Econ 240A, Fall 2018
Problem Set 4

Due date: Wednesday, Oct. 3

Review of hypothesis testing, Neyman-Pearson lemma, UMP test, p-value, interval estimation.
Note: Problems start with a star, *, are optional and don't count for grade. Of course, feel free to write them up if you want.

1. **Normal testing: one-sided**
   Suppose $X_i \overset{\text{iid}}{\sim} \mathrm{N}(\theta, \sigma^2),\ i = 1, 2, \ldots, n$, where $\theta \geq 0$ is unknown and σ^2 is known. Consider the problem of testing $H_0 : \theta = 0$ vs. $H_1 : \theta > 0$. Let $\alpha \in (0, 1)$.

   (a) Find the level-α UMP test. Write down the rejection region.

   (b) Express the power function of this test, $\beta_n(\theta)$, using standard normal cdf Φ. How does $\beta_n(\theta)$ depend on n and σ^2?

   (c) Find the limit $\gamma(\theta) = \lim_{n\to\infty} \beta_n(\theta)$ and plot it. Is it desirable?

   (d) Suppose the sample size n can be picked by the researcher. Find the smallest n, called n^*, such that the level-0.05 UMP test has $\beta_n(1) \geq 0.95$. How does n^* depend on σ^2?

   (e) Find a 1-α confidence interval of θ by inverting a family of level-α test of $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$ with test functions $\varphi_{\theta_0}(X) = \mathbf{1}\left(\frac{(\bar{X} - \theta_0)}{\sigma/\sqrt{n}} > \Phi^{-1}(1 - \alpha)\right)$.

2. *** Normal testing: two-sided**
   Suppose $X_i \overset{\text{iid}}{\sim} \mathrm{N}(\theta, \sigma^2),\ i = 1, 2, \ldots, n$, where $\theta \in \mathbb{R}$ is unknown and σ^2 is known. Consider the problem of testing $H_0 : \theta = 0$ vs. $H_1 : \theta \neq 0$. Let $\alpha \in (0, 1)$.

   (a) Find the level-α UMPU (unbiased) test (example 8.3.20 in Casella and Berger). Write down the rejection region.

   (b) Express the power function of this test, $\beta_n(\theta)$, using standard normal cdf Φ. How does $\beta_n(\theta)$ depend on n and σ^2?

   (c) Find the limit $\gamma(\theta) = \lim_{n\to\infty} \beta_n(\theta)$ and plot it. Is it desirable?

   (d) Suppose the sample size n can be picked by the researcher. Find the smallest n, called n^*, such that the level-0.05 UMPU test has $\beta_n(1) \geq 0.95$. How does n^* depend on σ^2?

   (e) Find a 1-α confidence interval of θ by inverting a family of level-α UMPU test for $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0,\ \theta_0 \in \mathbb{R}$.

3. **Normal Testing: nuisance parameter**
   Suppose $X_i \overset{\text{iid}}{\sim} \mathrm{N}(\theta, \sigma^2),\ i = 1, 2, \ldots, n$, where $\theta \in \mathbb{R}$ and $\sigma^2 \in (0, \infty)$ are both unknown. Consider the problem of testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$, where θ_0 is a constant.

   (a) Show that the test that rejects H_0 when $\frac{|\bar{X} - \theta_0|}{\sqrt{S^2}/\sqrt{n}} > t_{n-1,\frac{\alpha}{2}}$ is a level-α test. Here $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i,\ S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$, and $t_{n-1,\frac{\alpha}{2}}$ is the $1 - \frac{\alpha}{2}$ quantile of t distribution with degree of freedom $n - 1$.

(b) * Show that this test can be derived as a (generalized) likelihood ratio test.

4. Likelihood ratio test

(a) Consider testing for the model with exponential distributions $\mathcal{P} = \{p_\theta(x) = \theta e^{-\theta x}\mathbf{1}(x > 0), \theta \in [1, \infty)\}$. Let $\alpha \in (0, 1)$. Derive the level-α likelihood ratio test for $H_0 : \theta = 1$ versus $H_1 : \theta > 1$.

(b) * Consider testing for Cauchy location model $\mathcal{P} = \left\{p_\theta(x) = \frac{1}{\pi\left(1+(x-\theta)^2\right)}, \theta \in \{0,\ 1\}\right\}$. Let $\alpha = 0.05$. Numerically compute the rejection region of the likelihood ratio test for $H_0 : \theta = 0$ versus $H_1 : \theta = 1$. Does the test reject when X is extremely large?

5. * p-value

Suppose $\mathcal{P}$ is a family with monotone likelihood ratio in $T(X)$, and the distribution of $T(X)$ is continuous with common support for all θ. Let φ_α denote the UMP level-α test of $H_0 : \theta \leq \theta_0$ vs. $\theta > \theta_0$ that rejects when $T(X)$ is large. Let $p(X)$ denote the resulting p-value, defined as $p(X) = \inf\{\alpha : \varphi_\alpha(X) = 1\}$.

(a) Show that these test φ_α are "nested" in the sense that φ_α is nondecreasing as a function of α. The p-value defined as $p(X) = \inf\{\alpha : \varphi_\alpha(X) = 1\}$ is therefore the "attained significance" for the observed data X.

(b) Show that if $X = x$ is observed, the p-value $p(x) = P_{\theta_0}(T(X) > T(x))$.

(c) Show that $p(X) \sim \mathrm{U}[0, 1]$ if $\theta = \theta_0$, has non-increasing density on $[0, 1]$ if $\theta > \theta_0$, and has non-decreasing density on $[0, 1]$ if $\theta < \theta_0$.