ECON 139 - Intermediate Financial Economics

Lecture 18 - Arron-Debreu Pricing I

Yanhao Miao, Bohan Shen, Yuhao Weng, Patrick Hong

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Setup:

(i) two dates: $t = 0, t = 1$

(ii) N possible states of nature at $t = 1$;

− indeed by $\theta = 1, 2 \cdots N$

− each state has probability π_θ

− all agents have same probability beliefs

(iii) assume there is one perishable (non-storable) consumption good

(iv) there are K agents with preferences:

$$\underbrace{U_0^K(C_0^K)}_{t=0} + \underbrace{\delta^K \sum_{\theta=1}^N \pi_\theta U^K\left(C_\theta^K\right)}_{t=1} \rightarrow \text{can be written as } E_\pi[U^K(C^K)]$$

K: agent number

δ^K: agent specific discount rate

(v) each agent's endowment vector is given by

$$W^K = \left(\begin{matrix} W_0^K \\ t = 0 \end{matrix}, \underbrace{\left(W_1^K, \cdots W_N^K\right)}_{t=1}\right)$$

if state 1 happens, agent K will have W_1^K at $t = 1$

(vi) Consumption vector: $C^K = \left(C_0^K, \left(C_1^K, \cdots C_N^K\right)\right)$

Traded securities are called Arron-Debreu (AD) securities:

− AD security for state θ pays 1 if state θ occurs and 0 otherwise

− q_θ is price of the state θ AD; for simplicity, we assume the price of C_0^K is 0

Maximization problem:

$$\max_{C^K} U_0^K(C_0^K) + \delta^K \sum_{\theta=1}^N \pi_\theta U^K\left(C_\theta^K\right)$$

s.t. $C^K \geq 0$, and $C_0^K + \sum_{\theta=1}^N q_\theta C_\theta^K \leq W_0^K + \sum_{\theta=1}^N q_\theta W_\theta^K$,

generally, we can write "="

F.O.C:

For C_0^K: $(U_0^K)'(C_0^K) - \lambda = 0$, where λ is the Langrange multiplier

$\Rightarrow \delta^K \pi_\theta (U^K)'\left(C_\theta^K\right) - q_\theta \lambda = 0$, for $\theta = 1, 2 \cdots N$

$\Rightarrow q_\theta = \frac{\delta^K \pi_\theta (U^K)'(C_\theta^K)}{(U_0^K)'(C_0^K)}$

Numerical Example:

– two agents, two periods

– endowments: $W^1 = \left(W_0^1, (W_1^1, W_2^1)\right) = \left(10, (1, 2)\right)$

$W^2 = \left(W_0^2, (W_1^2, W_2^2)\right) = \left(5, (4, 6)\right)$

with $\pi_1 = \frac{1}{3}, \pi_2 = \frac{2}{3}$

– agents have the same preferences: $U_0(C) = \frac{1}{2}C, U(C) = \ln(C)$

– agents have same time discount factors: $\delta^1 = \delta^2 = 0.9$

For agent 1:

$C^1 = \left(C_0^1, (C_1^1, C_2^1)\right)$

$\max_{C^1} \frac{1}{2}C_0^1 + 0.9\left[\frac{1}{3}\ln(C_1^1) + \frac{2}{3}\ln(C_2^1)\right]$

s.t. $C_0^1 + q_1 C_1^1 + q_2 C_2^1 = W_0^1 + q_1 W_1^1 + q_2 W_2^1$

$\Rightarrow \; C_0^1 = W_0^1 + q_1 W_1^1 + q9_2 W_2^1 - q_1 C_1^1 - qC_2^1$

$\Rightarrow \; C_0^1 = 10 + q_1 + 29_2 - q_1 C_1^1 - q_2 C_2^1$

$\Rightarrow \max_{C_1^1 C_2^1} \frac{1}{2}(10 + q_1 + 2q_2 - q_1 C_1^1 - q_2 C_2^1) + 0.9\left[\frac{1}{3}\ln(C_1^1) + \frac{2}{3}\ln(C_2^1)\right]$

Similarly, for agent 2:

$\Rightarrow \max_{C_1^2 C_2^2} \frac{1}{2}(5 + 4q_1 + 6q_2 - q_1 C_1^2 - q_2 C_2^2) + 0.9\left[\frac{1}{3}\ln(C_2^1) + \frac{2}{3}\ln(C_2^2)\right]$

F.O.Cs:

Agent 1:

C_1^1: $-\frac{1}{2}q_1 + 0.9\frac{1}{3} \cdot \frac{1}{C_1^1} = 0 \Rightarrow \; \frac{0.3}{C_1^1} = \frac{1}{2}q_1$

$\Rightarrow \; C_1^1 = \frac{0.6}{q_1}$

C_2^1: $-\frac{1}{2}q_2 + 0.9 \cdot \frac{2}{3} \cdot \frac{1}{C_2^1} = 0$

$\Rightarrow\ C_2^1 = \frac{1.2}{q_2}$

Agent 2:

C_1^2: $C_1^2 = \frac{0.6}{q_1}$,

C_2^2: $C_2^2 = \frac{1.2}{q_2}$ (the same as 1)

Market clearing conditions: $C_1^1 + C_1^2 = W_1^1 + W_1^2 = 5$

also, $C_2^1 + C_2^2 = W_2^1 + W_2^2 = 8$

So we have $C_1^1 = C_1^2 = 2.5, C_2^1 = C_2^2 = 4$

$q_1 = \frac{0.6}{C_1^1} = 0.24, q_2 = \frac{1.2}{C_2^1} = 0.3$

Back to General Case: $q_\theta = \frac{\pi_\theta \delta^K (U^K)'(C_\theta^K)}{(U_0^K)'(C_0^K)}$ holds for all θ

$(U^K)'(C_0^K) = \left(\frac{\pi_\theta}{q_\theta}\right)\delta^K (U^K)'\left(C_\theta^K\right)$ for all states $\theta = 1, \cdots N$

and all agents $K = 1, 2 \cdots \overline{K}$

(It can be compared to the Euler Equation: $U'(C_0^*) = \delta(1 + r_f)E[U'(C_1^*)]$)

Stochastic discount factor: $\frac{q_\theta}{\pi_\theta} = \frac{\delta^K (U^K)'(C_\theta^K)}{(U_0^K)'(C_0^K)}$

define: $m_\theta = \frac{q_\theta}{\pi_\theta}$

Names: (1) stochastic discount factor (SDF)

(2) price kernel

(3) state price density

expected gross rate of return for an AD security: $\frac{E[\tilde{X}]}{P_X} = \frac{1-\pi_\theta + 0\cdot(1-\pi_\theta)}{q_\theta} = \frac{\pi_\theta}{q_\theta} = {m_\theta}^{-1}$

$q_\theta = m_\theta \pi_\theta \Rightarrow$ expected payoff $\times$ price kernel $=$ price

(That is where the term "price kernel" comes from.)

We can price any portfolio in the economy:

$P_X = \sum_{\theta=1}^{N} \pi_\theta m_\theta X_\theta = E_\pi\left[\tilde{m}\tilde{X}\right]$

$= \sum_{\theta=1}^{N} q_\theta X_\theta \qquad \uparrow$

$\tilde{X} = (X_1, X_2 \cdots X_N)$, where X_θ is the payoff at state θ

(m_θ is a random variable because it can take different values from $t = 0$)

Back to our example:

Consumption at $t = 0$: $C_0^1 = 10 + q_1 + 2q_2 - q_1 C_1^1 - q_2 C_2^1 = 9.04;\ C_0^2 = 5.96$

$\Rightarrow C^1 = \big(9.04, (2.5, 4)\big), C^2 = \big(5.96, (2.5, 4)\big)$

Since $W^1 = \big(10, (1, 2)\big), W^2 = \big(5, (4, 6)\big)$,

Agent 1 sold 0.96 of the perishable consumption good and purchased 1.5 of AD_1 and 2 of AD_2

Inversely for agent 2.

Check IR condition: $U(W^1) = \frac{1}{2} \times 10 + 0.9\left(\frac{1}{3}\ln(1) + \frac{2}{3}\ln(2)\right) = 5.42$

Since $U(C^1) = 5.63, U(C^1) > U(W^1)$ IR holds for agent 1

Since $U(W^2) = 3.99, U(C^2) = 4.09,$ IR also holds for agent 2