Homework Assignment #1
Due: In class, two weeks after distribution

§1 Normal and Log-normal Random Variables

1. Show that the moment generating function of normal $X \sim N(\mu, \sigma^2)$ is
$$\phi(\vartheta) = \mathbb{E}e^{\vartheta X} = \exp\left\{\frac{1}{2}\sigma^2\vartheta^2 + \mu\vartheta\right\}.$$
2. By differentiating the moment generating function, find the skewness and kurtosis of X, i.e. compute $\mathbb{E}(X - EX)^3$ and $\mathbb{E}(X - EX)^4$, respectively.
3. Suppose that Y is log-normally distributed, i.e. $\log Y$ has normal distribution $N(\mu, \sigma^2)$, compute the $\mathbb{E}Y$ and $\mathrm{Var}Y$.
4. For $K > 0$, explicitly compute the expectation $\mathbb{E}[Y 1_{\{Y>K\}}]$, where $1_{\{y>K\}}$ is the indicator function of the set $\{y > K\}$. The function $1_{\{y>K\}}$ takes value 1, if $y > K$, and 0, otherwise. You may express your results using the cummulative standard normal distribution function $\Phi(x)$.
5. Further, prove that
$$\begin{aligned}\mathbb{E}(Y-K)^+ &= e^{\mu+\frac{1}{2}\sigma^2}\Phi\left(\frac{\mu - \log K + \sigma^2}{\sigma}\right) - K\Phi\left(\frac{\mu - \log K}{\sigma}\right),\\ \mathbb{E}(K-Y)^+ &= -e^{\mu+\frac{1}{2}\sigma^2}\Phi\left(-\frac{\mu - \log K + \sigma^2}{\sigma}\right) + K\Phi\left(-\frac{\mu - \log K}{\sigma}\right),\end{aligned}$$
where the functions $(y-K)^+ = (y-K)1_{\{y>K\}}$ and $(K-y)^+ = (K-y)1_{\{y<K\}}$.

§2 Bivariate Normal Variables

Let (X, Y) be a bivariate normal variable with correlation ρ. Denote $\mathrm{Var}X = \sigma_X^2$ and $\mathrm{Var}Y = \sigma_Y^2$. Show that X and $W = Y - \frac{\rho\sigma_Y}{\sigma_X}X$ are uncorrelated, and thus independent.

§3 Risk Minimization

Suppose you invest in m stocks with returns $R_1, R_2, ..., R_m$. If you regard the variance of your portfolio as a risk mreasure. Please find the optimal portfolio weight $(w_1, w_2, \cdots, w_m)$ such that the risk is minimized, i.e. find the min $\mathrm{Var}(w_1R_1 + w_2R_2 + \cdots + w_mR_m)$ given that $w_1 + w_2 + \cdots + w_m = 1$. Here we assume that $ER_i = \mu_i$, $\mathrm{Var}R_i = \sigma_i^2$ for $i = 1, 2, ..., m$ and $\mathrm{Corr}(R_i, R_j) = \rho_{ij}$ for $i, j = 1, 2, ..., m$.

§4 Roll a Dice

You roll a single dice with six sides three times without pause, and the payoff to the player is the maximum of the three rolls. What is the expected payoff to the player? Thus, by the law of large number this amount should be charged.