Lec 2 How to solve ODE? A review of some techniques from Calc/54.

Warm up (Revisit of concepts from last lecture)

	Dep var	ind. var	order	auto	linear	homo? (if linear)
1. $x^2 \frac{d^4x}{dt^4} + \left(\frac{dx}{dt}\right)^5 = \cos x$.	x	t	4	✓	✗	
$y'' = ty + t^2$	y	t	2	✗	✓	✗

$$\begin{cases} \dot{y} = u \\ \dot{u} = ty + t^2 \end{cases} \qquad \begin{pmatrix} \dot{y} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} u \\ ty + t^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}\begin{pmatrix} y \\ u \end{pmatrix} + \begin{pmatrix} 0 \\ t^2 \end{pmatrix}$$

$$\dot{\vec{y}} = A(t)\vec{y} + \vec{g}(t)$$

2. Free fall (IVP) $\begin{cases} \ddot{x} = -g \\ x(0) = x_0 \\ \dot{x}(0) = v_0 \end{cases}$ Solve the eq. Explain why 2 conditions are needed in IVP?

$x(t) = -\frac{1}{2}gt^2 + C_1 t + C_2$ "general soln"

$x(t) = -\frac{1}{2}gt^2 + v_0 t + x_0$ "soln"

Generally, how many initial conditions should be given in an IVP w/ n-th order ODE?

<u>Ans</u>: n.

Today: To solve ODE

Q1: what does it mean to solve ODE?

Q2: How?

[BN 1.3]

Ans to Q1: From last lecture, we saw the def of soln. Assume $f_1, \dots, f_n$ are given fns $D \subseteq \mathbb{R}^{n+1} \to \mathbb{R}$.

To solve $\begin{cases} \dot{y}_1 = f_1(t, y_1, \dots, y_n) \\ \vdots \\ \dot{y}_n = f_n(t, y_1, \dots, y_n) \end{cases}$ means to find an open interval I on the t axis and n fns

$\phi_1, \dots, \phi_n$ defined on I such that

(i) $\phi_1'(t), \phi_2'(t), \dots, \phi_n'(t)$ exist $\forall t \in I$

(ii) $(t, \phi_1(t), \dots, \phi_n(t)) \in D$ $\forall t \in I$.

(iii) Eq is satisfied.

- Why is I important?

Ex. $\begin{cases} \dot{y} = y^2 \\ y(1) = 1 \end{cases}$ ① Is $y = \frac{1}{2-t}$ a soln on the interval $(0,3)$?
② Determine the largest interval that the soln is valid. (interval of validity)

Ans ① NO!! Although Eq is satisfied. y is not defined on $(0,3)$
let alone its derivative.

② $f(t,y) = y^2$, $D = \mathbb{R}^2 \to \mathbb{R}^2$

$(2-x)^{-1}$ defined on $(-\infty, 2)$, $(2, +\infty)$

interval of validity : $(-\infty, 2)$

Quick Exercise $\begin{cases} \dot{y} = y^2 \\ y(1) = -1 \end{cases}$ $y = -\frac{1}{x}$ is a soln on $(0, +\infty)$.

Q2: How to solve? scalar ODE. Today: 1st order scalar ODE. [1B]

$\frac{dy}{dt} = f(t,y)$

Sep of var.

$$\frac{dy}{dt} = F(t)\, G(y)$$

One can do: $\frac{dy}{G(y)} = F(t)\, dt$

Note these two Eqs are not completely equivalent.
Need to pay attention to $G(y) = 0$

Ex. $\dot{y}(t) = t\, y^3 + y^3$

① Find General soln

$\dot{y} = (t+1)\, y^3$

$y^{-3}\, dy = t+1\, dt$

$-\frac{1}{2} y^{-2} = \frac{t^2}{2} + t + C$ "Equilibrium soln"

$y = \pm \frac{1}{\sqrt{-t^2 - 2t + C}}$ or $y = 0$.

② soln to IVP w./ $y(-1) = \frac{1}{2}$: largest valid interval?

$y = \frac{1}{\sqrt{-t^2 - 2t + 3}}$ $t \in (-3, 1)$

$t^2 + 2t - 3 \leq 0$

$(t+3)(t-1) \leq 0$

I lost my apple pencil, so have to type…

Two great questions in the class:

1. Concave up/ down?
Check increasing or decreasing of the first derivative.

2.Why the solution starting in between 0 and 1 can not touch the line P=1?

THIS IS A WONDERFUL QUESTION that everyone should remember!
We will be able to give an effortless answer after discussing the theorems of ODEs!!!

Ex2. logistic Eq. Population

$$\frac{dP}{dt} = k(1-\frac{P}{N})P$$

N: environmental carrying capacity, N=1 for simplicity

Ans $\frac{dp}{(1-p)p} = k\,dt$ or $p=0,1$

$$\frac{P}{1-P} = Ae^{kt}$$

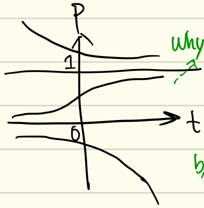
$$\left(\frac{1}{1-p} + \frac{1}{p}\right)dp = k\,dt$$

$$\Rightarrow P = \frac{Ae^{kt}}{1+Ae^{kt}} \text{ or } P=1$$

$$\ln\left|\frac{P}{1-P}\right| = kt + C$$

Qualitative Behavior of soln.

$\frac{dP}{dt}$: $-$ | 0 | $+$ | 1 | $-$



why cannot cross?

b/c need $\frac{dP}{dt}>0$ to cross, but if $P=1$, we get $\frac{dP}{dt}=0$.

$$P(0) = P_0$$

$$\Rightarrow A = \frac{P_0}{1-P_0}$$

$$P(t) = \frac{P_0e^{kt}}{1-P_0+P_0e^{kt}}$$

1st order linear ODE (Integrating factor)

$$\dot{y}(t) + a(t)y = b(t)$$

If $b(t)=0$, separable. $y(t) = Ce^{-\int a(t)dt}$

Generally. integrating factor $m(t) \triangleq e^{\int a(t)dt}$

$$m(t)\dot{y}(t) + a(t)m(t)y = m(t)b(t)$$

$$\frac{d}{dt}(m(t)y(t)) = m(t)b(t)$$

$$\Rightarrow y(t) = \frac{1}{m(t)}\left(\int m(t)b(t)\,dt + C\right)$$

(will be proved later in course)

"linearity Principle"

Observation General soln of non-homo Eq = General soln of homo part ⊕ particular soln of non-homo Eq.

Ex. $\dot{y}(t) = t + y$, $y(1) = 0$

Ans $m(t) = e^{-t}$

$$e^{-t}\dot{y} - e^{-t}y = te^{-t}$$

$$(e^{-t}y)^{\cdot} = te^{-t}$$

$$\int te^{-t}dt = -\int t\,de^{-t} = -te^{-t} - e^{-t} + C$$

$$y(t) = e^{t}(-te^{-t} - e^{-t} + C)$$

$$= -(t+1) + Ce^{t}$$

$$\Rightarrow y(t) = -(t+1) + 2e^{t-1}$$

*. Other Eqs that can be converted into above form

Bernoulli Eq

$$\frac{dy}{dt} + a(t)y = b(t)y^n \qquad n \geq 0.$$

When $n=0$. 1st order linear.

$n=1$. 1st order linear homogeneous.

WLOG assume $n \neq 0,1$. not necessarily integer.

$$\underbrace{y^{-n}\frac{dy}{dt}}_{\frac{1}{1-n}\frac{d\,y^{1-n}}{dt}} + a(t)\,y^{1-n} = b(t)$$

change of var $\quad z = y^{1-n}$

$$\frac{1}{1-n}\,dz + a(t)z = b(t) \qquad \text{1st order linear!}$$

Ex

$$\frac{dy}{dt} = -\frac{3}{t}y + t^2y^2 \quad (t>0) \qquad \text{Find its general solution.}$$

$z = y^{-1}$

$$\frac{dz}{dt} = -y^{-2}\frac{dy}{dt} = \frac{3}{t}z - t^2$$

$$m(t) = e^{-3\ln|t|} = t^{-3}$$

$$z = t^3\left(\int -t^{-1}\,dt + c\right)$$

$$= t^3(c - \ln t)$$

$$\Rightarrow y = t^{-3}(c - \ln t)^{-1} \quad \text{or } y = 0.$$