Econ 139 Lecture 7

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1 Introduction

1. **Expected Utility Theorem**
2. **Allais Paradox**
3. **Risk Aversion**

Assumptions of Expected Utility Theorem A1 - A5:

A1. Rationality
There is a rational preference relation $\succeq$ defined on L.

A2. Continuity
The preference relation is continuous in the following sense:
for any $L_{xy}, L_{vz}, L_{st} \in L$ where $L_{xy} \succeq L_{vz} \succeq L_{st}$:
$\exists\alpha \in [0, 1]$ s.t. $L_{vz} \sim \alpha L_{xy} + (1 - \alpha)L_{st}$

A3. Independence Axiom
The preference relation $\succeq$ on L is such that for all $L_{xy}, L_{vz}, L_{st} \in$ L and all $\alpha \in$ (0,1), we have $\alpha L_{xy} + (1 - \alpha)L_{st} \succeq \alpha L_{vz} + (1 - \alpha)L_{st}$.

If $\succeq$ satisfies the independence axiom, then it can be shown:
$L_{xy} \succ L_{vz} \iff \alpha L_{xy} + (1 - \alpha)L_{st} \succ \alpha L_{vz} + (1 - \alpha)L_{st}$
$L_{xy} \sim L_{vz} \iff \alpha L_{xy} + (1 - \alpha)L_{st} \sim \alpha L_{vz} + (1 - \alpha)L_{st}$

A4. L is bounded
There is a best and a worst lottery in L :
$\overline{L} = (b1, b2, \pi_{b_1})$ is the best lottery.
$\underline{L} = (w1, w2, \pi_{w_1})$ is the worst lottery.

A5. For all payoff $X \in \mathcal{X}$, we make the following identification
$\mathcal{U}(x, y, 1) \equiv u(x)$
$\mathcal{U} : \mathcal{L} \longrightarrow \mathbb{R}$, $u : \mathbb{X} \longrightarrow \mathbb{R}$

Proof:
Step 1:
By A2, there exist $\alpha_{xy}, \alpha_{vz} \in [0, 1]$ such that

$L_{xy} \sim \alpha_{xy}\overline{L} + (1-\alpha_{xy})\underline{L}$
$L_{vz} \sim \alpha_{vz}\overline{L} + (1-\alpha_{vz})\underline{L}$

Step 2:
We need to show that $L_{xy} \succeq L_{vz} \iff \alpha_{xy} \geq \alpha_{vz}$

Suppose $\L_{xy} \succeq \L_{vz}$,
$\Rightarrow \alpha_{xy}\overline{L} + (1-\alpha_{xy})\underline{L} \succeq \alpha_{vz}\overline{L} + (1-\alpha_{vz})\underline{L}$
$\Rightarrow (\alpha_{xy}-\alpha_{vz})\overline{L} \succeq (\alpha_{xy}-\alpha_{vz})\underline{L}$
$\Rightarrow \alpha_{xy} \geq \alpha_{vz}$
Suppose $\alpha_{xy} \geq \alpha_{vz}$,
If $\alpha_{xy} = \alpha_{vz}$, we have indifference.
If $\alpha_{xy} > \alpha_{vz}$,
$L_{xy} \sim \alpha_{xy}\overline{L} + (1-\alpha_{xy})\underline{L}$
Define $\gamma = (\alpha_{xy} - \alpha_{vz})/(1-\alpha_{vz}) \in [0,1]$.
$L_{xy} \sim \alpha_{xy}\overline{L} + (1-\alpha_{xy})\underline{L} = \gamma\overline{L} + (1-\gamma)(\alpha_{vz}\overline{L} + (1-\alpha_{vz})\underline{L}) \succeq \gamma(\alpha_{vz}\overline{L} + (1-\alpha_{vz})\underline{L}) + (1-\gamma)(\alpha_{vz}\overline{L} + (1-\alpha_{vz})\underline{L}) = \alpha_{vz}\overline{L} + (1-\alpha_{vz})\underline{L} \sim \L_{vz}$
By transitivity, $L_{xy} \succeq L_{vz}$

Step 3: Since $L_{xy} \succeq L_{vz} \iff \alpha_{xy} \geq \alpha_{vz}$, define function $\mathcal{U}$ such that
$\mathcal{U}(L_{xy}) \equiv \alpha_{xy}, \mathcal{U}(L_{vz}) \equiv \alpha_{vz}$

Step 4:
By (A2) There exists scalars α_1, α_0 such that
$L_1 = (x,y,1) \sim \alpha_1\overline{L} + (1-\alpha_1)\underline{L}$
$L_0 = (x,y,0) \sim \alpha_0\overline{L} + (0-\alpha_0)\underline{L}$

Step 5: Observe that $L_{xy} = \pi_x L_1 + (1-\pi_x)L_0$
$L_{xy} = \pi_x L_1 + (1-\pi_x)L_0 \sim \pi_x[\alpha_1\overline{L} + (1-\alpha_1)\underline{L}] + (1-\pi_x)[\alpha_0\overline{L} + (1-\alpha_0)\underline{L}] \sim (\pi_x\alpha_1 + (1-\pi x)\alpha 0)\underline{L} + (\pi_x(1-\alpha_1) + (1-\pi_x)(1-\alpha_0))\underline{L}$

Step 6:
$\mathcal{U}(L_{xy}) = \alpha_{xy} = \pi_x\alpha_1 + (1-\pi x)\alpha_0 = \pi_x\mathcal{U}((x,y,1)) + (1-\pi_x)\mathcal{U}((x,y,0)) = \pi_x u(x) + (1-\pi_x)u(y)$

Where $\mathcal{U} : \mathcal{L} \longrightarrow \mathbb{R}$
$\mathcal{U}((x,y,1)) = u(x)$
$\mathcal{U}((x,y,0)) = u(y)$
$\mathcal{W} \subseteq \mathbb{R} \longrightarrow \mathbb{R}$

Thus, $L_{xy} \succeq L_{vz} \iff \mathcal{U}(L_{xy}) \geq \mathcal{U}(L_{vz})$

2 Corollary

Suppose $U[(X, Y, \Pi x)]$ represents $\succsim$ over $\hbar$, then $V[(X, Y, \Pi x] = \Pi x V(x) + (1 - \Pi x)V(X)$ also represents $\succeq$ iff there exists an a ¿ 0 and B $\in$ R such that V(x) = aU(x)+b

Jensen's inequality

$E[f(u)] \leq f(E[u])$ if f is concave E[f(u)] = f(E[u]) iff f"=0

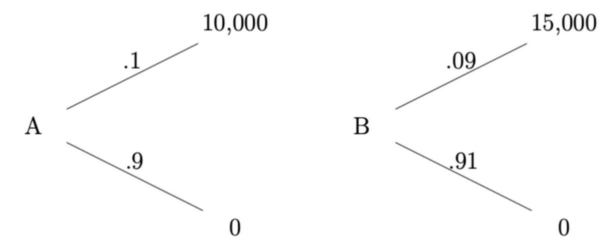
$R_A = -\frac{U"(x)}{U'(x)}$ thus for V(x)= f(u(x))

$R_A = -\frac{V"(x)}{V'(x)} = -\frac{U"(x)}{U'(x)} \iff f" = 0$

$v'(x) = f'(u(x))u"(x)$

$v"(x) = f'(x))u'(x) + f'(x)u"(x0)$

$-\frac{v"(x)}{v'(x)} = -\frac{f"(u(x))u'(x)+f'(u(x))u"(x)}{f'(u(x))u'(x)}$



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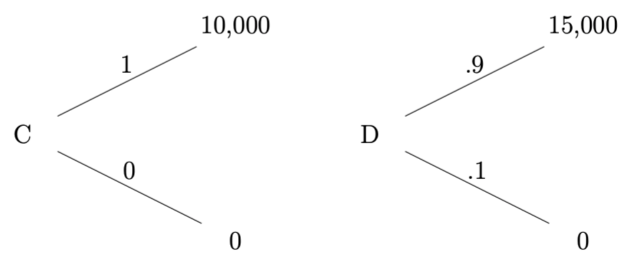
In this case, B is preferred to A

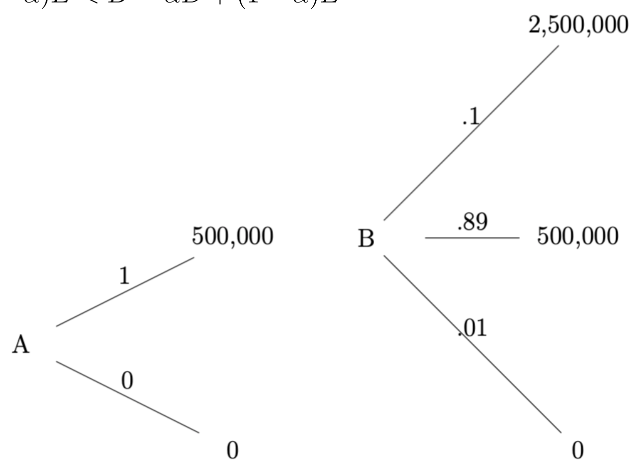


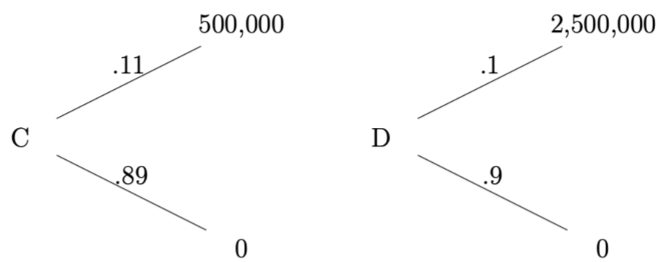
In this case, C is preferred to D

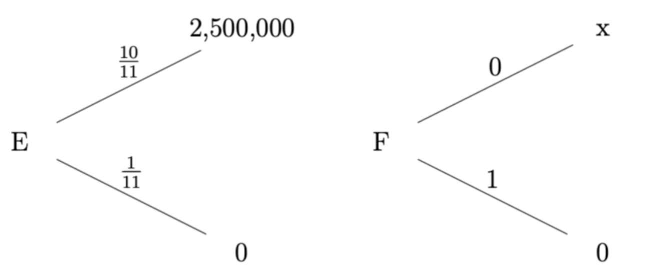
Independence says for any $\alpha \in (0,1)$ $C > D \iff \alpha C + (1-\alpha)E > \alpha D + (1-\alpha)$

let E be a degenerate lotteries that pays 0 for sure and let $\alpha = 0.1$

$A = \alpha$ C + (1 - $\alpha)E < B = \alpha D + (1-\alpha)E$







Suppose $A > B, D > C$,
But
$0.11A + 0.89A = A > B = 0.11E + 0.89A$
$0.11A + 0.89F = C < D = 0.11E + 0.89F$
which is also a violation of independence

Two measures:

1, absolute risk aversion

$R_A = -\frac{U"(x)}{U'(x)}$

2, relative risk aversion

$R_R(x) = -\frac{U"(x)}{U'(x)} * x = R_A(x) * x$

3 Interpreting measures of risk aversion

Let x represent current wealth and consider an investment that pays off +h with probability π and -h with probability $1 - \pi$

Let $\pi = \pi(x, h)$ be the probability that makes me indifferent between entering investment or not, we can show that $\pi(x, h) \approx 1/2 + 1/4hR_A(x)$
Note: $1/4hR_A(x) = o$ for risk neutral

Consider:
$u(x) = -1/ve^{-vx}$
$R_A(x) = -(-ve^{-vx})/e^{-vx} = v$
$U(x) = \pi(x, h) - u(x + h) + (1 - \pi)(x, h) - u(x - h)$

By Taylor's Theorom:
$u(x + h) = u(x) + hu'(x) + \frac{h^2}{2}u"(x) + H_1$
$u(x - h) = u(x) - hu'(x) + \frac{h^2}{2}u"(x) + H_2$
$u(x) \approx \pi(x, h)[u(x) + hu'(x) + \frac{h^2}{2}u"(x)] + (1 - \pi(x, h))[u(x) - hu'(x) + \frac{h^2}{2}u"(x)])$
Rearrangement gives:
$u(x) \approx u(x) + (2\pi(x, h) - 1)hu'(x) + \frac{h^2}{2}u"(x)$
$2\pi(x, h)hu'(x) = hu'(x) - \frac{h^2}{2}u"(x)$
$\pi(x, h) \approx 1/2 + 1/4h[-\frac{u"(x)}{u'(x)}]$

$u(x) \approx \pi(x, h)[u(x) + hu'(x) + \frac{h^2}{2}u"(x)] + (1 - \pi(x, h)[u(x) - hu'(x) + \frac{h^2}{2}u"(x)])$