Economics 139 Scribe Notes (Spring 2019)

Lecture 17 - 1. Efficient set mathematics 2. Zero-beta CAPM 3. Standard CAPM

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1. Efficient set mathematics

Proposition #5:

- Let p and r be any two mv frontier portfolios, then the covariance of their returns is given by $\boxed{cov(\tilde{r_p}, \tilde{r_r}) = \frac{C}{D}\left(\mu_p - \frac{A}{C}\right)\left(\mu_r - \frac{A}{C}\right) + \frac{1}{C}}$
- Proof:

$$\begin{aligned} cov(\tilde{r_p}, \tilde{r_r}) &= W_p^T\, \Sigma\, W_r \\ &= W_p^T\, \Sigma\, (\mathrm{g} + \mathrm{h}\mu_r) \\ &= W_p^T\, (\frac{1}{D}\, (B_e - A_\mu) + \frac{1}{D}\, ((\mathrm{C}\mu - A_e)\mu_r)) \\ &= \frac{C}{D}(\frac{B}{C} + \mu_p\mu_r - \frac{A}{C}\mu_p - \frac{A}{C}\mu_r) \\ &= \frac{C}{D}\left(\mu_p - \frac{A}{C}\right)\left(\mu_r - \frac{A}{C}\right) + \frac{1}{C} \end{aligned}$$

Proposition #6:

- The covariance of the return of the global mv portfolio & the return of any mv frontier portfolio a is given by $\boxed{cov(\widetilde{r_{mv}}, \tilde{r_q}) = \frac{1}{C}} \rightarrow \sigma_{mv}^2 = \frac{1}{C}$

Proposition #7:

- Let p be a mv frontier portfolio. The covariance of the returns of p and any portfolio r is given by $\boxed{cov(\tilde{r_p}, \tilde{r_r}) = \lambda\mu_r + \gamma}$

$$\lambda = \frac{\mu_p C - A}{D}, \qquad \gamma = \frac{B - \mu_r A}{D}$$

- Proof:

$$\begin{aligned} cov(\tilde{r_p}, \tilde{r_r}) &= W_p^T\, \Sigma\, W_r \\ &= (\lambda\Sigma^{-1}\mu + \gamma\Sigma^{-1}e)^T\, \Sigma\, W_r \\ &= (\mu^T\Sigma^{-1}\lambda + e^T\Sigma^{-1}\gamma)\, \Sigma\, W_r \end{aligned}$$

$$= \lambda \, \mu^T W_r + \gamma e^T \, W_r$$

Recall that $\mu^T W_r = \mu_r$, $e^T \, W_r = 1$, then we have

$cov(\tilde{r}_p \,, \tilde{r}_r) = \lambda \mu_r + \gamma$

- r is not necessary to be the mv frontier portfolio.

<u>Proposition #8:</u>

- For any mv frontier portfolio p (except the global mv portfolio), there is an unique mv frontier portfolio has zero covariance with p, denoted z_p
- Proof:

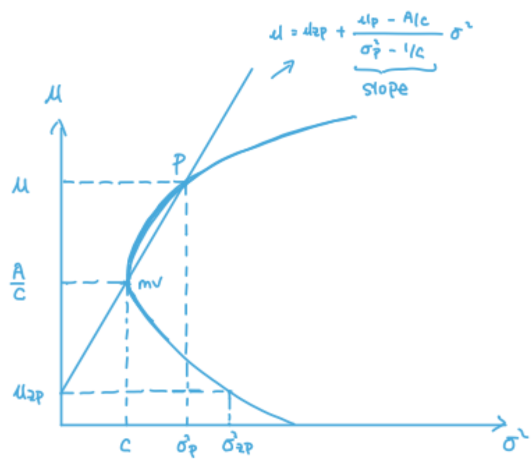
  Using proposition #5, $cov\left(\tilde{r}_p \,, \widetilde{r_{z_p}}\right) = \frac{C}{D}\left(\mu_p - \frac{A}{C}\right)\left(\mu_{z_p} - \frac{A}{C}\right) + \frac{1}{C} = 0$

  expected return over z_p: $\mu_{z_p} = \frac{A}{C} - \frac{D/C^2}{\mu_p - (A/C)}$

  weight of z_p: $W_{z_p} = g + h\mu_{z_p}$

  variance: $\sigma_{z_p}^2 = W_{z_p}^T \, \Sigma \, W_{z_p}$

  $$\mu = \mu_{z_p} + \frac{\mu_p - A/C}{\sigma_p^2 - {}^{1}\!/_{C}} \sigma^2$$



  From proposition #5, $cov\left(\tilde{r}_p \,, \widetilde{r_{z_p}}\right) = \lambda \mu_{z_p} + \gamma = 0$

  $$\gamma = -\lambda \mu_{z_p}$$

  $$cov(\tilde{r}_p \,, \tilde{r}_r) = \lambda \mu_r - \lambda \mu_{z_p} = \lambda(\mu_r - \mu_{z_p})$$

  $$cov(\tilde{r}_p \,, \tilde{r}_p) = \sigma_p^2 = \lambda(\mu_p - \mu_{z_p})$$

  $$\frac{cov(\tilde{r}_p \,, \tilde{r}_r)}{\sigma_p^2} = \frac{\lambda(\mu_r - \mu_{z_p})}{\lambda(\mu_p - \mu_{z_p})}$$

Where $\frac{cov(\widetilde{r_p},\widetilde{r_r})}{\sigma_p^2}$ is β of the portfolio r w.r.t portfolio p, denoted by β_{rp}

$$\frac{cov(\widetilde{r_p},\widetilde{r_r})}{\sigma_p^2} = \frac{(\mu_r - \mu_{z_p})}{(\mu_p - \mu_{z_p})} = \beta_{rp}$$

$$\mu_r - \mu_{z_p} = \beta_{rp}(\mu_p - \mu_{z_p})$$
$$E[\widetilde{r_r}] - E\left[\widetilde{r_{z_p}}\right] = \beta_{rp}(E[\widetilde{r_p}] - E\left[\widetilde{r_{z_p}}\right])$$

2.Zero-Beta CAPM

paper on bcourses (Black, 1972)
-Black: Same assumptions as CAPM, but dealing with risky assets ONLY
-Same assumptions as CAPM, but there is no risk-free asset.
-Market portfolio is convex combination of all investor portfolios
-By corollary to proposition#4, market portfolio is efficient (efficient observable portfolio)
Let p be the market portfolio (pick p = M), and from proposition #8, we have

$$E[\widetilde{r_r}] - E\left[\widetilde{r_{z_m}}\right] = \beta_{rm}(E[\widetilde{r_m}] - E\left[\widetilde{r_{z_m}}\right])$$

where $\beta_{rm} = \frac{cov(\widetilde{r_m},\widetilde{r_r})}{var(\widetilde{r_m})}$ (same as CAPM)

3.Standard CAPM

$$\min_W \ \frac{1}{2} W^T \Sigma\, W$$
$$s.t. W^T\mu + W_{r_f} r_f = \mu_p$$
$$W^T e + W_{r_f} = 1 \ \ \rightarrow W_{r_f} = 1 - W^T e$$

$$\min_W \ \frac{1}{2} W^T \Sigma\, W$$
$$s.t. W^T\mu + (1 - W^T e)\, r_f = \mu_p$$

$$\mathcal{L}(W,\lambda) = \frac{1}{2}\, W^T \Sigma\, W + \lambda(\mu_p - W^T\mu - (1 - W^T e)\, r_f)$$

FOC: $\Sigma\, W - \ \lambda(\mu + \mathrm{e}r_f) = 0$

$W_p = \ \lambda\Sigma^{-1}(\mu - er_f)$

plug into constraint to find λ

$W_p^T(\mu - er_f) = \ \mu_p - r_f$

$\lambda(\mu - er_f)^T \Sigma^{-1}(\mu - er_f) = \ \mu_p - r_f$

let $\left(\mu - er_f\right)^T \Sigma^{-1}\left(\mu - er_f\right) = H$ (since it is numeric)

$$\lambda = \frac{\mu_p - r_f}{H}$$

$$W_p = \frac{\mu_p - r_f}{H}\Sigma^{-1}(\mu - er_f)$$

Where $\frac{\mu_p - r_f}{H} = c(\,\mu_p)$ (input),

And $\Sigma^{-1}\left(\mu - er_f\right) = \overline{W}$(a fixed vector of weight for risky asset)

$$W_p = c(\,\mu_p)\overline{W}$$

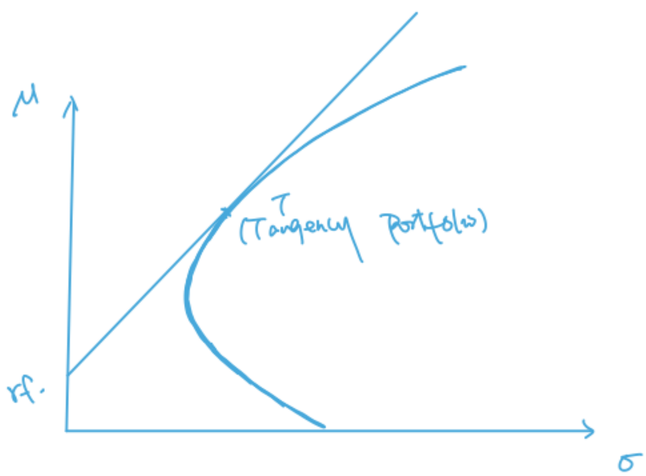
<u>Tangency Portfolio</u>

no lending or borrowing going on, and investing all on risky asset.

$$\sigma_p^2 = W_p^T\,\Sigma\,W_p$$

$$= \left(\frac{\mu_p - r_f}{H}\right)\left(\mu - er_f\right)^T \Sigma^{-1}\,\Sigma\,\Sigma^{-1}\,\left(\mu - er_f\right)\left(\frac{\mu_p - r_f}{H}\right)$$

$$= \frac{(\,\mu_p - r_f)^2}{H}$$



<u>Covariance of a mv portfolio p and any portfolio q</u>

* q is not necessary to be a mv portfolio

$$cov\left(\widetilde{r_p}\,,\widetilde{r_q}\right) = \frac{(\,\mu_p - r_f)(\,\mu_q - r_f)}{H}$$

$$\beta_{qp} = \frac{cov\left(\widetilde{r_p}\,,\widetilde{r_q}\right)}{\sigma_p^2} = \frac{\mu_q - r_f}{\mu_p - r_f}$$

$$\mu_q - r_f = \beta_{qm}\left(\,\mu_p - r_f\right)$$

substitute m, $\mu_q - r_f = \beta_{qm}\left(\,\mu_m - r_f\right)$

$$
\begin{aligned}
W_p^T \, \Sigma \, W_q &= \left(\frac{\mu_p - r_f}{H}\right)\left(\mu - e r_f\right)^T \Sigma^{-1} \, \Sigma \, W_q \\
&= \left(\tfrac{\mu_p - r_f}{H}\right)\left(\mu - e r_f\right)^T W_q
\end{aligned}
$$