Problem Set 4
Econ 139, Fall 2019
Due by 5pm on Tu Dec 9. No late Problem Sets accepted, sorry!

General rules for problem sets: show your work, write down the steps that you use to get a solution (no credit for right solutions without explanation), write legibly. If you cannot solve a problem fully, write down a partial solution. We give partial credit for partial solutions that are correct. Do not forget to write your name on the problem set!

Problem 1. Options and Risk-Neutral Probabilities. Consider an economy where there are three possible states of the world at $t = 1$. A security with payoffs $\tilde{x}(\theta, 1) = (5, 10, 15)$ in states 1, 2, and 3, respectively, has a price of 8 at $t = 0$. A risk-free security has a gross return of 1.1 and a price of 1 at $t = 0$. A call option on the risky security has a strike price of 12 and a price of 1 at $t = 0$. The probabilities of states 1, 2, and 3, repsectively, are $\pi = (0.3, 0.4, 0.3)$.

1. Is the market complete?, If yes, what are the state prices?
2. Price a put option with a strike price of 8.
3. Derive the risk-neutral probabilities.
4. Determine the values of the stochastic discount factor for each state.
5. Determine the $t = 0$ value of the payoffs $\tilde{x}(\theta, 1) = (7, 3, 5)$ using:
   (i) The state prices.
   (ii) Risk-neutral valuation.
   (iii) The stochastic discount factor.

   Do all of these valuations agree?

Problem 2. Arbitrage Pricing Theory. Suppose there are three assets with the following characteristics:

Asset	$\mathbb{E}[r_i]$	b_i
A	0.07	0.5
B	0.09	1.0
C	0.17	1.5

1. Plot the three assets with b_i on the horizontal axis and $E[r_i]$ on the vertical axis.
2. Using assets A and B, construct a portfolio with no systematic risk. Do the same using assets B and C.
3. Construct an arbitrage portfolio and compute its expected return.
4. Describe how prices will adjust as a result of the arbitrage opportunity.
5. No assume the following data for the three assets:

Asset	$\mathbb{E}[r_i]$	b_i
A	0.06	0.5
B	0.10	1.0
C	0.14	1.5

   Plot the three assets on the graph from point 1. Is it still possible to construct an arbitrage portfolio with positive expected return? Explain.

Problem 3. A two-factor arbitrage pricing theory.

Consider an economy in which the random return $\tilde{r}_j$ on each individual asset j follows the equation

$$\tilde{r}_j = \alpha_j + \beta_{j,M}(\tilde{r}_M - \mathbb{E}[\tilde{r}_M]) + \beta_{j,V}(\tilde{r}_V - \mathbb{E}[\tilde{r}_V]) + \varepsilon_j,$$

where, as we discussed in class, $\tilde{r}_M$ is the random return on the market portfolio, $\tilde{r}_V$ is the random return on a "value" portfolio that takes a long position in shares of stock issued by smaller, overlooked companies, or companies with high book-to-market values, ε_j is an idiosyncratic, firm-specific component, and $\beta_{j,M}$ and $\beta_{j,V}$ are the "factor loadings" that measure the extent to which the return on asset j is correlated with the return on the market and value portfolios. Assume $\mathbb{E}[\varepsilon_j] = 0$ and $\mathbb{E}[\tilde{r}_M \varepsilon_j] = 0$ for each individual asset j, and $\mathbb{E}[\varepsilon_j \varepsilon_k] = 0$ for all asset pairs j and k. Further assume, as Stephen Ross did when developing the arbitrage pricing theory (APT), that there are enough individual assets for investors to form many well-diversified portfolios and that investors act to eliminate all arbitrage opportunities that may arise across all well-diversified portfolios.

(a) In the context of APT, define the term *well-diversified portfolio*. Show that we can write the random return of a well-diversified portfolio as

$$\tilde{r}_p = \mathbb{E}[\tilde{r}_p] + \beta_{p,M}(\tilde{r}_M - \mathbb{E}[\tilde{r}_M]) + \beta_{p,V}(\tilde{r}_V - \mathbb{E}[\tilde{r}_V]).$$

What is the key *statistical* assumption that gives this result?

(b) Consider two well-diversified portfolios such that:

$$\begin{aligned} \tilde{r}_p^1 &= \mathbb{E}[\tilde{r}_p^1] + \beta_{p,M}(\tilde{r}_M - \mathbb{E}[\tilde{r}_M]) + \beta_{p,V}(\tilde{r}_V - \mathbb{E}[\tilde{r}_V]), \\ \tilde{r}_p^2 &= \mathbb{E}[\tilde{r}_p^2] + \beta_{p,M}(\tilde{r}_M - \mathbb{E}[\tilde{r}_M]) + \beta_{p,V}(\tilde{r}_V - \mathbb{E}[\tilde{r}_V]), \end{aligned}$$

meaning they have the same factor loadings on the market and value portfolios. Suppose that

$$\mathbb{E}[\tilde{r}_p^2] = \mathbb{E}[\tilde{r}_p^1] + \Delta.$$

Show the absence of arbitrage opportunities requires that $\Delta = 0$.

(c) Now consider a well-diversified portfolio that has $\beta_{p,M} = \beta_{p,V} = 0$. Write down the equation, implied by this version of the APT, that links the expected return $\mathbb{E}[\tilde{r}_p^1]$ of this portfolio to the return r_f on a portfolio of risk-free assets.

(d) Next, consider two more well-diversified portfolios: portfolio 2, with $\beta_{p,M} = 1$ and $\beta_{p,V} = 0$, and portfolio 3, with $\beta_{p,M} = 0$ and $\beta_{p,V} = 1$. Write down the equations implied by this version of the APT, that link the expected returns $\mathbb{E}[\tilde{r}_p^2]$ and $\mathbb{E}[\tilde{r}_p^3]$ on each of these two portfolios to $\mathbb{E}[\tilde{r}_M]$ and $\mathbb{E}[\tilde{r}_V]$.

(e) By construction, the expected returns of all three portfolios satisfy the equation

$$\mathbb{E}[\tilde{r}_p] = r_f + \beta_{p,M}(\mathbb{E}[\tilde{r}_M] - r_f) + \beta_{p,V}(\mathbb{E}[\tilde{r}_V] - r_f).$$

Suppose you find a fourth well-diversified portfolio that has non-zero values of both $\beta_{p,M}$ and $\beta_{p,V}$, with expected return

$$\mathbb{E}[\tilde{r}_p^4] = r_f + \beta_{p,M}(\mathbb{E}[\tilde{r}_M] - r_f) + \beta_{p,V}(\mathbb{E}[\tilde{r}_V] - r_f) + \Delta,$$

where $\Delta \neq 0$. Explain how you could use this portfolio, together with the first three from parts (c) and (d), in a trading strategy that involves no risk, requires no money down, but yields a certain future profit.

(f) Based on your answer to part (e), argue that the absence of arbitrage opportunities, the expected return of *any* well-diversified portfolio must satisfy

$$\mathbb{E}[\tilde{r}_p] = r_f + \beta_{p,M}(\mathbb{E}[\tilde{r}_M] - r_f) + \beta_{p,V}(\mathbb{E}[\tilde{r}_V] - r_f).$$

Why does this result only apply to well-diversified portfolios, and not to individual assets?

Problem 4. Derivatives.

Consider an economy in three periods, $t = 0$, $t = 1$ and $t = 2$. At $t = 0$, the market index is trading at a value of 100. At t=1, the index either rises to 115 with 50% probability, or falls to 95 with 50% probability. Following either of these outcomes, the index either rises by 15 or falls by 5, with equal probabilities, at $t = 2$. Thus the highest possible index value at $t = 2$ is 130, and the lowest is 90. The index pays no dividends during this time, and the riskfree rate is $r_f = 0$.

(a) Our first goal is to compute the price of a European put option on the index, with exercise price $X = 120$ and expiration date $t = 2$. Draw the event tree for the economy. For each node at $t = 1$ and $t = 2$, write the index value. For each node at $t = 2$, write the payoff of the put.

(b) Consider the node at $t = 1$ where the index has gone up to 115. We begin by focusing on events subsequent to this node only. Our goal is to compute the price of the option, P_1, in this event. To do so, construct a portfolio at this node that replicates the payoff of the option in both possible subsequent states at $t = 2$. Specifically, assume that you purchase x shares of the market index and invest y dollars in the riskfree asset. Solve for x and y from the assumption that this is a replicating portfolio. What is the price of this portfolio at $t = 1$ (in the event when the index is at 115)? What is the put price P_1 in this event?

(c) Following a similar procedure as in (b), now solve for the price of the option at $t = 1$ in the event when the market index has gone down to 95.

(d) Now go back to period $t = 0$. To compute P_0, construct a portfolio of the market index and the riskfree asset that pays P_1 in period $t = 1$ (that is, the number you obtained in (b) if the price goes up in period 1, and the number you obtained in (c) otherwise). What is the price of this portfolio? What is the price of the put option?

(e) Suppose that you buy one put option at $t = 0$ and sell it at $t = 1$. What is the expected return between $t = 0$ and $t = 1$ of this investment? Is it higher or lower than the riskfree rate? If CAPM holds, what must be the sign of the beta of the put between $t = 0$ and $t = 1$?

(f) Consider the market index. What is its expected return between $t = 0$ and $t = 1$? What is the variance of the return? What is the covariance of the return on the market index and the return on investing in the put between $t = 0$ and $t = 1$?

(g) Compute the beta of the put with respect to the market index using your results in (f). Does it have the sign you predicted in (e)? If CAPM holds, what must be the expected return of the put implied by the CAPM equation between $t = 0$ and $t = 1$? Is this the same expected return you got in (e)? Does CAPM hold in this example?

Problem 5. The Black-Scholes differential equation and options.

Suppose the time t price S_t of a non-dividend paying stock follows geometric Brownian motion, i.e.,

$$dS = \mu S_t dt + \sigma S_t dz$$

where μ is the expected return of the stock, σ is the volatility of the stock, and $dz = \tilde{\epsilon}\sqrt{dt}$ with $\tilde{\epsilon} \sim N(0,1)$.

As we saw in class, under some additional assumptions (which you may assume are satisfied) any derivative written on this stock must satisfy the following differential equation:

$$\frac{\partial f}{\partial t} + rS_t \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2} = rf_t$$

where t is time, f_t is the price of the derivative at time t, and r is the risk-free rate. This is the Black-Scholes differential equation.

In particular, a European style call option written on this stock must satisfy the Black-Scholes differential equation. The formula for the time $0 \leq t < T$ price of a European style call option with strike price K and expiration T is given by

$$c_t = S_t N(d_1) - e^{-r(T-t)} K N(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

and

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz, \quad n(x) = \frac{dN(x)}{dx} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

that is, $N(x)$ and $n(x)$ are the CDF and PDF, respectively, of a standard normal random variable.

In this exercise, you will verify that the formula for the price of a call option satisfies the Black-Scholes differential equation.

In what follows, you may use the following fact (without proof):

$$S_t n(d_1) = e^{-r(T-t)} K n(d_2)$$

1. Calculate $\partial d_1/\partial S$ and $\partial d_2/\partial S$ for $t = 0$.

2. For $t = 0$ we have
$$c_0 = S_0 N(d_1) - e^{-rT} K N(d_2)$$
Using this, show that $\partial c/\partial S = N(d_1)$.

3. Calculate $\partial^2 c/\partial S^2$ for $t = 0$.

4. Show that for $0 < t < T$
$$\frac{\partial c}{\partial t} = -S_t n(d_1) \frac{\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)} N(d_2)$$
which for $t = 0$ yields
$$\frac{\partial c}{\partial t} = -S_0 n(d_1) \frac{\sigma}{2\sqrt{T}} - rKe^{-rT} N(d_2)$$
[Hint:
$$\left(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}\right) = \frac{\partial (d_1 - d_2)}{\partial t} = -\frac{\sigma}{2\sqrt{T-t}}$$
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5. Show that the formula for the price of a European style call option satisfies the Black-Scholes differential equation. Do this for $t = 0$.