Problem Set 3
Econ 139, Fall 2019
Due in class on Tu November 5. No late Problem Sets accepted, sorry!

General rules for problem sets: show your work, write down the steps that you use to get a solution (no credit for right solutions without explanation), write legibly. If you cannot solve a problem fully, write down a partial solution. We give partial credit for partial solutions that are correct. Do not forget to write your name on the problem set!

Problem 1. Risk Aversion and Portfolio Choice. Consider an economy with two types of financial assets: one risk-free asset and one risky asset. The rate of return of the risk-free asset is r_f and the rate of return of the risky asset is $\tilde{r}$, where $\mathbb{E}[\tilde{r}] > r_f$.

Agents are risk averse. Let W_0 be the initial wealth. The purpose of this exercise is to determine the optimal amount a^* to be invested in the risky asset as a function of the *Absolute Risk Aversion Coefficient*.

The objective of the agents is to maximize the expected utility of terminal wealth $\mathbb{E}\left[U\left(\tilde{W}_1\right)\right]$ with respect to the amount a to be invested in the risky asset.

1. Write down an expression for final wealth $\tilde{W}_1$ as a function of W_0, a, r_f, and $\tilde{r}$.
2. Compute the first order condition of the expected utility maximization problem. Check the second order condition. Is it satisfied for a maximum?
3. We are interested in determining the sign of da^*/dW_0. First, calculate the total differential of the first order condition with respect to W_0. Second, derive an expression for da^*/dW_0. Show that the sign of this expression depends on the sign of its numerator.
4. The absolute risk-aversion coefficient is given by $r_A(W) = -u''(W)/u'(W)$. What does it mean if
$$\frac{dr_A(W)}{dW} = r'_A(W) < 0?$$
5. Assume $r'_A(W) < 0$ for the remainder of the exercise. Compare $r_A(\tilde{W}_1)$ and $r_A(W_0(1+r_f))$. Is $r_A(\tilde{W}_1) > r_A(W_0(1+r_f))$ or vice-versa? Note there are two possible cases: $\tilde{r} \geq r_f$ and $\tilde{r} < r_f$.
6. Show that
$$U''(W_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f) > -r_A(W_0(1+r_f))U'(W_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)$$
for both cases in point 5.

7. Argue that the following epxression

$$\mathbb{E}[U''(W_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)] > -r_A(W_0(1+r_f))\mathbb{E}[U'(W_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)]$$

holds, based on the results from point 6. Use the first order condition to determine the sign of the left-hand side. What do you conclude about the sign of da^*/dW_0? What was the key assumption for arriving at this result?

Problem 2. (Yet Another) Derivation of the CAPM. Consider an economy with K agents, J risky assets, and one risk-free asset with net return r_f. Agent k's future wealth can be written as:

$$\tilde{W}_1^k = \left(W_0^k - \sum_{j=1}^{J} a_j^k \right)(1 + r_f) + \sum_{j=1}^{J} a_j^k (1 + \tilde{r}_j),$$

where W_0^k is the initial wealth of agent k, a_j^k is the amount agent k invests in the jth asset, and $\tilde{r}_j$ is the risky net return of the jth asset.

As usual, agent k's choice problem is given by:

$$\max_{a^k} \ \mathbb{E}\left[U^k(\tilde{W}_1^k) \right],$$

where $(U^k)' > 0, (U^k)'' < 0$ for all agents, and $a^k = ((a_j^k)_{j=1,\ldots,J})$.

1. Show that agent k's first order condition (with respect to asset j) can be written as:

$$\mathbb{E}\left[(U^k)'(\tilde{W}_1^k)(\tilde{r}_j - r_f) \right] = 0.$$

2. Show that agent k's first order condition can be rewritten as:

$$\mathbb{E}\left[(U^k)'(\tilde{W}_1^k) \right] \mathbb{E}\left[(\tilde{r}_j - r_f) \right] = -\mathbf{C}ov\left((U^k)'(\tilde{W}_1^k), \tilde{r}_j \right).$$

3. Use the property of covariance $\mathbf{C}ov(g(x), y) = \mathbb{E}[g'(x)]\mathbf{C}ov(x, y)$ to rewrite the first order condition as:

$$\mathbb{E}\left[(U^k)'(\tilde{W}_1^k) \right] \mathbb{E}\left[(\tilde{r}_j - r_f) \right] = -\mathbb{E}\left[(U^k)''(\tilde{W}_1^k) \right] \mathbf{C}ov\left(\tilde{W}_1^k, \tilde{r}_j \right).$$

4. Aggregating over all agents (summing over k) show that the first order condition can be rewritten as:

$$\mathbb{E}\left[(\tilde{r}_j - r_f) \right] = \frac{W_0^M}{\sum_{k=1}^{K} \left[R_A^k(W) \right]^{-1}} \mathbf{C}ov(\tilde{r}_M, \tilde{r}_j),$$

where

$$R_A^k(W) = -\frac{\mathbb{E}\left[(U^k)''(\tilde{W}_1^k) \right]}{\mathbb{E}\left[(U^k)'(\tilde{W}_1^k) \right]},$$

which is reminiscent of the absolute risk aversion coefficient, $W_0^M = \sum_{k=1}^K W_0^k$, and

$$W_0^M(1 + \tilde{r}_M) = \sum_{k=1}^K \tilde{W}_1^k.$$

5. Show that the equation from point 4 implies:

$$\mathbb{E}\left[(\tilde{r}_M - r_f)\right] = \frac{W_0^M}{\sum_{k=1}^K \left[R_A^k(W)\right]^{-1}} \mathbf{V}ar(\tilde{r}_M).$$

6. Derive the standard CAPM equation. [Hint: combine the equations from point 4 and point 5.]

Problem 3. General Equilibrium and Uncertainty. Consider an economy with two agents, both with the following utility function:

$$U(c) = c_0 + \mathbb{E}[c_\theta] - 2\mathbf{V}ar(c_\theta),$$

where c_0 is consumption at $t = 0$ and c_θ is consumption at $t = 1$, if state θ occurs. There are two possible states of nature at $t = 1$, with probabilities $\pi_1 = 1/4$ and $\pi_2 = 3/4$. The agents endowments are given by

$$\begin{aligned} \omega^1 &= (\omega_0^1, (\omega_1^1, \omega_2^1)) = (4, (1, 5)) \\ \omega^2 &= (\omega_0^2, (\omega_1^2, \omega_2^2)) = (4, (5, 1)), \end{aligned}$$

where ω_0^k is the $t = 0$ endowment of agent k, $(\omega_\theta^k)_{\theta=1,2}$ is the $t = 1$ endowment of agent k in state θ.

1. Intuitively, what should a Pareto optimal allocation satisfy in this particular setup?

2. Is there more than one Pareto optimal allocation? If so, characterize the set of Pareto optimal allocations.

3. Is the following allocation Pareto optimal?

   $$\begin{aligned} \omega^1 &= (4, (2, 2)) \\ \omega^2 &= (4, (4, 4)). \end{aligned}$$

   Explain.

For the remainder of the problem, assume the agents both have the following utility function:

$$U(c) = c_0 + \mathbb{E}[\ln(c_\theta)],$$

with the same endowments and state probabilities as before.

4. Describe the set of Pareto optimal allocations for the new utility function. [Hint: set up and solve the utility maximization problem of a benevolent social planner. Consider the three cases $\lambda > 1, \lambda = 1$, and $0 < \lambda < 1$.]

5. Let z_θ^k denote the quantity of the Arrow-Debreu security with payoff in state θ demanded by agent k, where q_θ denotes the price of the Arrow-Debreu security with payoff in state θ. In both states, the Arrow-Debreu securities are in zero net supply, i.e.,

   $$\begin{aligned} z_1^1 + z_1^2 &= 0 \\ z_2^1 + z_2^2 &= 0. \end{aligned}$$

Compute the competitive equilibrium allocation. Is it Pareto optimal? Discuss intuitively its characteristics (the determinants of the prices of the two securities and the post-trade allocation).

6. Suppose only one Arrow-Debreu security with payoff in state 1 is traded. It is in zero net supply. Compute the competitive equilibrium allocation. Is it Pareto optimal?

Problem 4. Efficient Set Mathematics. You will probably want to use Excel, R, Stata, Python, or Matlab to do this exercise. Consider the following three assets:

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 9 \end{pmatrix}.$$

Note that

$$\Sigma^{-1} = \begin{pmatrix} 1.346154 & 0.346154 & 0.038462 \\ 0.346154 & 0.346154 & 0.038462 \\ 0.038462 & 0.038462 & 0.115385 \end{pmatrix}.$$

1. Recall that for any target portfolio expected return μ_p, we can write the variance minimizing portfolio weights as
$$w_p = g + h\mu_p.$$
Determine g and h.

2. Determine the global minimum variance portfolio weights. What is the standard deviation and expected return of the global minimum variance portfolio?

3. Using your result from point 1, let μ_p range from 0 to 3 in increments of 0.10, and plot the resulting minimum variance frontier of risky assets (in expected-return standard deviation space).

4. Now introduce a risk-free asset with gross return R_f and let $R_f = 1$. Find the tangency portfolio weights. What is the standard deviation and expected return of the tangency portfolio? Identify the tangency portfolio on your plot from point 3.

5. Let μ_p range from 1 (i.e., R_f) to 3 in increments of 0.10, and plot the resulting global efficient frontier (on your plot from point 3).

6. Determine the zero-covariance portfolio weights for the tangency portfolio (i.e., the weights of the portfolio that has zero-covariance with the tangency portfolio). What is the standard deviation and expected return of this portfolio? Identify this portfolio on your plot from point 3. Verify that the covariance of this portfolio and the tangency portfolio is indeed zero.

7. What is the Sharpe ratio of the tangency portfolio? Calculate the Sharpe ratios for all of the portfolios from point 3. Verify that they are all less than or equal to the Sharpe ratio of the tangency portfolio.

Problem 5. Efficient Frontier and the CAPM. Short answer questions.

1. Explain why the efficient frontier must be concave.
2. Suppose that there are N risky assets in an economy, each being the single claim to a different firm (hence, there are N firms). Then suppose that some firms go bankrupt, i.e., their single stock disappears; how is the efficient frontier altered?
3. How is the efficient frontier altered if the borrowing rate is higher than the lending (risk-free) rate? Draw a picture.
4. Suppose you believe that the CAPM holds and you notice that an asset plots above the Security Market Line (SML). How can you take advantage of this situation? What will happen to the price and expected return of the asset in the long run?