Econ 240A Econometrics **Fall 2018**

Problem Set 1 Solutions

Due date: Sept. 5, 2018 *Fengshi Niu*

1. Conditional probability

(a) i. Notice that

$$P(A) = P(A \cap (B \cup B^c)) = P((A \cap B) \cup (A \cap B^c))$$
$$= P(A \cap B) + P(A \cap B^c).$$

The last equality holds by $(A \cap B) \cap (A \cap B^c) = \emptyset$ and countable additivity of probability for disjoint sets. Since $P(B) = 1$, we have $0 \le P(A \cap B^c) \le P(B^c) = 1 - P(B) = 0$ and thus $P(A \cap B^c) = 0$. Combining these we get $P(A) = P(A \cap B)$. Then, $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{1} = P(A)$.

ii. $A \subset B \implies A \cap B = A$. Then, $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)}{P(A)} = 1$ and $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$.

iii. $A \cap B = \emptyset \implies P(A \cup B) = P(A) + P(B)$. Then, $P(A|A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A)+P(B)}$.

iv. $P(A \cap B \cap C) = P(A \cap (B \cap C)) = P(A|B \cap C)P(B \cap C) = P(A|B \cap C)P(B|C)P(C)$.

(b) First, $P(A|B) = \frac{P(A \cap B)}{P(B)} \ge 0$, for all $A \in \mathcal{F}$. Second, since $B \subset \Omega$, $P(\Omega|B) = 1$. Third, if $A_1, A_2, \ldots \in \mathcal{F}$ are pairwise disjoint, then $P(\cup_{i=1}^{\infty} A_i|B) = \frac{P((\cup_{i=1}^{\infty} A_i) \cap B)}{P(B)} = \frac{P(\cup_{i=1}^{\infty}(A_i \cap B))}{P(B)} = \frac{\sum_{i=1}^{\infty} P(A_i \cap B)}{P(B)} = \sum_{i=1}^{\infty} \frac{P(A_i \cap B)}{P(B)} = \sum_{i=1}^{\infty} P(A_i|B)$. $P(\cdot|B)$ is thus a probability function over $(\Omega, \mathcal{F})$ by definition.

2. Boole's inequality and Bonferroni's method

(a) i. $A_n \subset A_{n+1} \implies P(A_n) \le P(A_{n+1})$. To prove $P(A) = \lim_{n\to\infty} P(A_n)$, consider sets $B_n = A_n \setminus A_{n-1}$ (take $A_0 = \emptyset$). Notice that $\cup_{i=1}^{n} B_n = \cup_{i=1}^{n} A_n$. Since $A_n \uparrow A$, $\cup_{i=1}^{n} A_n = A_n$ and $B_1, B_2, \ldots$ are mutually exclusive. It follows that $P(A) = P(\cup_{i=1}^{\infty} A_i) = P(\cup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} P(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} P(B_n) = \lim_{n\to\infty} P(\cup_{i=1}^{n} B_n) = \lim_{n\to\infty} P(\cup_{i=1}^{n} A_n) = \lim_{n\to\infty} P(A_n)$. Hence $P(A_n) \uparrow P(A)$.

ii. Consider sets $S_n = \cup_{i=1}^{n} A_n$. Since $S_n \uparrow \cup_{n=1}^{\infty} A_n$, we have $P(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} P(S_n)$. Now remember $P(A \cup B) = P(A) + P(B \setminus A) \le P(A) + P(B)$. It follows inductively that $P(\cup_{i=1}^{n} A_i) \le \sum_{i=1}^{n} P(A_i)$ for any $n \in \mathbb{N} \setminus \{0\}$. Therefore, $P(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} P(S_n) \le \lim_{n\to\infty} \sum_{i=1}^{n} P(A_i) = \sum_{i=1}^{\infty} P(A_i)$.

Note: we can also use indicators to prove these inequalities.

(b) i. $P(\min_{1\le i\le n} p_i \le \alpha/n) = P(\cup_{i=1}^{n}\{p_i \le \alpha/n\}) \le \sum_{i=1}^{n} P(p_i \le \alpha/n) = \alpha$.

ii. $P(\min_{1\le i\le n} p_i \le \alpha/n) = 1 - P(\min_{1\le i\le n} p_i > \alpha/n) = 1 - P(\cap_{i=1}^{n}\{p_i > \alpha/n\}) = 1-(1-\alpha/n)^n$. The last equality holds because $p_1, p_2, \ldots, p_n$ are jointly independent. $1 - (1 - \alpha/n)^n \to 1 - e^{-\alpha}$, as $n \to \infty$.

iii. $1-e^{-0.05} \approx 0.04877$. The upper bound 0.05 from Boole's inequality is not too much larger than it.

3. cdf, pdf, and transformations

(a) To find the density of X, we take the derivative of F. $f(x) = \frac{dF(x)}{dx} = \frac{e^x}{(1+e^x)^2}$.

(b) Remember that pdf must integrate to one, $\int f(x)dx = 1$. This constraint implies $c\int_0^1 x(1-x)dx = 1$, where $\int_0^1 x(1-x)dx = \int_0^1 xdx - \int_0^1 x^2dx = \frac{1}{2}x^2|_0^1 - \frac{1}{3}x^3|_0^1 = \frac{1}{6}$. Hence $c = 6$.

(c) i. The constraint $\int f_X(x;\theta)dx = 1$ implies $k(\theta) = \int x\mathbf{1}(0 \le x \le \theta)dx = \int_0^\theta xdx = \theta^2/2$.

ii. To find the cdf of X, we take the integral of f_X. For $0 \le x \le \theta$, $F_X(x;\theta) = \int_{-\infty}^x f_X(t;\theta)dt = \frac{2}{\theta^2}\int_0^x tdt = \frac{x^2}{\theta^2}$.

$$F_X(x;\theta) = \begin{cases} 0 & x \le 0 \\ \frac{x^2}{\theta^2} & 0 < x < \theta \\ 1 & x \ge \theta \end{cases}.$$

iii. $F_Y(y;\theta) = P(Y \le y) = P(X^2 \le y)$. For $y < 0$, $P(X^2 \le y) = 0$. For $y \ge 0$, $P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}^-)$. Hence the cdf of Y is

$$F_Y(y;\theta) = \begin{cases} 0 & y \le 0 \\ \frac{y}{\theta^2} & 0 < y < \theta^2 \\ 1 & y \ge \theta^2 \end{cases}.$$

Take derivative to get $f_Y(y;\theta) = \frac{dF_Y(y;\theta)}{dy}$ for $y \ne 0, \theta^2$.

$$f_Y(y;\theta) = \begin{cases} 0 & y < 0 \\ \frac{1}{\theta^2} & 0 < y < \theta^2 \\ 0 & y > \theta^2 \end{cases}.$$

(The density at countable points, $0, \theta^2$ here, doesn't affect the distribution.) The distribution of Y is $\mathrm{U}[0,\theta^2]$, uniform over $[0,\theta^2]$!

4. Markov's inequality

(a) See Figure 1.

(b) It is clear from Figure 1 that $\mathbf{1}(x \ge b) \le \frac{x}{b}, \forall x \ge 0$. Since random varible $X \ge 0$, we have $\mathbf{1}(X \ge b) \le \frac{X}{b}$.

(c) Take expectation on both sides of the inequality from part (b). We get $P(X \ge b) = \mathbb{E}(\mathbf{1}(X \ge b)) \le \mathbb{E}\left(\frac{X}{b}\right) = \frac{\mathbb{E}X}{b}$.