Numerical Methods in Economics and Finance
Lecture 3: Perturbation

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Review and Overview of this lecture

- Linear Policy Function
- LQ approximation
- Policy function with more accuracy

Implicit Function Theorem

- The relationship between x, y is implicitly or explicitly determined by

$$0_{dim=?} = h(x_{n\times 1}, y_{m\times 1}) \tag{1}$$

where $h : U \subset R^{n\times m} \to V \subset R^{?}$, implicit function theorem allows us to compute the derivatives of y respect to x.

- Let $h : R^{n+m} \to R^{?}$ be a continuously differentiable function. Or continuously differentiable function on a open set U and mapping it into V, (question: continuous function will map an open set into ?), If there is $\bar{x}, \bar{y} \in U,\ V,\ s.t\ h(\bar{x}, \bar{y}) = 0$, I assume that $Dg_y(x, y)$ is invertible at $\bar{x}, \bar{y}$, and there is a uniquely defined continuously differentiable around $\bar{x}$, such that $h(x, f(x)) = 0$

Implicit Function Theorem Continued

- And the function f has derivatives, or Jacobian matrix, BTW, what is Hessian?

$$\frac{\partial f}{\partial x^T} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & & & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} = -[\frac{\partial\, h(x,y)}{\partial\, y^T}]^{-1}[\frac{\partial h(x,y)}{\partial x^T}]|_{x,y=\bar{x},\bar{y}} \tag{2}$$

- Why? Total Differentiate it

$$dh(x,f(x)) = \frac{\partial\, h(x,y)}{\partial\, x^T}(x-\bar{x}) + \frac{\partial\, h(x,y)}{\partial\, y^T}(y-\bar{y}) \tag{3}$$

$$= \frac{\partial\, h(x,y)}{\partial\, x^T}dx + \frac{\partial\, h(x,y)}{\partial\, y^T}\frac{\partial\, f(x)}{\partial\, x^T}dx \tag{4}$$

$$= 0 \tag{5}$$

PROBLEM TO BE SOLVED

- Euler Equation with Budget constraint

$$K_{t+1} + C_t - g(K_t) = 0 = h_1(\begin{bmatrix} K_t \\ C_t \end{bmatrix}, \begin{bmatrix} K_{t+1} \\ C_{t+1} \end{bmatrix}) \tag{6}$$

$$\beta\, u'(C_{t+1})g'(K_{t+1}) - u'(C_t) = 0 = h_2(\begin{bmatrix} K_t \\ C_t \end{bmatrix}, \begin{bmatrix} K_{t+1} \\ C_{t+1} \end{bmatrix}) \tag{7}$$

- What would be my candidate of $\bar{x}, \bar{y}$?
- My objective is to get a function

$$\begin{bmatrix} K_{t+1} \\ C_{t+1} \end{bmatrix} = g(\begin{bmatrix} K_t \\ C_t \end{bmatrix}) \tag{8}$$

Implement Implicit Function Theorem

- Jacobian of g would be

$$-[\frac{\partial h}{\partial y^T}]^{-1}[\frac{\partial h}{\partial x^T}] \tag{9}$$

- The first term

$$\frac{\partial h}{\partial y^T} = \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \beta u'(\bar{c})g''(\bar{k}) & \beta u''(\bar{c})g'(\bar{k}) \end{bmatrix} \tag{10}$$

- The Second term

$$\frac{\partial h}{\partial x^T} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -g'(k) & 1 \\ 0 & u''(\bar{c}) \end{bmatrix} \tag{11}$$

Result

- Jacobian of g would be

$$-\begin{bmatrix} 1 & 0 \\ \beta\, u'g'' & u'' \end{bmatrix}^{-1} \begin{bmatrix} -\frac{1}{\beta} & 1 \\ 0 & -u'' \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta} & -1 \\ -\frac{u'g''}{u''} & 1 + \frac{\beta\, u'g''}{u''} \end{bmatrix}$$

- I used a identity

$$\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Hence there will be eigenvalues λ_1, λ_2:

$$\lambda_1 \lambda_2 = \frac{1}{\beta} > 1 \tag{12}$$

$$\lambda_1 + \lambda_2 = 1 + \frac{1}{\beta} + \frac{\beta\, u'g''}{u''} > 1 + \frac{1}{\beta} \tag{13}$$

SCHUR DECOMPOSITION

- Any square matrix can be decomposed $A = QUQ^{-1}$, U is upper Triangle with eigenvalues on diagonal.
- Hence I will represent

$$\frac{\partial g}{\partial \left[k_t\ c_t\right]} = Q \begin{bmatrix} \lambda_1 & s_{12} \\ 0 & \lambda_2 \end{bmatrix} Q^{-1},\ \lambda_1 < \lambda_2 \tag{14}$$

- Hence we have

$$Q^{-1} \frac{\partial g}{\partial \left[k_t\ c_t\right]} \begin{bmatrix} dk_t \\ dc_t \end{bmatrix} = \begin{bmatrix} \lambda_1 & s_{12} \\ 0 & \lambda_2 \end{bmatrix} Q^{-1} \begin{bmatrix} dk_t \\ dc_t \end{bmatrix} \tag{15}$$

- Or

$$Q^{-1} \begin{bmatrix} dk_{t+1} \\ dc_{t+1} \end{bmatrix} = \begin{bmatrix} \lambda_1 & s_{12} \\ 0 & \lambda_2 \end{bmatrix} Q^{-1} \begin{bmatrix} dk_t \\ dc_t \end{bmatrix} \tag{16}$$

Implication

- See Sims(2001) for the elegant argument.
- I have proved that in the Deterministic case that our economy will converge to s.s

$$Q^{-1} = \begin{bmatrix} b11 & b_{12} \\ b21 & b_{22} \end{bmatrix}, \; b_{21}dk_t + b_{22}dc_t = 0 \tag{17}$$

- More explicitly

$$C_t = \bar{C} - \frac{b_{21}}{b_{22}}(K_t - \bar{K}), dC_t = constant dK_t \tag{18}$$

- Hence from the dynamic equation, you will have

$$dK_{t+1} = \lambda_1 dK_t$$

$$[b_{11} \; b_{12}] \begin{bmatrix} dK_{t+1} \\ constant \times dK_{t+1} \end{bmatrix} = \lambda [b_{11} \; b_{12}] \begin{bmatrix} dK_t \\ constant \times dK_t \end{bmatrix}$$

Implication

- Why do I only care the first Derivative? And the implicit assumption?
- Because the function we define has a fixed point, you really should take Topologic Class

$$x^* = f(x^*) \tag{19}$$

- Hence, by Taylor expansion:

$$y = f(x) = f(x^*) + f'(x^*)(x - x^*) \tag{20}$$

- Implicit assumption? Extension?

Another way to look

- Now let me use
  $K_{t+1} = h(K_t) \approx K^* + h'(K^*)(K_t - K^*), C_t = g(K_t) - h(K_t)$,
  and use it in my Euler Equation

$$u'(C_t) = \beta\, u'(U(C_{t+1}))g'(K_{t+1}) \quad (21)$$
$$u'(g(K_t) - h(K_t)) = \beta\, u'(g(h(K_t)) - h(h(K_t)))g'(h(K_t)) \quad (22)$$
$$u''(g' - h') = \beta\, u'g''h' + \beta\, u''g'(g'h' - h'^2) \quad (23)$$
$$u''h'^2 + u''g' = u''h' + \beta\, u'g''h' + u''h' \quad (24)$$
$$h'^2 + \frac{1}{\beta} = (g' + \beta\, u'g''/u'' + 1)h' \quad (25)$$

- Does this seems similar to u? There will be two solution
  $x_1x_2 = 1/\beta, x_1 + x_2 = (g' + \beta\, u'g''/u'' + 1) = (\frac{1}{\beta} + \beta\, u'g''/u'' + 1)$
- Surely my previous routine can be pretty general, does not need to be 2×1 dimensional

Linear Quadratic

- Linear Exogenous Dynamics:
  $x_{t+1} = A_{n\times n}x_t + B_{n\times m}u_t + \epsilon,\ E[\epsilon\epsilon^T] = \Sigma$
- Let's also assume that Agent would like to
  $\max\ E[\sum_t \beta^t(x_t'Qx_t + u_t'Ru_t + 2u_t'Sx_t)]$
- Dynamic Programming Method:

$$x'Px + d = \max_u\ x'Qx + 2u'Sx + u'Ru + \beta E[x_{t+1}'Px_{t+1} + d] \tag{26}$$

$$= \max_u\ x'Qx + 2u'Sx + u'Ru \tag{27}$$

$$+ \beta E[(Ax + Bu + \epsilon)'P(Ax + Bu + \epsilon) + d] \tag{28}$$

$$= x'Qx + \beta x'A'PAx + 2u'Sx + u'Ru + 2\beta x'A'PBu \tag{29}$$

$$+ \beta u'B'PBu + \beta tr(P\Sigma) + \beta d \tag{30}$$

LINEAR QUADRATIC

- F.O.C

$$Sx + Ru + \beta B'PAx + \beta B'PBu = 0 \tag{31}$$
$$u = -(R + \beta B'PB)^{-1}(S + \beta B'PA)x \tag{32}$$

- Get control back you will have

$$P = Q + \beta A'PA - (S + \beta B'PA)'[R + \beta B'PB]^{-1}(S + \beta B'PA) \tag{33}$$
$$d = \frac{\beta tr(P\Sigma)}{1-\beta} \tag{34}$$

- BTW, does Σ affect control problem? So Bear in mind that if I use LQ to approximate a solution, I implicitly impose this Certainty Equivalence Property.
- See Tom Sargent's Website for more details.

LINEAR QUADRATIC EXTENDED

- For Economics Problem, I have to separate two kind of s.v, exogenous and endogenous. See Heer and Maußner(2011) and Sims(2001); And Objective Would be

$$
x_{t+1} \leq A_x x_t + A_z z_t + Bu_t \tag{35}
$$

$$
z_t = \rho z_{t-1} + \epsilon,\ E[\epsilon\epsilon^T] = \Sigma \tag{36}
$$

$$
\max\ E \sum_t \beta^t \begin{bmatrix} x_t \\ u_t \\ z_t \end{bmatrix}^T \begin{bmatrix} A_{xx} & A_{xu} & A_{xz} \\ A_{xu} & A_{uu} & A_{uz} \\ A_{xz} & A_{uz} & A_{zz} \end{bmatrix} \begin{bmatrix} x_t \\ u_t \\ z_t \end{bmatrix} \tag{37}
$$

- HW problem, due next week, use Dynamic Problem Method to solve it
- I will use KKT instead

Linear Quadratic Extended Continues

- Lagrange would be, note my shadow price is a vector.

$$L = E\beta^t \{ \begin{bmatrix} x_t \\ u_t \\ z_t \end{bmatrix}^T \begin{bmatrix} A_{xx} & A_{xu} & A_{xz} \\ A_{xu} & A_{uu} & A_{uz} \\ A_{xz} & A_{uz} & A_{zz} \end{bmatrix} \begin{bmatrix} x_t \\ u_t \\ z_t \end{bmatrix} + 2\lambda_t'[A_x x_t + A_z z_t + B u_t - x_{t+1}]\}$$

- F.O.C:

$$2A_{uu}u_t + 2A_{ux}x_t + 2A_{uz}z_t + 2B'\lambda = 0 \quad (38)$$

$$\lambda_t = E\ \beta[A_{xx}x_{t+1} + A_{xu}u_{t+1} + A_{xz}z_{t+1} + A_x'\lambda_{t+1}] \quad (39)$$

- Treat shadow price as an endogenous state variable

Matrix Solution

- Summarize what I have derived with Dynamics into

$$A_{uu}u_t = -[A_{ux}\ B'] \begin{bmatrix} x_t \\ \lambda_t \end{bmatrix} - A_{uz}z_t \tag{40}$$

$$\begin{bmatrix} \beta A_{xx} & \beta A'_x \\ I & 0 \end{bmatrix} E_t \begin{bmatrix} x_{t+1} \\ \lambda_{t+1} \end{bmatrix} + \begin{bmatrix} 0 & -I \\ -A_x & 0 \end{bmatrix} \begin{bmatrix} x_t \\ \lambda_t \end{bmatrix}$$
$$= \begin{bmatrix} -A_{xu} \\ 0 \end{bmatrix} Eu_{t+1} + \begin{bmatrix} 0 \\ A_u \end{bmatrix} u_t + \begin{bmatrix} -\beta A_{xz} \\ 0 \end{bmatrix} E[z_{t+1}] + \begin{bmatrix} 0 \\ A_z \end{bmatrix} z_t$$

- First equation tells u how to do optimal controling, while the latter tells u how does the s.v moves. Please note all the coefficient matrix are pinned down by my problem setting. BTW, the latter is stochastic difference equation.

Hansen and Prescott Method: An Example

- Read Hansen and Prescott(1995)
- Let me solve a pretty simple version of problem

$$\max_{c_t,k_{t+1}} \sum_t \beta^t \ln(c_t)$$

$$s.t$$

$$k_{t+1} + c_t \leq k_t^\alpha$$

- s.s

$$k_{t+1} + c_t = k_t^\alpha \tag{41}$$

$$\frac{\beta c_t}{c_{t+1}} \alpha k_{t+1}^{\alpha-1} = 1 \tag{42}$$

- Approximate my objective function $c_t = k_t^\alpha - I_t$, $I_t = k_{t+1}$, $\ln(c_t) = \ln(k_t^\alpha - I_t)$

Approximation

- Quadratic Approximation of the objective function

$$\begin{aligned}
\ln(k_t^\alpha - I_t) &= f(\bar{k}, \bar{I}) + f_1(k_t - \bar{k}) + f_2(I_t - \bar{I}) \\
&+ \frac{1}{2} f_{11}(k_t - \bar{k})^2 + \frac{1}{2} f_{22}(I_t - \bar{I})^2 + f_{12}(I_t - \bar{I})(k_t - \bar{k}) \\
&= \begin{bmatrix} 1 \\ K \\ I \end{bmatrix}^T Q \begin{bmatrix} 1 \\ K \\ I \end{bmatrix}
\end{aligned}$$

and Q is?

RECURSIVE STEP

- Since from LQ step we know value function would be $k'Qk + d = \begin{bmatrix} 1 \\ k \end{bmatrix}^T V \begin{bmatrix} 1 \\ k \end{bmatrix}$, though I do not know what is V matrix is, maybe you know. But hold on for 1 second.
- Bellman equation method tells that our Current Value is

$$\begin{bmatrix} 1 \\ k \end{bmatrix}^T V \begin{bmatrix} 1 \\ k \end{bmatrix} = \ln(K_t^\alpha - I_t) + \beta \begin{bmatrix} 1 \\ k_{t+1} \end{bmatrix}^T V \begin{bmatrix} 1 \\ k_{t+1} \end{bmatrix} \tag{43}$$

which can be approximated by

$$\begin{bmatrix} 1 \\ K \\ I \\ 1 \\ K' \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} 1 \\ K \\ I \\ 1 \\ K' \end{bmatrix} \tag{44}$$

Recursive Step

- However, I have set $I = K'$, hence

$$\begin{bmatrix} 1 \\ K \\ I \\ 1 \\ K' \end{bmatrix} = \begin{bmatrix} & & I_{4\times 4} & & \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K \\ I \\ 1 \end{bmatrix} \tag{45}$$

- Hence the original problem reduced to

$$\begin{bmatrix} 1 \\ K \\ I \\ 1 \end{bmatrix}^T D_1^T \begin{bmatrix} Q & 0 \\ 0 & V \end{bmatrix} D_1 \begin{bmatrix} 1 \\ K \\ I \\ 1 \end{bmatrix} \equiv \begin{bmatrix} 1 \\ K \\ I \\ 1 \end{bmatrix}^T S_1 \begin{bmatrix} 1 \\ K \\ I \\ 1 \end{bmatrix} \tag{46}$$

Recursive Step

- Again, I have set $1 = 1$, hence

$$\begin{bmatrix} 1 \\ K \\ I \\ 1 \end{bmatrix} = \begin{bmatrix} & I_{3\times 3} & \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K \\ I \end{bmatrix} \tag{47}$$

- Hence the original problem reduced to

$$\begin{bmatrix} 1 \\ K \\ I \end{bmatrix}^T D_2^T S_1 D_3 \begin{bmatrix} 1 \\ K \\ I \end{bmatrix} \equiv \begin{bmatrix} 1 \\ K \\ I \end{bmatrix}^T S_2 \begin{bmatrix} 1 \\ K \\ I \end{bmatrix} \tag{48}$$

Optimize Step

- Ok We have reduced our problem into

$$\begin{bmatrix} 1 \\ K \\ I \end{bmatrix}^T S_2 \begin{bmatrix} 1 \\ K \\ I \end{bmatrix} = s_2^{11} + s_2^{22} K^2 + s_2^{33} I^2 + 2 s_2^{12} K + 2 s_2^{13} I + 2 s_2^{23} K I \tag{49}$$

Optimal Choice would be

$$I_t = -\frac{s_2^{13}}{s_2^{33}} - \frac{s_2^{23}}{s_2^{33}} K \tag{50}$$

- Therefore

$$\begin{bmatrix} 1 \\ K \\ I \end{bmatrix} = \begin{bmatrix} I_{2\times 2} & \\ -\frac{s_2^{13}}{s_2^{33}} & -\frac{s_2^{23}}{s_2^{33}} \end{bmatrix} \begin{bmatrix} 1 \\ K \end{bmatrix} = D_3 \begin{bmatrix} 1 \\ K \end{bmatrix} \tag{51}$$

Recursive Step

- Ok We have reduced our problem further into

$$\begin{bmatrix} 1 \\ K \end{bmatrix}^T D_3^T S_2 D_3 \begin{bmatrix} 1 \\ K \end{bmatrix} \tag{52}$$

- Which by definition should equal to

$$\begin{bmatrix} 1 \\ K \end{bmatrix}^T D_3^T S_2 D_3 \begin{bmatrix} 1 \\ K \end{bmatrix} \equiv \begin{bmatrix} 1 \\ K \end{bmatrix}^T V \begin{bmatrix} 1 \\ K \end{bmatrix} \tag{53}$$

- Iterate my steps until V matrix converges
- Homework: Set $V^0 = -.2I_{2\times2}$, $V^0 = .2I_{2\times2}$ and $V^0 = 0 \times I_{2\times2}$, see how does your solution converge.

LINEAR QUADRATIC APPROXIMATION GENERAL

- Line s.v and c.v as a vector say $x = [1 \; s.v \; c.v]$, and approximate periodic gain function as quadratic form

$$g(x) = x'Qx \tag{54}$$

- Guess value function is $\begin{bmatrix} 1 \\ s.v. \end{bmatrix}^T V \begin{bmatrix} 1 \\ s.v. \end{bmatrix}$
- Hence bellman equation told us that current value is

$$\begin{bmatrix} 1 \\ s.v. \\ c.v. \\ 1 \\ s.v.' \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & \beta V \end{bmatrix} \begin{bmatrix} 1 \\ s.v. \\ c.v. \\ 1 \\ s.v.' \end{bmatrix} \tag{55}$$

Linear Quadratic Approximation General

- Model Dynamic will tell us that $s.v.' = As.v. + Bc.v.$
- Hence

$$\begin{bmatrix} 1 \\ s.v. \\ c.v. \\ 1 \\ s.v.' \end{bmatrix} = \left[\begin{array}{ccc} \multicolumn{3}{c}{I_{n\times n}} \\ 1 & 0 & 0 \\ 0 & A & B \end{array}\right] \begin{bmatrix} 1 \\ s.v. \\ c.v. \end{bmatrix} \tag{56}$$

- Optimize control $c.v = D + Es.v.$

$$\begin{bmatrix} 1 \\ s.v. \\ c.v. \end{bmatrix} = \left[\begin{array}{cc} \multicolumn{2}{c}{I_{k\times k}} \\ D & E \end{array}\right] \begin{bmatrix} 1 \\ s.v. \end{bmatrix} \tag{57}$$

- Update V matrix

Simplest Case

- Let me move into stochastic version. This is a typical way how New Keyesian or DSGEers solve problem. PS: I ignore the fact that most of them are using Bayesian Approach, which I think is not essential by itself.
- Consider Small Variation

$$0 = k_{t+1} + c_t - z_t f(k_t) - (1-\delta)k_t \tag{58}$$

$$0 = u'(c_t) - \beta E[u'(c_{t+1})(z_{t+1}f'(k_{t+1}) + 1 - \delta)] \tag{59}$$

$$\ln z_t = \rho \ln z_{t-1} + \epsilon \tag{60}$$

$$\bar{z}_t = z_t - 1 \tag{61}$$

$$\bar{z}_t \approx \rho \bar{z}_{t-1} + \epsilon_t \tag{62}$$

- Linearized dynamics of this system

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \bar{k}_t \\ \bar{c}_t \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} E \begin{bmatrix} \bar{k}_{t+1} \\ \bar{c}_{t+1} \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} z_t + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} E[z_{t+1}] \tag{63}$$

where $\bar{y} = y - y^{ss}$
After Linearization Step

Bonus Question

- Use Taylor Expansion to prove

$$\ln F_T = \ln F_t + \frac{1}{F_t}(F_T - F_t) - \int_0^{F_t} \frac{1}{K^2}(K - F_T)^+ \, dK - \int_{F_t}^{\infty} \frac{1}{K^2}(F_T - K)^+ \, dK$$

- Proof

$$\begin{aligned}
\int_0^{F_t} \frac{1}{K^2}(K - F_T)^+ \, dK &= \int_{F_T}^{F_t} \frac{1}{K^2}(K - F_T) \, dK \\
&= \ln(F_t) - \ln(F_T) - F_T\left(\frac{1}{F_t} - \frac{1}{F_F}\right)
\end{aligned}$$

$$\begin{aligned}
\int_{F_t}^{\infty} \frac{1}{K^2}(F_T - K)^+ \, dK &= \int_{F_t}^{F_T} \frac{1}{K^2}(F_T - K) \\
&= F_T\left(\frac{1}{F_t} - \frac{1}{F_T}\right) - \ln\left(\frac{F_T}{F_t}\right)
\end{aligned}$$

MATRIX FORM

- Consider the first two equation

$$0 = \bar{k}_{t+1} + \bar{c}_t - \bar{z}_t f(k^*) - (1-\delta+f'(k^*))\bar{k}_t \tag{64}$$

$$0 = u''(c^*)\bar{c}_t - \beta\{[u''(c^*)f'(k^*) + u''(c^*)(1-\delta)]\bar{c}_{t+1} + u'(c^*)f'(k^*)\bar{z}_{t+1} \tag{65}$$

$$+ u'(c^*)f''(k^*)\bar{k}_{t+1}\} \tag{66}$$

- Hence you will get

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\beta u' f'' & -\beta u''(1-\delta+f') \end{bmatrix} E \begin{bmatrix} \bar{k}_{t+1} \\ \bar{c}_{t+1} \end{bmatrix} + \begin{bmatrix} -(1-\delta)-f' & 1 \\ 0 & u'' \end{bmatrix} \begin{bmatrix} \bar{k}_t \\ \bar{c}_t \end{bmatrix} \tag{67}$$

$$+ \begin{bmatrix} -f \\ 0 \end{bmatrix} \bar{z}_t + \begin{bmatrix} 0 \\ -\beta u' f' \end{bmatrix} E[\bar{z}_{t+1}] \tag{68}$$

- BTW, I assume $\bar{z}_{t+1} = \rho \bar{z}_t$

Direct Solution

- I will also use linear policy function to solve it. Consider

$$
\begin{aligned}
k_{t+1} &= p_{kk}k_t + p_{kz}z_t && (69)\\
c_t &= p_{ck}k_t + p_{cz}z_t && (70)\\
& && (71)
\end{aligned}
$$

- Substitute these two equation into above function
- We will solve

$$
\begin{bmatrix} coff_1 \\ coff_2 \end{bmatrix} k_t + \begin{bmatrix} coff_3 \\ coff_4 \end{bmatrix} z_t = 0 \tag{72}
$$

IMPLICIT FUNCTION THEOREM

- Actually Expectation operator is just a linear operator, nothing funcy, hence $E[f(x,y)] = 0$ can also pin down a relationship between y and x
- Think about the GMM method in Econometrics
- Now consider $x = x_t = [k_t \; c_t]^T, y = x_{t+1} = [k_{t+1} \; c_{t+1}]^T$, and Consider policy function

$$k_{t+1} = g^k(k_t, z_t, \sigma) \tag{73}$$

$$c_t = g^c(k_t, z_t, \sigma) \tag{74}$$

Question: what is steady state value?
- From previous Slides or you can derive now that

$$E\begin{bmatrix}\bar{k}_{t+1}\\ \bar{c}_{t+1}\end{bmatrix} = \begin{bmatrix}1/\beta & -1\\ -u'f''/u'' & 1+\beta u'f''/u''\end{bmatrix}\begin{bmatrix}\bar{k}_t\\ \bar{c}_t\end{bmatrix} + \begin{bmatrix}f\\ -\frac{\beta u'ff''+\rho\beta u'f'}{u''}\end{bmatrix}\bar{z}_t \tag{75}$$

Diverge and Converge

- Let me write the previous equation like $E[x_{t+1}] = Wx_t + Sz_t$, and We know that the eigenvalues of W lie on both sides of 1, Hence

$$E\begin{bmatrix}\tilde{k}_{t+1}\\ \tilde{c}_{t+1}\end{bmatrix} \equiv Q^{-1}E\begin{bmatrix}\bar{k}_{t+1}\\ \bar{c}_{t+1}\end{bmatrix} = \begin{bmatrix}\lambda_1 & s_{12}\\ 0 & \lambda_2\end{bmatrix} Q^{-1}E\begin{bmatrix}\bar{k}_{t}\\ \bar{c}_{t}\end{bmatrix} + Q^{-1}R\bar{z}_t \tag{76}$$

Or

$$E\begin{bmatrix}\tilde{k}_{t+1}\\ \tilde{c}_{t+1}\end{bmatrix} = \begin{bmatrix}\lambda_1 & s_{12}\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}\tilde{k}_{t}\\ \tilde{c}_{t}\end{bmatrix} + Q^{-1}R\bar{z}_t \tag{77}$$

- And From the second equation we have

$$E[\tilde{c}_{t+1}] = \underbrace{\lambda_2}_{>1}\tilde{c}_t + q_2 z_t$$

$$\tilde{c}_t = -\frac{1}{\lambda_2}E[\tilde{c}_{t+1}] - \frac{q_2}{\lambda_2}\bar{z}_t$$

Recursive Formula

- Iterate the formula forward

$$\tilde{c}_t = -\frac{\frac{q_2}{\lambda_2}}{1 - \frac{\rho}{\lambda_2}} \bar{z}_t$$

- Later is a policy function for consumption, combing with $\bar{k}_{t+1} = \frac{1}{\beta}\bar{k}_t - \bar{c}_t + f\bar{z}$, we have another policy function.
-

HOMEWORK

- Set utility function as $\frac{c^{1-\eta}}{1-\eta}$, production function $f(k) = k^{\alpha}$
- Parameter $\delta = 1$, $\alpha = .3$, $\beta = 0.99$, $\rho = 0.89$, $\sigma = 0.0072$
- Derive your policy function
- Simulate 100,000 Shock and discard the first 50000 points, plot out the distribution of capital stock.
- BTW, you normally counter log-linearizion, how did they get it?

Generally

- According to my LQ approximation with shocks, rewrite as

$$
A_{uu}u_t = -[A_{ux}\; B'] \begin{bmatrix} x_t \\ \lambda_t \end{bmatrix} - A_{uz}z_t
$$

$$
\begin{bmatrix} \beta A_{xx} & \beta A_x' \\ I & 0 \end{bmatrix} E_t \begin{bmatrix} x_{t+1} \\ \lambda_{t+1} \end{bmatrix} + \begin{bmatrix} 0 & -I \\ -A_x & 0 \end{bmatrix} \begin{bmatrix} x_t \\ \lambda_t \end{bmatrix}
$$

$$
= \begin{bmatrix} -A_{xu} \\ 0 \end{bmatrix} Eu_{t+1} + \begin{bmatrix} 0 \\ A_u \end{bmatrix} u_t + \begin{bmatrix} -\beta A_{xz} \\ 0 \end{bmatrix} E[z_{t+1}] + \begin{bmatrix} 0 \\ A_z \end{bmatrix} z_t
$$

- Becasue of our current and future control $u_{t,t+1}$ is pinned down by the first equation, hence one can derive the following with exogenous transition z_t

$$
E_t \begin{bmatrix} x_{t+1} \\ \lambda_{t+1} \end{bmatrix} = W \begin{bmatrix} x_t \\ \lambda_t \end{bmatrix} + Rz_t \tag{78}
$$

GENERAL SLIDES II

- Schur Decompostion agein

$$W = Q \begin{bmatrix} \lambda_1 & s_{12} \\ 0 & \lambda_2 \end{bmatrix} Q^{-1} \tag{79}$$

$$Q^{-1}E_t \begin{bmatrix} x_{t+1} \\ \lambda_{t+1} \end{bmatrix} = E \begin{bmatrix} \tilde{x}_{t+1} \\ \tilde{\lambda}_{t+1} \end{bmatrix} = \begin{bmatrix} \lambda_1 & s_{12} \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_t \\ \tilde{\lambda}_t \end{bmatrix} + Q^{-1}Rz_t \tag{80}$$

- Then I will use the Later equation to iterate forward. λ_2 denotes a upperright triangle-matrix, and I assume that $z' = \Pi z_t$, here z is a vector, hence the second equation would be

$$\tilde{\lambda}_t = \Phi z \tag{81}$$

$$\Phi\Pi = \lambda_2\Phi + QR_2 \tag{82}$$

$$(\Pi' \otimes I)vec(\Phi) = (I \otimes \lambda)vec(\Phi) + vec(QR_2) \tag{83}$$

$$vec(\Phi) = [\Pi' \otimes I - I \otimes \lambda]^{-1} vec(QR_2) \tag{84}$$

General Slides III

- Given I have solved $\tilde{\lambda}_t = \Phi z_t = Q_{21} x_t + Q_{22} \lambda_t$, I have the policy function for λ_t
- Because of $E_t \begin{bmatrix} x_{t+1} \\ \lambda_{t+1} \end{bmatrix} = W \begin{bmatrix} x_t \\ \lambda_t \end{bmatrix} + R z_t$,
  $E[x_{t+1}] = W_{11} x_t + W12 \lambda_t + R_1 z_t = f(x_t, z_t)$
- Control u_t is a functon of x_t, z_t, λ_t, so ...

INTUITION

- I will follow Physics Literature to follow Taylor's Rule frequently; Implicit function

$$H(x,f(x))=0 \tag{85}$$

$$\frac{\partial H}{\partial x^T}dx+\frac{\partial H}{\partial y^T}\frac{\partial f}{\partial x^T}dx=0 \tag{86}$$

$$\textit{From here hard to use matrix} \tag{87}$$

$$\forall\, j \tag{88}$$

$$0=H^j_{x_k}(x,f(x))+\sum_i H^j_{y_i}(x,y)f^i_{x_k}(x) \tag{89}$$

$$0=H^j_{x_k,x_m}(x,f(x))+\sum_i H^j_{x_k,y_i}(x,f(x))f^i_{x_m}(x)+\sum_i H^j_{y_i}(x,y)f^i_{x_k,x_m}(x \tag{90}$$

$$+\sum_i H^j_{y_i,x_m}(x,y)f^i_{x_k}(x)+\sum_i\sum_h H^j_{y_i,y_h}(x,y)f^i_{x_k}(x)f^h_{x_k}(x) \tag{91}$$

DETERMINISTIC EXAMPLE

- I will follow use policy function like

$$\bar{k}_{t+1} = p_{11}\bar{k}_t + p_{12}\bar{k}_t^2 \tag{92}$$

$$\bar{c}_{t+1} = p_{21}\bar{k}_t + p_{22}\bar{k}_t^2 \tag{93}$$

- Again I have two equations like $H^i(k, c, k', c') = 0, \ i = 1, 2$

$$H^i(k, f(k), h(k), f(h(k))) = 0 \tag{94}$$

$$H_1^i + H_2^i f' + H_3^i h' + H_4^i f' h' = 0 \tag{95}$$

This part is same for linear approximation, 2 equations and 2 parameters.

- Differentiate the two equation again, and suppose I have got h' and f' or $p_{11}, \ p_{21}$

Deterministic Example II

- Differntiate again will give

$$H^i_{11} + H^i_{12}f' + H^i_{13}h' + H^i_{14}f'h' + (H^i_{21} + H^i_{22}f' + H^i_{23}h' + H^i_{24}f'h')f'$$
$$+ H^i_2 f'' +$$
$$(H^i_{31} + H^i_{32}f' + H^i_{33}h' + H^i_{34}f'h')h'$$
$$+ H^i_3 h''$$
$$(H^i_{41} + H^i_{42}f' + H^i_{43}h' + H^i_{44}f'h')f'h' +$$
$$H^i_4 f''h' + H^i_4 f'h'' = 0$$

Also Two parameter two equation

- HW: Represent it as matrix form. I will allow you one piece of paper to present your result.

Stochastic Example II

- Schmitt-Grohe and Martin Uribe(2004, NBER), Lombardo and Sutherland (2007, JEDC)
- State Variable now is z_t, k_t and σ_t ,which is absent when we are using LQ.

$$E\begin{bmatrix} k' + c - (1-\delta)k - e^z k^\alpha \\ c^{-\gamma} - \beta c'^{-\gamma}(1-\delta + e^{z'}\alpha k'^{\alpha-1}) \end{bmatrix} \equiv E[H(k,c,k',c',z,z')] = 0 \tag{96}$$

- I am looking for a Quadratic Policy function:

$$f^i(k,z,\sigma) = f^i(k^*, z^*, \sigma = 0) + f_1^i \bar{k} + f_2^i \bar{z} + f_3^i \bar{\sigma} + \\ + [\bar{k} \;\; \bar{z} \;\; \bar{\sigma}] H^i \begin{bmatrix} \bar{k} \\ \bar{z} \\ \bar{\sigma} \end{bmatrix}, \; i = C,\; K$$

Stochastic Algorithm

- One can represent

$$\begin{bmatrix} c \\ k' \\ c' \end{bmatrix} = \begin{bmatrix} f^c(k,z,\sigma) \\ f^k(k,z,\sigma) \\ f^c(f^k(k,z,\sigma),\rho z+\sigma\epsilon',\sigma) \end{bmatrix} = G(k,z,z'|_{=\rho z+\sigma\epsilon},\sigma) \tag{97}$$

A function of random variable; This is why the expectation operater has effect.

- First Differentiate the two conditions will have

$$E[H_1 + H_2 f_k^c + H_3 f_k^k + H_4 f_k^c f_k^k] = 0 \tag{98}$$

$$E[H_5 + H_6\rho + H_2 f_z^c + H_3 f_1^z + H_4 f_k^c f_z^k + H_4 f_z^c \rho] = 0 \tag{99}$$

$$E[H_2 f_\sigma^c + H_3 f_\sigma^k + H_4 f_\sigma^c + H_4 f_k^c f_\sigma^k + H_4 f_z^c \rho\epsilon + H_6\epsilon] = 0 \tag{100}$$

6 equations and 6 parameters.

- Hessian? Differentiate again...

Stochastic Algorithm Matrix Form

- One can represent $x = (k, c, k', c', z, z')$

$$E\frac{\partial H}{\partial(k,c,k',c')}\begin{bmatrix}1\\ \frac{\partial G}{\partial k}\end{bmatrix} = 0 \tag{101}$$

$$E\frac{\partial H}{\partial(c,k',c',z,z')}\begin{bmatrix}\frac{\partial G}{\partial z}\\ 1\\ \rho\end{bmatrix} = 0 \tag{102}$$

$$E\frac{\partial H}{\partial(c,k',c',z')}\begin{bmatrix}\frac{\partial G}{\partial \sigma}\\ \epsilon\end{bmatrix} = 0 \tag{103}$$

- Second Order equation: 12 parameters to solve or estimate, below is two equations. How did I get it?

$$E\frac{\partial H}{\partial(k,c,k',c')}\begin{bmatrix}0\\ \frac{\partial^2 G}{\partial k^2}\end{bmatrix} + [1 \quad \frac{\partial G}{\partial k}]\frac{\partial^2 H}{\partial(k,c,k',c')\partial(k,c,k',c')^T}\begin{bmatrix}1\\ \frac{\partial G}{\partial k}\end{bmatrix} = 0 \tag{104}$$

Stochastic Algorithm Matrix Form II

- For kz, differential the first F.O.C to z, providing two more equations

$$E\frac{\partial H}{\partial(k,c,k',c')}\begin{bmatrix}0\\ \frac{\partial^2 G}{\partial k\partial z}\end{bmatrix}+[1\ \ \frac{\partial G}{\partial k}]\frac{\partial^2 H}{\partial(k,c,k',c')^T\partial(c,k',c',z,z')}\begin{bmatrix}\frac{\partial G}{\partial z}\\ 1\\ \rho\end{bmatrix}=0 \tag{105}$$

- For $k\sigma$ partal, differential the first F.O.C to σ, providing two more equations

$$E\frac{\partial H}{\partial(k,c,k',c')}\begin{bmatrix}0\\ \frac{\partial^2 G}{\partial k\partial\sigma}\end{bmatrix}+[1\ \ \frac{\partial G}{\partial k}]\frac{\partial^2 H}{\partial(k,c,k',c')^T\partial(c,k',c',z')}\begin{bmatrix}\frac{\partial G}{\partial\sigma}\\ \epsilon\end{bmatrix}=0 \tag{106}$$

Stochastic Algorithm Matrix Form III

- For $z\sigma$, differential the second F.O.C to σ, providing two more equations

$$E\frac{\partial H}{\partial(c,k',c',z,z')}\begin{bmatrix}\frac{\partial G}{\partial z\sigma}\\0\\0\end{bmatrix}+[\frac{\partial G}{\partial z}\ \ 1\ \ \rho]\frac{\partial^2 H}{\partial(c,k',c',z,z')^T\partial(c,k',c',z')}\begin{bmatrix}\frac{\partial G}{\partial\sigma}\\\epsilon\end{bmatrix} \tag{107}$$

- For zz partal, differential the second F.O.C to z, providing two more equations

$$E\frac{\partial H}{\partial(c,k',c',z,z')}\begin{bmatrix}\frac{\partial^2 G}{\partial z^2}\\0\\0\end{bmatrix}+[\frac{\partial G}{\partial z}\ \ 1\ \ \rho]\frac{\partial^2 H}{\partial(c,k',c',z,z')^T\partial(same)}\begin{bmatrix}\frac{\partial G}{\partial z}\\1\\\rho\end{bmatrix}= \tag{108}$$

Stochastic Algorithm Matrix Form III

- For σ^2, differential the third F.O.C to σ, providing two more equations

$$E\frac{\partial H}{\partial(c,k',c',z')}\begin{bmatrix}\frac{\partial^2 G}{\partial\sigma^2}\\0\end{bmatrix}+[\frac{\partial G}{\partial\sigma}\quad\epsilon]\frac{\partial^2 H}{\partial(c,k',c',z')^T\partial(c,k',c',z')}\begin{bmatrix}\frac{\partial G}{\partial\sigma}\\\epsilon\end{bmatrix}=0 \tag{109}$$

- Then you get 12 equations and 12 parameters. Furthermore, you may asked what if you take differentiates on other equations.
- Overidentification, See GMM.

HOW TO SOLVE EQUATIONS NUMERICALLY?

- See next lecture