金融经济学

第三讲

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Consumer Optimization: The Risk Dimension

What makes it possible to go back and forth between traded assets, like stocks and bonds, and contingent claims is that there are the same number of traded assets as there are possible states of the world next year.

More generally, asset markets are complete if there are as many assets (with linearly independent payoffs) as there are states next year.

有多少个状态就有多少个资产

Consumer Optimization: The Risk Dimension

If asset markets are complete, then we can use the prices of traded assets to infer the prices of contingent claims.

Then we can use the contingent claims prices to infer the price of any newly-introduced asset.

2 Overview of Asset Pricing Theory

2 Overview of Asset Pricing Theory

A Pricing Safe Cash Flows

B Pricing Risky Cash Flows

C Two Perspectives on Asset Pricing

Pricing Safe Cash Flows

A *T-year discount bond* is an asset that pays off \$1, for sure, T years from now.

If this bond sells for \$ P_T today, the annualized return from buying the bond today and holding it to maturity is

$$1 + r_T = \left(\frac{1}{P_T}\right)^{1/T}.$$

Hence, the bond price and the interest rate are related via

$$P_T = \frac{1}{(1 + r_T)^T}.$$

Pricing Safe Cash Flows

Since, for a T-period discount bond,

$$P_T = \frac{1}{(1 + r_T)^T},$$

the interest rate equates today's price of the bond to the present discounted value of the future payments made by the bond.

US Treasury bills, that is, US government bonds with maturities less than one year, are structured as discount bonds.

Pricing Safe Cash Flows

A T-year coupon bond is an asset that makes an annual interest (coupon) payment of $\$C$ each year, every year, for the next T years, and then pays off $\$F$ (face or par value), for sure, T years from now.

US Treasury notes and bonds, with maturities of more than one year, are structured as coupon bonds.

Pricing Safe Cash Flows

Notice that a coupon bond can be viewed as a bundle, or portfolio of discount bonds, since the cash flows from a T-year coupon bond can be replicated by buying

C one-year discount bonds

C two-year discount bonds

...

C T-year discount bonds

F more T-year discount bonds

Pricing Safe Cash Flows

And if both discount and coupon bonds are traded, then the price of the coupon bond must equal the price of the portfolio of discount bonds.

If the coupon bond was cheaper than the portfolio of discount bonds, one could sell the discount bonds, buy the coupon bond, and thereby profit.

If the coupon bond was more expensive than the portfolio of discount bonds, one could sell the coupon bond, buy the discount bonds, and thereby profit.

Pricing Safe Cash Flows

Building on this insight, the price P_T^C of the coupon bond must satisfy

$$\begin{aligned} P_T^C &= CP_1 + CP_2 + \ldots + CP_T + FP_T \\ &= \frac{C}{1+r_1} + \frac{C}{(1+r_2)^2} + \ldots + \frac{C}{(1+r_T)^T} + \frac{F}{(1+r_T)^T} \end{aligned}$$

Today's price of the coupon bond equals the present discounted value of the future payments made by the bond.

Pricing Safe Cash Flows

$$P_T^C = \frac{C}{1+r_1} + \frac{C}{(1+r_2)^2} + \ldots + \frac{C}{(1+r_T)^T} + \frac{F}{(1+r_T)^T}$$

Note that the interest rates used to compute the present value are those on the discount bonds

The yield to maturity defined by the value r that satisfies

$$P_T^C = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \ldots + \frac{C}{(1+r)^T} + \frac{F}{(1+r)^T}$$

is a measure of the interest rate on the coupon bond.

Pricing Safe Cash Flows

In fact, the US Treasury allows financial institutions to break US Treasury coupon bonds down into portfolios of separately-traded discount bonds.

These securities are called US Treasury STRIPS (Separate Trading of Registered Interest and Principal of Securities).

Pricing Safe Cash Flows

Next, consider an asset that generates an arbitrary stream of safe (riskless) cash flows $C_1, C_2, \ldots, C_T$, over the next T years.

To simplify the task of "pricing" this asset, we might view it as a portfolio of more basic assets: one that pays C_1 for sure in one year, one that pays C_2 for sure in two years, . . ., and one that pays C_T for sure in T years.

The price of the multi-period asset must equal the sum of the prices of the more basic assets.

Pricing Safe Cash Flows

We've now reduced the problem of pricing any riskless asset to the simpler problem of pricing a more basic asset that pays C_t for sure t years from now.

But this more basic asset has the same payoff as C_t t-year discount bonds. Its price P_t^A today must equal

$$P_t^A = C_t P_t = \frac{C_t}{(1 + r_t)^t},$$

the present discounted value of its cash flow.

Pricing Risky Cash Flows

Now consider a risky asset, with cash flows $\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_T$ over the next T years that are random variables with values that are unknown today.

Again, we might simplify the task of pricing this asset, by viewing it as a portfolio of more basic assets, each of which makes a random payment $\tilde{C}_t$ after t years, then summing up the prices of all of these more basic assets.

Pricing Risky Cash Flows

But we still have to deal with the fact that the payoff $\tilde{C}_t$ is risky.

And that is what the modern theory of asset pricing, on which this course is based, is really all about.

Pricing Risky Cash Flows

In probability theory, if a random variable X can take on n possible values, $X_1, X_2, \ldots, X_n$, with probabilities $\pi_1, \pi_2, \ldots, \pi_n$, then the expected value of X is

$$E(X) = \pi_1 X_1 + \pi_2 X_2 + \ldots + \pi_n X_n.$$

Pricing Risky Cash Flows

One approach to asset pricing replaces the random payoff $\tilde{C}_t$ with its expected value $E(\tilde{C}_t)$ and then "penalizes" the fact that the payoff is random by either discounting it at a higher rate

$$P_t^A = \frac{E(\tilde{C}_t)}{(1 + r_t + \psi_t)^t}$$

or by reducing its value more directly as

$$P_t^A = \frac{E(\tilde{C}_t) - \Psi_t}{(1 + r_t)^t}$$

Pricing Risky Cash Flows

$$P_t^A = \frac{E(\tilde{C}_t)}{(1 + r_t + \psi_t)^t}$$

$$P_t^A = \frac{E(\tilde{C}_t) - \Psi_t}{(1 + r_t)^t}$$

The capital asset pricing model (CAPM), the consumption capital asset pricing model (CCAPM), and the arbitrage pricing theory (APT) will give us ways of determining values for the risk premium ψ_t or Ψ_t.

Pricing Risky Cash Flows

Another possibility is to break down the random payoff $\tilde{C}_t$ into separate components $C_{t,1}, C_{t,2}, \ldots, C_{t,n}$ delivered in n different "states of the world" that can prevail t years from now.

The risky asset that delivers the random payoff $\tilde{C}_t$ t years from now can itself be viewed as a portfolio of contingent claims: $C_{t,1}$ contingent claims for state 1, $C_{t,2}$ contingent claims for state 2, ..., and $C_{t,n}$ contingent claims for state n.

Pricing Risky Cash Flows

This Arrow-Debreu approach to asset pricing then computes

$$P_t^A = q_{t,1}C_{t,1} + q_{t,2}C_{t,2} + \ldots + q_{t,n}C_{t,n}$$

where $q_{t,i}$ is the price today of a contingent claim that delivers one dollar if state i occurs t years from now and zero otherwise.

This approach uses contingent claims as the "basic building blocks" for risky assets, in the same way that discount bonds can be viewed as the building blocks for coupon bonds.

Pricing Risky Cash Flows

Yet another possibility is to "distort" the probabilities so as to down-weight favorable outcomes and over-weight adverse outcomes, to use these distorted probabilities to re-calculate

$$\hat{E}(C_t) = \hat{\pi}_1 C_{t,1} + \hat{\pi}_2 C_{t,2} + \ldots + \hat{\pi}_n C_{t,n},$$

and then to price the asset based on the distorted expectation:

$$P_t^A = \frac{\hat{E}(C_t)}{(1 + r_t)^t}.$$

The martingale approach to asset pricing will tell us how to do this.

Two Perspectives on Asset Pricing

Although all are designed to accomplish the same basic goal – to value risky cash flows – these different theories of asset pricing can be grouped under two broad headings.

No-arbitrage theories take the prices of some assets as given and use those to determine the prices of other assets.

Equilibrium theories price all assets based on the principles of microeconomic theory.

Two Perspectives on Asset Pricing

No-arbitrage theories require fewer assumptions and are sometimes easier to use.

We've already used no-arbitrage arguments, for example, to price stocks and bonds as portfolios of contingent claims and to price coupon bonds as portfolios of discount bonds.

Two Perspectives on Asset Pricing

But no-arbitrage theories raise questions that only equilibrium theories can answer.

Where do the prices of the basic securities come from?

And how do asset prices relate to economic fundamentals?

Two Perspectives on Asset Pricing

	Equilibrium	No-Arbitrage
Risk Premia	CAPM, CCAPM	APT
Contingent Claims	A-D	A-D
Distorted Probabilities		Martingale

Two Perspectives on Asset Pricing

Two Perspectives on Asset Pricing

- D Classic Asset Pricing Models
  - 7 The Capital Asset Pricing Model
  - 8 Arbitrage Pricing Theory
- E Arrow-Debreu Pricing
  - 9 Arrow-Debreu Pricing: Equilibrium
  - 10 Arrow-Debreu Pricing: No-Arbitrage
- F Extensions
  - 11 Martingale Pricing
  - 12 The Consumption Capital Asset Pricing Model

3 Making Choices in Risky Situations

3 Making Choices in Risky Situations

A Criteria for Choice Over Risky Prospects

B Preferences and Utility Functions

C Expected Utility Functions

Criteria for Choice Over Risky Prospects

In the broadest sense, "risk" refers to uncertainty about the future cash flows provided by a financial asset.

A more specific way of modeling risk is to think of those cash flows as varying across different states of the world in future periods . . .

. . . that is, to describe future cash flows as random variables.

Criteria for Choice Over Risky Prospects

Consider three assets:

		Payoffs Next Year in	
	Price Today	Good State	Bad State
Asset 1	-1000	1200	1050
Asset 2	-1000	1600	500
Asset 3	-1000	1600	1050

where the good and bad states occur with equal probability ($\pi = 1 - \pi = 1/2$).

Criteria for Choice Over Risky Prospects

		Payoffs Next Year in	
	Price Today	Good State	Bad State
Asset 1	-1000	1200	1050
Asset 2	-1000	1600	500
Asset 3	-1000	1600	1050

Asset 3 exhibits state-by-state dominance over assets 1 and 2. Any investor who prefers more to less would always choose asset 3 above the others.

Criteria for Choice Over Risky Prospects

		Payoffs Next Year in	
	Price Today	Good State	Bad State
Asset 1	-1000	1200	1050
Asset 2	-1000	1600	500
Asset 3	-1000	1600	1050

But the choice between assets 1 and 2 is not as clear cut. Asset 2 provides a larger gain in the good state, but exposes the investor to a loss in the bad state.

Criteria for Choice Over Risky Prospects

It can often be helpful to convert prices and payoffs to percentage returns:

		Payoffs Next Year in	
	Price Today	Good State	Bad State
Asset 1	-1000	1200	1050
Asset 2	-1000	1600	500
Asset 3	-1000	1600	1050

	Percentage Return in	
	Good State	Bad State
Asset 1	20	5
Asset 2	60	-50
Asset 3	60	5

Criteria for Choice Over Risky Prospects

In probability theory, if a random variable X can take on n possible values, $X_1, X_2, \ldots, X_n$, with probabilities $\pi_1, \pi_2, \ldots, \pi_n$, then the expected value of X is

$$E(X) = \pi_1 X_1 + \pi_2 X_2 + \ldots + \pi_n X_n,$$

the variance of X is

$$\begin{aligned}\sigma^2(X) &= \pi_1[X_1 - E(X)]^2 \\ &\quad + \pi_2[X_2 - E(X)]^2 + \ldots + \pi_n[X_n - E(X)]^2,\end{aligned}$$

and the standard deviation of X is $\sigma(X) = [\sigma^2(X)]^{1/2}$.

Criteria for Choice Over Risky Prospects

	Percentage Return in	
	Good State	Bad State
Asset 1	20	5
Asset 2	60	-50
Asset 3	60	5

$$E(R_1) = (1/2)20 + (1/2)5 = 12.5$$

$$\sigma(R_1) = [(1/2)(20 - 12.5)^2 + (1/2)(5 - 12.5)^2]^{1/2} = 7.5$$

Criteria for Choice Over Risky Prospects

	Percentage Return in			
	Good State	Bad State	$E(R)$	$\sigma(R)$
Asset 1	20	5	12.5	7.5
Asset 2	60	-50		
Asset 3	60	5		

$$E(R_2) = (1/2)60 + (1/2)(-50) = 5$$

$$\sigma(R_2) = [(1/2)(60-5)^2 + (1/2)(-50-5)^2]^{1/2} = 55$$

Criteria for Choice Over Risky Prospects

	Percentage Return in			
	Good State	Bad State	$E(R)$	$\sigma(R)$
Asset 1	20	5	12.5	7.5
Asset 2	60	-50	5	55
Asset 3	60	5		

$$E(R_3) = (1/2)60 + (1/2)5 = 32.5$$

$$\sigma(R_3) = [(1/2)(60 - 32.5)^2 + (1/2)(5 - 32.5)^2]^{1/2} = 27.5$$

Criteria for Choice Over Risky Prospects

	Percentage Return in			
	Good State	Bad State	$E(R)$	$\sigma(R)$
Asset 1	20	5	12.5	7.5
Asset 2	60	-50	5	55
Asset 3	60	5	32.5	27.5

Asset 1 exhibits mean-variance dominance over asset 2, since it offers a higher expected return with lower variance.

Criteria for Choice Over Risky Prospects

	Percentage Return in			
	Good State	Bad State	$E(R)$	$\sigma(R)$
Asset 1	20	5	12.5	7.5
Asset 2	60	-50	5	55
Asset 3	60	5	32.5	27.5

But notice that by the mean-variance criterion, asset 3 dominates asset 2 but not asset 1, even though on a state-by-state basis, asset 3 is clearly to be preferred.

Criteria for Choice Over Risky Prospects

Consider two more assets:

	Percentage Return in			
	Good State	Bad State	$E(R)$	$\sigma(R)$
Asset 4	5	3		
Asset 5	8	2		

Again, neither exhibits state-by-state dominance, so let's try to use the mean-variance criterion again.

Criteria for Choice Over Risky Prospects

	Percentage Return in			
	Good State	Bad State	$E(R)$	$\sigma(R)$
Asset 4	5	3		
Asset 5	8	2		

$$E(R_4) = (1/2)5 + (1/2)3 = 4$$

$$\sigma(R_4) = [(1/2)(5-4)^2 + (1/2)(3-4)^2]^{1/2} = 1$$

$$E(R_5) = (1/2)8 + (1/2)2 = 5$$

$$\sigma(R_5) = [(1/2)(8-5)^2 + (1/2)(2-5)^2]^{1/2} = 3$$

Criteria for Choice Over Risky Prospects

	Percentage Return in			
	Good State	Bad State	$E(R)$	$\sigma(R)$
Asset 4	5	3	4	1
Asset 5	8	2	5	3

Neither asset exhibits mean-variance dominance either.

Criteria for Choice Over Risky Prospects

	Percentage Return in			
	Good State	Bad State	$E(R)$	$\sigma(R)$
Asset 4	5	3	4	1
Asset 5	8	2	5	3

William Sharpe (US, b.1934, Nobel Prize 1990) suggested that in these circumstances, it can help to compare the two assets' Sharpe ratios, defined as $E(R)/\sigma(R)$.

Criteria for Choice Over Risky Prospects

	Percentage Return in				
	Good State	Bad State	$E(R)$	$\sigma(R)$	E/σ
Asset 4	5	3	4	1	4
Asset 5	8	2	5	3	1.67

Comparing Sharpe ratios, asset 4 is preferred to asset 5.

Criteria for Choice Over Risky Prospects

	Percentage Return in Good State	Bad State	$E(R)$	$\sigma(R)$	E/σ
Asset 4	5	3	4	1	4
Asset 5	8	2	5	3	1.67

But using the Sharpe ratio to choose between assets means assuming that investors "weight" the mean and standard deviation equally, in the sense that a doubling of $\sigma(R)$ is adequately compensated by a doubling of $E(R)$. Investors who are more or less averse to risk will disagree.

Criteria for Choice Over Risky Prospects

1. State-by-state dominance is the most robust criterion, but often cannot be applied.
2. Mean-variance dominance is more widely-applicable, but can sometimes be misleading and cannot always be applied.
3. The Sharpe ratio can always be applied, but requires a very specific assumption about consumer attitudes towards risk.

We need a more careful and comprehensive approach to comparing random cash flows.

Preferences and Utility Functions

Of course, economists face a more general problem of this kind.

Even if we accept that more (of everything) is preferred to less, how do consumers compare different "bundles" of goods that may contain more of one good but less of another?

Microeconomists have identified a set of conditions that allow a consumer's preferences to be described by a utility function.

Expected Utility Functions

Under certainty, the “goods” are described by consumption baskets with known characteristics.

Under uncertainty, the “goods” are random (state-contingent) payoffs.

The problem of describing preferences over these state-contingent payoffs, and then summarizing these preferences with a utility function, requires some efforts. What do we think are good rules ("axioms") that a utility function over random (state-contingent) payoffs should satisfy?

Expected Utility Functions

Consider shares of stock in two companies:

		Price Next Year in	
	Price Today	Good State	Bad State
AT&T	-100	150	100
Verizon	-100	150	100

where the good state occurs with probability π and the bad state occurs with probability $1-\pi$.

Expected Utility Functions

		Price Next Year in	
	Price Today	Good State	Bad State
AT&T	-100	150	100
Verizon	-100	150	100
	probability	π	$1-\pi$

We will assume that if the two assets provide exactly the same state-contingent payoffs, then investors will be indifferent between them.

Expected Utility Functions

		Price Next Year in	
	Price Today	Good State	Bad State
AT&T	-100	150	100
Verizon	-100	150	100
	probability	π	$1-\pi$

Rule 1. Investors care only about payoffs and probabilities.

Expected Utility Functions

Consider another comparison:

		Price Next Year in	
	Price Today	Good State	Bad State
AT&T	-100	150	100
Apple	-100	160	110
	probability	π	$1-\pi$

We will also assume that investors will prefer any asset that exhibits state-by-state dominance over another.

Expected Utility Functions

		Price Next Year in	
	Price Today	Good State	Bad State
AT&T	-100	150	100
Apple	-100	160	110
	probability	π	$1-\pi$

Rule 2. If $u(p)$ measures utility from the payoff p in any particular state, then u is increasing.

Expected Utility Functions

Consider a third comparison:

		Price Next Year in	
	Price Today	Good State	Bad State
AT&T	-100	150	100
IBM	-100	160	90
	probability	π	$1-\pi$

Here, there is no state-by-state dominance, but it seems reasonable to assume that a higher probability π will make investors tend to prefer IBM, while a higher probability $1-\pi$ will make investors tend to prefer AT&T.

Expected Utility Functions

		Price Next Year in	
	Price Today	Good State	Bad State
AT&T	-100	150	100
IBM	-100	160	90
	probability	π	$1-\pi$

Rule 3. Investors should care more about states of the world that occur with greater probability.

Expected Utility Functions

A criterion that has all three of these properties was suggested by Blaise Pascal (France, 1623-1662): base decisions on the expected payoff,

$$E(p) = \pi p_G + (1 - \pi) p_B,$$

where p_G and p_B, with $p_G > p_B$, are the payoffs in the good and bad states.

Expected Utility Functions

Expected payoff satisfy the first three rules

$$E(p) = \pi p_G + (1 - \pi) p_B$$

1. Depends only on payoffs and probabilities.
2. Increases whenever p_G or p_B rises.
3. Attaches higher weight to states with higher probabilities.

Expected Utility Functions

Nicolaus Bernoulli (Switzerland, 1687-1759) pointed to a problem with basing investment decisions exclusively on expected payoffs: it ignores risk. To see this, specialize the previous example by setting $\pi = 1 - \pi = 1/2$ but add, as well, a third asset:

		Price Next Year in	
	Price Today	Good State	Bad State
AT&T	-100	150	100
IBM	-100	160	90
US Gov't Bond	-100	125	125
	probability	$\pi = 1/2$	$1 - \pi = 1/2$

Expected Utility Functions

		Price Next Year in	
	Price Today	Good State	Bad State
AT&T	-100	150	100
IBM	-100	160	90
US Gov't Bond	-100	125	125
	probability	$\pi = 1/2$	$1-\pi = 1/2$

AT&T: $E(p) = (1/2)150 + (1/2)100 = 125$

IBM $E(p) = (1/2)160 + (1/2)90 = 125$

Gov't Bond: $E(p) = (1/2)125 + (1/2)125 = 125$

Expected Utility Functions

$$\text{AT\&T: } E(p) = (1/2)150 + (1/2)100 = 125$$

$$\text{IBM } E(p) = (1/2)160 + (1/2)90 = 125$$

$$\text{Gov't Bond: } E(p) = (1/2)125 + (1/2)125 = 125$$

All three assets have the same expected payoff, but the bond is less risky than both stocks and AT&T stock is less risky than IBM stock.

Expected Utility Functions

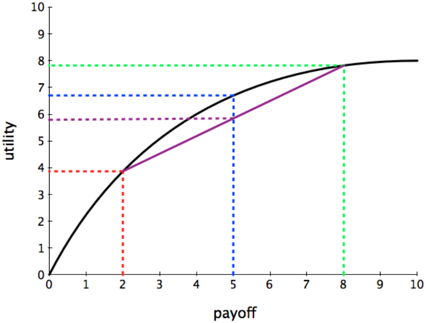
About two centuries later, John von Neumann (Hungary, 1903-1957) and Oskar Morgenstern (Germany, 1902-1977) suggest that investors' preferences over risky payoffs (p) could be described by an expected utility function such as

$$U(p) = E\left[u(p)\right] = \pi u(p_G) + (1 - \pi) u(p_B),$$

where the Bernoulli utility function over payoffs u is increasing and concave and the aggregation over states is linear in the probabilities π.

This is called the von Neumann-Morgenstern (vNM) expected utility framework.

Expected Utility Functions



When u is concave, a payoff of 5 for sure is preferred to a payoff of 8 with probability $1/2$ and 2 with probability $1/2$.

4 Measuring Risk and Risk Aversion

4 Measuring Risk and Risk Aversion

A Measuring Risk Aversion

B Interpreting the Measures of Risk Aversion

(Next lecture)

C Risk Premium and Certainty Equivalent

D Assessing the Level of Risk Aversion

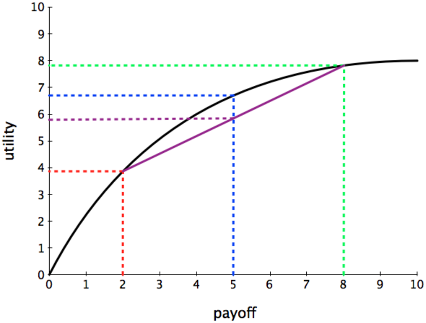
E The Concept of Stochastic Dominance

F Mean Preserving Spreads

Measuring Risk Aversion

We've already seen that within the von Neumann-Morgenstern expected utility framework, risk aversion enters through the concavity of the Bernoulli utility function.

Expected Utility Functions



When u is concave, a payoff of 5 for sure is preferred to a payoff of 8 with probability $1/2$ and 2 with probability $1/2$.

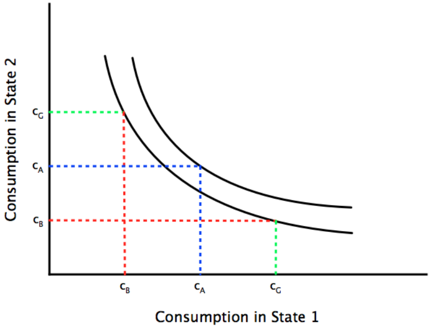
Measuring Risk Aversion

We've also seen previously that concavity of the utility function is related to convexity of indifference curves.

In standard microeconomic theory, this feature of preferences represents a "taste for diversity."

Under uncertainty, it represents a desire to smooth consumption across future states of the world.

Expected Utility Functions



A risk averse consumer prefers $c_A = (c_G + c_B)/2$ in both states to c_G in one state and c_B in the other.

Measuring Risk Aversion

Mathematically, $u'(p) > 0$ means that an investor prefers higher payoffs to lower payoffs, and $u''(p) < 0$ means that the investor is risk averse.

But is there a way of quantifying an investor's degree of risk aversion?

And is there a criterion according to which we might judge one investor to be more risk averse than another?

Measuring Risk Aversion

Since $u''(p) < 0$ makes an investor risk averse, one conjecture would be to say that an investor with Bernoulli utility function $v(p)$ is more risk averse than another investor with Bernoulli utility function $u(p)$ if $v''(p) < u''(p)$ for all payoffs p.

Measuring Risk Aversion

Recall, however, that the preference ordering of an investor with vN-M utility function

$$U(z) = U(x, y, \pi) = \pi u(x) + (1 - \pi)u(y)$$

is also represented by the vN-M utility function

$$V(z) = \alpha U(z) + \beta = \pi v(x) + (1 - \pi)v(y),$$

where

$$v(x) = \alpha u(x) + \beta \text{ and } v(y) = \alpha u(y) + \beta.$$

Measuring Risk Aversion

And with

$$v(p) = \alpha u(p) + \beta,$$

for any payoff p,

$$v'(p) = \alpha u'(p)$$

$$v''(p) = \alpha u''(p),$$

所以 $u''(x)$
不是一个好的
measure

By making α larger or smaller, the Bernoulli utility function can be made "more" or "less" concave without changing the underlying preference ordering.

Measuring Risk Aversion

等比例的放大
与缩小

Two alternative measures of risk aversion are

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{ coefficient of absolute risk aversion}$$

绝对赌博：50%赢 1000元
50%输 1000元

$u''(\cdot) < 0$

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \text{ coefficient of relative risk aversion}$$

同样[illegible]

where Y measures the investor's income level.

Since $v(p) = \alpha u(p) + \beta$ implies $v'(p) = \alpha u'(p)$ and $v''(p) = \alpha u''(p)$, these measures are invariant to linear transformations of the Bernoulli utility function.

Measuring Risk Aversion

Two alternative measures of risk aversion are

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{ coefficient of absolute risk aversion}$$

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \text{ coefficient of relative risk aversion}$$

where Y measures the investor's income level.

And since both measures have a minus sign out in front, both are positive and increase when risk aversion rises.

Interpreting the Measures of Risk Aversion

To interpret the two measures of risk aversion, it is helpful to recall from calculus the theorem stated by Brook Taylor (England, 1685-1731), regarding the approximation of a function f using its derivatives: the "first-order" approximation

$$f(x+a) \approx f(x) + f'(x)a$$

$E(u(x+\tilde{a}))$

and the "second-order" approximation

$$f(x+a) \approx f(x) + f'(x)a + \frac{1}{2}f''(x)a^2.$$

The second-order approximation is more accurate than the first, and both become more accurate as a becomes smaller.

Interpreting the Measures of Risk Aversion

$$Y \begin{cases} Y+h & \pi \\ Y-h & 1-\pi \end{cases}$$

$\pi < \frac{1}{2}$ 接受(买lottery)

Focusing first on the measure of absolute risk aversion, consider an investor with initial income Y who is offered a bet: win h with probability π and lose h with probability $1-\pi$.

A risk-averse investor with vN-M expected utility would never accept this bet if $\pi = 1/2$.

The question is: how much higher than $1/2$ does π have to be to get the investor to accept the bet?

Interpreting the Measures of Risk Aversion

Let π^* be the probability that is just high enough to get the investor to accept the bet.

Then π^* must satisfy

$$u(Y) = \pi^* u(Y + h) + (1 - \pi^*) u(Y - h).$$

Interpreting the Measures of Risk Aversion

It turns out that under second-order approximation, we have a clear relationship between the probability that makes the risk-averse investor indifferent and the absolute risk aversion of the investor

会记住

考试：给公式、给 h、给 $R_A(Y)$

$$\pi^* \approx \frac{1}{2} + \frac{1}{4}\left[-\frac{u''(Y)}{u'(Y)}\right] h = \frac{1}{2} + \frac{1}{4} h R_A(Y) > \frac{1}{2}.$$

算 π^*

The boost in π above $1/2$ required for an investor with income Y to accept a bet of plus or minus h relates directly to the coefficient of absolute risk aversion.

(bonus: derive the above formula)

附加题：推导公式

$$W(Y) = \pi^* u(Y+h) + (1-\pi^*) u(Y-h)$$

$$= \pi^* \left(u(Y) + u'(Y) h + \frac{1}{2} u''(Y) h^2 \right)$$

$$+ (1-\pi^*) \left(u(Y) - u'(Y) h + \frac{1}{2} u''(Y) h^2 \right)$$

$$= u(Y) + u'(Y)(2\pi^* - 1) h + \frac{1}{2} u''(Y) h^2$$

$$\therefore \quad 2\pi^* - 1 = \frac{1}{2}\left(-\frac{u''(Y)}{u'(Y)} \right) h$$

$$\therefore \quad \pi^* = \frac{1}{2} + \frac{1}{4} h R_A(Y)$$

Interpreting the Measures of Risk Aversion

考试!

As an example, suppose that we ask an investor: What value of π^* would you need to accept a bet of plus-or-minus $h = \$1000$?

And the investor says: I'll take it if $\pi^* = 0.75$.

Interpreting the Measures of Risk Aversion

With $h = \$1000$ and $\pi^* = 0.75$,

$$\pi^* \approx \frac{1}{2} + \frac{1}{4} h R_A(Y)$$

implies

$$0.75 \approx 0.50 + \frac{1000}{4} R_A(Y)$$

$$0.25 \approx 250 R_A(Y)$$

$$R_A(Y) \approx \frac{0.25}{250} = 0.001.$$

Interpreting the Measures of Risk Aversion

Realistically, a bet over \$1000 is probably going to seem more risky to someone who starts out with less income.

In general, we are allowing for that. Since

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \text{ coefficient of absolute risk aversion,}$$

it also follows that investors with different income levels generally display different levels of absolute risk aversion.

假设CARA不随收入变化（简单起来方便）.

Interpreting the Measures of Risk Aversion

Suppose, however, that the Bernoulli utility function takes the form (exponential utility function)

$$u(Y) = -\frac{1}{\nu} e^{-\nu Y},$$ 或 $u(Y) = -\exp(\nu Y)$

where $\nu > 0$ and e^x is the exponential function ($e \approx 2.718$).

Recall that exponential function has the special property that

$$f(x) = e^x \Rightarrow f'(x) = e^x$$

and by the chain rule

$$g(x) = e^{\alpha x} \Rightarrow g'(x) = \alpha e^{\alpha x}$$

Interpreting the Measures of Risk Aversion

With

$$u(Y) = -\frac{1}{\nu} e^{-\nu Y},$$

it follows that

$$u'(Y) = -\frac{1}{\nu} e^{-\nu Y}(-\nu) = e^{-\nu Y}$$

$$u''(Y) = -\nu e^{-\nu Y}$$

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \frac{\nu e^{-\nu Y}}{e^{-\nu Y}} = \nu,$$

so that this utility function displays constant absolute risk aversion, which does not depend on income.

Interpreting the Measures of Risk Aversion

So if we were willing to make the assumption of constant absolute risk aversion, we could use the results from our example, where an investor requires $\pi^* = 0.75$ to accept a bet with $h = \$1000$ to set $\nu = 0.001$ in

$$u(Y) = -\frac{1}{\nu}e^{-\nu Y},$$

and thereby tailor portfolio decisions specifically for this investor.

Interpreting the Measures of Risk Aversion

Absolute risk aversion describes an investor's attitude towards absolute bets of plus or minus h.

Relative risk aversion describes an investor's attitude towards relative bets of plus or minus kY, i.e. bets that are fractions of total income.

Interpreting the Measures of Risk Aversion

$$\begin{array}{lll} & (1+k)Y & \pi \\ Y & & \\ & (1-k)Y & 1-\pi \end{array}$$

Consider an investor with initial income Y who is offered a bet: win kY with probability π and lose kY with probability $1-\pi$.

A risk-averse investor with vN-M expected utility would never accept this bet if $\pi = 1/2$.

The question is: how much higher than $1/2$ does π have to be to get the investor to accept the bet?

Interpreting the Measures of Risk Aversion

Let π^* be the probability that is just high enough to get the investor to accept the bet.

Now π^* must satisfy

$$u(Y) = \pi^* u(Y + Yk) + (1 - \pi^*) u(Y - Yk).$$

Interpreting the Measures of Risk Aversion

For relative bets, under second-order approximation, we have a clear relationship between the probability that makes the risk-averse investor indifferent and the relative risk aversion of the investor

$$\pi^* \approx \frac{1}{2} + \frac{1}{4}\left[-\frac{Yu''(Y)}{u'(Y)}\right]k = \frac{1}{2} + \frac{1}{4}kR_R(Y) > \frac{1}{2}.$$

The boost in π above $1/2$ required for an investor with income Y to accept a bet of plus or minus kY relates directly to the coefficient of relative risk aversion.

(bonus: derive the above formula)

Interpreting the Measures of Risk Aversion

Suppose that we ask an investor: What value of π^* would you need to accept a bet of plus-or-minus one percent ($k = 0.01$) of your income?

And the investor says: I'll take it if $\pi^* = 0.75$.

Interpreting the Measures of Risk Aversion

With $k = 0.01$ and $\pi^* = 0.75$,

$$\pi^* \approx \frac{1}{2} + \frac{1}{4} k R_R(Y)$$

implies

$$0.75 \approx 0.50 + \frac{0.01}{4} R_R(Y)$$

$$0.25 \approx 0.0025 R_R(Y)$$

$$R_A(Y) \approx \frac{0.25}{0.0025} = 100.$$

Interpreting the Measures of Risk Aversion

Since

$$R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = \text{ coefficient of relative risk aversion,}$$

it also follows that investors with different income levels generally display different levels of relative risk aversion.

On the other hand, since the coefficient of relative risk aversion describes aversion to risk over bets that are expressed relative to income, it is more plausible to assume that investors have constant relative risk aversion.

Interpreting the Measures of Risk Aversion

Suppose, therefore, that the Bernoulli utility function takes the form (power utility function)

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma}$$

where $\gamma > 0$. For this function, l'Hôpital's rule implies that when $\gamma = 1$

$$\frac{Y^{1-\gamma} - 1}{1-\gamma} = \ln(Y),$$

where ln is the natural logarithm (log utility function).

Homework: Verify that power utility function displays constant relative risk aversion, which does not depend on income.

Interpreting the Measures of Risk Aversion

So if we were willing to make the assumption of constant relative risk aversion, we could use the results from our example, where an investor requires $\pi^* = 0.75$ to accept a bet with $k = 0.01$ to set $\gamma = 100$ in

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}$$

and thereby tailor portfolio decisions specifically for this investor.

Interpreting the Measures of Risk Aversion

Finally, suppose that we do away with the concavity of the Bernoulli utility function and simply assume that

$$u(p) = \alpha p + \beta,$$

$\gamma \to 0$, utility function [illegible]线性

where $\alpha > 0$, so that higher payoffs are preferred to lower payoffs. For this utility function,

$$u'(Y) = \alpha \text{ and } u''(Y) = 0$$

$$R_A(Y) = -\frac{u''(Y)}{u'(Y)} = 0 \text{ and } R_R(Y) = -\frac{Yu''(Y)}{u'(Y)} = 0.$$

This investor is risk-neutral and cares only about expected payoffs.