Lec 8 Existence Thm of linear system [BN 2.2]

Warm up. Existence? f cts

Uniqueness? f Lip & bdd

Extension of soln? A priori estimate. (needs RHS to be bdd)

Thm. $\frac{dy}{dt} = A(t)y + g(t)$ $y(t_0) = y_0$

If $A(t), g(t)$ are cts on some interval $[a,b]$, and $t_0 \in [a,b]$, $y_0 < \infty$.

Then system has a unique soln $\phi(t)$ satisfying $\phi(t_0) = y_0$ and existing on the interval $[a,b]$.

Only need to show $|\phi(t)| \le M$ indep of t. Wait. $\phi(t)$ is a vector, what does $|\phi(t)|$ mean?

Tools (norm) [BN 14, 2.1]

recall when $x \in \mathbb{R}$ (or $\mathbb{C}$), we use $|x|$ to measure the "length" of x.

Idea of norm: want to present "length"

1. **Vector** $\mathbb{R}^n$ n-dim linear (vector) sp over $\mathbb{R}$. (E_n in book)

$\vec{y} \in \mathbb{R}^n$ $\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

- Euclidean norm (2 norm)

$$\|\vec{y}\| = \left[|y_1|^2 + \cdots + |y_n|^2 \right]^{\frac{1}{2}} = \left[\sum_i |y_i|^2 \right]^{\frac{1}{2}} \quad \|\cdot\|_2$$ natural / physical way.

- 1-norm (more convenient for us)

$$|\vec{y}| = |y_1| + \cdots + |y_n| = \sum_i |y_i| \quad \|\cdot\|_1$$

Q: Def Norm is not unique. Generalize? What is a norm, generally speaking?

Norm $\forall x \in X$.

- $\|x\| \ge 0$ and $\|x\| = 0$ iff $x = 0$
- $\|cx\| = |c| \, \|x\|$ $\forall$ c const.
- $\|x_1 + x_2\| \le \|x_1\| + \|x_2\|$, $\forall x_1, x_2 \in X$ (triangle Ineq)

Easy to check: $\|\cdot\|_2$, $\|\cdot\|_1$ are norms.

Rmk: 1. In fact, one can def any p-norm $\|\cdot\|_p = \left(\sum_i |y_i|^p\right)^{1/p}$ $(p \geq 1)$

Even when p is not integer.

In particular, $p=\infty$, $\|\cdot\|_\infty$ is defined as $\max_{1\leq i\leq n} |y_i|$

(one could check the norm definition)

2. For finite-D vector sp, norms are equivalent!

In the sense, $\forall a, b$.

$$C_1\|x\|_b \leq \|x\|_a \leq C_2\|x\|_b \quad \text{for some const } C_1, C_2.$$

Why is this useful? It tells us we could work w/ any norm we want in a priori estimates.

ex: $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are equivalent.

$$\max_{1\leq i\leq n}|y_i| \leq |y_1| + \cdots + |y_n| \leq n \max_{1\leq i\leq n}|y_i|$$

pf $\|\cdot\|_2$ and $\|\cdot\|$ equivalent (HW)

3. In fact, def of norms can also be generalized to "∞-D vector sp", namely, sp of seqs. $y = (y_1, y_2, \cdots)$

Then one can define p-norm. $\|\cdot\|_p = \left(\sum_{i\in\mathbb{N}^+} |y_i|^p\right)^{1/p}$ l^p sp

But note that for "∞-D", we do NOT have the norm equivalence.

4. Metric sp? A norm can be turned into a metric.

(X, d) $d(x,y)$ is metric "induced metric"

One could def $d(x,y) = \|x-y\|$ to construct a metric from norm.

$\Rightarrow$ normed sp is a metric sp.

The converse is not true. In general, a metric can not be turned into a norm.

Metric $d(x,y)$

- $d(x,y) \geq 0$ and $d(x,y)=0$ iff $x=y$
- $d(x,y) = d(y,x)$
- $d(x,y) \leq d(x,z) + d(z,y)$

Ex Discrete metric on $\mathbb{R}$

$$d(x,y) = \begin{cases} 1 & x \neq y. \\ 0 & x = y \end{cases}$$

[Metric satisfying 2 more conditions can have induced norm

$d(x,y) = d(x+z, y+z)$ translation inv.

$d(cx, cy) = |c|\, d(x,y)$. homogeneus.]

can define $\|x\| \triangleq d(x,0)$

2. matrix. $\mathbb{R}^{n\times n}$ or $M_n(\mathbb{R})$

Intuitively, we could reshape a matrix to a vector, s.t. all the $\|\cdot\|_p$ can still be defined.

For us, the most useful one:

$$|A| = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|$$

Properties: $\forall A, B \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$

(i) $|A+B| \le |A| + |B|$

(ii) $|AB| \le |A|\,|B|$ (HW)

☆ (iii) $|Ab| \le |A|\cdot|b|$ (will be used later)

pf of (iii) $Ab = \begin{pmatrix} \sum_{k=1}^{n} a_{1k} b_k \\ \vdots \\ \sum_{k=1}^{n} a_{nk} b_k \end{pmatrix}$

$$|Ab| = \left|\sum_{k=1}^{n} a_{1k} b_k\right| + \cdots + \left|\sum_{k=1}^{n} a_{nk} b_k\right| \le \sum_{i=1}^{n}\sum_{k=1}^{n} |a_{ik}|\,|b_k| = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|\,|b_j|$$

$\le |A|\,|b| = \left(\sum_i\sum_j |a_{ij}|\right)\left(\sum_k |b_k|\right)$ more terms, b/c

fixed i,j, LHS is $|a_{ij}|\,|b_j|$. RHS is $|a_{ij}|\left(\sum_k |b_k|\right)$.

Time to prove a priori estimate.

1-norm

WTS: $|\phi(t)| \le M$ for some const M indep of t.

Pf: $\phi'(t) = A(t)\phi(t) + g(t)$

$\phi(t) = y_0 + \int_{t_0}^{t} A(s)\phi(s)\,ds + \int_{t_0}^{t} g(s)\,ds$ $\qquad t \in [a,b]$

$\Rightarrow |\phi(t)| \le |y_0| + \left|\int_{t_0}^{t} A(s)\phi(s)\,ds\right| + \left|\int_{t_0}^{t} g(s)\,ds\right|$ triangle Ineq (def of norm)

$\le |y_0| + \int_{t_0}^{t} |A(s)|\,|\phi(s)|\,ds + \int_{t_0}^{t} |g(s)|\,ds$ properties of matrix norm.

$|g(s)|$, $s \in [a,b]$ has max, Denote $\max_{s\in[a,b]} |g(s)| = K_0$ Continous fcn on closed interval has max.

$y_0 + \int_{t_0}^{t} |g(s)|\,ds \le |y_0| + K_0(t-t_0)$

$\le |y_0| + K_0(b-a) \triangleq K_1$ const.

scalar (1-D)

$|A(s)|$ $s\in[a,b]$ has max, $\max_{s\in[a,b]}|A(s)| = K_2$

$$\int_{t_0}^{t} |A(s)|\,|\phi(s)|\,ds \le K_2 \int_{t_0}^{t} |\phi(s)|\,ds$$

Hence, we have

$$|\phi(t)| \le K_1 + K_2 \int_{t_0}^{t} |\phi(s)|\,ds.$$

By Gronwall Ineq.

$$|\phi(t)| \le K_1 e^{K_2(t-t_0)} \le K_1 e^{K_2(b-a)}$$ const indep of time! □.

Rmk. The result is true for $I=(a,b)$, $[a,b]$, $(-\infty,b)$, $(a,+\infty)$, $(-\infty,+\infty)$.

namely, needs $A(t)$, $g(t)$ to be defined for I.

IVP admits global soln. if $\forall$ open interval $I \subseteq \mathbb{R}$ s.t.

$t_0 \in I$, $\quad [x\in\mathbb{R}^n : (t,x)\in D] \neq \phi$ for $\forall t\in I$.

$\exists$ a fcn $y: I \to \mathbb{R}^n$ which is a soln of (IVP) on I.

Now know $\exists!$ soln, next Q: what is the structure of soln sp. (is a vector sp) Start w./ homo linear ODE.

"Dictionary" of vector sp. / linear alg.

- linear subspace

  Let S be a set of vectors in $\mathbb{R}^n$, then S is a linear sbsp of $\mathbb{R}^n$ if.

  1. $\forall v_1, v_2 \in S$, $\quad v_1+v_2 \in S$
  2. $\forall v \in S$ and any number c, vector $cv \in S$.

  (closed under the formation of "+" and "c*")

- To Span

  Vectors $v_1, \cdots, v_k$ span a linear sbsp S of $\mathbb{R}^n$ if every vector in S is a linear combo of vectors $v_1, \cdots, v_k$

- linear dep. of vectors

  $v_1, \cdots, v_k$ are linearly dep if $\exists$ $c_1, \cdots, c_k$ not all zero s.t. $c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$.

linear indep. $v_1, \cdots, v_k$ are linearly indep. if the eq $c_1v_1+\cdots+c_kv_k=0$ has only soln $c_1 = \cdots = c_k = 0$.

- Basis

Vectors $v_1, \cdots, v_k$ form a basis for a linear subsp S if
(i) they are indep.
(ii) they span S.

- Dim

Dim of linear subsp $S \in \mathbb{R}^n$ is the number of vectors in a basis of S.

- Eval / E-vec

If $\exists$ vector $\vec{v} \neq 0$ s.t. $A\vec{v} = \lambda\vec{v}$ for some scalar λ.
then λ is called eigenvalue, $\vec{v}$ is called evec associated w/ eigenval. λ.
Note that eigenvalues are roots of characteristic poly $\det(A-\lambda I)$.

Thm. $A \in \mathbb{R}_{n\times n}$. $1 \le k \le n$

- If $v_1, \cdots, v_k$ are e-vecs for a matrix A whose corresponding eval $\lambda_1, \cdots, \lambda_k$ are distinct.
  then $\{v_1, \cdots, v_k\}$ is linearly indep.
- If $n\times n$ matrix A has n distinct e-val, then any set of corresponding e-vectors $\{v_1, \cdots, v_n\}$ is a basis of $\mathbb{R}^n$.

Note that $C(\mathbb{R})$ sp of cts fcn on $\mathbb{R}$ is ∞-dim, since $1, t, \cdots, t^n, \cdots$ is an infinite set in $C(\mathbb{R})$ and linearly indep. on any interval.

- det (Review $\rightarrow$ HW)

If $A = [a_1 \cdots a_n]$
column vectors are linearly indep. $\det A \neq 0$.

HW Review det, AB, Ax.