Econ 139 02/07 Lecture Scribe Notes

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Figures from Danthine and Donaldson, 3rd Edition

February 7: Expected Utility Functions, Expected Utility Theorem

$U(x) = E[u(x)] = \pi_1 u(x_1) + \pi_2 u(x_2)$

Expected Utility Functions

$U(x) = E[u(x)] = \pi_1 u(x_1) + \pi_2 u(x_2)$
(x = payoff, π = probability) where,
1. u(x): Bernoulli Utility Function is increasing and concave
2. U(x): Neumann & Morgenstern Utility Function is linear in the probabilities

1. Considered when making rational choice decisions.
2. Assumptions:
    a. If I have two assets with payoffs (not returns):

	Asset 1	Asset 2	π
State 1	150	150	π_1
State 2	100	100	π_2
Price	100	100	

i. An investor should be indifferent between two assets with the same payoffs and price.
ii. An investor only considers the payoffs and costs, nothing else.

b. Now change the payoffs slightly:

	Asset 1	Asset 2	π
State 1	150	160	π_1
State 2	100	110	π_2
Price	100	100	

i. An investor should prefer Asset 2 due to state by state dominance.
    1. Regardless of which state we face, Asset 2 has better payoffs than Asset 1.
ii. Investors prefer more to less.

c. Now change the payoffs again:

	Asset 1	Asset 2	π
State 1	150	160	π_1
State 2	100	90	π_2
Price	100	100	

i. An investor tends to prefer Asset 2 when π_1 increases, and tends to prefer Asset 1 when π_2 increases.
ii. Here, the state by state dominance ceases, and investors will prefer whichever state occurs with higher probability

3. Blaise Pascal (France, 1623 - 1662)
    a. Suggested basing decisions on expected payoff $E[X] = \pi_1 x_1 + \pi_2 x_2.$
    A criterion that had all the properties discussed above in the assumptions:
        i. Attach higher weights to states with higher probabilities/payoffs.
        ii. But the expected return does not consider variance.
4. Nicolaus Bernoulli (Switzerland, 1687 - 1759)
    a. Suggested that Pascal's criterion does not consider risks i.e. assets could have same expected payoffs but wildly different risks
    b. Suppose we have three assets:
        i. All three assets have the same $E[X] = 125$. $risk_3 < risk_1 < risk_2$

	Asset 1	Asset 2	Asset 3	π
State 1	150	160	125	0.5
State 2	100	90	125	0.5
Price	100	100	100	

5. Daniel Bernoulli (1700 - 1782) & Gabriel Cramer (1704 - 1752)
    a. Suggested using a concave utility function over payoffs I.e. assuming u is increasing and concave, we can now rank the asset by risk
    b. Recall: this implies
        i. Investors prefer more to less, but with diminishing marginal utility.
        ii. Investors prefer smoother payoff factors.
            1. Or payoff vectors with less variability.
    c. Following the example above: Asset 3 > Asset 1 > Asset 2
6. John von Neumann (1903 - 1957) & Oskar Morganstern (1902 - 1977): Two centuries later developed expected utility criterion
    a. Proposed a utility function
        i. $U(x) = E[u(x)] = \pi_1 u(x_1) + \pi_2 u(x_2)$
            1. U(x): Neumann & Morgenstern Utility Function, linear combination of utility functions.
            2. u(x): Bernoulli Utility Function, concave increasing.
            3. Suggesting a linear combination of the utility of individual payoffs
    b. The derivations are found in Theory of Games and Behavior (1947)
7. Simple Lotteries introduction: $(x, y, \pi_x) = L_{xy}$
    a. Two possible outcomes, x (payoff in state 1) and y (payoff in state 2).
    b. Probability of x payoff is π_x and y payoff is $1 - \pi_x$.
        i.

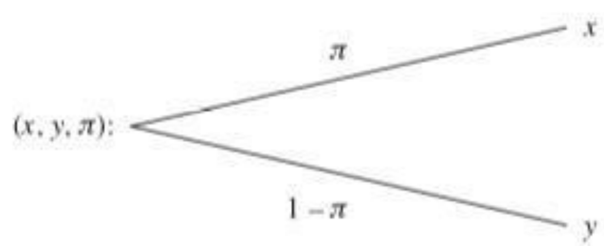

Figure 3.1
A simple lottery (x, y are monetary payoffs).

    c. Compound lottery: If x is simple lottery itself:

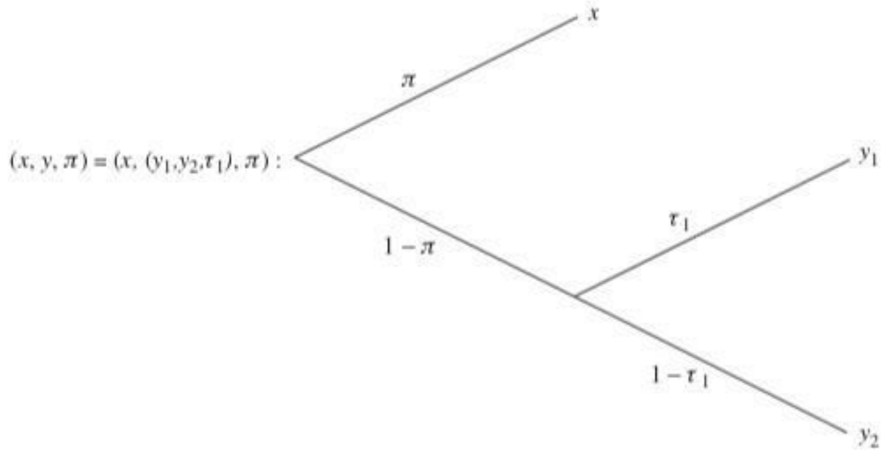

Figure 3.2
A compound lottery (y is itself a lottery).

i.

d. Compound lottery: If x and y are simple lotteries themselves:

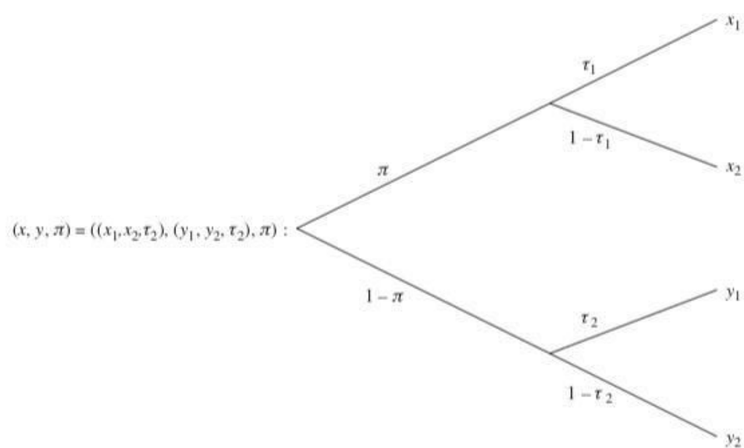

Figure 3.3
A compound lottery (both x and y are themselves lotteries).

i.

e. Thus we can use simple lotteries to get compound lotteries

Expected Utility Theorem Under some assumptions, the preference relation $\succsim$ on the space of simple lotteries $\mathcal{L}$ can be represented by a function $\mathcal{U} : \mathcal{L} \longrightarrow \mathbb{R}$ that is linear in probabilities.

That is, for any two lotteries L_{xy}, $L_{vz} \in \mathcal{L}$ such that $L_{xy} \succsim L_{vz}$ if and only if $\mathcal{U}(L_{xy}) \geq \mathcal{U}(L_{vz})$,

$$\mathcal{U}(L_{xy}) = \pi_x u(x) + (1 - \pi_x) u(y)$$
$$\mathcal{U}(L_{vz}) = \pi_v u(v) + (1 - \pi_v) u(z)$$

Expected Utility Theorem: Assumptions

Assumption 1 (Rationality): There exists a rational (complete and transient i.e. fully informed) preference relation $\succsim$ defined on $\mathcal{L}$

Assumption 2 (Continuity): The preference relation $\succsim$ is continuous in the following sense:

For any (3 lotteries) $L_{xy}, L_{vz}, L_{st} \in \mathcal{L}$ where $L_{xy} \succsim L_{vz} \succsim L_{st}$, there exists an α ∈ [0, 1] such that:

$$L_{vz} \sim \alpha L_{xy} + (1-\alpha) L_{st}$$

Assumption 3 (Independence): For any $L_{xy}, L_{vz}, L_{st} \in \mathcal{L}$, and $\alpha \in [0, 1]$:

$$L_{xy} \succsim L_{vz} \text{ if and only if } \alpha L_{xy} + (1-\alpha) L_{st} \succsim \alpha L_{vz} + (1-\alpha) L_{st}$$

Assumption 4: Suppose there are best (most preferred) and worst (least preferred) lotteries in $\mathcal{L}$:

$$\overline{L} = (b_1, b_2, \pi_{b_1}) \quad \text{is the best lottery}$$
$$\underline{L} = (w_1, w_2, \pi_{w_1}) \quad \text{is the worst lottery}$$

Assumption 5: For any specific payoff x, U((x, y, 1)) ≡ u(x), where U is the sample space of the lottery and u is a subset of U

Proof Idea: We want to construct a function of the form $\mathcal{U}((L_{xy})) = \pi_x u(x) + (1-\pi_x)u(y)$

Step 1: By continuity (Assumption 2), there exists $\alpha_{xy}, \alpha_{vz} \in [0, 1]$ such that:

$$\begin{aligned} L_{xy} &\sim \alpha_{xy}\overline{L} + (1-\alpha_{xy})\underline{L} \\ L_{vz} &\sim \alpha_{vz}\overline{L} + (1-\alpha_{vz})\underline{L} \end{aligned} \quad (\overline{L} \succsim L_{xy},\, L_{vz} \succsim \underline{L})$$

Step 2: Observe that $L_{xy} \succsim L_{vz}$ if and only if $\alpha_{xy} \geq \alpha_{vz}$.

($\Longrightarrow$) Suppose $L_{xy} \succsim L_{vz}$,

$$\begin{aligned} &\Longrightarrow \alpha_{xy}\overline{L} + (1-\alpha_{xy})\underline{L} \succsim \alpha_{vz}\overline{L} + (1-\alpha_{vz})\underline{L} \text{ (by Step 1)} \\ &\Longrightarrow (\alpha_{xy} - \alpha_{vz})\overline{L} \succsim (\alpha_{xy} - \alpha_{vz})\underline{L} \text{ (by rearranging)} \\ &\Longrightarrow \alpha_{xy} \geq \alpha_{vz} \end{aligned}$$

($\Longleftarrow$) Suppose $\alpha_{xy} \geq \alpha_{vz}$. Let $\gamma = \frac{\alpha_{xy}-\alpha_{vz}}{1-\alpha_{vz}} \in [0,1]$.

$$\begin{aligned} L_{xy} \sim \alpha_{xy}\overline{L} + (1-\alpha_{xy})\underline{L} &= \gamma\overline{L} + (1-\gamma)(\alpha_{vz}\overline{L} + (1-\alpha_{vz})\underline{L}) \\ &\succsim \gamma(\alpha_{vz}\overline{L} + (1-\alpha_{vz})\underline{L}) + (1-\gamma)(\alpha_{vz}\overline{L} + (1-\alpha_{vz})\underline{L}) \\ &= \alpha_{vz}\overline{L} + (1-\alpha_{vz})\underline{L} \sim L_{vz} \end{aligned}$$

Therefore, by transitivity, we have

$$L_{xy} \succsim L_{vz}$$

Step 3: Since $L_{xy} \succsim L_{vz}$ if and only if $\alpha_{xy} \geq \alpha_{vz}$, we define our function $\mathcal{U}$ to be:

$$\mathcal{U}(L_{xy}) \equiv \alpha_{xy}, \mathcal{U}(L_{vz}) \equiv \alpha_{vz}$$

Note: Assumption 3 is often violated by the Allais Paradox

Step 4: By continuity (Assumption 2), there exist scalars $\alpha_1, \alpha_0 \in [0, 1]$ such that

$$\begin{aligned} L_1 = (x, y, 1) \sim \alpha_1\overline{L} + (1-\alpha_1)\underline{L} \\ L_0 = (x, y, 0) \sim \alpha_0\overline{L} + (1-\alpha_0)\underline{L} \end{aligned}$$

Step 5: Observe that $L_{xy} = \pi_x L_1 + (1-\pi_x)L_0$ since,

$$\begin{bmatrix} \pi_x \\ 1-\pi_x \end{bmatrix} = \pi_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (1-\pi_x) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then we can write

$$\begin{aligned} L_{xy} &\sim \pi_x L_1 + (1-\pi_x)L_0 \\ &\sim \pi_x[\alpha_1\overline{L} + (1-\alpha_1)\underline{L}] + (1-\pi_x)[\alpha_0\overline{L} + (1-\alpha_0)\underline{L}] \\ &\sim \underbrace{(\pi_x\alpha_1 + (1-\pi_x)\alpha_0)}_{\alpha_{xy}}\overline{L} + \underbrace{(\pi_x(1-\alpha_1) + (1-\pi_x)(1-\alpha_0))}_{1-\alpha_{xy}}\underline{L} \end{aligned}$$