Time Series Analysis Y.D. Tu

2019 Spring

# Homework #2

**(Due Wednesday 27 March)**

1. At time series is called covariance stationary if

$$E(Z_t) = \mu, \text{ cov}(Z_t, Z_{t+k}) = \gamma_k, \text{ for all } t, k > 0.$$

Show that any series which has a Wold decomposition, i.e.

$$Z_t = c_0 a_t + c_1 a_{t-1} + c_2 a_{t-2} + \cdots$$

where $a_t$ is a white noise and $\sum_i c_i^2 < \infty$ is covariance stationary. In particular, all MA processes are covariance stationary.

2. Assume $Z_t$ follows an MA(2) model,

$Z_t = a_t + 2.2\, a_{t-1} + 0.4\, a_{t-2}$, where the $a_t \sim i.i.d.N(0,2)$.

(a) Write down the auto covariance generating function, and derive the variance and the autocorrelations $\{r_i, i \geq 1\}$ of $Z_t$.

(b) Is the model invertible? If not invertible, find the equivalent invertible representation and calculate the corresponding parameters.

3. For an ARMA(1, 2) model

$(1 - \phi B) Y_t = (1 - \theta_1 B - \theta_2 B^2) a_t$, where the $a_t \sim i.i.d.N(0, \sigma^2)$.

(a) When can this model be reduced to an MA(1) model?

(b) If it is reducible, when is it invertible? Also show that $|\rho_1| < 0.5$, where $\rho_1$ is the autocorrelation between $Y_t$ and $Y_{t+1}$ for invertible MA(1) model.

Unless otherwise expressed explicitly, assuming it is not reducible in the following.

(c) Show that $\theta_1 + \theta_2 < 1, \theta_2 - \theta_1 < 1$ and $|\theta_2| < 1$ are the conditions for invertibility.

(d) When is the model stationary?

(e) Derive the $\rho$-weight and $\gamma$-weights.

(f) Derive the variance and autocorrelations $\{r_k,\ k \geq 1\}$ of $Y_t$ in terms of $(\phi, \theta_1, \theta_2, \sigma^2)$.

4. Consider the following two independent processes $X_t$ and $Y_t$:

$$X_t = (1-\theta_1 B)a_t, \quad (1-\phi B)Y_t = (1-\theta_2 B)b_t,$$

where $a_t$ and $b_t$ are $i.i.d.\ N(0, \sigma^2)$, and $|\phi|<1, |\theta_1|<1, |\theta_2|<1$. Let $Z_t = X_t + Y_t$.

(a) Show that $Z_t$ follows an ARMA ($p$, $q$) model with finite $p$ and $q$, find the values of $p$ and $q$.

(b) Express their parameters in terms of $(\phi, \theta_1, \theta_2, \sigma^2)$.

5. Derive the conditional likelihood and the exact likelihood functions for an AR (2) model and those for an MA(1) model. Be sure to state any assumptions you have made about the process.