Econ 240A, Fall 2018
Problem Set 2

Due date: Monday, Sept. 17

Review of moments, moment generating functions, distribution families, exponential family, multiple random varibles, independence, covariance, conditional probabilities, distributions, expectations, inequalities.
Note: Problems start with a star, *, are optional and don't count for grade. Of course, feel free to write them up if you want.

1. **Probability integral transform**
   Suppose X is a continuous random variable with strictly increasing cdf F. Show that the random variable $Y = F(X) \sim \mathrm{U}[0,1]$.

2. **Inverse transform sampling**
   Suppose $F : \mathbb{R} \to [0,1]$ is a strictly increasing continuous function satisfying $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$ and a random varible $Y \sim \mathrm{U}[0,1]$. Show that $X = F^{-1}(Y) \sim F$.

3. **Moments and moment generating functions**

   (a) Let X be a random variable. Show that if $\mathbb{E}(|X|^r) < \infty$ for some $r \in (0,\infty)$, then $\mathbb{E}(|X|^l) < \infty$ for any $l \in (0,\infty)$, $l \le r$.
   Hint: try Jensen's inequality or show $|x|^l < |x|^r + 1$.

   (b) Let $X \sim \mathrm{N}(0,1)$. Find the moment generating function of X, $M_X(t)$. Use it to find $\mathbb{E}X$, $\mathbb{E}X^2$, $\mathbb{E}X^3$, $\mathbb{E}X^4$.

4. **Covariance**

   (a) Suppose the random vector $(X,Y)'$ has uniform distribution on the square $(-1,1) \times (-1,1)$. Find $\mathrm{Cov}(X,Y)$. Are X and Y independent?

   (b) Suppose the random vector $(X,Y)'$ has uniform distribution on the ball $\{(x,y)' \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$. Find $\mathrm{Cov}(X,Y)$. Are X and Y independent?

   (c) Let the random variable $X \sim F$. Show that if $g(\cdot)$ and $h(\cdot)$ are non-decreasing functions on $\mathbb{R}$, then
   $$\mathrm{Cov}(g(X), h(X)) \ge 0.$$
   Hint: Show that if $X_1, X_2 \overset{\text{iid}}{\sim} F$ then $\mathbb{E}[(g(X_1) - g(X_2))(h(X_1) - h(X_2))] \ge 0$.

5. **The gamma distribution**
   The gamma function is defined for $\alpha > 0$ by
   $$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

   (a) Use integration by parts to show that $\Gamma(x+1) = x\Gamma(x)$, $\forall x > 0$. Show that $\Gamma(x+1) = x!$ for $x = 0, 1, \ldots$.

(b) * Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

(c) Show that the function

$$p(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} & x > 0 \\ 0 & x \le 0 \end{cases} .$$

is a probability density when $\alpha > 0$ and $\beta > 0$. This is called the gamma density with parameters α and β. The corresponding probability distribution is denoted $\text{Gamma}(\alpha, \beta)$.

(d) * Show that if $X \sim \text{Gamma}(\alpha, \beta)$, then $\mathbb{E}(X^r) = \beta^r \Gamma(\alpha + r)/\Gamma(\alpha)$. Use this formula to find the mean and variance of X.

(e) * Show that if $X \sim \text{Gamma}(\alpha, 1)$, then its moment generating function $M_X(t) = \left(\frac{1}{1-t}\right)^\alpha$, $t < 1$. Use it to show that if $X \sim \Gamma(a_1, 1)$, $Y \sim \Gamma(a_2, 1)$ and X, Y are independent, then $X + Y \sim \text{Gamma}(a_1 + a_2, 1)$.

(f) * The chi-squared distribution and the exponential distribution are special cases of the gamma distribution. Give the parametrization corresponding to these two useful distributions.

6. Best linear predictor

Let X, Y be two random variables with finite second moments. We know that

$$\mathbb{E}\left(Y \mid X\right) = g^*\left(X\right) = \arg\min_{g(X)\in\mathcal{G}} \mathbb{E}\left[\left(Y - g\left(X\right)\right)^2\right]$$
$$\mathcal{G} = \{g(X): \quad g: \mathbb{R} \to \mathbb{R},\ g \text{ measurable } ^1\}.$$

This says $\mathbb{E}\left(Y|X\right)$ is the best predictor of Y with minimal MSE among all (measurable) functions of X. Observe that the function g^* can be of any form. This question ask you to restrict attention to affine functions of X and derive the best linear predictor of Y using X. Show the following very useful result:
Let X, Y be two random variables with finite second moments and $\text{Var}(X) > 0$. Then

$$\beta^* X + \alpha^* = \tilde{g}\left(X\right) = \arg\min_{g(X)\in\mathcal{A}} \mathbb{E}\left[\left(Y - g\left(X\right)\right)^2\right]$$
$$\mathcal{A} = \{g(X): \quad g: \mathbb{R} \to \mathbb{R},\ g(x) = bx + a,\ a, b \in \mathbb{R}\},$$

where $\alpha^* = \mathbb{E}\left(Y\right) - \beta^* \mathbb{E}\left(X\right)$ and $\beta^* = \left(\text{Var}\left(X\right)\right)^{-1} \text{Cov}\left(X, Y\right)$.

7. Kullback-Leibler divergence

Suppose $p: \mathbb{R} \to (0, \infty)$ and $q: \mathbb{R} \to (0, \infty)$ are two probability density functions. Show that

$$-\int_{-\infty}^{\infty} p(x) \log\left(\frac{q(x)}{p(x)}\right) dx \ge 0$$

and the equality holds iff $p(x) = q(x), a.s.$ (with respect to p).
Hint: consider a random variable X with density p and use Jensen's inequality.

[1] The "measurable" constraint over the class of functions being considered is technical here.