Midterm Exam
Mathematical Methods in Finance
Fall 2014
Nov. 24th, 2014

Question #1 (20 *points*)
Let $\{W_1(t)\}$ and $\{W_2(t)\}$ be two independent standard one-dimensional Brownian motions.

1. *(12 points)* Find a constant c such that $\{B(t)\}$, where $B(t) = c(W_1(t) - W_2(t))$, is also a standard one-dimensional Brownian motion. And, use the definition of Brownian motion to carefully justify your answer, i.e., to show that $\{B(t)\}$ satisfy the definition of a standard one-dimensional Brownian motion in this case.

2. *(8 points)* Is it True or False that "with probability 1, the path of $\{W_1(t)\}$ and $\{W_2(t)\}$ intersect infinitely many times?" Why? Clearly use a property we learnt in class to briefly justify your claim.

Suggested solution:
1. To make B a standard Brownian motion, we need

$$\mathbb{E}B(t)^2 = t.$$

Because

$$\begin{aligned}
\mathbb{E}B(t)^2 &= c^2\mathbb{E}(W_1(t) - W_2(t))^2 \\
&= c^2\left[\mathbb{E}W_1(t)^2 + \mathbb{E}W_2(t)^2 - 2\mathbb{E}(W_1(t)W_2(t))\right] \\
&= c^2\left[\mathbb{E}W_1(t)^2 + \mathbb{E}W_2(t)^2 - 2\mathbb{E}W_1(t)\mathbb{E}W_2(t)\right] \\
&= 2c^2 t,
\end{aligned}$$

we need

$$2c^2 = 1.$$

Thus, we have

$$c = \frac{1}{\sqrt{2}}.$$

Then, use the four points in the definition of Brownian motion to justify that $B(t) = \frac{1}{\sqrt{2}}(W_1(t) - W_2(t))$, is a standard one-dimensional Brownian motion.

(1) It is obvious that $B(0) = 0$ since $W_1(0) = W_2(0) = 0$;

(2) $B(t)$ is a continuous function for any $\omega \in \Omega$. This can be easily obtained from continuity of $W_1(t)$ and $W_2(t)$.

(3) For any $s < t$, note that

$$B(t) - B(s) = \frac{1}{\sqrt{2}}\left((W_1(t) - W_1(s)) + (W_2(t) - W_2(s))\right),$$

and

$$W_1(t) - W_1(s) \stackrel{d}{=} W_2(t) - W_2(s) \sim N\left(0, t-s\right).$$

Since $W_1(t)$ and $W_2(t)$ are independent, so do $W_1(t) - W_1(s)$ and $W_2(t) - W_2(s)$. Thus, we claim $B(t) - B(s) \sim N\left(0, t-s\right)$. Then, the stationary increment is verified.

(4) Independent increment. This can be obtained from independent increment of $W_1(t)$ and $W_2(t)$, as well as their independence.

2. The claim is TRUE!

As a Brownian motion B visits zero infinitely often (recurrence), thus $B(t) = \frac{1}{\sqrt{2}}(W_1(t) - W_2(t))$ equals zero infinitely often. This results in that the path of $\{W_1(t)\}$ and $\{W_2(t)\}$ intersect infinitely many times.

Question #2 (20 *points*)

Poisson process is very useful in modeling arrivals of market shocks. Let $\{N(t)\}$ be a Poisson process with intensity λ and suppose $\{\mathcal{F}(t)\}$ is the a filtration generated by $\{N(t)\}$ itself. i.e. $\mathcal{F}(t) = \sigma(N(s), s \leq t)$. Intuitively, $\mathcal{F}(t)$ is the information of the Poisson process accumulated up to time t. Let $t_2 > t_1 > 0$.

1. *(8 points)* Compute the conditional probability $\mathbb{P}(N(t_2) = n_2 | N(t_1) = n_1)$ for some integers $n_2 > n_1 > 0$. Thus, are the two random variables $N(t_2)$ and $N(t_1)$ independent? Why?

2. *(12 points)* Find conditional expectations $\mathbb{E}(N(t_2)|N(t_1))$, $\mathbb{E}(N(t_2)|\mathcal{F}(t_1))$, $\mathbb{E}(N(t_1)|N(t_2))$, and $\mathbb{E}(N(t_1)|\mathcal{F}(t_2))$.

Suggested solution:

1. Since Poisson process admit independent and stationary increment, we have

$$\begin{aligned}
\mathbb{P}(N(t_2) = n_2 | N(t_1) = n_1) &= \mathbb{P}(N(t_2) - N(t_1) = n_2 - n_1) \\
&= \mathbb{P}(N(t_2 - t_1) = n_2 - n_1) \\
&= \frac{(\lambda\left(t_2 - t_1\right))^{n_2 - n_1}}{(n_2 - n_1)!} e^{-\lambda(t_2 - t_1)}.
\end{aligned}$$

We note that

$$\mathbb{P}(N(t_2) = n_2 | N(t_1) = n_1) \neq \mathbb{P}(N(t_2) = n_2),$$

i.e.,

$$\mathbb{P}(N(t_2) = n_2, N(t_1) = n_1) \neq \mathbb{P}(N(t_1) = n_1)\mathbb{P}(N(t_2) = n_2).$$

Thus, it is obvious that the two random variables $N(t_2)$ and $N(t_1)$ are not independent.

2. Using the definition and basic properties of conditional expectation, we obtain the following results. Because the Markov property, we have

$$\mathbb{E}\left(N(t_2)|N(t_1)\right) = \mathbb{E}(N(t_2)|\mathcal{F}(t_1)).$$

Because the compensated Poisson process $M(t) = N(t) - \lambda t$ is a martingale, we have

$$\mathbb{E}\left(M(t_2)|\mathcal{F}(t_1)\right) = M(t_1).$$

Thus, we have

$$\mathbb{E}\left(N(t_2) - \lambda t_2 | \mathcal{F}(t_1)\right) = N(t_1) - \lambda t_1.$$

So,

$$\mathbb{E}\left(N(t_2) | \mathcal{F}(t_1)\right) = N(t_1) + \lambda(t_2 - t_1).$$

Thus, we obtained

$$\mathbb{E}\left(N(t_2) | N(t_1)\right) = \mathbb{E}\left(N(t_2) | \mathcal{F}(t_1)\right) = N(t_1) + \lambda(t_2 - t_1).$$

To calculate $\mathbb{E}(N(t_1)|N(t_2))$, note that given $N(t_2)$, the arrivial times are order statistics of $N(t_2)$ random varianbles uniformly distributed on $(0, t_2)$. So, we conclude

$$\mathbb{E}\left(N(t_1) | N(t_2)\right) = \frac{t_1}{t_2} N(t_2).$$

Or we can either use the joint density of $(N(t_1)|N(t_2))$. Note that by definition of condition probability, we have

$$\mathbb{P}(N(t_1) = n_1 | N(t_2) = n_2) = \frac{\mathbb{P}(N(t_1) = n_1, N(t_2) = n_2)}{\mathbb{P}(N(t_2) = n_2)}.$$

Since $N(t_1)$ and $N(t_2) - N(t_1)$ are independent, we further derive

$$\begin{aligned}
\mathbb{P}\left(N(t_1) = n_1 | N(t_2) = n_2\right) &= \frac{\mathbb{P}(N(t_1) = n_1)\mathbb{P}\left(N(t_2) - N(t_1) = n_2 - n_1\right)}{\mathbb{P}(N(t_2) = n_2)} \\
&= \frac{(\lambda t_1)^{n_1} \left(\lambda\left(t_2 - t_1\right)\right)^{n_2 - n_1} n_2!}{(\lambda t_2)^{n_2} n_1! \left(n_2 - n_1\right)!} = \frac{t_1^{n_1} \left(t_2 - t_1\right)^{n_2 - n_1} n_2!}{t_2^{n_2} n_1! \left(n_2 - n_1\right)!}.
\end{aligned}$$

Then, the conditional expectation follows

$$\begin{aligned}
\mathbb{E}\left(N(t_1) | N(t_2) = n_2\right) &= \sum_{n_1=0}^{\infty} \frac{t_1^{n_1} \left(t_2 - t_1\right)^{n_2 - n_1} n_2!}{t_2^{n_2} \left(n_1 - 1\right)! \left(n_2 - n_1\right)!} \\
&= n_2 \left(\frac{t_2 - t_1}{t_2}\right)^{n_2} \sum_{n_1=0}^{\infty} \left(\frac{t_1}{t_2 - t_1}\right)^{n_1} \frac{(n_2 - 1)!}{(n_1 - 1)! \left(n_2 - n_1\right)!} \\
&= n_2 \left(\frac{t_2 - t_1}{t_2}\right)^{n_2} \left(1 + \frac{t_1}{t_2 - t_1}\right)^{n_2 - 1} = \frac{t_1 n_2}{t_2}.
\end{aligned}$$

So, we also obtain $\mathbb{E}\left(N(t_1)|N(t_2)\right) = t_1 n_2 / t_2$.Because $N(t_1)$ is $\mathcal{F}(t_1)$ measurable (known to the information), it is thus $\mathcal{F}(t_2)$ measurable. So

$$\mathbb{E}\left(N(t_1) | \mathcal{F}(t_2)\right) = N(t_1).$$

Note that other methods arriving to the same results are also acceptable!

Question #3 (20 *points*)

Consider a two-period binomial lattice model with $S_0 = 6,\ u = 2,\ d = \frac{1}{2}$. Suppose that the real-world probability for the stock to go up at each period is $p = \frac{5}{6}$. We assume the risk-free rate as $r = \frac{1}{4}$.

1. *(4 points)* What is the risk-neutral probability measure?

2. *(8 points)* Find the initial no-arbitrage price of a put option with strike $K = 6$. You may use either the backward induction or the forward pricing method to calculate it.

3. *(8 points)* Find the corresponding Delta-hedging strategy, i.e., the number of stock shares in the replicating portfolio, and briefly describe how the replication is done.

Suggested solution:

1. Risk-neutral measure is an equivalent probability measure under which the discounted security prices of the market are martingales.

2. According to binomial lattice pricing, the risk-neutral probability can be given as

$$\begin{aligned} \tilde{p} &= \frac{1+r-d}{u-d} = \frac{1}{2}, \\ \tilde{q} &= \frac{u-(1+r)}{u-d} = \frac{1}{2}. \end{aligned}$$

Also, the ternimal value of option is

$$\begin{aligned} V_2(HH) &= (6 \times 2 \times 2 - 6)^+ = 18, \\ V_2(HT) &= V_2(TH) = \left(6 \times 2 \times \frac{1}{2} - 6\right)^+ = 0, \\ V_2(TT) &= \left(6 \times \frac{1}{2} \times \frac{1}{2} - 6\right)^+ = 0. \end{aligned}$$

Thus, backward induction implies

$$\begin{aligned} V_1(H) &= 18 \times \frac{1}{2} \times \frac{4}{5} = \frac{36}{5}, \ V_1(T) = 0, \\ V_0 &= \frac{36}{5} \times \frac{1}{2} \times \frac{4}{5} = \frac{72}{25}. \end{aligned}$$

3. We have

$$\begin{aligned} \Delta_0 &= \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{4}{5}, \ V_0 - \Delta_0 S_0 = -\frac{48}{25}, \text{(borrow)} \\ \Delta_1(H) &= \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = 1, \ V_1(H) - \Delta_1(H) S_1(H) = -\frac{24}{5}, \text{(borrow)} \\ \Delta_1(T) &= \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)} = 0, \ V_1(T) - \Delta_1(T) S_1(T) = 0. \end{aligned}$$

Question #4 (20 *points*)

Suppose a geometric Brownian motion is employed to model the USD/CNY foreign exchange rate, i.e.

$$R(t) = R_0 \exp\left(\sigma W(t) + \mu t\right),$$

where W is a standard one-dimensional Brownian motion; R_0, μ and σ are positive constants representing the initial exchange rate, the drift and the volatility, respectively.

1. *(12 points)* In general cases, do we have $\mathbb{E}(1/R(t)) = 1/(\mathbb{E}R(t))$? Please justify your answer!

2. *(8 points)* Under what relation between μ and σ, the CNY/USD exchange rate process $\{1/R(t)\}$ is a martingale?

Suggested solution:

1. No, it doesn't hold. Note that $R(t)$ is log-normal distributed whose mean is $\exp\left(\log R_0 + \mu t + \sigma^2 t/2\right)$. On the other hand, $1/R(t)$ is also log-normal distributed whose mean is $\exp\left(-\log R_0 - \mu t + \sigma^2 t/2\right)$. Then, we can see the relation doesn't hold.

2. From the last question, it is necessary and sufficient that

$$\mu = \frac{1}{2}\sigma^2.$$

Or, we can see

$$\frac{1}{R(t)} = \frac{1}{R_0}\exp\left(-\sigma W(t) - \mu t\right),$$

which is a geometric Brownian motion. Since we know

$$\exp\left(-\sigma W(t) - \frac{1}{2}\sigma^2 t\right)$$

is an expoenntial martingale, we can also obtain the relation.

Question #5 (*plus additional* 20 *points*)

The price of an asset with an initial value S_0 follows a binomial lattice model

$$S_{k+1} = S_k X_{k+1},$$

for $k = 0, 1, 2, \cdots$, where $X_1, X_2, \cdots$, are independently and identically distributed according to a Bernoulli distribution taking value $u > 1$ with probability p and taking value $1/u$ with probability $1-p$. Suppose $0 < p < \frac{1}{2}$.

1. Let

$$M_k = \left(\prod_{l=1}^{k} X_l\right)^{\frac{\log(1-p)-\log p}{\log u}}.$$

   And, denote by $\{\mathcal{F}_k\}$ is a filtration generated by $\{X_k\}$ itself, i.e., $\mathcal{F}_k = \sigma\left(X_l, l \le k\right)$. Intuitively, $\mathcal{F}_k$ is the information of the process $\{X_k\}$ accumulated up to the time k. Prove that $\{M_k\}$ is a martingale adapted to the filtration $\{\mathcal{F}_k\}$.

2. Find the probability that the stock price hit the level $S_0 u^m$ before hitting $S_0 u^n$. Here, $m > 0$ and $n < 0$ are two integers.

Suggested solution:

1. It suffice to prove that

$$\mathbb{E}\left[M_{k+1}|M_k\right] = M_k,$$

i.e.,

$$\mathbb{E}\left[\left(\prod_{l=1}^{k} X_l\right)^{\frac{\log(1-p)-\log p}{\log u}} X_{k+1}^{\frac{\log(1-p)-\log p}{\log u}}\right] = \left(\prod_{l=1}^{k} X_l\right)^{\frac{\log(1-p)-\log p}{\log u}}.$$

Since $X_1, X_2, \cdots$, are independently and identically distributed, we have

$$\mathbb{E}\left[\left(\prod_{l=1}^{k} X_l\right)^{\frac{\log(1-p)-\log p}{\log u}} X_{k+1}^{\frac{\log(1-p)-\log p}{\log u}}\right] = \mathbb{E}\left[\left(\prod_{l=1}^{k} X_l\right)^{\frac{\log(1-p)-\log p}{\log u}}\right]\mathbb{E}\left[X_{k+1}^{\frac{\log(1-p)-\log p}{\log u}}\right].$$

Thus, it is equivalent to veirfy that

$$\mathbb{E}\left[X_{k+1}^{\frac{\log(1-p)-\log p}{\log u}}\right] = 1.$$

By definition, we deduce

$$\mathbb{E}\left[X_{k+1}^{\frac{\log(1-p)-\log p}{\log u}}\right] = pu^{\frac{\log(1-p)-\log p}{\log u}} + (1-p)\,u^{\frac{\log p-\log(1-p)}{\log u}}.$$

Note that

$$u = e^{\log u},$$

which implies

$$u^{\frac{1}{\log u}} = e.$$

Thus, we have

$$\begin{aligned} u^{\frac{\log(1-p)-\log p}{\log u}} &= e^{\log(1-p)-\log p} = e^{\log\left(\frac{1-p}{p}\right)} = \frac{1-p}{p}, \\ u^{\frac{\log p-\log(1-p)}{\log u}} &= e^{\log p-\log(1-p)} = e^{\log\left(\frac{p}{1-p}\right)} = \frac{p}{1-p}. \end{aligned}$$

Finally, we have

$$\mathbb{E}\left[X_{k+1}^{\frac{\log(1-p)-\log p}{\log u}}\right] = p\frac{1-p}{p} + (1-p)\,\frac{p}{1-p} = 1-p+p = 1.$$

2. Define

$$\tau = \min\left\{k \in \mathbb{N} : S_k = S_0 u^m \text{ or } S_k = S_0 u^n\right\}.$$

It is obvious that τ is a stopping time. Note that

$$M_k = \left(\frac{S_k}{S_0}\right)^{\frac{\log(1-p)-\log p}{\log u}}$$

Since for any integer $t > 0$, $|M_{k\wedge t}|$ is uniformly bounded by a constant $\max\{|u^{\frac{m(\log(1-p)-\log p)}{\log u}}|, |u^{\frac{n(\log(1-p)-\log p)}{\log u}}|\}$ almost surely, optional sampling theorem implies M_τ is a martingale. Thus, we have $\mathbb{E}\left[M_\tau\right] = M_1 = 1$. On the other hand, assuming the probability we want is p_0, we can rewrite $\mathbb{E}\left[M_\tau\right]$ by definition as follow

$$\begin{aligned} 1 &= \mathbb{E}\left[M_\tau\right] = u^{\frac{m(\log(1-p)-\log p)}{\log u}} p_0 + u^{\frac{n(\log(1-p)-\log p)}{\log u}} \left(1 - p_0\right) \\ &= \left(\frac{1-p}{p}\right)^m p_0 + \left(\frac{1-p}{p}\right)^n \left(1-p_0\right), \end{aligned}$$

which implies

$$p_0 = \frac{1 - \left(\frac{1-p}{p}\right)^n}{\left(\frac{1-p}{p}\right)^m - \left(\frac{1-p}{p}\right)^n}.$$