Econ 139 – Midterm Suggested Solutions
Fall 2018

As a general rule for exams, you may use any theorems stated in lecture without proof, unless you are explicitly directed otherwise.

Problem 1. True or false.
Are the following statements true or false? Explain your answer as concisely as possible. You will recieve no credit without an explanation.

1. When the risk-free rate r_f increases, the optimal portfolio share of risky assets for a mean-variance investor increases. (7 points)

   Solution: False. The optimal portfolio share of risky assets is given by $w^* = (\mathbb{E}[\tilde{r}] - r_f)/(A\sigma^2)$, hence when r_f increases w^* declines.

2. The following situation is an arbitrage opportunity:

State	Asset 1	Asset 2
State 1	4	6
State 2	1	2
Price	2	3

   (9 points)

   Solution: True. Construct an arbitrage portfolio by going long 2 shares of asset 2 and short 3 shares of asset 1. This gives the following payoffs and prices:

State	-3*(Asset 1)	2*(Asset 2)	Portfolio
State 1	-12	12	0
State 2	-3	4	1
Price	-6	6	0

   This is a zero-cost portfolio that pays off 0 in state 1 and 1 in state 2.

3. Suppose the independence axiom holds for a preference relation $\succcurlyeq$ on $\mathcal{L}$ (the space of simple lotteries). Then for all L_{xy}, L_{vz}, L_{pq}, $L_{st} \in \mathcal{L}$, and for all $\alpha \in [0, 1]$, we have

$$L_{xy} \succcurlyeq L_{vz},$$

if and only if

$$\alpha L_{xy} + (1-\alpha)L_{pq} \succcurlyeq \alpha L_{vz} + (1-\alpha)L_{st}.$$

In words, if we mix each of the two lotteries L_{xy} and L_{vz} with a third lottery, the preference ordering of the two resulting mixtures is *independent* of the particular third lottery used. (9 points)

Solution: **False.** Under the independence axiom, this is only true provided the third lottery is the *same* for both mixtures.

Problem 2. Evaluating risky investments. (21 points, 7 points each)

Consider an economic environment in which there are two possible future states: a good state that occurs with probability $\pi = 1/2$ and a bad state that occurs with probability $1 - \pi = 1/2$. Three assets are traded, with percentage returns in the two states tabulated below:

State	Asset 1	Asset 2	Asset 3
Good	10	20	30
Bad	10	10	10

1. Rank these assets, or determine that they can not be ranked, in terms of

    (a) State-by-state dominance

    (b) First order stochastic dominance

    (c) Second order stochastic dominance

    Solution: Asset 2 dominates asset 1 on a state-by-state basis, and asset 3 dominates both asset 1 and asset 2 on a state-by-state basis. Therefore, using state-by-state dominance, asset 3 is preferred to asset 2 and asset 1, and asset 2 is preferred to asset 1.

    Since state-by-state dominance implies both first order stochastic dominance and second order stochastic dominance, the rankings using these criteria are the same as the ranking using state-by-state dominance.

2. Rank these assets, or determine that they can not be ranked, by the mean-variance criterion.

    Solution: The expected returns and return variances are tabulated below:

	Asset 1	Asset 2	Asset 3
Good state	10	20	30
Bad state	10	10	10
Expected return	10	15	20
Return variance	0	25	100
Return std dev	0	5	10

In this case, no asset mean-variance dominates another, since the return variances among the assets rise along with expected return. Hence, these assets can not be ranked by the mean-variance criterion.

3. How will a VNM expected utility investor who prefers more to less allocate their investable wealth among these three assets (assuming they allocate all of their investable wealth among these assets and they are not allowed to borrow)?

   Solution: Any VNM investor who prefers more to less will always choose an asset that exhibits state-by-state dominance over all others. In this case, the investor will allocate all of their investable wealth to asset 3.

Problem 3. Portfolio allocation and gains from diversification. (24 points, 6 points each)

Consider an investor that would like to allocate their investable wealth among the following three risky assets:

	Expected Return (μ)	Return Variance (σ^2)
Asset 1	1	1
Asset 2	2	4
Asset 3	3	9

Assume the return correlations among all of the assets is 0.

1. The investor will allocate w_1, w_2, and w_3 of their investable wealth to assets 1, 2, and 3, respectively. Write down expressions for the expected return (μ_p) and variance (σ_p^2) of the portfolio as a function of w_1, w_2, and w_3.

   Solution: The expected return and variance of the investor's portfolio are given by

$$\begin{aligned} \mu_p &= w_1 + 2w_2 + 3w_3 \\ \sigma_p^2 &= w_1^2 + 4w_2^2 + 9w_3^2 \end{aligned}$$

   Some students may have solved the problem assuming that the numbers given in the far right column of the table were standard deviations instead of variances. In this case the expected return and variance of the portfolio would be written as

$$\begin{aligned} \mu_p &= w_1 + 2w_2 + 3w_3 \\ \sigma_p^2 &= w_1^2 + 16w_2^2 + 81w_3^2 \end{aligned}$$

   (4 out of 6 points are awarded for this solution)

2. The investor hopes to choose the portfolio weights (w_1, w_2, and w_3) optimally, however, in order to minimize the variance of the portfolio, subject to the constraint that the expected return of the portfolio equals 2 and the constraint that the portfolio weights sum to 1 (i.e., the "full investment constraint").

   Write down the optimization problem facing the investor and write down the Lagrangian (using λ as the Lagrange multiplier) and the first order conditions. Please use the full investment constraint to subsitute for w_2, reducing the problem to a minimization over two variables (w_1 and w_3) subject to an expected return constraint.

Solution: The optimization problem can be written as

$$\begin{aligned} \min_{w_1,w_3} \quad & w_1^2 + 4(1-w_1-w_3)^2 + 9w_3^2 \\ \text{s.t.} \quad & w_1 + 2(1-w_1-w_3) + 3w_3 = 2 \end{aligned}$$

The Lagrangian is

$$\mathcal{L}(w_1, w_3, \lambda) = w_1^2 + 4(1-w_1-w_3)^2 + 9w_3^2 - \lambda(w_1 + 2(1-w_1-w_3) + 3w_3 - 2)$$

The first order conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial w_1} &= 2w_1^* - 8(1-w_1^*-w_3^*) + \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial w_3} &= 18w_3^* - 8(1-w_1^*-w_3^*) - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= w_1^* + 2(1-w_1^*-w_3^*) + 3w_3^* - 2 = 0 \end{aligned}$$

These can be simplified to

$$\begin{aligned} 10w_1^* + 8w_3^* - 8 + \lambda &= 0 \\ 8w_1^* + 26w_3^* - 8 - \lambda &= 0 \\ w_3^* - w_1^* &= 0 \end{aligned}$$

Note that this simplification is not necessary to receive full credit for this portion.

Carrying through the confusion between standard deviation and variance from point 1, the optimization problem is

$$\begin{aligned} \min_{w_1,w_3} \quad & w_1^2 + 16(1-w_1-w_3)^2 + 81w_3^2 \\ \text{s.t.} \quad & w_1 + 2(1-w_1-w_3) + 3w_3 = 2 \end{aligned}$$

The Lagrangian is

$$\mathcal{L}(w_1, w_3, \lambda) = w_1^2 + 16(1-w_1-w_3)^2 + 81w_3^2 - \lambda(w_1 + 2(1-w_1-w_3) + 3w_3 - 2)$$

The first order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial w_1} &= 2w_1^* - 32(1 - w_1^* - w_3^*) + \lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial w_3} &= 162w_3^* - 32(1 - w_1^* - w_3^*) - \lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= w_1^* + 2(1 - w_1^* - w_3^*) + 3w_3^* - 2 = 0
\end{aligned}$$

These can be simplified to

$$\begin{aligned}
34w_1^* + 32w_3^* - 32 + \lambda &= 0 \\
32w_1^* + 194w_3^* - 32 - \lambda &= 0 \\
w_3^* - w_1^* &= 0
\end{aligned}$$

(4 out of 6 points are awarded for this solution)

3. Find the numerical values of w_1^*, w_2^*, and w_3^* that solve the investor's problem.

Solution: The last first order condition tells us that $w_1^* = w_3^*$. Thus using the first two first order conditions we have

$$\begin{aligned}
10w_1^* + 8w_1^* - 8 + \lambda &= 0 \\
8w_1^* + 26w_1^* - 8 - \lambda &= 0
\end{aligned}$$

Combining these gives

$$52w_1^* - 16 = 0,$$

then

$$w_1^* = \frac{4}{13} = w_3^*$$

Finally, using the full investment constraint gives

$$w_1^* = \frac{4}{13}, \quad w_2^* = \frac{5}{13}, \quad w_3^* = \frac{4}{13}$$

Carrying through the confusion between standard deviation and variance from point 1, the solution is

$$w_1^* = \frac{16}{73}, \quad w_2^* = \frac{41}{73}, \quad w_3^* = \frac{16}{73}$$

(4 out of 6 points are awarded for this solution)

4. Verify that your solution from the previous point satisfies the expected return con-

straint. Calculate the portfolio variance and verify that it is less than the variance of asset 2 (hence, the optimal portfolio has the same expected return and lower variance than a portfolio that consists entirely of asset 2).

Solution: Looking at the portfolio expected return we have

$$\mu_p = 1 \cdot \frac{4}{13} + 2 \cdot \frac{5}{13} + 3 \cdot \frac{4}{13} = \frac{26}{13} = 2,$$

as required. For the portfolio variance we have

$$\begin{aligned}\sigma_p^2 &= 1 \cdot \left(\frac{4}{13}\right)^2 + 4 \cdot \left(\frac{5}{13}\right)^2 + 9 \cdot \left(\frac{4}{13}\right)^2 \\ &= \frac{16}{169} + \frac{100}{169} + \frac{144}{169} \\ &= \frac{260}{169} \approx 1.5,\end{aligned}$$

which is considerably less than 4 (the variance of asset 2).

Carrying through the confusion between standard deviation and variance from point 1, for the portfolio expected return we have

$$\mu_p = 1 \cdot \frac{16}{73} + 2 \cdot \frac{41}{73} + 3 \cdot \frac{16}{73} = \frac{146}{73} = 2,$$

as required. For the portfolio variance we have

$$\begin{aligned}\sigma_p^2 &= 1 \cdot \left(\frac{16}{73}\right)^2 + 16 \cdot \left(\frac{41}{73}\right)^2 + 81 \cdot \left(\frac{16}{73}\right)^2 \\ &= \frac{256}{5329} + \frac{26896}{5329} + \frac{20736}{5329} \\ &= \frac{47888}{5329} \approx 9,\end{aligned}$$

which is considerably less than 16.

(4 out of 6 points are awarded for this solution)

Problem 4. Hyperbolic absolute risk aversion (HARA) utility. (30 points, 6 points each)

The HARA utility function takes the following form

$$u(w) = \frac{1-\gamma}{\gamma}\left(\frac{\alpha w}{1-\gamma} + \beta\right)^{\gamma}$$

with restrictions on the parameters such that $\alpha > 0$, $\gamma \neq 1$, and $\beta + \alpha w/(1-\gamma) > 0$.

We can use this utility function to represent the preferences of investors that exhibit decreasing absolute risk aversion (DARA) and increasing relative risk aversion (IRRA). None of the utility functions you have seen so far, either in class or on the problem sets, allows us to represent the preferences of such investors.

To simplify, please assume that $\alpha = 1$ throughout this problem.

1. Show that the utility function is increasing and concave in w.

   Solution: For this utility function (with $\alpha = 1$), we have

$$\begin{aligned} u'(w) &= \left(\frac{w}{1-\gamma} + \beta\right)^{\gamma-1} > 0 \\ u''(w) &= -\left(\frac{w}{1-\gamma} + \beta\right)^{\gamma-2} < 0, \end{aligned}$$

   Since

$$\left(\frac{w}{1-\gamma} + \beta\right) > 0,$$

   by definition of the utility function. Therefore, the utility function is increasing and concave in w.

2. Show that the absolute and relative risk aversion coefficients can be written as

$$r_A(w) = \frac{1-\gamma}{w + (1-\gamma)\beta}, \quad r_R(w) = \frac{(1-\gamma)w}{w + (1-\gamma)\beta}$$

Solution: The risk aversion coefficients are given by

$$\begin{aligned} r_A(w) &= -\frac{u''(w)}{u'(w)} \\ &= -\frac{-\left(\frac{w}{1-\gamma}+\beta\right)^{\gamma-2}}{\left(\frac{w}{1-\gamma}+\beta\right)^{\gamma-1}} \\ &= \left(\frac{w}{1-\gamma}+\beta\right)^{-1} \\ &= \frac{1-\gamma}{w+(1-\gamma)\beta} \end{aligned}$$

and

$$\begin{aligned} r_R(w) &= r_A(w)\cdot w \\ &= \frac{(1-\gamma)w}{w+(1-\gamma)\beta} \end{aligned}$$

3. What restrictions on the parameters γ and β are required to ensure that $dr_A(w)/dw < 0$ and $dr_R(w)/dw > 0$? What does it mean if $dr_A(w)/dw < 0$ and $dr_R(w)/dw > 0$?

   Solution: The derivatives of the risk aversion coefficients are given by

$$\begin{aligned} \frac{dr_A(w)}{dw} &= \frac{\gamma-1}{(w+(1-\gamma)\beta)^2} \\ \frac{dr_R(w)}{dw} &= \frac{\beta(1-\gamma)^2}{(w+(1-\gamma)\beta)^2} \end{aligned}$$

   Hence, to ensure that $dr_A(w)/dw < 0$ and $dr_R(w)/dw > 0$ we need $\gamma < 1$ and $\beta > 0$.

   If $dr_A(w)/dw < 0$ and $dr_R(w)/dw > 0$, the investor has declining absolute risk aversion and increasing relative risk aversion. Therefore, as the investor's wealth increases they will be willing to invest more money in the risky asset, but at the same time the *fraction* of their wealth invested in the risky asset will decline.

4. Assuming the restrictions on γ and β that you found in the previous point, suppose a VNM expected utility investor with investable wealth W_0 wants to construct an optimal portfolio of a risk-free asset with net return r_f and a risky asset with net return $\tilde{r}$, where $\tilde{r}$ has a continuous probability density function over $[-1, \infty)$.

Write down the utility maximization problem for this investor, where the decision variable is a, the (absolute) amount of money to be invested in the risky asset, and find the first order condition with respect to a. Check that the second order condition for a maximum is satisfied.

Solution: The utility maximization problem is given by

$$\max_a \quad \mathbb{E}\left[\frac{1-\gamma}{\gamma}\left(\frac{\tilde{W}_1}{1-\gamma}+\beta\right)^{\gamma}\right]$$

where

$$\tilde{W}_1 = W_0(1+r_f) + a(\tilde{r}-r_f).$$

The first order condition with respect to a is given by

$$\mathbb{E}\left[\left(\frac{\tilde{W}_1}{1-\gamma}+\beta\right)^{\gamma-1}(\tilde{r}-r_f)\right] = 0,$$

where

$$\tilde{W}_1 = W_0(1+r_f) + a^*(\tilde{r}-r_f).$$

The second order condition with respect to a is given by

$$\mathbb{E}\left[-\left(\frac{\tilde{W}_1}{1-\gamma}+\beta\right)^{\gamma-2}(\tilde{r}-r_f)^2\right] < 0,$$

where

$$\tilde{W}_1 = W_0(1+r_f) + a^*(\tilde{r}-r_f),$$

since

$$\left(\frac{\tilde{W}_1}{1-\gamma}+\beta\right) > 0$$

with the parameter restrictions $\gamma < 1$ and $\beta > 0$. Therefore, the second order condition for a maximum is satisfied.

5. Suppose $a^* > 0$, where a^* is the solution to the optimization problem from point 4. Determine the signs of the following quantities:

   (a) $\mathbb{E}[\tilde{r}] - r_f$

   (b) da^*/dW_0

   (c) $(da^*/dW_0)(W_0/a^*) - 1$

Justify your answers (no credit without justification).

Solution:

(a) $\mathbb{E}[\tilde{r}] - r_f > 0$

(b) $da^*/dW_0 > 0$

(c) $(da^*/dW_0)(W_0/a^*) - 1 < 0$

(a) follows by the theorem stated in lecture which says that

$$\mathbb{E}[\tilde{r}] - r_f > 0 \text{ if and only if } a^* > 0$$

(b) follows by the theorem stated in lecture which says that

$$\text{if } \frac{dr_A(W_0)}{dW_0} < 0 \text{ then } \frac{da^*}{dW_0} > 0$$

(c) follows by the theorem stated in lecture which says that

$$\text{if } \frac{dr_R(W_0)}{dW_0} > 0 \text{ then } \eta = \frac{da^*}{dW_0} \cdot \frac{W_0}{a^*} < 1$$