Lec7 Dependence on IC and parameter. [BN 3.5]

(wellposedness)

Dependence of IC.

Thm Spse f cts bdd in D. $|f| \le M$

and Lip cts in D w/ Lip const L.

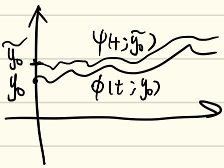
Let ϕ be the soln of $\begin{cases} \frac{dy}{dt} = f(t,y) \\ y(t_0) = y_0 \end{cases}$ and ψ be the soln of $\begin{cases} \frac{dy}{dt} = f(t,y) \\ y(t_0) = \tilde{y}_0 \end{cases}$.

Spse ϕ and ψ exist on some interval $a<t<b$.

Then $\forall \varepsilon>0$ $\exists \delta>0$ s.t. if $|y_0-\tilde{y}_0|<\delta$

then $|\phi(t)-\psi(t)|<\varepsilon$. $t\in(a,b)$

Understanding



As $\psi(t_0)\approx\phi(t_0)$, we have $\psi(t)\approx\phi(t)$ on (a,b).

pf:

$$\phi(t) = y_0 + \int_{t_0}^{t} f(s,\phi(s))\,ds$$

$$\psi(t) = \tilde{y}_0 + \int_{t_0}^{t} f(s,\psi(s))\,ds$$

$$\Rightarrow |\phi(t)-\psi(t)| \le |y_0-\tilde{y}_0| + \int_{t_0}^{t} |f(s,\phi(s))-f(s,\psi(s))|\,ds$$

$$< \delta + \int_{t_0}^{t} L|\phi(s)-\psi(s)|\,ds$$

Use Gronwall Ineq.

$$|\phi(t)-\psi(t)| \le \delta \exp\left(L(t-t_0)\right)$$

$$\le \delta \exp(L(b-a)).$$

Take $\delta < \dfrac{\varepsilon}{\exp(L(b-a))}$ $\square$.

Rmk [BN] Thm 3.7 P135 has a more general version, discuss (t_0,y_0) $(\tilde{t}_0,\tilde{y}_0)$ close.

Idea is the same.

Counter-Ex Lorentz system

$$\begin{cases} \frac{dx}{dt} = \sigma(y-x) \\ \frac{dy}{dt} = x(\rho - z) - y \\ \frac{dz}{dt} = xy - \beta z \end{cases} \qquad \begin{cases} \sigma = 10 \\ \rho = 28 \\ \beta = \frac{8}{3} \end{cases}$$

Chaos:
present predicts future
approx. present does not predict future.

Dependence of parameter

Thm Let f, g def on domain D. both bdd

$$|f| \le M, \quad |g| \le M$$

and both cts satisfying Lip condition w/ the same Lip const.

Let ϕ be soln $\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$ and ψ be soln $\begin{cases} y' = g(t,y) \\ y(t_0) = y_0 \end{cases}$

existing on a common interval (a,b).

Suppose $|f(t,y) - g(t,y)| \le \varepsilon \qquad \forall (t,y) \in D$.

Then solns ϕ and ψ satisfy the estimate

$$|\phi(t) - \psi(t)| \le \varepsilon(b-a)\exp(L|t-t_0|)$$

pf: (HW)

$$\phi(t) = y_0 + \int_{t_0}^{t} f(s,\phi(s))\,ds$$

$$\psi(t) = y_0 + \int_{t_0}^{t} g(s,\psi(s))\,ds$$

$$\Rightarrow |\phi(t) - \psi(t)| \le \int_{t_0}^{t} |f(s,\phi(s)) - f(s,\psi(s))|\,ds + \int_{t_0}^{t} |f(s,\psi(s)) - g(s,\psi(s))|\,ds$$

$$\le \int_{t_0}^{t} L|\phi(s) - \psi(s)|\,ds + \varepsilon(b-a)$$

By Gronwall Ineq.

$$|\phi(t) - \psi(t)| \le \varepsilon(b-a)\exp(L|t-t_0|). \qquad \square$$

Why is it useful?

related w/ sensitivity analysis of Uncertainty Quantification

Understanding:

$$\begin{cases} \frac{dy}{dt} = f(t, y, \alpha) \\ y(t_0) = y_0 \end{cases}$$

where α is a parameter. We assume f is cdly diff fxn of (t, x, α).

Note now the soln depends on α. $y(t;\alpha)$.

Then $z \triangleq \frac{\partial y}{\partial \alpha}$ satisfies some ODE as well.

$$\frac{dz}{dt} = f_\alpha(t, y, \alpha) + f_y(t, y, \alpha)\frac{\partial y}{\partial \alpha}$$

$$\Rightarrow \begin{cases} \frac{dz}{dt} = f_\alpha(t, y(t), \alpha) + f_y(t, y(t), \alpha)\, z \\ z(0) = 0 \end{cases}$$

non-homo linear ODE.

b/c $y(t_0)$ does not dep on α.

"variational equation" (of parameter)

If $z = \frac{\partial y}{\partial \alpha}$ remain smooths, we can expand our soln in α using it.

Ex $\dot{y} = y(1-y) + \alpha \sin t$, $y(0) = 0$

Solve it? Not linear, not sep'ble. Hard...

$\alpha = 0$ we can!

$y(t) \equiv 0$. $y(t;0) = 0$

☆ Idea. To expand $y(t;\alpha)$ around $\alpha = 0$.

$$y(t;\alpha) = y(t;0) + \frac{\partial y}{\partial \alpha}(t;0)\alpha + \frac{\partial^2 y}{\partial \alpha^2}(t;0)\frac{\alpha^2}{2} + \cdots$$

Consider $z(t) \triangleq \frac{\partial y}{\partial \alpha}(t;0)$

$$\dot{z} = y_\alpha(1-y) - y\, y_\alpha + \sin t$$

$$\dot{z} = (1 - 2y(t;0))z + \sin t$$

$$\dot{z} = z + \sin t \qquad z(0) = 0$$

linear Eq. $\Rightarrow z(t) = \frac{-\sin t - \cos t + e^t}{2}$

So we get an approx. of $y(t;\alpha) = \alpha \frac{-\sin t - \cos t + e^t}{2}$!

How parameter affect equilibrium soln (fixed pts).

An intro to bifurcation analysis 1D

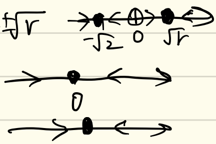
Ex1 Pitchfork (supercritical)

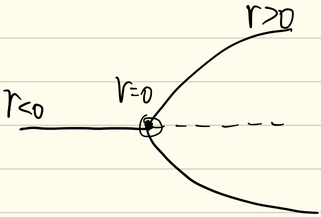
$$\dot{y} = ry - y^3 = y(r - y^2)$$

$r>0$ $\quad y=0,\ y=\pm\sqrt{r}$

$r=0$ $\quad y=0$

$r<0$ $\quad y=0$





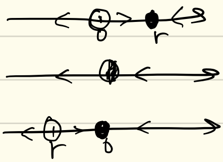
Ex2 transcritical

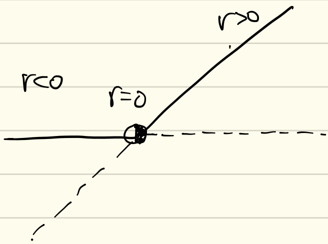
$$\dot{y} = ry - y^2 = y(r-y)$$

$r>0$ $\quad y=0, r$

$r=0$ $\quad y=0$

$r<0$ $\quad y=0, r$





Ex3 subcritical Pitchfork

$$\dot{y} = ry + y^3. \quad \text{(HW)} \quad = y(r+y^2)$$

$r>0$ $\quad y=0$

$r=0$ $\quad y=0$

$r<0$ $\quad y=0,\ \pm\sqrt{-r}$