Midterm Exam

Suggested Answers

Instructions: This is a closed book exam, but you may refer to one sheet of notes. You have 80 minutes for the exam. Answer as many questions as possible. Partial answers get partial credit. Please write legibly. *Good luck!*

Problem 1 (5 points). Determine whether or not the statement below is correct and give a *brief* (e.g., a bluebook page or less) justification for your answer.

Suppose $X_1, \ldots, X_n$ is a random sample from a continuous distribution with pdf $f(\cdot|\theta)$, where $\theta \in \Theta \subseteq \mathbb{R}^k$ is unknown. If the $\mathbb{R}^m$-valued statistic $T = T(X_1, \ldots, X_n)$ is sufficient, then $m \geq k$.

The statement is incorrect. For instance, the $\mathbb{R}^n$-valued statistic $(X_1, \ldots, X_n)'$ is sufficient for any value of n.

Problem 2 (45 points, each part receives equal weight). Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with mean $\sqrt{\theta}$ and pdf

$$f_X(x|\theta) = c(\theta) \exp(-\frac{1}{2}x^2) \exp[w(\theta)x],$$

where $\theta \in \Theta = [0, \infty)$ is an unknown parameter while $c(\cdot)$ and $w(\cdot)$ are some functions.

(a) Show that

$$c(\theta) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}\theta), \qquad w(\theta) = \sqrt{\theta}.$$

For any $w(\theta)$,

$$\exp(-\frac{1}{2}x^2) \exp[w(\theta)x] = \exp(-\frac{1}{2}[x - w(\theta)]^2) \exp[\frac{1}{2}w(\theta)^2]$$

and, by the properties of the normal distribution,

$$\int_{-\infty}^{\infty} \exp(-\frac{1}{2}[x - w(\theta)]^2)dx = \sqrt{2\pi}.$$

As a consequence, $f_X(\cdot|\theta)$ is a pdf if and only if

$$c(\theta) = \frac{1}{\sqrt{2\pi}} \exp[-\frac{1}{2}w(\theta)^2].$$

For this choice of $c(\theta)$ (and for any $w(\theta)$), $f_X(\cdot|\theta)$ is a pdf of the $\mathcal{N}[w(\theta), 1]$ distribution, so $E_\theta(X_i) = \sqrt{\theta}$ implies that $w(\theta) = \sqrt{\theta}$.

(b) As an estimator of θ, consider $\bar{X}^2$, where $\bar{X} = n^{-1}\sum_{i=1}^n X_i$. Give conditions (on $X_1, \ldots, X_n$) under which a method moments estimator of θ exists and equals $\bar{X}^2$. Is $\bar{X}^2$ an unbiased estimator of θ?

Because $X_i \sim \mathcal{N}(\sqrt{\theta}, 1)$, $E_\theta(X_i) = \sqrt{\theta}$ *and* $Var_\theta(X_i) = 1$. *As a consequence, the estimating equation* $\sqrt{\hat{\theta}_{MM}} = \bar{X}$ *admits a solution (namely,* $\hat{\theta}_{MM} = \bar{X}^2$*) if and only if* $\bar{X} \geq 0$.

The estimator $\bar{X}^2$ *is biased because*

$$E_\theta(\bar{X}^2) = E_\theta(\bar{X})^2 + Var_\theta(\bar{X}) = E_\theta(X_i)^2 + Var_\theta(X_i)/n = \theta + 1/n \neq \theta.$$

(c) Find the log likelihood function. Does θ admit a scalar sufficient statistic?

Because

$$\log f_X(X_i|\theta) = \log\{\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}\theta)\exp(-\frac{1}{2}X_i^2)\exp[\sqrt{\theta}X_i]\} = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\theta - \frac{1}{2}X_i^2 + \sqrt{\theta}X_i,$$

the log likelihood function is given by

$$\ell(\theta|X_1, \ldots, X_n) = \sum_{i=1}^n \log f_X(X_i|\theta) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\theta - \frac{1}{2}\sum_{i=1}^n X_i^2 + \sqrt{\theta}\sum_{i=1}^n X_i.$$

It follows from the factorization criterion that the scalar $\sum_{i=1}^n X_i$ *is a sufficient statistic for* θ.

(d) Find the maximum likelihood estimator $\hat{\theta}_{ML}$ of θ.

If $\bar{X} \leq 0$, *then* $\ell(\cdot|X_1, \ldots, X_n)$ *is decreasing on* Θ *and therefore* $\hat{\theta}_{ML} = \min\{\theta : \theta \in \Theta\} = 0$. *On the other hand, if* $\bar{X} > 0$, *then* $\hat{\theta}_{ML} = \bar{X}^2$ *because*

$$\frac{\partial}{\partial\theta}\ell(\theta|X_1, \ldots, X_n) = -\frac{n}{2} + \frac{1}{2\sqrt{\theta}}\sum_{i=1}^n X_i = 0 \qquad \Leftrightarrow \qquad \theta = \bar{X}^2$$

and

$$\frac{\partial^2}{\partial\theta^2}\ell(\theta|X_1, \ldots, X_n) = -\frac{1}{4\theta^{3/2}}\sum_{i=1}^n X_i < 0.$$

As a consequence, $\hat{\theta}_{ML} = \max(\bar{X}, 0)^2$.

(e) Find a uniform minimum variance unbiased estimator of θ.

The sufficient statistic $\sum_{i=1}^n X_i$ *is complete because*

$$\{w(\theta) : \theta \in \Theta\} = \{\sqrt{\theta} : \theta \geq 0\} = [0, \infty)$$

contains an open set. It was shown in (b) that $E_\theta(\bar{X}^2) = \theta + 1/n$. *Therefore,*

$$\hat{\theta}_{UMVU} = \bar{X}^2 - 1/n$$

is an unbiased estimator based on the complete, sufficient statistic $\sum_{i=1}^n X_i$. *In particular,* $\hat{\theta}_{UMVU}$ *is a uniform minimum variance unbiased estimator of* θ.

(f) Show that the Cramér-Rao bound (on the variance of unbiased estimators of θ) is given by $4\theta/n$.

The Cramér-Rao bound is

$$\frac{1}{nE_\theta([\partial \log f_X(X_i|\theta)/\partial\theta]^2)}.$$

Because

$$\frac{\partial}{\partial\theta} \log f_X(X_i|\theta) = \frac{\partial}{\partial\theta}[-\frac{1}{2}\log(2\pi) - \frac{1}{2}\theta - \frac{1}{2}X_i^2 + \sqrt{\theta}X_i] = -\frac{1}{2} + \frac{1}{2\sqrt{\theta}}X_i$$

and $X_i \sim \mathcal{N}(\sqrt{\theta}, 1)$, *we have*

$$\frac{\partial}{\partial\theta} \log f_X(X_i|\theta) \sim \mathcal{N}(0, \frac{1}{4\theta}),$$

implying in particular that $E_\theta([\partial \log f_X(X_i|\theta)/\partial\theta]^2) = 1/(4\theta)$, *as was to be shown.*

(g) Is the bound from (f) attained by the estimator from (e)?
[Hint: If $Y \sim \mathcal{N}(\mu, \sigma^2)$, *then* $Var(Y^2) = 4\mu^2\sigma^2 + 2\sigma^4$.*]*

Because $\bar{X} \sim \mathcal{N}(\sqrt{\theta}, 1/n)$, *it follows from the hint that*

$$Var_\theta(\hat{\theta}_{UMVU}) = Var_\theta(\bar{X}^2) = \frac{4\theta}{n} + \frac{2}{n^2}.$$

In particular, $\hat{\theta}_{UMVU}$ *does not attain the bound from (e).*

Let $\theta_0 > 0$ be some constant and consider the one-sided testing problem

$$H_0 : \theta = \theta_0 \qquad \text{vs.} \qquad H_1 : \theta > \theta_0.$$

(h) Consider a test which rejects H_0 if (and only if) $\bar{X} > c$, where c is some constant (possibly depending on θ_0). Find c such that the test has 5% size.

[Hint: If $Z \sim \mathcal{N}(0,1)$, then $P(Z \leq 1.645) \approx 0.95$.]

The size of the test $P_{\theta_0}(\bar{X} > c)$. Because $\bar{X} \sim \mathcal{N}(\sqrt{\theta}, 1/n)$,

$$P_{\theta_0}(\bar{X} > c) = 1 - P_{\theta_0}(\bar{X} \leq c) = 1 - P_{\theta_0}[\sqrt{n}(\bar{X} - \sqrt{\theta_0}) \leq \sqrt{n}(c - \sqrt{\theta_0})] = 1 - \Phi[\sqrt{n}(c - \sqrt{\theta_0})],$$

where $\Phi(\cdot)$ is the standard normal cdf. As a consequence, the test has size 5% if and only if

$$\sqrt{n}(c - \sqrt{\theta_0}) = 1.645 \qquad \Leftrightarrow \qquad c = \sqrt{\theta_0} + 1.645/\sqrt{n}.$$

(i) Show that the test derived in (h) is uniformly most powerful (within the class of tests of the same level).

For any $\theta_1 > \theta_0$, the log likelihood ratio

$$\ell(\theta_1 | X_1, \ldots, X_n) - \ell(\theta_0 | X_1, \ldots, X_n) = -\frac{n}{2}(\theta_1 - \theta_0) + n(\sqrt{\theta_1} - \sqrt{\theta_0})\bar{X}$$

is an increasing function of $\bar{X}$, so it follows from the Neyman-Pearson lemma that the test derived in (h) is the (uniformly) most powerful 5% test of $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$. As a consequence, the test is (also) the uniformly most powerful 5% test of $H_0 : \theta = \theta_0$ vs. $H_1 : \theta > \theta_0$.