夹逼原理及例题

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本段内容要点:

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夹逼原理(两边夹原理)
二项式公式用于不等式放缩
系列例题,包括 \lim_{n \to \infty} \sqrt[n]{n} = 1, \lim_{n \to \infty} n^k q^n = 0, \ (k \in \mathbb{N}, |q| < 1).
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定理:[夹逼原理]:

设有三个数列 $\{x_n\},\{y_n\},\{z_n\}$ 满足:

- (a) $x_n \leqslant y_n \leqslant z_n, n = 1, 2, \cdots$
- (b) $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = a$,

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- (b) $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = a$,

则必有 $\lim_{n\to\infty} y_n = a$.

例 $_{1:}\ x_n=rac{\sin n}{n},\ n\in\mathbb{N}.$

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$$0 < \left| \frac{\sin n}{n} \right| < \frac{1}{n}, \quad \overline{\prod}_{n \to \infty} \frac{1}{n} = 0,$$

所以
$$\lim_{n\to\infty} \frac{\sin n}{n} = 0.$$

例 $_{2:}\lim_{n o\infty}n\left(rac{1}{n^2+\pi}+rac{1}{n^2+2\pi}+\cdots+rac{1}{n^2+n\pi}
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解:记
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 $n \rightarrow \infty$

因为
$$\lim_{n \to \infty} \frac{n^2}{n^2 + n\pi} = \lim_{n \to \infty} \frac{n^2}{n^2 + \pi} = 1.$$
故 $\lim_{n \to \infty} x_n = 1.$

例 $_{3:}\lim_{n o\infty}(a_1^n+a_2^n+\cdots+a_k^n)^{rac{1}{n}},$

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例3: $\lim_{n o\infty}(a_1^n+a_2^n+\cdots+a_k^n)^{rac{1}{n}},$

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則
$$(a^n)^{rac{1}{n}}\leqslant x_n\leqslant (ka^n)^{rac{1}{n}},\ n\in\mathbb{N}$$

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 因为 $\lim_{n o\infty}\sqrt[n]{k}=1,$ 所以据夹逼原理, $\lim_{n o\infty}x_n=a$

例
$$_3$$
: $\lim_{n o\infty}(a_1^n+a_2^n+\cdots+a_k^n)^{\frac{1}{n}},$ 其中 $a_i>0,\ i=1,\cdots,k,\ k\in\mathbb{N}$ 是确定的数.

$$\exists x_n = (a_1^n + a_2^n + \dots + a_k^n)^{\frac{1}{n}}, \quad \underline{a} = \max\{a_1, a_2, \dots, a_k\}.$$

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例 $_4$: $x_n=rac{(2n-1)!!}{(2n)!!}=rac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \cdots \cdot (2n)}$

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故
$$\dfrac{n-1}{n}< \left|\dfrac{n}{n+1}\right|$$

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解: 因为
$$(n+1)(n-1) = n^2 - 1 < n^2$$
,

故
$$\left| \frac{n-1}{n} < \frac{n}{n+1} \right|$$
 $x_n = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \cdots \cdot \frac{2n-1}{2n}$

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$$=rac{1}{x_n}\cdotrac{1}{2n+1}$$

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例4:
$$x_n = \frac{(2n-1)!!}{(2n)!!} = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}$$

$$|x| = \frac{1}{n} < \frac{n}{n+1}, \quad x_n = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \dots \cdot \frac{2n-1}{2n}$$
 $x_n^2 < \frac{1}{2n+1}, \quad x_n = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \dots \cdot \frac{2n-2}{2n-1}, \quad x_{n-1} = \frac{1}{2n}$

$$x_n^2 < rac{1}{2n+1}$$
 $= rac{1}{x_n} \cdot rac{1}{2n+1}$

$$0 < x_n < rac{1}{\sqrt{2n+1}}$$
 所以 $\lim_{n o \infty} rac{(2n-1)!!}{(2n)!!} = 0.$

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时, $a^n = [1 + (a-1)]^n = \sum_{k=0}^n C_n^k (a-1)^k$

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这里使用了
$$n>2$$
时, $n-1>\frac{n}{2}$.

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事实上

$$a^n > n$$
的 k 次多项式 $*(a-1)^k, n > N_0$

例
$$_{5:}\lim_{n o\infty}n\overline{n}=\lim_{n o\infty}\sqrt[n]{n}=1.$$
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证明:

因为
$$\sqrt[n]{n} > 1$$
,故 $(\sqrt[n]{n})^n > \frac{n^2}{4}(\sqrt[n]{n}-1)^2$

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故
$$0 < \sqrt[n]{n} - 1 \leqslant rac{2}{\sqrt{n}}$$

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$$0 < \sqrt[n]{n} - 1 \leqslant \frac{2}{\sqrt{n}} \rightarrow 0 \ (n \rightarrow \infty).$$

据夹逼原理, $\lim_{n\to\infty} \sqrt[n]{n} - 1 = 0$,

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

$$a^n\geqslant rac{n^2}{4}(a-1)^2,\ (n>2).$$

例6:
$$\lim_{n \to \infty} nq^n = 0 \quad (0 < |q| < 1)$$
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$$a^n\geqslant rac{n^2}{4}(a-1)^2,\ (n>2).$$

例6:
$$\lim_{n\to\infty} nq^n = 0 \ (0 < |q| < 1)$$
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$$|nq^n| = n|q|^n = nrac{1}{(|q|^{-1})^n}$$

$$a_{n}^{n} \geqslant \frac{n^{2}}{4}(a-1)^{2}, \ (n>2).$$

例
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: $\lim_{n o\infty}nq^n=0 \ \ (0<|q|<1)$.

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 $\rightarrow 0 (n \rightarrow \infty)$.

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$$|nq^n|\leqslant n\cdot rac{4}{n^2}\cdot rac{1}{(|q|^{-1}-1)^2}=rac{4}{n}\cdot rac{1}{(|q|^{-1}-1)^2}$$

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据夹逼原理, $\lim_{n \to \infty} nq^n = 0 \ (0 < |q| < 1)$.

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$$\lim_{n o \infty} nq^n = 0 \ \ (0 < |q| < 1)$$
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例7:
$$\lim_{n \to \infty} n^2 q^n = 0, \ 0 < |q| < 1$$
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$$(|q|^{-1})^n\geqslant rac{n(n-1)(n-2)}{3!}(|q|^{-1}-1)^3$$

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$$\lim_{n\to\infty} n^2 q^n = 0, \ 0 < |q| < 1.$$

$$|n^2q^n|=n^2|q|^n=n^2\cdotrac{1}{(|q|^{-1})^n}$$

-

$$(|q|^{-1})^n\geqslant \frac{n(n-1)(n-2)}{3!}(|q|^{-1}-1)^3 \qquad \geqslant \frac{n}{6}\cdot\frac{n}{2}\cdot\frac{n}{3}\cdot(|q|^{-1}-1)^3$$

$$\geqslant rac{n}{6} \cdot rac{n}{2} \cdot rac{n}{3} \cdot (|q|^{-1} - 1)^{\frac{n}{2}}$$

例6: $\lim nq^n = 0 \ (0 < |q| < 1)$. $n \rightarrow \infty$

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$$|n^2q^n|=n^2|q|^n=n^2\cdotrac{1}{(|q|^{-1})^n}$$

这里使用了
$$n>3$$
时, $n-1>rac{n}{2}, \ n-2>rac{n}{3}.$

$$(|q|^{-1})^n\geqslant \frac{n(n-1)(n-2)}{3!}(|q|^{-1}-1)^3 \qquad \geqslant \frac{n}{6}\cdot\frac{n}{2}\cdot\frac{n}{3}\cdot(|q|^{-1}-1)^3$$

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$$\geqslant rac{n}{6} \cdot rac{n}{2} \cdot rac{n}{3} \cdot (|q|^{-1} - 1)^3$$

$$= \frac{n^3}{36}(|q|^{-1} - 1)^3 \qquad (n > 3)$$

$$(|q|^{-1})^n\geqslant rac{n(n-1)(n-2)}{3!}(|q|^{-1}-1)^3 \qquad \geqslant rac{n}{6}\cdotrac{n}{2}\cdotrac{n}{3}\cdot(|q|^{-1}-1)^3$$

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$$|n^2q^n|=n^2|q|^n=n^2\cdotrac{1}{(|q|^{-1})^n}$$

所以
$$|n^2q^n| \leqslant n^2 \frac{36}{n^3} \frac{|q|^3}{(1-|q|^3)}$$

$$\geqslant rac{n}{6} \cdot rac{n}{2} \cdot rac{n}{3} \cdot (|q|^{-1}-1)^3$$

$$= \frac{n^3}{36}(|q|^{-1} - 1)^3 \qquad (n > 3)$$

$$(|q|^{-1})^n\geqslant \frac{n(n-1)(n-2)}{3!}(|q|^{-1}-1)^3 \qquad \geqslant \frac{n}{6}\cdot\frac{n}{2}\cdot\frac{n}{3}\cdot(|q|^{-1}-1)^3$$

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$$\lim_{n o\infty} nq^n=0 \ \ (0<|q|<1)$$
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例7:
$$\lim_{n\to\infty} n^2 q^n = 0, \ 0 < |q| < 1.$$

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当
$$a>1$$
时, $a^n=[1+(a-1)]^n=\sum_{k=0}^n C_n^k(a-1)^k$
$$=1+n(a-1)+\frac{n(n-1)}{2}(a-1)^2+\cdots+(a-1)^n$$

$$\forall k \in \mathbb{N}, \quad \lim_{n \to \infty} n^k q^n = 0, \ 0 < |q| < 1.$$

本段知识要点:

夹逼原理

二项式公式用于不等式放缩

$$\lim_{n o \infty} \sqrt[n]{n} = 1, \ \lim_{n o \infty} n^k q^n = 0, \ (k \in \mathbb{N}, |q| < 1). \ \lim_{n o \infty} (a_1^n + a_2^n + \dots + a_k^n)^{\frac{1}{n}} = \max\{a_1, a_2, \dots, a_k\}.$$

