Module 3:
Term Structure of Interest Rate

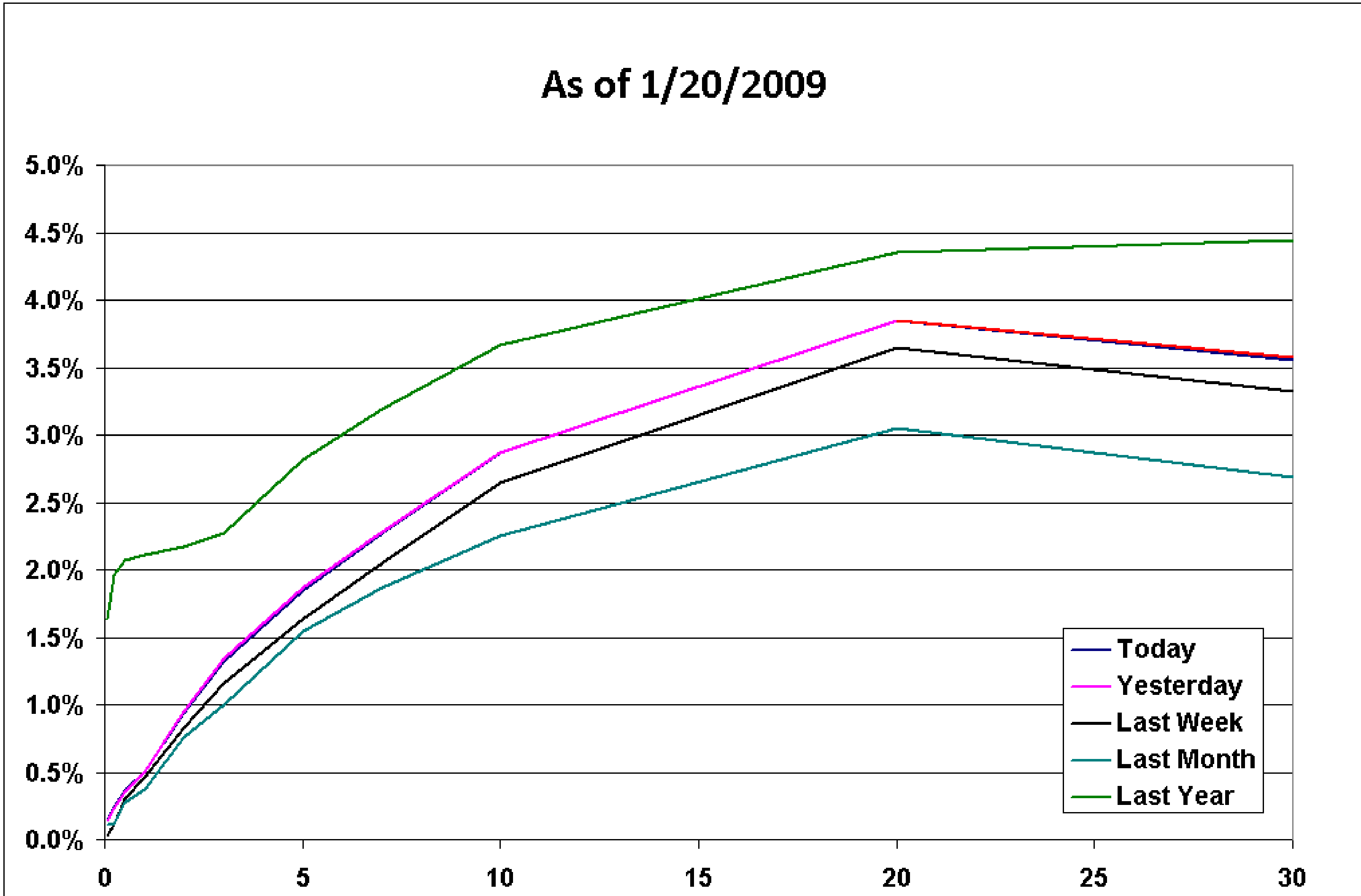
Last Time

- Price changes are in the opposite direction of yield changes.
- Duration is a measure of how much price changes when yield changes.
- Duration should ***not*** be thought of as a weighted average maturity.

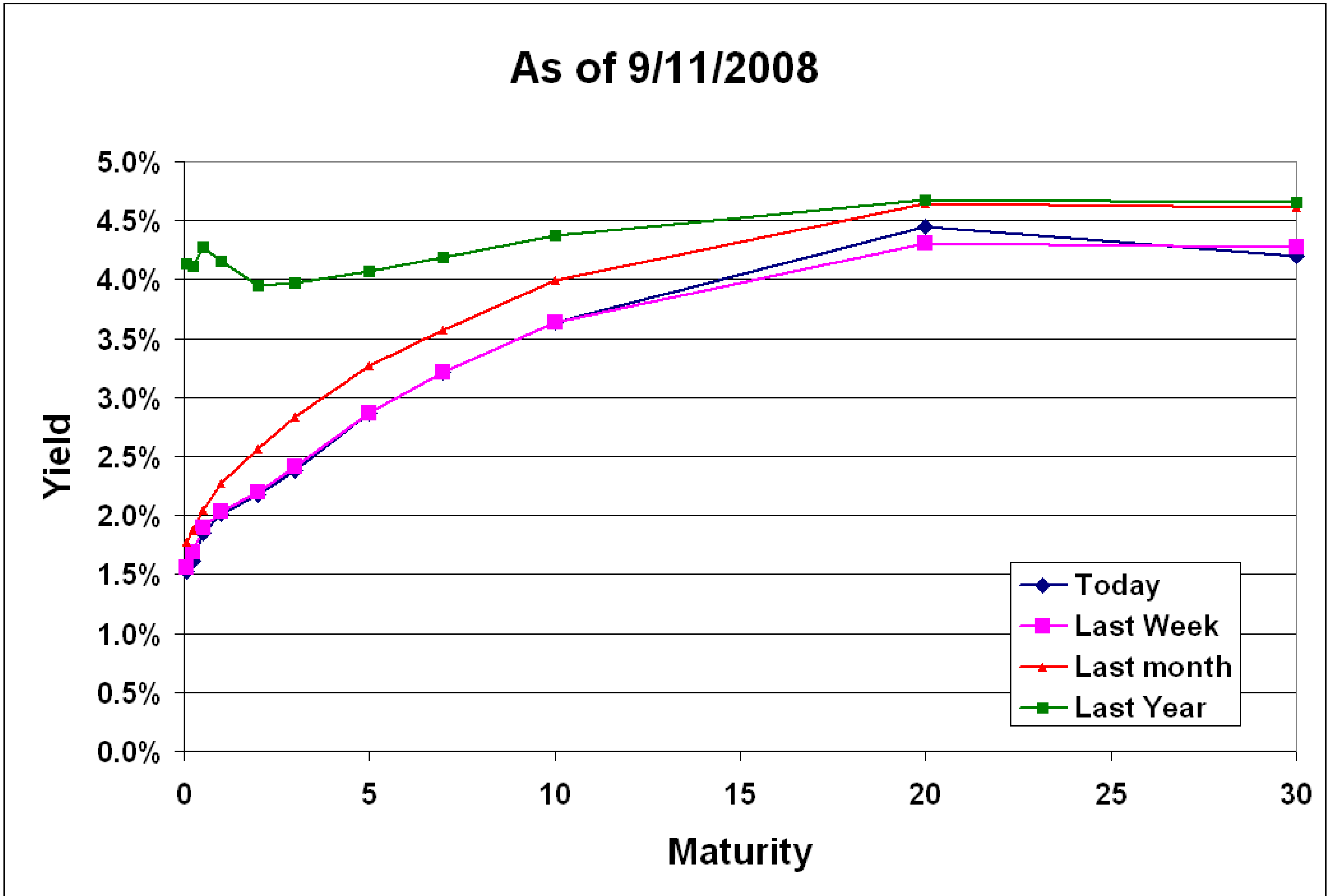
Term Structure

- Term Structure of Interest Rates is relationship between Yield and Maturity.
- Bonds of different maturities have different yields.
  - Why?
- A graph of yield vs. maturity is called a Yield Curve.

Example


As of 1/20/2009
5.0%
4.5%
4.0%
3.5%
3.0%
2.5%
2.0%
1.5%
1.0%
0.5%
0.0%
0
5
10
15
20
25
30
Today
Yesterday
Last Week
Last Month
Last Year

Example



As of 9/11/2008
Yield
5.0%
4.5%
4.0%
3.5%
3.0%
2.5%
2.0%
1.5%
1.0%
0.5%
0.0%
0
5
10
15
20
25
30
Maturity
Today
Last Week
Last month
Last Year

Spot Rates

- A Spot Rate is the yield of a single cash flow of a given maturity. Think of this as Zero-Coupon Bond Yields.
- Spot Rate Curve is a graph of Zero Coupon Bond yields vs. maturity.

Relationship Between Spot Curve and Yield Curve

- If you know Yield Curve, can you build a Spot Curve?
- If you know Spot Curve, can you build a yield curve?
  - We can just price bonds with a portfolio of zero coupon bonds that have the same cash flows – then find yield.
- Claim: one-to-one relationship between curves.

Theoretical Spot Curve

- Theoretical spot curve is a graph of yields of bonds with one cash flow. It can be built up from prices/yields of coupon bearing bonds.
- If you know that a one-year 7% coupon bond has a yield of 6% and six-month bill has a yield of 5.5%, can you find yield of one year bill?

Theoretical Spot Curve

Price of one-year bond is:

$$PV = \frac{3.5}{1.03} + \frac{103.5}{1.03^2} = 100.95673$$

But we can find value of first flow using six month yield of 5.5%

$$PV = \frac{3.5}{1.0275} = 3.40633$$

This means that second cash flow must be worth $100.95673 - 3.40633 = 97.55040$

If 103.5 is worth 97.55040, then

$$97.55040 = \frac{103.5}{\left(1 + \frac{y_{12}}{2}\right)^2}$$

$$\left(1 + \frac{y_{12}}{2}\right)^2 = \frac{103.5}{97.55040} = 1.06099$$

$$1 + \frac{y_{12}}{2} = \sqrt{1.06099} = 1.030044$$

$$y_{12} = 6.00874\%$$

Forward Rates

- Suppose you were guaranteed to receive a large amount of money in one year that you plan to reinvest for another year (in a 1-year zero or bill).
- You hate risk, and want to lock in a yield now. Suppose that an investment bank is willing to enter into an agreement to lock in a rate on an investment starting in one year.
- What rate can you get?
- This rate is called one-year forward rate in one year.

Forward Rates

- How could you lock in this rate on your own?
  - Borrow money for one year
  - Buy two-year zero coupon bond now
  - Pay back loan in one year.
  - By Law of One-Price (no arbitrage), this must have the same price as if you agreed on a price to purchase one-year zero in one year.

Forward Rates

- What does Math look like?

$$\left(1+\frac{y_2}{2}\right)^4=\left(1+\frac{y_1}{2}\right)^2\left(1+\frac{f}{2}\right)^2$$

y_2 - Yield of two year zero coupon bond

y_1 - Yield of one year zero coupon bond

f - Forward Rate for one year zero coupon bond starting in one year

- Investing in two-year must be the same as investing in one year & then in one-year forward.

Forward Rates

- In general, we could look at forward rate of an *m* period bond after *n* periods (e.g., what is ten-year forward rate starting five years from now).
- Following relationship must hold:

$$\left(1+\frac{y_{n+m}}{2}\right)^{n+m}=\left(1+\frac{y_n}{2}\right)^{n}\left(1+\frac{f_m}{2}\right)^{m}$$

Notation

- Note that notation *f* is sloppy. To do it right, we would need to be able to distinguish between forward rate in one year and forward rate in two years -- as well as one-year forward rate in one year and two year forward rate in one year.
- We could use $f_{1,1}$ and $f_{1,2}$ when there are ambiguities.
- In this class we will avoid most of these details.

Spot Rates and Forward Rates

- There is a one-to-one relationship between spot curve and forward curve.
- If you know spot rates, you can find forward rates.
- Example: Suppose you know one-year spot rate is 8% and two year spot rate is 10%. Find one-year forward rate in a year.

Spot Rates and Forward Rates

$$\left(1+\frac{y_2}{2}\right)^4 = \left(1+\frac{y_1}{2}\right)^2\left(1+\frac{f}{2}\right)^2$$

$$\left(1+\frac{0.10}{2}\right)^4 = \left(1+\frac{0.08}{2}\right)^2\left(1+\frac{f}{2}\right)^2$$

$$(1.05)^4 = (1.04)^2\left(1+\frac{f}{2}\right)^2$$

$$1.21551 = 1.08160\left(1+\frac{f}{2}\right)^2$$

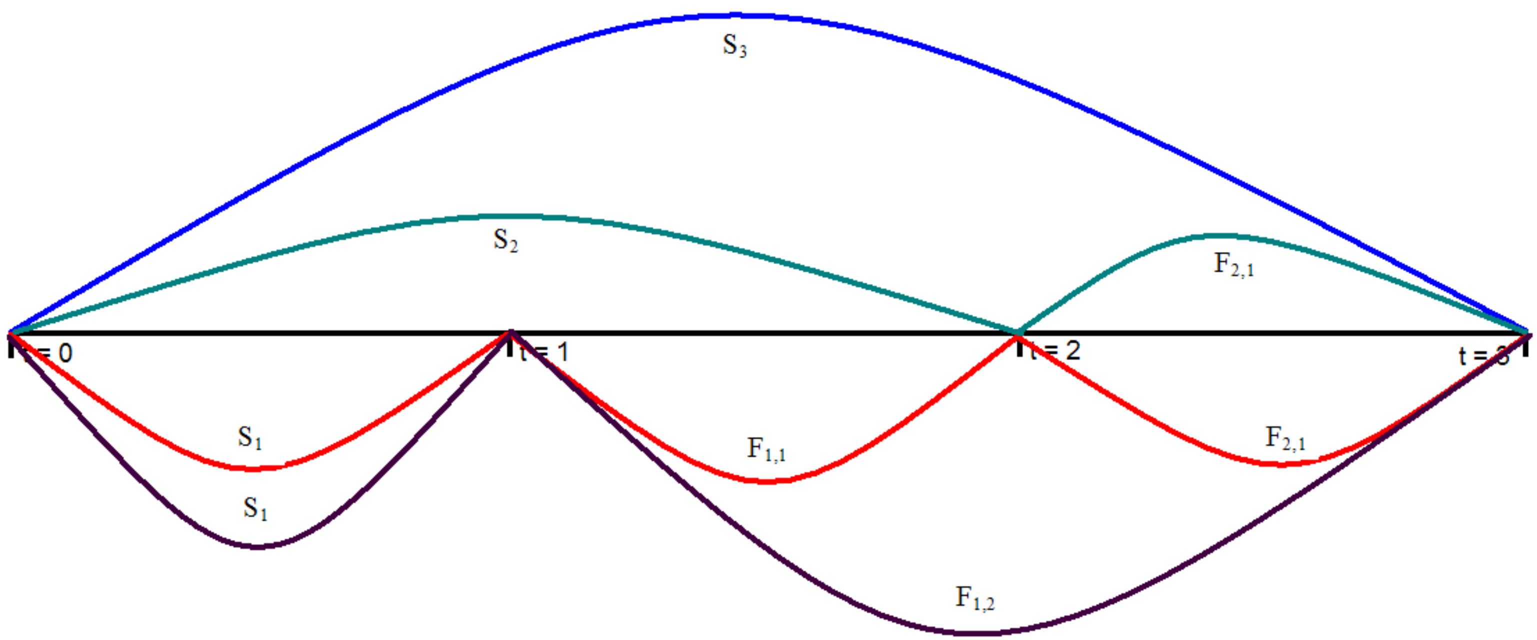
$$\left(1+\frac{f}{2}\right)^2 = 1.23804$$

$$1+\frac{f}{2} = 1.06010$$

$$f = 12.019\%$$

Forward Rates

- Forward rates must be consistent
  - Discounting a cash flow in the future should be the same if you use forward rates or spot rates.



Interrelationships of Yield Curves, Spot Rates and Forward Rates

- There is a one-to-one relationship between Yield Curve and Spot Curve
  - Spot Curve can be used to price bonds – and get yields.
  - Bootstrap method can be used to get Spot Curve if you have Yield Curve.

Interrelationships of Yield Curves, Spot Rates and Forward Rates

- There is a one-to-one relationship between Spot Curve and Forward Rates
  - Spot Curve can be used to calculate forward rates.
  - Forward Rates can be used to discount any cash flow – giving price. Yield can be calculated from price.
- There is a one-to-one relationship between Yield Curve and Forward Rates
  - This is implied by their relationship with Spot Curve.

Yield Curve Shapes

- Normal Shape:
  - Slope upward to ten years.
  - More gentle upward to 15 years.
  - Downward slope 25-30 years.
  - Long term rates greater than short-term rates.
- Inverted Shape
  - Long term rates lower than short-term rates.
- Flat
  - All rates are about the same.

Yield Curve Shapes

- What does it mean?
  - Expectations hypothesis.
    - Forward Rates are expected (average) values .
    - Ignores other risks in bond market.
    - Generally, financial economists dismiss this theory.
    - However, they do believe that certain expectations about where rates will be and about inflation factor into shape.
  - Liquidity Theory (Risk Premium)
    - Long-term investors demand a premium because of the price risk. Does this mean that when the curve is inverted people are more worried about reinvestment risk?
  - Market Segmentation
    - Matching liabilities creates more demand in one segment than another.

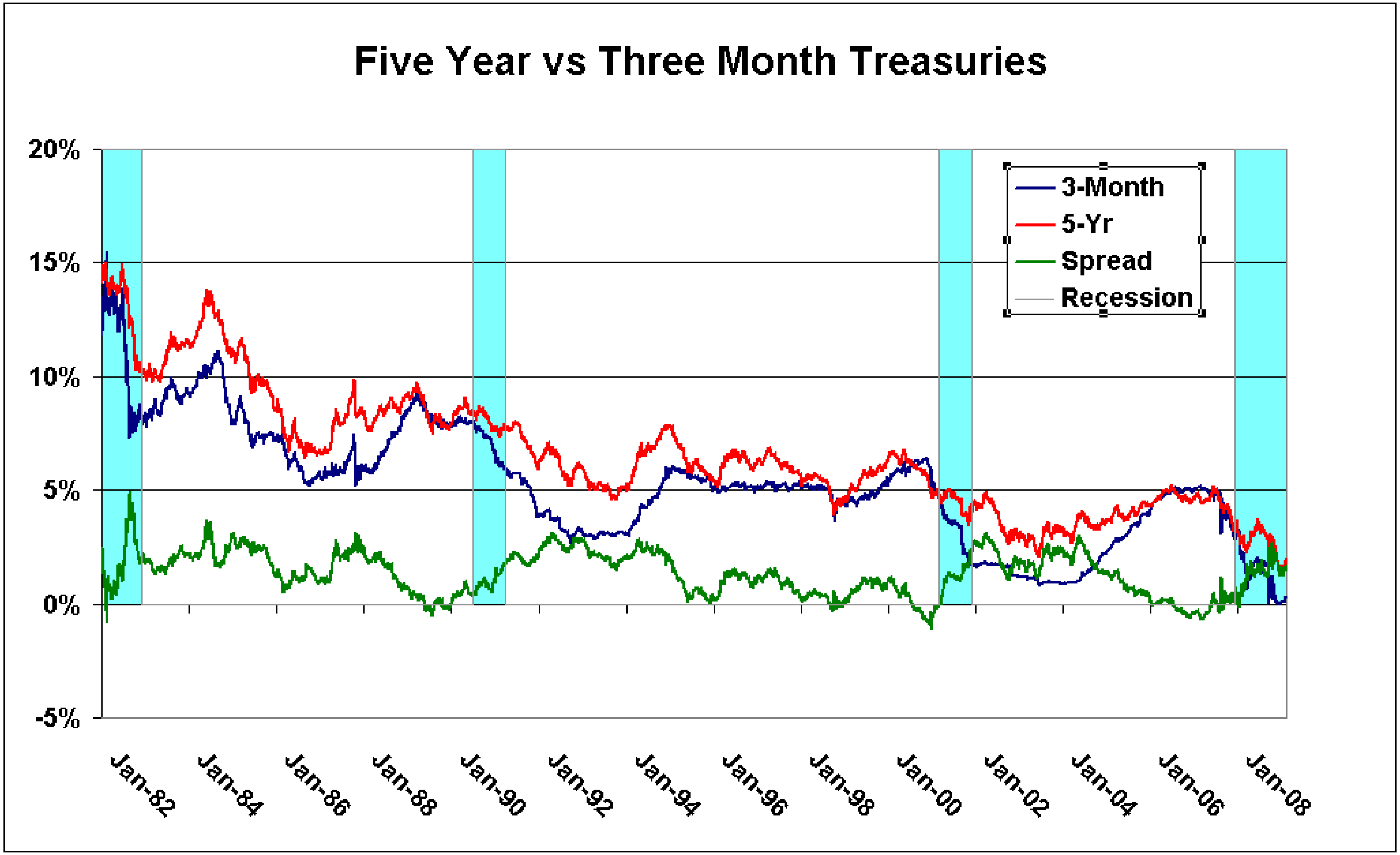
Yield Curve and Economic Growth

- Link to Growth in the Economy (Harvey, 1986)
  - Expectations of future growth affects prices. Fears about slowdown may cause firms and individuals to hedge by shifting from short-term investments to long-term Treasuries.
  - This causes yield curve to invert when growth is expected to decline.

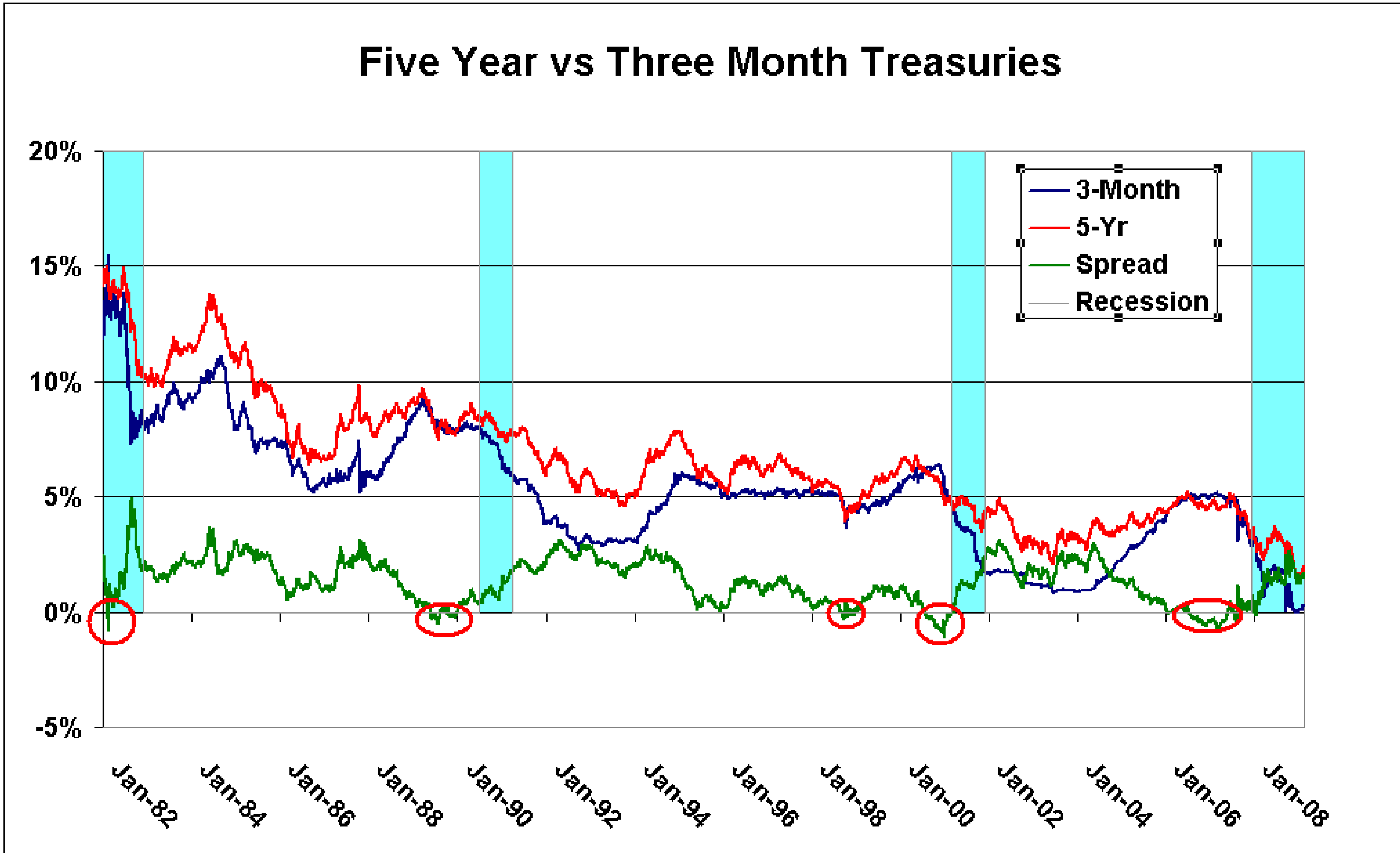
Yield Curve and Economic Growth

- Evidence:
  - Inversions predict recessions
    - 3Q 1968 – recession four quarters later.
    - 2Q 1973 – recession two quarters later.
    - 4Q 1978 – recession five quarters later.
    - 4Q 1980 – recession four quarters later.
    - 2Q 1989 – recession follows in five quarters.
    - 3Q 2000 – recession follows.
    - 3Q 2006 – recession follows.
  - Degree of inversion predicts length and severity of recessions
    - Flat curve 1995 – slowdown, but no recession.

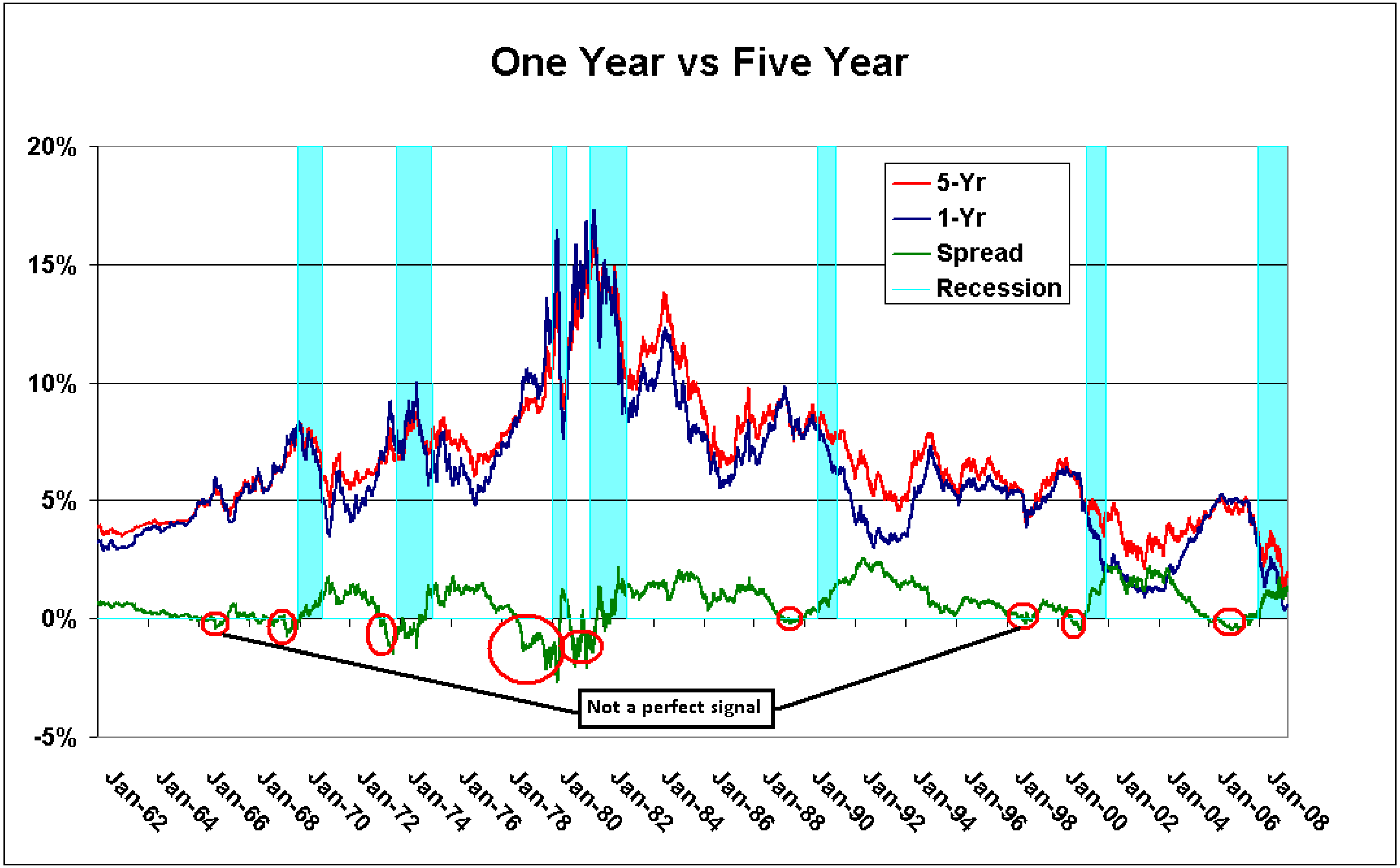
Yield Curve and Economic Growth



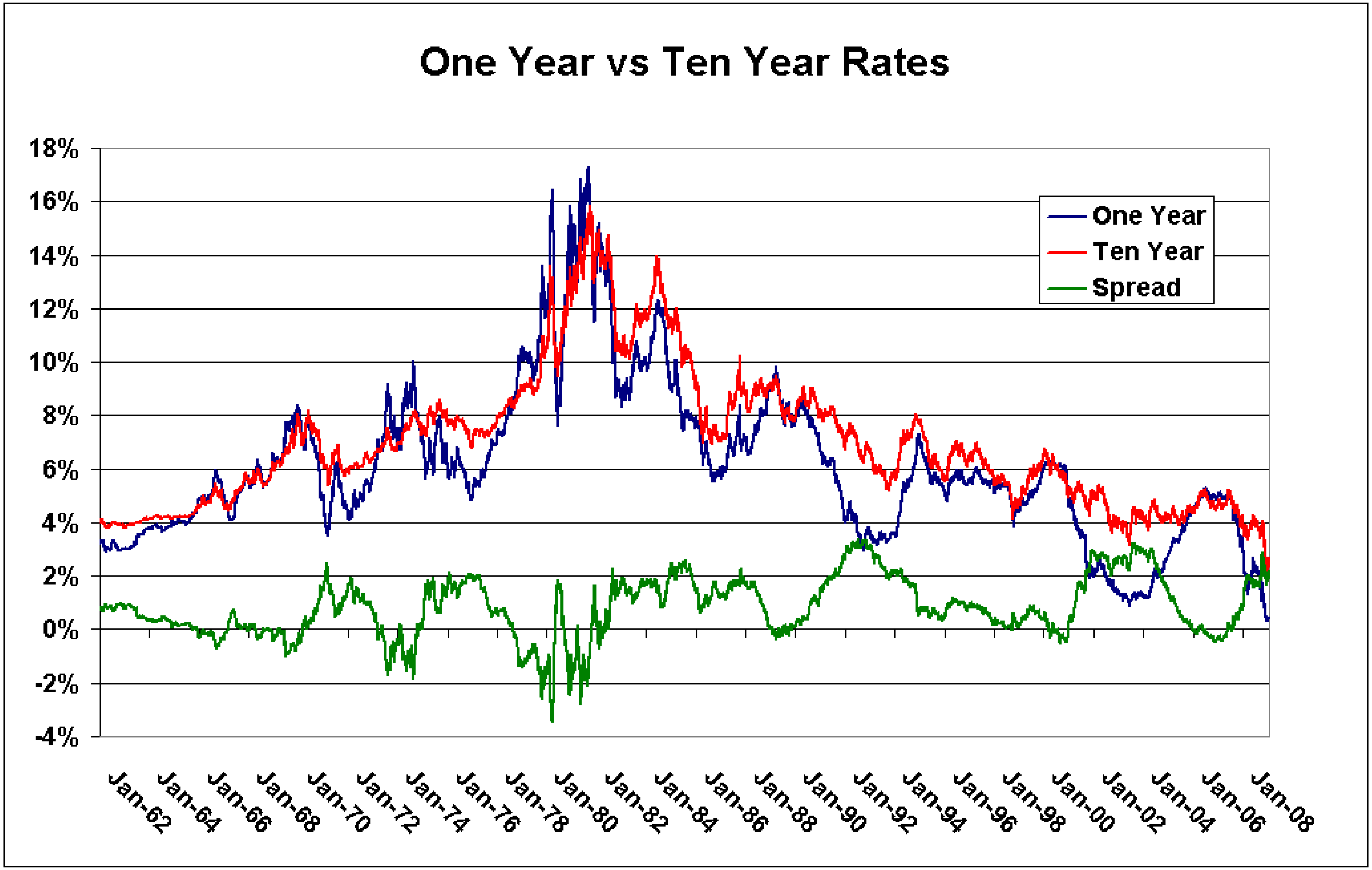
Yield Curve and Economic Growth



Yield Curve and Economic Growth



Yield Curve and Economic Growth



Problem Set

Forward Rate Problems

- Assume that all rates are stated annually and compound twice per year.
- Use following relationship where ***y*** is a spot rate and ***f*** is a forward rate:

$$\left(1+\frac{y_{long}}{2}\right)^{n+m} = \left(1+\frac{y_{short}}{2}\right)^{n}\left(1+\frac{f}{2}\right)^{m}$$

- Where *n* and *m* are number of periods

Problem 1

- Assume that two year spot rate (y_2) is 3%.
- Assume that five year spot (y_5) rate is 3.5%.
- Find three year forward rate (f) starting at year two.

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$$\left(1+\frac{y_5}{2}\right)^{10}=\left(1+\frac{y_2}{2}\right)^4\left(1+\frac{f}{2}\right)^6$$

$$\left(1+\frac{0.035}{2}\right)^{10}=\left(1+\frac{0.03}{2}\right)^4\left(1+\frac{f}{2}\right)^6$$

$$(1.0175)^{10}=(1.015)^4\left(1+\frac{f}{2}\right)^6$$

$$1.18944=1.06136\left(1+\frac{f}{2}\right)^6$$

$$f=3.834\%$$

Problem 2

- Assume that five year spot (y_5) rate is 3.5%.
- Assume that one year forward rate (f) starting at year five is 4%.
- Find six year spot (y_6) rate.

Problem 2

- Assume that five year spot (y_5) rate is 3.5%
- Assume that one year forward rate (f) starting at year five is 4%.
- Find six year spot (y_6) rate

$$\left(1+\frac{y_6}{2}\right)^{12} = \left(1+\frac{y_5}{2}\right)^{10}\left(1+\frac{f}{2}\right)^{2}$$

$$\left(1+\frac{y_6}{2}\right)^{12} = \left(1+\frac{0.035}{2}\right)^{10}\left(1+\frac{0.04}{2}\right)^{2}$$

$$\left(1+\frac{y_6}{2}\right)^{12} = (1.0175)^{10}(1.02)^{2}$$

$$\left(1+\frac{y_6}{2}\right)^{12} = 1.18944 \cdot 1.04040 = 1.23750$$

$$y_6 = 3.583\%$$

Problem 3

- Assume that two year spot rate (y_2) is 3%
- Assume that one year forward rate (f) starting at year one is 3.3%.
- Find one year spot rate (y_1)

Problem 3

- Assume that two year spot rate (y_2) is 3%
- Assume that one year forward rate (f) starting at year one is 3.3%.
- Find one year spot rate (y_1)

$$\left(1+\frac{y_2}{2}\right)^4=\left(1+\frac{y_1}{2}\right)^2\left(1+\frac{f}{2}\right)^2$$

$$\left(1+\frac{0.03}{2}\right)^4=\left(1+\frac{y_1}{2}\right)^2\left(1+\frac{0.033}{2}\right)^2$$

$$(1.015)^4=\left(1+\frac{y_1}{2}\right)^2(1.0165)^2$$

$$1.06136=\left(1+\frac{y_1}{2}\right)^2\cdot 1.03327$$

$$y_1=2.700\%$$

Problem 4

- Assume that one year spot rate (y_1) is 2.5%
- Assume that one year forward rate ($f_{1,1}$) starting at year one is 2.8%.
- Assume that one year forward rate ($f_{2,1}$) starting at year two is 3.0%.
- Find present value of $1000 received three years.

Problem 4

- Assume that one year spot rate (y_1) is 2.5%
- Assume that one year forward rate ($f_{1,1}$) starting at year one is 2.8%.
- Assume that one year forward rate ($f_{2,1}$) starting at year two is 3.0%.
- Find present value of $1000 received in three years.
- Value(2) = Value(3) discounted by $f_{2,1}$
- Value(1) = Value(2) discounted by $f_{1,1}$
- Value(0) = Value(1) discounted by y_1

Problem 4

- Assume that one year spot rate (y_1) is 2.5%
- Assume that one year forward rate ($f_{1,1}$) starting at year one is 2.8%.
- Assume that one year forward rate ($f_{2,1}$) starting at year two is 3.0%.
- Find present value of $1000 received in three years.

$$PV = \frac{C}{\left(1+\frac{y_1}{2}\right)^2 \left(1+\frac{f_{1,1}}{2}\right)^2 \left(1+\frac{f_{2,1}}{2}\right)^2}$$

$$PV = \frac{1000}{\left(1+\frac{0.025}{2}\right)^2 \left(1+\frac{0.028}{2}\right)^2 \left(1+\frac{0.03}{2}\right)^2}$$

$$PV = \frac{1000}{1.0125^2 \cdot 1.014^2 \cdot 1.015^2} = \frac{1000}{1.02515 \cdot 1.02820 \cdot 1.03023}$$

$$PV = \frac{1000}{1.08592} = \$920.88$$

Review: Brownian Motion

Introduction

- Textbook uses notation that involves continuous stochastic processes – involving Brownian Motion.
- An in-depth discussion of Brownian Motion is more appropriate for a PhD level Finance class.
- But it is useful for you to be exposed to idea and notation.

Random Walk

- A random walk is one where each new value is equal to old value plus a random piece.
  - Consider a random walk where a new step is taken after even time steps of length *t*. At each step, you add random piece ***x***

$$S_{k+1} = S_k + x$$

$$x \text{ is either } \sqrt{\text{t}} \text{ or } -\sqrt{\text{t}} \text{ chosen randomly with equal probability.}$$

  - We could rewrite random piece to indicate that it is change in S,

$$\Delta S = x = \pm\sqrt{t}$$

Random Walk

- Imagine what would happen if we broke up a time period into a large number of equal steps.
  - At each step, S would take on a value equal to its previous value plus or minus $\sqrt{t}$, so it has Binomial Distribution.
- What would happen if we took limit as time period between steps goes to zero?
  - Distribution converges to Normal Distribution. For a time period of length *h* we get:

$$S_{t_0+h} = S_{t_0} + x$$
$$x \sim N(0,h)$$

Brownian Motion

- Brownian motion is a random walk occurring in continuous time.
- A Standard Brownian Motion has following properties:
  - $Z(0) = 0$.
  - $dZ = (Z(t+h)-Z(t)) \sim N(0,h)$.
  - $(Z(t+h_1)-Z(t))$ is independent of $(Z(t+h_2)-Z(t+h_1))$ where $h_2>h_1>0$.
  - $Z(t)$ is continuous in t.

Brownian Motion

- We can formalize this process.
- Brownian Motion arises from Binomial Distribution:
  - Time between steps is ‘h’.
  - dZ = Z(t+h)-Z(t) = Binary(t+h)*SQRT(h).
    - Binary is chosen randomly to be 1 or -1 (with equal probability).
  - Take limit as h→0.
  - As number of steps gets large, binomial distribution converges to Normal Distribution.

Brownian Motion

- As h→0, we refer to dZ as dZ(t).
- We will use dZ(t) as standard building block for arithmetic and geometric Brownian motion
  - These are also referred to as Wiener Processes.
- We use dZ(t) to indicate Standard Brownian Motion. This is referred to as "Differential Form."
- There is an Integral Form – but it is not often used. Following relation holds:

$$Z(T) = Z(0) + \int_0^T dZ(t)$$

- This integral is called a stochastic integral.

Arithmetic Brownian Motion

- We can generalize this to allow for a non-zero mean and an arbitrary variance

$$\text{If } \Delta X = X(t+s) - X(t) \sim N\left(\mu s, s\sigma^2\right)$$

$$\text{Then we can write}: dX(t) = \mu \cdot dt + \sigma \cdot dZ(t)$$

  - We will refer to *mu* as drift and *sigma* as noise term.

Brownian Motion

- Drift and Noise portions can be functions of value and time.

$$dS = \mu(S,t)dt + \sigma(S,t)dZ$$

- A function of a Brownian Motion is sometimes referred to as an Itô process.

Ornstein-Uhlenbeck Process

- We may want drift term to change, based on asset value.
- For example, if interest rates are high, they are more likely to go down than up (and vice versa)
- We can incorporate this into process

$$dX(t) = \lambda(\mu - X(t))dt + \sigma dZ(t)$$

  - Here, X will drift back towards long-term mean *mu*.
  - Value *lambda* tells us how fast it reverts to mean.

Functions of Brownian Motion

- Do functions of Brownian motions look like Brownian motions?
  - Yes
  - There is a formula in Mathematics that follows from Itô’s Lemma that tells us how to create new process
    - You do not need to know Itô’s Lemma, or know how to use it.
    - It can be used to get a new Brownian motion as a function of an old one
  - How does this help us in FI world?
    - If we know that interest rates follow Brownian motions, then we can use Itô’s Lemma to show that Present values and future values of cash flows also follow a Brownian motion.

Geometric Brownian Motion

- If logarithm of a process follows arithmetic Brownian motion, then we say that process follows Geometric Brownian motion
- In this case, changes are multiplicative:

$$dS = \alpha S dt + \sigma S dZ(t)$$

$$\frac{dS}{S} = \alpha dt + \sigma dZ(t)$$

- Geometric Brownian Motion with a constant drift and volatility leads to distributions of returns that have lognormal distribution.

Approximating Change of Brownian Motion

- We often model a process using Brownian Motion and continuous changes.
- But computer models usually want discrete changes.
- We approximate Brownian motion using following:

$$dS = \mu(S,t)dt + \sigma(S,t)dZ$$

$$\Delta S = \mu(S,t)\Delta t + \sigma(S,t)\sqrt{\Delta t} \cdot z$$

$$z \sim N(0,1)$$