Solutions for Homework 1

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1. Without loss of generality, we assume $X_0 = 0$, otherwise we replace X_t with $X_t - X_0$ in what follows. Let $\Pi = \{t_0, \ldots, t_n\}$, with $0 = t_0 \le t_1 \le \cdots \le t_n = t$, be a partition of $[0,t]$. Then the second variation of X_t is given by

$$\begin{aligned} V_t^{(2)}(\Pi) &= \sum_{k=1}^{n} \left| X_{t_k} - X_{t_{k-1}} \right|^2 \\ &\le \max_{1\le k\le n} \left| X_{t_k} - X_{t_{k-1}} \right| \sum_{k=1}^{m} \left| X_{t_k} - X_{t_{k-1}} \right| \\ &= \max_{1\le k\le n} \left| X_{t_k} - X_{t_{k-1}} \right| V_t^{(1)}(\Pi). \end{aligned}$$

$V_t^{(2)}(\Pi)$ converges to zero as $\|\Pi\| \to 0$ since $V_t^{(1)}(\Pi)$ converges to a constant. Therefore, $\langle X \rangle_t = \lim_{\|\Pi\|\to 0} V_t^{(2)}(\Pi) = 0$. According to Doob-Meyer decomposition, X^2 is a martingale, then $\operatorname{Var} X_t = \mathbb{E}X_t^2 = 0$. In addition, $\mathbb{E}X_t = 0$ since X is a martingale. For every $0 \le t < \infty$, we obtain $\mathbb{E}X_t = 0$ and $\operatorname{Var} X_t = 0$, then $X_t = 0$.

2. (1.5.7 Problem from p. 31 of Karatzas and Shreve (1991))

(i) We know that $XZ - \langle X, Z\rangle$ and $YZ - \langle Y, Z\rangle$ are both martingales, then $\alpha XZ - \alpha \langle X, Z\rangle$ and $\beta YZ - \beta \langle Y, Z\rangle$ are martingales, and therefore so is their summation $\alpha XZ + \beta YZ - [\alpha \langle X, Z\rangle + \beta \langle Y, Z\rangle]$. Note that $\alpha XZ + \beta YZ = (\alpha X + \beta Y)Z$, and $(\alpha X + \beta Y)Z - \langle \alpha X + \beta Y, Z\rangle$ is a martingale. By the uniqueness of cross-variation (Theorem 1.5.13),

$$\langle \alpha X + \beta Y, Z\rangle = \alpha \langle X, Z\rangle + \beta \langle Y, Z\rangle .$$

(ii) $XY - \langle X, Y\rangle$ and $YX - \langle Y, X\rangle$ are both martingales, and $XY = YX$. By the uniqueness of cross-variation, $\langle X, Y\rangle = \langle Y, X\rangle$.

(iii) According to (i) and (ii),

$$\begin{aligned} \langle \alpha X + Y\rangle &= \langle \alpha X + Y, \alpha X + Y\rangle \\ &= \alpha \langle X, \alpha X + Y\rangle + \langle Y, \alpha X + Y\rangle \\ &= \alpha \langle \alpha X + Y, X\rangle + \langle \alpha X + Y, Y\rangle \end{aligned}$$

$$\begin{aligned} &= \alpha^2 \langle X, X\rangle + 2\alpha \langle X, Y\rangle + \langle Y, Y\rangle \\ &= \alpha^2 \langle X\rangle + 2\alpha \langle X, Y\rangle + \langle Y\rangle , \end{aligned}$$

and $\langle \alpha X + Y\rangle \geq 0$, for every $\alpha \in \mathbb{R}$. Then $4\left|\langle X, Y\rangle\right|^2 - 4\langle X\rangle \langle Y\rangle \leq 0$, which leads to the conclusion.

(iv) Let $\Pi = \{t_0, \ldots, t_n\}$, with $s = t_0 \leq t_1 \leq \cdots \leq t_n = t$, be a partition of $[s, t]$. Then

$$\check{\xi}_t(\omega) - \check{\xi}_s(\omega) = \sup_{\Pi} \sum_{k=1}^{n} \left|\xi_{t_k}(\omega) - \xi_{t_{k-1}}(\omega)\right| .$$

It suffices to prove that

$$|\xi_t(\omega) - \xi_s(\omega)| \leq \frac{1}{2}[\langle X\rangle_t (\omega) - \langle X\rangle_s (\omega) + \langle Y\rangle_t (\omega) - \langle Y\rangle_s (\omega)],$$

for $0 \leq s < t < \infty$. Note that

$$\begin{aligned} \xi_t(\omega) &= \langle X, Y\rangle_t (\omega) = \frac{1}{4}[\langle X+Y\rangle_t (\omega) - \langle X-Y\rangle_t (\omega)], \\ \xi_s(\omega) &= \langle X, Y\rangle_s (\omega) = \frac{1}{4}[\langle X+Y\rangle_s (\omega) - \langle X-Y\rangle_s (\omega)], \end{aligned}$$

then

$$\begin{aligned} & |\xi_t(\omega) - \xi_s(\omega)| \\ = \ & \frac{1}{4}\left|(\langle X+Y\rangle_t (\omega) - \langle X+Y\rangle_s (\omega)) - (\langle X-Y\rangle_t (\omega) - \langle X-Y\rangle_s (\omega))\right| \\ \leq \ & \frac{1}{4}[(\langle X+Y\rangle_t (\omega) - \langle X+Y\rangle_s (\omega)) + (\langle X-Y\rangle_t (\omega) - \langle X-Y\rangle_s (\omega))]. \end{aligned}$$

The second inequality holds since $\langle X+Y\rangle_t (\omega) - \langle X+Y\rangle_s (\omega)$ and $\langle X-Y\rangle_t (\omega) - \langle X-Y\rangle_s (\omega)$ are both greater than or equal to 0. According to (i) and (ii),

$$\begin{aligned} \langle X+Y\rangle_t (\omega) &= \langle X\rangle_t (\omega) + 2\langle X, Y\rangle_t (\omega) + \langle Y\rangle_t (\omega), \\ \langle X+Y\rangle_s (\omega) &= \langle X\rangle_s (\omega) + 2\langle X, Y\rangle_s (\omega) + \langle Y\rangle_s (\omega), \end{aligned}$$

so

$$\begin{aligned} & \langle X+Y\rangle_t (\omega) - \langle X+Y\rangle_s (\omega) \\ = \ & \langle X\rangle_t (\omega) - \langle X\rangle_s (\omega) + \langle Y\rangle_t (\omega) - \langle Y\rangle_s (\omega) \\ & +2\langle X, Y\rangle_t (\omega) - 2\langle X, Y\rangle_s (\omega). \end{aligned}$$

Similarly,

$$\begin{aligned} & \langle X-Y\rangle_t (\omega) - \langle X-Y\rangle_s (\omega) \\ = \ & \langle X\rangle_t (\omega) - \langle X\rangle_s (\omega) + \langle Y\rangle_t (\omega) - \langle Y\rangle_s (\omega) \\ & -2\langle X, Y\rangle_t (\omega) + 2\langle X, Y\rangle_s (\omega). \end{aligned}$$

Therefore,

$$
\begin{aligned}
&((\langle X+Y\rangle_t(\omega) - \langle X+Y\rangle_s(\omega)) + (\langle X-Y\rangle_t(\omega) - \langle X-Y\rangle_s(\omega)) \\
= \;& 2(\langle X\rangle_t(\omega) - \langle X\rangle_s(\omega) + \langle Y\rangle_t(\omega) - \langle Y\rangle_s(\omega)),
\end{aligned}
$$

which amounts to the following inequality

$$
|\xi_t(\omega) - \xi_s(\omega)| \le \frac{1}{2}(\langle X\rangle_t(\omega) - \langle X\rangle_s(\omega) + \langle Y\rangle_t(\omega) - \langle Y\rangle_s(\omega)).
$$

Remark 1. If X, Y, Z are also continous, we can proof (i) and (ii) by using the conclusion in Problem 5.14. For instance, let $\Pi = \{t_0, \dots, t_n\}$, with $0 = t_0 \le t_1 \le \cdots \le t_n = t$, be a partition of $[0, t]$. Then

$$
\begin{aligned}
\langle X, Y\rangle_t &= \lim_{\|\Pi\|\to 0} \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})(Y_{t_k} - Y_{t_{k-1}}) \\
&= \lim_{\|\Pi\|\to 0} \sum_{k=1}^{n} (Y_{t_k} - Y_{t_{k-1}})(X_{t_k} - X_{t_{k-1}}) \\
&= \langle Y, X\rangle_t.
\end{aligned}
$$

3. (1.5.20 Exercise from p. 37 Karatzas and Shreve (1991))

We are going to prove $X_t^2 - t\mathbb{E}X_1^2$ is a martingale, then the conclusion follows from the uniqueness of Doob-Meyer decomposition.

For $0 \le s < t < \infty$,

$$
\begin{aligned}
&\mathbb{E}[X_t^2 - t\mathbb{E}X_1^2|\mathcal{F}_s] \\
=\;& \mathbb{E}[X_t^2|\mathcal{F}_s] - t\mathbb{E}X_1^2 \\
=\;& \mathbb{E}[(X_t - X_s + X_s)^2|\mathcal{F}_s] - t\mathbb{E}X_1^2 \\
=\;& \mathbb{E}[(X_t - X_s)^2] + X_s^2 - t\mathbb{E}X_1^2 \\
=\;& X_s^2 - s\mathbb{E}X_1^2 + \mathbb{E}[(X_t - X_s)^2] - (t-s)\mathbb{E}X_1^2 \\
=\;& X_s^2 - s\mathbb{E}X_1^2 + \mathbb{E}[X_{t-s}^2 - (t-s)\mathbb{E}X_1^2].
\end{aligned}
$$

The last equality holds since X has stationary increments, then it suffices to prove that $\mathbb{E}M_{t-s} = 0$, where $M_t = X_t^2 - t\mathbb{E}X_1^2$. Define $f(t) = \mathbb{E}M_t$, and taking expection on both sides of equation, we have

$$
f(t) = f(s) + f(t-s)
$$

with f right-countinous and $f(1) = 0$. We can prove that $f(t) = tf(1)$, and under the condition $f(1) = 0$, we have $\boxed{f(t) = 0,}$ for every $t \ge 0$.

Let $t = ns, n \in \mathbb{Z}^+$, we have $f(ns) = f(s) + f((n-1)s)$, then $f(ns) = nf(s)$. Moreover, $s = 1$ implies $f(n) = nf(1)$. If $s \in \mathbb{Q}^+$, there exists $p, q \in \mathbb{Z}^+$ such that $s = q/p$. Then,

$$
qf(1) = f(q) = f(p\frac{q}{p}) = pf(\frac{q}{p}),
$$

which is equivalent to

$$f(\frac{q}{p}) = \frac{q}{p} f(1).$$

If $s \in \mathbb{R}^+$, there exists a sequence $\{s_k\}_{k=1}^{\infty}$ of rational numbers such that s_k converges to s from right, and $f(s_k) = s_k f(1)$, for every $k \geq 1$. According to the right-continuity of f, we have $f(s) = sf(1)$.

4. (2.8.12 Problem from p. 100 of Karatzas and Shreve (1991))

   The optional sampling theorem gives

$$e^{\lambda x} = \mathbb{E}^x X_0 = \mathbb{E}^x X_{t \wedge T_0 \wedge T_a} = \mathbb{E}^x[\exp\{\lambda W_{t \wedge T_0 \wedge T_a} - \frac{1}{2}\lambda^2 (t \wedge T_0 \wedge T_a)\}].$$

   Since $W_{t \wedge T_0 \wedge T_a}$ is bounded, we may let $t \to \infty$ to obtain

$$\begin{aligned} e^{\lambda x} &= \mathbb{E}^x[\exp\{\lambda W_{T_0 \wedge T_a} - \frac{1}{2}\lambda^2 (T_0 \wedge T_a)\}] \\ &= \mathbb{E}^x[1_{\{T_0 < T_a\}} e^{-\lambda^2 T_0/2}] + e^{\lambda a}\mathbb{E}^x[1_{\{T_a < T_0\}} e^{-\lambda^2 T_a/2}]. \end{aligned}$$

   By choosing $\lambda = \pm\sqrt{2\alpha}$, we obtain two equations

$$\begin{aligned} e^{x\sqrt{2\alpha}} &= \mathbb{E}^x[1_{\{T_0 < T_a\}} e^{-\alpha T_0}] + e^{a\sqrt{2\alpha}}\mathbb{E}^x[1_{\{T_a < T_0\}} e^{-\alpha T_a}], \\ e^{-x\sqrt{2\alpha}} &= \mathbb{E}^x[1_{\{T_0 < T_a\}} e^{-\alpha T_0}] + e^{-a\sqrt{2\alpha}}\mathbb{E}^x[1_{\{T_a < T_0\}} e^{-\alpha T_a}], \end{aligned}$$

   which can be solved simultaneously and yileds

$$\begin{aligned} \mathbb{E}^x[1_{\{T_0 < T_a\}} e^{-\alpha T_0}] &= \frac{\sinh((a-x)\sqrt{2\alpha})}{\sinh(a\sqrt{2\alpha})}, \\ \mathbb{E}^x[1_{\{T_a < T_0\}} e^{-\alpha T_a}] &= \frac{\sinh(x\sqrt{2\alpha})}{\sinh(a\sqrt{2\alpha})}. \end{aligned}$$

   (2.8.13 Exercise from p. 100 of Karatzas and Shreve (1991))

   Let $\alpha \to 0$ in the results of Problem 2.8.12 and by L'Hôpital's rule,

$$\begin{aligned} \mathbb{E}^x[1_{\{T_0 < T_a\}}] &= \lim_{\alpha \to 0} \frac{\sinh((a-x)\sqrt{2\alpha})}{\sinh(a\sqrt{2\alpha})} \\ &= \lim_{\alpha \to 0} \frac{(a-x)\cosh((a-x)\sqrt{2\alpha})}{a\cosh(a\sqrt{2\alpha})} \\ &= \frac{a-x}{a}, \end{aligned}$$

   and

$$\begin{aligned} \mathbb{E}^x[1_{\{T_a < T_0\}}] &= \lim_{\alpha \to 0} \frac{\sinh(x\sqrt{2\alpha})}{\sinh(a\sqrt{2\alpha})} \\ &= \lim_{\alpha \to 0} \frac{x\cosh((a-x)\sqrt{2\alpha})}{a\cosh(a\sqrt{2\alpha})} \end{aligned}$$

$$= \frac{x}{a}.$$

The conclusion follows from $\mathbb{E}^x[1_{\{T_0<T_a\}}] = P^x[T_0 < T_a]$ and $\mathbb{E}^x[1_{\{T_a<T_0\}}] = P^x[T_a < T_0]$.

(2.8.14 Problem from p. 100 of Karatzas and Shreve (1991))

Since $e^{-\alpha(T_0\wedge T_a)}$ is bounded, we may take derivatives w.r.t α in (8.29) to obtain

$$\begin{aligned}
& \mathbb{E}^x[-(T_0\wedge T_a)e^{-\alpha(T_0\wedge T_a)}] \\
= \; & \cosh^{-2}(a/2\sqrt{2\alpha})\Big(\frac{x-a/2}{\sqrt{2\alpha}}\sinh((x-a/2)\sqrt{2\alpha})\cosh(a/2\sqrt{2\alpha}) \\
& -\frac{a/2}{\sqrt{2\alpha}}\cosh((x-a/2)\,\sqrt{2\alpha})\sinh(a/2\sqrt{2\alpha})\Big).
\end{aligned}$$

Let $\alpha\to 0$, and note that

$$\begin{aligned}
\lim_{\alpha\to 0}\frac{\sinh((x-a/2)\,\sqrt{2\alpha})}{\sqrt{2\alpha}} &= x-\frac{a}{2}, \\
\lim_{\alpha\to 0}\frac{\sinh(a/2\sqrt{2\alpha})}{\sqrt{2\alpha}} &= \frac{a}{2},
\end{aligned}$$

we obtain

$$\begin{aligned}
-\mathbb{E}^x[T_0\wedge T_a] &= \Big(x-\frac{a}{2}\Big)^2+\Big(\frac{a}{2}\Big)^2 \\
&= x^2-ax,
\end{aligned}$$

then

$$\mathbb{E}^x[T_0\wedge T_a] = x(a-x).$$

5. (3.3.18 Problem p. 158 of Karatzas and Shreve (1991))

For any $1\le i\le d$, $\tilde W_t^{(i)} = \sum_{k=1}^d q_{ik}W_t^{(k)}$, where $Q=(q_{ik})_{1\le i,k\le d}$. It is easy to see $\tilde W_0^{(i)}=0$ and $\tilde W_t^{(i)}(\omega)$ is continous because $W_0^{(k)}=0$ and $W_t^{(k)}(\omega)$ is continous.

For $1\le i,j\le d$,

$$\begin{aligned}
d\tilde W_t^{(i)}d\tilde W_t^{(j)} &= \left(\sum_{k=1}^d q_{ik}dW_t^{(k)}\right)\left(\sum_{l=1}^d q_{jl}dW_t^{(l)}\right) \\
&= \sum_{k=1}^d\sum_{l=1}^d q_{ik}q_{jl}dW_t^{(k)}dW_t^{(l)} \\
&= \sum_{k=1}^d q_{ik}q_{jk}dt \\
&= (QQ^\top)_{ij}dt = \delta_{ij}dt.
\end{aligned}$$

According to Lévy theorem, $\tilde W$ is a d-dimensional Brownian motion.

6. (3.3.14 from p. 156 Karatzas and Shreve (1991))

At first, we will consider the situation with $d = 1$. According to Itô rules,

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dM_s + \int_0^t f'(X_s)dB_s + \frac{1}{2}\int_0^t f''(X_s)d\langle M\rangle_s .$$

and

$$f'(X_s) = f'(X_0) + \int_0^s f''(X_u)dM_u + \int_0^t f'(X_s)dB_s + \frac{1}{2}\int_0^s f'''(X_u)d\langle M\rangle_u ,$$

where $M_u \in \mathcal{M}^{c,loc}$, $\int_0^s f''(X_u)dM_u \in \mathcal{M}^{c,loc}$. Because $B_u = A_u^+ - A_u^-$ and both $A_u^+, A_u^-, \langle M\rangle_u$ are nondecreasing adapted process. (A_u^+, A_u^- is according to their definition, $\langle M\rangle_u$ is according to the course notes: P10 Theorem 2.6). $\int_0^s f''(X_u)dB_u$ and $\int_0^s f'''(Xu)d\langle M\rangle_u$ are Lebesgue-Stieljes and they are continuous process with bounded variations. Then $f'(X_s)$ is a semimartingale.

Use Def 3.13 in Karatzas and Shreve (1991) and Course notes Remark 3.5, we have

$$\begin{aligned}\int_0^t f'(X_s)\circ dX_s &= \int_0^t f'(X_s)dM_s + \int_0^t f'(X_s)dB_s + \frac{1}{2}\left\langle \int_0^\cdot f''(X_s)dM_s, M\right\rangle_t . \\ &= \int_0^t f'(X_s)dM_s + \int_0^t f'(X_s)dB_s + \frac{1}{2}\int_0^t f''(X_s)d\langle M\rangle_s .\end{aligned}$$

Then,

$$\int_0^t f'(X_s)dM_s + \int_0^t f'(X_s)dB_s = \int_0^t f'(X_s)\circ dX_s - \frac{1}{2}\int_0^t f''(X_s)d\langle M\rangle_s .$$

Plugging into the first equation, we have

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\circ dX_s.$$

For the multi-dimensional case,

$$\begin{aligned}f(X_t) &= f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial}{\partial x_i} f(X_s)dM_s^{(i)} + \sum_{i=1}^d \int_0^t \frac{\partial}{\partial x_i} f(X_s)dB_s^{(i)} \\ &\quad + \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(X_s)d\left\langle M^{(i)}, M^{(j)}\right\rangle_s ,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial x_i} f(X_s) &= \frac{\partial}{\partial x_i} f_i(X_0) + \sum_{j=1}^d \int_0^s \frac{\partial^2}{\partial x_i \partial x_j} f(X_u)dM_u^{(j)} + \sum_{j=1}^d \int_0^s \frac{\partial^2}{\partial x_i \partial x_j} f(X_u)dB_u^{(j)} \\ &\quad + \frac{1}{2}\sum_{j=1}^d\sum_{k=1}^d \int_0^s \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} f(X_u)d\left\langle M^{(j)}, M^{(k)}\right\rangle_u .\end{aligned}$$

Similarly,

$$
\begin{aligned}
& \int_0^t \frac{\partial}{\partial x_i} f(X_s) \circ dX_s^{(i)} \\
= \quad & \int_0^t \frac{\partial}{\partial x_i} f(X_s) dM_s^{(i)} + \int_0^t \frac{\partial}{\partial x_i} f(X_s) dB_s^{(i)} + \frac{1}{2} \left\langle \sum_{j=1}^d \int_0^{\cdot} \frac{\partial^2}{\partial x_i \partial x_j} f(X_s) dM_s^{(i)}, M^{(j)} \right\rangle_t \\
= \quad & \int_0^t \frac{\partial}{\partial x_i} f(X_s) dM_s^{(i)} + \int_0^t \frac{\partial}{\partial x_i} f(X_s) dB_s^{(i)} + \frac{1}{2} \sum_{j=1}^d \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(X_s) d \left\langle M^{(i)}, M^{(j)} \right\rangle_s .
\end{aligned}
$$

Then we have

$$
f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial}{\partial x_i} f(X_s) \circ dX_s^{(i)}.
$$

7. (Exercise 11.3 from p. 526 of Shreve (2004))

   In order to proof that $(\sigma+1)^{N(t)} e^{-\lambda\sigma t}$ is a martingale. We should prove:

$$
\mathbb{E}[(\sigma+1)^{N(t)} e^{-\lambda\sigma t} | \mathcal{F}(s)] = (\sigma+1)^{N(s)} e^{-\lambda\sigma s}.
$$

   Consider the independence and stationary of increment for Poisson Process, the left hand side can be written as

$$
\begin{aligned}
& \mathbb{E}[(\sigma+1)^{N(t)} e^{-\lambda\sigma t} | \mathcal{F}(s)] \\
= \quad & \mathbb{E}[(\sigma+1)^{N(t)-N(s)} (\sigma+1)^{N(s)} e^{-\lambda\sigma t} | \mathcal{F}(s)] \\
= \quad & (\sigma+1)^{N(s)} e^{-\lambda\sigma t} \mathbb{E}[(\sigma+1)^{N(t)-N(s)}] \\
= \quad & (\sigma+1)^{N(s)} e^{-\lambda\sigma t} \mathbb{E}[e^{(N(t)-N(s)) \ln(\sigma+1)}].
\end{aligned}
$$

   So the key point is to calculate the moment generating function of $N(t)$.

$$
\begin{aligned}
\mathbb{E} e^{\theta N(t)} &= \sum_{k=0}^{+\infty} e^{\theta k} \mathbb{P}(N(t) = k) \\
&= \sum_{k=0}^{+\infty} e^{\theta k} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\
&= \sum_{k=0}^{+\infty} \frac{(\lambda t e^\theta)^k}{k!} e^{-\lambda t}.
\end{aligned}
$$

   Note that

$$
\sum_{k=0}^{+\infty} \frac{(\lambda t e^\theta)^k}{k!} e^{-\lambda t e^\theta} = 1,
$$

   we have

$$
\mathbb{E} e^{\theta N(t)} = e^{-\lambda t} \cdot e^{\lambda t e^\theta} = e^{\lambda t (e^\theta - 1)}.
$$

Now let $\theta = \ln(\sigma+1)$, we have

$$\mathbb{E}[e^{N(t-s)\ln(\sigma+1)}] = e^{\lambda(t-s)(\sigma+1-1)} = e^{\lambda\sigma(t-s)}.$$

Thus,

$$\mathbb{E}[(\sigma+1)^{N(t)}e^{-\lambda\sigma t}|\mathcal{F}(s)] = (\sigma+1)^{N(s)}e^{-\lambda\sigma t}\cdot e^{\lambda\sigma(t-s)} = (\sigma+1)^{N(t)}e^{-\lambda\sigma s}.$$

8. (Exercise 4.13 from p. 197 of Shreve (2004))

$$\begin{aligned} dB_1(t) &= dW_1(t), \\ dB_2(t) &= \rho(t)dW_1(t) + \sqrt{1-\rho^2(t)}dW_2(t). \end{aligned}$$

Obviously, $W_1(t)$ is a Brownian Motion, then we have to show that $W_2(t)$ is a BM and independent with $W_1(t)$.

$$dW_2(t) = \frac{1}{\sqrt{1-\rho^2(t)}}dB_2(t) - \frac{\rho(t)}{\sqrt{1-\rho^2(t)}}dB_1(t),$$

then

$$\begin{aligned} dW_2(t)dW_2(t) &= \frac{1}{1-\rho^2(t)}dt + \frac{\rho^2(t)}{1-\rho^2(t)}dt - \frac{2\rho^2(t)}{1-\rho^2(t)}dt \\ &= dt. \end{aligned}$$

On the one hand, $W_2(t)$ has continuous paths. On the other hand, $W_2(t)$ is a martingale since

$$W_2(t) = W_2(0) + \int_0^t \frac{1}{\sqrt{1-\rho^2(t)}}dB_2(t) - \int_0^t \frac{\rho(t)}{\sqrt{1-\rho^2(t)}}dB_1(t).$$

In addition, it is easy to see $dW_1(t)dW_2(t) = 0$. According to Lévy theorem, $W_1(t)$ and $W_2(t)$ are independent Brownian motions.

9. (Exercise 4.15 from p. 199 of Shreve (2004))

(1) It is obviously that $B_i(t)$ is a martingale with $B_i(0) = 0$ and continuous sample paths, according to the properties of Itô integral. So the only thing we have to find is $[B_i, B_j](t)$. We use the box poly rings to simplify the calculation. Obviously,

$$dB_i(t) = \sum_{j=1}^{d}\frac{\sigma_{ij}(t)}{\sigma_i(t)}dW_j(t),$$

then we have

$$dB_i(t)dB_i(t) = \sum_{j=1}^{d}\sum_{k=1}^{d}\frac{\sigma_{ij}(t)}{\sigma_i(t)}\frac{\sigma_{ik}(t)}{\sigma_i(t)}dW_j(t)dW_k(t).$$

Since $dW_j(t)dW_k(t) = \delta_{jk}dt$, then

$$dB_i(t)dB_i(t) = \sum_{j=1}^{d} \frac{\sigma_{ij}^2(t)}{\sigma_i^2(t)}dt = \frac{1}{\sigma_i^2(t)} \sum_{j=1}^{d} \sigma_{ij}^2(t)dt = dt.$$

According to Lévy theorem, $B_i(t)$ is a Brownian motion.

(2)

$$\begin{aligned}
dB_i(t)dB_k(t) &= \left(\sum_{j=1}^{d} \frac{\sigma_{ij}^2(t)}{\sigma_i^2(t)}dW_j(t)\right)\left(\sum_{l=1}^{d} \frac{\sigma_{kl}^2(t)}{\sigma_k^2(t)}dW_l(t)\right) \\
&= \sum_{j=1}^{d}\sum_{l=1}^{d} \frac{\sigma_{ij}(t)\sigma_{kl}(t)}{\sigma_i(t)\sigma_k(t)}dW_j(t)dW_l(t) \\
&= \sum_{j=1}^{d} \frac{\sigma_{ij}(t)\sigma_{kj}(t)}{\sigma_i(t)\sigma_k(t)}dt = \rho_{ik}(t)dt.
\end{aligned}$$

10. (Exercise 4.16 from p. 200 of Shreve (2004))

Let $dW_j(t) = \sum\limits_{k=1}^{m} \alpha_{jk}(t)dB_k(t)$. We verify that $dB_i(t) = \sum_{j=1}^{m} a_{ij}(t)dW_j(t)$.

$$\begin{aligned}
\sum_{j=1}^{m} a_{ij}(t)dW_j(t) &= \sum_{j=1}^{m}\left(\sum_{k=1}^{m} a_{ij}(t)\alpha_{jk}(t)dB_k(t)\right) \\
&= \sum_{k=1}^{m}\left(\sum_{j=1}^{m} a_{ij}(t)\alpha_{jk}(t)\right)dB_k(t) \\
&= \sum_{k=1}^{m} \delta_{ik}dB_k(t) = dB_i(t).
\end{aligned}$$

This implies

$$B_i(t) = \sum_{j=1}^{m} \int_0^t a_{ij}(u)dW_j(u).$$

Now it's sufficient to prove that $W(t) = (W_1(t), \ldots, W_m(t))$ are m independent Brownian motions. Note that

$$\begin{aligned}
dW_i(t)dW_k(u) &= \left(\sum_{j=1}^{m} \alpha_{ij}(t)dB_j(t)\right)\left(\sum_{l=1}^{m} \alpha_{kl}(t)dB_l(t)\right) \\
&= \sum_{j=1}^{m}\sum_{l=1}^{m} \alpha_{ij}(t)\alpha_{kl}(t)dB_j(t)dB_l(t) \\
&= \sum_{j=1}^{m}\sum_{l=1}^{m} \alpha_{ij}(t)\alpha_{kl}(t)\rho_{jl}(t)dt
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{m}\sum_{l=1}^{m}\alpha_{ij}(t)\alpha_{kl}(t)\sum_{s=1}^{m}a_{js}(t)a_{ls}(t)dt \\
&= \sum_{s=1}^{m}\left(\sum_{j=1}^{m}\alpha_{ij}(t)a_{js}(t)\right)\left(\sum_{l=1}^{m}\alpha_{kl}(t)a_{ls}(t)\right)dt \\
&= \sum_{s=1}^{m}\delta_{is}\delta_{ks}dt = \delta_{ik}dt.
\end{aligned}
$$

According to Lévy theorem, we complete the proof.

References

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