复合函数及隐函数的高阶偏导数

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解： $\dfrac{1}{2}\ln(x^2+y^2)=\arctan\dfrac{y}{x}$， $\dfrac{1}{2}\cdot\dfrac{2x\,\mathrm{d}x+2y\,\mathrm{d}y}{x^2+y^2}=\dfrac{\frac{x\,\mathrm{d}y-y\,\mathrm{d}x}{x^2}}{1+(\frac{y}{x})^2}$，

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$$=\frac{-2y(x-y)+2x(x+y)}{(x-y)^3}=\frac{2(x^2+y^2)}{(x-y)^3}$$

例 $e^{-xy}-2z+e^{z}=0,\quad z=z(x,y)$,求各二阶偏导.

例 $e^{-xy} - 2z + e^z = 0,\quad z = z(x, y)$,求各二阶偏导.

解:

$$-e^{-xy}(x\,\mathrm{d}y + y\,\mathrm{d}x) - 2\,\mathrm{d}z + e^z\,\mathrm{d}z = 0,$$

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$$\frac{\partial^2 z}{\partial x^2}=\frac{-y^2e^{-xy}(e^{z}-2)-e^{z}\frac{\partial z}{\partial x}ye^{-xy}}{(e^{z}-2)^2}=\cdots,$$

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例 $z=f(u,v)$, $u=\varphi(x,y)$, $v=\psi(x,y)$,求$\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial y^2}$, $\dfrac{\partial^2 z}{\partial x\partial y}$.

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解:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x},$$

例 $z=f(u,v)$, $u=\varphi(x,y)$, $v=\psi(x,y)$,求$\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial y^2}$, $\dfrac{\partial^2 z}{\partial x\partial y}$.

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例 $z=f(u,v),\quad u=\varphi(x,y),\quad v=\psi(x,y)$,求$\dfrac{\partial^2 z}{\partial x^2},\ \dfrac{\partial^2 z}{\partial y^2},\ \dfrac{\partial^2 z}{\partial x\partial y}$.

解:

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x},\qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y},$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial u^2}(\frac{\partial u}{\partial y})^2+2\frac{\partial^2 z}{\partial u\partial v}\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\frac{\partial^2 z}{\partial v^2}(\frac{\partial v}{\partial y})^2+\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2}+\frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\cdots$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 z}{\partial u^2}(\frac{\partial u}{\partial x})^2+2\frac{\partial^2 z}{\partial u\partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial^2 z}{\partial v^2}(\frac{\partial v}{\partial x})^2+\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2}+\frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

例 $z = f(u, v)$, $u = \varphi(x)$, $v = \psi(x)$. 求 $\dfrac{d^2 z}{dx^2}$.

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$$\frac{d^2z}{dx^2}=\frac{\mathrm{d}}{\mathrm{d}x}(\frac{\partial z}{\partial u})\varphi'(x)+\frac{\partial z}{\partial u}\varphi''(x)+\frac{\mathrm{d}}{\mathrm{d}x}(\frac{\partial z}{\partial v})\psi'(x)+\frac{\partial z}{\partial v}\psi''(x)$$

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$$\frac{d^2z}{dx^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial z}{\partial u}\right)\varphi'(x)+\frac{\partial z}{\partial u}\varphi''(x)+\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial z}{\partial v}\right)\psi'(x)+\frac{\partial z}{\partial v}\psi''(x)$$

$$=\boxed{\frac{\partial^2 z}{\partial u^2}\cdot\varphi'(x)^2+\frac{\partial^2 z}{\partial u\partial v}\varphi'(x)\psi'(x)}+\frac{\partial z}{\partial u}\varphi''(x)$$

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$$=\boxed{\frac{\partial^2 z}{\partial u^2}\cdot\varphi'(x)^2+\frac{\partial^2 z}{\partial u\partial v}\varphi'(x)\psi'(x)}+\frac{\partial z}{\partial u}\varphi''(x)$$

$$+\boxed{\frac{\partial^2 z}{\partial v^2}\cdot\psi'(x)^2+\frac{\partial^2 z}{\partial u\partial v}\varphi'(x)\psi'(x)}+\frac{\partial z}{\partial v}\psi''(x)$$

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$$\frac{d^2z}{dx^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial z}{\partial u}\right)\varphi'(x)+\frac{\partial z}{\partial u}\varphi''(x)+\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial z}{\partial v}\right)\psi'(x)+\frac{\partial z}{\partial v}\psi''(x)$$

$$=\boxed{\frac{\partial^2 z}{\partial u^2}\cdot\varphi'(x)^2+\frac{\partial^2 z}{\partial u\partial v}\varphi'(x)\psi'(x)}+\frac{\partial z}{\partial u}\varphi''(x)$$

$$+\boxed{\frac{\partial^2 z}{\partial v^2}\cdot\psi'(x)^2+\frac{\partial^2 z}{\partial u\partial v}\varphi'(x)\psi'(x)}+\frac{\partial z}{\partial v}\psi''(x)$$

$$=\frac{\partial^2 z}{\partial u^2}\cdot\varphi'(x)^2+2\frac{\partial^2 z}{\partial u\partial v}\varphi'(x)\psi'(x)+\frac{\partial^2 z}{\partial v^2}\cdot\psi'(x)^2+\frac{\partial z}{\partial u}\varphi''(x)+\frac{\partial z}{\partial v}\psi''(x)$$

简单情况： 例

$z = f(x,y)$, $x = x_0 + t\cos\alpha$, $y = y_0 + t\sin\alpha$, x_0, y_0, α是常数.

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$$\frac{d^2z}{dx^2}=\frac{\partial^2 z}{\partial u^2}\cdot\varphi'(x)^2+2\frac{\partial^2 z}{\partial u\partial v}\varphi'(x)\psi'(x)+\frac{\partial^2 z}{\partial v^2}\cdot\psi'(x)^2+\frac{\partial z}{\partial u}\varphi''(x)+\frac{\partial z}{\partial v}\psi''(x)$$

$$\frac{\mathrm{d}^2z}{\mathrm{d}t^2}=\cos^2\alpha\frac{\partial^2 z}{\partial x^2}+2\cos\alpha\sin\alpha\frac{\partial^2 z}{\partial x\partial y}+\sin^2\alpha\frac{\partial^2 z}{\partial y^2}$$

简单情况：例

$$z=f(x,y),\quad x=x_0+t\cos\alpha,\quad y=y_0+t\sin\alpha,\quad x_0,y_0,\alpha\text{是常数}.$$

$$z=f(u,v),\quad u=\varphi(x),\quad v=\psi(x).\ \text{求}\ \frac{d^2z}{dx^2}.$$

$$\frac{d^2z}{dx^2}=\frac{\partial^2 z}{\partial u^2}\cdot\varphi'(x)^2+2\frac{\partial^2 z}{\partial u\partial v}\varphi'(x)\psi'(x)+\frac{\partial^2 z}{\partial v^2}\cdot\psi'(x)^2+\frac{\partial z}{\partial u}\varphi''(x)+\frac{\partial z}{\partial v}\psi''(x)$$

$$\begin{aligned}\frac{\mathrm{d}^2z}{\mathrm{d}t^2}&=\cos^2\alpha\frac{\partial^2 z}{\partial x^2}+2\cos\alpha\sin\alpha\frac{\partial^2 z}{\partial x\partial y}+\sin^2\alpha\frac{\partial^2 z}{\partial y^2}\\&=(\cos^2\alpha\frac{\partial^2}{\partial x^2}+2\cos\alpha\sin\alpha\frac{\partial^2}{\partial x\partial y}+\sin^2\alpha\frac{\partial^2}{\partial y^2})z\end{aligned}$$

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$z=f(x,y),\quad x=x_0+t\cos\alpha,\quad y=y_0+t\sin\alpha,\quad x_0,y_0,\alpha$是常数.

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约定算子相乘就是复合!

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$z=f(x,y),\quad x=x_0+t\cos\alpha,\quad y=y_0+t\sin\alpha,\quad x_0,y_0,\alpha$是常数.

$$\boxed{\frac{\partial^n z}{\partial t^n}=(\cos\alpha\frac{\partial}{\partial x}+\sin\alpha\frac{\partial}{\partial y})^n z}$$

$$\begin{aligned}\frac{\mathrm{d}^2 z}{\mathrm{d}t^2}&=\cos^2\alpha\frac{\partial^2 z}{\partial x^2}+2\cos\alpha\sin\alpha\frac{\partial^2 z}{\partial x\partial y}+\sin^2\alpha\frac{\partial^2 z}{\partial y^2}\\&=(\cos^2\alpha\frac{\partial^2}{\partial x^2}+2\cos\alpha\sin\alpha\frac{\partial^2}{\partial x\partial y}+\sin^2\alpha\frac{\partial^2}{\partial y^2})z\\&=(\cos\alpha\frac{\partial}{\partial x}+\sin\alpha\frac{\partial}{\partial y})^2 z\end{aligned}$$

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