函数极限的运算

本段内容要点:

函数极限的四则运算法则

复合函数的极限运算

若干函数极限的例题

定理: $\lim\limits_{x \to x_0} f(x) = A \Leftrightarrow$

$f(x) - A$是$x \to x_0$时的无穷小量.

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$f(x) - A$是$x \to x_0$时的无穷小量.

这是因为

$$f(x) = A + [f(x) - A],$$

故$\lim\limits_{x \to x_0} f(x) = A \Leftrightarrow f(x) = A + \alpha(x),$

其中$\alpha(x)$是一个$x \to x_0$时的无穷小量.

$$\lim_{x}$$

定理: 设在某一个极限过程中,

$$\lim_{x} u(x) = A,\ \lim_{x} v(x) = B$$ 则有:

(1) $$\lim_{x}(u(x) \pm v(x)) = \lim_{x} u(x) \pm \lim_{x} v(x) = A \pm B$$

(和差的极限等于极限的和差)

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因为$\lim_{x} u(x) = A$, 所以$u(x) = A + \alpha(x)$;

因为$\lim_{x} v(x) = B$, 所以$v(x) = B + \beta(x)$;

其中$\alpha(x), \beta(x)$为该过程中的无穷小量.

$$\boxed{\begin{array}{l} (1)\ \ u(x) \pm v(x) = (A \pm B) + (\alpha(x) \pm \beta(x)) \\ \Rightarrow\ \ \lim_{x}(u(x) \pm v(x)) = A \pm B. \end{array}}$$

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(乘积的极限等于极限的乘积)

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(2) $u(x)v(x) = AB + (A\beta(x) + B\alpha(x) + \alpha(x)\beta(x))$,

$\Rightarrow \lim_{x}(u(x) \cdot v(x)) = AB.$

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(3) if $B \neq 0$, then $\lim_{x} \dfrac{u(x)}{v(x)} = \dfrac{\lim_{x} u(x)}{\lim_{x} v(x)} = \dfrac{A}{B}$

(商的极限等于极限的商)

因为$\lim\limits_{x} u(x) = A$, 所以$u(x) = A + \alpha(x)$;
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(3) $\dfrac{u(x)}{v(x)} = \dfrac{A + \alpha(x)}{B + \beta(x)}$

$$= \frac{A}{B} + \left(\frac{A + \alpha(x)}{B + \beta(x)} - \frac{A}{B}\right)$$

$$= \frac{A}{B} + \frac{B\alpha(x) - A\beta(x)}{B(B + \beta(x))}$$

$$\Rightarrow \lim_{x} \frac{u(x)}{v(x)} = \frac{A}{B}.$$

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(推广:有限四则运算均成立!)

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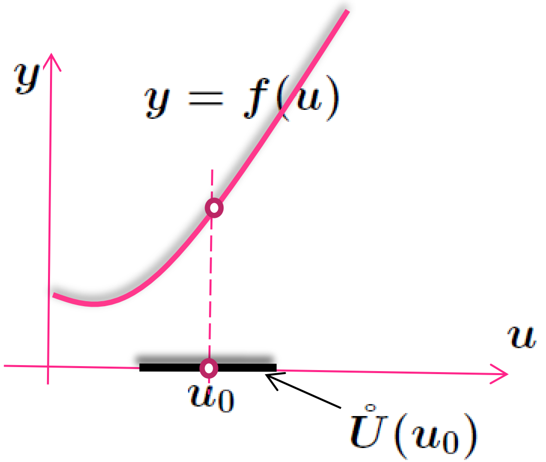

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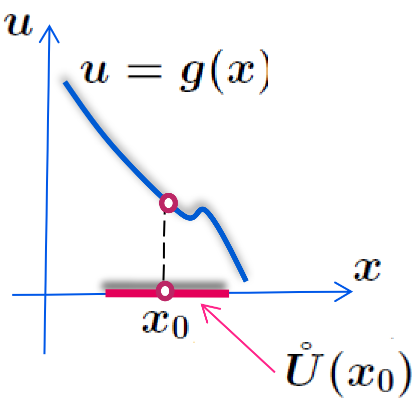

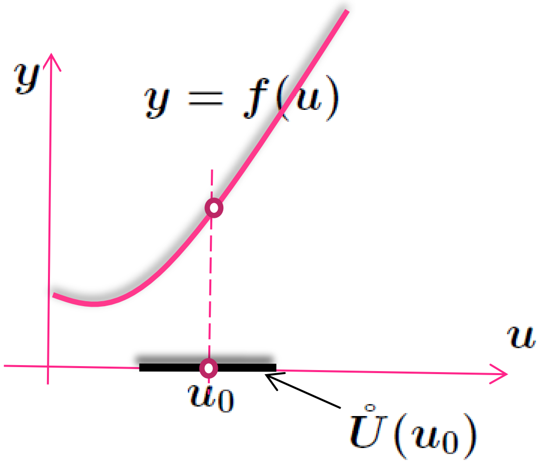

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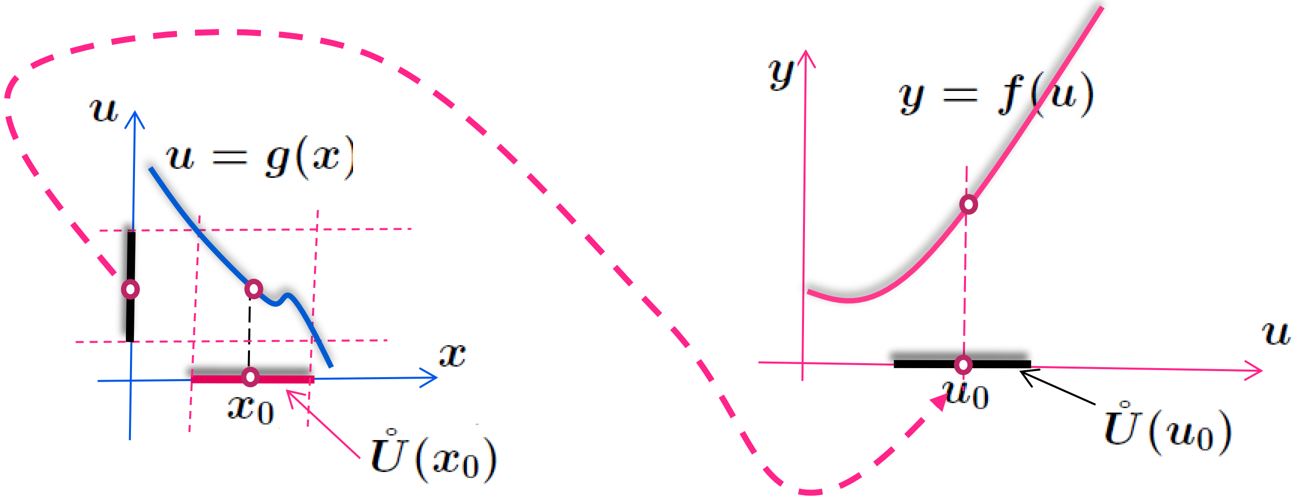

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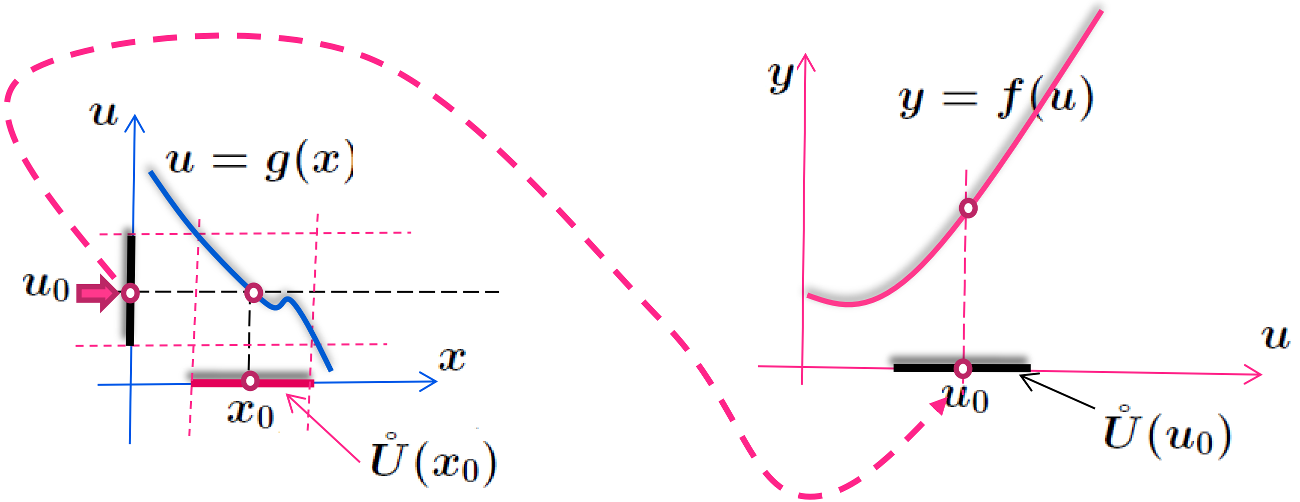

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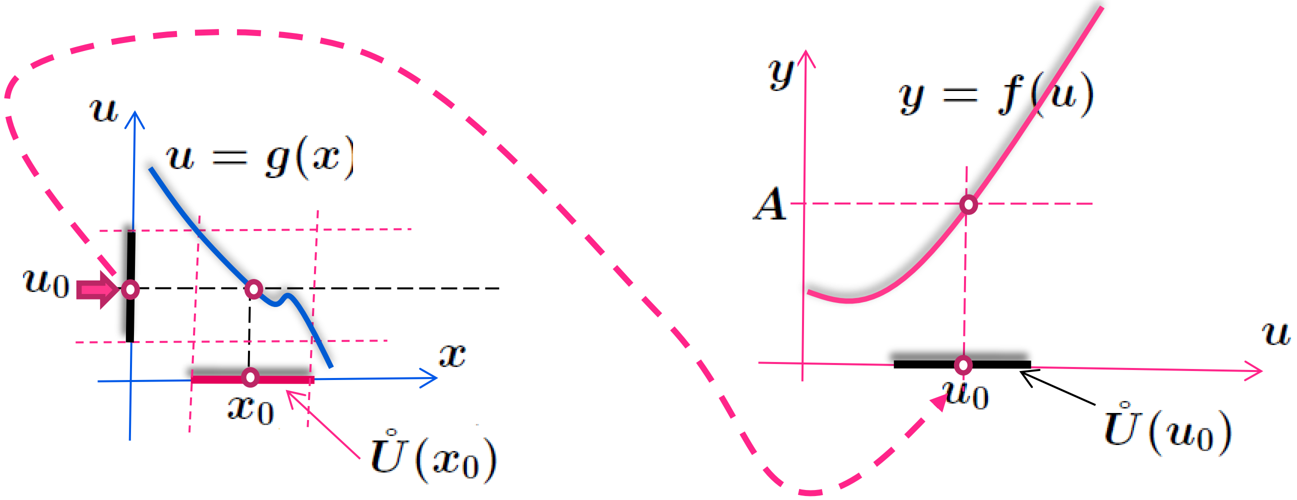

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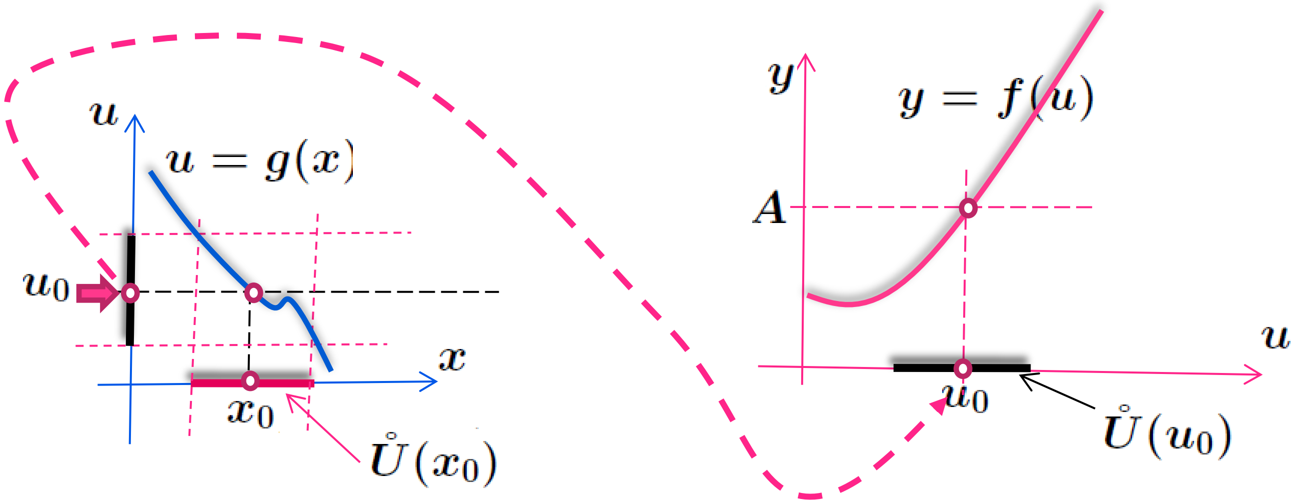

证明: 往证 $\forall\varepsilon > 0,\ \exists\delta > 0\ s.t.\ 0 < |x - x_0| < \delta \Rightarrow |f(g(x)) - A| = |f(u) - A| < \varepsilon.$

若有 $\lim\limits_{x\to x_0} g(x) = u_0,\quad \lim\limits_{u\to u_0} f(u) = A,$

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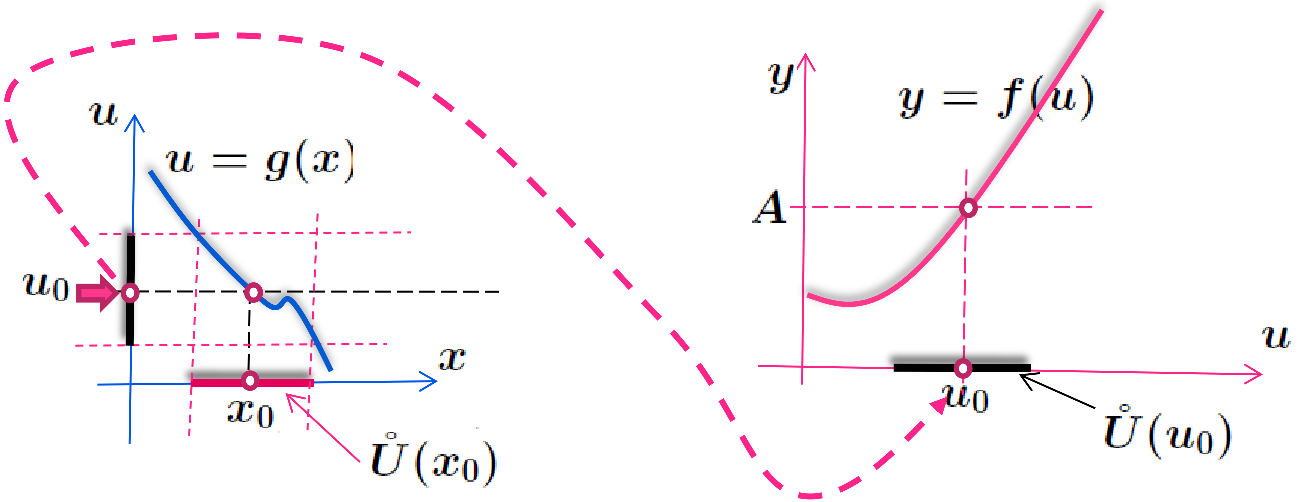

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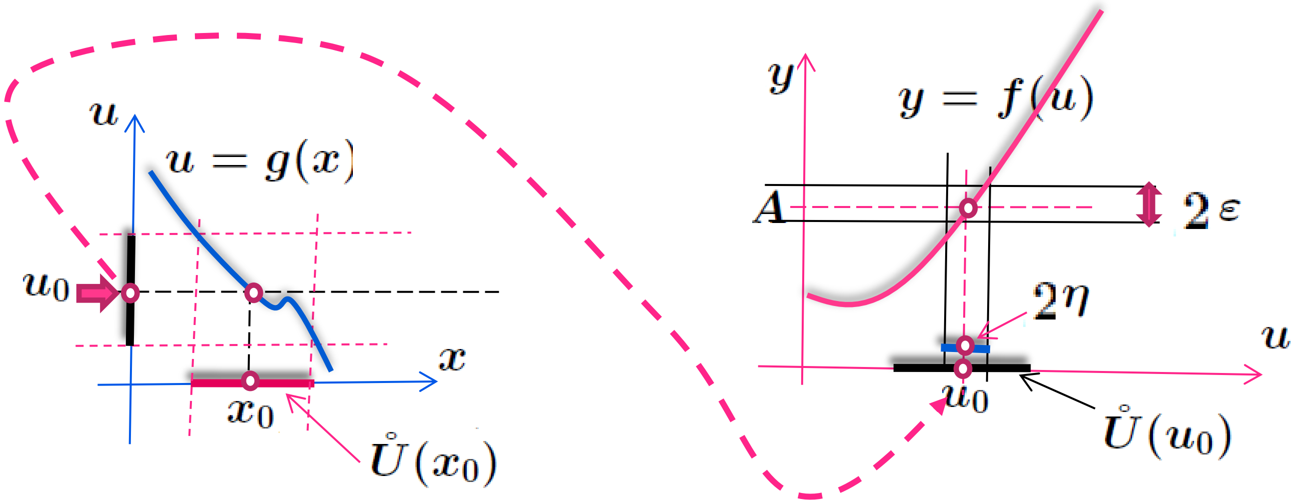

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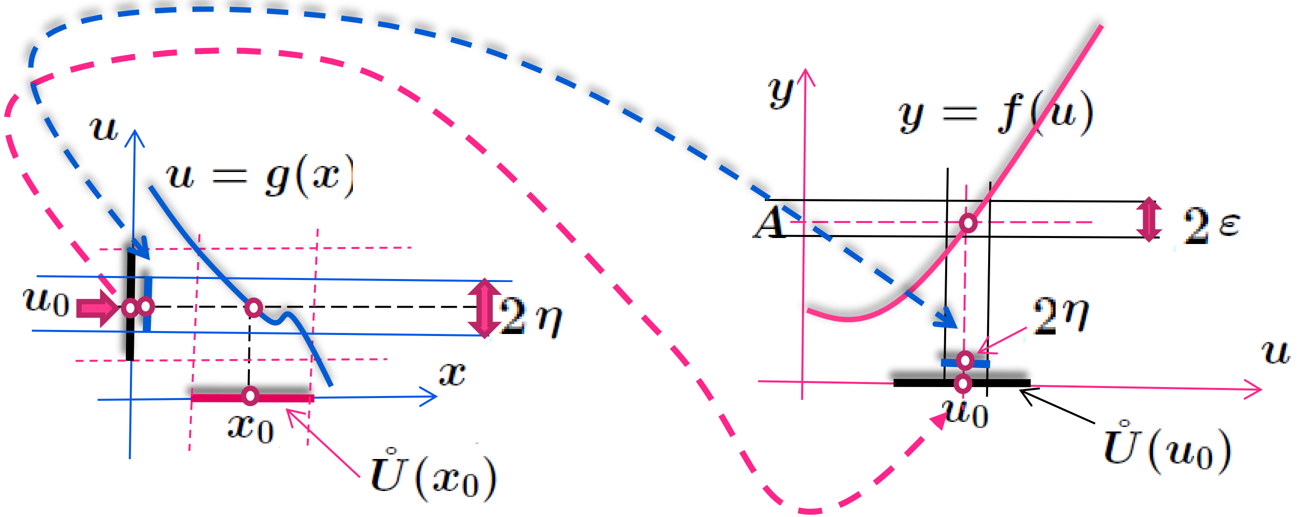

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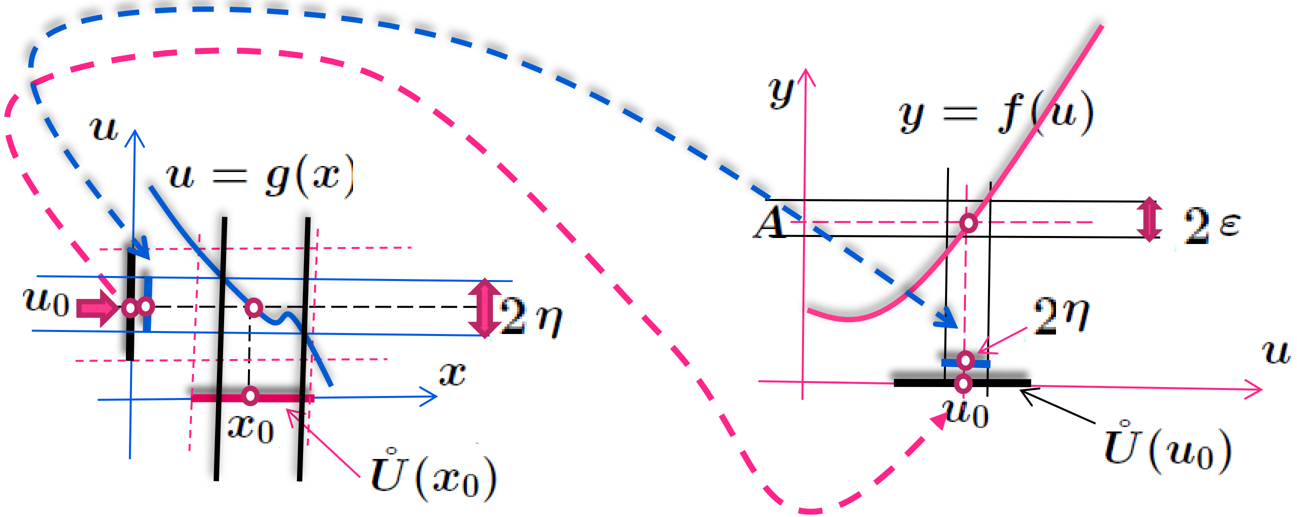

证明: 往证 $\forall\varepsilon>0,\ \exists\delta>0\ s.t.\ 0<|x-x_0|<\delta \Rightarrow |f(g(x))-A|=|f(u)-A|<\varepsilon.$

若有 $\lim\limits_{x\to x_0} g(x)=u_0,\quad \lim\limits_{u\to u_0} f(u)=A,$

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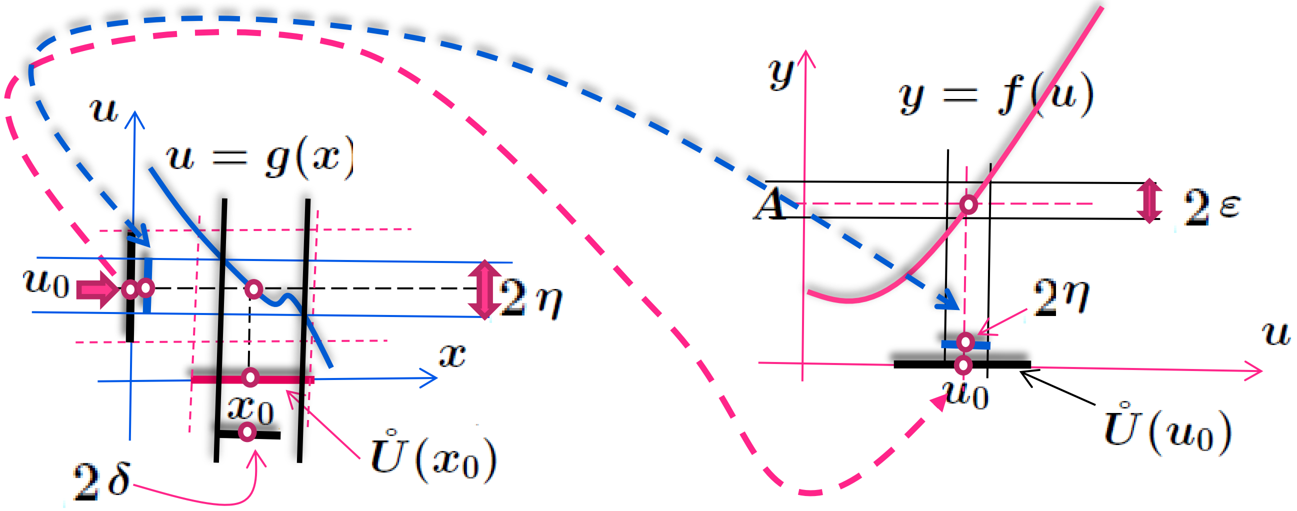

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对上述的 $\varepsilon > 0$, 因为 $\lim\limits_{u \to u_0} f(u) = A$, 所以

$\exists \eta > 0 \ s.t. \ 0 < |u - u_0| < \eta \Rightarrow |f(u) - A| < \varepsilon;$

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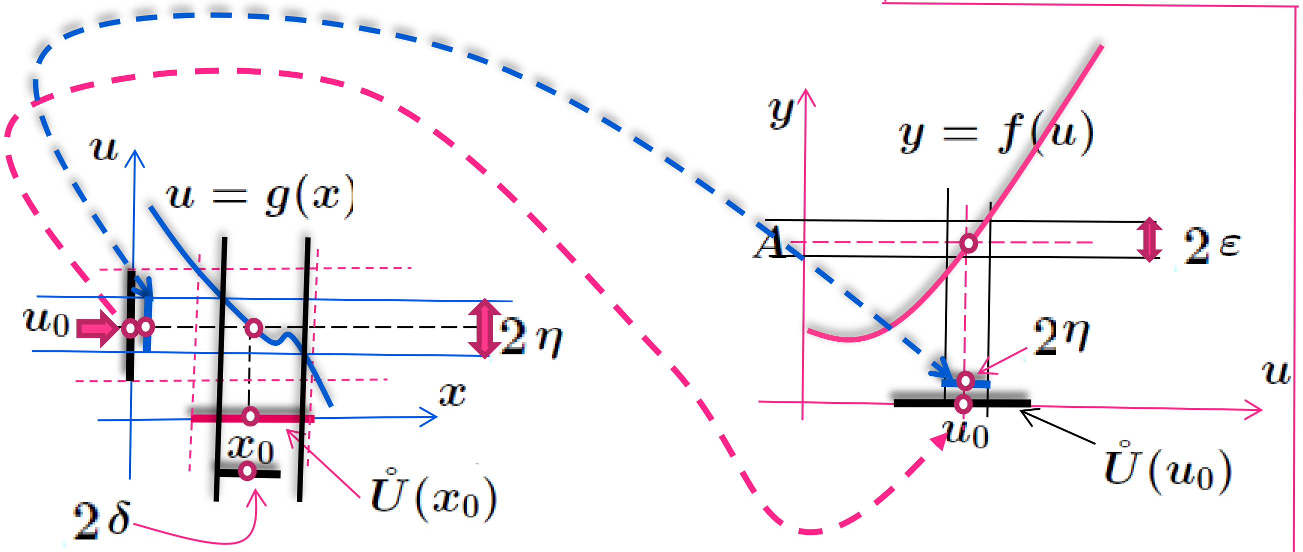

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对上述的 $\varepsilon > 0$，因为 $\lim\limits_{u \to u_0} f(u) = A$，所以

$\exists \eta > 0 \ s.t. \ 0 < |u - u_0| < \eta \Rightarrow |f(u) - A| < \varepsilon;$

又由于 $\lim\limits_{x \to x_0} g(x) = u_0$，故对于 $\eta > 0$，

$\exists \delta > 0 \ s.t. \ 0 < |x - x_0| < \delta$

$\Rightarrow |u - u_0| = |g(x) - u_0| < \eta.$

若有 $\lim\limits_{x \to x_0} g(x) = u_0$， $\lim\limits_{u \to u_0} f(u) = A$，

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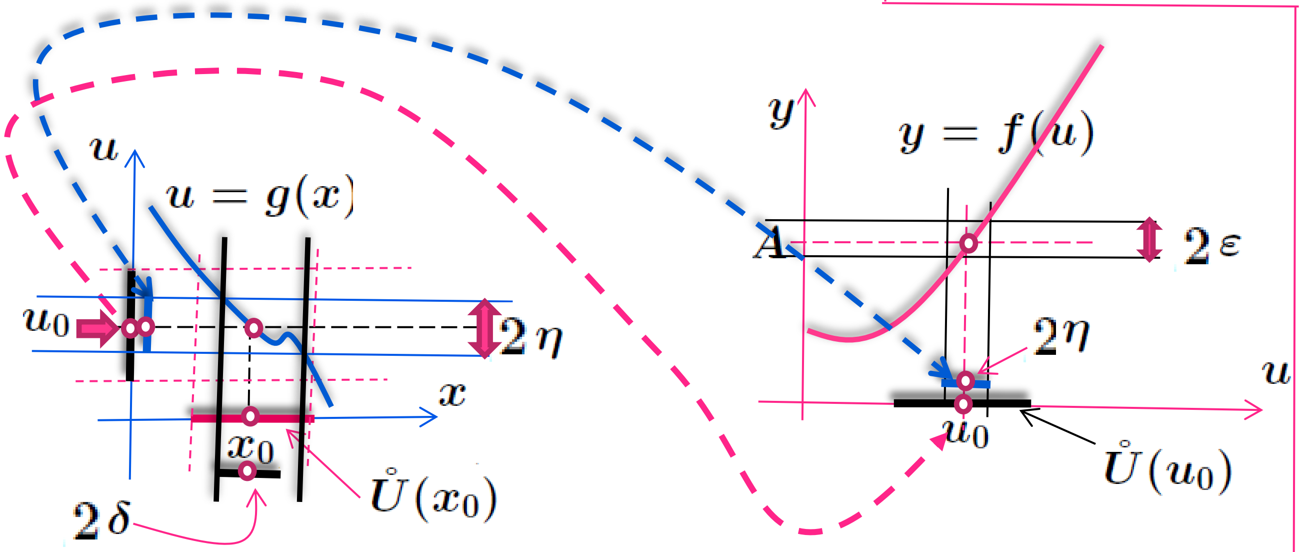

证明: 往证 $\forall\varepsilon>0,\ \exists\delta>0\ \ s.t.\ \ 0<|x-x_0|<\delta\ \Rightarrow\ |f(g(x))-A|=|f(u)-A|<\varepsilon.$

对上述的 $\varepsilon>0$, 因为 $\lim\limits_{u\to u_0} f(u)=A$, 所以

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从而

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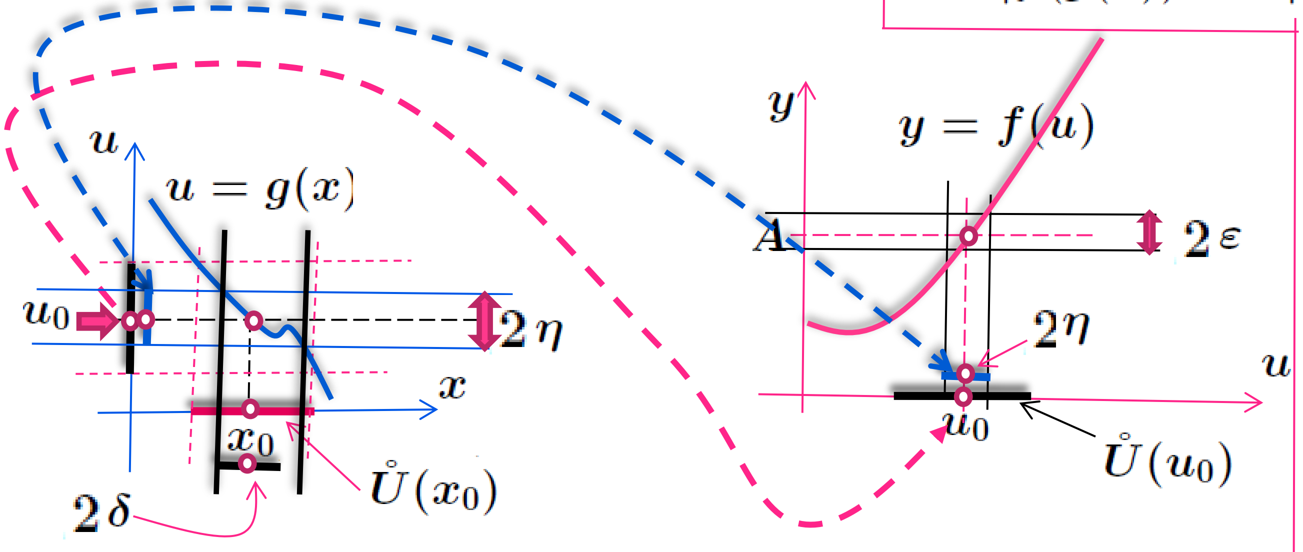
$\Rightarrow|f(g(x))-A|=|f(u)-A|<\varepsilon.$



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此结论对所有的极限过程都成立.

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注意: $\lim\limits_{x\to x_0}g(x)=u_0$, $\lim\limits_{u\to u_0}f(u)=A$时,

一般只能导致

$$\lim_{x\to x_0}f(g(x))=\lim_{u\to u_0}f(u)=A.$$

不能导致

$$\lim_{x\to x_0}f(g(x))=f(\lim_{x\to x_0}g(x))=f(u_0).$$

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若有$\lim\limits_{x \to x_0} g(x) = u_0$, $\lim\limits_{u \to u_0} f(u) = A$,

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不能导致

$$\lim_{x \to x_0} f(g(x)) \cancel{=} f(\lim_{x \to x_0} g(x)) = f(u_0).$$

这是因为$\lim\limits_{u \to u_0} f(u)$不一定$= f(u_0)$.

例如: $f(u)=\begin{cases} u\sin u, & u\neq 0, \\ 1, & u=0, \end{cases}$ $\qquad g(x)=(x-1)^2,$

$\lim\limits_{x\to 1} g(x)=0,\qquad$ 则$f(g(x))=\begin{cases} (x-1)^2\sin(x-1)^2, & x\neq 1, \\ 1, & x=1, \end{cases}$

$$\lim_{x\to 1} f(g(x))=\lim_{u\to 0} f(u)=0\neq f(0)$$

$$f(0)=f(\lim_{x\to 1} g(x))=1.$$

注意: $\lim\limits_{x\to x_0} g(x)=u_0,\ \lim\limits_{u\to u_0} f(u)=A$时,

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$$\lim_{x\to x_0} f(g(x))=\lim_{u\to u_0} f(u)=A.$$

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$$\lim_{x\to x_0} f(g(x))\cancel{=}f(\lim_{x\to x_0} g(x))=f(u_0).$$

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$$\lim_{x} u(x) = A > 0 \quad \Rightarrow \lim_{x} \ln u(x) = \lim_{u \to A} \ln u = \ln A$$

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推论:

若 $\lim\limits_{x \to x_0} f(x) = A$, 则 $\lim\limits_{x \to x_0} \sin f(x) = \sin A$

推论:

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若 $\lim\limits_{x \to x_0} f(x) = A$, 则 $\lim\limits_{x \to x_0} \sin f(x) = \sin A$

$$(|\sin f(x) - \sin A| \leqslant |f(x) - A|).$$

推论:

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$$\lim_{x} u(x) = A \geqslant 0 \quad \Rightarrow \lim_{x} u^{\alpha}(x) = \lim_{u \to A} u^{\alpha} = A^{\alpha}$$

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$$\lim_{x} u(x) = A > 0,\ \lim_{x} v(x) = B \quad \Rightarrow \lim_{x} u(x)^{v(x)} = A^{B}.$$

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而$\lim_{x} u(x) = A > 0$

$$\Rightarrow \lim_{x} \ln u(x) = \lim_{u \to A} \ln u = \ln A$$

所以

$$\lim_{x} e^{v(x)\ln u(x)} = e^{B\ln A} = A^B.$$

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注意, 只要$A \neq 1$,
上述公式对$B = +\infty, B = -\infty$的情况也成立.

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$$\lim_{n\to\infty}(1+\frac{1}{n})^{n} = e,\ \lim_{n\to\infty}(1+\frac{1}{n})^{-n} = e^{-1}$$

例: $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{2x^2 - x - 1}$

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$$=\frac{\lim_{x\to 1}(x+1)}{\lim_{x\to 1}(2x+1)}=\frac{2}{3}.$$

例: $\lim\limits_{x \to -1} \left(\frac{1}{x+1} - \frac{3}{x^3+1} \right)$

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$$= \lim_{x \to -1} \frac{x^2 - x - 2}{(x+1)(x^2 - x + 1)}$$

$$= \lim_{x \to -1} \frac{(x+1)(x-2)}{(x+1)(x^2 - x + 1)}$$

$$= \lim_{x \to -1} \frac{x-2}{x^2 - x + 1}$$

$$= \frac{\lim\limits_{x \to -1}(x-2)}{\lim\limits_{x \to -1}(x^2 - x + 1)} = \frac{-3}{3} = -1.$$

例: $\lim\limits_{x\to+\infty} x(\sqrt{x^2+1}-x)$

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$$= \lim_{x\to+\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}+1}$$

$$= \frac{1}{\lim\limits_{x\to+\infty}(\sqrt{1+\dfrac{1}{x^2}}+1)} = \frac{1}{2}.$$

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例: $\lim\limits_{x\to\infty}\left(\frac{x^2+1}{x+1}-x\right)$

$$\begin{aligned}
&= \lim_{x\to\infty}\frac{x^2+1-x^2-x}{x+1} \\
&= \lim_{x\to\infty}\frac{1-x}{x+1} \\
&= \lim_{x\to\infty}\frac{1/x-1}{1+1/x} = \frac{-1}{1} = -1.
\end{aligned}$$

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$$= \lim_{x \to 8} \frac{(\sqrt{9+2x}-5)(\sqrt{9+2x}+5)}{\sqrt{9+2x}+5} \frac{\sqrt[3]{x^2}+\sqrt[3]{x}+4}{(\sqrt[3]{x}-2)(\sqrt[3]{x^2}+\sqrt[3]{x}+4)}$$

例: $\lim\limits_{x\to 8}\dfrac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2}$

$$=\lim_{x\to 8}\frac{(\sqrt{9+2x}-5)(\sqrt{9+2x}+5)}{\sqrt{9+2x}+5}\frac{\sqrt[3]{x^2}+\sqrt[3]{x}+4}{(\sqrt[3]{x}-2)(\sqrt[3]{x^2}+\sqrt[3]{x}+4)}$$

$$=\lim_{x\to 8}\frac{2x-16}{\sqrt{9+2x}+5}\frac{\sqrt[3]{x^2}+\sqrt[3]{x}+4}{x-8}$$
$$=\lim_{x\to 8}\frac{2(\sqrt[3]{x^2}+\sqrt[3]{x}+4)}{\sqrt{9+2x}+5}$$
$$=\frac{20}{10}=2.$$

本段知识要点:

和差积商的极限等于极限的和差积商

复合函数的极限等于极限的复合

不能直接使用极限运算规则的函数要进行变形

分式求极限的关键是约分掉无穷小量或无穷大量