光华管理学院
Guanghua School of Management

Mathematical Methods in Finance

Lecture 5: Brownian Motion

Fall 2013

Overview

- Definition of Brownian motion and its construction
- Basic properties
- Applications

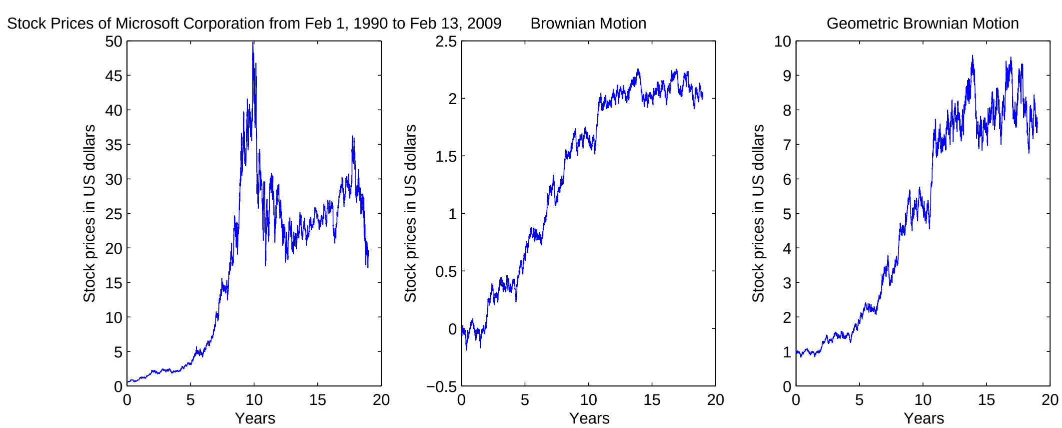
Definition of Standard Brownian Motion

Definition: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Stochastic process $\{W(t)\}$ is a (one-dimensional) standard Brownian motion (BM) if it satisfies that

- $W(0) = 0$;
- for each $\omega \in \Omega$, the realization (path) $W(t)(\omega)$ is a continuous function of $t \geq 0$;
- it has stationary increments with normal distribution $W(t) - W(s) \sim \mathrm{N}(0, t-s)$, and
- it has independent increments. More precisely, for all $0 = t_0 < t_1 < \cdots < t_m$, the increments $W(t_1) - W(t_0)$, $W(t_2) - W(t_1), \cdots, W(t_m) - W(t_{m-1})$ are independent.

Motivation to study Brownian motion

Empirical features of stock prices



Observations: Greatly volatile sample paths

- Brownian motion can serve as important building block
- We can even assume discontinuity or other features

Motivation to study Brownian motion

A Question: asset prices are observed at discrete time, why using Brownian motion (a continuous time stochastic process)?

- As the time increment is usually small, Brownian motion is a proper approximation
- Incorporate high frequency trading data
- Mathematically and numerically tractable (as we shall see)
- Easy to build on Brownian motion to obtain favorable features
- Many other important features as we shall see

Motivation to study Brownian motion

- Under the continuity assumption, it is reasonable to assume that
  - (i) the sample path $S(t)$ is non-differentiable everywhere.
  - (ii) for any $T > 0$,

$$\sum_{j=0}^{n-1} |S(t_{j+1}) - S(t_j)| \to +\infty$$

    as $||\Pi|| := \max_{0 \leq j \leq n-1}(t_{j+1} - t_j) \to 0$, where $\Pi := \{t_0, t_1, \cdots, t_n\}$ is a partition of $[0, T]$. It implies that within a finite interval, there exist an infinite number of ups and downs.
- Geometric Brownian Motion (GBM)

$$\exp\{\sigma W(t) + at\}$$

  is a good candidate because
  - every path of $W(t)$ is non-differentiable everywhere.
  - it satisfies (ii) (discussed in detail later).
  - it is always positive.

Construction of Brownian Motion from Random Walk

Consider a symmetric random walk
$M_n := \sum_{j=1}^{n} X_j$ for $n = 1, 2, ...;\ M_0 := 0$, where X_j are i.i.d. random variables such that

$$P(X_j = 1) = P(X_j = -1) = 0.5.$$

- $\{M_n\}$ is a martingale.
- Independent increments: $(M_{k_1} - M_{k_0}), (M_{k_2} - M_{k_1}), \cdots, (M_{k_m} - M_{k_{m-1}})$ are independent where $0 = k_0 < k_1 < \cdots < k_m$. Moreover,

$$\mathrm{Var}(M_{k_{i+1}} - M_{k_i}) = k_{i+1} - k_i.$$

  In particular, we have $\mathrm{Var}(M_k) = k$.

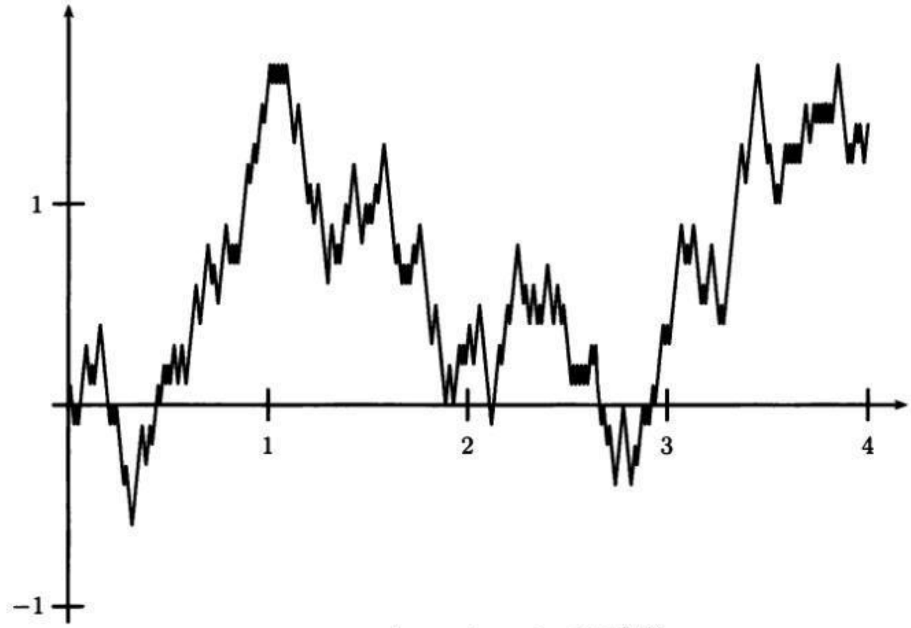
Construction of Brownian Motion from Random Walk

- Divide every unit time into n periods and define the scaled symmetric random walk:

$$W^{(n)}(t) = \frac{M_{nt}}{\sqrt{n}} \quad \text{if } nt \text{ is an integer.}$$

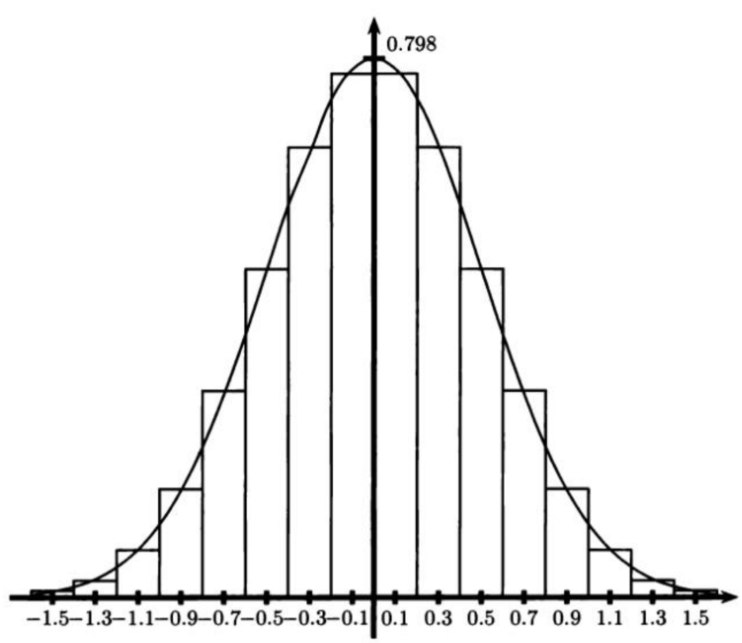
- **Magnify the local behavior** by nt and then **scale** it by $\frac{1}{\sqrt{n}}$.
- If nt and ns are integers, we have: $E(W^{(n)}(t) - W^{(n)}(s)) = 0$ and $Var(W^{(n)}(t) - W^{(n)}(s)) = t - s$
- **Theorem 3.2.1 (Central Limit)** Fix $t \geq 0$. As $n \to +\infty$, the distribution of the scaled random walk $W^{(n)}(t)$ evaluated at time t converges to $\mathrm{N}(0, t)$.

Construction of Brownian Motion from Random Walk



A sample path of $W^{(100)}$.

Construction of Brownian Motion from Random Walk



Distribution of $W^{(100)}(0.25)$ and normal curve $y = \frac{2}{\sqrt{2\pi}} e^{-2x^2}$.

Construction of Brownian Motion from Random Walk

- **Sketch of the Proof:** It suffices to show that the moment generating function $\phi_n(u) := Ee^{uW^{(n)}(t)}$ goes to

$$\phi(u) := Ee^{u\mathrm{N}(0,t)} \equiv e^{\frac{u^2 t}{2}}.$$

- Linear interpolation of $\frac{M_{\lfloor nt \rfloor}}{\sqrt{n}}$ and $\frac{M_{\lfloor nt \rfloor+1}}{\sqrt{n}}$):

$$W^{(n)}(t) = \frac{M_{\lfloor nt \rfloor}}{\sqrt{n}} + \left(\frac{M_{\lfloor nt \rfloor+1}}{\sqrt{n}} - \frac{M_{\lfloor nt \rfloor}}{\sqrt{n}} \right) (nt - \lfloor nt \rfloor).$$

- Not hard to calculate $\phi_n(u)$. Then basic algebra yields the results. $\square$

Brownian motion (BM)–distribution

- **Definition:** $X(t)$ is Gaussian process if for any $0 = t_0 < t_1 < \cdots < t_m$ and $m \in \mathcal{N}$, $(X(t_1), X(t_2), \cdots, X(t_m))$ assumes multivariate normal distribution.
- BM is a Gaussian process
- Its mean and covariance function:
  - $EW(t) = 0$.
  - $E[W(t)W(s)] = t \wedge s := \min\{t, s\}$.
- What is the correlation function?
- Characterization of the distribution of $(W(t_1), W(t_2), \cdots, W(t_m))$
  - the moment generating function

$$\phi(u_1, u_2, \cdots, u_n) := \mathbb{E}e^{\sum_{i=1}^{m} u_i W(t_i)}.$$

  - the closed form expression of $\phi(u_1, u_2, \cdots, u_n)$ can be derived using property of independent and stationary increments.

Brownian motion – Filtration

- **Definition:** A filtration for the BM is a collection of σ-algebra $\mathcal{F}(t)$ such that
  - (Information accumulates) $\mathcal{F}(s) \subseteq \mathcal{F}(t)$ if $s < t$.
  - (Adaptivity) For any $t \geq 0$, $W(t)$ is $\mathcal{F}(t)$-measurable.
  - (Independence of future increments) If $u > t \geq 0$, $W(u) - W(t)$ is independent of $\mathcal{F}(t)$.

- Two possibilities of the filtration $\mathcal{F}(t)$.
  - $\mathcal{F}(t)$ contains only the information by observing the BM itself up to time t.
  - $\mathcal{F}(t)$ contains the information by observing the BM as well as other processes up to time t. In this case, the information of other processes cannot give any clues of the future increments of the BM.

Brownian motion – Some Fundamental Properties

- BM is a Markov process.

$$\mathbb{E}(f(W(t))|\mathcal{F}(s)) = \mathbb{E}(f(W(t))|W(s)), \quad \text{for } 0 < s < t.$$

- BM is a **strong Markov** process (generalize the Markov property to stopping times):
  Let τ be a finite stopping time ("known" to the Brownian filtration, i.e. $\{\tau < t\} \in \mathcal{F}(t)$, then

$$\mathbb{E}(f(W(\tau + t))|\mathcal{F}(\tau)) = \mathbb{E}(f(W(\tau + t))|W(\tau)).$$

- Implication of the strong Markov property: **Brownian motion refreshes after a stopping time**!

$$B(t) := W(\tau + t) - W(\tau)$$

  is again a Brownian motion independent of $\mathcal{F}(\tau)$.

Brownian motion – Some Fundamental Properties

- Brownian motion is a **martingale**:

$$\mathbb{E}(W(t)|\mathcal{F}(s)) = W(s), \text{ for } s < t.$$

- Invariance under time translation (a special case of "Brownian motion refreshing after a stopping time"): $B(t) = W(t+T) - W(T)$ is a Brownian motion independent of $\mathcal{F}(T)$
- Invariance under scaling: $B(t) = \frac{1}{\sqrt{c}} W(ct)$ is a BM for any given $c > 0$
- Invariance under symmetry: $B(t) = -W(t)$ is a BM
- Invariance under time-reversal: $B(t) = W(T) - W(T-t)$ is a BM for $0 \le t \le T$.

Brownian motion – Some Fundamental Properties

- BM is unbounded:

$$\mathbb{P}\left(\sup_{0\le t<\infty} W(t) = \infty\right) = 1, \quad \mathbb{P}\left(\inf_{0\le t<\infty} W(t) = -\infty\right) = 1.$$

- BM is recurrent; it visits every site on the real line and keeps returning to it **over and over again**. (this can be explained by the strong Markov property)
- The BM path is nowhere differentiable (very zigzag).
- Several related martingales: $W(t)$, $W^2(t) - t$, and $Z(t) := e^{\sigma W(t) - \frac{\sigma^2}{2} t}$.

Brownian motion – Quadratic variation

- **Quadratic variation** (the total variation of the second order) up to time k is

$$[M, M]_k := \sum_{j=1}^{k} (M_j - M_{j-1})^2 = k$$

- Both the variance and the quadratic variation of the random walk accumulate at rate one per unit time. However, the difference is that the former is deterministic whereas the latter is random.

- Quadratic variation of the scaled random walk: $[W^{(n)}, W^{(n)}](t) = \sum_{j=1}^{nt} \left[W^{(n)}\left(\frac{j}{n}\right) - W^{(n)}\left(\frac{j-1}{n}\right)\right]^2 = \sum_{j=1}^{nt} \left[\frac{X_j}{\sqrt{n}}\right]^2 = \sum_{j=1}^{nt} \frac{1}{n} = t.$

Brownian motion – Quadratic variation

Recall

- $W(t)$ seems to fluctuate very frequently (extreme zig-zagness)
- The scaled random walk $W^{(n)}(t)$ has a quadratic variation t

We anticipate that

- the first-order variation $FV_T(W)$ of the BM $W(t)$ is $+\infty$, i.e., for any $T > 0$,

$$\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)| = +\infty,$$

  where $||\Pi|| := \max_{0 \le j \le n-1} (t_{j+1} - t_j)$ and $\Pi := \{t_0, t_1, \cdots, t_n\}$ is a **partition** of $[0, T]$.
- the quadratic variation $[W, W](t)$ of the BM $W(t)$ is t, i.e., it accumulates at rate 1 per unit time, i.e., for any $T > 0$,

$$\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} [W(t_{j+1}) - W(t_j)]^2 = T.$$

Brownian motion – Quadratic variation

- **Proposition from Calculus:** If $f(t)$ is continuously differentiable (derivatives exist and smooth enough),
  - its first-order variation: $\sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)| = \sum_{j=0}^{n-1} |f'(t_j^*)(t_{j+1} - t_j)| \to \int_0^T |f'(t)|dt < +\infty$, as $||\Pi|| \to 0$.
  - its quadratic variation: $\sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2 = \sum_{j=0}^{n-1} f'^2(t_j^*)(t_{j+1} - t_j)^2 \leq ||\Pi|| \sum_{j=0}^{n-1} f'^2(t_j^*)(t_{j+1} - t_j) \to 0 \times \int_0^T f'^2(t)dt = 0$, as $||\Pi|| \to 0$.
- However, $W(t)$ is **non-differentiable everywhere** (extremely zigzag).
- **Theorem** For a BM $W(t)$, we have that
  - its first-order variation is: $FV_T(W) = +\infty$
  - its quadratic variation is: $[W, W](T) = T$ for all $T \geq 0$ almost surely

Brownian motion – Quadratic variation

- We write (ii) informally as

$$dW(t)dW(t) = dt,$$

meaning that

$$\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} [W(t_{j+1}) - W(t_j)]^2 = T.$$

- Similarly, we can obtain that

$$dW(t)dt = 0, \text{ and } dtdt = 0.$$

- Implications of the Theorem: The sample path of the BM must have an **infinite number of ups and downs**, each of which, however, is **infinitesimal**. So the **extreme zig-zagness** of the path implies its non-differentiability.

Brownian motion – First Passage Time Distribution

- The first passage time (FPT) of a process $Y(t)$ to a level m from below is defined to be

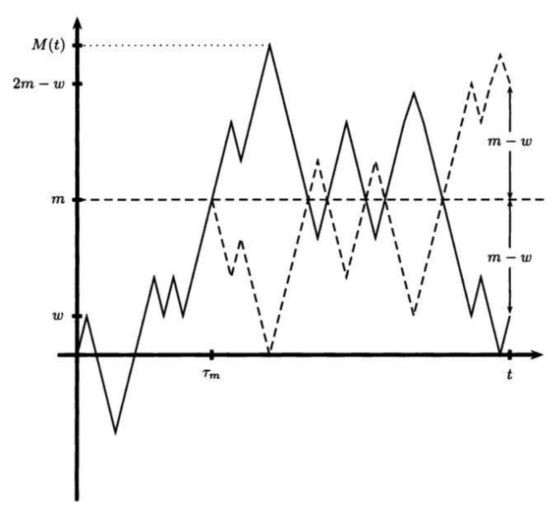
$$\tau_m := \inf\{t \geq 0 : Y(t) \geq m\},$$

  where $\inf \emptyset := +\infty$.
- What is the distribution of the first passage time of $W(t)$ to m: τ_m
- Potential application: prediction of the behavior of an asset
- Potential application: modeling the credit default
- Two approaches to find this distribution
  - Calculate the distribution from the Reflection Principle
  - Calculate the Laplace transform of the probability density function (optional for self-reading)

Brownian motion – Reflect Principle and the FPT

Reflection Principle: Assume $m > 0$. If we "reflect the path after τ_m with respect to level m", we get a Brownian motion again! i.e.

$$\begin{aligned} \widetilde{W}(t) =& W(t),\ 0 \leq t \leq \tau_m; \\ =& 2m - W(t),\ t > \tau_m. \end{aligned} \tag{1}$$



Brownian path and reflected path.

Brownian motion – Reflect Principle and the FPT

Let $w \leq m$. We obtain that

$$P(\tau_m \leq t, W(t) \leq w) = P(W(t) \geq 2m - w).$$

- Let $w = m$,

$$\begin{aligned} P(\tau_m \leq t) =& P(\tau_m \leq t, W(t) \leq m) + P(\tau_m \leq t, W(t) \geq m) \\ =& 2P(\tau_m \leq t, W(t) \geq m) = 2P(W(t) \geq m) \\ =& \frac{2}{\sqrt{2\pi t}} \int_m^{+\infty} e^{-\frac{x^2}{2t}} dx. \end{aligned} \tag{2}$$

- The Brownian motion goes up or down with the same probability symmetrically
- Taking the derivative w.r.t. t yields the pdf

Brownian motion – The Historical Maximum

- Define the historical maximum $M(t) = \max_{0 \leq s \leq t} W(s)$, we have

$$P(\tau_m \leq t) = P(M(t) \geq m).$$

- Potential application: prediction of the maximum of stock price! How?
- $$\begin{aligned} P(M(t) \geq m, W(t) \leq w) =& P(W(t) \geq 2m - w) \\ =& \frac{2}{\sqrt{2\pi t}} \int_{2m-w}^{+\infty} e^{-\frac{x^2}{2t}} dx. \end{aligned} \tag{3}$$
- Taking the derivative w.r.t. m and w and multiplying the result by -1 yields the joint pdf

$$f_{M(t),W(t)}(m, w) = \frac{2(2m - w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}.$$

Brownian motion – Laplace Transform of the FPT

- Without loss of generality, we assume $m > 0$
- A common method to study a stopping time is to construct a martingale. Here we use $Z(t) := e^{\sigma W(t) - \frac{\sigma^2}{2} t}$.
- Then $Z(t \wedge \tau_m)$ is also a martingale, as implies that
  $1 = Z(0) = EZ(t \wedge \tau_m) = Ee^{\sigma W(t\wedge\tau_m) - \frac{\sigma^2}{2} t\wedge\tau_m} =$
  $E\left[e^{\sigma W(t\wedge\tau_m) - \frac{\sigma^2}{2} t\wedge\tau_m} I_{\{\tau_m < +\infty\}}\right] +$
  $E\left[e^{\sigma W(t\wedge\tau_m) - \frac{\sigma^2}{2} t\wedge\tau_m} I_{\{\tau_m = +\infty\}}\right] =$
  $E\left[e^{\sigma W(t\wedge\tau_m) - \frac{\sigma^2}{2} t\wedge\tau_m} I_{\{\tau_m < +\infty\}}\right] + E\left[e^{\sigma W(t) - \frac{\sigma^2}{2} t} I_{\{\tau_m = +\infty\}}\right].$
- Letting $t \to +\infty$ and applying the dominated convergence theorem, we obtain that
  $1 = E\left[e^{\sigma W(\tau_m) - \frac{\sigma^2}{2}\tau_m} I_{\{\tau_m < +\infty\}}\right] = E\left[e^{\sigma m - \frac{\sigma^2}{2}\tau_m} I_{\{\tau_m < +\infty\}}\right].$

Brownian motion – Laplace Transform of the FPT

- So we have that $E\left[e^{-\frac{\sigma^2}{2}\tau_m} I_{\{\tau_m < +\infty\}}\right] = e^{-\sigma m}$. Note that $E\left[e^{-\frac{\sigma^2}{2}\tau_m} I_{\{\tau_m = +\infty\}}\right] = 0$. We have $E\left[e^{-\frac{\sigma^2}{2}\tau_m}\right] = e^{-\sigma m}$
- Letting $\sigma \to 0+$ yields

$$P(\tau_m < +\infty) = 1.$$

- For $m \in \mathcal{R}$, the first passage time of the BM to m is finite almost surely, and the Laplace transform of its pdf is given by

$$E\left[e^{-\alpha\tau_m}\right] = e^{-|m|\sqrt{2\alpha}}.$$

- Taking derivative of $E\left[e^{-\alpha\tau_m}\right]$ with respect to α, we get

$$E\left[\tau_m e^{-\alpha\tau_m}\right] = \frac{|m|}{\sqrt{2\alpha}} e^{-|m|\sqrt{2\alpha}}.$$

- Letting $\alpha \to 0+$ leads to $E\tau_m = +\infty$ if $m \neq 0$.

Some Processes Derived from Brownian Motion

Building more processes from Brownian motion towards the goal of modeling financial market!

- Brownian motion with drift:

$$X(t) = \sigma W(t) + \mu t$$

  Allow arbitrary "volatility" and a "trend".

- Geometric Brownian motion (the celebrated Black-Schole-Merton (1973) model):

$$S(t) = \exp\{\sigma W(t) + \alpha t\}.$$

  A fundamental candidate for describing the financial asset price.

- Brownian Bridge: a process equivalent in law to a Brownian motion given a terminal value. example: $B(t) = W(t) - tW(1)$ is a Brownian bridge on $[0, 1]$ with terminal value 0.

An Application: Modeling Credit Default Risk

- Merton (1974) Structural Credit Model
- Assumption: the firm is financed by equity and a zero coupon bond with face value K and maturity date T.

  Firm's value $V(t) =$ Firm's equity $S(t) +$ Firm's debt $B(t)$.

- If the firm cannot fulfil its payment obligation K, then bond holders will immediately take over the firm.
- Corporate liabilities are contingent claims on the assets of a firm
  - Firm's debt (e.g., bond)

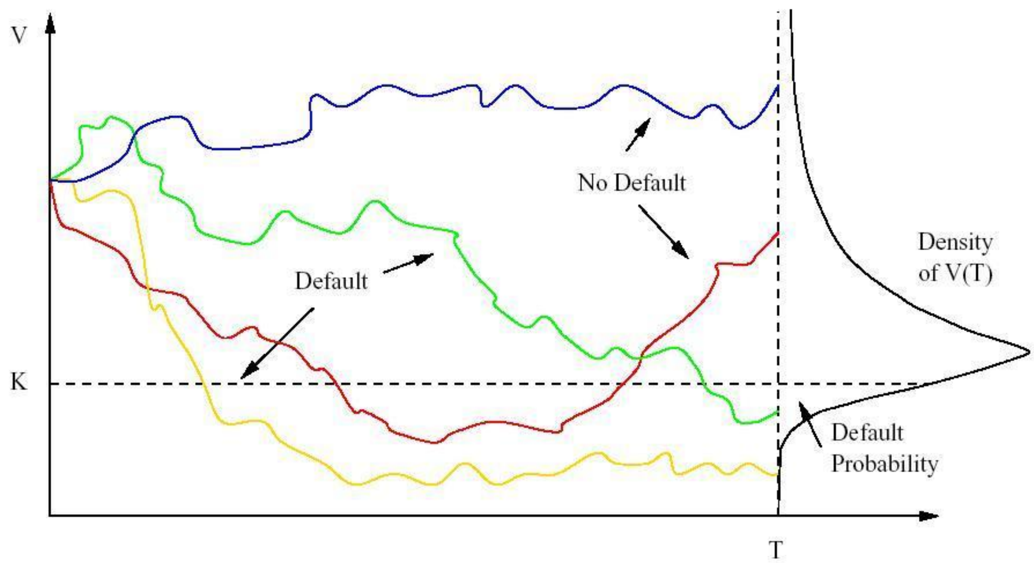
$$B(T) = \min\{K, V(T)\} = V(T) - \max\{0, V(T) - K\}.$$

  - Firm's equity

$$S(T) = \max\{0, V(T) - K\}.$$

- In Merton's classical approach (1974) default is seen as $V(T) < K$.

Merton's Classical Approach for Modeling Default



Calculating Default Probability

- Question: if we model $V(t)$ as a geometric Brownian motion

$$V(t) = V_0 e^{\sigma W(t) + mt},$$

  What is the probability of default? i.e.

$$\mathbb{P}(V(T) < K) = ?$$

- Moody's KMV has developed an industry standards for measuring default probabilities
- Their so-called expected default frequency (EDF) are used for computations of VaR and various risk-measures as well as for simple defaultable asset valuation.
- Moody's idea is based on the simple Merton model. But, Moody's own technology is proprietary.

Alternative Approach: Modeling Default by First Passage Time

- Suppose default Barrier D is a constant value in $(0, V_0)$. The default time is defined as a stopping time

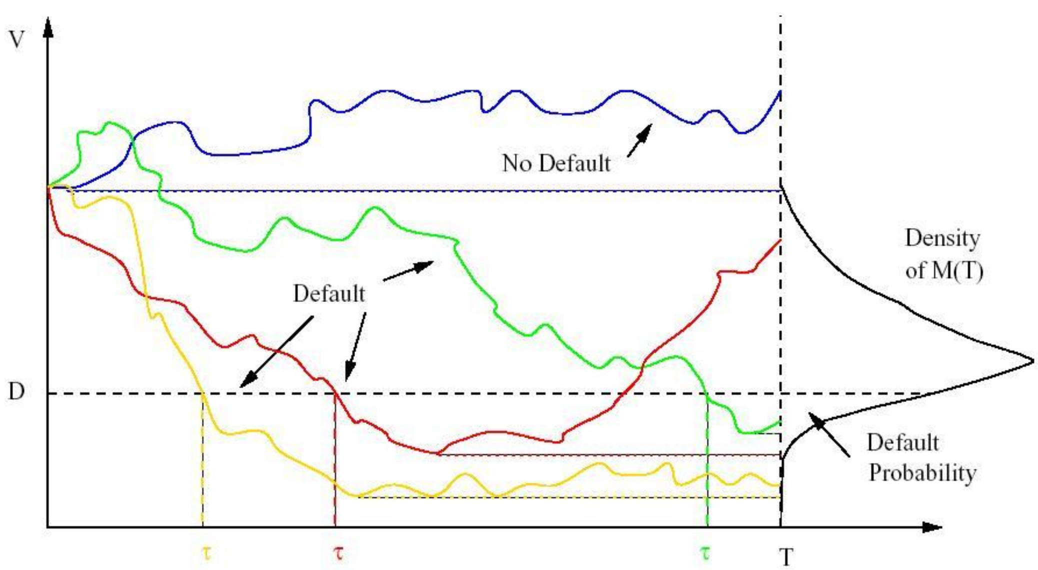
$$\tau = \inf\{t > 0, V(t) < D\}.$$

- Default happens during $[0, T]$ if $\tau < T$.
- What is the probability of default?

$$\mathbb{P}(\tau < T) = ?$$

- Need the distribution of $M(T) = \min_{s \le T} V(s)$
- We had the distribution of the historical maximum /minimum of a Brownian motion.
- Here, we need the distribution of the historical maximum /minimum of a Brownian motion with drift. Try Exercise 3.7 in Shreve Vol. II if you are interested

Modeling Default by First Passage Time



Multidimensional Brownian Motion: Standard Case

Intuitively, a standard d-dimensional Brownian motion is d independent copies of standard one-dimensional Brownian motion.

Formal definition: a $d-$dimensional stochastic process $W(t) = (W_1(t), ..., W_d(t))$

- $W(0) = 0$;
- Independent increment
- For any $t > s$, $W(t) - W(s)$ has a joint normal distribution with mean 0 and covariance matrix $(t-s)I$.
- For any $i = 1, 2, ..., d$, $W_i(t)$ is a continuous function of t.

Multidimensional Brownian Motion: Correlated Case

Question: How about the correlated Brownian motions?
Answer: Change the covariance matrix to $(t-s)\Sigma$, where $\Sigma = (\rho_{ij})$. Here

$$\rho_{ij} = \text{Corr}(W_i(t), W_j(t)).$$

Connection with independent Brownian motion:
Cholesky decomposition: We can always find a standard $d-$dimensional Brownian motion $Z(t)$ such that

$$W(t) = AZ(t),$$

where A is sub-triangular matrix satisfying that $AA^T = \Sigma$.
An example:
for $d = 2$,

$$W_1(t) = Z_1(t),\ W_2(t) = \rho_{12} Z_1(t) + \sqrt{1-\rho_{12}^2} Z_2(t).$$

Suggested Reading Material (We only need to focus on the material parallel to our course slides):

- Selected Material from Mikosch: section 1.3, some examples from section 1.4, 1.5

  Or you can find equivalent material from

- Shreve Vol. II: some parts from chapter 4 (Note: We don't need those lengthy mathematical proofs! Just understand the material and be able to apply the tools)

Suggested Exercises (Do Not Hand In; For Your Deeper Understanding Only)

- Shreve Vol. II: Exercise 3.2, 3.5, 3.3, 3.7