Econ 139: Intermediate Financial Economics

Lecture 4 Scribe Notes

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1. Social Planner's Problem

Constrained Optimization:

$$
\begin{aligned}
& Max_{x_1^1, x_2^1} \;\; u^1(x_1^1, x_2^1) \\
& \text{s.t.} \;\; u^2(x_1^2, x_2^2) = \bar{u} \qquad x_1^1 + x_1^2 = w^1 \qquad x_2^1 + x_2^2 = w^2 && (1) \\
& && (2) \\
& \text{can rewrite as:} && (3) \\
& Max_{x_1^1, x_2^1} \;\; u^1(x_1^1, x_2^1) \\
& \text{s.t.} \;\; u^2(w^1 - x_1^1, w^2 - x_2^1) = \bar{u} && (4) \\
& && (5) \\
& \mathcal{L}(x_1^1, x_2^1, \lambda) = u^1(x_1^1, x_2^1) + \lambda(u^2(w^1 - x_1^1, w^2 - x_2^1) - \bar{u}) && (6) \\
& && (7) \\
& \text{using FOCs with } x_1^1 \text{ and } x_2^1\text{, we get} && (8) \\
& \lambda = \frac{u_1^1}{u_1^2} = \frac{u_2^1}{u_2^2} \qquad \text{at optimum} && (9) \\
& u_1^1 = \frac{\partial u^1}{\partial x_1^1} \qquad u_2^1 = \frac{\partial u^1}{\partial x_2^1} && (10) \\
& u_1^2 = \frac{\partial u^2}{\partial x_1^2} \qquad u_2^2 = \frac{\partial u^2}{\partial x_2^2} && (11) \\
& && (12) \\
& \text{optimality condition can be rewritten as} && (13) \\
& -\frac{u_1^1}{u_2^1} = -\frac{u_1^2}{u_2^2} && (14) \\
& MRS_{1,2}^1 = MRS_{1,2}^2 && (15) \\
& && (16)
\end{aligned}
$$

Unconstrained Optimization

$$
\begin{aligned}
&Max_{x_1^1,x_2^1}\;\; u^1(x_1^1,x_2^1)+\lambda u^2(x_1^2,x_2^2) \\
&\text{s.t.}\;\; x_1^1+x_1^2=w^1 \qquad x_2^1+x_2^2=w^2 && (17)\\
& && (18)\\
&\text{can rewrite as:} && (19)\\
&Max_{x_1^1,x_2^1}\;\; u^1(x_1^1,x_2^1)+\lambda u^2(w^1-x_1^1,w^2-x_2^1) \\
& && (20)\\
&\mathcal{L}(x_1^1,x_2^1,\lambda)=u^1(x_1^1,x_2^1)+\lambda(u^2(w^1-x_1^1,w^2-x_2^1)-\bar{u}) && (21)\\
& && (22)\\
&\text{using FOCs with } x_1^1 \text{ and } x_2^1\text{, we get} && (23)\\
&x_1^1: u_1^1-\lambda u_1^2=0 && (24)\\
&x_2^1: u_2^1-\lambda u_2^2=0 && (25)\\
&\text{optimality condition can be rewritten as} && (26)\\
&-\frac{u_1^1}{u_2^1}=-\frac{u_1^2}{u_2^2} && (27)\\
&MRS_{1,2}^1=MRS_{1,2}^2 && (28)\\
& && (29)
\end{aligned}
$$

Optimization with Three Goods

$$
\begin{aligned}
&Max_{x_1^1,x_2^1,x_3^1}\;\; u^1(x_1^1,x_2^1,x_3^1)+\lambda u^2(x_1^2,x_2^2,x_3^2) \\
&\text{s.t.}\;\; x_1^1+x_1^2=w^1 \qquad x_2^1+x_2^2=w^2 \qquad x_3^1+x_3^2=w^2 && (30)\\
& && (31)\\
&\text{can rewrite as:} && (32)\\
&Max_{x_1^1,x_2^1,x_3^1}\;\; u^1(x_1^1,x_2^1,x_3^1)+\lambda u^2(w^1-x_1^1,w^2-x_2^1,w^3-x_3^1) \\
& && (33)\\
&\text{using FOCs with } x_1^1 \text{ and } x_2^1\text{, we get} && (34)\\
&x_1^1: u_1^1-\lambda u_1^2=0 && (35)\\
&x_2^1: u_2^1-\lambda u_2^2=0 && (36)\\
&x_3^1: u_3^1-\lambda u_3^2=0 && (37)\\
&\lambda=\frac{u_1^1}{u_1^2}=\frac{u_2^1}{u_2^2}=\frac{u_3^1}{u_3^2} && (38)\\
&-\frac{u_1^1}{u_2^1}=-\frac{u_1^2}{u_2^2} \qquad -\frac{u_1^1}{u_3^1}=-\frac{u_1^2}{u_3^2} \qquad -\frac{u_2^1}{u_3^1}=-\frac{u_2^2}{u_3^2} && (39)\\
&MRS_{1,2}^1=MRS_{1,2}^2 \qquad MRS_{1,2}^1=MRS_{1,2}^3 \qquad MRS_{1,2}^2=MRS_{1,2}^3 && (40)\\
&\text{Pareto Efficient} && (41)\\
& && (42)
\end{aligned}
$$

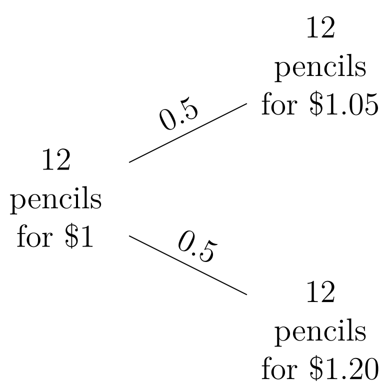
2. Arbitrage

Arbitrage opportunity is a profit opportunity that is risk-less in the following sense:

- Opportunity never looses money neither today nor in the future.
- The opportunity makes money today or **may** make money in the future (positive probability of making money in the future).

Ex: (making money today) Suppose we have a package of 12 pencils that sells for \$1 today. and individual pencils sell for 10 cents.

Ex: Suppose we have a package of 12 pencils that sells for \$1 today, money can be borrowed at 5 % interest, tomorrow 12 pencils sell for \$1.05 or \$1.20.



State	Asset 1	Asset 2
S1	4	6
S2	0	0
Price	2	4

This is an arbitrage opportunity, because I can buy/long 1.5 shares of Asset 1 and sell/short 1 share of Asset 2, and end up pocketing 1 dollar while keeping the same payout in the future.

State	1.5 shares of A1	1 share of A2
S1	6	6
S2	0	0
Price	3	4

People would be selling Asset 2 and buying Asset 1 putting a downward pressure on Asset 2 and an upward pressure on Asset 1 until the point where there is no opportunity for arbitrage.

Form an arbitrage portfolio sell(short) 1 share of Asset 2 and buy(long) 1.5 shares of Asset 1.

State	Asset 1	Asset 2	Portfolio
S1	4	6	0
S2	0	0	0
Price	P_1	P_2	1.5P_1-P_2

If 1.5P_1-P_2 <0 we have an arbitrage opportunity. This means that we sell the expensive asset and buy the cheaper asset.

State	Asset 1	Asset 2
S1	2	3
S2	4	6
Price	3	4.5

In the example above, we do not have an arbitrage opportunity. However, in the example that follows, we will see an arbitrage opportunity, but of a different kind than what we have seen before.

State	Asset 1	Asset 2	Asset 3	Portfolio
S1	1	5	0	0
S2	0.1	9	1	0.5
Price	0.3	6.9	0.6	0

In the example above, we can long 5 shares of Asset 1, long 9 shares of Asset 3, and short 1 share of Asset 2. In this example, we do not pocket money in the present. Previous examples where type 1 arbitrage opportunities, while this is a type 2 arbitrage opportunity.

Type 1 - get money today, no money in the the future
Type 2 - no money today, positive probability of making money

The example above is a type 2 arbitrage opportunity.

State	Asset 1	Asset 2	Asset 3	Asset 4
S1	1	0	0	7
S2	0	1	0	3
S2	0	0	1	5
Price	0.3	0.5	0.4	5

In the example above, can short 7 shares of Asset 1, 3 shares of Asset 2, and 5 shares of Asset 3; long 1 share of Asset 4 and pocket 0.6 dollars today.

In the example above, Asset 1, 2 and 3 are Arrow Debreu securities. They have value of 1 only in 1 stats, in all other states their value is 0. These securities define prices for other securities. Asset 4 would have to cost 5.6 dollars for there not to be an arbitrage opportunity because Asset 1, 2 and 3 can replicated the payout of Asset 4.

3. Law of One Price

Let's begin the discussion of Law of One Price with the following example.

State	Asset 1	Asset 2	Asset 3
S1	0	1	5
S2	2	0	6
Price	P_1	P_2	P_3

Notice that we can replicated the payoff of Asset 3 by forming a portfolio of 3 shares of Asset 1 and 5 shares of Asset 2.

$$P_3 = 3P_1 + 5P_2$$

Thus, Asset 3 is called a redundant asset.

LOOP: Asset with an identical payoff in the future must have the same price today in the absence of arbitrage. Law of One Price is frequently abbreviated to LOOP.

4. Put - Call Parity

Consider European style call and put options on the same stock, with the same exercise price X and the same maturity T. Let P_0 and C_0 represent current prices of put and call options. Let

$$\frac{X}{1 + r_f}$$

be price of a risk free zero coupon bond with maturity T.

Claim: call price + ZCB price = stock price + put price

$$C_0 + \frac{X}{1 + r_f} = S_0 + P_0$$

Price of ZCB can also be expressed as $X^{-r_f T}$ if we use continuous compounding.