Homework Assignment #5
Due: In class, two weeks after distribution

# §1 Scaling Property of Brownian Motion

Given that $\{W(t)\}$ is a standard Brownian motion, prove that $B(t) = \frac{1}{\sqrt{a}}W(at)$ is also a standard Brownian motion.

# §2 Finite Dimensional Distribution of a Brownian Motion

Let $\{W(t)\}$ be a standard Brownian motion.

1. For $0 < s_1 < s_2$, prove that the joint density of $(W(s_1), W(s_2))$ is given by
$$p(s_1, 0, y_1)p(s_2 - s_1, y_1, y_2),$$
where
$$p(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}.$$

2. Suppose $p(w_2, s_2|w_1, s_1)$ denotes the transition density of the Brownian motion, i.e.,
$$P(W(s_2) \in dw_2 | W(s_1) \in dw_1) = p(w_2, s_2|w_1, s_1)dw_2. \tag{2.1}$$
Write the conditional denity $p(w_2, s_2|w_1, s_1)$ explicitly.

3. (a bonus question) Suppose $0 < s_1 < s_2 < \cdots < s_m$. Denote by $p(s_1, w_1; s_2, w_2; \cdots; s_m, w_m)$ the joint density of $(W(s_1), W(s_2), \cdots, W(s_m))$, i.e.,
$$P(W(s_1) \in dw_1, W(s_2) \in dw_2, \cdots, W(s_m) \in dw_m) = p(s_1, w_1; s_2, w_2; \cdots; s_m, w_m)dw_1 dw_2 \cdots dw_m.$$
Use the Markov property of Brownian motion to prove that
$$p(s_1, w_1; s_2, w_2; \cdots; s_m, w_m) = \prod_{i=1}^{m-1} p(w_{i+1}, s_{i+1}|w_i, s_i)p(w_1, s_1|0, 0).$$

4. (a bonus question) In this exercise, we prepare for being an financial econometrician. Let $X(t) = \sigma W(t) + \mu t$ be a Brownian motion with drift $\mu$ and volatility $\sigma$. Denote by $p_X(s_1, x_1; s_2, x_2; \cdots; s_m, x_m)$ the joint density of $(X(s_1), X(s_2), \cdots, X(s_m))$, i.e.,
$$P(X(s_1) \in dx_1, X(s_2) \in dx_2, \cdots, X(s_m) \in dx_m) = p(s_1, x_1; s_2, x_2; \cdots; s_m, x_m)dx_1 dx_2 \cdots dx_m.$$
Please use the transition density of Brownian motion given in (2.1) to express $p(s_1, x_1; s_2, x_2; \cdots; s_m, x_m)$ in closed-form. Suppose we observe $X(s_1), X(s_2), \cdots, X(s_m)$ as a series of data. By maximizing the log-likelihood function
$$l(s_1, x_1; s_2, x_2; \cdots; s_m, x_m) = \log p(s_1, x_1; s_2, x_2; \cdots; s_m, x_m),$$
find the maximum-likelihood estimator for $\sigma$ and $\mu$. Is is possible to get an explicit expression?

## §3 Brownian Motion with Drift

Let $X(t) = \sigma W(t) + \mu t$, where $\{W(t)\}$ is a standard Brownian motion; $\sigma$ and $\mu$ are both real constant.

1. Find the auto correlation function for $\{X(t)\}$, i.e. $\mathrm{Corr}(X(t), X(s))$.
2. Find the quadratic variation of $\{X(t)\}$, i.e. $[X, X](t)$.

## §4 Geometric Brownian Motion

Suppose that the Microsoft stock is modeled by a geometric Brownian motion:

$$S(t) = S_0 \exp\{\sigma W(t) + G(t)\},$$

where $G(t)$ is a deterministic function.

1. Find the probability that $S(T)$ is above a level $K > 0$.
2. Find all functions $G(t)$, such that $S(t)$ is a martingale adapted to the filtration generated by Brownian motion $W$.

## §5 Multidimensional Brownian Motion

Use a standard three-dimensional Brownian motion $\{(Z_1(t), Z_2(t), Z_3(t))\}$ to construct a three-dimensional correlated Brownian motion $\{(W_1(t), W_2(t), W_3(t))\}$ with given correlations:

$$\mathrm{Corr}(W_i(t), W_j(t)) = \rho_{ij},$$

for $i, j = 1, 2, 3$. Note that $\rho_{11} = \rho_{22} = \rho_{33} = 1$.

## §6 Brownian Bridge

Let $\{W(t)\}$ be a standard Brownian motion. Define a "Brownian bridge" over time interval $[0, 1]$ as $\{B(t)\}$ with $B(t) = W(t) - tW(1)$.

1. Prove that $\{B(t)\}$ follows the law of a Brownian bridge, i.e., it is enough to show that
$$\mathbb{P}(B(t) \leq x) = \mathbb{P}(W(t) \leq x | W(1) = 0).$$
2. Find the mean and variance of $B(t)$.
3. By generalizing the express of $B(t)$, give a construction of a Brownian bridge starting from $a$ at time 0 and arriving eventually at $b$ at time $T$.

Note: Indeed, for Question 1, it is more rigorous to prove, for $0 \leq t < 1$ and some $\Delta$ such that $t + \Delta \leq 1$, we have

$$\mathbb{P}(B(t+\Delta) \leq x | B(t) = x_0) = \mathbb{P}(W(t+\Delta) \leq x | W(t) = x_0, W(1) = 0).$$