Econ 139 – Midterm Suggested Solutions
Spring 2017

Problem 1. True or false. (18 points, 6 points each)
Are the following statements true or false? Explain your answer in no more than two sentences. You will be graded on your explanation.

1. **False**. The optimal share of the tangency portfolio in the portfolio of a mean-variance optimizer is $w_T = (\mathbb{E}[R_T] - R_f)/A\sigma_T^2$ which is positive for any $A > 0$. Hence even a highly risk-averse mean-variance optimizer will choose to hold some stocks in their portfolio.

2. **True**. All Walrasian equilibria satisfy the individual rationality condition and are Pareto efficient. Therefore, all Walrasian equilibria are on the contract curve.

3. **False**. With the specified utility function we have

$$\max_{s \in (0, M_0)} \quad M_0 - s + \gamma s \mathbb{E}[\tilde{R}] - \frac{1}{2} b s^2 \sigma_{\tilde{R}}^2.$$

The first order condition is

$$-1 + \gamma \mathbb{E}[\tilde{R}] - b s^* \sigma_{\tilde{R}}^2 = 0.$$

Solving for s^* gives

$$s^* = \frac{\gamma \mathbb{E}[\tilde{R}] - 1}{b \sigma_{\tilde{R}}^2}.$$

Since $\tilde{R}_A$ second order stochastically dominates $\tilde{R}_B$, with $\mathbb{E}[\tilde{R}_A] = \mathbb{E}[\tilde{R}_B]$, $\tilde{R}_B$ is a mean preserving spread of $\tilde{R}_A$ and we have $\sigma_{\tilde{R}_B}^2 > \sigma_{\tilde{R}_A}^2$. This implies $s_B^* < s_A^*$.

Problem 2. Portfolio choice with expected utility. (42 points, 6 points each)

1.
$$\tilde{W}_1 = (W_0 - a)R_f + a\tilde{R},$$
which can be re-written as
$$\tilde{W}_1 = W_0 R_f + a(\tilde{R} - R_f).$$

2. The probability distribution for $\tilde{W}_1$ is
$$\begin{aligned} \tilde{W}_1 &= 2 - a \text{ w.p. } p_1 \\ \tilde{W}_1 &= 2 \text{ w.p. } p_2 \\ \tilde{W}_1 &= 2 + a \text{ w.p. } p_3 \end{aligned}$$

3. The agent's optimization problem is
$$\max_{a \geq 0} \ \mathbb{E}[U(\tilde{W}_1)] = \mathbb{E}[U(W_0 R_f + a(\tilde{R} - R_f))]$$
This can be re-written as
$$\max_{a \geq 0} \ p_1 U(2 - a) + p_2 U(2) + p_3 U(2 + a).$$

4. The first order condition is
$$-p_1 U'(2 - a^*) + p_3 U'(2 + a^*) = 0.$$

5. The FOC can be written as
$$\frac{p_3}{p_1} = \frac{U'(2 - a^*)}{U'(2 + a^*)}$$
Note that if $a^* > 0$ then the right hand side is greater than 1, which implies that we must have $p_3 > p_1$. If $p_3 > p_1$, then $\mathbb{E}[\tilde{R}] > 2$.

6. (i) $U(W) = 1 - e^{-bW}$, with $b > 0$:
$$-p_1 b e^{-b(2-a^*)} + p_3 b e^{-b(2+a^*)} = 0,$$
or
$$p_3 e^{-b(2+a^*)} = p_1 e^{-b(2-a^*)}.$$

Taking logs gives

$$\ln(p_3) - b(2 + a^*) = \ln(p_1) - b(2 - a^*),$$

or

$$a^* = \frac{\ln(p_3) - \ln(p_1)}{2b}.$$

Since a^* is independent of W_0, we have $da^*/dW_0 = 0$.

(ii) $U(W) = \frac{W^{1-\gamma}}{1-\gamma}$, with $0 < \gamma < 1$:

$$-p_1(2 - a^*)^{-\gamma} + p_3(2 + a^*)^{-\gamma} = 0,$$

or

$$p_3(2 + a^*)^{-\gamma} = p_1(2 - a^*)^{-\gamma}.$$

Taking both sides to the power $-1/\gamma$ gives

$$p_3^{-1/\gamma}(2 + a^*) = p_1^{-1/\gamma}(2 - a^*),$$

or

$$a^* = 2\left(\frac{p_1^{-1/\gamma} - p_3^{-1/\gamma}}{p_1^{-1/\gamma} + p_3^{-1/\gamma}}\right) = W_0 R_f \left(\frac{p_1^{-1/\gamma} - p_3^{-1/\gamma}}{p_1^{-1/\gamma} + p_3^{-1/\gamma}}\right).$$

Since $p_3 > p_1$ and $0 < \gamma < 1$, we have $da^*/dW_0 > 0$.

7. (i) $U(W) = 1 - e^{-bW}$, with $b > 0$:

$$r_A(W) = -\frac{U''(W)}{U'(W)} = -\frac{-b^2 e^{-bW}}{be^{-bW}} = b.$$

(i) $U(W) = \frac{W^{1-\gamma}}{1-\gamma}$, with $0 < \gamma < 1$:

$$r_A(W) = -\frac{U''(W)}{U'(W)} = -\frac{-\gamma W^{-\gamma-1}}{W^{-\gamma}} = \frac{\gamma}{W}.$$

Problem 3. Welfare effects of risk. (20 points, 5 points each)

1. (See figure 1) By CAPM the market portfolio is the tangency portfolio. The optimal share of the market portfolio is $w_m = (\mathbb{E}[R_m] - R_f)/(A\sigma_m^2) = 0.08/(2 \cdot 0.16) = 0.25 =$ 25%, while 75% is invested in the risk-free asset. The expected return is $0.75 \cdot 1.02 + 0.25 \cdot 1.10 = 1.04$ and the standard deviation is $w_m\sigma_m = 0.25 \cdot 0.40 = 0.1$. Expected utility is $1.04 - (2/2) \cdot 0.1^2 = 1.03$.

2. (See figure 1) The optimal portfolio share of the market is now $w_m = (\mathbb{E}[R_m] - R_f)/(A\sigma_m^2) = 0.08/(4 \cdot 0.16) = 0.125 = 12.5\%$, while 87.5% is invested in the risk-free asset. The expected return is $0.875 \cdot 1.02 + 0.125 \cdot 1.10 = 1.03$ and the standard deviation is $w_m\sigma_m = 0.125 \cdot 0.40 = 0.05$. Expected utility is $1.03 - (4/2) \cdot 0.05^2 = 1.03 - 0.005 = 1.025$.

3. (See figure 2) The new optimal portfolio share of the market portfolio for the first investor is $w_m = (\mathbb{E}[R_m] - R_f)/(A\sigma_m^2) = 0.08/(2 \cdot 0.04) = 1 = 100\%$ and zero holdings in the risk-free asset. The expected return is 1.10 and the standard deviation is 0.2 = 20%. For the second investor we have $w_m = (\mathbb{E}[R_m] - R_f)/(A\sigma_m^2) = 0.08/(4 \cdot 0.04) = 0.5 = 50\%$ and another 50% in the risk-free asset. The expected return is $0.50 \cdot 1.02 + 0.50 \cdot 1.10 = 1.06$ and the standard deviation is $w_m\sigma_m = 0.50 \cdot 0.20 = 0.10$. The first investor's portfolio share changes by more, because he is less risk averse and hence more willing to take advantage of the reduced risk.

4. The expected utility of the first investor is $1.10 - (2/2) \cdot 0.20^2 = 1.06$, and hence the utility gain is 0.03. The expected utility of the second investor is $1.06 - (4/2) \cdot 0.1^2 = 1.04$ and the utility gain is 0.015. Thus the first investor gains more then the second. This is in part because of the larger portfolio response. The statement ignores investors' response to changing conditions and hence is incorrect.

Problem 4. Certainty equivalent. (20 points, 5 points each)

1. The certainty equivalent must satissfy:

$$U(W + CE) = \mathbb{E}[U(W + \tilde{X})].$$

   Using logarithmic utility and plugging in the given numbers gives:

$$\ln(100000 + CE) = 0.01 \ln(1) + 0.04 \ln(50000) + 0.95 \ln(100000).$$

   Hence, the CE of the gamble (i.e., going without insurance) is $-13,312$. With insurance we have

$$\ln(100000 + CE) = 0.01 \ln(100000) + 0.04 \ln(100000) + 0.95 \ln(100000) = \ln(100000),$$

   giving $CE = 0$. Thus, the maximum the CEO would pay for the policy is $-CE$, or 13,312.

2. With these probabilities we have

$$\ln(100000 + CE) = 0.01 \ln(1) + 0.05 \ln(50000) + 0.94 \ln(100000),$$

   giving $CE = -13,911$. The maximum the CEO would pay for the policy is $-CE$, or 13,911.

3. With these probabilities we have

$$\ln(100000 + CE) = 0.02 \ln(1) + 0.04 \ln(50000) + 0.94 \ln(100000),$$

   giving $CE = -22,739$. The maximum the CEO would pay for the policy is $-CE$, or 22,739.

4. Starting from point 1, in points 2 and 3 we have transferred 1 percent from state 3 to another state (to state 2 in point 2 and to state 1 in point 3). However, the outcome is very different: the maximum willingness to pay is slightly higher in point 2, while it is a lot higher in point 3. This could have been expected because the logarithmic utility function is very curved at low values, and flattens out rapidly, i.e., ln(1) is very different from ln(100000), but ln(50000) is only slightly different from ln(100000). Also, the logarithmic utility function is DARA.

Figure 1

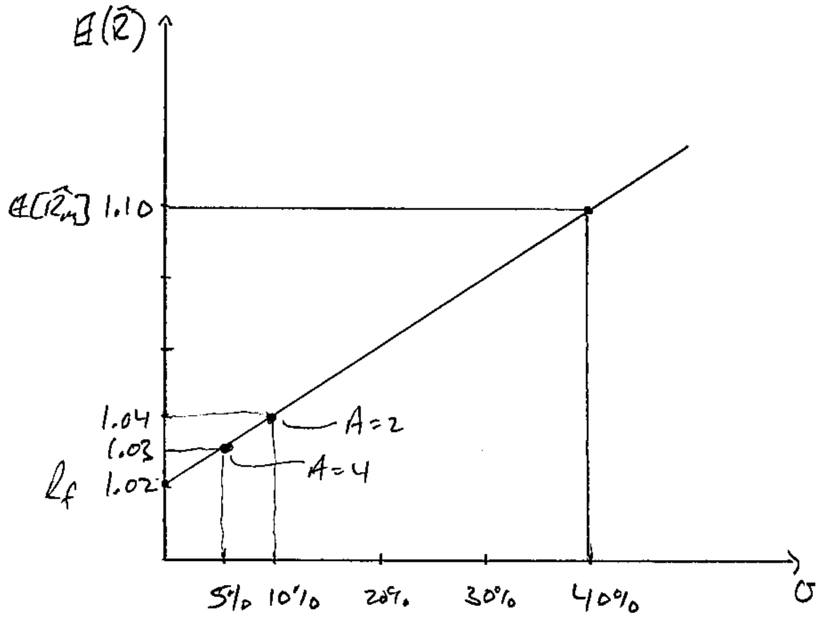

Figure 2