复合函数及隐函数的高阶偏导数

$$\ln \sqrt{x^2 + y^2} = \arctan \frac{y}{x}, \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = ?$$

例
$$\ln \sqrt{x^2 + y^2} = \arctan \frac{y}{x}, \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = ?$$
 解:
$$\frac{1}{2} \ln(x^2 + y^2) = \arctan \frac{y}{x},$$

解:
$$\frac{1}{2}\ln(x^2 + y^2) = \arctan\frac{y}{x}, \quad \frac{1}{2} \cdot \frac{2x\,\mathrm{d}x + 2y\,\mathrm{d}y}{x^2 + y^2} = \frac{\frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2}}{1 + (\frac{y}{2})^2},$$

$$\ln \sqrt{x^2 + y^2} = \arctan \frac{y}{x}, \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = ?$$

$$\frac{1}{x} \ln(x^2 + y^2) = \arctan \frac{y}{x}, \quad \frac{1}{x} \cdot \frac{2x \, \mathrm{d}x}{x^2}$$

$$\ln \sqrt{x^2 + y^2} = \arctan \frac{y}{x}, \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = ?$$

解:
$$\frac{1}{2}\ln(x^2+y^2) = \arctan\frac{y}{x}, \quad \frac{1}{2} \cdot \frac{2x\,\mathrm{d}x + 2y\,\mathrm{d}y}{x^2+y^2} = \frac{\frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2}}{1 + (\frac{y}{x})^2},$$

- x dx + y dy = x dy y dx

$$\ln \sqrt{x^2 + y^2} = \arctan \frac{y}{x}, \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = ?$$

 $\text{$\widehat{H}$:} \quad \frac{1}{2}\ln(x^2+y^2) = \arctan\frac{y}{x}, \quad \frac{1}{2} \cdot \frac{2x\,\mathrm{d}x + 2y\,\mathrm{d}y}{x^2+y^2} = \frac{\frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2}}{1 + (\frac{y}{2})^2},$

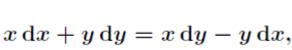
- x dx + y dy = x dy y dx
- $dy = \frac{x+y}{x-u} dx, \quad \therefore y' = \frac{x+y}{x-u},$

$$\ln \sqrt{x^2 + y^2} = \arctan \frac{y}{x}, \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = ?$$

解:
$$\frac{1}{2}\ln(x^2+y^2) = \arctan\frac{y}{x}$$
, $\frac{1}{2}\cdot\frac{2x\,\mathrm{d}x+2y\,\mathrm{d}y}{x^2+y^2} = \frac{\frac{x\,\mathrm{d}y-y\,\mathrm{d}x}{x^2}}{1+(\frac{y}{2})^2}$,





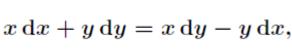


 $dy = \frac{x+y}{x-y} dx, \quad \therefore y' = \frac{x+y}{x-y}, \quad y'' = (\frac{x+y}{x-y})'$



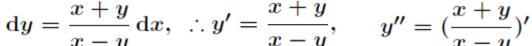
例
$$\ln \sqrt{x^2 + y^2} = \arctan \frac{y}{x}, \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = ?$$
解: 1 $\frac{y}{x} = \frac{y}{x}$ 1 $\frac{2x}{x} = \frac{y}{x}$

解:
$$\frac{1}{2}\ln(x^2+y^2) = \arctan\frac{y}{x}$$
, $\frac{1}{2} \cdot \frac{2x\,\mathrm{d}x + 2y\,\mathrm{d}y}{x^2+y^2} = \frac{\frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2}}{1 + (\frac{y}{2})^2}$,









 $=\frac{(1+y')(x-y)-(1-y')(x+y)}{(x-y)^2}$

例
$$\ln \sqrt{x^2 + y^2} = \arctan \frac{y}{x}, \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = ?$$
解: 1 $(2x + 3)$ $(2x + 3)$ $(2x + 3)$ $(3x + 3)$ $(3x + 3)$ $(3x + 3)$

解:
$$\frac{1}{2}\ln(x^2+y^2) = \arctan\frac{y}{x}, \quad \frac{1}{2} \cdot \frac{2x\,\mathrm{d}x + 2y\,\mathrm{d}y}{x^2+y^2} = \frac{\frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2}}{1 + (\frac{y}{x})^2},$$

$$x \, \mathrm{d}x + y \, \mathrm{d}y = x \, \mathrm{d}y - y \, \mathrm{d}x,$$

$$x \, \mathrm{d}x + y \, \mathrm{d}y = x \, \mathrm{d}y - y \, \mathrm{d}x,$$

$$x dx + y dy = x dy - y dx,$$
$$x + y dy = x dy - y dx,$$
$$x + y dy = x dy - y dx,$$

$$x + y dy = x dy - y dx,$$

 $x + y$ $x + y$

 $=\frac{(1+y')(x-y)-(1-y')(x+y)}{(x-y)^2}$

 $= \frac{-2y + 2xy'}{(x-y)^2} = \frac{-2y + 2x\frac{x+y}{x-y}}{(x-y)^2}$

$$y = x \, \mathrm{d} y \qquad y \, \mathrm{d} x, + y$$

$$y = x + y$$

$$dy = \frac{x+y}{x-y} dx, \quad \therefore y' = \frac{x+y}{x-y}, \qquad y'' = (\frac{x+y}{x-y})'$$

$$x + y$$

$$x^2$$

$$x^2 +$$

$$x^{2} +$$

例
$$\ln \sqrt{x^2 + y^2} = \arctan \frac{y}{x}, \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = ?$$
解:
$$\frac{1}{2} \ln(x^2 + y^2) = \arctan \frac{y}{x}, \quad \frac{1}{2} \cdot \frac{2x \, \mathrm{d}x + y^2}{2}$$

解: $\frac{1}{2}\ln(x^2+y^2) = \arctan\frac{y}{x}$, $\frac{1}{2}\cdot\frac{2x\,\mathrm{d}x+2y\,\mathrm{d}y}{x^2+y^2} = \frac{\frac{x\,\mathrm{d}y-y\,\mathrm{d}x}{x^2}}{1+(\frac{y}{2})^2}$,

x dx + y dy = x dy - y dx

 $dy = \frac{x+y}{x-y} dx, \quad \therefore y' = \frac{x+y}{x-y}, \quad y'' = (\frac{x+y}{x-y})'$

 $=\frac{(1+y')(x-y)-(1-y')(x+y)}{(x-y)^2}$

 $= \frac{-2y + 2xy'}{(x-y)^2} = \frac{-2y + 2x\frac{x+y}{x-y}}{(x-y)^2}$

 $=\frac{-2y(x-y)+2x(x+y)}{(x-y)^3}=\frac{2(x^2+y^2)}{(x-y)^3}$

解: $-e^{-xy}(x dy + y dx) - 2 dz + e^{z} dz = 0,$

$$-e^{-xy}(x \, dy + y \, dx) - 2 \, dz + e^z \, dz = 0,$$

$$dz = \frac{e^{-xy}(x dy + y dx)}{e^z - 2},$$

$$-e^{-xy}(x\,\mathrm{d}y + y\,\mathrm{d}x) - 2\,\mathrm{d}z + e^z\,\mathrm{d}z = 0,$$

$$\mathrm{d}z = \frac{e^{-xy}(x\,\mathrm{d}y + y\,\mathrm{d}x)}{e^z - 2}, \quad \frac{\partial z}{\partial x} = \frac{ye^{-xy}}{e^z - 2}, \quad \frac{\partial z}{\partial y} = \frac{xe^{-xy}}{e^z - 2},$$

$$-e^{-xy}(x \, dy + y \, dx) - 2 \, dz + e^z \, dz = 0,$$

$$-e^{-xy}(x\,\mathrm{d}y + y\,\mathrm{d}x) - 2\,\mathrm{d}z + e^z\,\mathrm{d}z = 0,$$

$$\mathrm{d}z = \frac{e^{-xy}(x\,\mathrm{d}y + y\,\mathrm{d}x)}{e^z - 2}, \quad \frac{\partial z}{\partial x} = \frac{ye^{-xy}}{e^z - 2}, \quad \frac{\partial z}{\partial y} = \frac{xe^{-xy}}{e^z - 2},$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{-y^2 e^{-xy} (e^z - 2) - e^z \frac{\partial z}{\partial x} y e^{-xy}}{(e^z - 2)^2} = \cdots,$$

$$-e^{-xy}(x \, dy + y \, dx) - 2 \, dz + e^z \, dz = 0,$$

$$-e^{-xy}(x\,\mathrm{d}y + y\,\mathrm{d}x) - 2\,\mathrm{d}z + e^z\,\mathrm{d}z = 0,$$

$$\mathrm{d}z = \frac{e^{-xy}(x\,\mathrm{d}y + y\,\mathrm{d}x)}{e^z - 2}, \quad \frac{\partial z}{\partial x} = \frac{ye^{-xy}}{e^z - 2}, \quad \frac{\partial z}{\partial y} = \frac{xe^{-xy}}{e^z - 2},$$

$$\frac{\partial^2 z}{\partial y} = -y^2 e^{-xy}(e^z - 2) - e^z \frac{\partial z}{\partial y} y e^{-xy}$$

$$z = \frac{e^{-xy}(x dy + y dx)}{e^z - 2}, \quad \frac{\partial z}{\partial x} = \frac{ye^{-xy}}{e^z - 2}, \quad \frac{\partial z}{\partial y} = \frac{xe^{-xy}}{e^z - 2}$$

$$z = \frac{e^{-xy}(x dy + y dx)}{e^z - 2}, \quad \frac{\partial z}{\partial z} = \frac{xe^{-xy}}{e^z - 2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{-y^2 e^{-xy} (e^z - 2) - e^z \frac{\partial z}{\partial x} y e^{-xy}}{(e^z - 2)^2} = \cdots,$$

$$\frac{z}{e^2} = \frac{-y^2 e^{-xy} (e^z - 2) - e^z \frac{\partial z}{\partial x} y e^{-xy}}{(e^z - 2)^2} = \cdots,$$

$$\frac{(e^{z}-2)^{2}}{z} = \frac{(e^{-xy}-xye^{-xy})(e^{z}-2)-e^{z}\frac{\partial z}{\partial y}ye^{-xy}}{(-z-2)^{2}} = \cdots,$$

$$\frac{\partial x^2}{\partial x \partial y} = \frac{(e^z - 2)^2}{(e^z - 2)(e^z - 2) - e^z \frac{\partial z}{\partial y} y e^{-xy}}{(e^z - 2)^2} = \cdots,$$

解:

 $-e^{-xy}(x \, dy + y \, dx) - 2 \, dz + e^z \, dz = 0,$

 $\frac{\partial^2 z}{\partial x^2} = \frac{-y^2 e^{-xy} (e^z - 2) - e^z \frac{\partial z}{\partial x} y e^{-xy}}{(e^z - 2)^2} = \cdots,$

 $\frac{\partial^2 z}{\partial x \partial y} = \frac{(e^{-xy} - xye^{-xy})(e^z - 2) - e^z \frac{\partial z}{\partial y} ye^{-xy}}{(e^z - 2)^2}$

 $\frac{\partial^2 z}{\partial y^2} = \frac{-x^2 e^{-xy} (e^z - 2) - e^z \frac{\partial z}{\partial y} x e^{-xy}}{(e^z - 2)^2} = \cdots$

 $dz = \frac{e^{-xy}(x dy + y dx)}{e^z - 2}, \quad \frac{\partial z}{\partial x} = \frac{ye^{-xy}}{e^z - 2}, \quad \frac{\partial z}{\partial y} = \frac{xe^{-xy}}{e^z - 2},$

例
$$z = f(u, v), \quad u = \varphi(x, y), \quad v = \psi(x, y),$$
 $\frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial y^2}, \quad \frac{\partial^2 z}{\partial x \partial y}.$ 解:
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x},$$

解:
$$\frac{\partial z}{\partial x}$$

例
$$z = f(u, v), \quad u = \varphi(x, y), \quad v = \psi(x, y),$$
 $\frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial y^2}, \quad \frac{\partial^2 z}{\partial x \partial y}.$ 解:
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial u} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial u}.$$

解:
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y},$$

$$=rac{\partial z}{\partial u}$$

例
$$z = f(u, v), \quad u = \varphi(x, y), \quad v = \psi(x, y),$$
 $\frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial y^2}, \quad \frac{\partial^2 z}{\partial x \partial y}.$ 解:
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y},$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} (\frac{\partial z}{\partial u}) \frac{\partial u}{\partial x} + \frac{\partial z}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x} (\frac{\partial z}{\partial v}) \frac{\partial v}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial^2 v}{\partial x^2}$$

例
$$z = f(u, v), \quad u = \varphi(x, y), \quad v = \psi(x, y),$$
 $\frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial y^2}, \quad \frac{\partial^2 z}{\partial x \partial y}.$ 解:
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y},$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial u}\right) \frac{\partial u}{\partial x} + \frac{\partial z}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial v}\right) \frac{\partial v}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial^2 v}{\partial x^2}$$

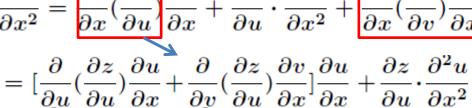
 $= \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u}\right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u}\right) \frac{\partial v}{\partial x}\right] \frac{\partial u}{\partial x}$

解:
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y},$$

 $\text{Fig. } z=f(u,v), \quad u=\varphi(x,y), \quad v=\psi(x,y), \\ \text{$\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, $\frac{\partial^2 z}{\partial x \partial y}$.}$

$$\frac{2z}{z} =$$

$$\frac{\partial^2 z}{\partial x^2} = \underbrace{\frac{\partial}{\partial x} (\frac{\partial z}{\partial u})}_{\partial x} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2} + \underbrace{\frac{\partial}{\partial x} (\frac{\partial z}{\partial v})}_{\partial x} \frac{\partial v}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial^2 v}{\partial x^2}$$







解:
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y},$$

 $\text{Fig. } z=f(u,v), \quad u=\varphi(x,y), \quad v=\psi(x,y), \\ \text{$\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, $\frac{\partial^2 z}{\partial x \partial y}$.}$

$$\frac{\partial x}{\partial x^2} = \frac{\partial u}{\partial x} (\frac{\partial z}{\partial u}) \frac{\partial u}{\partial x} + \frac{\partial z}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial u} (\frac{\partial z}{\partial v}) \frac{\partial v}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial u} \right) \frac{\partial u}{\partial x} + \frac{\partial z}{\partial u} \cdot \frac{\partial}{\partial x}$$

$$\frac{\partial z}{\partial u} \frac{\partial u}{\partial u} + \frac{\partial z}{\partial u} \frac{\partial z}{\partial u} \frac{\partial z}{\partial u}$$

$$\left(\frac{\partial x}{\partial u}\right)\frac{\partial u}{\partial u} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial u}$$

- $= \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u}\right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u}\right) \frac{\partial v}{\partial x}\right] \frac{\partial u}{\partial x} + \frac{\partial z}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2}$ $+\big[\frac{\partial}{\partial u}(\frac{\partial z}{\partial v})\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}(\frac{\partial z}{\partial v})\frac{\partial v}{\partial x}\big]\frac{\partial v}{\partial x}$

解:
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y},$$

 $\text{Fig. } z=f(u,v), \quad u=\varphi(x,y), \quad v=\psi(x,y), \\ \text{$\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, $\frac{\partial^2 z}{\partial x \partial y}$.}$

 $+\big[\frac{\partial}{\partial u}(\frac{\partial z}{\partial v})\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}(\frac{\partial z}{\partial v})\frac{\partial v}{\partial x}\big]\frac{\partial v}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial^2 v}{\partial x^2}$

$$\frac{\partial x}{\partial x^2} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial u} \frac{\partial v}{\partial x} + \frac{\partial z}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial u} \frac{\partial v}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial^2 v}{\partial x^2}$$

$$= \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u}\right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u}\right) \frac{\partial v}{\partial x} \right] \frac{\partial u}{\partial x} + \frac{\partial z}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2}$$

$$= \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u}\right) \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \left(\frac{\partial z}{\partial u}\right) \frac{\partial v}{\partial x} \right] \frac{\partial v}{\partial x} + \frac{\partial z}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2}$$



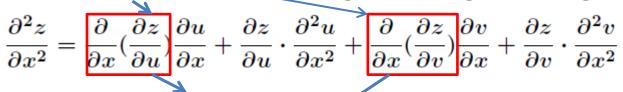


$$= \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u}\right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u}\right) \frac{\partial v}{\partial x}\right] \frac{\partial u}{\partial x} + \frac{\partial z}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2}$$









 $=\frac{\partial^2 z}{\partial u^2}(\frac{\partial u}{\partial x})^2+\frac{\partial^2 z}{\partial u\partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial x^2}$

 $\text{Fig. } z=f(u,v), \quad u=\varphi(x,y), \quad v=\psi(x,y), \\ \text{$\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, $\frac{\partial^2 z}{\partial x \partial y}$.}$

 $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y},$

 $+\big[\frac{\partial}{\partial u}(\frac{\partial z}{\partial v})\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}(\frac{\partial z}{\partial v})\frac{\partial v}{\partial x}\big]\frac{\partial v}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial^2 v}{\partial x^2}$

例
$$z = f(u, v), \quad u = \varphi(x, y), \quad v = \psi(x, y),$$
 $\frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial y^2}, \quad \frac{\partial^2 z}{\partial x \partial y}.$ 解:
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y},$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial u} \right) \frac{\partial u}{\partial x} + \frac{\partial z}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial v} \right) \frac{\partial v}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial^2 v}{\partial x^2}$$

$$= \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u}\right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u}\right) \frac{\partial v}{\partial x}\right] \frac{\partial u}{\partial x} + \frac{\partial z}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2}$$

$$rac{\partial u \ \partial u' \ \partial x' \ \partial v' \ \partial u' \ \partial x' \ \partial x' \ \partial u \ \partial x^2}{\partial + [rac{\partial}{\partial u} (rac{\partial z}{\partial v}) rac{\partial u}{\partial x} + rac{\partial}{\partial v} (rac{\partial z}{\partial v}) rac{\partial v}{\partial x}] rac{\partial v}{\partial x} + rac{\partial z}{\partial v} \cdot rac{\partial^2 v}{\partial x^2}}$$

$$+\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]\frac{\partial v}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial^{2} v}{\partial x^{2}}$$

$$\frac{\partial^{2} z}{\partial x^{2}}\left(\frac{\partial u}{\partial x}\right)^{2} + \frac{\partial^{2} z}{\partial x^{2}}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial z}{\partial x} \cdot \frac{\partial^{2} u}{\partial x} + \frac{\partial^{2} z}{\partial x^{2}}\left(\frac{\partial v}{\partial x}\right)^{2} + \frac{\partial^{2} z}{\partial x^{2}}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}$$

$$=\frac{\partial^2 z}{\partial u^2}(\frac{\partial u}{\partial x})^2+\frac{\partial^2 z}{\partial u\partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 z}{\partial v^2}(\frac{\partial v}{\partial x})^2+\frac{\partial^2 z}{\partial u\partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial x^2}$$

例
$$z = f(u, v), \quad u = \varphi(x, y), \quad v = \psi(x, y),$$
 $\frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial y^2}, \quad \frac{\partial^2 z}{\partial x \partial y}.$ 解:
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y},$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial u} \right) \frac{\partial u}{\partial x} + \frac{\partial z}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial v} \right) \frac{\partial v}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial^2 v}{\partial x^2}$$

$$= \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u}\right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u}\right) \frac{\partial v}{\partial x}\right] \frac{\partial u}{\partial x} + \frac{\partial z}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2}$$

$$= \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u}\right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u}\right) \frac{\partial v}{\partial x}\right] \frac{\partial u}{\partial x} + \frac{\partial z}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2}$$

$$= \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u}\right) \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \left(\frac{\partial z}{\partial u}\right) \frac{\partial v}{\partial x}\right] \frac{\partial v}{\partial x} + \frac{\partial z}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2}$$

$$\begin{split} &+[\frac{\partial}{\partial u}(\frac{\partial z}{\partial v})\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}(\frac{\partial z}{\partial v})\frac{\partial v}{\partial x}]\frac{\partial v}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial x^2}\\ &=\frac{\partial^2 z}{\partial u^2}(\frac{\partial u}{\partial x})^2+\frac{\partial^2 z}{\partial u\partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 z}{\partial v^2}(\frac{\partial v}{\partial x})^2+\frac{\partial^2 z}{\partial u\partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial x^2} \end{split}$$

$$\frac{\partial u^2}{\partial x^2} \frac{\partial x}{\partial x} \frac{\partial u}{\partial x} \frac{\partial x}{\partial x} \frac{\partial u}{\partial x} \frac{\partial x}{\partial x} \frac{\partial u}{\partial x} \frac{\partial x^2}{\partial x} \frac{\partial v^2}{\partial x} \frac{\partial x}{\partial x} \frac{\partial u}{\partial x} \frac{\partial x}{\partial x} \frac{\partial x}{$$

 $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y},$

 $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2} (\frac{\partial u}{\partial x})^2 + 2 \frac{\partial^2 z}{\partial u \partial v} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2} (\frac{\partial v}{\partial x})^2 + \frac{\partial z}{\partial u} \frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v} \frac{\partial^2 v}{\partial x^2}$

 $\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2} (\frac{\partial u}{\partial y})^2 + 2 \frac{\partial^2 z}{\partial u \partial v} \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial^2 z}{\partial v^2} (\frac{\partial v}{\partial y})^2 + \frac{\partial z}{\partial u} \frac{\partial^2 u}{\partial y^2} + \frac{\partial z}{\partial v} \frac{\partial^2 v}{\partial y^2}$

 $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y},$

 $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2} (\frac{\partial u}{\partial x})^2 + 2 \frac{\partial^2 z}{\partial u \partial x} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial u^2} (\frac{\partial v}{\partial x})^2 + \frac{\partial z}{\partial u} \frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial u} \frac{\partial^2 v}{\partial x^2}$

例 $z = f(u,v), \quad u = \varphi(x,y), \quad v = \psi(x,y), \\ \ \frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial x^2}.$

 $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y},$

 $\frac{\partial^2 z}{\partial u^2} = \frac{\partial^2 z}{\partial u^2} (\frac{\partial u}{\partial u})^2 + 2 \frac{\partial^2 z}{\partial u \partial v} \frac{\partial u}{\partial u} \frac{\partial v}{\partial u} + \frac{\partial^2 z}{\partial v^2} (\frac{\partial v}{\partial u})^2 + \frac{\partial z}{\partial u} \frac{\partial^2 u}{\partial u^2} + \frac{\partial z}{\partial v} \frac{\partial^2 v}{\partial u^2}$

 $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2} (\frac{\partial u}{\partial x})^2 + 2 \frac{\partial^2 z}{\partial u \partial v} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial u^2} (\frac{\partial v}{\partial x})^2 + \frac{\partial z}{\partial u} \frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial u} \frac{\partial^2 v}{\partial x^2}$

例 $z = f(u, v), \quad u = \varphi(x), \quad v = \psi(x).$ 求 $\frac{d^2z}{dx^2}$.

例 $z = f(u, v), \quad u = \varphi(x), \quad v = \psi(x).$ 求 $\frac{d^2z}{dx^2}$.

$$\frac{u}{z} + \frac{\partial z}{\partial u} \frac{\mathrm{d}v}{\mathrm{d}z}$$

解:
$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial z}{\partial u} \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial z}{\partial v} \frac{\mathrm{d}v}{\mathrm{d}x}$$

$$\frac{u}{x} + \frac{\partial z}{\partial v} \frac{\mathrm{d}v}{\mathrm{d}x}$$

例
$$z = f(u, v), \quad u = \varphi(x), \quad v = \psi(x).$$
 求 $\frac{d^2z}{dx^2}.$

解:
$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial z}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial z}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\partial z}{\partial u}\varphi'(x) + \frac{\partial z}{\partial v}\psi'(x),$$

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial z}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial z}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\partial z}{\partial u}\varphi'(x) + \frac{\partial z}{\partial v}\psi'(x),$$

例
$$z = f(u, v), \quad u = \varphi(x), \quad v = \psi(x). \$$
 $\frac{d^2z}{dx^2}.$
 $\frac{dz}{dx} = \frac{\partial z}{\partial u} \frac{du}{dx} + \frac{\partial z}{\partial v} \frac{dv}{dx} = \frac{\partial z}{\partial u} \varphi'(x) + \frac{\partial z}{\partial v} \psi'(x)$

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial z}{\partial u} \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial z}{\partial v} \frac{\mathrm{d}v}{\mathrm{d}x} =$$

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial z}{\partial u} \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial z}{\partial v} \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\partial z}{\partial u} \varphi'(x) + \frac{\partial z}{\partial v} \psi'(x),$$

$$\frac{d^2 z}{dx^2} = \frac{\mathrm{d}}{\mathrm{d}x} (\frac{\partial z}{\partial u}) \varphi'(x) + \frac{\partial z}{\partial u} \varphi''(x) + \frac{\mathrm{d}}{\mathrm{d}x} (\frac{\partial z}{\partial v}) \psi'(x) + \frac{\partial z}{\partial v} \psi''(x)$$

例
$$z = f(u, v), \quad u = \varphi(x), \quad v = \psi(x). \quad \dot{x} \frac{d^2 z}{dx^2}.$$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} \frac{du}{dx} + \frac{\partial z}{\partial x} \frac{dv}{dx} = \frac{\partial z}{\partial x} \varphi'(x) + \frac{\partial z}{\partial x} \psi'(x)$$

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial z}{\partial u} \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial z}{\partial v} \frac{\mathrm{d}z}{\mathrm{d}x}$$

$$\frac{d^2z}{dx^2} = \frac{\mathrm{d}z}{\mathrm{d}x} (\frac{\partial z}{\partial u}) \varphi'(x) + \frac{\partial z}{\partial v} \frac{\mathrm{d}z}{\mathrm{d}x}$$

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial z}{\partial u} \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial z}{\partial v} \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\partial z}{\partial u} \varphi'(x) + \frac{\partial z}{\partial v} \psi'(x),$$

$$\frac{d^2 z}{dx^2} = \frac{\mathrm{d}}{\mathrm{d}x} (\frac{\partial z}{\partial u}) \varphi'(x) + \frac{\partial z}{\partial u} \varphi''(x) + \frac{\mathrm{d}}{\mathrm{d}x} (\frac{\partial z}{\partial v}) \psi'(x) + \frac{\partial z}{\partial v} \psi''(x)$$

$$\frac{d^2 z}{dx^2} = \frac{d}{dx} \left(\frac{\partial z}{\partial u} \right) \varphi'(x) + \frac{\partial z}{\partial u} \varphi''(x) + \frac{d}{dx} \left(\frac{\partial z}{\partial v} \right) \psi'(x) + \frac{\partial z}{\partial v} \psi''(x)
= \frac{\partial^2 z}{\partial u^2} \cdot \varphi'(x)^2 + \frac{\partial^2 z}{\partial u \partial v} \varphi'(x) \psi'(x) + \frac{\partial z}{\partial u} \varphi''(x)
= \frac{\partial^2 z}{\partial u^2} \varphi''(x)^2 + \frac{\partial^2 z}{\partial u \partial v} \varphi'(x) \psi'(x) + \frac{\partial z}{\partial u} \varphi''(x)$$

例
$$z = f(u, v), \quad u = \varphi(x), \quad v = \psi(x). \quad \dot{x} \frac{d^2z}{dx^2}.$$

解:
$$\frac{dz}{dx} = \frac{\partial z}{\partial x} \frac{du}{dx} + \frac{\partial z}{\partial x} \frac{dv}{dx} = \frac{\partial z}{\partial x} \varphi'(x) + \frac{\partial z}{\partial x} \psi'(x)$$

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial z}{\partial u} \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial z}{\partial v} \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\partial z}{\partial u} \varphi'(x) + \frac{\partial z}{\partial v} \psi'(x),$$

$$\frac{d^2 z}{dx^2} = \frac{\mathrm{d}}{\mathrm{d}x} (\frac{\partial z}{\partial u}) \varphi'(x) + \frac{\partial z}{\partial u} \varphi''(x) + \frac{\mathrm{d}}{\mathrm{d}x} (\frac{\partial z}{\partial v}) \psi'(x) + \frac{\partial z}{\partial v} \psi''(x)$$

$$= \frac{\partial^2 z}{\partial u^2} \cdot \varphi'(x)^2 + \frac{\partial^2 z}{\partial u \partial v} \varphi'(x) \psi'(x) + \frac{\partial z}{\partial u} \varphi''(x)$$

$$+ \frac{\partial^2 z}{\partial v^2} \cdot \psi'(x)^2 + \frac{\partial^2 z}{\partial u \partial v} \varphi'(x) \psi'(x) + \frac{\partial z}{\partial v} \psi''(x)$$

例 $z = f(u, v), \quad u = \varphi(x), \quad v = \psi(x).$ 菜 $\frac{d^2z}{dx^2}$.

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial z}{\partial u} \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial z}{\partial v} \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\partial z}{\partial u} \varphi'(x) + \frac{\partial z}{\partial v} \psi'(x),$$

$$\frac{d^2 z}{dx^2} = \frac{\mathrm{d}}{\mathrm{d}x} (\frac{\partial z}{\partial u}) \varphi'(x) + \frac{\partial z}{\partial u} \varphi''(x) + \frac{\mathrm{d}}{\mathrm{d}x} (\frac{\partial z}{\partial v}) \psi'(x) + \frac{\partial z}{\partial v} \psi''(x)$$

$$= \frac{\partial^2 z}{\partial u^2} \cdot \varphi'(x)^2 + \frac{\partial^2 z}{\partial u \partial v} \varphi'(x) \psi'(x) + \frac{\partial z}{\partial u} \varphi''(x)$$

$$+ \frac{\partial^2 z}{\partial v^2} \cdot \psi'(x)^2 + \frac{\partial^2 z}{\partial u \partial v} \varphi'(x) \psi'(x) + \frac{\partial z}{\partial v} \psi''(x)$$

$$= \frac{\partial^2 z}{\partial u^2} \cdot \varphi'(x)^2 + 2 \frac{\partial^2 z}{\partial u \partial v} \varphi'(x) \psi'(x) + \frac{\partial^2 z}{\partial v^2} \cdot \psi'(x)^2 + \frac{\partial z}{\partial u} \varphi''(x) + \frac{\partial z}{\partial v} \psi''(x)$$

简单情况: 例 $z=f(x,y), \quad x=x_0+t\cos\alpha, \quad y=y_0+t\sin\alpha, \quad x_0,y_0,\alpha$ 是常数.

简单情况: 例 $z = f(x,y), \quad x = x_0 + t \cos \alpha, \quad y = y_0 + t \sin \alpha, \quad x_0, y_0, \alpha$ 是常数.

$$z = f(x, y), \quad x = x_0 + t \cos \alpha, \quad y = y_0 + t \sin \alpha, \quad x_0, y_0, \alpha \succeq \pi \mathsf{y}.$$

$$z = f(u, v), \quad u = \varphi(x), \quad v = \psi(x). \quad \stackrel{\cdot}{x} \frac{d^2z}{dx^2}.$$

$$\frac{d^2z}{dx^2} = \frac{\partial^2z}{\partial u^2} \cdot \varphi'(x)^2 + 2\frac{\partial^2z}{\partial u\partial v} \varphi'(x)\psi'(x) + \frac{\partial^2z}{\partial v^2} \cdot \psi'(x)^2 + \frac{\partial z}{\partial u} \varphi''(x) + \frac{\partial z}{\partial v} \psi''(x)$$

简单情况: 例 $z = f(x,y), \quad x = x_0 + t \cos \alpha, \quad y = y_0 + t \sin \alpha, \quad x_0, y_0, \alpha$ 是常数.

$$z = f(u, v), \quad u = \varphi(x), \quad v = \psi(x). \quad \stackrel{\circ}{R} \frac{d^2z}{dx^2}.$$

$$\frac{d^2z}{dx^2} = \frac{\partial^2z}{\partial u^2} \cdot \varphi'(x)^2 + 2\frac{\partial^2z}{\partial u\partial v} \varphi'(x)\psi'(x) + \frac{\partial^2z}{\partial v^2} \cdot \psi'(x)^2 + \frac{\partial z}{\partial u} \varphi''(x) + \frac{\partial z}{\partial v} \psi''(x)$$

$$\frac{\mathrm{d}x^{2}}{\mathrm{d}t^{2}} = \cos^{2}\alpha \frac{\partial^{2}z}{\partial x^{2}} + 2\cos\alpha\sin\alpha \frac{\partial^{2}z}{\partial x\partial y} + \sin^{2}\alpha \frac{\partial^{2}z}{\partial y^{2}}$$

简单情况: 例 $z = f(x, y), \quad x = x_0 + t \cos \alpha, \quad y = y_0 + t \sin \alpha, \quad x_0, y_0, \alpha$ 是常数.

$$z = f(u, v), \quad u = \varphi(x), \quad v = \psi(x). \quad \stackrel{?}{R} \frac{d^2z}{dx^2}.$$

$$\frac{d^2z}{dx^2} = \frac{\partial^2z}{\partial u^2} \cdot \varphi'(x)^2 + 2\frac{\partial^2z}{\partial u\partial v} \varphi'(x)\psi'(x) + \frac{\partial^2z}{\partial v^2} \cdot \psi'(x)^2 + \frac{\partial z}{\partial u} \varphi''(x) + \frac{\partial z}{\partial v} \psi''(x)$$

$$\frac{z}{z^2} = \frac{\partial^2 z}{\partial u^2} \cdot \varphi'(x)^2 + 2\frac{\partial^2 z}{\partial u \partial v} \varphi'(x) \psi'(x) + \frac{\partial^2 z}{\partial v^2} \cdot \psi'(x)^2 + \frac{\partial z}{\partial u} \varphi''(x) + \frac{\partial z}{\partial v} \psi''(x)$$

$$z \qquad \partial^2 z \qquad \partial^2 z \qquad \partial^2 z \qquad \partial^2 z$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \cos^2 \alpha \frac{\partial^2 z}{\partial x^2} + 2\cos \alpha \sin \alpha \frac{\partial^2 z}{\partial x \partial y} + \sin^2 \alpha \frac{\partial^2 z}{\partial y^2}$$
$$= (\cos^2 \alpha \frac{\partial^2}{\partial x^2} + 2\cos \alpha \sin \alpha \frac{\partial^2}{\partial x \partial y} + \sin^2 \alpha \frac{\partial^2}{\partial y^2})z$$

简单情况: 例 $z = f(x,y), \quad x = x_0 + t \cos \alpha, \quad y = y_0 + t \sin \alpha, \quad x_0, y_0, \alpha$ 是常数.

$$z = f(u, v), \quad u = \varphi(x), \quad v = \psi(x). \quad \stackrel{?}{R} \frac{d^2z}{dx^2}.$$

$$\frac{d^2z}{dx^2} = \frac{\partial^2z}{\partial u^2} \cdot \varphi'(x)^2 + 2\frac{\partial^2z}{\partial u\partial v} \varphi'(x)\psi'(x) + \frac{\partial^2z}{\partial v^2} \cdot \psi'(x)^2 + \frac{\partial z}{\partial u} \varphi''(x) + \frac{\partial z}{\partial v} \psi''(x)$$

$$= (\cos \alpha \frac{\partial}{\partial x} + \sin \alpha \frac{\partial}{\partial y})^2 z$$

 $= (\cos^2 \alpha \frac{\partial^2}{\partial x^2} + 2\cos \alpha \sin \alpha \frac{\partial^2}{\partial x \partial y} + \sin^2 \alpha \frac{\partial^2}{\partial y^2})z$

 $\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \cos^2 \alpha \frac{\partial^2 z}{\partial x^2} + 2\cos \alpha \sin \alpha \frac{\partial^2 z}{\partial x \partial y} + \sin^2 \alpha \frac{\partial^2 z}{\partial y^2}$

约定算子相乘就是复合!

简单情况: 例

同年 同元: 例 $z = f(x,y), \quad x = x_0 + t \cos \alpha, \quad y = y_0 + t \sin \alpha, \quad x_0, y_0, \alpha$ 是常数.

$$\frac{\partial^n z}{\partial t^n} = (\cos \alpha \frac{\partial}{\partial x} + \sin \alpha \frac{\partial}{\partial y})^n z$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \cos^2 \alpha \frac{\partial^2 z}{\partial x^2} + 2\cos\alpha\sin\alpha \frac{\partial^2 z}{\partial x \partial y} + \sin^2 \alpha \frac{\partial^2 z}{\partial y^2}$$
$$= (\cos^2 \alpha \frac{\partial^2}{\partial x^2} + 2\cos\alpha\sin\alpha \frac{\partial^2}{\partial x \partial y} + \sin^2 \alpha \frac{\partial^2}{\partial y^2})z$$
$$= (\cos\alpha \frac{\partial}{\partial x} + \sin\alpha \frac{\partial}{\partial y})^2 z$$

约定算子相乘就是复合!