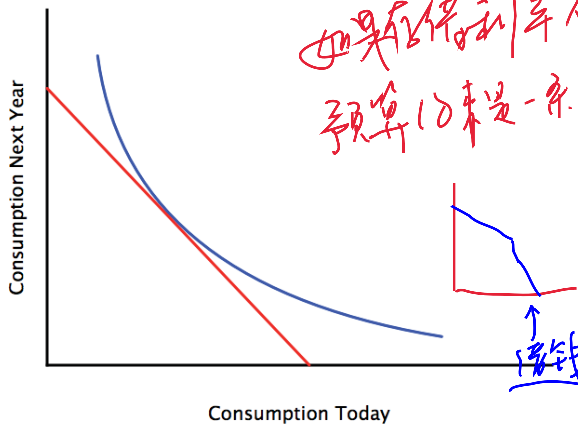
$$\frac{p_0}{p_1} = 1 + r.$$

# Consumer Optimization: The Time Dimension




The slope of the intertemporal budget constraint is $-(1 + r)$.

# Consumer Optimization: The Time Dimension




如果存贷利率不同，那么预算约束是一条折线

借钱利率高、更陡峭

At the optimum, the intertemporal marginal rate of substitution equals the slope of the intertemporal budget constraint.

# Consumer Optimization: The Time Dimension

We now know the answer ahead of time: if we take an algebraic approach to solve the consumer's problem, we will find that the IMRS equals the slope of the intertemporal budget constraint:

$$\frac{u'(c_0)}{\beta u'(c_1)} = 1 + r.$$

But let's use calculus to derive the same result.

# Consumer Optimization: The Time Dimension

The problem is to choose $c_0$ and $c_1$ to maximize utility

$$u(c_0) + \beta u(c_1)$$

subject to the budget constraint

$$Y_0 + \frac{Y_1}{1+r} \geq c_0 + \frac{c_1}{1+r}.$$

The Lagrangian is

$$L = u(c_0) + \beta u(c_1) + \lambda \left( Y_0 + \frac{Y_1}{1+r} - c_0 - \frac{c_1}{1+r} \right).$$

## Consumer Optimization: The Time Dimension

$$L = u(c_0) + \beta u(c_1) + \lambda \left( Y_0 + \frac{Y_1}{1+r} - c_0 - \frac{c_1}{1+r} \right).$$

The first-order conditions

$$u'(c_0^*) - \lambda^* = 0$$

$$\beta u'(c_1^*) - \lambda^* \left( \frac{1}{1+r} \right) = 0.$$

lead directly to the graphical result

$$\frac{u'(c_0^*)}{\beta u'(c_1^*)} = 1 + r.$$

# Consumer Optimization: The Time Dimension

At first glance, Fisher's model seems unrealistic, especially in its assumption that the consumer can borrow at the same interest rate $r$ that he or she receives on his or her savings.

A reinterpretation of saving and borrowing in this framework, however, can make it more applicable, at least for some consumers.

# Investment Strategies and Cash Flows

| Investment Strategy | Cash Flow at $t = 0$ | Cash Flow at $t = 1$ |
|---|---|---|
| Saving | $-1$ | $+(1+\mathrm{r})$ |
| Buying a bond (long position in bonds) | $-1$ | $+(1+\mathrm{r})$ |

# Investment Strategies and Cash Flows

| Investment Strategy | Cash Flow at $t = 0$ | Cash Flow at $t = 1$ |
|---|---|---|
| Borrowing | $+1$ | $-(1 + r)$ |
| Issuing a bond | $+1$ | $-(1 + r)$ |
| Short selling a bond (short position in bonds) | $+1$ | $-(1 + r)$ |
| Selling a bond (out of inventory) | $+1$ | $-(1 + r)$ |

# Investment Strategies and Cash Flows

| Investment Strategy | Cash Flow at $t = 0$ | Cash Flow at $t = 1$ |
|---|---|---|
| Buying a stock (long position in stocks) | $-P_0^s$ | $+P_1^s$ |
| Short selling a stock (short position in stocks) | $+P_0^s$ | $-P_1^s$ |
| Selling a stock (out of inventory) | $+P_0^s$ | $-P_1^s$ |

# Consumer Optimization: The Time Dimension

Someone who already owns bonds can "borrow" by selling a bond out of inventory. In fact, theories like Fisher's work better when applied to consumers who already own stocks and bonds.

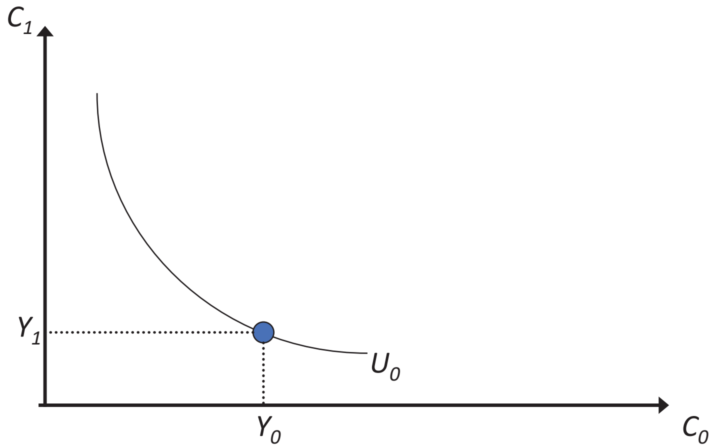
Greg Mankiw and Stephen Zeldes, "The Consumption of Stockholders and Nonstockholders," *Journal of Finance*, 1991.

Annette Vissing-Jorgensen, "Limited Asset Market Participation and the Elasticity of Intertemporal Substitution," *Journal of Political Economy*, 2002.

# How Capital Market Can Improve Welfare: An Illustrative Example

# No Intertemporal Wealth Allocation

- Recall that an individual maximizes his happiness subject to constraints. What are the constraints?
- It depends on the options available for the individual to allocate his wealth across different time periods.
- We study two options: [1] Production opportunity and [2] participation of capital market.
- We assume the individual has endowment of $Y_0$ and $Y_1$ in the current and future periods respectively. So, we can plot the endowment point on the diagram.
- Constraint A: "With no wealth allocation across periods", his utility is $U_0$.

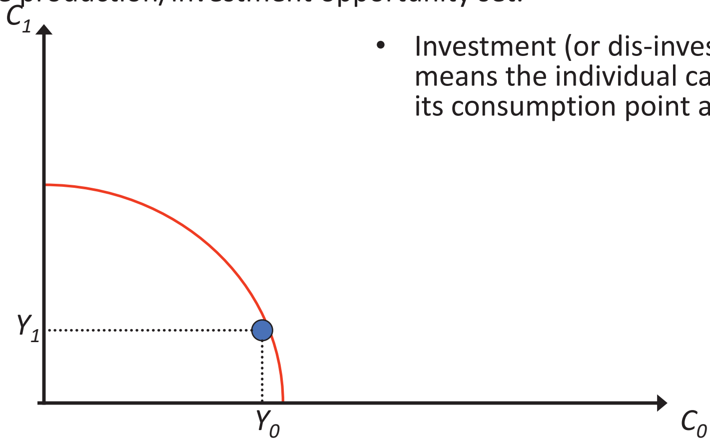



Source: Travis Ng, Ibid.

# Production Opportunity

Constraint B: “The individual can only invest in production opportunities to allocate wealth across periods”

- Now, we introduce production opportunities that allow a unit of current savings/investment to be turned into more than one unit of future consumption.
- The production opportunities are subject to decreasing returns to scale. With this in mind, we can plot the constraint on the $C_0$-$C_1$ space.
- We call this constraint the production opportunity set (POS).
- The slope of the POS is now called the Marginal Rate of Transformation (MRT) offered by the production/investment opportunity set.




- Investment (or dis-investment) means the individual can move its consumption point along POS.

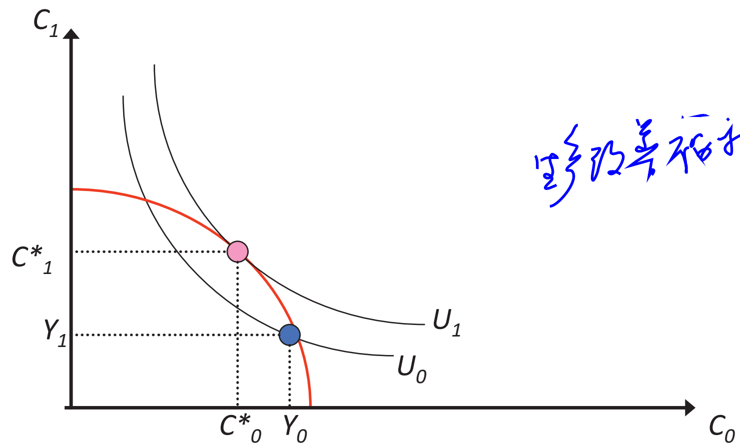
Source: Travis Ng, Ibid.

# Production Opportunity

- At the endowment point, the individual is not maximizing its utility subject to the constraint B. He can do better by investing more (i.e., move north-west along the POS
- The equilibrium is when he invests until the return offered by the marginal investment is just equal to its subjective rate of time preference. In math, we have:

  (slope of POS) *MRT = MRS* (slope of indifference curve)

- The existence of production opportunity makes the individual better off.
- Can the individual do even better?




增多的幸福利

Source: Travis Ng, Ibid.

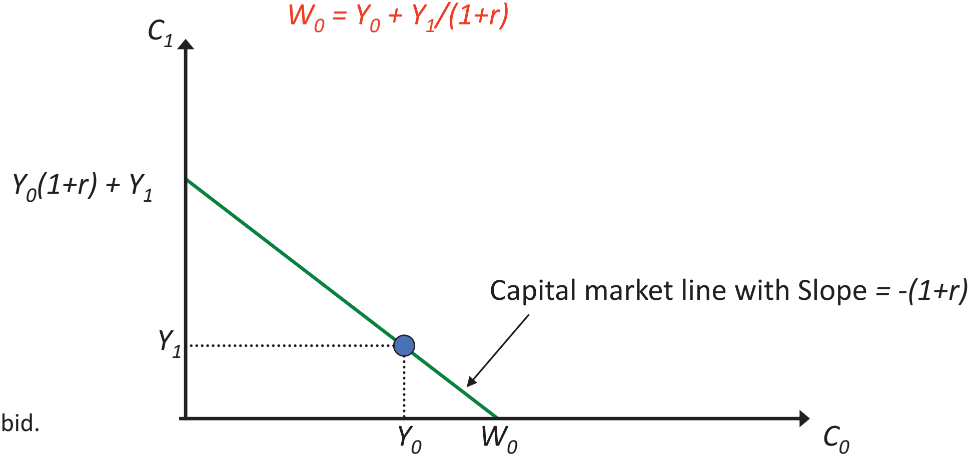
# Capital market

- Now, instead of one individual, let's assume there are many individuals in the economy. Some are lenders, while others are borrowers. Among them, there are opportunities to borrow and lend at the market-determined interest rate (r).

Constraint C: No production opportunity. But individuals can lend/borrow at r.

- We can then graph the borrowing and lending opportunities along the capital market line.
- Now, we introduce the concept of wealth. Wealth of an individual is the present value of his current and future endowment. Thus,

$$W_0 = Y_0 + Y_1/(1+r)$$

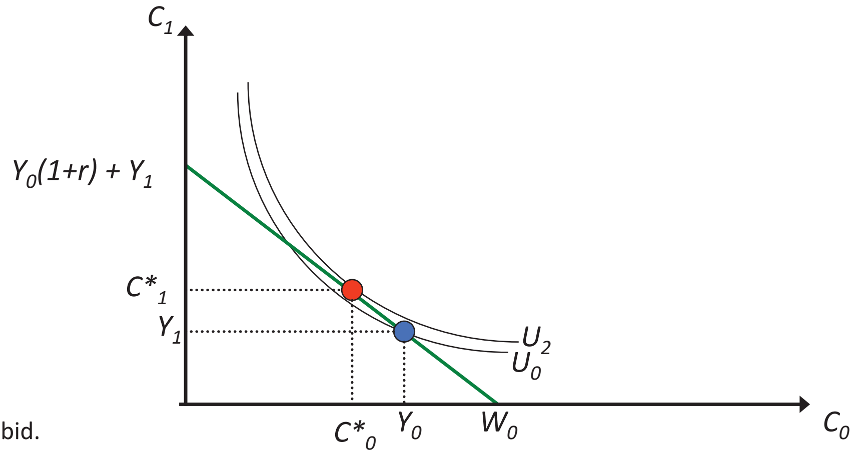



Source: Travis Ng, Ibid.

# Capital market

- The feasible consumption set is now all the points along the capital market line.
- Moving North-west along the capital market line, the individual can achieve a higher utility ($U_2 > U_0$)
- This individual is now lending ($Y_0 - C^*_0$) amount of money, and will get back ($(1+r)(Y_0 - C^*_0)$) in the next period so that he can consume a total of $C^*_1 = Y_1 + (1+r)(Y_0 - C^*_0)$.
- Equilibrium condition is:

$$MRS = (1+r)$$

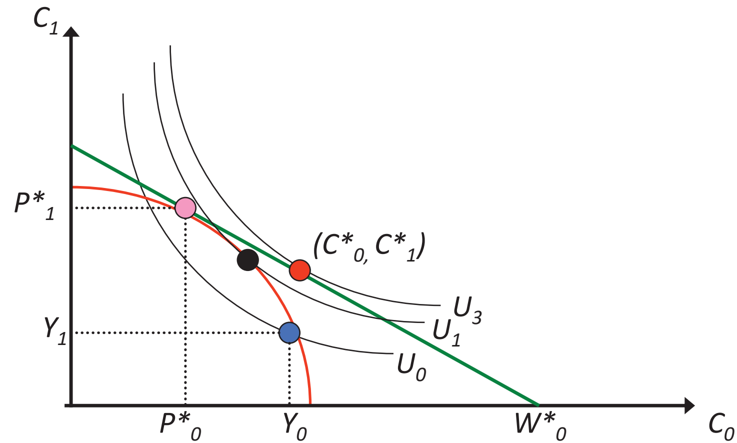



Source: Travis Ng, Ibid.

# Production and capital market

Constraint D: Individuals can now borrow/lend at r & invest in production opportunities.

- With only production opportunity, the individual achieve $U_1$ only. But if capital market is introduced, he can actually do better.
- At point ●, the individual can borrow more money at rate *r* from the capital market, and be able to invest more and get return higher than *r*. Until in equilibrium, he reaches ●, where the return on the marginal investment is equal to the market interest rate *r*. At this point, his wealth is maximized.
- Now his wealth is $W^*_0 = P^*_0 + P^*_1/(1+r)$ which is larger than $W_0$.
- With his wealth maximized, he chooses $(C^*_0, C^*_1)$ to consume and yield him $U_3$.



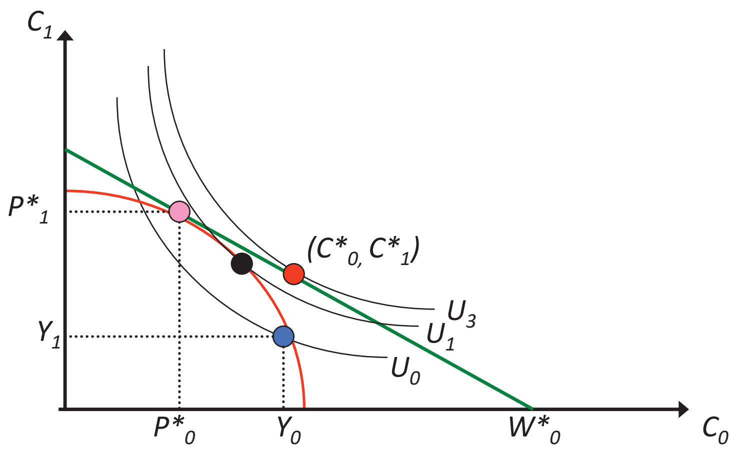

Source: Travis Ng, Ibid.

# Fisher Separation Theorem

[illegible] target

<u>Message:</u>

- The decisions of production and consumption involve 2 distinct steps.
  - <u>1st step:</u> Choosing production point by moving along POS and produce at the point where MRT = (1+ r), this means return on the marginal investment is just equal to market interest rate.
  - <u>2nd step:</u> Choosing consumption point by moving along the capital market line and consume at the point where MRS = (1 + r), this means subjective rate of time preference is equal to market interest rate.
- We call this the "FISHER SEPARATION THEOREM". The important point is the production point is governed solely by objective criteria, namely, the set of opportunities available and the market interest rate. This is independent of individual's subjective rate of time preferences.

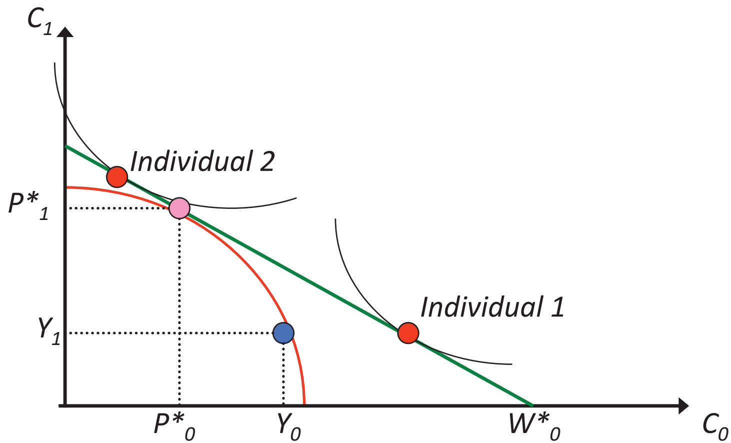


# Fisher Separation Theorem

Implication 1:

- As the graph below shows, with two different individuals that differs only in their subjective preferences, given the same opportunity set, both of them would choose the exact same point of production regardless of the difference of their preferences.

*Question: Read the formal definition of Fisher Separation Theorem on page 10. What are the required assumptions and why?*

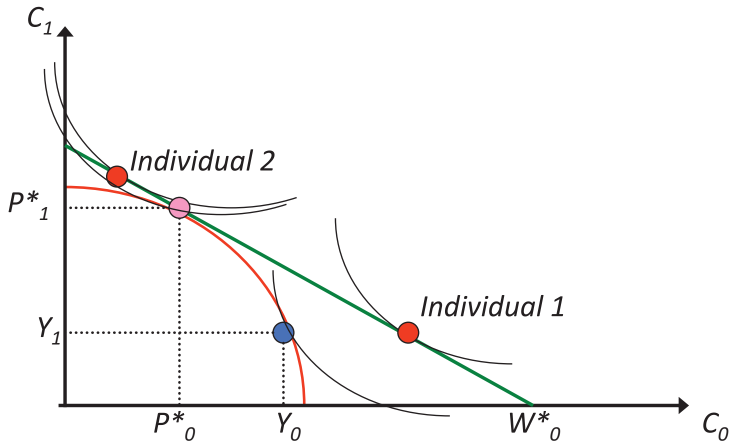



Source: Travis Ng, Ibid.

# Fisher Separation Theorem

Implication 2:

- The role of capital market: the market channel funds from the lenders to the borrowers. Setting demand = supply, we have a market-determined r. Given the individual's exposure to his own production opportunities, he may decide whether to lend or borrow money. By allowing lending and borrowing, those who need money can get financed, while those have excess fund will be able to lend out and earn interest. Everyone is made better off.
- In short, as shown below, capital market is important because everyone can be happier with it.
- The sources of happiness enhancement: 1) measurement: capital market gives a precise benchmark for what to invest and what not to invest. 2) financing: individuals can actually borrow and lend to invest in what ought to be invest.



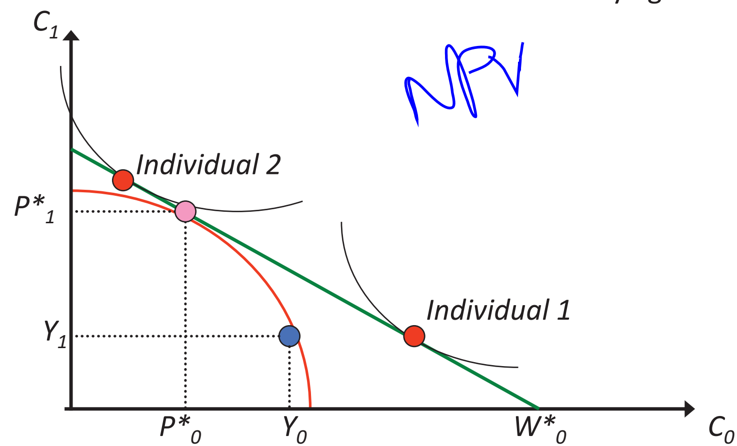

Source: Travis Ng, Ibid.

# Fisher Separation Theorem

Implication 3:

- Consider the following two investors investing all their money on the stocks of a single firm. Their well-being is thus tied to the well-being of the firm. Consider the firm is making decision of what to produce.
- Fisher Separation Theorem implies even the two investors differ in their subjective perception of how to consume between now and future, they both has one unified objective, i.e, to maximize their current wealth.
- Doing so means the firm can maximize its value. This is the same as investing until the return on the marginal investment is just equal to the cost of capital, i.e, the market interest rate. And the firm knows that its shareholders will unanimously agree on what it does.

NPV



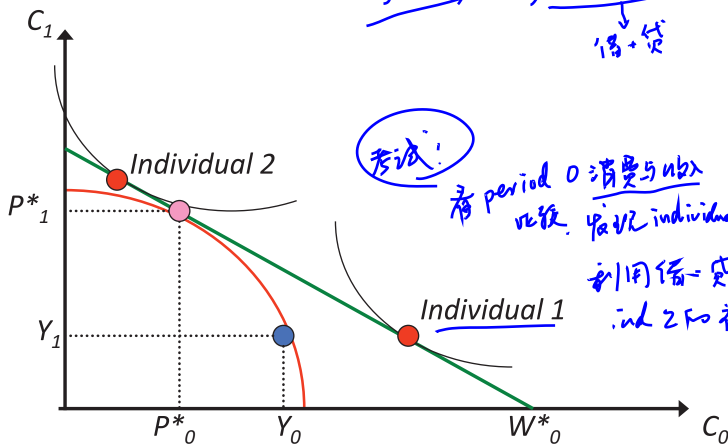

Source: Travis Ng, Ibid.

# Fisher Separation Theorem

Implication 3:

- MRT = (1 + r) is the point where both of the two individuals would agree for the firm to produce.
- This is exactly the famous project selection rule, the "positive Net Present Value rule". The firm value is maximized by taking all projects that have positive NPV.
- NPV = -initial investment + present value of future payout discounted by cost of capital.
- Cost of capital = r

参与生产 + 参与金融市场 → 借+贷




考试：看 period 0 消费与收入比较，发现 individual 1 借钱，利用借+贷，back out ind 2 的初始禀赋

Source: Travis Ng, Ibid.

# How to max shareholder's wealth?

- We again uses Fisher Separation Theorem
- Given perfect and complete capital markets, the owners of the firm (shareholders) will unanimously support the acceptance of all projects until the least favourable project has return the same as the cost of capital.
- In the presence of capital markets, the cost of capital is the market interest rate.
- The project selection rule, i.e., equate

  marginal rate of return of investment = cost of capital (market interest rate)
- Is exactly the same as the positive net present value rule:

**Net Present Value Rule**

- Calculate the NPV for all available (independent) projects. Those with positive NPV are taken.

  At the optimum:

  NPV of the least favourable project ~= zero
- This is a rule of selecting projects of a firm that no matter how individual investors of that firm differ in their own opinion (preferences), such rule is still what they are willing to direct the manager to follow.

Source: Travis Ng, Ibid.

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