Econ 139 Lecture 20 Notes

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1 CCAPM

$$\max_c \ U(C_0) + \delta \mathbb{E}[U(C_\theta)]$$
$$\text{s.t. } C_0 = W_0 - P_x a \text{ (to consumption),}$$
$$C_\theta = W_\theta + aX_\theta, \theta = 1, ..., N(t_1 \text{ consumption but multiple } \theta \text{ states})$$

$$\max_a \ U(W_0 - P_x a) + \delta \mathbb{E}[U(W_\theta + aX_\theta)]$$

$$P_x = \mathbb{E}_\pi[m_\theta x_\theta]$$
$$m_\theta = \frac{\delta U'(C_\theta)}{U'(C_0)}$$
$$q_\theta = \frac{\pi\theta\delta U'(C_\theta)}{U'(C_0)}$$
$$m_\theta = \frac{q_\theta}{\pi_\theta}$$

$$P_x = \mathbb{E}_\pi[m_\theta x_\theta] = \mathbb{E}_\pi[\frac{q_\theta}{\pi_\theta}x_\theta] = \sum_{\theta=1}^{N} \pi_\theta(\frac{q_\theta}{\pi_\theta}x_\theta) = \sum_{\theta=1}^{N} q_\theta x_\theta$$

(pricing AD asset in different states)

$$\mathbb{E}_\pi[m_\theta(1+r_\theta)] = 1, \mathbb{E}_\pi[m_\theta(1+r_f)] = 1, \mathbb{E}_\pi[m_\theta] = \frac{1}{1+r_f}$$

$$\sum_{\theta=1}^{N} \pi_\theta m_\theta = \sum_{\theta=1}^{N} \pi_\theta(\frac{q_\theta}{\pi_\theta}) = \sum_{\theta=1}^{N} q_\theta = \frac{1}{1+r_f}$$

$(\sum_{\theta=1}^{N} q_\theta :$ if I own all AD security, I earn the r_f at t=1.)

$$P_x = \frac{\sum_{\theta=1}^{N} q_\theta x_\theta}{\sum_{\theta=1}^{N} q_\theta} \sum_{\theta=1}^{N} q_\theta = \frac{1}{1+r_f} (\sum_{\theta=1}^{N} (\frac{q_\theta}{\sum_{\theta=1}^{N} q_\theta}) x_\theta)$$

$$\text{define } \pi_\theta^{RN} = \frac{q_\theta}{\sum_{\theta=1}^{N} q_\theta} \text{ (RN - risk neutral)}$$

$$\sum_{\theta=1}^{N} \pi_\theta^{RN} = \sum_{\theta=1}^{N} \frac{q_\theta}{\sum_{\theta=1}^{N} q_\theta} = 1$$

$$q_\theta > 0 \implies \pi_\theta^{RN} > 0$$

$$P_x = \frac{1}{1+r_f} \sum_{\theta=1}^{N} \pi_\theta^{RN} x_\theta$$

$$P_x = \frac{1}{1+r_f} \mathbb{E}_\pi^{RN}[x_\theta]$$

$$\mathbb{E}[m_\theta(1+r_\theta)] = 1$$

$$\text{since } Cov(x,y) = \mathbb{E}(x,y) - \mathbb{E}(x)\mathbb{E}(y),$$

$$\mathbb{E}(x,y) = \mathbb{E}(x)\mathbb{E}(y) + Cov(x,y),$$

$$\mathbb{E}[m_\theta(1+r_\theta)] = \mathbb{E}[m_\theta]\mathbb{E}[1+r_\theta] + Cov(m_\theta, r_\theta)$$

$$\text{since } \mathbb{E}[m_\theta] = \frac{1}{1+r_f},$$

$$\mathbb{E}[m_\theta(1+r_\theta)] = \frac{1}{1+r_f}\mathbb{E}[1+r_\theta] + Cov(m_\theta, r_\theta)$$

$$\frac{1}{1+r_f}\mathbb{E}[1+r_\theta] + Cov(m_\theta, r_\theta) = 1$$

$$\mathbb{E}[1+r_\theta] + (1+r_f)Cov(m_\theta, r_\theta) = 1 + r_f$$

$$\mathbb{E}[r_\theta] - r_f = -(1 + r_f)Cov(m_\theta, r_\theta)$$
$$\mathbb{E}[r_\theta] = r_f - (1 + r_f)Cov(m_\theta, r_\theta)$$
$$\mathbb{E}[r_\theta] = r_f - (1 + r_f)Cov(\frac{\delta U'(C_0)}{U'(C_0)}, r_\theta)$$
$$\mathbb{E}[r_\theta] = r_f - (1 + r_f)(\frac{\delta}{U'(C_0)})Cov(U'(C_\theta), r_\theta)$$

here $(1 + r_f)(\frac{\delta}{U'(C_0)})Cov(U'(C_\theta), r_\theta)$ is the risk adjustment,

since $(1 + r_f)$ is positive, $(\frac{\delta}{U'(C_0)})$ is positive, $Cov(U'(C_\theta), r_\theta)$ is negative,

the entire risk adjustment is positive, consumption is more risky

$$\mathbb{E}[r_\theta] = r_f - \frac{Cov(m_\theta, r_\theta)}{\mathbb{E}[m_\theta]}$$
$$\mathbb{E}[r_\theta] = r_f + (\frac{Cov(-m_\theta, r_\theta)}{Var[m_\theta]})(\frac{Var[m_\theta]}{\mathbb{E}[m_\theta]})$$

(this is the CCAPM Relationship)

$(\frac{Cov(-m_\theta, r_\theta)}{Var[m_\theta]})$ is $B_{r,m}$ which is the quantity of risk,

$(\frac{Var[m_\theta]}{\mathbb{E}[m_\theta]})$ is $\lambda_{r,m}$ which is the price of risk

Hansen-Jagannathan Bound

- upper bound for the Sharpe ratio of all assets in the economy
- gives us a way to empirically test CCAPM model

$$\begin{aligned}
\mathbb{E}[m_\theta]\mathbb{E}[1+r_\theta] + \mathbb{C}(m_\theta, r_\theta) &= 1 \\
\mathbb{E}[m_\theta]\mathbb{E}[1+r_\theta] + \rho_{m,r}\sigma_m\sigma_r &= 1 \\
\frac{1}{1+r_f}\mathbb{E}[1+r_\theta] - 1 &= -\rho_{m,r}\sigma_m\sigma_r \\
\mathbb{E}[1+r_\theta] - (1+r_f) &= -\rho_{m,r}\sigma_m\sigma_r(1+r_f) \\
\mathbb{E}[r_\theta] - r_f &= -\rho_{m,r}\sigma_r\frac{\sigma_m}{\mathbb{E}[m_\theta]} \\
\left|\frac{\mathbb{E}[r_\theta] - r_f}{\sigma_r}\right| &= |-\rho_{m,r}|\frac{\sigma_m}{\mathbb{E}[m_\theta]} \\
\left|\frac{\mathbb{E}[r_\theta] - r_f}{\sigma_r}\right| &\leq \frac{\sigma_m}{\mathbb{E}[m_\theta]}
\end{aligned}$$

Equity Premium Puzzle

- assume power utility
$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$
- assume consumption growth rate
$$Cg = ln\left(\frac{c_{t+1}}{c_t}\right)$$
  is normally distributed
$$Cg \sim N(\mu_{Cg}, \sigma_{Cg}^2)$$
- under this assumption
$$\frac{\sigma_m}{\mathbb{E}[m_\theta]} \approx \gamma\sigma_{Cg}$$
- between 1995 and 2005 in the United States
  1. real stock returns averaged 9% with a standard deviation of 16%
  2. real return on Treasury bills was 1%
  3. aggregate consumption growth rate had mean and st. dev of of 1%

$$S = \frac{0.09 - 0.01}{0.16} = 0.5$$
$$\frac{\sigma_m}{\mathbb{E}[m_\theta]} \approx \gamma\sigma_{Cg} = 0.01\gamma \implies 0.5 \leq 0.01\gamma$$

For this to hold, risk aversion coefficient would have to be at least 50 (unreasonable). To illustrate how high this number is, consider a gamble that pays 100,000 with probability 0.5, and 50,000 w.p. 0.5. The following show the certainty equivalents for various gammas.

γ	2	3	50
CE	66,667	63,246	50,712

Arrow-Debreu Pricing II

	AD1	AD2	AD3	Asset 1	Asset X
State 1	1	0	0	3	X_1
State 2	0	1	0	2	X_2
State 3	0	0	1	1	X_3
P	q_1	q_2	q_3	$3q_1 + 2q_2 + q_3$	$q_1X_1 + q_2X_2 + q_3X_3$

Let's look at a market:

	Asset 1	Asset 2	AD1	AD2
State 1	3	5	1	0
State 2	1	5	0	1
P	2	4	0.6	0.2

<u>Create AD1</u>
Pays off 1 in state 1 and 0 in state 2.

$$3a_1 + 5a_2 = 1$$

$$a_1 + 5a_2 = 0$$

$a_1 = 0.5$ and $a_2 = -0.1$
Buy 0.5 share of a_1 and short 0.1 share of a_2.
Price of portfolio is $0.5 \times 2 - 0.1 \times 4 = 0.6$

<u>Create AD2</u>
Pays off 1 in state 1 and 0 in state 2.

$$3a_1 + 5a_2 = 0$$

$$a_1 + 5a_2 = 1$$

$a_1 = -0.5$ and $a_2 = 0.3$
Short 0.5 share of a_1 and buy 0.3 share of a_2.
Price of portfolio is $0.3 \times 4 - 0.5 \times 2 = 0.2$

This is a complete market because you can construct AD securities as portfolios of assets.

More assets than states example:

	Asset 1	Asset 2	Asset 3
State 1	3	5	4
State 2	1	5	3
P	2	4	3

Let $P = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 3 \end{pmatrix}^T$

Price of AD security $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} q_1 & q_2 \end{pmatrix}^T$

Payoff matrix $X = \begin{pmatrix} 3 & 5 & 4 \\ 1 & 5 & 3 \end{pmatrix}$

$$
\begin{aligned}
P &= X^T q \\
P &= \begin{pmatrix} 3 & 1 \\ 5 & 5 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \\
&= \begin{pmatrix} 3q_1 + q_2 \\ 5q_1 + 5q_2 \\ 4q_1 + 3q_3 \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}
\end{aligned}
$$

Starting with $P = X^T q$ and multiplying both sides by X:

$$
\begin{aligned}
XP &= XX^T q \\
q &= (XX^T)^{-1} XP
\end{aligned}
$$

If this equation can be solved, the market is complete.