ECON 139 Lecture 25

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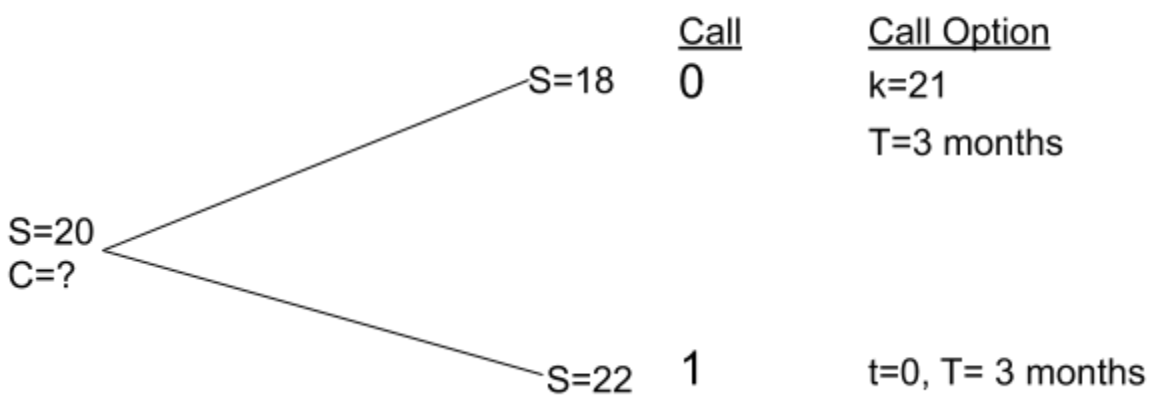
One Step Binomial Model

Example: Suppose we have a stock with current price of \$20. At the end of 3 months, the stock can either be valued at \$18 or \$22. The asset is risk-free that pays 12% per year compounding continuously.



<u>Continuous Compounding:</u> If you invest \$1, you will get e^{rT} returned where r= risk free rate and T= time (in years).
From our example, $e^{rT} = e^{(0.12)(0.25)} = e^{0.03}$

<u>Two Methods to find C:</u>

1) Form a Replicating Portfolio

$x22 + ye^{(0.12)(0.25)} = 1$

$x18 + ye^{(0.12)(0.25)} = 0$

If you subtract the second equation from the first one, you get

$4x = 1 \Rightarrow x = 0.25$

Plugging the value of x into the first equation gives us

$(0.25)22 + ye^{(0.12)(0.25)} = 1 \Rightarrow y = \frac{[1-(0.25)(22)]}{e^{0.03}} \Rightarrow y = -4.367$

We will short 4.367 shares of y, long 0.25 shares of x

$C = (0.25)(20) - 4.367 \Rightarrow C = 0.633$

2) Set up a portfolio of stock and call that has no uncertainty

Consider a portfolio that is long x shares of a stock and short one call option

Stock goes up: 22x-1

Stock goes down: 18x-0

$22x - 1 = 18x - 0 \Rightarrow x = 0.25$

Therefore,

Stock goes up: 22(0.25)-1=4.5

Stock goes down: 18(0.25)-0=4.5

We will get a payoff 4.5 whether the stock goes up or down.

Risk Free rate:

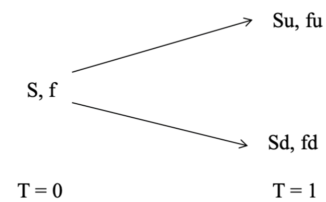
$V e^{rT} = 4.5 \Rightarrow V e^{(0.12)(0.25)} = 4.5 \Rightarrow V = 4.5e^{-0.03} \Rightarrow V = 4.367$

According to the Law of One Price,

$4.367 = (0.25)(20) - C \Rightarrow C = 0.633$

Generalization of Binomial Model

- Non-divided paying stock
- Derivative on stock with price $= f$
- Current time is zero
- Derivative pays off at time $T > 0$
- Stock price can move up to Su or down to Sd at time T, where $d < e^{rT} < u$
- If stock goes up, derivative pays off f_u
- If stock goes down, derivative pays off f_d



Now, set up portfolio that is long Δ shares of stock and short one derivative:

- If stock moves up, our pay off is $\Delta Su - f_u$;
- If stock moves down, our pay off is $\Delta Sd - f_d$

Let us make $\Delta Su - f_u = \Delta Sd - f_d$, then $\Delta = \dfrac{f_u - f_d}{su - sd}$; the cost of setting up this portfolio is $\Delta S - f$. There is $(\Delta S - f)e^{rT} = \Delta Su - f_u$, that is, $\Delta S - f = (\Delta Su - f_u)e^{-rT}$

let $\pi = \dfrac{e^{rT} - d}{u - d}$ and $f = e^{-rT}[\pi f_u + (1 - \pi)f_d)]$; then f is the price of derivative under measurement of π, assuming that LOOP holds.

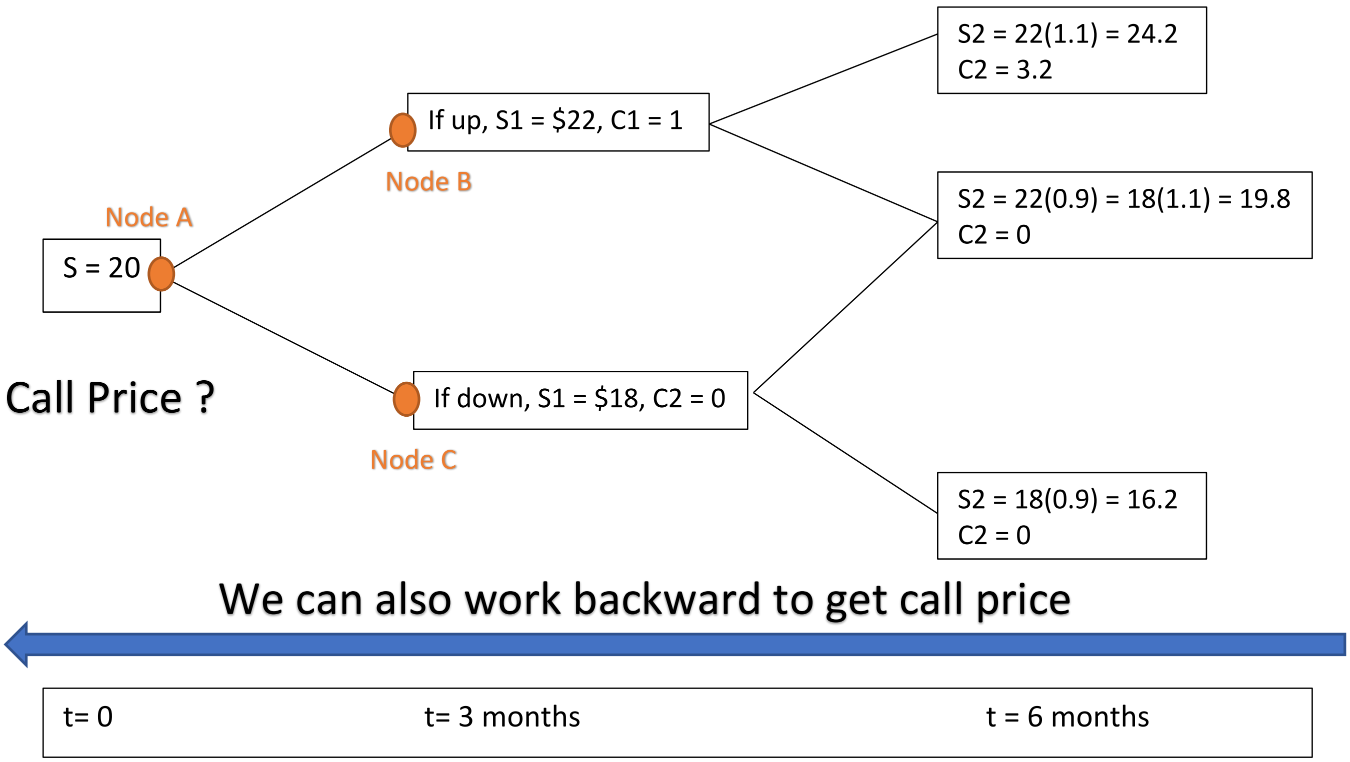
For example, let $u = 1.1$, $d = 0.9$, $r = 0.12$, $T = 0.25$, $f_u = 1$, and $f_d = 0$

So, $\pi = \dfrac{e^{0.03} - 0.9}{1.1 - 0.9} = 0.6523$, and
$f = e^{-0.03}(0.6523 * 1 + (1 - 0.6523) * 0) = 0.633$

Assume that probability of stock moving up is π, then
$E(S_T) = \pi Su + (1 - \pi)Sd = Se^{rT}$
Then, there is $\pi = \dfrac{e^{0.03} - 0.9}{1.1 - 0.9} = 0.6523$

Two-Step Binomial Model

Example 1: Assume there is a stock with current price at \$20, u is equal to 1.1, and d is equal to 0.9. The length of each time step is 3 months. The interest rate is 12% per year compounding continuously. The call option with strike price K = \$21 here. The total time length is 6 months.



Example 2: Backward Induction

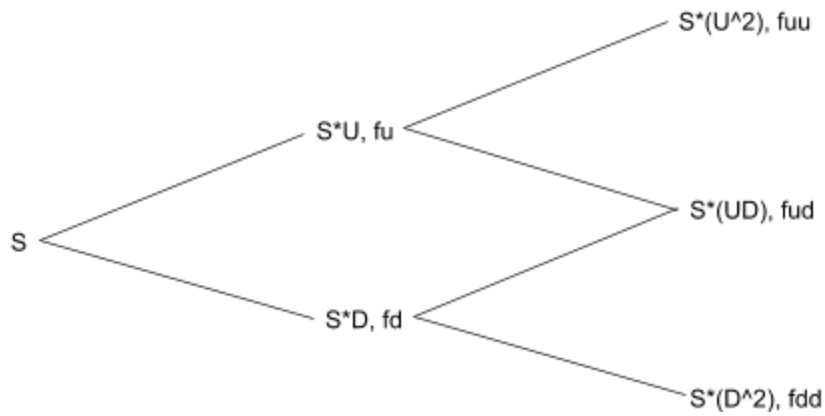
- At node C, the call option value is 0
- At node B, $\pi = 0.6523, \mathrm{f} = e^{-(0.12)(0.25)}[(0.6523)(3.2) + (0.3477)(0)]$ = 2.0257
- At node A, Call price = $e^{-0.03}[(0.62523)(2.0257) + (0.3477)(0)] = 1.2823$
- Therefore, we pay 1.2823 per call option in this example

Two-Step Generalization

As before, let the net rate-of-return for "good" states be U, and that of the "bad" states be D, such that

$$D < rf < U$$

The generalization for the first step remains the same, and the second step extends outward to the right:



Using the same backwards-inductive procedure as in the example, (let $\boldsymbol{\pi}$ be defined as it was in the one-step generalization):

$$f_u = e^{-rt}(\boldsymbol{\pi} f_{uu} + (1 - \boldsymbol{\pi})f_{ud})$$
$$f_d = e^{-rt}(\boldsymbol{\pi} f_{ud} + (1 - \boldsymbol{\pi})f_{dd})$$

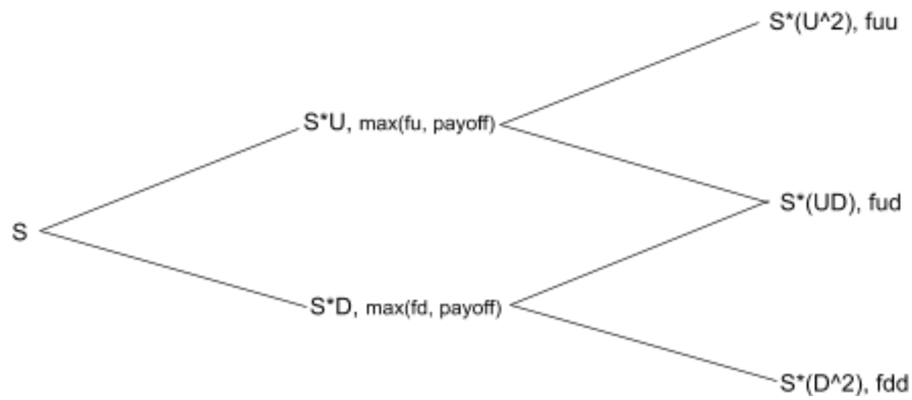
Therefore,

$$f = e^{-rt}(\boldsymbol{\pi} f_u + (1 - \boldsymbol{\pi})f_d)$$
$$f = e^{-2rt}(\boldsymbol{\pi}^2 f_{uu} + 2\boldsymbol{\pi}(1 - \boldsymbol{\pi})f_{ud} + (1 - \boldsymbol{\pi})^2 f_{dd})$$

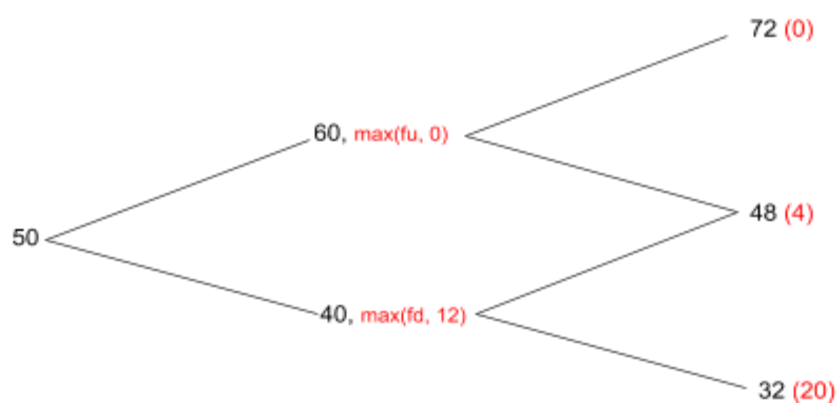
Application of the Two-Step Model: American Put Options

Remember that American-style options may be exercised at any time, up until and including their maturity. At each time step, the value of the option is the maximum of its back-induced value, and the payoff of exercising the option at that time step.

Update the binomial model to reflect this:



Look at a Put Option as a prime example of this. Specifically, consider an example of a stock that has the following binomial payoff model, and a put with strike price of $52 (the value of the put is in red):



Given a risk-free rate (r) of 0.05, and a time step (t) of 1, we calculate f_u and f_d as before, and get 1.415, and 9.464 respectively.

Notice that exercising the option in the lower node yields a greater payoff than holding onto the option. Therefore, when calculating the value of the put at t = 0, we use the payoff in exercising the option, rather than f_d:

$$f = e^{-0.05}(0.628 * f_u + (1 - 0.628)12) = 5.089$$