8 Arbitrage Pricing Theory

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Overview and Comparisons

The Arbitrage Pricing Theory (APT) was developed by Stephen Ross (US, b.1944) in the mid-1970s.

Stephen Ross, "The Arbitrage Theory of Capital Asset Pricing," *Journal of Economic Theory* Vol.13 (December 1976): pp.341-360.

Overview and Comparisons

The APT bears a close resemblance to the CAPM.

In fact, a special case of the APT implies a relationship between expected returns on arbitrary, but well-diversified, portfolios and the return on the market portfolio that is identical to the relationship implied by the CAPM.

Overview and Comparisons

But the APT allows for a more flexible, or general, depiction of aggregate risk than does the CAPM.

And, as its name suggests, the APT is a no-arbitrage theory of asset pricing, which does not require the strong assumptions imposed by the CAPM as an equilibrium theory.

The weakness of the APT – the cost, if you will, of its added flexibility and generality – is that it applies only to "well-diversified portfolios" and "most" individual securities: there is no guarantee that it applies to all individual assets.

Overview and Comparisons

The APT can also be compared to the no-arbitrage variant of Arrow-Debreu theory.

The APT replaces the A-D model's abstract description of "states of the world" with a more practical, or empirically-motivated depiction of the underlying sources of aggregate risk and then prices portfolios of assets based on their exposures to those sources of risk.

The Market Model

The simplest version of the APT is also the one that can best be thought of as the "no-arbitrage" variant of the CAPM.

It is based on the market model, which by itself is simply a statistical model according to which the random return $\tilde{r}_j$ on each asset $j = 1, 2, \ldots, J$ is related to the random return $\tilde{r}_M$ on the market portfolio via

$$\tilde{r}_j = \alpha_j + \beta_j [\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_j$$

where $E(\varepsilon_j) = 0$, $Cov(\tilde{r}_M, \varepsilon_j) = 0$, and $Cov(\varepsilon_j, \varepsilon_k) = 0$ for all $j = 1, 2, \ldots, J$ and $k = 1, 2, \ldots, J$ with $k \neq j$.

The Market Model

independent to ε_i - $\forall i$

Assumption!

不同股票有不同的 ε_j

$$\tilde{r}_j = \alpha_j + \beta_j[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_j$$

where $E(\varepsilon_j) = 0$, $Cov(\tilde{r}_M, \varepsilon_j) = 0$, and $Cov(\varepsilon_j, \varepsilon_k) = 0$ for all $j = 1, 2, \ldots, J$ and $k = 1, 2, \ldots, J$ with $k \neq j$.

market factor 还是全部相关因子

The market model is called a single-factor model since it breaks the random return on each asset down into two uncorrelated or "orthogonal" components:

1. $\beta_j[\tilde{r}_M - E(\tilde{r}_M)]$, which depends on the model's single aggregate variable or "factor," in this case the market return.

2. ε_j, which is a purely idiosyncratic effect.

The Market Model

强假设

$$\tilde{r}_j = \alpha_j + \beta_j[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_j$$

where $E(\varepsilon_j) = 0$, $Cov(\tilde{r}_M, \varepsilon_j) = 0$, and $Cov(\varepsilon_j, \varepsilon_k) = 0$ for all $j = 1, 2, \ldots, J$ and $k = 1, 2, \ldots, J$ with $k \neq j$.

The parameter's α_j and β_j can be estimated through a statistical regression of each asset's return on the market return, imposing the additional statistical assumption that the regression error ε_j is uncorrelated across assets as well as with the market portfolio.

The Market Model

$$\tilde{r}_j = \alpha_j + \beta_j[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_j$$

where $E(\varepsilon_j) = 0$, $Cov(\tilde{r}_M, \varepsilon_j) = 0$, and $Cov(\varepsilon_j, \varepsilon_k) = 0$ for all $j = 1, 2, \ldots, J$ and $k = 1, 2, \ldots, J$ with $k \neq j$.

As we will see, it is partly this added assumption that ε_j is purely idiosyncratic that gives the APT its "bite."

The Market Model

$$\tilde{r}_j = \alpha_j + \beta_j[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_j$$

where $E(\varepsilon_j) = 0$, $Cov(\tilde{r}_M, \varepsilon_j) = 0$, and $Cov(\varepsilon_j, \varepsilon_k) = 0$ for all $j = 1, 2, \ldots, J$ and $k = 1, 2, \ldots, J$ with $k \neq j$.

Before moving on to see how the APT makes use of the market model, let's take note of two other purposes that the market model can serve.

The Market Model

First, since

$$\tilde{r}_j = \alpha_j + \beta_j[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_j$$

can be rewritten as

$$\tilde{r}_j = \bar{\alpha}_j + \beta_j \tilde{r}_M + \varepsilon_j$$

where

$$\bar{\alpha}_j = \alpha_j - \beta_j E(\tilde{r}_M)$$

the regressions from the market model yield slope coefficients

$$\beta_j = \frac{Cov(\tilde{r}_j, \tilde{r}_M)}{\sigma_M^2}$$

that can be viewed as estimates of the CAPM beta for asset j.

The Market Model

Second, since

$$\tilde{r}_j = \alpha_j + \beta_j[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_j$$

implies

$$E(\tilde{r}_j) = \alpha_j$$

the market model implies that for any asset j

$$\begin{aligned}
\sigma_j^2 &= E\{[\tilde{r}_j - E(\tilde{r}_j)]^2\} \\
&= E\left(\{\beta_j[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_j\}^2\right) \\
&= \beta_j^2 E\{[\tilde{r}_M - E(\tilde{r}_M)]^2\} + 2\beta_j E\{[\tilde{r}_M - E(\tilde{r}_M)]\varepsilon_j\} + E(\varepsilon_j^2) \\
&= \beta_j^2\sigma_M^2 + 2\beta_j Cov(\tilde{r}_M, \varepsilon_j) + \sigma_{\varepsilon_j}^2 \\
&= \beta_j^2\sigma_M^2 + \sigma_{\varepsilon_j}^2
\end{aligned}$$

The Market Model

$$\tilde{r}_j = \alpha_j + \beta_j[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_j$$

Similarly, the market model implies that for any two assets j and $k \neq j$

$$\begin{aligned}
Cov(\tilde{r}_j, \tilde{r}_k) &= E\{[\tilde{r}_j - E(\tilde{r}_j)][\tilde{r}_k - E(\tilde{r}_k)]\} \\
&= E\left(\{\beta_j[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_j\}\{\beta_k[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_k\}\right) \\
&= \beta_j\beta_k E\{[\tilde{r}_M - E(\tilde{r}_M)]^2\} + \beta_j E\{[\tilde{r}_M - E(\tilde{r}_M)]\varepsilon_k\} \\
&\quad + \beta_k E\{[\tilde{r}_M - E(\tilde{r}_M)]\varepsilon_j\} + E(\varepsilon_j\varepsilon_k) \\
&= \beta_j\beta_k\sigma_M^2 + \beta_j Cov(\tilde{r}_M, \varepsilon_k) \\
&\quad + \beta_k Cov(\tilde{r}_M, \varepsilon_j) + Cov(\varepsilon_j, \varepsilon_k) \quad = 0 \\
&= \beta_j\beta_k\sigma_M^2
\end{aligned}$$

The Market Model

Recall one difficulty with MPT: it requires estimates of the $J(J+1)/2$ variances and covariances for J individual asset returns.

But since the market model implies

$$\sigma_j^2 = Var(\tilde{r}_j) = \beta_j^2 \sigma_M^2 + \sigma_{\varepsilon_j}^2$$

$$Cov(\tilde{r}_j, \tilde{r}_k) = \beta_j \beta_k \sigma_M^2$$

it allows these variances and covariances to be estimated based on only $2J+1$ underlying "parameters:" the J β_j's, the J $\sigma_{\varepsilon_j}^2$'s, and the 1 σ_M^2.

The Market Model

$$\sigma_j^2 = Var(\tilde{r}_j) = \beta_j^2 \sigma_M^2 + \sigma_{\varepsilon_j}^2$$

$$Cov(\tilde{r}_j, \tilde{r}_k) = \beta_j \beta_k \sigma_M^2$$

With $J = 100$, $J(J+1)/2 = 5050$ but $2J + 1 = 201$.

With $J = 500$, $J(J+1)/2 = 125250$ but $2J + 1 = 1001$.

The "savings" are considerable!

The APT

We have already noted that the market model

$$\tilde{r}_j = \alpha_j + \beta_j[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_j$$

implies that the parameter α_j measures the expected return on asset j:

$$E(\tilde{r}_j) = \alpha_j$$

Recognizing this fact, let's rewrite the equations for the market model as

$$\tilde{r}_j = E(\tilde{r}_j) + \beta_j[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_j$$

for all $j = 1, 2, \ldots, J$.

The APT

$$\tilde{r}_j = E(\tilde{r}_j) + \beta_j[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_j$$

By itself, however, the market model tells us nothing about how expected returns are determined or how they are related across different assets.

The APT

The APT starts by assuming that asset returns are governed by a factor model such as the market model.

To derive implications for expected returns, the APT makes two additional assumptions:

1. There are enough individual assets to create many well-diversified portfolios.
2. Investors act to eliminate all arbitrage opportunities across all well-diversified portfolios.

The APT

Before moving all the way to well-diversified portfolios, let's consider an "only somewhat" diversified portfolio.

Consider, in particular, a portfolio consisting of only two assets, j and k, both of which have returns described by the market model.

Let w be the share of the portfolio allocated to asset j and $1 - w$ the share allocated to asset k.

The APT

$$\tilde{r}_j = E(\tilde{r}_j) + \beta_j[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_j$$

$$\tilde{r}_k = E(\tilde{r}_k) + \beta_k[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_k$$

imply that the return $\tilde{r}_w$ on the portfolio is

$$\begin{aligned}
\tilde{r}_w &= wE(\tilde{r}_j) + (1-w)E(\tilde{r}_k) \\
&\quad + w\beta_j[\tilde{r}_M - E(\tilde{r}_M)] + (1-w)\beta_k[\tilde{r}_M - E(\tilde{r}_M)] \\
&\quad + w\varepsilon_j + (1-w)\varepsilon_k \\
&= E(\tilde{r}_w) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_w
\end{aligned}$$

The APT

$$
\begin{aligned}
\tilde{r}_w &= wE(\tilde{r}_j) + (1-w)E(\tilde{r}_k) \\
&\quad + w\beta_j[\tilde{r}_M - E(\tilde{r}_M)] + (1-w)\beta_k[\tilde{r}_M - E(\tilde{r}_M)] \\
&\quad + w\varepsilon_j + (1-w)\varepsilon_k \\
&= E(\tilde{r}_w) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_w
\end{aligned}
$$

The first implication is that the expected return on the portfolio is just a weighted average of the expected returns on the individual assets, with weights corresponding to those in the portfolio itself:

$$E(\tilde{r}_w) = wE(\tilde{r}_j) + (1-w)E(\tilde{r}_k)$$

The APT

$$
\begin{aligned}
\tilde{r}_w &= wE(\tilde{r}_j) + (1-w)E(\tilde{r}_k) \\
&\quad + w\beta_j[\tilde{r}_M - E(\tilde{r}_M)] + (1-w)\beta_k[\tilde{r}_M - E(\tilde{r}_M)] \\
&\quad + w\varepsilon_j + (1-w)\varepsilon_k \\
&= E(\tilde{r}_w) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_w
\end{aligned}
$$

The second implication is that the portfolio's beta is the same weighted average of the betas of the individual assets:

$$\beta_w = w\beta_j + (1-w)\beta_k$$

The APT

$$\begin{aligned}
\tilde{r}_w &= wE(\tilde{r}_j) + (1-w)E(\tilde{r}_k) \\
&\quad + w\beta_j[\tilde{r}_M - E(\tilde{r}_M)] + (1-w)\beta_k[\tilde{r}_M - E(\tilde{r}_M)] \\
&\quad + w\varepsilon_j + (1-w)\varepsilon_k \\
&= E(\tilde{r}_w) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_w
\end{aligned}$$

The third implication is subtle but very important. We can see that the idiosyncratic component of the portfolio's return is a weighted average of the idiosyncratic components of the individual asset returns:

$$\varepsilon_w = w\varepsilon_j + (1-w)\varepsilon_k$$

The APT

$$\varepsilon_w = w\varepsilon_j + (1-w)\varepsilon_k$$

But the variance of the idiosyncratic component of the portfolio's return is not a weighted average of the variances of the idiosyncratic components of the individual asset returns:

$$\begin{aligned}
\sigma^2_{\varepsilon_w} &= E(\varepsilon^2_w) = E\{[w\varepsilon_j + (1-w)\varepsilon_k]^2\} \\
&= w^2 E(\varepsilon^2_j) + 2w(1-w)E(\varepsilon_j\varepsilon_k) + (1-w)^2 E(\varepsilon^2_k) \\
&= w^2\sigma^2_{\varepsilon_j} + (1-w)^2\sigma^2_{\varepsilon_k}
\end{aligned}$$

The APT

$$\sigma^2_{\varepsilon_w} = w^2\sigma^2_{\varepsilon_j} + (1-w)^2\sigma^2_{\varepsilon_k}$$

For example, if the individual returns have idiosyncratic components with equal variances

$$\sigma^2_{\varepsilon_j} = \sigma^2_{\varepsilon_k} = \sigma^2$$

and the portfolio gives equal weight to the two assets

$$w = 1 - w = 1/2$$

then

$$\sigma^2_{\varepsilon_w} = (1/2)^2\sigma^2 + (1/2)^2\sigma^2 = (1/2)\sigma^2$$

The APT

$$\sigma^2_{\varepsilon_w} = (1/2)^2\sigma^2 + (1/2)^2\sigma^2 = (1/2)\sigma^2$$

Even with just two assets, the portfolio's idiosyncratic risk is cut in half.

Of course, this is just a special case of the gains from diversification exploited by MPT. But the APT pushes the idea to its logical and mathematical limits.

The APT

Let's head in the same direction, by considering a portfolio with a large number N of individual assets.

Let w_i, $i = 1, 2, \ldots, N$, denote the share of the portfolio allocated to asset i.

And consider, in particular, the "equal weighted" case, where

$$w_i = 1/N \text{ for all } i = 1, 2, \ldots, N$$

The APT

Since each of the individual asset returns are generated by the market model, the return on this equal-weighted portfolio will be

$$\tilde{r}_w = E(\tilde{r}_w) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_w$$

where ...

...the expected return on the equal-weighted portfolio is just the average of the expected returns on the individual assets:

$$E(\tilde{r}_w) = \sum_{i=1}^{N} w_i E(\tilde{r}_i) = (1/N) \sum_{i=1}^{N} E(\tilde{r}_i)$$

The APT

Since each of the individual asset returns are generated by the market model, the return on this equal-weighted portfolio will be

$$\tilde{r}_w = E(\tilde{r}_w) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_w$$

where ...

... the equal-weighted portfolio's beta is just the average of the individual assets' betas:

$$\beta_w = \sum_{i=1}^{N} w_i \beta_i = (1/N) \sum_{i=1}^{N} \beta_i$$

The APT

Since each of the individual asset returns are generated by the market model, the return on this equal-weighted portfolio will be

$$\tilde{r}_w = E(\tilde{r}_w) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_w$$

where ...

... and the idiosyncratic component of the return on the equal-weighted portfolio is an average of the idiosyncratic components of the individual asset returns:

$$\varepsilon_w = \sum_{i=1}^{N} w_i \varepsilon_i = (1/N) \sum_{i=1}^{N} \varepsilon_i$$

The APT

$$\varepsilon_w = \sum_{i=1}^{N} w_i \varepsilon_i = (1/N) \sum_{i=1}^{N} \varepsilon_i$$

Once again, however, the variance of ε_w will not equal the average of the variances of the individual ε_i's. In fact,

$$\begin{aligned}
\sigma^2_{\varepsilon_w} &= (1/N)^2 \sum_{i=1}^{N} E(\varepsilon_i^2) + (1/N)^2 \sum_{i=1}^{N} \sum_{h \neq i} E(\varepsilon_i \varepsilon_h) \\
&= (1/N)^2 \sum_{i=1}^{N} E(\varepsilon_i^2) \\
&= \left(\frac{1}{N}\right) \left[\frac{1}{N} \sum_{i=1}^{N} \sigma^2_{\varepsilon_i}\right]
\end{aligned}$$

The APT

有很多股票时，
个股 [illegible]
well diversified

$$\sigma^2_{\varepsilon_w} = \left(\frac{1}{N}\right)\left[\frac{1}{N}\sum_{i=1}^{N}\sigma^2_{\varepsilon_i}\right] = \frac{1}{N}\bar{\sigma}^2_{\varepsilon_i}$$

where $\bar{\sigma}^2_{\varepsilon_i}$ denotes the average the variances of the individual ε_i

Hence, the variance of ε_w is not the average of the variances of the individual ε_i's.

The variance of ε_w is $1/N$ times the average of the variances of the individual ε_i's.

The APT

$$\sigma^2_{\varepsilon_w} = \left(\frac{1}{N}\right)\left[\frac{1}{N}\sum_{i=1}^{N}\sigma^2_{\varepsilon_i}\right] = \frac{1}{N}\bar{\sigma}^2_{\varepsilon_i}$$

The variance of ε_w is $1/N$ times the average of the variances of the individual ε_i's.

Hence, the amount of idiosyncratic risk in the portfolio's return can be made arbitrarily small by making N, the number of assets included in the portfolio, sufficiently large.

The APT

More generally, for portfolio's with unequal weights

$$\varepsilon_w = \sum_{i=1}^{N} w_i \varepsilon_i$$

implies

$$\begin{aligned} \sigma^2_{\varepsilon_w} &= \sum_{i=1}^{N} w_i^2 E(\varepsilon_i^2) + \sum_{i=1}^{N} \sum_{h \neq i} w_i w_h E(\varepsilon_i \varepsilon_h) \\ &= \sum_{i=1}^{N} w_i^2 E(\varepsilon_i^2) = \sum_{i=1}^{N} w_i^2 \sigma^2_{\varepsilon_i} \end{aligned}$$

The APT

$\tilde{r}_j = E(\tilde{r}_j) + \beta[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_j$

diversified

$\tilde{r}_w = E(\tilde{r}_w) + \beta(\tilde{r}_m - E(\tilde{r}_m))$

$$\sigma^2_{\varepsilon_w} = \sum_{i=1}^{N} w_i^2 \sigma^2_{\varepsilon_i}$$

Any portfolio in which the share of each individual asset w_i becomes arbitrarily small as the number of assets N grows larger and larger will have negligible idiosyncratic risk.

From a practical perspective, this means: holding the dollar value of your portfolio fixed, buy some shares in more and more individual companies by buying fewer and fewer shares in each individual company.

The APT

$$\sigma^2_{\varepsilon_w} = \sum_{i=1}^{N} w_i^2 \sigma^2_{\varepsilon_i}$$

Any portfolio in which the share of each individual asset w_i becomes arbitrarily small as the number of assets N grows larger and large will have negligible idiosyncratic risk.

This, specifically, is what is meant by a well-diversified portfolio: one in which the number of assets included is sufficiently large to make the portfolio's idiosyncratic risk vanish.

The APT

In his 1976 paper, Stephen Ross notes that while the assumption

$$Cov(\varepsilon_j, \varepsilon_k) = 0 \text{ for all } j, k = 1, 2, \dots, J \text{ with } k \neq j$$

helps in making a portfolio's idiosyncratic risk vanish as the number of assets included becomes larger, this can still happen with positive correlations between individual assets' idiosyncratic returns provided the correlations are not too large and the number of assets is sufficiently large.

The APT

Hence, the APT's first assumption – that individual asset returns are generated by a factor model – implies that the return on any portfolio is

$$\tilde{r}_w = E(\tilde{r}_w) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)] + \varepsilon_w$$

And the APT's second assumption – that there are a sufficient number of assets J to create well-diversified portfolios – implies that the return on any portfolio well-diversified portfolio is

$$\tilde{r}_w = E(\tilde{r}_w) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$

The APT

APT：

假设在 market model ① 和 no arbitrage ② 的前提下，well diversified 的资产组合的 $E(\tilde{r}_w)$

$$\tilde{r}_w = E(\tilde{r}_w) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$

We still have not said anything about what determines the expected return $E(\tilde{r}_w)$.

But we can now, based on the APT's third assumption: that investors act to eliminate arbitrage opportunities across all well-diversified portfolios.

The APT

Our first no-arbitrage argument applies to well-diversified portfolios with the same betas.

Proposition 1 The absence of arbitrage opportunities requires well-diversified portfolios with the same betas to have the same expected returns.

The APT

To see why this proposition must be true, consider two well-diversified portfolios, one with

$$\tilde{r}_w^1 = E(\tilde{r}_w) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$

and the other with

$$\tilde{r}_w^2 = E(\tilde{r}_w) + \Delta + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$

where $\Delta > 0$. These portfolios have the same beta, but the second has a higher expected return.

The APT

$$\tilde{r}_w^1 = E(\tilde{r}_w) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$
$$\tilde{r}_w^2 = E(\tilde{r}_w) + \Delta + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$

Now consider a strategy of taking a "long position" worth x in portfolio 2 and a "short position" worth $-x$ in portfolio 1.

Note that the two portfolios may contain some of the same individual assets; hence, in practice, this strategy may involve taking appropriate long and short positions in just some of those assets so as to capture the net differences between the two portfolios.

The APT

$$\tilde{r}_w^1 = E(\tilde{r}_w) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$
$$\tilde{r}_w^2 = E(\tilde{r}_w) + \Delta + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$

Now consider a strategy of taking a long position worth x in portfolio 2 and a short position worth $-x$ in portfolio 1.

This strategy is self-financing, in that it requires "no money down" at $t = 0$.

The APT

$$\tilde{r}_w^1 = E(\tilde{r}_w) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$
$$\tilde{r}_w^2 = E(\tilde{r}_w) + \Delta + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$

Now consider a strategy of taking a long position worth x in portfolio 2 and a short position worth $-x$ in portfolio 1.

But the strategy yields a payoff at $t = 1$ of

$$\begin{aligned} x(1 + \tilde{r}_w^2) - x(1 + \tilde{r}_w^1) &= xE(\tilde{r}_w) + x\Delta + x\beta_w[\tilde{r}_M - E(\tilde{r}_M)] \\ &\quad - xE(\tilde{r}_w) - x\beta_w[\tilde{r}_M - E(\tilde{r}_M)] \\ &= x\Delta > 0 \end{aligned}$$

The APT

$$\tilde{r}_w^1 = E(\tilde{r}_w) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$
$$\tilde{r}_w^2 = E(\tilde{r}_w) + \Delta + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$

Now consider a strategy of taking a long position worth x in portfolio 2 and a short position worth $-x$ in portfolio 1.

Since this strategy is self-financing at $t = 0$ but yields a payoff of $x\Delta$ at $t = 1$, $\Delta > 0$ is inconsistent with the absence of arbitrage opportunities.

The APT

So suppose instead that

$$\tilde{r}_w^1 = E(\tilde{r}_w) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$

and the other with

$$\tilde{r}_w^2 = E(\tilde{r}_w) + \Delta + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$

where now $\Delta < 0$. These portfolios have the same beta, but the second has a lower expected return.

The APT

$$\tilde{r}_w^1 = E(\tilde{r}_w) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$
$$\tilde{r}_w^2 = E(\tilde{r}_w) + \Delta + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$

With $\Delta < 0$, the profitable strategy involves taking a long position worth x in portfolio 1 and a short position worth $-x$ in portfolio 2.

The APT

$$\tilde{r}_w^1 = E(\tilde{r}_w) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$
$$\tilde{r}_w^2 = E(\tilde{r}_w) + \Delta + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$

With $\Delta < 0$, take a long position worth x in portfolio 1 and a short position worth $-x$ in portfolio 2.

This strategy is self-financing at $t = 0$, but yields a payoff at $t = 1$ of

$$\begin{aligned} x(1+\tilde{r}_w^1) - x(1+\tilde{r}_w^2) &= xE(\tilde{r}_w) + x\beta_w[\tilde{r}_M - E(\tilde{r}_M)] \\ &\quad - xE(\tilde{r}_w) - x\Delta - x\beta_w[\tilde{r}_M - E(\tilde{r}_M)] \\ &= -x\Delta > 0 \end{aligned}$$

The APT

$$\tilde{r}_w^1 = E(\tilde{r}_w) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$

$$\tilde{r}_w^2 = E(\tilde{r}_w) + \Delta + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$

With $\Delta < 0$, take a long position worth x in portfolio 1 and a short position worth $-x$ in portfolio 2.

Since this strategy is self-financing at $t = 0$, but yields a payoff at $t = 1$ of $-x\Delta$ at $t = 1$, $\Delta < 0$ is also inconsistent with the absence of arbitrage opportunities.

The APT

$$\tilde{r}_w^1 = E(\tilde{r}_w) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$
$$\tilde{r}_w^2 = E(\tilde{r}_w) + \Delta + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$

Since Δ cannot be positive or negative, it must equal zero.

This proves the first proposition: in the absence of arbitrage opportunities, well-diversified portfolios with the same betas must have the same expected returns.

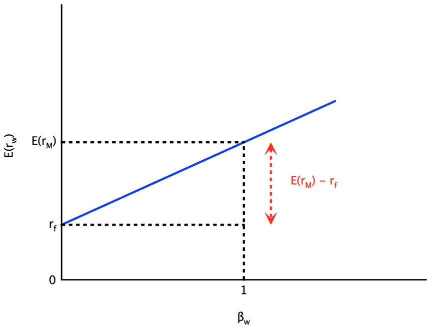
The APT

Now that we've seen how the APT's arguments work, we can extend the results to apply to well-diversified portfolios with different betas.

Proposition 2 The absence of arbitrage opportunities requires the expected returns on all well-diversified portfolios to satisfy

$$E(\tilde{r}_w) = r_f + \beta_w [E(\tilde{r}_M) - r_f]$$

The APT



The APT implies that expected returns on well-diversified portfolios line up along $E(\tilde{r}_w) = r_f + \beta_w[E(\tilde{r}_M) - r_f]$.

The APT

$$E(\tilde{r}_w) = r_f + \beta_w[E(\tilde{r}_M) - r_f]$$

This is the same relationship implied by the CAPM.

But the APT derives this relationship without assuming that utility is quadratic and without assuming that returns are normally distributed (although the returns do have to behave in accordance with the market model).

The APT

$$E(\tilde{r}_w) = r_f + \beta_w[E(\tilde{r}_M) - r_f]$$

On the other hand, the APT holds only for well-diversified portfolios, not necessarily for individual assets.

Hence, the APT's added generality comes at a cost, in terms of less widely-applicable results.

The APT

To interpret this result, consider a well-diversified portfolio, call it portfolio 1, with $\beta_w = 0$, so that

$$\tilde{r}_w^1 = E(\tilde{r}_w^1) + 0 \times [\tilde{r}_M - E(\tilde{r}_M)] = E(\tilde{r}_w^1)$$

But if the return on portfolio 1 always equals its own expected return, either:

1. Portfolio 1 is a well-diversified portfolio that forms a "synthetic" risk-free asset ...
2. ... or portfolio 1 consists entirely of risk-free assets.

Either way, its return and expected return must equal the risk-free rate:

$$\tilde{r}_w^1 = E(\tilde{r}_w^1) = r_f$$

The APT

Next, consider a second well-diversified portfolio, call it portfolio 2, with $\beta_w = 1$, so that

$$\tilde{r}_w^2 = E(\tilde{r}_w^2) + 1 \times [\tilde{r}_M - E(\tilde{r}_M)]$$

Portfolio 2 is well diversified and has the same beta as the market portfolio, so it must also have the same expected return as the market portfolio. Hence,

$$\begin{aligned} \tilde{r}_w^2 &= E(\tilde{r}_w^2) + [\tilde{r}_M - E(\tilde{r}_M)] \\ &= E(\tilde{r}_M) + [\tilde{r}_M - E(\tilde{r}_M)] \\ &= \tilde{r}_M \end{aligned}$$

The APT

But if portfolio 2 is such that

$$\tilde{r}_w^2 = \tilde{r}_M$$

then either:

1. Portfolio 2 is a well-diversified portfolio that always has the same return as the market portfolio or ...
2. ... portfolio 2 actually is the market portfolio.

The APT

So "by construction,"

$$E(\tilde{r}_w) = r_f + \beta_w[E(\tilde{r}_M) - r_f]$$

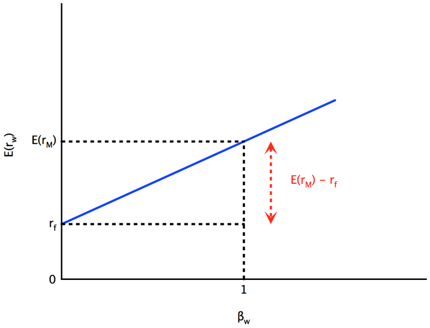
Must hold for portfolio 1, with $\beta_w = 0$:

$$E(\tilde{r}_w^1) = r_f$$

And for portfolio 2, with $\beta_w = 1$:

$$E(\tilde{r}_w^2) = E(\tilde{r}_M).$$

The APT



But to prove the proposition, we still need to show that all other well-diversified portfolios, with values of β_w different from zero or one, have expected returns that lie along $E(\tilde{r}_w) = r_f + \beta_w[E(\tilde{r}_M) - r_f]$.

The APT

Consider, therefore, a third well-diversified portfolio, with

$$\tilde{r}_w^3 = E(\tilde{r}_w^3) + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$

and β_w different from zero and one. We need to show that

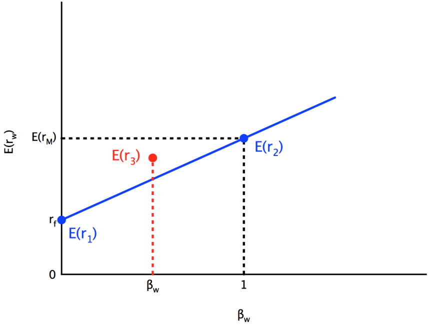
$$E(\tilde{r}_w^3) = r_f + \beta_w[E(\tilde{r}_M) - r_f]$$

so, paralleling the argument from before, suppose instead that

$$E(\tilde{r}_w^3) = r_f + \beta_w[E(\tilde{r}_M) - r_f] + \Delta$$

where $\Delta > 0$.

The APT



If $E(\tilde{r}_w^3)$ lies above the line $r_f + \beta_w[E(\tilde{r}_M) - r_f]$, then a strategy that takes a long position in portfolio 3 and short positions in portfolios 1 and 2 will constitute an arbitrage opportunity.

The APT

Since all returns are described by the market model:

$$\tilde{r}_w^1 = r_f$$

$$\tilde{r}_w^2 = E(\tilde{r}_M) + [\tilde{r}_M - E(\tilde{r}_M)]$$

$$\tilde{r}_w^3 = r_f + \beta_w[E(\tilde{r}_M) - r_f] + \Delta + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$

where the color coding distinguishes between the expected and random components of each well-diversified portfolio's return.

The no-arbitrage argument requires two steps.

The APT

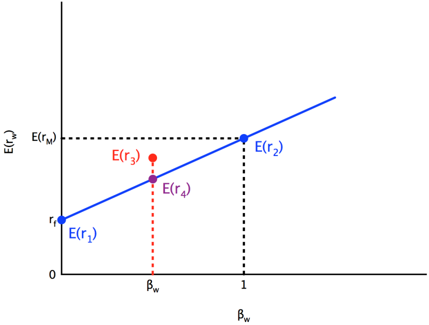
$$\tilde{r}_w^1 = r_f$$

$$\tilde{r}_w^2 = E(\tilde{r}_M) + [\tilde{r}_M - E(\tilde{r}_M)]$$

First, form a fourth well-diversified portfolio from the first two, by allocating the shares $1 - \beta_w$ to portfolio 1 and β_w to portfolio 2. This portfolio has

$$\begin{aligned}
\tilde{r}_w^4 &= (1-\beta_w)\tilde{r}_w^1 + \beta_w \tilde{r}_w^2 \\
&= (1-\beta_w) r_f + \beta_w E(\tilde{r}_M) + \beta_w [\tilde{r}_M - E(\tilde{r}_M)] \\
&= r_f + \beta_w [E(\tilde{r}_M) - r_f] + \beta_w [\tilde{r}_M - E(\tilde{r}_M)]
\end{aligned}$$

The APT



Second, observe that portfolio 4 has the same beta, but a lower expected return, than portfolio 3. Proposition 1 implies that this is inconsistent with the absence of arbitrage.

The APT

$$\tilde{r}_w^3 = r_f + \beta_w[E(\tilde{r}_M) - r_f] + \Delta + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$

$$\tilde{r}_w^4 = r_f + \beta_w[E(\tilde{r}_M) - r_f] + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$

Since portfolios 3 and 4 have the same beta, a strategy that allocates x to portfolio 3 and $-x$ to portfolio 4 is self-financing at $t = 0$ and yields a payoff of $x\Delta > 0$ at $t = 1$. This confirms that $\Delta > 0$ is inconsistent with the absence of arbitrage.

The APT

Similarly, if

$$\tilde{r}_w^1 = r_f$$

$$\tilde{r}_w^2 = E(\tilde{r}_M) + [\tilde{r}_M - E(\tilde{r}_M)]$$

but portfolio 3 has

$$\tilde{r}_w^3 = r_f + \beta_w[E(\tilde{r}_M) - r_f] + \Delta + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$

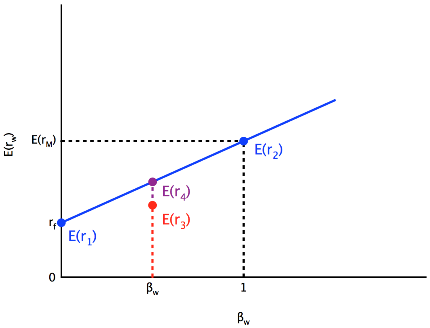
with $\Delta < 0$, we would begin by constructing portfolio 4 as before, with shares $1 - \beta_w$ allocated to portfolio 1 and β_w to portfolio 2.

The APT

$$\tilde{r}_w^3 = r_f + \beta_w[E(\tilde{r}_M) - r_f] + \Delta + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$
$$\tilde{r}_w^4 = r_f + \beta_w[E(\tilde{r}_M) - r_f] + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$

Since portfolios 3 and 4 have the same beta, a strategy that allocates $-x$ to portfolio 3 and x to portfolio 4 is self-financing at $t = 0$ and yields a payoff of $-x\Delta > 0$ at $t = 1$. This confirms that $\Delta < 0$ is not consistent with the absence of arbitrage either.

The APT



With $\Delta < 0$, portfolio 4 has the same beta, but a higher expected return, than portfolio 3. Proposition 1 again implies that this is inconsistent with the absence of arbitrage.

The APT

Hence, if all returns are described by the market model and

$$\tilde{r}_w^3 = r_f + \beta_w[E(\tilde{r}_M) - r_f] + \Delta + \beta_w[\tilde{r}_M - E(\tilde{r}_M)]$$

then $\Delta = 0$. All expected returns on well-diversified portfolios must satisfy

$$E(\tilde{r}_w) = r_f + \beta_w[E(\tilde{r}_M) - r_f]$$

APT不能直接应用于个股，
但不可能有大量个股不符合
否则组合结果存在系统性偏差

The APT

According to the APT, all expected returns on well-diversified portfolios must satisfy

$$E(\tilde{r}_w) = r_f + \beta_w[E(\tilde{r}_M) - r_f]$$

One might argue, as well, that the APT must apply "most" individual securities, since if a large number of individual assets violated the APT relationship, it would be possible to construct a well-diversified portfolio of those assets and arbitrage away the higher or lower expected returns.

Multifactor Models and the APT

We've now seen how the APT can reproduce the main implications of the CAPM, linking expected returns on well-diversified portfolios to the correlations between the actual returns on those portfolios and the return on the market portfolio.

In that sense, we can think of the APT as a "no-arbitrage variant" of the equilibrium CAPM theory, which has the advantage of requiring fewer assumptions.

Multifactor Models and the APT

The APT goes well beyond the CAPM in another direction, however, by allowing returns to follow multifactor models that are more general and more flexible than the market model.

In the literature, two famous multifactor models highlight further advantages of the APT.

Multifactor Models and the APT

As noted previously, evidence against the CAPM is presented by Eugene Fama and Kenneth French, "Common Risk Factors in the Returns on Stocks and Bonds," *Journal of Financial Economics* Vol.33 (February 1993): pp.3-56.

In particular, Fama and French find that returns on a portfolio that takes long positions in small firms and short positions in large firms and a portfolio that takes long positions in firms with high book value relative to market value and short positions in firms with low book-to-market value are as important as the overall market return in explaining expected returns on individual stocks.

Multifactor Models and the APT

Thus, Fama and French suggest replacing the market model with a three-factor model in which the random return on each individual asset $j = 1, 2, \ldots, J$ is governed by

4J+3

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$$
\begin{aligned}
\tilde{r}_j \;&=\; \alpha_j + \beta_{j,m}[\tilde{r}_M - E(\tilde{r}_M)] \\
&\quad + \underbrace{\beta_{j,s}[\tilde{r}_{SMB} - E(\tilde{r}_{SMB})]}_{\text{公司规模}} + \beta_{j,h}[\tilde{r}_{HML} - E(\tilde{r}_{HML})] + \varepsilon_j
\end{aligned}
$$

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where $\tilde{r}_{SMB}$ is the return on the "small-minus-big" portfolio, $\tilde{r}_{HML}$ is the return on the "high-minus-low" book-to-market value portfolio, $\beta_{j,m}$, $\beta_{j,s}$ and $\beta_{j,h}$ measure the exposure (correlation) of the return on asset j to each of these sources of risk, and the idiosyncratic term ε_j has the same properties as in the market model.

Multifactor Models and the APT

Fama and French's success in explaining expected returns using the betas on the small-minus-big and high-minus-low portfolios is evidence against the CAPM, which implies that only the beta on the market portfolio should matter.

But, as we will see, the APT extends readily to the three-factor model.

Multifactor Models and the APT

A famous paper by Chen, Roll and Ross, takes us even further away from the CAPM, by constructing a multifactor version of the APT in which a set of macroeconomic variables measure alternative sources of aggregate risk.

Nai-Fu Chen, Richard Roll, and Stephen Ross, "Economic Forces and the Stock Market," *Journal of Business* Vol.59 (July 1986): 383-403.

Chen, Roll, and Ross experiment with a variety of specifications before settling on a five-factor macroeconomic model.

Multifactor Models and the APT

In Chen, Roll, and Ross' multifactor model, the random return on each individual asset $j = 1, 2, \ldots, J$ is governed by

$$\begin{aligned} \tilde{r}_j &= \alpha_j + \beta_{j,IP}\tilde{IP} + \beta_{j,UI}\tilde{UI} \\ &\quad + \beta_{j,EI}\tilde{EI} + \beta_{j,TS}\tilde{TS} + \beta_{j,RP}\tilde{RP} + \varepsilon_j \end{aligned}$$

where

1. $\tilde{IP}$ = industrial production
2. $\tilde{UI}$ = unexpected inflation
3. $\tilde{EI}$ = expected inflation
4. $\tilde{TS}$ = a "term structure" variable defined as long minus short-term interest rates
5. $\tilde{RP}$ = a "risk premium" variable defined as return from holding on low versus high grade bonds

Multifactor Models and the APT

$$\begin{aligned}\tilde{r}_j &= \alpha_j + \beta_{j,IP}\tilde{IP} + \beta_{j,UI}\tilde{UI} \\ &\quad + \beta_{j,EI}\tilde{EI} + \beta_{j,TS}\tilde{TS} + \beta_{j,RP}\tilde{RP} + \varepsilon_j\end{aligned}$$

All of the factors are expressed as deviations from their expected values, so that α_j continues to measure the expected return on asset j.

The idiosyncratic term ε_j has the same properties as in the market model.

Multifactor Models and the APT

Again, it is important to stress that the multifactor models, by themselves, say nothing about expected returns.

To derive those implications, the APT must again assume that

1. There are enough individual assets to create many well-diversified portfolios.
2. Investors act to eliminate all arbitrage opportunities across all well-diversified portfolios.

Multifactor Models and the APT

To see how the APT works with a multifactor model without getting overwhelmed by notation, let's consider a two-factor model.

Let's consider, in particular, a simplified version of the Fama-French model in which the return on the market portfolio and the return on a portfolio that takes long positions in "value" stocks, that is, small or underfollowed instead of big or well-known companies and overlooked or old-fashioned companies that have high book-to-market values.

Multifactor Models and the APT

Our two-factor model then implies that the return on each individual asset $j = 1, 2, \ldots, J$ is

$$\tilde{r}_j = E(\tilde{r}_j) + \beta_{j,m}[\tilde{r}_M - E(\tilde{r}_M)] + \beta_{j,v}[\tilde{r}_V - E(\tilde{r}_V)] + \varepsilon_j$$

well diversified

and that the return on any well-diversified portfolio is

$$\tilde{r}_w = E(\tilde{r}_w) + \beta_{w,m}[\tilde{r}_M - E(\tilde{r}_M)] + \beta_{w,v}[\tilde{r}_V - E(\tilde{r}_V)]$$

Multifactor Models and the APT

We can extend our previous two no-arbitrage arguments to this multifactor case.

Proposition 3 The absence of arbitrage opportunities requires well-diversified portfolios with identical betas on both factors to have the same expected returns.

Multifactor Models and the APT

To see why this proposition must be true, consider two well-diversified portfolios, one with

$$\tilde{r}_w^1 = E(\tilde{r}_w) + \beta_{w,m}[\tilde{r}_M - E(\tilde{r}_M)] + \beta_{w,v}[\tilde{r}_V - E(\tilde{r}_V)]$$

and the other with

$$\tilde{r}_w^2 = E(\tilde{r}_w) + \Delta + \beta_{w,m}[\tilde{r}_M - E(\tilde{r}_M)] + \beta_{w,v}[\tilde{r}_V - E(\tilde{r}_V)]$$

where $\Delta > 0$. These portfolios have identical betas, but portfolio 2 has a higher expected return.

Multifactor Models and the APT

$$\tilde{r}_w^1 = E(\tilde{r}_w) + \beta_{w,m}[\tilde{r}_M - E(\tilde{r}_M)] + \beta_{w,v}[\tilde{r}_V - E(\tilde{r}_V)]$$

$$\tilde{r}_w^2 = E(\tilde{r}_w) + \Delta + \beta_{w,m}[\tilde{r}_M - E(\tilde{r}_M)] + \beta_{w,v}[\tilde{r}_V - E(\tilde{r}_V)]$$

Now consider a strategy of taking a long position worth x in portfolio 2 and a short position worth $-x$ in portfolio 1.

This strategy is self-financing at $t = 0$ but yields a $t = 1$ payoff of

$$x(1 + \tilde{r}_w^2) - x(1 + \tilde{r}_w^1) = x\Delta > 0.$$

Hence, the absence of arbitrage opportunities is inconsistent with $\Delta > 0$.

Multifactor Models and the APT

$$\tilde{r}_w^1 = E(\tilde{r}_w) + \beta_{w,m}[\tilde{r}_M - E(\tilde{r}_M)] + \beta_{w,v}[\tilde{r}_V - E(\tilde{r}_V)]$$

$$\tilde{r}_w^2 = E(\tilde{r}_w) + \Delta + \beta_{w,m}[\tilde{r}_M - E(\tilde{r}_M)] + \beta_{w,v}[\tilde{r}_V - E(\tilde{r}_V)]$$

With $\Delta < 0$, take a long position worth x in portfolio 1 and a short position worth $-x$ in portfolio 2.

This strategy is self-financing at $t = 0$ but yields a $t = 1$ payoff of

$$x(1 + \tilde{r}_w^1) - x(1 + \tilde{r}_w^2) = -x\Delta > 0.$$

Hence, the absence of arbitrage opportunities is also inconsistent with $\Delta < 0$.

Multifactor Models and the APT

This proves proposition 3: the absence of arbitrage opportunities requires well-diversified portfolios with identical betas on both factors to have the same expected returns. The extension of proposition 2 applies to well-diversified portfolios with different betas or loadings on one or both of the factors.

Proposition 4 The absence of arbitrage opportunities requires the expected returns on all well-diversified portfolios to satisfy

$$E(\tilde{r}_w) = r_f + \beta_{w,m}[E(\tilde{r}_M) - r_f] + \beta_{w,v}[E(\tilde{r}_V) - r_f]$$

Multifactor Models and the APT

$$E(\tilde{r}_w) = r_f + \beta_{w,m}[E(\tilde{r}_M) - r_f] + \beta_{w,v}[E(\tilde{r}_V) - r_f]$$

Drawing a first analogy to the CAPM, the APT implies a multidimensional version of the security market line that reflects multiple sources of aggregate risk.

Multifactor Models and the APT

$$E(\tilde{r}_w) = r_f + \beta_{w,m}[E(\tilde{r}_M) - r_f] + \beta_{w,v}[E(\tilde{r}_V) - r_f]$$

Drawing a second analogy to the Arrow-Debreu model, the APT replaces contingent claims with portfolios that track the assumed factors.

Since every well-diversified portfolio is itself a bundle of these pure factor, tracking, or mimicking portfolios, they earn a risk-premium to the extent that they are exposed to any or all of those multiple sources of aggregate risk.

Multifactor Models and the APT

With two factors, we need to start by considering three tracking portfolios, all well-diversified. The first has $\beta_{w,m} = \beta_{w,v} = 0$, so that

$$\tilde{r}_w^1 = E(\tilde{r}_w^1)$$

the second has $\beta_{w,m} = 1$ and $\beta_{w,v} = 0$, so that

$$\tilde{r}_w^2 = E(\tilde{r}_w^2) + [\tilde{r}_M - E(\tilde{r}_M)]$$

and the third has $\beta_{w,m} = 0$ and $\beta_{w,v} = 1$, so that

$$\tilde{r}_w^3 = E(\tilde{r}_w^3) + [\tilde{r}_V - E(\tilde{r}_V)]$$

Multifactor Models and the APT

Since portfolio one, with $\beta_{w,m} = \beta_{w,v} = 0$, has return

$$\tilde{r}_w^1 = E(\tilde{r}_w^1)$$

it is either a portfolio of risk-free assets or a "synthetic" risk-free asset. Either way, its return and expected return must equal the risk-free rate:

$$\tilde{r}_w^1 = E(\tilde{r}_w^1) = r_f$$

Multifactor Models and the APT

Since portfolio two, with $\beta_{w,m} = 1$ and $\beta_{w,v} = 0$, has return

$$\tilde{r}_w^2 = E(\tilde{r}_w^2) + [\tilde{r}_M - E(\tilde{r}_M)]$$

and expected return

$$E(\tilde{r}_w^2) = E(\tilde{r}_M)$$

it follows that

$$\tilde{r}_w^2 = \tilde{r}_M.$$

It either is the market portfolio or it always has the same return as the market portfolio.

Multifactor Models and the APT

And since portfolio three, with $\beta_{w,m} = 0$ and $\beta_{w,v} = 1$, has return

$$\tilde{r}_w^3 = E(\tilde{r}_w^3) + [\tilde{r}_V - E(\tilde{r}_V)]$$

and expected return

$$E(\tilde{r}_w^3) = E(\tilde{r}_V)$$

it follows that

$$\tilde{r}_w^3 = \tilde{r}_V.$$

It either is the "value" portfolio or it always has the same return as the value portfolio.

Multifactor Models and the APT

So by construction,

$$E(\tilde{r}_w) = r_f + \beta_{w,m}[E(\tilde{r}_M) - r_f] + \beta_{w,v}[E(\tilde{r}_V) - r_f]$$

must hold for portfolio 1, with $\beta_{w,m} = \beta_{w,v} = 0$:

$$E(\tilde{r}_w^1) = r_f$$

For portfolio 2, with $\beta_{w,m} = 1$ and $\beta_{w,v} = 0$:

$$E(\tilde{r}_w^2) = E(\tilde{r}_M)$$

And for portfolio 3, with $\beta_{w,m} = 0$ and $\beta_{w,v} = 1$:

$$E(\tilde{r}_w^3) = E(\tilde{r}_V)$$

Multifactor Models and the APT

Next, let's consider a fourth and fifth well-diversified portfolios. The fourth has

$$\tilde{r}_w^4 = E(\tilde{r}_w^4) + \beta_{w,m}[\tilde{r}_M - E(\tilde{r}_M)] + \beta_{w,v}[\tilde{r}_V - E(\tilde{r}_V)]$$

and features an arbitrary configuration of $\beta_{w,m}$ and $\beta_{w,v}$. Since we want to show that

$$E(\tilde{r}_w^4) = r_f + \beta_{w,m}[E(\tilde{r}_M) - r_f] + \beta_{w,v}[E(\tilde{r}_V) - r_f]$$

we obtain a contradiction by assuming instead that

$$E(\tilde{r}_w^4) = r_f + \beta_{w,m}[E(\tilde{r}_M) - r_f] + \beta_{w,v}[E(\tilde{r}_V) - r_f] + \Delta$$

where $\Delta \neq 0$.

Multifactor Models and the APT

The fifth well-diversified portfolio has the same configuration of $\beta_{w,m}$ and $\beta_{w,v}$ but is itself built as a portfolio of the first three portfolios:

1. Since portfolio 2 "loads" exclusively on the market portfolio, allocate the share $\beta_{w,m}$ to obtain the appropriate exposure to the source of aggregate risk associated with the market as a whole.
2. Since portfolio 3 loads exclusively on the value portfolio, allocate the share $\beta_{w,v}$ to obtain the appropriate exposure to the source of aggregate risk associated with value stocks.
3. Since portfolio 1 is free from aggregate risk, allocate the remaining share $1 - \beta_{w,m} - \beta_{w,v}$ to avoid any additional exposure to risk.

Multifactor Models and the APT

Notice again how the construction of this fifth portfolio draws an analogy between the APT and the Arrow-Debreu model.

Here, we are using the factors – in this case, the risk-free, market, and value portfolios – as "basic" or "fundamental" securities, like the A-D model's contingent claims, then building other portfolios up as baskets of the basic securities.

Since the factor model implies that all well-diversified portfolios are baskets of the basic securities, no-arbitrage arguments imply that prices for those well-diversified portfolios can be derived from the prices of those same basic securities.

Multifactor Models and the APT

$$\tilde{r}_w^1 = r_f$$

$$\tilde{r}_w^2 = E(\tilde{r}_M) + [\tilde{r}_M - E(\tilde{r}_M)]$$

$$\tilde{r}_w^3 = E(\tilde{r}_V) + [\tilde{r}_V - E(\tilde{r}_V)]$$

Since portfolio 5 allocates shares $1 - \beta_{w,m} - \beta_{w,v}$, $\beta_{w,m}$, and $\beta_{w,v}$ to portfolios 1 through 3,

$$\begin{aligned}
\tilde{r}_w^5 &= (1 - \beta_{w,m} - \beta_{w,v})r_f + \beta_{w,m}E(\tilde{r}_M) + \beta_{w,v}E(\tilde{r}_V) \\
&\quad + \beta_{w,m}[\tilde{r}_M - E(\tilde{r}_M)] + \beta_{w,v}[\tilde{r}_V - E(\tilde{r}_V)] \\
&= r_f + \beta_{w,m}[E(\tilde{r}_M) - r_f] + \beta_{w,v}[E(\tilde{r}_V) - r_f] \\
&\quad + \beta_{w,m}[\tilde{r}_M - E(\tilde{r}_M)] + \beta_{w,v}[\tilde{r}_V - E(\tilde{r}_V)]
\end{aligned}$$

Multifactor Models and the APT

Now we have two well-diversified portfolios, portfolio 4 with

$$\begin{aligned}\tilde{r}_w^4 &= r_f + \beta_{w,m}[E(\tilde{r}_M) - r_f] + \beta_{w,v}[E(\tilde{r}_V) - r_f] + \Delta \\ &\quad + \beta_{w,m}[\tilde{r}_M - E(\tilde{r}_M)] + \beta_{w,v}[\tilde{r}_V - E(\tilde{r}_V)]\end{aligned}$$

and portfolio 5 with

$$\begin{aligned}\tilde{r}_w^5 &= r_f + \beta_{w,m}[E(\tilde{r}_M) - r_f] + \beta_{w,v}[E(\tilde{r}_V) - r_f] \\ &\quad + \beta_{w,m}[\tilde{r}_M - E(\tilde{r}_M)] + \beta_{w,v}[\tilde{r}_V - E(\tilde{r}_V)]\end{aligned}$$

that have identical betas on both factors but different expected returns.

Multifactor Models and the APT

$$\begin{aligned}\tilde{r}_w^4 &= r_f + \beta_{w,m}[E(\tilde{r}_M) - r_f] + \beta_{w,v}[E(\tilde{r}_V) - r_f] + \Delta \\ &+ \beta_{w,m}[\tilde{r}_M - E(\tilde{r}_M)] + \beta_{w,v}[\tilde{r}_V - E(\tilde{r}_V)]\end{aligned}$$

$$\begin{aligned}\tilde{r}_w^5 &= r_f + \beta_{w,m}[E(\tilde{r}_M) - r_f] + \beta_{w,v}[E(\tilde{r}_V) - r_f] \\ &+ \beta_{w,m}[\tilde{r}_M - E(\tilde{r}_M)] + \beta_{w,v}[\tilde{r}_V - E(\tilde{r}_V)]\end{aligned}$$

Proposition 3 implies that $\Delta \neq 0$ is inconsistent with the absence of arbitrage opportunities, thereby completing the proof of proposition 4.

Multifactor Models and the APT

Hence, if individual asset returns are generated by our two-factor model, the expected return on any well-diversified portfolio must be

$$E(\tilde{r}_w) = r_f + \beta_{w,m}[E(\tilde{r}_M) - r_f] + \beta_{w,v}[E(\tilde{r}_V) - r_f]$$

Multifactor Models and the APT

Hence, the APT can be considered an extension of the CAPM that allows for multiple sources of aggregate risk, albeit one that applies to well-diversified portfolios but not necessarily to all individual assets.

And the APT can be viewed as a more empirically-motivated alternative to A-D no-arbitrage pricing theory, where the basic securities are associated with returns on specific portfolios like Fama and French's or even macroeconomic fundamentals like Chen, Roll, and Ross', instead of a more abstract notion of "states of the world."

Multifactor Models and the APT

For the Fama-French model, the same type of no-arbitrage arguments that led us to propositions 3 and 4 imply that the expected return on any well-diversified portfolio is

$$\begin{aligned} E(\tilde{r}_w) &= r_f + \beta_{w,m}[E(\tilde{r}_M) - r_f] \\ &\quad + \beta_{w,s}E(\tilde{r}_{SMB}) + \beta_{w,h}E(\tilde{r}_{HML}) \end{aligned}$$

where $\tilde{r}_M$, $\tilde{r}_{SMB}$, and $\tilde{r}_{HML}$ are returns on the market, small-minus-big, and high-minus-low book-to-market portfolios.

Multifactor Models and the APT

For the Fama-French model, the same type of no-arbitrage arguments that led us to propositions 3 and 4 imply that the expected return on any well-diversified portfolio is

$$\begin{aligned} E(\tilde{r}_w) &= r_f + \beta_{w,m}[E(\tilde{r}_M) - r_f] \\ &\quad + \beta_{w,s}E(\tilde{r}_{SMB}) + \beta_{w,h}E(\tilde{r}_{HML}) \end{aligned}$$

In this case, the risk-free rate r_f does not have to be subtracted from the expected returns on the SMB and HML portfolio's since these portfolios are constructed to be "self-financing."

Multifactor Models and the APT

$$\begin{aligned} E(\tilde{r}_w) &= r_f + \beta_{w,m}[E(\tilde{r}_M) - r_f] \\ &\quad + \beta_{w,s}E(\tilde{r}_{SMB}) + \beta_{w,h}E(\tilde{r}_{HML}) \end{aligned}$$

This version of the APT implies that a portfolio will have a higher expected returns when its own return is

1. Positively correlated with the market return (consistent with the CAPM)
2. Positively correlated with the SMB and/or HML return (inconsistent with the CAPM).

Correlation with SMB and/or HML may be an indicator of financial vulnerability, over and above macroeconomic risk.

Multifactor Models and the APT

Likewise, for the Chen-Roll-Ross model, no-arbitrage arguments imply that the expected return on any well-diversified portfolio is

$$
\begin{aligned}
E(\tilde{r}_w) &= r_f + \beta_{w,IP}[E(\tilde{r}_{IP}) - r_f] \\
&\quad + \beta_{w,UI}[E(\tilde{r}_{UI}) - r_f] + \beta_{w,EI}[E(\tilde{r}_{EI}) - r_f] \\
&\quad + \beta_{w,TS}[E(\tilde{r}_{TS}) - r_f] + \beta_{w,RP}[E(\tilde{r}_{RP}) - r_f]
\end{aligned}
$$

where $\tilde{r}_{IP}$, $\tilde{r}_{UI}$, $\tilde{r}_{EI}$, $\tilde{r}_{TS}$, and $\tilde{r}_{RP}$ are the returns on tracking portfolios for the macroeconomic factors: industrial production, unexpected inflation, expected inflation, the term structure, and bond risk premium.

Multifactor Models and the APT

Likewise, for the Chen-Roll-Ross model, no-arbitrage arguments imply that the expected return on any well-diversified portfolio is

$$\begin{aligned} E(\tilde{r}_w) &= r_f + \beta_{w,IP}[E(\tilde{r}_{IP}) - r_f] \\ &\quad + \beta_{w,UI}[E(\tilde{r}_{UI}) - r_f] + \beta_{w,EI}[E(\tilde{r}_{EI}) - r_f] \\ &\quad + \beta_{w,TS}[E(\tilde{r}_{TS}) - r_f] + \beta_{w,RP}[E(\tilde{r}_{RP}) - r_f] \end{aligned}$$

For this model, tracking portfolios for each of the macroeconomic variables must be constructed to make the no-arbitrage arguments.

Multifactor Models and the APT

$$\begin{aligned} E(\tilde{r}_w) &= r_f + \beta_{w,IP}[E(\tilde{r}_{IP}) - r_f] \\ &+ \beta_{w,UI}[E(\tilde{r}_{UI}) - r_f] + \beta_{w,EI}[E(\tilde{r}_{EI}) - r_f] \\ &+ \beta_{w,TS}[E(\tilde{r}_{TS}) - r_f] + \beta_{w,RP}[E(\tilde{r}_{RP}) - r_f] \end{aligned}$$

This version of the APT implies that a portfolio will have a higher expected return when its own return is

1. Positively correlated with IP and/or RP, which are high in good times.
2. Negatively correlated with UI (especially) and/or EI and/or TS, which are high in bad times.

The TS and RP variables may once again indicate a role for financial vulnerability.

Multifactor Models and the APT

$$\begin{aligned}\tilde{r}_w &= E(\tilde{r}_w) + \beta_{w,IP}\tilde{IP} + \beta_{w,UI}\tilde{UI} \\ &\quad + \beta_{w,EI}\tilde{EI} + \beta_{w,TS}\tilde{TS} + \beta_{w,RP}\tilde{RP} + \varepsilon_w\end{aligned}$$

The "Arrow-Debreu analogy" is also helpful in interpreting the Chen-Roll-Ross model, since this model implies that one can "hedge" against a specific form of macroeconomic risk – say, an unexpected shift in the term structure – by building a portfolio with $\beta_{w,TS} = 1$ and all of the other loadings equal to zero.

Multifactor Models and the APT

Antti Ilmanen, *Expected Returns: An Investor's Guide to Harvesting Market Rewards*, John Wiley & Sons, 2011.

Provides an exhaustive overview of expected returns on stocks, bonds, currencies, commodities, and other asset classes, emphasizing that expected returns on all those assets depend systematically on exposures to sources of risk that go beyond the CAPM's market return, but still remain limited in number.

Multifactor Models and the APT

Clifford S. Asness, Antti Ilmanen, Ronen Israel, and Tobias J. Moskowitz, "Investing With Style," *Journal of Investment Management* Vol.13 (First Quarter 2015): pp.27-63.

Shows how higher expected returns on value and other portfolios can be captured in practice by "smart beta" fund management strategies.

The APT and Risk Premia

Since the APT's formula for expected returns applies to well-diversified portfolios and not necessarily to individual assets, one should take care in using it to value risky cash flows from individual investment projects.

Still, partly for the sake of completeness and also to see how we can extend our previous valuation exercise using the CAPM to allow for the APT's multiple sources of aggregate risk, let's go ahead and see how it works.

The APT and Risk Premia

The valuation problem involves attaching a price P_0^C today (at $t = 0$) to a risky cash-flow $\tilde{C}_1$ received in the future (at $t = 1$).

After taking the expected value $E(\tilde{C}_1)$ of the cash flow, we want to find the appropriate risk premium ψ to add to the risk-free rate r_f so that

$$P_0^C = \frac{E(\tilde{C}_1)}{1 + r_f + \psi}$$

provides an accurate assessment of the project's value today.

The APT and Risk Premia

Since the APT, like the CAPM, is cast in terms of returns, we can start by computing the random return on the project as

$$1 + \tilde{r}_C = \frac{\tilde{C}_1}{P_0^C}$$

or

$$\tilde{r}_C = \frac{\tilde{C}_1}{P_0^C} - 1 = \frac{\tilde{C}_1 - P_0^C}{P_0^C}.$$

The APT and Risk Premia

Next, we need to choose a factor model to use with the APT.

Let's use a simpler, two-factor version of the Chen-Roll-Ross macroeconomic model where the return on each asset i is given by

$$\tilde{r}_i = E(\tilde{r}_i) + \beta_{i,IP}\tilde{IP} + \beta_{i,INF}\tilde{INF} + \varepsilon_i$$

where $\tilde{IP}$ is industrial production and $\tilde{INF}$ is inflation.

The APT and Risk Premia

$$\tilde{r}_i = E(\tilde{r}_i) + \beta_{i,IP}\tilde{IP} + \beta_{i,INF}\tilde{INF} + \varepsilon_i$$

Here is where another advantage of the APT becomes apparent.

It might be easier to estimate (or guess!) how the return on a project will vary with macroeconomic output and inflation then to estimate how it will vary with the return on the market portfolio.

The APT and Risk Premia

$$\tilde{r}_i = E(\tilde{r}_i) + \beta_{i,IP}\tilde{IP} + \beta_{i,INF}\tilde{INF} + \varepsilon_i$$

Boldly setting aside the distinction between well-diversified portfolios and individual assets or cash flows, our two-factor macroeconomic model and the APT imply that the expected return on the project should be

$$E(\tilde{r}_C) = r_f + \beta_{c,IP}[E(\tilde{r}_{IP}) - r_f] + \beta_{c,INF}[E(\tilde{r}_{INF}) - r_f]$$

where $\tilde{r}_{IP}$ and $\tilde{r}_{INF}$ are returns on the tracking or mimicking portfolios for industrial production and inflation.

The APT and Risk Premia

Finally, we can combine

$$\tilde{r}_C = \frac{\tilde{C}_1}{P_0^C} - 1$$

and

$$E(\tilde{r}_C) = r_f + \beta_{c,IP}[E(\tilde{r}_{IP}) - r_f] + \beta_{c,INF}[E(\tilde{r}_{INF}) - r_f]$$

to obtain

$$E\left(\frac{\tilde{C}_1}{P_0^C} - 1\right) = r_f + \beta_{c,IP}[E(\tilde{r}_{IP}) - r_f] + \beta_{c,INF}[E(\tilde{r}_{INF}) - r_f]$$

The APT and Risk Premia

$$E\left(\frac{\tilde{C}_1}{P_0^C} - 1\right) = r_f + \beta_{c,IP}[E(\tilde{r}_{IP}) - r_f] + \beta_{c,INF}[E(\tilde{r}_{INF}) - r_f]$$

implies

$$\begin{aligned} \left(\frac{1}{P_0^C}\right) E(\tilde{C}_1) &= 1 + r_f \\ &+ \beta_{c,IP}[E(\tilde{r}_{IP}) - r_f] + \beta_{c,INF}[E(\tilde{r}_{INF}) - r_f] \end{aligned}$$

The APT and Risk Premia

$$\left(\frac{1}{P_0^C}\right) E(\tilde{C}_1) = 1 + r_f + \beta_{c,IP}[E(\tilde{r}_{IP}) - r_f] + \beta_{c,INF}[E(\tilde{r}_{INF}) - r_f]$$

The APT implies

$$P_0^C = \frac{E(\tilde{C}_1)}{1 + r_f + \beta_{c,IP}[E(\tilde{r}_{IP}) - r_f] + \beta_{c,INF}[E(\tilde{r}_{INF}) - r_f]}$$

The APT and Risk Premia

The APT implies

$$P_0^C = \frac{E(\tilde{C}_1)}{1 + r_f + \beta_{c,IP}[E(\tilde{r}_{IP}) - r_f] + \beta_{c,INF}[E(\tilde{r}_{INF}) - r_f]}$$

or, more simply,

$$P_0^C = \frac{E(\tilde{C}_1)}{1 + r_f + \psi}$$

where the risk premium

$$\psi = \beta_{c,IP}[E(\tilde{r}_{IP}) - r_f] + \beta_{c,INF}[E(\tilde{r}_{INF}) - r_f]$$

compensates for the project's exposure to the risk of recession (falling IP) or inflation (rising INF).