Homework Solutions

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The homework §1

§1.1 Normal and Log-normal Random Variables

1. First prove the following equation

$$M_{a+bX}(\vartheta) = e^{\vartheta a} M_X(b\vartheta) \tag{1}$$

According to the definition of moment generating function

$$M_{a+bX}(\vartheta) = Ee^{\vartheta(a+bX)} = E(e^{\vartheta a}e^{\vartheta bX}) = Ee^{\vartheta a}Ee^{\vartheta bX} = e^{\vartheta a}M_X(b\vartheta)$$

Let $X = \mu + \sigma Z, Z \sim N(0,1)$, then

$$M_Z(\vartheta) = Ee^{\vartheta Z} = \int_{-\infty}^{+\infty} e^{\vartheta z} f(z)dz$$
$$= \int_{-\infty}^{+\infty} e^{\vartheta z}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz = e^{\frac{\vartheta^2}{2}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(z-\vartheta)^2}{2}}dz$$
$$= e^{\frac{\vartheta^2}{2}}$$

So that

$$\phi(\vartheta) = M_X(\vartheta) = M_{\mu+\sigma Z}(\vartheta) = e^{\vartheta\mu}M_Z(\sigma\vartheta) = e^{\vartheta\mu}e^{\frac{\sigma^2\vartheta^2}{2}} = exp\{\frac{1}{2}\sigma^2\vartheta^2 + \vartheta\mu\} \tag{2}$$

2. Using equationn (1)and (2)

$$M_{X-\mu}(\vartheta) = e^{-\mu\vartheta}M_X(\vartheta) = e^{-\mu\vartheta}e^{\vartheta\mu}e^{\frac{\sigma^2\vartheta^2}{2}}$$
$$= exp\{\frac{1}{2}\sigma^2\vartheta^2\} = \sum_{k=0}^{\infty}\frac{\vartheta^{2k}\sigma^{2k}}{2^k k!}$$
$$= \sum_{k=0}^{\infty}\frac{\vartheta^{2k}}{(2k)!}\frac{(2k)!}{k!}\frac{\sigma^{2k}}{2^k} \tag{3}$$

One property of moment generating function is

$$E(X^k) = \frac{d^k}{d\vartheta^k}M_X(\vartheta)|_{\vartheta=0} \tag{4}$$

So

$$E(X-\mu)^k = \frac{d^k}{d\vartheta^k}\sum_{k=0}^{\infty}\frac{\vartheta^{2k}}{(2k)!}\frac{(2k)!}{k!}\frac{\sigma^{2k}}{2^k}|_{\vartheta=0}$$

$$= \begin{cases} 0 & \text{for } k=1,3,5\cdots \\ \frac{\sigma^k (k)!}{2^{\frac{k}{2}}\frac{k}{2}!} & \text{for } k=2,4,6\cdots \end{cases}$$

So

$$E(X-EX)^3 = 0$$

$$E(X-EX)^4 = \frac{\sigma^4 4!}{2^2 2!} = 3\sigma^4$$

3. First, get the CDF of log-normally distribution

   For any $y>0$

$$F(y) = Pr(Y \le y) = Pr(lnY \le lny) = \int_{-\infty}^{lny}\frac{1}{\sqrt{2\pi}\sigma}exp\{-\frac{1}{2\sigma^2}(y-\mu)^2\}dy$$

   So the PDF of log-normal distribution is

$$f(y) = \frac{dF(y)}{dy} = \frac{1}{\sqrt{2\pi}\sigma y}exp\{-\frac{1}{2\sigma^2}(ln\ y-\mu)^2\}, y>0$$

   The expectation is

$$EY = \int_0^{+\infty} yf(y)dy = \int_0^{+\infty}\frac{1}{\sqrt{2\pi}\sigma}exp\{-\frac{1}{2\sigma^2}(ln\ y-\mu)^2\}dy$$

   Let$x = ln\ y$

$$EY = \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\sigma}exp\{x-\frac{1}{2\sigma^2}(x-\mu)^2\}dx$$

$$= \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\sigma}exp\{\mu+\frac{\sigma^2}{2}-\frac{1}{2\sigma^2}(x-\mu-\sigma^2)^2\}dx$$

$$= exp\{\mu+\frac{\sigma^2}{2}\}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\sigma}exp\{-\frac{1}{2\sigma^2}(x-\mu-\sigma^2)^2\}dx$$

$$= exp\{\mu+\frac{\sigma^2}{2}\}$$

   The variance is

$$VarY = EY^2 - E^2Y = \int_0^{+\infty}\frac{y}{\sqrt{2\pi}\sigma}exp\{-\frac{1}{2\sigma^2}(ln\ y-\mu)^2\}dy - exp\{2\mu+\sigma^2\}$$

   Let$x = ln\ y$

$$VarY = \int_0^{+\infty}\frac{1}{\sqrt{2\pi}\sigma}exp\{2x-\frac{1}{2\sigma^2}(x-\mu)^2\}dx - exp\{2\mu+\sigma^2\}$$

$$= \int_0^{+\infty}\frac{1}{\sqrt{2\pi}\sigma}exp\{2\mu+2\sigma^2-\frac{1}{2\sigma^2}(x-\mu-2\sigma^2)^2\}dx - exp\{2\mu+\sigma^2\}$$

$$= exp\{2\mu+2\sigma^2\}\int_0^{+\infty}\frac{1}{\sqrt{2\pi}\sigma}exp\{-\frac{1}{2\sigma^2}(x-\mu-2\sigma^2)^2\}dx - exp\{2\mu+\sigma^2\}$$

$$= exp\{2\mu+2\sigma^2\} - exp\{2\mu+\sigma^2\}$$

$$= e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$$

4.

$$E(Y1_{\{Y>K\}}) = \int_{K}^{+\infty} yf(y)dy$$
$$= \int_{K}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} exp\{-\frac{1}{2\sigma^2}(ln\ y-\mu)^2\}dy$$

Let$x = ln\ y$

$$E(Y1_{\{Y>K\}}) = \int_{lnK}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} exp\{x-\frac{1}{2\sigma^2}(x-\mu)^2\}dx$$
$$= exp\{\mu+\frac{\sigma^2}{2}\}\int_{lnK}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} exp\{-\frac{1}{2\sigma^2}(x-\mu-\sigma^2)^2\}dx$$
$$= exp\{\mu+\frac{\sigma^2}{2}\}\int_{-\infty}^{2\mu+2\sigma^2-lnK} \frac{1}{\sqrt{2\pi}\sigma} exp\{-\frac{1}{2\sigma^2}(x-\mu-\sigma^2)^2\}dx$$
$$= e^{\mu+\frac{\sigma^2}{2}}\Phi(\frac{\mu-lnK+\sigma^2}{\sigma})$$

5.

$$E(Y-K)^+ = \int_{K}^{+\infty}(y-K)f(y)dy = \int_{K}^{+\infty} yf(y)dy - K\int_{K}^{+\infty} f(y)dy$$
$$= \int_{K}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} exp\{-\frac{1}{2\sigma^2}(ln\ y-\mu)^2\}dy - K\int_{K}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma y} exp\{-\frac{1}{2\sigma^2}(ln\ y-\mu)^2\}dy$$

Let$x = ln\ y$

$$E(Y-K)^+ = exp\{\mu+\frac{\sigma^2}{2}\}\int_{lnK}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} exp\{-\frac{1}{2\sigma^2}(\mu-x+\sigma^2)^2\}dx - K\int_{lnK}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} exp\{-\frac{1}{2\sigma^2}(x-\mu)^2\}dx$$

$$= e^{\mu+\frac{\sigma^2}{2}}\int_{-\infty}^{2\mu+2\sigma^2-lnK} \frac{1}{\sqrt{2\pi}\sigma} exp\{-\frac{1}{2\sigma^2}(x-\mu-\sigma^2)^2\}dx - K\int_{-\infty}^{2\mu-lnK} \frac{1}{\sqrt{2\pi}\sigma} exp\{-\frac{1}{2\sigma^2}(x-\mu)^2\}dx$$
$$= e^{\mu+\frac{\sigma^2}{2}}\Phi(\frac{\mu-lnK+\sigma^2}{\sigma}) - K\Phi(\frac{\mu-lnK}{\sigma})$$

Use the same method

$$E(K-Y)^+ = \int_{0}^{k}(K-y)f(y)dy = -\int_{0}^{k} yf(y)dy + K\int_{-\infty}^{k} f(y)dy$$

Let$x = ln\ y$

$$E(K-Y)^+ = -exp\{\mu+\frac{\sigma^2}{2}\}\int_{-\infty}^{lnK} \frac{1}{\sqrt{2\pi}\sigma} exp\{-\frac{1}{2\sigma^2}(\mu-x+\sigma^2)^2\}dx + K\int_{-\infty}^{lnk} \frac{1}{\sqrt{2\pi}\sigma} exp\{-\frac{1}{2\sigma^2}(x-\mu)^2\}dx$$

$$= -e^{\mu+\frac{\sigma^2}{2}}\Phi(-\frac{\mu-lnK+\sigma^2}{\sigma}) + K\Phi(-\frac{\mu-lnK}{\sigma})$$

§1.2 Bivariate Normal Variables

X and W are uncorrelated if

$$Cov(X,\ W) = 0$$

Proof:

$$\begin{aligned} Cov(X,\ W) &= E(X)E(W) - E(XW) \\ &= E(X)E(Y - \frac{\rho\sigma_Y}{\sigma_X}X) - E(XY - \frac{\rho\sigma_Y}{\sigma_X}X^2) \\ &= E(X)E(Y) - \frac{\rho\sigma_Y}{\sigma_X}E(X)^2 - E(XY) + \frac{\rho\sigma_Y}{\sigma_X}E(X^2) \\ &= Cov(X,\ Y) - \frac{\rho\sigma_Y}{\sigma_X}Var(X) \\ &= \rho\sigma_Y\sigma_X - \frac{\rho\sigma_Y}{\sigma_X}\sigma_X^2 = 0 \end{aligned}$$

X and W are independent if

$$E(XW) = E(X)E(W)$$

The result I have shown above, so they are independent.

§1.3 Risk Minimization

$$\begin{aligned} Var(w_1R_1 + w_2R_2) &= w_1^2 VarR_1 + w_2^2 VarR_2 + 2w_1w_2\sqrt{Var(R_1)Var(R_2)}Corr(R_1, R_2) \\ &= w_1^2\sigma_1^2 + (1-w_1)^2\sigma_2^2 + 2w_1(1-w_1)\sigma_1\sigma_2\rho \end{aligned}$$

Minimize $Var(w_1R_1 + w_2R_2)$, the first order condition is

$$\frac{\partial Var}{\partial w_1} = 0$$

Solve this equation,we can get the optimal portfolio weight is

$$(w_1 = \frac{\sigma_2^2 - \sigma_1\sigma_2\rho}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}, w_2 = \frac{\sigma_1^2 - \sigma_1\sigma_2\rho}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho})$$

§1.4 Roll a Dice

Payoff has 6 outcome1,2,3,4,5,6.So, first, I calculate the distribution of the payoff.
When payoff is 1, which means all of the three outcome is 1,the probability is

$$Pr(payoff = 1) = (\frac{1}{6})^3 = \frac{1}{216}$$

When payoff is 2, which means at least one outcome is 2,and the rest is 1, the probability is

$$Pr(payoff = 2) = (\frac{1}{6})^3 + (\frac{1}{6})^2 \times \frac{1}{6} \times 3 + (\frac{1}{6}) \times (\frac{1}{6})^2 \times 3 = \frac{7}{216}$$

Use the same method, we can calculate

$$Pr(payoff=3)=(\frac{1}{6})^3+(\frac{1}{6})^2\times\frac{2}{6}\times 3+(\frac{1}{6})\times(\frac{2}{6})^2\times 3=\frac{19}{216}$$

$$Pr(payoff=4)=(\frac{1}{6})^3+(\frac{1}{6})^2\times\frac{3}{6}\times 3+(\frac{1}{6})\times(\frac{3}{6})^2\times 3=\frac{37}{216}$$

$$Pr(payoff=5)=(\frac{1}{6})^3+(\frac{1}{6})^2\times\frac{4}{6}\times 3+(\frac{1}{6})\times(\frac{4}{6})^2\times 3=\frac{61}{216}$$

$$Pr(payoff=6)=(\frac{1}{6})^3+(\frac{1}{6})^2\times\frac{5}{6}\times 3+(\frac{1}{6})\times(\frac{5}{6})^2\times 3=\frac{91}{216}$$

So the distribution is

Payoff	1	2	3	4	5	6
Probability	$\frac{1}{216}$	$\frac{7}{216}$	$\frac{19}{216}$	$\frac{37}{216}$	$\frac{61}{216}$	$\frac{91}{216}$

$$E(Payoff)=\sum_{i=1}^{6} iPr(Payoff=i)=\frac{119}{24}$$

The homework §2

§2.1 Bernoulli Trials and Conditional Expectation

1. $E(X_i|S_n)$ means given the condition that there are S_n outcomes are 1, $n-S_n$ outcomes are 0 in n trials, the expectation of i^{th} trial's outcome.

$$E(X_i|S_n)=1\times Pr(X_i=1|S_n)+0\times Pr(X_i=0|S_n)$$

$$=\frac{Pr(X_i=1,S_n)}{Pr(S_n)}=p\frac{C_{n-1}^{S_n-1}p^{S_n-1}(1-p)^{n-S_n}}{C_n^{S_n}p^{S_n}(1-p)^{n-S_n}}=\frac{S_n}{n}$$

Totally, there are S_n outcomes are 1 in the n trials. So in any trial, the expectation getting 1 is $\frac{S_n}{n}$

2. $E(S_m|S_n)$ means given the condition that there are S_n outcomes are 1, $n-S_n$ outcomes are 0 in n trials,the expectation of the outcome that the first m trials have S_m outcomes are 1, and $m-S_m$ outcomes are 0.

$$E(S_m|S_n)=\sum_{i=0}^{m} iPr(S_m=i|S_n)$$

$$\sum_{i=0}^{m} iPr(S_m=i|S_n)=\sum_{i=0}^{m} i\frac{Pr(S_m=i,S_n)}{Pr(S_n)}$$

$$=\sum_{i=0}^{m} i\frac{C_m^i p^i(1-p)^{m-i}C_{n-m}^{S_n-i}p^{S_n-i}(1-p)^{n-m-S_n+i}}{C_n^{S_n}p^{S_n}(1-p)^{n-S_n}}=\frac{m}{n}S_n$$

Each trail, the expectation is $\frac{S_n}{n}$. Now, we have m trials. It is natural to get the expectation is $\frac{m}{n}S_n$.

§2.2 Conditional Variance Formula

$$\begin{aligned}
&Var[E(X|Y)] + E[Var(X|Y)] \\
&= E[E^2(X|Y)] - E^2[E(X|Y)] + E[E(X^2|Y) - E^2(X|Y)] \\
&= E[E^2(X|Y)] - E^2(X) + E(X^2) - E[E^2(X|Y)] \\
&= -E^2(X) + E(X^2) \\
&= Var(X)
\end{aligned}$$

§2.3 Conditional Distribution from Normal Vectors

According to the definition of conditional probability density function

$$f(x|y) = \frac{f(x,y)}{f(y)}$$

Now

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} exp\{-\frac{1}{2(1-\rho^2)}((\frac{x-\mu_X}{\sigma_X})^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + (\frac{x-\mu_Y}{\sigma_Y})^2)\}$$

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} exp\{-\frac{1}{2\sigma_Y^2}(y-\mu_Y)^2\}$$

So

$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{1}{\sqrt{2\pi}\sigma_X\sqrt{1-\rho^2}} exp[-\frac{1}{2\sigma_X(1-\rho^2)}(x-(\mu_X+\rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)))^2]$$

So given Y, the conditional distribution of X is a normal distribution

$$N(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y), \sigma_X^2(1-\rho^2))$$

§2.4 Poisson Process and Conditional Expectation

$$\begin{aligned}
E(N(1)|N(2)) &= \sum_{i=0}^{N(2)} iPr(N(1)|N(2)) = \sum_{i=0}^{N(2)} i\frac{Pr(N(1),N(2))}{Pr(N(2))} \\
&= \sum_{i=0}^{N(2)} i\frac{\frac{1^i}{i!}e^{-1}\frac{1^{N(i)-i}}{(N(i)-i)!}e^{-1}}{\frac{2^{N2}}{N(2)}e^{-2}} = \sum_{i=1}^{N(2)} C_{N(2)-i}^{i-1}\frac{N(2)}{2^{N(2)}} \\
&= \frac{N(2)}{2^{N(2)}} \times (1+1)^{N(2)-1} \\
&= \frac{N(2)}{2}
\end{aligned}$$

$$\begin{aligned}
E(N(2)|N(1)) &= E(N(2)-N(1)+N(1)|N(1)) \\
&= E(N(2)-N(1)|N(1)) + E(N(1)|N(1)) \\
&= \lambda + E(N(1)) \\
&= N(1)+1
\end{aligned}$$

§2.5 Roll a Dice Again

Using backward induction, the expectation of third roll is 3.5, so if the second roll's outcome is 1, 2 or 3, you should continue to roll; if the outcome is 4, 5, or 6, you should stop immediately. Using this strategy, the distribution of second and third roll is

Payoff	1	2	3	4	5	6
Probability	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

The expectation is 4.25. Using backward induction again, if the first roll's outcome is 1, 2, 3 and 4, you should roll again because you will be better off; otherwise, you should stop immediately.

Using this strategy

$$Pr(payoff = 1) = Pr(payoff = 2) = Pr(payoff = 3) = \frac{4}{6} \times \frac{1}{2} \times \frac{1}{6} = \frac{1}{18}$$

$$Pr(payoff = 4) = \frac{4}{6} \times \frac{1}{6} + \frac{4}{6} + \frac{1}{2} \times \frac{1}{6} = \frac{1}{6}$$

$$Pr(payoff = 5) = Pr(payoff = 6) = \frac{1}{6} + \frac{4}{6} \times \frac{1}{6} + \frac{4}{6} \times \frac{1}{2} \times \frac{1}{6} = \frac{1}{3}$$

So the distribution of payoff is

Payoff	1	2	3	4	5	6
Probability	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$

$$E(payoff) = \sum_{i=1}^{6} iPr(payoff = i) = \frac{14}{3}$$

§2.6 Poisson Process: Conditional Distribution of Arrival Times

Let

$$0 < t_1 < t_2 < \cdots < t_{n+1} = t$$

and h_i which satisfies

$$t_i + h_i < t_{i+1}, i = 1, 2, \cdots, n$$

$$Pr(t_i \leq S_i \leq t_i + h_i, i = 1, 2, \cdots, n | N(t) = n)$$

$$= \frac{\lambda h_1 e^{-\lambda h_1} \lambda h_2 e^{-\lambda h_2} \cdots \lambda h_n e^{-\lambda h_n} e^{-\lambda(t - h_1 - h_2 - \cdots - h_n)}}{\frac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

$$= \frac{n!}{t^n} h_1 h_2 \cdots h_n$$

So

$$\frac{Pr(t_i \leq S_i \leq t_i + h_i, i = 1, 2, \cdots, n | N(t) = n)}{h_1 h_2 \cdots h_n} = \frac{n!}{t^n}$$

Let$h_i \to 0$

We can get the conditional distribution of arrival time $S_1, S_2 \cdots, S_n$ is

$$f(t_1, t_2 \cdots, t_n) = \frac{n!}{t^n}, \quad 0 < t_1 < t_2 < \cdots < t_n$$

The homework §3

§3.1 Random Walk and Martingales

1.

$$\begin{aligned}
& E(M_{n+1} - (n+1)\mu)|\mathcal{F}_n) \\
&= E(M_n + X_{n+1} - n\mu - \mu)|\mathcal{F}_n) \\
&= E(M_n - n\mu|\mathcal{F}_n) + E(X_{n+1} - \mu)|\mathcal{F}_n) \\
&= M_n - n\mu + E(X_{n+1} - \mu)) = M_n - n\mu + \mu - \mu \\
&= M_n - n\mu
\end{aligned}$$

2.

$$\begin{aligned}
& E(M_{n+1}^2 - (n+1)\sigma^2)|\mathcal{F}_n) \\
&= E((M_n + X_{n+1})^2 - (n+1)\sigma^2)|\mathcal{F}_n) \\
&= E(M_n^2 + 2M_n X_{n+1} + X_{n+1}^2 - n\sigma^2 - \sigma^2)|\mathcal{F}_n) \\
&= E(M_n^2 - n\sigma^2|\mathcal{F}_n) + E(2M_n X_{n+1}|\mathcal{F}_n) + E(X_{n+1}^2 - \sigma^2)|\mathcal{F}_n) \\
&= M_n^2 - n\sigma^2 + 2M_n E(X_{n+1}|\mathcal{F}_n) + E(X_{n+1}^2|\mathcal{F}_n) - \sigma^2 \\
&= M_n^2 - n\sigma^2 + Var(X_{n+1}) - E^2(X_{n+1}) - \sigma^2 = M_n^2 - n\sigma^2 + \sigma^2 - \sigma^2 \\
&= M_n^2 - n\sigma^2
\end{aligned}$$

§3.2 Wald Martingale

For $n \geq 1$

$$\begin{aligned}
E(W_{n+1}|\mathcal{F}_n) &= E(\frac{e^{\theta \sum_{j=1}^{n+1} X_j}}{(\phi(\theta))^{n+1}}|\mathcal{F}_n) \\
&= E(\frac{e^{\theta \sum_{j=1}^{n} X_j} \cdot \theta e^{X_{n+1}}}{(\phi(\theta))^{n+1}}|\mathcal{F}_n) \\
&= E(\frac{e^{\theta \sum_{j=1}^{n} X_j}}{(\phi(\theta))^{n+1}}|\mathcal{F}_n) \cdot E(\theta e^{X_{n+1}}|\mathcal{F}_n) = \frac{W_n}{\phi(\theta)} E(\theta e^{X_{n+1}}|\mathcal{F}_n) \\
&= \frac{W_n}{\phi(\theta)} \phi(\theta) = W_n
\end{aligned}$$

For $n = 0$

$$E(W_1|\mathcal{F}_0) = E(\frac{e^{\theta X_1}}{\phi(\theta)}|\mathcal{F}_0)$$
$$= \frac{E(e^{\theta X_1})}{\phi(\theta)} = \frac{\phi(\theta)}{\phi(\theta)} = 1 = W_0$$

§3.3 Compensated Poisson Process as a Martingale

For any $0 \le s \le t \le T$

$$E(N(t) - \lambda t|\mathcal{F}_s) = E(N(t) - N(s) + N(s) - \lambda t|\mathcal{F}_s)$$
$$= E(N(t) - N(s)|\mathcal{F}_s) + N(s) - \lambda t$$
$$= \lambda(t - s) + N(s) - \lambda t$$
$$= N(s) - \lambda s$$

§3.4 Asymmetric Random Walk and Gamblers Problem

1. Let $T_n = (\frac{1-p}{p})^{S_n}$

$$E(T_{n+1}|\mathcal{F}_n) = E((\frac{1-p}{p})^{S_{n+1}}|\mathcal{F}_n) = E((\frac{1-p}{p})^{S_n} \cdot (\frac{1-p}{p})^{X_{n+1}}|\mathcal{F}_n)$$
$$= E((\frac{1-p}{p})^{S_n}|\mathcal{F}_n)E((\frac{1-p}{p})^{X_{n+1}}|\mathcal{F}_n) = T_n(p(\frac{1-p}{p}) + (1-p)(\frac{1-p}{p})^{-1})$$
$$= T_n(1 - p + p) = T_n$$

2. T_n is a martingale, so

$$E((\frac{1-p}{p})^{S_\tau}|\mathcal{F}_0) = (\frac{1-p}{p})^{S_0} = (\frac{1-p}{p})^a$$

So

$$E((\frac{1-p}{p})^{S_\tau}|\mathcal{F}_0) = (\frac{1-p}{p})^N Pr(S_\tau = N) + (\frac{1-p}{p})^0 Pr(S_\tau = 0) = (\frac{1-p}{p})^a$$

We also know that

$$Pr(S_\tau = N) + Pr(S_\tau = 0) = 1$$

Solve equations

$$\begin{cases} (\frac{1-p}{p})^N Pr(S_\tau = N) + (\frac{1-p}{p})^0 Pr(S_\tau = 0) = (\frac{1-p}{p})^a \\ \\ Pr(S_\tau = N) + Pr(S_\tau = 0) = 1 \end{cases}$$

we can get

$$Pr(S_\tau = N) = \frac{1 - (\frac{1-p}{p})^{-a}}{(\frac{1-p}{p})^{N-a} - (\frac{1-p}{p})^{-a}}$$

The homework §4

§4.1 One-Period Binomial Lattice Model

1. Consider a self-financing strategy while invest M in stock market and borrow M from money market. The initial investment is 0, at time 1 we have:

$$\begin{aligned} V_u(1) &= uM-(1+r)M=(u-1-r)M \quad \text{if the stock price rises} \\ V_d(1) &= dM-(1+r)M=(d-1-r)M \quad \text{if the stock price goes down} \end{aligned}$$

   No arbitrage requires that $(u-1-r)(d-1-r)<0$, i.e.

$$0<d<1+r<u$$

2. Consider the following portfolios

   (a) Portfolio 1: One Call option and $\frac{K}{1+r}$ Bond at time 0.

   (b) Portfolio 2: One Put option and S_0 Stock.

   At time 1, the value of Portfolio 1 is

$$P_1 = max(S_1-K,0)+K = max(S_1,K)$$

   While that of Portfolio 2 is

$$P_2 = max(K-S_1,0)+S_1 = max(K,S_1)$$

   Since $P_1=P_2$, and they should be equal to each other at time 0, i.e.

$$C_0+\frac{K}{1+r}=P_0+S_0 \Rightarrow C_0-P_0=S_0-\frac{K}{1+r}$$

3. From the Put-Call parity, we have

$$C_0+\frac{K}{1+r}=P_0+S_0$$

   However $C_0^{'}<C_0$, namely, $C_0^{'}+\frac{K}{1+r}<P_0+S_0$. We do as the followings:

   (a) Short one Put option and one stock and get P_0+S_0

   (b) Get one Call option and $\frac{K}{1+r}$ Bonds, the paid $C_0^{'}+\frac{K}{1+r}$

   (c) Hold $P_0+S_0-C_0^{'}+\frac{K}{1+r}$ Bond and paid $P_0+S_0-C_0^{'}+\frac{K}{1+r}$

   The initial investment is 0 at time 0. At time 1, we have different payoffs with respect to the three circumstances:

   (a) Short one Put option and one stock engender the cash outflow $max(K,S_1)$

   (b) One Call option and $\frac{K}{1+r}$ Bonds derives $max(S_1,K)$ cash inflow

   (c) $P_0+S_0-C_0^{'}+\frac{K}{1+r}$ Bond and get $(P_0+S_0-C_0^{'}+\frac{K}{1+r})(1+r)$ cash inflow

   Therefore the net revenue at time 1 is$(P_0+S_0-C_0^{'}+\frac{K}{1+r})(1+r)>0$, arbitrage.

§4.2 Multi-Period Binomial Lattice Model

1.
$$\widetilde{p} = \frac{1+r-d}{u-d} = \frac{1+0-\frac{1}{2}}{2-\frac{1}{2}} = \frac{1}{3}$$
$$V_2(HH) = (4\times 2\times 2-6)^+ = 10, \quad Q(HH) = \widetilde{p}^2 = \frac{1}{9}$$
$$V_2(HT) = V_2(TH) = (4\times 2\times \frac{1}{2}-6)^+ = 0, \quad Q(HT) = Q(TH) = \widetilde{p}(1-\widetilde{p}) = \frac{2}{9}$$
$$V_2(TT) = (4\times \frac{1}{2}\times \frac{1}{2}-6)^+ = 0, \quad Q(TT) = (1-\widetilde{p})^2 = \frac{4}{9}$$
$$V_0 = (\frac{1}{1+r})^2 E^Q V_N = 10\times\frac{1}{9} + 0\times\frac{2}{9} + 0\times\frac{2}{9} + 0\times\frac{4}{9} = \frac{10}{9}$$

2.
$$\Delta_1(H) = \frac{V_2(HH)-V_2(HT)}{S_2(HH)-S_2(HT)} = \frac{10-0}{16-4} = \frac{5}{6}$$
$$\Delta_1(T) = \frac{V_2(TH)-V_2(TT)}{S_2(TH)-S_2(TT)} = \frac{0-0}{4-1} = 0$$
$$\Delta_0 = \frac{X_1(H)-X_1(T)}{S_1(H)-S_1(T)} = \frac{\frac{1}{3}(16-6)^+ + (1-\frac{1}{3})(4-6)^+ - \frac{1}{3}(4-6)^+ + (1-\frac{1}{3})(1-6)^+}{8-2} = \frac{\frac{10}{3}}{6} = \frac{5}{9}$$

At time i, we hedge with the following strategies:

(a) Hold Δ_i stock

(b) Invest $X_i - \Delta_i S_i$ in money market

At time 0, we need to hold $\frac{5}{9}$ stock and borrow $\frac{10}{9}$
At time 1, if $S_1 = S_1(H)$ we increase our share to $\frac{5}{6}$ and borrow $\frac{10}{3}$
If $S_1 = S_1(T)$ at time 0, we hold 0 stock and borrow nothing.

3. It is an open question. The higher p in our formulae seems irrelevant to the call option price, however, it is notable that we should consider people's irrationalities; you can explain from the perspective of behavioral finance or practice.

The homework §5

§5.1 Scaling Property of Brownian Motion

$B(t)$ satisfies

1.
$$B(0) = \frac{1}{\sqrt{a}} W(0) = 0$$

2. For each $\omega \in \Omega W(t)(\omega)$ is a continuous function of $t > 0$, so $B(t)$ is also a continuous function.

3.
$$B(t) - B(s) = \frac{1}{\sqrt{a}}(W(at) - W(as))$$
Because $W(at) - W(as) \sim N(0, a(t-s))$
$$B(t) - B(s) \sim N(0, \frac{a(t-s)}{a}) = N(0, t-s)$$

4. For all $0 = t_0 < t_1 < \cdots < t_m$, the increments $B(t_1) - B(t_0), B(t_2) - B(t_1), \cdots B(t_m) - B(t_{m-1})$ are independent.

§5.2 Finite Dimensional Distribution of a Brownian Motion

$$X = w(s_1) \sim N(0, s_1)$$
$$Y = w(s_2) - w(s_1) \sim N(0, s_2 - s_1)$$

Now let
$$\begin{cases} U = X = w(s_1) \\ V = Y + X = w(s_2) \end{cases}$$

We need to get the joint distribution of (U,V)

First, we can get
$$\begin{cases} x = u \\ y = v - u \end{cases}$$

So the $J = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, $|det(J)| = 1$ We know that
$$f_{x,y} = p(s_1, 0, x_1)p(s_2 - s_1, x_2, 0)$$

So the distribution of (U, V)
$$f_{u,v} = |det(J)|f_{x,y}(x(u,v), y(u,v)) = p(s_1, 0, y_1)p(s_2 - s_1, y_2 - y_1, 0)$$

§5.3 Brownian Motion with Drift

1.
$$Corr(X_t, X_s) = \frac{Cov(X_t, X_s)}{\sqrt{Var(X_t)Var(X(s))}}$$
$$= \frac{E(X_tX_s) - E(X_tE(X_s))}{\sigma^2\sqrt{ts}} = \frac{\sigma^2 min\{t, s\}}{\sigma^2\sqrt{ts}}$$
$$= min\{\sqrt{\frac{t}{s}}, \sqrt{\frac{s}{t}}\}$$

2. Let $0 = t_0 < t_1 < \cdots < t_n = t$

$$[X, X](t) = lim_{||\Pi||\to 0} \sum_{i=1}^{n} [X(t_i) - X(t_{i-1})]^2$$

$$= lim_{||\Pi||\to 0} \sum_{i=1}^{n} [\sigma^2 (W(t_i - W(t_{i-1})^2) + 2\sigma\mu(W(t_i - W(t_{i-1})(t_i - t_{i-1}) + \mu^2(t_i - t_{i-1})^2]$$

$$= \sigma^2 t$$

§5.4 Geometric Brownian Motion

1.

$$Pr(S(t) > K) = Pr(S_0 exp\{\sigma(W(t) + G(t)\})$$
$$= Pr(W(t) > \frac{lnK - lnS_0 - G(t)}{\sigma})$$

We know that $W_t \sim N(0, t)$ So

$$Pr(W(t) > \frac{lnK - lnS_0 - G(t)}{\sigma}) = 1 - \Phi(\frac{lnK - lnS_0 - G(t)}{\sigma\sqrt{t}})$$

2. For $0 \leq s < t$, if S(t) is a martingale, S(t) should satisfy

$$E(S(t)|W(s)) = S(s)$$

So

$$E(exp\sigma(W(t) - W(s)) + G(t) - G(s)) = 1$$
$$e^{G(t)-G(s)} = E(exp{-}\sigma(W(t) - W(s)) = exp{-}\frac{\sigma^2(t-s)}{2}$$

So

$$G(t) = -\frac{\sigma^2 t}{2} + C$$

§5.5 Multidimensional Brownian Motion

To construct a three-demensional correlated BM, we Let $\begin{pmatrix} W_1(t) \\ W_2(t) \\ W_3(t) \end{pmatrix} = A \begin{pmatrix} Z_1(t) \\ Z_2(t) \\ Z_3(t) \end{pmatrix}$ Since we have known the correlations

$$Corr \begin{pmatrix} W_1(t) \\ W_2(t) \\ W_3(t) \end{pmatrix} = Acorr \begin{pmatrix} Z_1(t) \\ Z_2(t) \\ Z_3(t) \end{pmatrix} A^t = AA^t = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix}$$

Solve the equation, we can get

$$A = \begin{pmatrix} 1 & 0 & 0 \\ \rho_{12} & \sqrt{1-\rho_{12}^2} & 0 \\ \rho_{13} & \frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}} & \frac{\sqrt{1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23}}}{\sqrt{1-\rho_{12}^2}} \end{pmatrix}$$

The homework §6

§6.1 Shreve Vol. II Exercise 4.5

1. The differential form of Itô Formula is

$$\mathrm{d}f(t,W(t)) = f_t(t,W(t))\mathrm{d}t + f_x(t,W(t))\mathrm{d}W(t) + \frac{1}{2}f_{xx}(t,W(t))\mathrm{d}t.$$

So

$$\begin{aligned}\mathrm{d}\log S(t) &= \frac{1}{S(t)}\mathrm{d}t - \frac{1}{2S(t)^2}\mathrm{d}S(t)\mathrm{d}S(t)\\ &= \frac{1}{2S(t)^2}\left(2S(t)\mathrm{d}S(t) - \mathrm{d}S(t)\mathrm{d}S(t)\right)\\ &= \frac{1}{2S(t)^2}\left(2S(t)(\alpha(t)S(t)\mathrm{d}t + \sigma(t)S(t)\mathrm{d}W(t)) - \sigma(t)^2S(t)^2\right)\\ &= \sigma(t)\mathrm{d}W(t) + \left(\alpha(t) - \frac{\sigma(t)^2}{2}\right)\mathrm{d}t.\end{aligned}$$

2.

$$\log S(t) = \log S(0) + \int_0^t \sigma(s)\mathrm{d}W(s) + \int_0^t\left(\alpha(s) - \frac{\sigma(s)^2}{2}\right)\mathrm{d}s.$$

Then we have

$$S(t) = S(0)\exp\left\{\int_0^t \sigma(s)\mathrm{d}W(s) + \int_0^t\left(\alpha(s) - \frac{\sigma(s)^2}{2}\right)\mathrm{d}s\right\}.$$

§6.2 Continuing with Shreve Vol. II Exercise 4.5

$$\begin{aligned}\mathrm{d}\left(\frac{1}{S(t)}\right) &= -\frac{1}{S(t)^2}\mathrm{d}S(t) + \frac{1}{2}\frac{2}{S(t)^3}\sigma t^2 S(t)^2\mathrm{d}t\\ &= \frac{(\sigma(t)^2 - \alpha(t))\mathrm{d}t - \sigma(t)\mathrm{d}W(t)}{S(t)}.\end{aligned}$$

§6.3 Shreve Vol. II Exercise 4.6

$$\mathrm{d}S(t) = \sigma S(t)\mathrm{d}W(t) + \left(\alpha - \frac{\sigma^2}{2}\right)S(t)\mathrm{d}t + \frac{\sigma^2}{2}S(t)\mathrm{d}t = \alpha S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t).$$

So

$$\begin{aligned}\mathrm{d}S^p(t) &= pS^{p-1}(t)\mathrm{d}S(t) + \frac{1}{2}p(p-1)S^{p-2}(t)\sigma^2S(t)^2\mathrm{d}t\\ &= pS^{p-1}(t)\left(\alpha S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t)\right) + \frac{1}{2}p(p-1)S^{p-2}(t)\sigma^2S(t)^2\mathrm{d}t\\ &= pS^p(t)\left(\sigma\mathrm{d}W(t) + \alpha + \frac{p-1}{2}\sigma^2\mathrm{d}t\right).\end{aligned}$$

§6.4 Shreve Vol. II Exercise 4.7

1.
$$\mathrm{d}W(t) = 4W^3(t)\mathrm{d}W(t) + \frac{1}{2} \times W^2(t)\mathrm{d}t = 4W^3(t)\mathrm{d}t + 6W^2(t)\mathrm{d}t.$$

And
$$W^4(t) = 4\int_0^T W^3(t)\mathrm{d}W(t) + 6\int_0^T W^2(t)\mathrm{d}t.$$

2.
$$E\left(W^4(t)\right) = E\left(4\int_0^T W^3(t)\mathrm{d}W(t) + 6\int_0^T W^2(t)\mathrm{d}t\right) = 6\int_0^T t\mathrm{d}t = 3T^2.$$

3. Use the same method
$$\mathrm{d}W^6(t)6W^5(t)\mathrm{d}W(t) + \frac{1}{2} \times 6 \times W^4(t)\mathrm{d}t = 6W^5(t)\mathrm{d}W(t) + 15W^4(t)\mathrm{d}t.$$

So
$$W^6(T) = 6\int_0^T W^5(t)\mathrm{d}W(t) + 15\int_0^T W^4(t)\mathrm{d}t,$$
$$E(W^6(t)) = 15\int_0^T 3t^2\mathrm{d}t = 15T^3.$$

§6.5 Shreve Vol. II Exercise 4.15

1. $B_i(t) = \sum_{j=1}^d \int_0^t \frac{\sigma_{ij}(t)}{\sigma_i(t)}\mathrm{d}W_j(t)$

   (1) $B_i(0) = 0$;

   (2) Since $\int_0^t \frac{\sigma_{ij}(t)}{\sigma_i(t)}\mathrm{d}W_j(t)$ is *Itô* integral, $B_i(t)$ is continuous;

   (3) Since $\int_0^t \frac{\sigma_{ij}(t)}{\sigma_i(t)}\mathrm{d}W_j(t)$ is martingale, $B_i(t)$ is also a martingale;

   (4) Since $W_i(t)$ are independent,
$$\mathrm{d}B_i(t)\mathrm{d}B_i(t) = \left[\sum_{j=1}^d \frac{\sigma_{ij}(t)}{\sigma_i(t)}\mathrm{d}W_j(t)\right]^2 = \sum_{j=1}^d \left[\frac{\sigma_{ij}(t)}{\sigma_i(t)}\right]^2 \mathrm{d}W_j(t)\mathrm{d}W_j(t) = \sum_{j=1}^d \left[\frac{\sigma_{ij}(t)}{\sigma_i(t)}\right]^2 \mathrm{d}t = \mathrm{d}t.$$

   According to *Lévy* theorem, $B_i(t)$ is Brownian motion.

2.
$$\mathrm{d}B_i(t)\mathrm{d}B_k(t) = \left[\sum_{j=1}^d \frac{\sigma_{ij}(t)}{\sigma_i(t)}\mathrm{d}W_j(t)\right]\left[\sum_{j=1}^d \frac{\sigma_{kj}(t)}{\sigma_k(t)}\mathrm{d}W_j(t)\right]$$
$$= \sum_{j=1}^d \sum_{l=1}^d \frac{\sigma_{ij}(t)\sigma_{kl}(t)}{\sigma_i(t)\sigma_k(t)} = \rho_{ik}(t)dt.$$

§6.6 An Integral

1. $\mathrm{d}[tW(t)] = W(t)\mathrm{d}t + t\mathrm{d}W(t)$,

$$J(t) = \int_0^t W(s)\mathrm{d}s = tW(t) - \int_0^t s\mathrm{d}W(s) = tW(t) - I(t).$$

For $\forall t > 0$, $W(t) \sim N(0,t)$, hence $tW(t) \sim N\left(0, t^3\right)$.

$$I(t) = \int_0^t s\mathrm{d}W(s) \sim N\left(0, \int_0^t s^2\mathrm{d}s\right) = N\left(0, \frac{1}{3}t^3\right).$$

Therefore, $J(t)$ is the difference between two normal distribution variables, and $J(t)$ is also normal.

2. (By **Yuchi Zhang**) $E\int_0^t W_s\mathrm{d}s = \int_0^t EW_s\mathrm{d}s = 0$
$We\ know\ that\ :\ E\left[B_sB_u\right] = s \wedge u$
$E\left[\int_0^t B_s\mathrm{d}s\right]^2 = E\left[\int_0^t B_s\mathrm{d}s \times \int_0^t B_u\mathrm{d}u\right] = E\int_0^t \left[\int_0^t B_uB_s\mathrm{d}u\right]\mathrm{d}s = \int_0^t\int_0^t E\left[B_uB_s\right]\mathrm{d}u\mathrm{d}s$
$= \int_0^t\int_0^t s \wedge u\mathrm{d}u\mathrm{d}s = \int_0^t \mathrm{d}u\int_0^u s\mathrm{d}s + \int_0^t \mathrm{d}s\int_0^s u\mathrm{d}u = \frac{1}{3}t^3$
$So\ we\ get:\ \int_0^t W_s\mathrm{d}s\ \sim N\left(0, \frac{1}{3}t^3\right)$.

Another method for calculating $Var(J(t))$: (By **TA Moren Gao**)

$$Var\left(J(t)\right) = Var\left(tW(t) - I(t)\right) = t^2Var(W(t)) + Var(I(t)) - 2t\mathbb{E}(W(t)I(t))$$
$$= t^3 + \frac{1}{3}t^3 - 2t\mathbb{E}(W(t)I(t))$$

To calculate the covariance $\mathbb{E}(W(t)I(t))$, utilizing $It\hat{o}$'s lemma, we have

$$\mathrm{d}(W(t)I(t)) = I(t)\mathrm{d}W(t) + W(t)\mathrm{d}I(t) + \mathrm{d}W(t)\mathrm{d}I(t) = I(t)\mathrm{d}W(t) + W(t)t\mathrm{d}W(t) + \mathrm{d}W(t)\cdot t\mathrm{d}W(t)$$
$$= t\mathrm{d}t + [I(t) + tW(t)]\mathrm{d}W(t)$$

Hence, $W(t)I(t) = \int_0^t s\mathrm{d}s + \int_0^t \left[I(s) + sW(s)\right]\mathrm{d}W(s)$, and then $\mathbb{E}W(t)I(t) = \int_0^t s\mathrm{d}s = \frac{1}{2}t^2$.

Therefore, $Var(J(t)) = t^3 + \frac{1}{3}t^3 - 2t\cdot\frac{1}{2}t^2 = \frac{1}{3}t^3$.

The homework §7

§7.1 The Geometric Mean-Reversion Process

1.
$$\mathrm{d}(e^{kt}X(t)) = e^{kt}\mathrm{d}X(t) + ke^{kt}X(t)\mathrm{d}t = e^{kt}\left(k\theta - \frac{\sigma^2}{2} - kX(t)\right)\mathrm{d}t + ke^{kt}X(t)\mathrm{d}t + e^{kt}\sigma\mathrm{d}W(t)$$
$$= e^{kt}\left(k\theta - \frac{\sigma^2}{2}\right)\mathrm{d}t + e^{kt}\sigma\mathrm{d}W(t)$$
Since
$$e^{kt}X(t) - X(0) = \int_0^t e^{ku}\left(k\theta - \frac{\sigma^2}{2}\right)\mathrm{d}u + \int_0^t e^{ku}\sigma\mathrm{d}W(u)$$
$$= \frac{\left(k\theta - \frac{\sigma^2}{2}\right)(e^{kt}-1)}{k} + \int_0^t e^{ku}\sigma\mathrm{d}W(u)$$
So
$$S(t) = e^{X(t)} = exp\left\{e^{-kt}logS(0) + \left(\theta - \frac{\sigma^2}{2k}\right)(1-e^{-kt}) + \int_0^t e^{k(u-t)}\sigma dW(u)\right\}$$

2. According to the property of Itô Integrate, $\log S(t)$ is normal distribution
$$\mathbb{E}(\log S(t)) = e^{-kt}\log S(0) + \left(\theta - \frac{\sigma^2}{2k}\right)(1-e^{-kt})$$
$$Var(\log S(t)) = \int_0^t e^{2k(u-t)}\sigma^2 du = \frac{\sigma^2}{2k}(1-e^{-2kt})$$
So S(t) is log-normal distribution and
$$\mathbb{E}(S(t)) = \exp\left\{e^{-kt}\log S(0) + \left(\theta - \frac{\sigma^2}{2k}\right)(1-e^{-kt}) + \frac{\sigma^2}{4k}(1-e^{-2kt})\right\}$$

§7.2 A Double Mean-reversion Model

1. pass

2. Let $\mathrm{d}\widetilde{W}_2(t) = \rho\mathrm{d}W_1(t) + \sqrt{1-\rho^2}\mathrm{d}W_2(t)$, then
$$\mathrm{d}e^{\kappa_2 t}X_2(t) = \kappa_2 e^{\kappa_2 t}X_2(t)dt + e^{\kappa_2 t}\mathrm{d}X_2(t)$$
$$= e^{\kappa_2 t}\left\{\kappa_2\theta_2\mathrm{d}t + \sigma_2\mathrm{d}\widetilde{W}_2(t)\right\}$$
$$X_2(t) = x_2 e^{-\kappa_2 t} + \theta_2(1-e^{-\kappa_2 t}) + \sigma_2\int_0^t e^{\kappa_2(s-t)}\mathrm{d}\widetilde{W}_2(s)$$
Similarly,
$$X_1(t) = x_1 e^{-\kappa_1 t} + e^{-\kappa_1 t}\int_0^t e^{\kappa_1 s}X_2(s)\mathrm{d}s + e^{-\kappa_1 t}\int_0^t e^{\kappa_1 s}\mathrm{d}W_1(s)$$
$$= x_1 e^{-\kappa_1 t} + \frac{\kappa_1}{\kappa_1-\kappa_2}(x_2-\theta_2)(e^{-\kappa_2 t} - e^{-\kappa_1 t}) + \theta_2(1-e^{-\kappa_1 t})$$
$$+ \frac{\kappa_1}{\kappa_1-\kappa_2}\sigma_2\int_0^t e^{\kappa_2(s-t)}(1-e^{(\kappa_1-\kappa_2)(s-t)})\mathrm{d}\widetilde{W}_2(s)$$
$$+ \sigma_1\int_0^t e^{\kappa_1(s-t)}\mathrm{d}W_1(s)$$

§7.3 Correlated Assets

1. According to Itô formula,

$$\mathrm{d}\log S_i(t) = \frac{1}{S_i(t)}\mathrm{d}S_i(t) + \frac{1}{2}\left(-\frac{1}{S_i(t)^2}\right)\sigma_i^2 S_i(t)\mathrm{d}t$$

$$= \mu_i\mathrm{d}t + \sigma_i\mathrm{d}W_i(t) - \frac{1}{2}\sigma_i^2\mathrm{d}t = \left(\mu_i - \frac{1}{2}\sigma_i^2\right)\mathrm{d}t + \sigma_i\mathrm{d}W_i(t)$$

Then integrate

$$S_i(t) = S_i(0)\exp\left\{\left(\mu_i - \frac{1}{2}\sigma_i^2\right)t + \sigma_i W_i(t)\right\}$$

2. Let $\Sigma = (\rho_{ij})_{nm}$, there exist a lower triangular matrix $A = (a_{ij})_{nn}$ and $AA^t = \Sigma$ and

$$\begin{pmatrix} W_1(t) \\ . \\ . \\ W_n(t) \end{pmatrix} = A \begin{pmatrix} B_1(t) \\ . \\ . \\ B_n(t) \end{pmatrix}$$

where $\begin{pmatrix} B_1(t) \\ . \\ . \\ B_n(t) \end{pmatrix}$ is n dimension standard BM

$$\mathrm{d}W_i(t) = \sum_{j=1}^{i} a_{ij}\mathrm{d}B_j(t)$$

$$\mathrm{d}S_i(t) = \mu_i S_i(t)\mathrm{d}t + \sigma_i S_i(t)\sum_{j=1}^{i} a_{ij}\mathrm{d}B_j(t)$$

So

$$S_i(t) = S_i(0)exp\{(\mu_i - \frac{\sigma_i}{2}t + \sigma_i S_i(t)\sum_{j=1}^{i} a_{ij}B_j(t)\}$$

3. When Δt is small, we have

$$S_i(\Delta t) - S_i(0) = \mu_i S_i(0)\Delta t + \sigma_i S_i(0)W_i(\Delta t)$$

$$S_j(\Delta t) - S_j(0) = \mu_j S_j(0)\Delta t + \sigma_j S_j(0)W_j(\Delta t)$$

So

$$Cov(S_i(\Delta t) - S_i(0), S_j(\Delta t) - S_j(0)) = \sigma_i\sigma_j S_0 S_j(0)W_i(\Delta t)W_j(\Delta t)$$

$$Var(S_i(\Delta t) - S_i(0)) = \sigma_i^2 S_i(0)^2 Var(W_i(\Delta t))$$

$$lim_{\Delta t\to 0}Corr(S_i(\Delta t) - S_i(0), S_j(\Delta t) - S_j(0)) = lim_{\Delta t\to 0}\frac{Cov(S_i(\Delta t) - S_i(0), S_j(\Delta t) - S_j(0))}{\sqrt{Var(S_i(\Delta t) - S_i(0))Var(S_j(\Delta t) - S_j(0))}}$$

$$= \rho_{ij}$$

The homework §8

§8.1 Change of Discounted Values

Proof. Since

$$\mathrm{d}X(t) = \Delta(t)\mathrm{d}S(t) + r(X(t) - \Delta(t)S(t))\mathrm{d}t$$

$$\mathrm{d}[e^{-rt}X(t)] = -re^{-rt}X(t)\mathrm{d}t + e^{-rt}\mathrm{d}X(t) = -\Delta(t)re^{-rt}S(t)\mathrm{d}t + \Delta(t)e^{-rt}\mathrm{d}S(t) = \Delta(t)\mathrm{d}[e^{-rt}S(t)]$$

§8.2 Verification for Black-Scholes-Merton (1973)

Proof. We construct a self-financing portfolio with values $X(t) = c(t, S(t))$. $\Delta(t) = c_x(t, S(t))$, we have

$$\begin{aligned}
&\mathrm{d}[e^{-rt}X(t)] \\
=&\Delta(t)\mathrm{d}[e^{-rt}S(t)] \\
=&c_x(t, S(t))e^{-rt}\left[\mathrm{d}S(t) - rS(t)\mathrm{d}t\right] \\
=&c_x(t, S(t))S(t)e^{-rt}\left[(\alpha - r)\mathrm{d}t + \sigma \mathrm{d}W(t)\right]
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\mathrm{d}[e^{-rt}c(t, S(t))] = e^{-rt}\left[\mathrm{d}c(t, S(t)) - rc(t, S(t))\mathrm{d}t\right] \\
=&e^{-rt}\left[c_t\mathrm{d}t + c_x\mathrm{d}S(t) + \frac{1}{2}c_{xx}\mathrm{d}[S, S]_t - rc(t, S(t))\mathrm{d}t\right] \\
=&e^{-rt}\left[c_t + \alpha c_x S(t) + \frac{1}{2}\sigma^2 c_{xx}S(t)^2 - rc\right]\mathrm{d}t + e^{-rt}\sigma c_x S(t)\mathrm{d}W(t) \\
&\text{(using BSM PDE)} \\
=&e^{-rt}S(t)c_x\left[(\alpha - r)\mathrm{d}t + \sigma \mathrm{d}W(t)\right]
\end{aligned}$$

Therefore, we have $\mathrm{d}[e^{-rt}X(t)] = \mathrm{d}[e^{-rt}c(t, S(t))]$.
Which implies $e^{-rt}X(t) - X(0) = e^{-rt}c(t, S(t)) - c(0, S(0))$, that is $X(t) = c(t, S(t))$.

§8.3 Understanding Options Return

By Itô's formula,

$$\begin{aligned}
c(t+\Delta, S(t+\Delta)) - c(t, S(t)) &= \int_t^{t+\Delta} c_t(u, S_u)\mathrm{d}u + \int_t^{t+\Delta} c_x(u, S_u)\mathrm{d}S_u + \frac{1}{2}\int_t^{t+\Delta} c_{xx}(u, S_u)\mathrm{d}S_u\mathrm{d}S_u \\
&= \int_t^{t+\Delta}(c_t(u, S_u) + \alpha S_u c_x(u, S_u) + \frac{1}{2}\sigma^2 S_u^2 c_{xx}(u, S_u))\mathrm{d}u + \int_t^{t+\Delta}\sigma S_u \mathrm{d}W_u
\end{aligned}$$

$$\lim_{\Delta\to 0}\frac{1}{\Delta}\frac{\mathbb{E}[c(t+\Delta,S(t+\Delta))-c(t,S(t))|\mathcal{F}_t]}{c(t,S(t))}-r=\lim_{\Delta\to 0}\frac{1}{\Delta}\frac{\mathbb{E}[\int_t^{t+\Delta}(c_t(u,S_u)+\alpha S_u c_x(u,S_u)+\frac{1}{2}\sigma^2 S_u^2 c_{xx}(u,S_u))\mathrm{d}u|\mathcal{F}_t]}{c(t,S(t))}-r$$
$$=\frac{c_t(t,S_t)+\alpha S_t c_x(t,S_t)+\frac{1}{2}\sigma^2 S_t^2 c_{xx}(t,S_t)-rc(t,S(t))}{c(t,S(t))}$$
$$=\frac{\alpha S_t c_x(t,S_t)-rS_t c_x(t,S_t)}{c(t,S(t))}$$
$$=\frac{c_x(t,S_t)\sigma S_t}{c(t,S_t)}\lambda$$

§8.4 Option Pricing with the Local Volatility Model

Let $V(t)=X(t)=c(t,S(t)),t\in[0,T]$, on one hand,

$$\begin{aligned}\mathrm{d}X(t)&=\Delta(t)\mathrm{d}S(t)+r(X(t)-\Delta(t)S(t))\mathrm{d}t\\&=rX(t)\mathrm{d}t+\Delta(t)(\mu-r)S(t)\mathrm{d}t+\Delta(t)\sigma(S(t))S(t)\mathrm{d}W(t)\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathrm{d}c(t,S(t))&=c_t\mathrm{d}t+c_x\mathrm{d}S(t)+\frac{1}{2}c_{xx}\mathrm{d}S(t)\mathrm{d}S(s)\\&=[c_t+\mu S(t)c_x+\frac{1}{2}\sigma^2(S(t))S(t)^2c_{xx}]\mathrm{d}t+\sigma(S(t))S(t)c_x\mathrm{d}W(t)\end{aligned}$$

Since $\mathrm{d}X(t)=\mathrm{d}c(t,S(t))$ and let $S(t)=x$, we get the equation with terminal condition,

$$\begin{cases}rc(t,x)=c_t(t,x)+rc_x(t,x)+\frac{1}{2}\sigma^2(x)x^2c_{xx}(t,x)\\c(T,x)=(K-x)^+\end{cases}$$

§8.5 Exchange Option

Let $V(t)=X(t)=c(t,S_1(t),S_2(t)),t\in[0,T]$, on one hand,

$$\begin{aligned}\mathrm{d}X(t)&=\Delta_1(t)\mathrm{d}S_1(t)+\Delta_2(t)\mathrm{d}S_2(t)+r(X(t)-\Delta_1(t)S_1(t)-\Delta_2(t)S_2(t))\mathrm{d}t\\&=rX(t)\mathrm{d}t+[\Delta_1(t)(\mu_1-r)S_1(t)+\Delta_2(t)(\mu_2-r)S_2(t)]\,\mathrm{d}t\\&\quad+\Delta_1(t)\sigma_1S_1(t)\mathrm{d}W_1(t)+\Delta_2(t)\sigma_2S_2(t)\mathrm{d}W_2(t)\end{aligned}$$

On the other hand,

$$\begin{aligned}&\mathrm{d}c(t,S_1(t),S_2(t))\\&=\left[c_t+c_{x_1}\mu_1x_1+c_{x_2}\mu_2x_2+\frac{1}{2}c_{x_1x_1}\sigma_1^2S_1(t)^2+\frac{1}{2}c_{x_2x_2}\sigma_2^2S_2(t)^2+c_{x_1x_2}\sigma_1\sigma_2S_1(t)S_2(t)\rho\right]\mathrm{d}t\\&\quad+c_{x_1}\sigma_1S_1(t)\mathrm{d}W_1(t)+c_{x_2}\sigma_2S_2(t)\mathrm{d}W_2(t)\end{aligned}$$

Since $\mathrm{d}X(t) = \mathrm{d}c(t, S(t))$ and let $S_i(t) = x_i, i = 1, 2$, we get the equation with terminal condition,

$$\Delta_1(t) = c_{x_1}(t, x_1, x_2), \quad \Delta_2(t) = c_{x_2}(t, x_1, x_2)$$

$$\begin{cases} rc + c_{x_1}(\mu_1 - r)x_1 + c_{x_2}(\mu_2 - r)x_2 \\ = c_t + c_{x_1}\mu_1 x_1 + c_{x_2}\mu_2 x_2 + \frac{1}{2}c_{x_1x_1}\sigma_1^2 x_1^2 + \frac{1}{2}c_{x_2x_2}\sigma_2^2 x_2^2 + c_{x_1x_2}\sigma_1\sigma_2 x_1 x_2 \rho \\ c(T, x_1, x_2) = (x_1 - x_2)^+ \end{cases}$$

that is,

$$\begin{cases} r[c - c_{x_1}x_1 - c_{x_2}x_2] \\ = c_t + \frac{1}{2}c_{x_1x_1}\sigma_1^2 x_1^2 + \frac{1}{2}c_{x_2x_2}\sigma_2^2 x_2^2 + c_{x_1x_2}\sigma_1\sigma_2 x_1 x_2 \rho \\ c(T, x_1, x_2) = (x_1 - x_2)^+ \end{cases}$$

The homework §9

§9.1 Option Pricing under Bachelier's Arithmetic Brownian Motion Model

1. Because $\mathrm{d}S(t) = \mu\mathrm{d}t + \sigma\mathrm{d}W(t)$, we have

$$\mathrm{d}X(t) = \Delta(t)\mathrm{d}S(t) + r(X(t) - \Delta(t)S(t))\mathrm{d}t = [rX(t) + \Delta(t)\mu - r\Delta(t)S(t)]\mathrm{d}t + \sigma\Delta(t)\mathrm{d}W(t),$$

$$\mathrm{d}p(t, S(t)) = p_t\mathrm{d}t + p_x\mathrm{d}S(t) + \frac{1}{2}p_{xx}\mathrm{d}S(t)\mathrm{d}S(t) = [p_t + \mu p_x + \frac{1}{2}p_{xx}]\mathrm{d}t + \sigma p_x\mathrm{d}W(t).$$

Let $X(t) = p(t, S(t))$, $\mathrm{d}X(t) = \mathrm{d}p(t, S(t))$, then $\Delta(t) = p_x(t, S(t))$,

$$p_t(t, x) + rxp_x(t, x) + \frac{1}{2}\sigma^2 p_{xx}(t, x) = rp(t, x), \quad \forall t \in [0, T) \tag{5}$$

with a terminal condition $p(T, x) = (K - x)^+$.

2. By Feynman-Kac theorem,

$$p(t, S(t)) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(K - S(T))^+|\mathcal{F}(t)\right]. \tag{6}$$

satisfies the following PDE

$$p_t(t, x) + rxp_x(t, x) + \frac{1}{2}\sigma^2 p_{xx}(t, x) = rp(t, x) \quad for \quad all \quad t \in [0, T) \tag{7}$$

with a terminal condition $p(T, x) = (K - x)^+$, where $\beta(t, x) = rx, \gamma(t, x) = \sigma$

3. Under risk-neutral measure, we have

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sigma\mathrm{d}W^{\mathbb{Q}}(t) \tag{8}$$

Using the same method to Vasicek Model, we can deduce that

$$p(t, S(t)) = \sigma e^{rt}\sqrt{\tau}[d\Phi(d) + \phi(d)],$$

where

$$\sqrt{\tau} = \frac{e^{-2rt} - e^{-2rT}}{2r}, \quad d = \frac{Ke^{-rT} - S(t)e^{-rt}}{\sigma\sqrt{\tau}},$$

4.
$$\begin{aligned}\Delta(t) = p_x(t, S(t)) =& -\Phi(d) + [Ke^{-r(T-t)} - x]\Phi(d)\left(-\frac{e^{-rt}}{\sqrt{\tau}}\right)\\ &+ \sqrt{\tau}e^{rt}\left(-\frac{Ke^{-rT} - xe^{rt}}{\sqrt{\tau}}\right)\phi(d)\left(-\frac{e^{-rt}}{\sqrt{\tau}}\right) = -\Phi(d) < 0.\end{aligned}$$

Hence the hedging strategy is to short $\Phi(d)$ shares stocks, and invest $X(t) + \Phi(d)S(t)$ into money market.

5. **STEP 1: Eliminate the term on right side**

We mimic the way we solve Vasicek model, and let $u(t,x) = e^{-rt}p(t,x)$. Then (3) transforms into

$$\frac{\partial u}{\partial t} + rx\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} = 0 \tag{9}$$

STEP 2: Eliminate the drift term

Then it's natural to think of ways to eliminate the drift term. To do this, we have to make change of variable x. Let $z = f(t)x$ and $u(t,x) := v(t,z)$. Take derivative w.r.t. t and x,

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial y}f'(t)x$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}f(t), \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y^2}f(t).$$

Putting these equations into (5), and let the coeffecient of $\frac{\partial v}{\partial y}$ equal to 0, we derive the expression of $f(t)$ and z. And (5) transforms into

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial y^2}e^{-2rt} = 0 \tag{10}$$

STEP 3: Turn into heat equation

We are only one step away from victory. According to the characteristics of (6), we guess that this time we have to make change to variable t to eliminate the coeffient $\frac{1}{2}\sigma^2 e^{-2rt}$. Let $\tau = g(t)$ and $v(t,z) := s(\tau,z)$. By the same way as above, we can derive the expression of $g(t)$ or τ. Note you have to choose an appropriate form of $g(t)$ so that the terminal condition holds

$$s(0,z) = e^{-rT}p(T,x),$$

which means if $t = T$, then $\tau = 0$. Finally we have the heat equation

$$s_\tau(\tau,z) = s_{zz}(\tau,z). \tag{11}$$

Following the above three steps, we solve the heat equation

$$s(\tau,z) = \int_{-\infty}^{+\infty} s(0,y)G(z,y,\tau)dy, \quad G(z,y,\tau) = \frac{1}{\sqrt{4\pi\tau}}exp\left\{-\frac{(z-y)^2}{4\tau}\right\}.$$

$$\begin{aligned} s(\tau,z) = \int_{-\infty}^{+\infty} s(0,y)G(z,y,\tau)\mathrm{d}y &= \int_{-\infty}^{\frac{Ke^{-rT}-z}{\sqrt{\tau}}} \left(Ke^{-rT} - \sqrt{\tau}x - z\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\mathrm{d}x \\ &= \left(Ke^{-rT} - z\right)\Phi\left(\frac{Ke^{-rT}-z}{\sqrt{\tau}}\right) + \sqrt{\tau}\phi\left(\frac{Ke^{-rT}-z}{\sqrt{\tau}}\right) \end{aligned}$$

So

$$p(t,S(t)) = \sigma e^{rt}\sqrt{\tau}[d\Phi(d)+\phi(d)]$$

where

$$\sqrt{\tau} = \frac{e^{-2rt}-e^{-2rT}}{2r}, \quad d = \frac{Ke^{-rT}-S(t)e^{-rt}}{\sigma\sqrt{\tau}},$$

§9.2 Shreve Vol II. Exercise. 4.12 (i) and (ii)

1. For call option $c(t,x)$, we have deduced that

$$\begin{aligned} c_x(t,x) &= N(d_+(T-t,x)), \quad d_\pm(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{k} + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right], \\ c_t(t,x) &= -rKe^{-r(T-t)}N(d_-(T-t,x)) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+(T-t,x)), \\ c_{xx}(t,x) &= \frac{1}{\sigma x\sqrt{T-t}}N'(d_-(T-t,x)). \end{aligned}$$

By put-call parity $x - Ke^{-r(T-t)} = c(t,x) - p(t,x)$, the following hold

$$\begin{aligned} p_x(t,x) &= c_x(t,x) - 1 = N(d_+(T-t,x)) - 1, \\ p_t(t,x) &= c_t(t,x) + rKe^{-r(T-t)} = rKe^{-r(T-t)}\left[1 - N(d_-(T-t,x))\right] - \frac{\sigma x}{2\sqrt{T-t}}N(d_+(T-t,x)), \\ p_{xx}(t,x) &= c_{xx}(t,x) = \frac{1}{\sigma x\sqrt{T-t}}N(d_+(T-t,x)). \end{aligned}$$

2. Because $p_x(t,x) = N(d_+(T-t,x)) - 1 < 0$, the hedging strategy is to short $|p_x(t,S(t))|$ share stocks, and invest $X(t) + |p_x(t,x)|S(t)$ into money market.

§9.3 Shreve Vol II. Exercise. 4.11

Proof. First, we note $c(t,x)$ solves the Black-Scholes-Merton PDE with volatility σ_1:

$$\frac{\partial c(t,x)}{\partial t} + rx\frac{\partial c(t,x)}{\partial x} + \frac{1}{2}x^2\sigma_1^2\frac{\partial^2 c(t,x)}{\partial x^2} = rc(t,x),$$

so

$$c_t(t,S_t) + rS_t c_x(t,S_t) + \frac{1}{2}\sigma_1^2 S_t^2 c_{xx}(t,S_t) = rc(t,S_t). \tag{12}$$

And

$$\begin{aligned} \mathrm{d}c(t,S_t) &= c_t(t,S_t)\mathrm{d}t + c_x(t,S_t)\mathrm{d}S_t + \frac{1}{2}c_{xx}(t,S_t)\mathrm{d}S_t\mathrm{d}S_t \\ &= c_t(t,S_t)\mathrm{d}t + c_x(t,S_t)\left(\alpha S_t\mathrm{d}t + \sigma_2 S_t\mathrm{d}W_t\right) + \frac{1}{2}\sigma_2^2 S_t^2 c_{xx}(t,S_t)\mathrm{d}t \\ &= \left[c_t(t,S_t) + \alpha c_x(t,S_t)S_t + \frac{1}{2}\sigma_2^2 S_t^2 c_{xx}(t,S_t)\right]\mathrm{d}t + \sigma_2 S_t c_x(t,S_t)\mathrm{d}W_t \\ (Using\ formula\ (12)) &= \left[rc(t,S_t) + (\alpha - r)c_x(t,S_t)S_t + \frac{1}{2}(\sigma_2^2 - \sigma_1^2)S_t^2 c_{xx}(t,S_t)\right]\mathrm{d}t + \sigma_2 S_t c_x(t,S_t)\mathrm{d}W_t \end{aligned}$$

Therefore,

$$
\begin{aligned}
\mathrm{d}X_t &= \mathrm{d}c(t,S_t) - c_x(t,S_t)\mathrm{d}S_t + r\left[X_t - c(t,S_t) + S_t c_x(t,S_t)\right]\mathrm{d}t - \frac{1}{2}(\sigma_2^2-\sigma_1^2)S_t^2 c_{xx}(t,S_t)\mathrm{d}t \\
&= \Big[rc(t,S_t) + (\alpha - r)c_x(t,S_t)S_t + \frac{1}{2}(\sigma_2^2-\sigma_1^2)S_t^2 c_{xx}(t,S_t) + rX_t - rc(t,S_t) + rS_t c_x(t,S_t)\mathrm{d}t \\
&\quad - \frac{1}{2}(\sigma_2^2-\sigma_1^2)S_t^2 c_{xx}(t,S_t) - c_x(t,S_t)\alpha S_t\Big]\mathrm{d}t + \left[\sigma_2 S_t c_x(t,S_t) - c_x(t,S_t)\sigma_2 S_t\right]\mathrm{d}W_t \\
&= rX_t\mathrm{d}t
\end{aligned}
$$

Hence $X_t = X_0 e^{rt}$. By $X_0 = 0$, we conclude $X_t = 0$, $\forall t \in [0,T]$.

□

§9.4 Show that the implied volatility calculated from call and put options are the same.

Assume σ_{imp} is the implied volatility of call option, i.e. $c_{market} = c(\sigma_{imp})$, where c_{market} is the market price of call option and $c(\sigma)$ is the BSM option pricing formula. By put-call parity, we have

$$c_{market} - p_{market} = S_t - e^{-r(T-t)}K$$

$$c(\sigma_{imp}) - p(\sigma_{imp}) = S_t - e^{-r(T-t)}K$$

Therefore,

$$p_{market} - p(\sigma_{imp}) = c_{market} - c(\sigma_{imp}) = 0$$

so the implied volatility of call option σ_{imp} is the implied volatility of put option as well, and vice versa.