Econ 139 – Midterm Suggested Solutions
Fall 2017

Problem 1. True or false. (18 points, 6 points each)
Are the following statements true or false? Explain your answer in no more than two sentences. You will be graded on your explanation.

1. **True**. Put-Call parity says $C_0 + X/(1 + R_f) = P_0 + S_0$. If the right hand side $P_0 + S_0$ increases by \$2 - \$1, then so must the left-hand side. Since $X/(1 + R_f)$ is constant, this means C_0 must increase by a dollar.

2. **True**. This is the Second Fundamental Theorem of Welfare Economics.

3. **False**. Each bundle in the $\{(x_1^n, x_2^n)\}_{n=1}^{\infty}$ sequence is preferred to the corresponding bundle in the $\{(y_1^n, y_2^n)\}_{n=1}^{\infty}$ sequence, but that preference is reversed at the limit bundles, $(0, 0)$ for the $\{(x_1^n, x_2^n)\}_{n=1}^{\infty}$ sequence and $(0, 1)$ for the the $\{(y_1^n, y_2^n)\}_{n=1}^{\infty}$ sequence.

Problem 2. Portfolio choice with expected utility. (32 points, 4 points each)

1. $$\tilde{W}_1 = (W_0 - a)R_f + a\tilde{R},$$

   which can be re-written as

   $$\tilde{W}_1 = W_0 R_f + a(\tilde{R} - R_f).$$

2. The probability distribution for $\tilde{W}_1$ is

   $$\begin{aligned} \tilde{W}_1 &= 1.1 - 0.1a \;\text{ w.p. } p_1 \\ \tilde{W}_1 &= 1.1 \;\text{ w.p. } p_2 \\ \tilde{W}_1 &= 1.1 + 0.1a \;\text{ w.p. } p_3 \end{aligned}$$

3. The agent's optimization problem is

   $$\max_{a \geq 0} \; \mathbb{E}[U(\tilde{W}_1)] = \mathbb{E}[U(W_0 R_f + a(\tilde{R} - R_f))]$$

   This can be re-written as

   $$\max_{a \geq 0} \; p_1 U(1.1 - 0.1a) + p_2 U(2) + p_3 U(1.1 + 0.1a).$$

4. The first order condition is

   $$-0.1 p_1 U'(1.1 - 0.1a^*) + 0.1 p_3 U'(1.1 + 0.1a^*) = 0.$$

5. The second order condition is

   $$0.01 p_1 U''(1.1 - 0.1a^*) + 0.01 p_3 U''(1.1 + 0.1a^*) < 0,$$

   since $U'' < 0$ by supposition. Hence, the second order condition for a maximum is satisfied.

6. The FOC can be written as

   $$\frac{p_3}{p_1} = \frac{U'(1.1 - 0.1a^*)}{U'(1.1 + 0.1a^*)}$$

   For $a^* > 0$, the right hand side will be greater than 1, which implies $p_3 > p_1$. If $p_3 > p_1$, then $\mathbb{E}[\tilde{R}] > 1.1$. Hence, we must have $\mathbb{E}[\tilde{R}] > 1.1$ to ensure $a^* > 0$.

7. (i) $U(W) = \ln(W)$:

$$-p_1 \frac{1}{1.1 - 0.1a^*} + p_3 \frac{1}{1.1 + 0.1a^*} = 0,$$

or

$$p_3 \frac{1}{1.1 + 0.1a^*} = p_1 \frac{1}{1.1 - 0.1a^*}.$$

Cross multipying gives

$$p_3(1.1 - 0.1a^*) = p_1(1.1 + 0.1a^*).$$

or (recalling that $W_0 R_f = 1.1$)

$$a^* = 11 \frac{p_3 - p_1}{p_3 + p_1} = 10(W_0 R_f) \frac{p_3 - p_1}{p_3 + p_1}.$$

Since $p_3 > p_1$, we have $da^*/dW_0 > 0$.

(ii) $U(W) = \alpha W - \beta W^2$, where $\alpha, \beta > 0$ and $W < \frac{\alpha}{2\beta}$:

$$-p_1(\alpha - 2\beta(1.1 - 0.1a^*)) + p_3(\alpha - 2\beta(1.1 + 0.1a^*)) = 0,$$

or

$$p_3(\alpha - 2.2\beta - 0.2a^*) = p_1(\alpha - 2.2\beta + 0.2a^*),$$

or

$$0.2a^*(p_3 + p_1) = (p_3 - p_1)(\alpha - 2.2\beta),$$

or (recalling that $W_0 R_f = 1.1$)

$$a^* = 5(\alpha - 2.2\beta)\left(\frac{p_3 - p_1}{p_3 + p_1}\right) = 5(\alpha - 2(W_0 R_f)\beta)\left(\frac{p_3 - p_1}{p_3 + p_1}\right).$$

Since $p_3 > p_1$ and $\beta > 0$, we have $da^*/dW_0 < 0$.

8. (i) $U(W) = \ln(W)$:

$$r_A(W) = -\frac{U''(W)}{U'(W)} = -\frac{-1/W^2}{1/W} = \frac{1}{W}.$$

(ii) $U(W) = \alpha W - \beta W^2$, where $\alpha, \beta > 0$ and $W < \frac{\alpha}{2\beta}$:

$$r_A(W) = -\frac{U''(W)}{U'(W)} = -\frac{-2\beta}{\alpha - 2\beta W} = \frac{2\beta}{\alpha - 2\beta W}.$$

Problem 3. Portfolio choice and the efficient frontier. (30 points, 5 points each)

1. With $\rho_{A,B} = -1$, we have

$$\sigma_P^2 = (1 - w_B)^2\sigma_A^2 + w_B^2\sigma_B^2 - 2(1 - w_B)w_B\sigma_A\sigma_B.$$

   Hence,

$$\sigma_P = \pm((1 - w_B)\sigma_A - w_B\sigma_B).$$

   To maintain portfolio volatility at 0.10, we need to find w_B such that

$$0.10 = \pm((1 - w_B)0.10 - w_B 0.25),$$

   or

$$0.10 = \pm(0.10 - w_B 0.35).$$

   This gives us either $w_B = 0$, which leaves us where we started, or $w_B = 4/7$. Using $w_B = 4/7$, the new portfolio expected return is given by

$$\mu_P = (3/7) \cdot 0.10 + (4/7) \cdot 0.20 = 0.1571.$$

   So we could expect to receive additional return of 0.0571 by shifting 4/7 of the portfolio to asset B.

2. To achieve portfolio volatility of 0, we need to find w_B such that

$$0 = \pm((1 - w_B)0.10 - w_B 0.25),$$

   or

$$0 = \pm(0.10 - w_B 0.35).$$

   Hence, we can construct a portfolio with $\sigma_P = 0$, using $w_B = 2/7$. The expected return of this portfolio is

$$\mu_P = (5/7) \cdot 0.10 + (2/7) \cdot 0.20 = 0.1286.$$

   The risk-free rate in this economy must be 0.1286, or 12.86%.

3. Asset A is on the minimum variance frontier, since it plots on the curve. Asset A is not on the efficient frontier, since we can find portfolios on the minimum variance frontier with both the same or lower variance, and higher expected return.

4. As we saw in class, the weight of asset A in the minimum variance portfolio is given by

$$w_A^{MV} = \frac{\sigma_B^2 - \rho_{A,B}\sigma_A\sigma_B}{\sigma_A^2 + \sigma_B^2 - 2\rho_{A,B}\sigma_A\sigma_B}.$$

Plugging in our values gives

$$w_A^{MV} = \frac{0.25^2 - 0.1 \cdot 0.10 \cdot 0.25}{0.10^2 + 0.25^2 - 2 \cdot 0.1 \cdot 0.10 \cdot 0.25} = 0.89,$$

and $w_B^{MV} = 1 - w_A^{MV} = 0.11$. The expected return of the minimum variance portfolio is then

$$\mu_{MV} = 0.89 \cdot 0.10 + 0.11 \cdot 0.20 = 0.111.$$

Noting that

$$\sigma_{MV} = 0.0957 < 0.10 = \sigma_A,$$

and $\mu_{MV} = 0.111 > 0.10 = \mu_A$, the minimum variance portfolio mean-variance dominates asset A.

It is also sufficient to answer the last part of the question in words. Since the minimum variance portfolio lies at the vertex of the curve, which is above and to the left of asset A (meaning it has higher expected return and lower variance than asset A), the minimum variance portfolio mean-variance dominates asset A.

5. Going to the point on the efficient frontier that lies directly above asset A, we see that there is a portfolio such that $\sigma_P = 0.10 = \sigma_A$ and $\mu_P \approx 0.122$. Hence, the approximate increase in expected return is 0.022, or 2.2%. Note: any estimate of μ_P between 0.12 and 0.125 is fine.

6. This is bad advice. As long the asset is not perfectly positively correlated (i.e., $\rho = 1$) with the existing portfolio, it is often possible to reduce portfolio volatility, increase portfolio expected return, or both!

Problem 4. Certainty equivalent. (20 points, 5 points each)

1. The minimum selling price P_s must satisfy the following expression:

$$U(W + P_s) = \pi U(W + G) + (1 - \pi)U(W + B).$$

2. The maximum buying price P_b must satisfy the following expression:

$$U(W) = \pi U(W - P_b + G) + (1 - \pi)U(W - P_b + B).$$

3. P_s:

$$\begin{aligned}
\ln(10 + P_s) &= 0.5\ln(10 + 26) + 0.5\ln(10 + 6), \\
2\ln(10 + P_s) &= \ln(36) + \ln(16), \\
\ln((10 + P_s)^2) &= \ln(36 \cdot 16), \\
(10 + P_s)^2 &= 576, \\
10 + P_s &= 24, \\
P_s &= 14.
\end{aligned}$$

P_b:

$$\begin{aligned}
\ln(10) &= 0.5\ln(10 - P_b + 26) + 0.5\ln(10 - P_b + 6), \\
2\ln(10) &= \ln(36 - P_b) + \ln(16 - Pb), \\
\ln(100) &= \ln((36 - P_b)(16 - P_b)), \\
100 &= 576 - 52P_b + P_b^2, \\
0 &= 476 - 52P_b + P_b^2, \\
P_b &= 11.86 \ \text{(rounded to two decimal places)}.
\end{aligned}$$

The minimum selling price P_s and the maximum buying price P_b are not equal. In fact, $P_b < P_s$. If the individual already owns the lottery their minimum wealth is $10{+}6 = 16$. If they buy the lottery, their minimum wealth is $10 - P_b + 6 = 16 - P_b < 16$. In the former case, they are less risk averse and the lottery is worth more.

4. If the individual is risk-neutral (i.e, they have a linear utility function, where $U'' = 0$), then we will have $P_s = P_b$. For simplicity, let $U(W) = W$. Then

P_s:

$$\begin{aligned} 10 + P_s &= 0.5(10+26) + 0.5(10+6), \\ 10 + P_s &= 26, \\ P_s &= 16. \end{aligned}$$

P_b:

$$\begin{aligned} 10 &= 0.5(10 - P_b + 26) + 0.5(10 - P_b + 6), \\ 10 &= 26 - P_b, \\ P_b &= 16. \end{aligned}$$