Mathematical Methods in Finance

Lecture 11: Beyond Black-Scholes-Merton: Stochastic Volatility

Fall 2013

Overview

- Empirical evidence for the Black-Scholes model: historical and implied
- Possible explanations
- Derivative valuation under stochastic volatility models

Empirical Evidence

Is the Black-Scholes model an accurate description of the financial world?

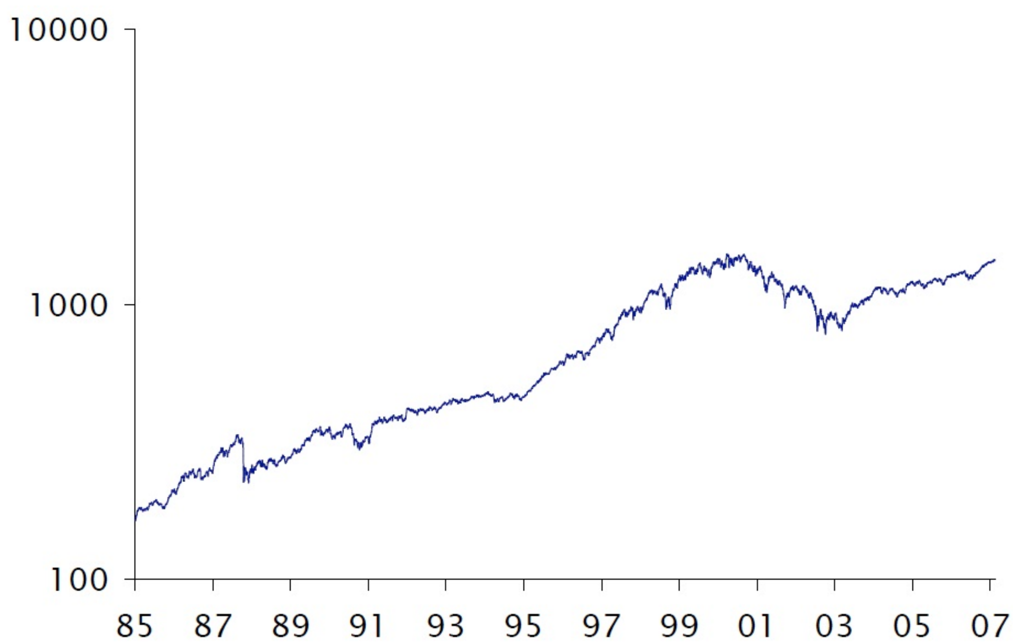
Historical (underlying asset)

- distribution of returns (check lognormal assumption)
- time series properties of returns (check independence and constant volatility assumptions)

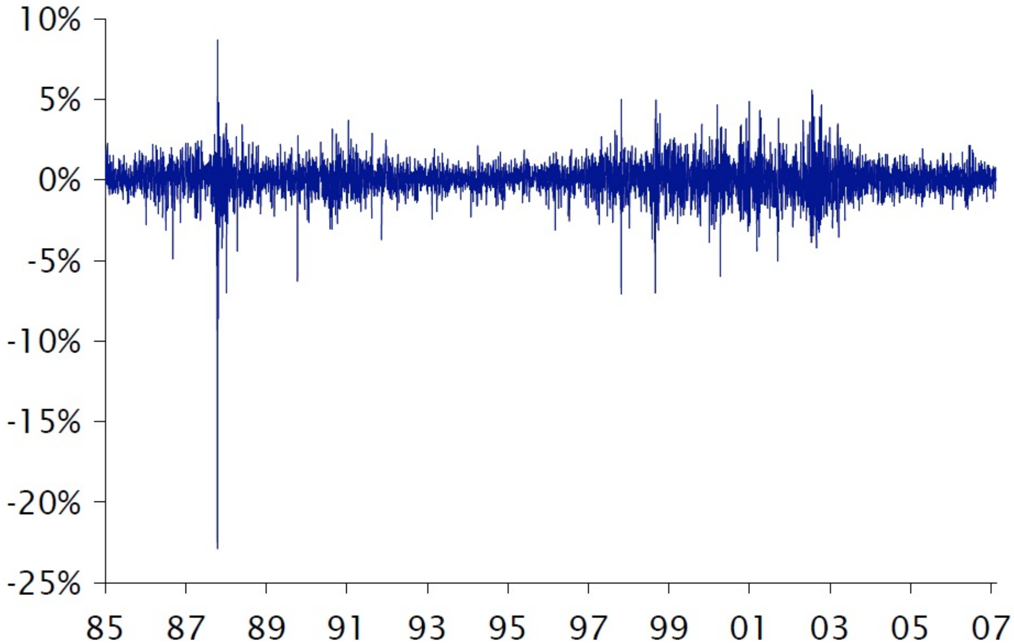
Implied (option)

- implied volatility (check dependence on strike and maturity; check change over time)

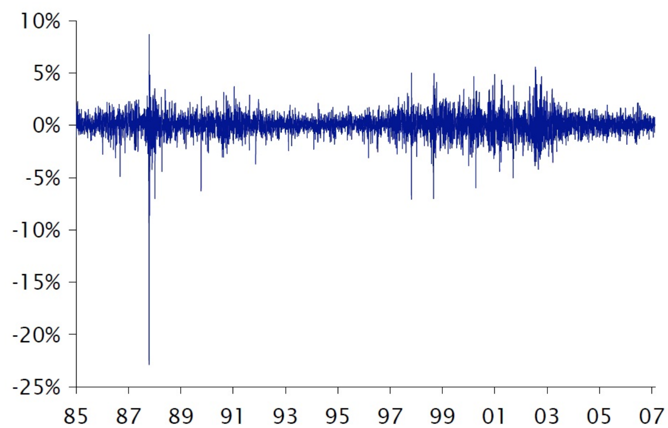
S&P500 Prices:1985-2007



Daily S&P500 Ln-Returns: 1985-2007

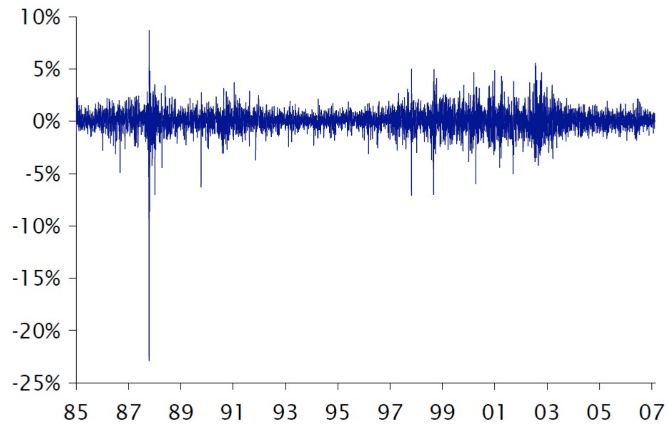


Non-Normality of Ln-Returns



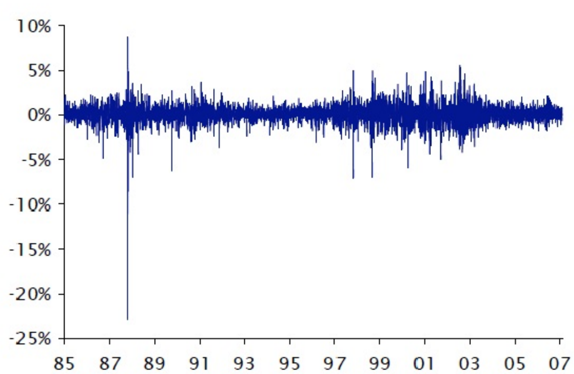
- Daily mean of Ln-returns: 0.04%
- Daily standard deviation of Ln-returns: 1.1%
- Daily Ln-return on 10/19/87: -22.9%
- Under normality, $\mathbb{P}(X \leq -22.9\%) = N(-20.9) \approx 10^{-96}$ (every 10^{93} years)

Non-Normality of Ln-Returns

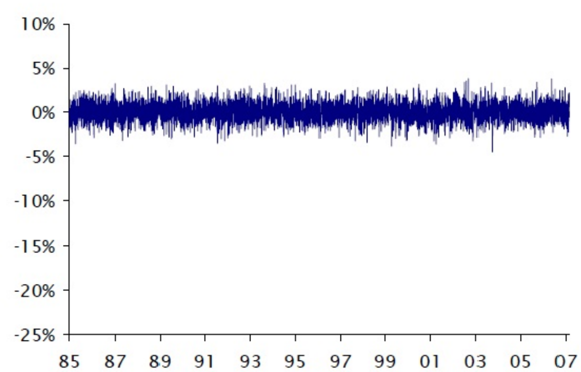


- 10/13/1989, $\mathbb{P}(X \leq -6.31\%) = N(-5.8) \approx 10^{-9}$ (every 1,000,000 years)
- Extreme movements are much more common than log-normal assumption suggests!

Non-Normality of Ln-Returns: Data versus Simulation



Data



Simulated returns

Non-Normality of Ln-Returns

$$\text{Kurtosis}(X) = \frac{E[(X - \bar{X})^4]}{\sigma^4} - 3.$$

Kurtosis is one measure of "fat tails", or the probability of extreme events.

$$\text{Kurtosis}(\text{normal random variable}) = 0.$$

$$\text{Sample Kurtosis}(\text{S\&P500 Ln-returns}) = 45.$$

Is this statistically significant?

Autocorrelation of Ln-Returns

Given a time series of Ln-returns $R_1, R_2, ..., R_n$, define the autocorrelation with a lag of k by

$$C(k) = \frac{E[(R_i - \bar{R})(R_{i-k} - \bar{R})]}{\sigma_R^2}$$

Time	Price	Ln-Return Lag 0	Ln-Return Lag 1	Ln-Return Lag 2
0	S_0			
Δt	$S_{\Delta t}$	$R_1 = \ln\left(\frac{S_{\Delta t}}{S_0}\right)$		
$2\Delta t$	$S_{2\Delta t}$	$R_2 = \ln\left(\frac{S_{2\Delta t}}{S_{\Delta t}}\right)$	R_1	
$3\Delta t$	$S_{3\Delta t}$	$R_3 = \ln\left(\frac{S_{3\Delta t}}{S_{2\Delta t}}\right)$	R_2	R_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n\Delta t$	$S_{n\Delta t}$	$R_n = \ln\left(\frac{S_{n\Delta t}}{S_{(n-1)\Delta t}}\right)$	R_{n-1}	R_{n-2}

Autocorrelation of Ln-Returns

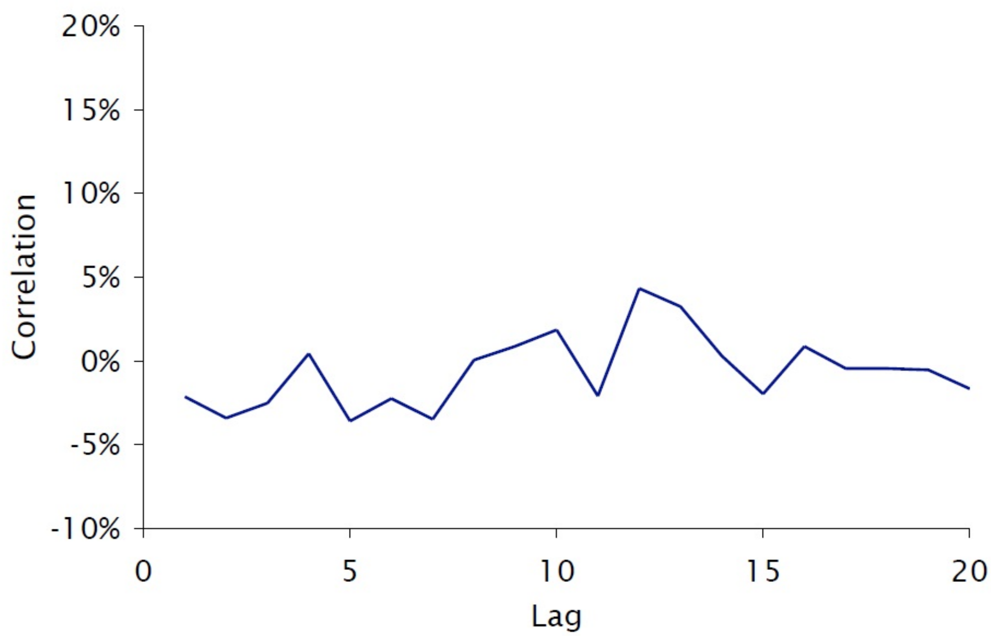
Under Black-Scholes assumptions,

$$R_i = \ln\left(\frac{S_{i\Delta t}}{S_{(i-1)\Delta t}}\right) = (\mu - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t} Z_i,$$

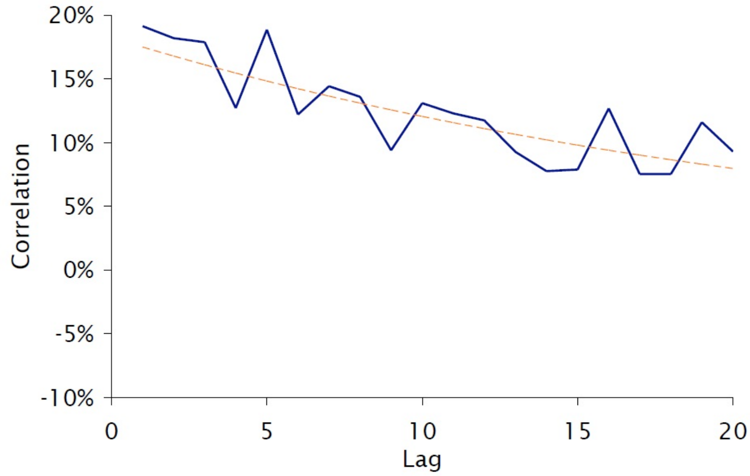
where $Z_1, Z_2, ...$are independent $N(0,1)$ random variables. Thus, $R_1, R_2, ...$should be independent, and

$$C(k) = 0 \quad \text{for } k \geq 1.$$

Autocorrelation of Ln-Returns



Autocorrelation of S&P500 Squared Ln-Returns

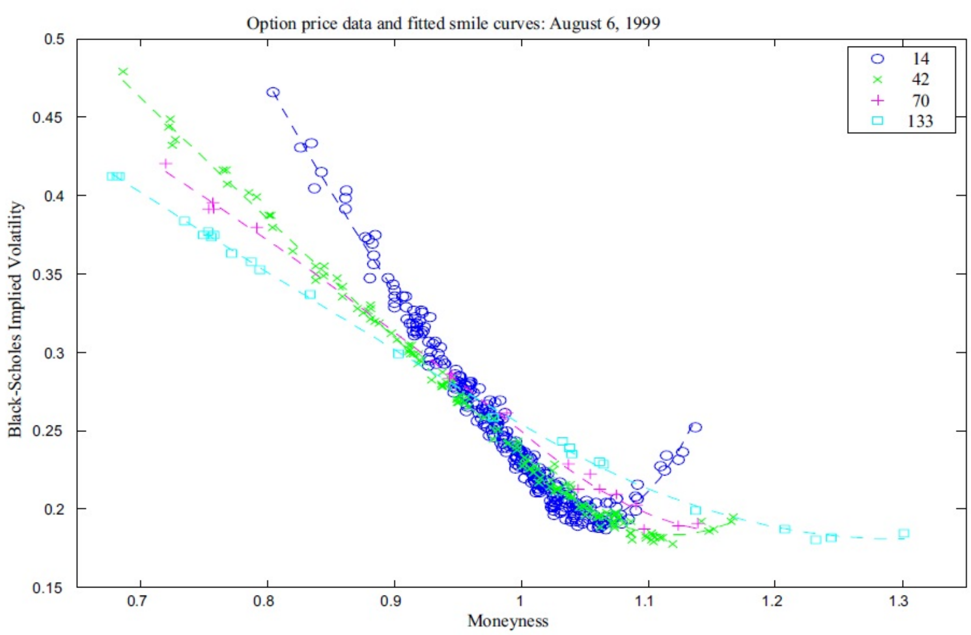


Autocorrelation of squared Ln-returns is hightly significant, even at a 20-day lag.

Big move today $\Rightarrow$ Big move tomorrow

$\Rightarrow$ volatility clustering!

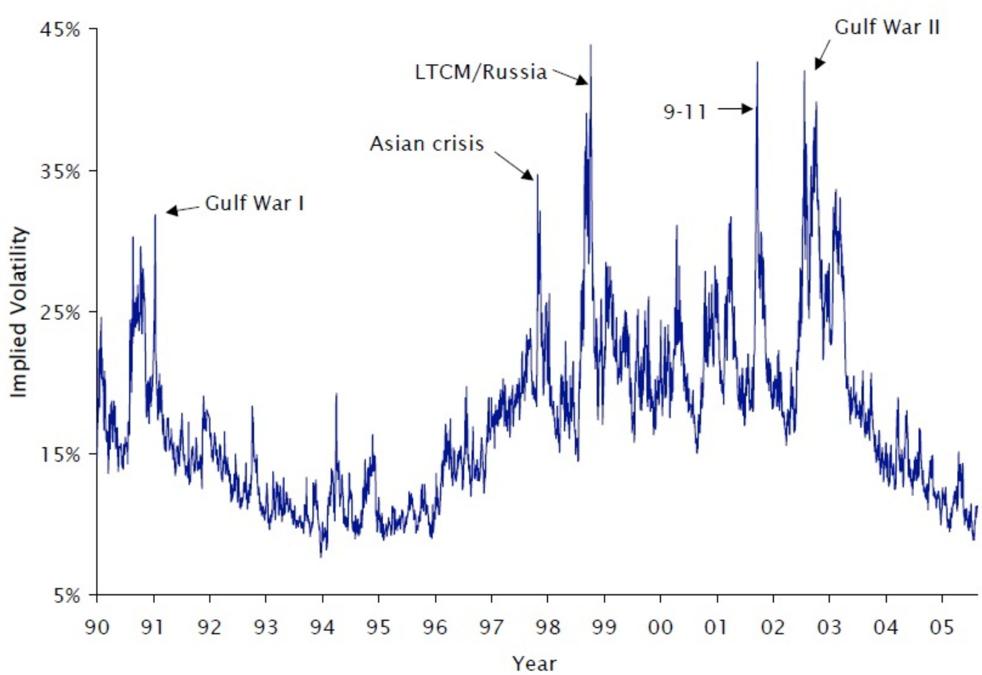
Implied Volatility Smile



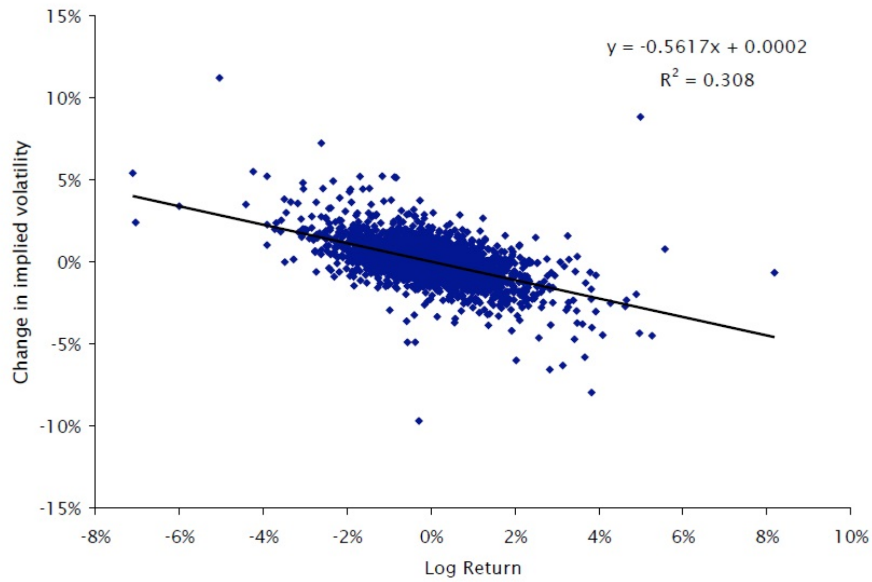
Implied Volatility Smile

- Option prices are often expressed in units of Black-Scholes implied volatility.
- Under the Blask-Scholes model, implied volatility should be constant as a function of strike price and maturity.
- Non-constant implied volatilities are direct evidence that the market does not price options with the Black-Sholes model.

S&P500 ATM Implied Volatility: 1990-2005



Change in Implied Volatility vs. Ln-Returns



The change in Implied volatility decreases when Ln-returns increases.

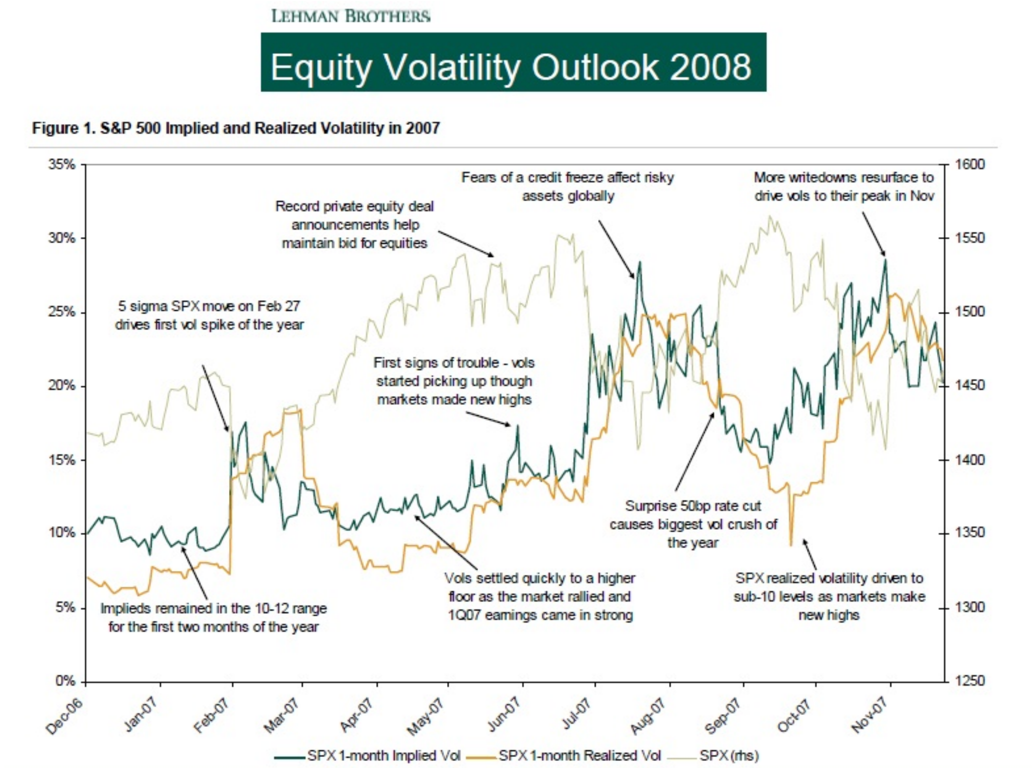
Empirical Evidence

- Ln-returns exhibit fatter tails that the normal distribution suggests
- Autocorrelation of squared Ln-returns implies dependence (volatility clustering)
- Downward sloping implied volatility curve (after 1987)
- Implied volatility changes over time
- Implied volatility changes are correlated with Ln-returns

What models are consistent with this?

$$(1) \text{ and } (3) \Rightarrow \text{jumps}$$
$$(1), (2), (3), (4), \text{ and } (5) \Rightarrow \text{stochastic volatility}$$

Empirical Evidence



Stochastic Volatility Models

So based on the empirical evidences, one of the directions is to generalize the Black-Scholes model by adding stochastic volatility.

Why not model volatility in the same way a stock price is modeled (i.e., log-normal distribution)?

Properties of a stochastic volatility model:

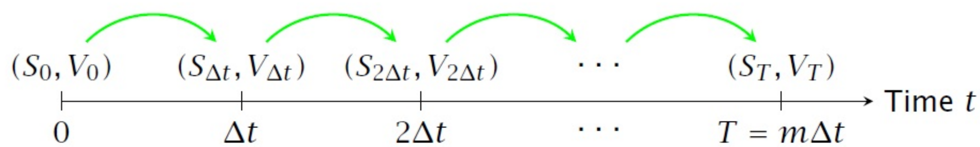
- Mean-reversion (long-run volatility parameter $\sqrt{\theta}$)
- Speed of mean-reversion (κ)
- Volatility of variance (σ_v)
- Correlation of variance and stock processes (ρ)

Formal Stochastic Volatility Model

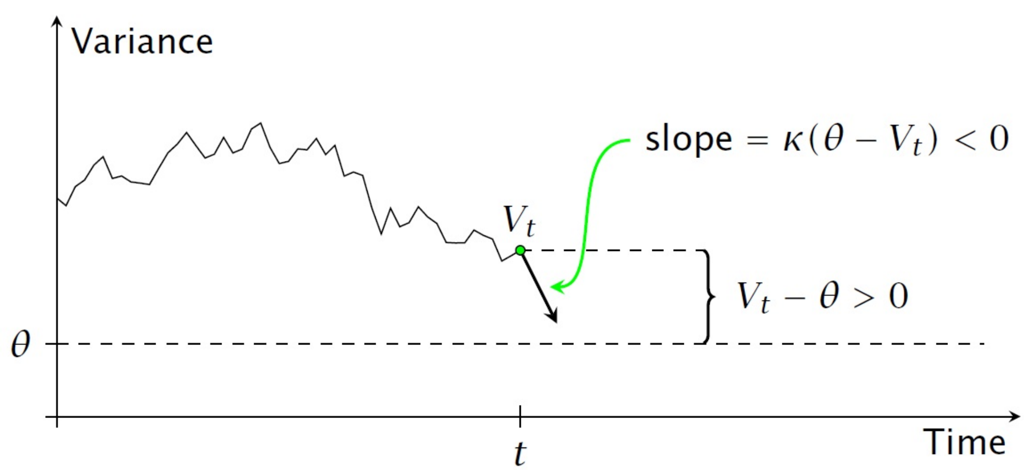
Heston (1993) Model

$$
\begin{aligned}
\frac{dS_t}{S_t} =& \mu dt + \sqrt{V_t} dW_t^1 \\
dV_t =& \kappa(\theta - V_t)dt + \sigma_v \sqrt{V_t} dW_t^2
\end{aligned}
$$

$S_t =$ stock price at time t $\quad V_t =$ variance at time t

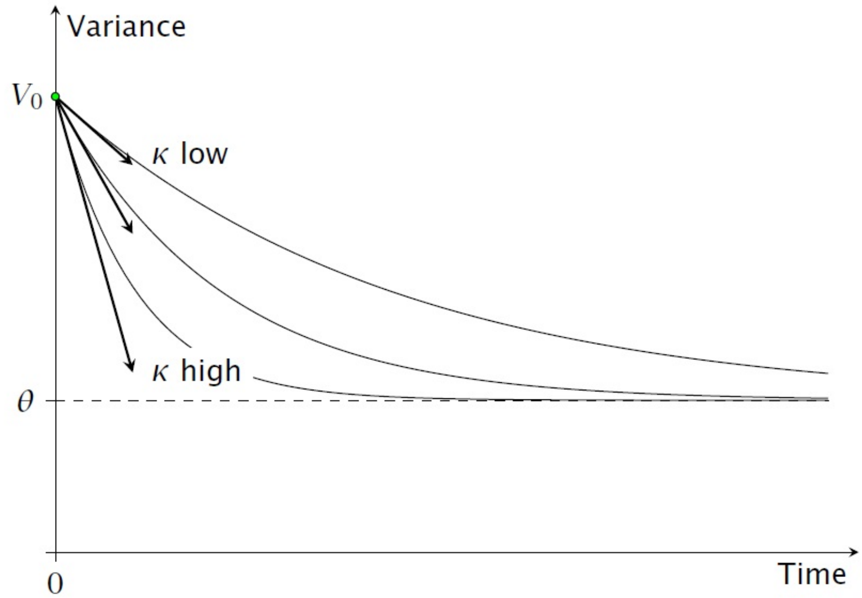


Formal Stochastic Volatility Model



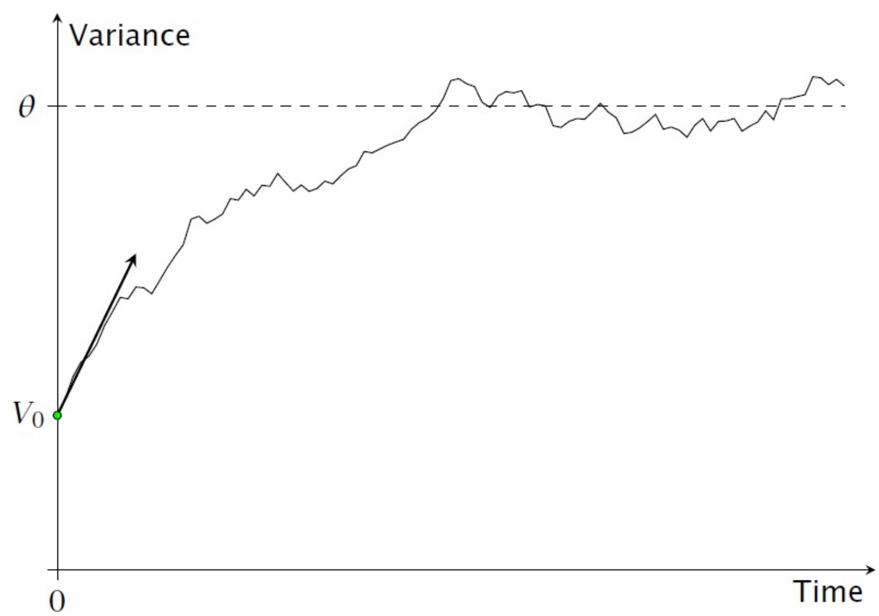
- Long-run variance θ (long-run volatility $\sqrt{\theta}$)
- Speed of mean reversion κ
- Volatility of variance σ_v
- $\sqrt{V_t}$ term guarantees positive variance (in the limit as $\Delta t \to 0$)

Mean Reversion Rate



$$\sigma_V = 0 \quad V_0 > \theta$$

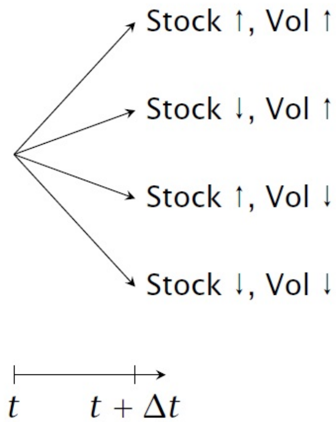
Mean Reversion Rate



$$\sigma_V = 0 \quad V_0 < \theta$$

Market Incompleteness

The stochastic volatility model is an incomplete market.



- Options cannot be replicated by dynamic trading of a stock and bond
- Can hedge volatility risk by trading options
- "Less" incomplete than jump models

How to price an option under a stochastic volatility model?

Review from Black-Sholes-Merton: An Understanding

We assume that real world dynamics of the underlying asset is

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW^P(t).$$

For a call option with maturity T and strike K, its price $v(t, S(t))$ satisfies

$$\begin{aligned} dv(t, S(t)) &= \frac{\partial v}{\partial t} dt + \frac{\partial v}{\partial x} dS(t) + \frac{1}{2}\frac{\partial^2 v}{\partial x^2} d[S, S](t) \\ &= \frac{\partial v}{\partial t} dt + \frac{\partial v}{\partial x}[\mu S(t) dt + \sigma S(t) dW^P(t)] + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}\sigma^2 S(t)^2 dt. \end{aligned}$$

Review from Black-Sholes-Merton: An Understanding

By the Black-Scholes-Merton equation, the above equation deduce to

$$dv(t,S(t)) = \left(rv(t,S(t)) - rS(t)\frac{\partial v}{\partial x}\right)dt + \frac{\partial v}{\partial x}[\mu S(t)dt + \sigma S(t)dW^P(t)].$$

Thus,

$$dv(t,S(t)) - rv(t,S(t))dt = \frac{\partial v}{\partial x}\sigma S(t)\left(\frac{\mu - r}{\sigma}dt + dW^P(t)\right), \qquad (1)$$

which is equivalent to

$$\frac{dv(t,S(t))}{v(t,S(t))} - rdt = \frac{\frac{\partial v}{\partial x}\sigma S(t)}{v(t,S(t))}\left(\frac{\mu - r}{\sigma}dt + dW^P(t)\right). \qquad (2)$$

The term $\frac{dv(t,S(t))}{v(t,S(t))} - rdt$ can be understood as an **excess return**.

Review from Black-Sholes-Merton: An Understanding

Integrating both sides of (1) and taking conditional expectation E_t,

$$E_t v(t+\Delta, S(t+\Delta)) - v(t,S(t)) - \int_t^{t+\Delta} rE_t v(u,S(u))du$$
$$= \int_t^{t+\Delta} E_t\left(\frac{\partial v}{\partial x}S(u)\right)(\mu - r)du.$$

Thus,

$$\frac{1}{\Delta}\left[E_t v(t+\Delta, S(t+\Delta)) - v(t,S(t))\right] - \frac{1}{\Delta}\int_t^{t+\Delta} rE_t v(u,S(u))du$$
$$= \frac{1}{\Delta}\int_t^{t+\Delta} E_t\left(\frac{\partial v}{\partial x}S(u)\right)(\mu - r)du.$$

Let $\Delta \to 0$,

$$\lim_{\Delta\to 0}\frac{1}{\Delta}\frac{E_t v(t+\Delta, S(t+\Delta)) - v(t,S(t))}{v(t,S(t))} - r = \frac{\frac{\partial v}{\partial x}\sigma S(t)}{v(t,S(t))}\frac{\mu - r}{\sigma} \qquad (3)$$

Market Price of Risk: the Black-Sholes-Merton Case

The Sharpe Ratio or the **market price of risk** of the underlying asset is

$$\lambda = \frac{\mu - r}{\sigma}.$$

Here, if we view

$$\frac{1}{\Delta} \frac{E_t v(t+\Delta, S(t+\Delta)) - v(t, S(t))}{v(t, S(t))}$$

as an expected return per time, (3) can be interpreted as a "CAPM" type result. Here $\frac{\frac{\partial v}{\partial x} S(t)}{v(t, S(t))}$ plays a role as the "beta".

The LHS of (3) is an instantenous excess return. We can somehow regard $\frac{\frac{\partial v}{\partial x} \sigma S(t)}{v(t, S(t))}$ as a percentage of "Brownian risk corresponding to $W^P(t)$" and $\frac{\mu - r}{\sigma}$ is the excess premium per unit of $dW^P(t)$.

A Mathematical Characterization of Market Price of Risk

We have

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW^P(t) = r dt + \sigma \left(\frac{\mu - r}{\sigma} dt + dW^P(t) \right).$$

No-arbitrage argument yields the risk-neutral probability measure, under which the dynamics of the underlying asset is written as

$$\frac{dS(t)}{S(t)} = r dt + \sigma dW^Q(t).$$

By the Girsanov theorem, $W^Q(t)$ can be constructed through

$$W^Q(t) = \lambda t + W^P(t) = \frac{\mu - r}{\sigma} t + W^P(t).$$

So, the market price of risk is exactly the "drift" in the Girsanov change of measure.

Stochastic Volatility Model

By analogy, we work on the stochastic volatility case in which the market with the underlying asset and a money market account is incomplete.

First, we assume the model under the physical probability measure as

$$\begin{aligned}\frac{dS(t)}{S(t)} &= \mu dt + \sigma(t) dW_1^P(t), \\ d\sigma(t) &= a(\sigma(t))dt + b(\sigma(t))\left[\rho dW_1^P(t) + \sqrt{1-\rho^2}dW_2^P(t)\right],\end{aligned}$$

where the $(W_1^P(t), W_2^P(t))$ is a two-dimensional standard Brownian motion.

Construction of Replicating Portfolio

There are underlying asset $S(t)$, a kind of asset $V_1(t)$ solely depending on volatility (e.g., a variance swap or a delta-hedged portfolio) and a money market account for us to replicate an option (with maturity T and strike K) with value $V(t)$.

To replicate an option, we use $\Delta(t)$ shares of the underlying asset with price $S(t)$, $\Delta_1(t)$ shares of an arbitrary asset with value $V_1(t)$. And put the rest in money market account. The change of the a self-financing replicating portfolio value satisfies

$$\begin{aligned}d\Pi(t) =& \Delta(t)dS(t) + \Delta_1(t)dV_1(t) \\ & + r(\Pi(t) - \Delta(t)S(t) - \Delta_1(t)V_1(t))dt.\end{aligned} \tag{4}$$

Construction of Replicating Portfolio

We can assume that $\Pi(t) = v(t, S(t), \sigma(t))$ for some smooth function $v(t, x, y)$ and $V_1(t) = v_1(t, \sigma(t))$ for some smooth function $v(t, y)$. Using the Ito formula on $v(t, S(t), \sigma(t))$,

$$\begin{aligned}
& dv(t,S(t),\sigma(t)) = \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial x}dS\left(t\right) + \frac{\partial v}{\partial y}d\sigma\left(t\right) \\
+ & \quad \frac{1}{2}\frac{\partial^2 v}{\partial x^2}d\left[S,S\right](t) + \frac{1}{2}\frac{\partial^2 v}{\partial y^2}d\left[\sigma,\sigma\right](t) + \frac{\partial^2 v}{\partial x \partial y}d\left[S,\sigma\right](t) \\
= & \quad \left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}\mu S\left(t\right) + \frac{\partial v}{\partial y}a\left(\sigma\left(t\right)\right) + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}\sigma^2\left(t\right)S^2\left(t\right)\right. \\
& \quad \left. +\frac{1}{2}\frac{\partial^2 v}{\partial y^2}\left(b^2\left(\sigma\left(t\right)\right)\right) + \frac{\partial^2 v}{\partial x \partial y}S(t)\sigma\left(t\right)b\left(\sigma\left(t\right)\right)\rho\right)dt \\
+ & \quad \left(\frac{\partial v}{\partial x}\sigma\left(t\right)S\left(t\right) + \frac{\partial v}{\partial y}b\left(\sigma\left(t\right)\right)\rho\right)dW_1^P\left(t\right) + \frac{\partial v}{\partial y}b\left(\sigma\left(t\right)\right)\sqrt{1-\rho^2}dW_2^P\left(t\right).
\end{aligned}$$

Construction of Replicating Portfolio

On the other hand, from (4), we also use Ito formula

$$\begin{aligned}
d\Pi(t) &= \Delta(t)[\mu S(t)dt + \sigma(t)S(t)dW_1^P(t)] + \Delta_1(t)dv_1(t,\sigma(t)) \\
&\quad +r(v(t,S(t),\sigma(t)) - \Delta(t)S(t) - \Delta_1(t)v_1(t,\sigma(t)))dt \\
&= \left(\Delta(t)\mu S\left(t\right) + \Delta_1(t)\left(\frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial y}a\left(\sigma\left(t\right)\right) + \frac{1}{2}\frac{\partial^2 v_1}{\partial y^2}b^2\left(\sigma\left(t\right)\right)\right)\right. \\
&\quad \left. +r(v(t,S(t),\sigma(t)) - \Delta(t)S(t) - \Delta_1(t)v_1(t,\sigma(t)))\right)dt \\
&\quad + \left(\Delta(t)\sigma(t)S(t) + \Delta_1(t)\frac{\partial v_1}{\partial y}b\left(\sigma\left(t\right)\right)\rho\right)dW_1^P\left(t\right) \\
&\quad +\Delta_1(t)\frac{\partial v_1}{\partial y}b\left(\sigma\left(t\right)\right)\sqrt{1-\rho^2}dW_2^P\left(t\right).
\end{aligned}$$

Then, replication requires to equate the above two equations.

Replicating Strategy

Thus, we should find the following two equations for the replicating strategy $(\Delta(t), \Delta_1(t))$ as

$$\begin{aligned} \frac{\partial v}{\partial x}\sigma\left(t\right)S\left(t\right)+\frac{\partial v}{\partial y}b\left(\sigma\left(t\right)\right)\rho &= \Delta(t)\sigma(t)S(t)+\Delta_1(t)\frac{\partial v_1}{\partial y}b\left(\sigma\left(t\right)\right)\rho, \\ \frac{\partial v}{\partial y}b\left(\sigma\left(t\right)\right)\sqrt{1-\rho^2} &= \Delta_1(t)\frac{\partial v_1}{\partial y}b\left(\sigma\left(t\right)\right)\sqrt{1-\rho^2}. \end{aligned}$$

Solving this equation system, we obtain the following replicating strategy

$$\Delta_1(t)=\frac{\partial v}{\partial y}(t,S(t),\sigma(t))/\frac{\partial v_1}{\partial y}(t,\sigma(t))$$

and

$$\Delta(t)=\frac{\partial v}{\partial x}(t,S(t),\sigma(t)).$$

PDE for Option Pricing

Equate the above two equations of $d\Pi(t)$ and $dv(t,S(t),\sigma(t))$, we can also get a PDE:

$$\begin{aligned} &\frac{\frac{\partial v}{\partial t}+\frac{1}{2}x^2y^2\frac{\partial^2 v}{\partial x^2}+\frac{1}{2}b^2\left(y\right)\frac{\partial^2 v}{\partial y^2}+\rho b\left(y\right)xy\frac{\partial^2 v}{\partial x\partial y}+rx\frac{\partial v}{\partial x}+a\left(y\right)\frac{\partial v}{\partial y}-rv}{\frac{\partial v}{\partial y}} \\ = &\frac{\frac{\partial v_1}{\partial t}+\frac{1}{2}b^2\left(y\right)\frac{\partial^2 v_1}{\partial y^2}+a\left(y\right)\frac{\partial v_1}{\partial y}-rv_1}{\frac{\partial v_1}{\partial y}}. \end{aligned}$$

Note that the RHS is only a function on the independent variable t and y. And if you have $\frac{\partial v}{\partial x}=0$, the left-hand side reduced to the right-hand side. We assume such a function to be

$$f(t,y)=\frac{\frac{\partial v_1}{\partial t}+\frac{1}{2}b^2\left(y\right)\frac{\partial^2 v_1}{\partial y^2}+a\left(y\right)\frac{\partial v_1}{\partial y}-rv_1}{\frac{\partial v_1}{\partial y}}.$$

PDE for Option Pricing

So, we have

$$\frac{\partial v}{\partial t}+\frac{1}{2}x^2y^2\frac{\partial^2 v}{\partial x^2}+\frac{1}{2}b^2(y)\frac{\partial^2 v}{\partial y^2}+\rho b(y)xy\frac{\partial^2 v}{\partial x\partial y} + rx\frac{\partial v}{\partial x}+a(y)\frac{\partial v}{\partial y}-f(t,y)\frac{\partial v}{\partial y}-rv=0 \tag{5}$$

and

$$\frac{\partial v_1}{\partial t}+\frac{1}{2}b^2(y)\frac{\partial^2 v_1}{\partial y^2}+a(y)\frac{\partial v_1}{\partial y}-f(t,y)\frac{\partial v_1}{\partial y}-rv_1=0. \tag{6}$$

- These PDEs are obtained from the replication procedure.
- Note that for option valuation, $f(t,y)$ has to be pre-specified as part of the real world model.

Feymann-Kac Representation for Option Pricing

For pricing an option with maturity T and payoff function $P(x)$, we impose the terminal condition $v(T,x,y)=P(x)$. We have

$$v(t,S(t),\sigma(t))=e^{-r(T-t)}E_t^Q P(S(T)).$$

Here, Q is the risk neutral measure under which the dynamics of $(S(t),\sigma(t))$ follows that

$$\begin{aligned}\frac{dS(t)}{S(t)} &= rdt+\sigma(t)dW_1^Q(t),\\ d\sigma(t) &= [a(\sigma(t))-f(t,\sigma(t))]dt+b(\sigma(t))\left[\rho dW_1^Q(t)+\sqrt{1-\rho^2}dW_2^Q(t)\right],\end{aligned}$$

where $(W_1^Q(t),W_2^Q(t))$ is a two-dimensional standard Brownian motion under the martingale pricing measure Q. This can be shown in a similar way to the one-dimensional case discussed in Lecture 9. How to obtain such a Q from P?

MPR and MPVR

$V_1(t) = v_1(t, \sigma(t))$ plays a role as the "Delta-hedged" option. Now, let us look at the excess return of such an asset. Use Ito Formula on $dv_1(t, \sigma(t))$, and apply

$$\frac{\partial v_1}{\partial t} + \frac{1}{2} b^2(y) \frac{\partial^2 v_1}{\partial y^2} + a(y) \frac{\partial v_1}{\partial y} - f(t, y) \frac{\partial v_1}{\partial y} - r v_1 = 0.$$

We have

$$dv_1(t, \sigma(t)) - r v_1(t, \sigma(t)) dt = b(\sigma(t)) \frac{\partial v_1}{\partial y} \left[\frac{f(t, \sigma(t))}{b(\sigma(t))} dt + dW_v^P(t) \right],$$

where

$$W_v^P(t) = \rho W_1^P(t) + \sqrt{1 - \rho^2} W_2^P(t)$$

represent a Brownian motion driving the volatility process.

MPR and MPVR

Analogy to (3), we have

$$\lim_{\Delta \to 0} \frac{1}{\Delta} \frac{E_t v_1(t + \Delta, \sigma(t + \Delta)) - v_1(t, \sigma(t))}{v_1(t, \sigma(t))} - r = \frac{b(\sigma(t)) \frac{\partial v_1}{\partial y}}{v_1(t, \sigma(t))} \frac{f(t, \sigma(t))}{b(\sigma(t))}.$$

- This is an analog to the CAPM as many people claimed.
- The LHS is an instantaneous excess return.
- We can somehow regard $\frac{b(\sigma(t)) \frac{\partial v_1}{\partial y}}{v_1(t, \sigma(t))}$ as a percentage of "Brownian risk corresponding to $W_v^P(t)$" and $\frac{f(t, \sigma(t))}{b(\sigma(t))}$ as the excess risk premium per unit of $dW_v^P(t)$.

MPR and MPVR

Now, we look at the excess return of the option. Based on Ito formula and the pricing equation (5), we can get

$$\begin{aligned} & dv(t,S(t),\sigma(t)) - rv(t,S(t),\sigma(t))dt \\ = \; & \sigma(t)\,S(t)\,\frac{\partial v}{\partial x}\left[\frac{\mu - r}{\sigma(t)}dt + dW_1^P(t)\right] + b(\sigma(t))\,\frac{\partial v_1}{\partial y}\left[\frac{f(t,\sigma(t))}{b(\sigma(t))}dt + dW_v^P(t)\right]. \end{aligned}$$

Thus,

$$\begin{aligned} & \lim_{\Delta\to 0}\frac{1}{\Delta}\frac{E_t v(t+\Delta,S(t+\Delta),\sigma(t+\Delta)) - v(t,S(t),\sigma(t))}{v(t,S(t),\sigma(t))} - r = \\ & \frac{S(t)\frac{\partial v}{\partial x}}{v(t,S(t),\sigma(t))}(\mu - r) + \frac{b(\sigma(t))\frac{\partial v_1}{\partial y}}{v(t,S(t),\sigma(t))}\frac{f(t,\sigma(t))}{b(\sigma(t))}. \end{aligned}$$

Similar to our previous discussion, we can give the above equation a very interesting economics interpretation from excess returns.

Now, we formally call

$$\lambda_1(t) = \frac{\mu - r}{\sigma(t)}, \quad \lambda_2(t) = \frac{f(t,\sigma(t))}{b(\sigma(t))}$$

as the **market price of risk**.

- We can call $\lambda_1(t)$ the market price of return risk (MPR) and call $\lambda_2(t)$ the market price of volatility risk (MPVR).
- Again, note that these two items render the drifts in the Girsanov change of measure (from the physical measure P to the risk-neutral measure Q).

MPR and MPVR

Similar to previous discussion in one-dimensional case, we just need to find two drifts such that

$$\begin{aligned}\frac{\mu - r}{\sigma(t)}dt + dW_1^P(t) &= \gamma_1(t)dt + dW_1^P(t) \\ \frac{f(t,\sigma(t))}{b(\sigma(t))}dt + dW_v^P(t) &= \rho\left[\gamma_1(t)dt + dW_1^P(t)\right] \\ &+ \sqrt{1-\rho^2}\left[\gamma_2(t)dt + dW_2^P(t)\right],\end{aligned}$$

i.e.,

$$\begin{aligned}\gamma_1(t) &= \frac{\mu - r}{\sigma(t)}, \\ \rho\gamma_1(t) + \sqrt{1-\rho^2}\gamma_2(t) &= \frac{f(t,\sigma(t))}{b(\sigma(t))}.\end{aligned}$$

The Girsanov Theorem: Multi-dimensional Case

Theorem. Let $W(t) = (W_1(t), ..., W_d(t), 0 \le t \le T$ be a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}(t);\ 0 \le t \le T\}$ be a filtration for this Brownian motion. Let $\Theta = (\Theta_1(t), ..., \Theta_d(t))$ is a d-dimensional adapted process. Define

$$Z(t) = \exp\left(-\int_0^t \Theta(u)\dot{d}W(u) - \frac{1}{2}\int_0^t \|\Theta(u)\|^2 du\right),$$

and

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)du.$$

Assume that

$$\mathbb{E}\int_0^T \|\Theta(u)\|^2 Z^2(u)du < \infty.$$

Then under the probability measure $\widetilde{\mathbb{P}}(A) = \int_A Z(\omega)d\widetilde{\mathbb{P}}(\omega),\ \forall A \in \mathcal{F}$, the process $\widetilde{W}(t),\ 0 \le t \le T$ is a Brownian motion.

MPR and MPVR under Girsanov Theorem

Thus, we can construct the probability measure Q through

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \exp\left(\int_0^t \gamma_1(s)dW_1^P(s) + \int_0^t \gamma_2(s)dW_2^P(s) - \frac{1}{2}\int_0^t \gamma_1(s)^2 ds - \frac{1}{2}\int_0^t \gamma_2(s)^2 ds\right).$$

So, under Q,

$$\begin{aligned} W_1^Q(t) &= \int_0^t \gamma_1(s)ds + W_1^P(t), \\ W_2^Q(t) &= \int_0^t \gamma_2(s)ds + W_2^P(t), \end{aligned}$$

is a standard two-dimensional Brownian motion.

MPR and MPVR under Girsanov Theorem

Now, we can see that under Q, the original model can be expressed as

$$\begin{aligned} \frac{dS(t)}{S(t)} &= rdt + \sigma(t)dW_1^Q(t), \\ d\sigma(t) &= [a(\sigma(t)) - f(t,\sigma(t))]dt + b(\sigma(t))\left[\rho dW_1^Q(t) + \sqrt{1-\rho^2}dW_2^Q(t)\right], \end{aligned}$$

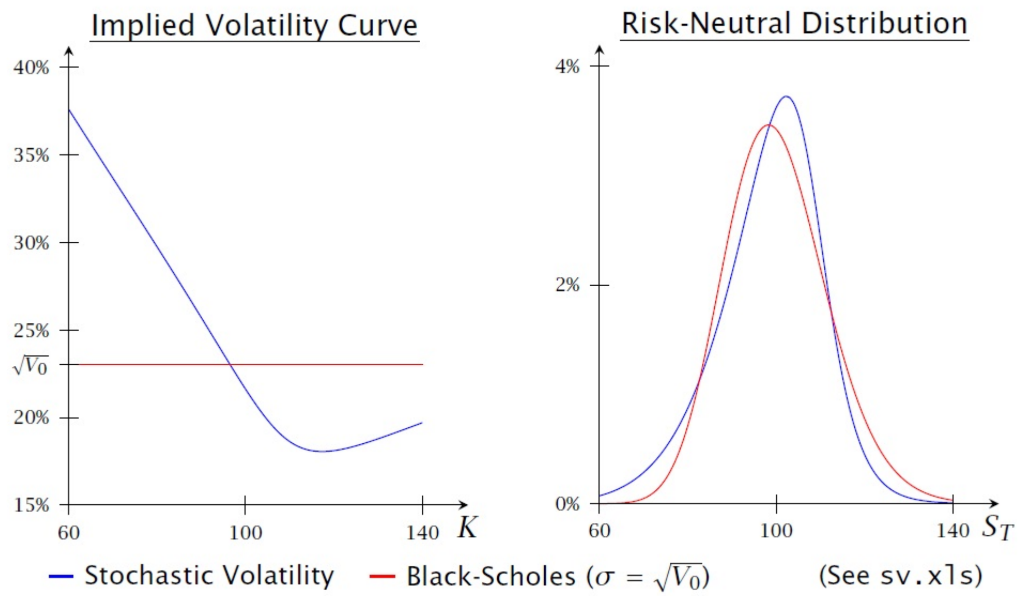
Further tasks:

- find a closed-form formula for option pricing, or,
- numerically evaluate the option price
- then, calibrate the model (find proper value of parameters) by fitting the formula to option price data

Risk-Neutral Distribution and Implied Volatility

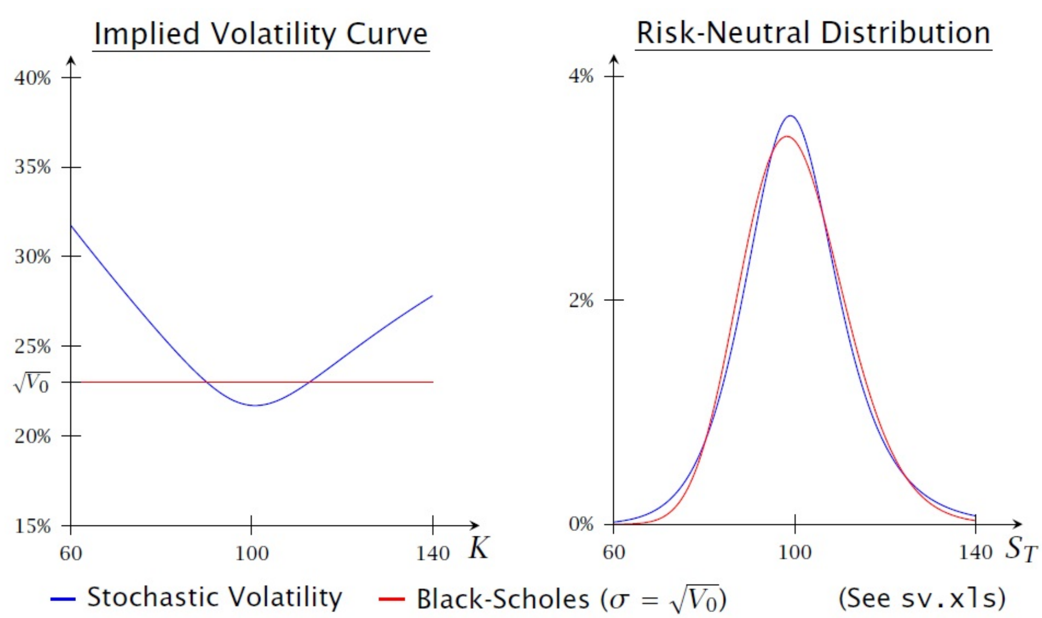
Possible calibration results for the Heston SV model:

$$r = 3\%,\ S_0 = 100,\ \sqrt{V_0} = \sqrt{\theta} = 23\%,\ \kappa = 4,\ \sigma_V = 0.8,\ \rho = -60\%, T = 0.25$$



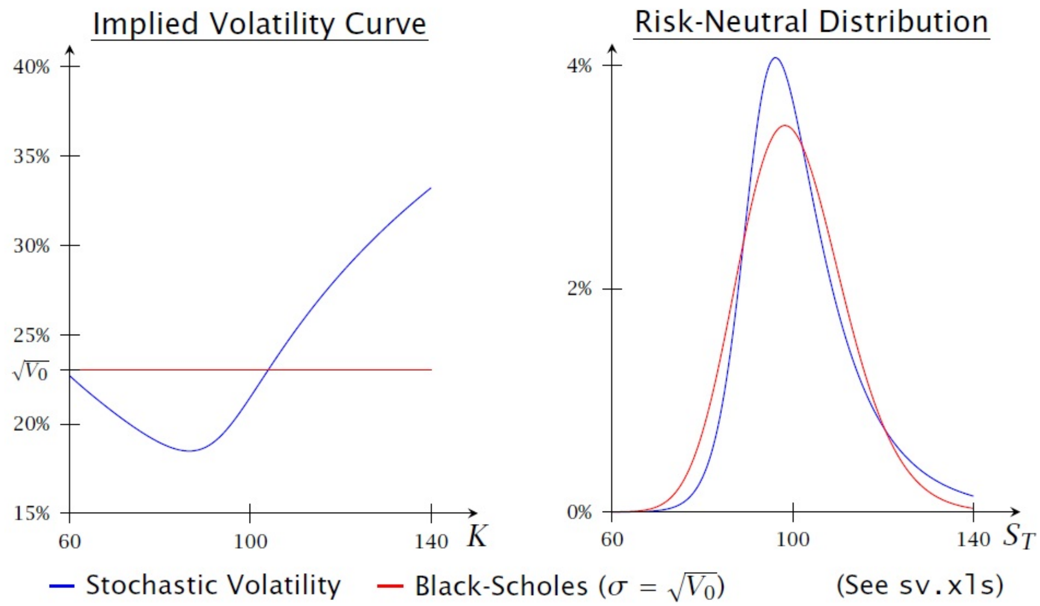
Risk-Neutral Distribution and Implied Volatility

$$r = 3\%,\ S_0 = 100,\ \sqrt{V_0} = \sqrt{\theta} = 23\%,\ \kappa = 4,\ \sigma_V = 0.8,\ \rho = 0\%, T = 0.25$$

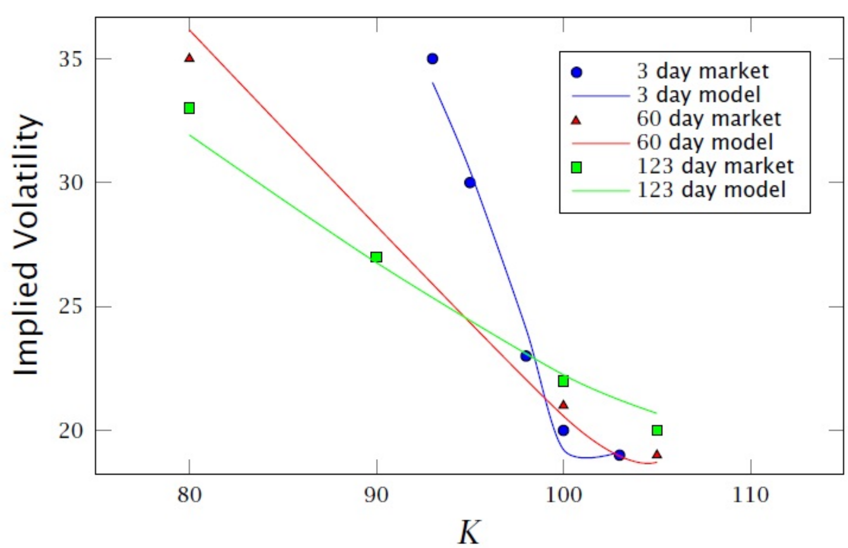


Risk-Neutral Distribution and Implied Volatility

$$r = 3\%,\ S_0 = 100,\ \sqrt{V_0} = \sqrt{\theta} = 23\%,\ \kappa = 4,\ \sigma_V = 0.8,\ \rho = 60\%, T = 0.25$$



Results of Fitting



$S_0 = 100$ $\sqrt{V_0} = 21.38\%$ $\sqrt{\theta} = 26.15\%$ $\kappa = 19.66$

$r = 5\%$ $\sigma_V = 4.25$ $\rho = -44.55\%$

Results of Fitting

$$S_0 = 100 \qquad \sqrt{V_0} = 21.38\% \qquad \sqrt{\theta} = 26.15\% \qquad \kappa = 19.66$$

$$r = 5\% \qquad \sigma_V = 4.25 \qquad \rho = -44.55\%$$

Resulting Sample Path: