Economics 139 Scribe Notes

Lecture 14 - Mutual Fund Theorem, Single Index Model, and Single Factor Model

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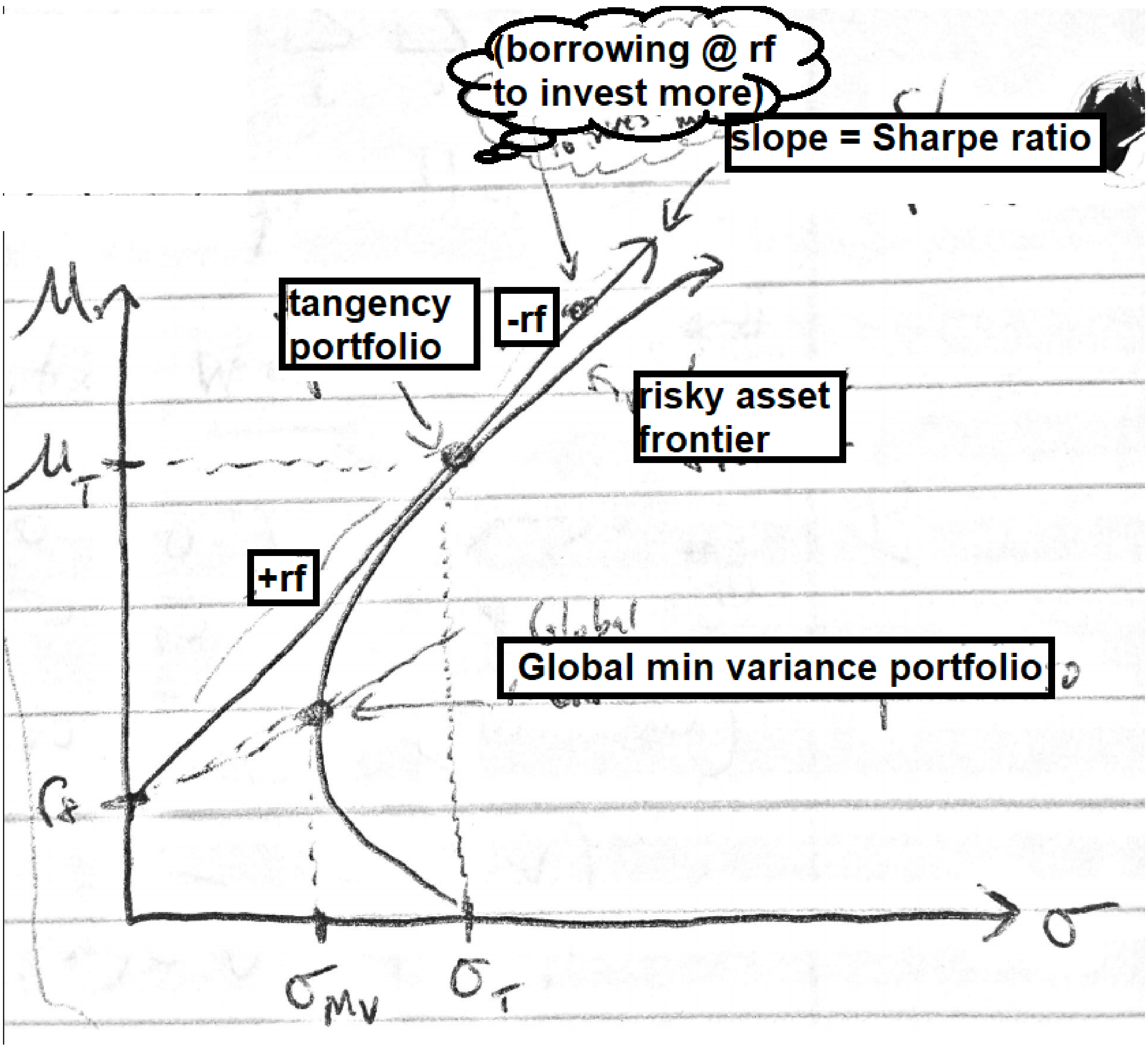
1 Tangency Portfolio (Two Risky Assets):

$$\max_{w_1} = \frac{\mu_p - r_f}{\sigma_p} \tag{1}$$

$$w_1* = \frac{\mu_1\sigma_2^2 - \mu_2\sigma_1\sigma_2\rho_{1,2}}{\sigma_p} \tag{2}$$



2 Mutual Fund Theorem:

Note 1. James Tobin (1958) (aka Separation Theorem; Two-Fund Theorem)

"All efficient portfolios are tangency portfolios $\pm r_f$ "

It features: $\frac{N(N-1)}{2}$ covariances; N expected returns; and N variances.

3 Single Index Model:

$$E(\tilde{r_i} - r_f) = \alpha_i + B_i E(\tilde{r_m} - r_f) \tag{3}$$

Where:

$E(\tilde{r_i} - r_f)$ represents "security risk premium";

$\tilde{r_i}$ represents the risk return of some asset i;

α_i represents some non-market risk premium;

$E(\tilde{r_m} - r_f)$ represents the market risk premium;

and B_i represents the "sensitivity" of asset i to the market risk premium, defined as:

$$B_i = \frac{cov(\tilde{r_i}, \tilde{r_m})}{var(\tilde{r_m})} \tag{4}$$

This leads to:

$$\tilde{r_i} = E(\tilde{r_i}) + (\tilde{r_i} - E(\tilde{r_i})) \tag{5}$$

where $E(\tilde{r_i})$ is expected return and $(\tilde{r_i} - E(\tilde{r_i}))$ is the "return surprise"; and

$$\tilde{\eta_i} = r_i - E(\tilde{r_i}) \tag{6}$$

$$E(\tilde{\eta_i}) = E(\tilde{r_i} - E(\tilde{r_i})) = 0 \tag{7}$$

$$\sigma^2_{\tilde{\eta_i}} = var(\tilde{r_i} - E(\tilde{r_i})) = var(\tilde{r_i}) = \sigma^2_i \tag{8}$$

$$\tilde{r_i} = E(\tilde{r_i}) + \tilde{\eta_i} \tag{9}$$

* Let $\tilde{m}$ be a macroeconomic variable that affects all firms, where $E(\tilde{m}) = 0$; $var(\tilde{m}) = \sigma^2_{\tilde{m}}$

Then, we can decompose return surprises: $\tilde{\eta_i} = \tilde{m} + \tilde{\epsilon_i}$; $E(\tilde{m}, \tilde{\epsilon_i}) = 0$

Where $\tilde{m}$ is the common component and $\tilde{\epsilon_i}$ is the firm-specific component.

Then:

$$\tilde{r_i} = E(\tilde{r_i}) + \tilde{m} + \tilde{\epsilon_i} \tag{10}$$

$$\sigma_i^2 = \sigma_{\tilde{m}}^2 + \sigma_{\tilde{\epsilon_i}}^2 \tag{11}$$

(We can assume $E(\tilde{\epsilon_i}, \tilde{\epsilon_j}) = 0$ for all $i = / = j$)

Then:

$$cov(\tilde{r_i}, \tilde{r_j}) = cov(\tilde{m} + \tilde{\epsilon_i}, \tilde{m} + \tilde{\epsilon_j}) = cov(\tilde{m}, \tilde{m}) + cov(\tilde{\epsilon_i}, \tilde{m}) + cov(\tilde{\epsilon_j}, \tilde{m}) + cov(\tilde{\epsilon_i}, \tilde{\epsilon_j}) \tag{12}$$

$$= \sigma_{\tilde{m}}^2 \tag{13}$$

Because:

$$cov(\tilde{\epsilon_i}, \tilde{m}) = cov(\tilde{\epsilon_j}, \tilde{m}) = cov(\tilde{\epsilon_i}, \tilde{\epsilon_j}) = 0 \tag{14}$$

Thus, the Single-Factor Model is:

$$\tilde{r_i} = E(\tilde{r_i}) + B_i\tilde{m} + \tilde{\epsilon_i} \tag{15}$$

$$\sigma_i^2 = B_i\sigma_{\tilde{m}}^2 + \sigma_{\tilde{\epsilon_i}}^2 \tag{16}$$

Where $B_i\sigma_{\tilde{m}}^2$ is systemic variance and $\sigma_{\tilde{\epsilon_i}}^2$ is idiosyncratic variance. Then:

$$cov(\tilde{r_i}, \tilde{r_j}) = cov(B_i\tilde{m} + \tilde{\epsilon_i}, B_j\tilde{m} + \tilde{\epsilon_j}) \tag{17}$$

$$= B_iB_j\sigma_{\tilde{m}}^2 \tag{18}$$

3.1 Continued:

$$\tilde{r_i} = E(\tilde{r_i}) + B_i\tilde{r_m} + \tilde{\epsilon_i} \tag{19}$$

(and $\tilde{r_i} - r_f = E(\tilde{r_i} - r_f) + B_i(\tilde{r_m} - r_f) + \tilde{\epsilon_i}$)

and let $\alpha_i = E(\tilde{r_i} - r_f)$

$$E(\tilde{r_i} - r_f) = \alpha_i + B_iE(\tilde{r_m} - r_f) \tag{20}$$

This is the "Single Index Model", suggested by Sharpe in 1963.

$$(\alpha_i, B_i) = arg\max_{a,b} E((\tilde{r_i}^e - a - b\tilde{r_m}^e)^2) \tag{21}$$

Where excess risky return of asset i is: $\tilde{r_i}^e = \tilde{r_i} - r_f$;

and excess risky return of the market m is: $\tilde{r_m}^e = \tilde{r_m} - r_f$

First Order Conditions: 1. $E(\tilde{r_i}^e - \alpha_i - B_i\tilde{r_m}^e) = 0$; 2. $E(\tilde{r_m}^e(\tilde{r_i}^e - \alpha_i - B_i\tilde{r_m}^e)) = 0$; thus:

$$E(\tilde{\epsilon_i}) = 0; E(\tilde{r_m}^e, \tilde{\epsilon_i}) = 0; \tag{22}$$

$$B_i = \frac{cov(\tilde{r_i}^e, \tilde{r_m}^e)}{var(\tilde{r_m}^e)} = \frac{cov(\tilde{r_i}, \tilde{r_m})}{var(\tilde{r_m})} \tag{23}$$

Thus; total risk (of an asset i) = systemic risk + idiosyncratic risk, or:

$$\sigma_i^2 = \sigma_{\tilde{m}}^2 + \sigma_{\epsilon_i}^2 \tag{24}$$

Covariance (of assets i and j) = (product of beta terms) * (systemic risk), or:

$$cov(\tilde{r_i}, \tilde{r_j}) = B_i B_j \sigma_{\tilde{m}}^2 \ \ \& \ \ corr(\tilde{r_i}, \tilde{r_j}) = \frac{B_i B_j \sigma_{\tilde{m}}^2}{\sigma_{\tilde{i}} \sigma_{\tilde{j}}} \tag{25}$$

$$corr(\tilde{r_i}, \tilde{r_j}) = (\frac{B_i \sigma_{\tilde{m}}^2}{\sigma_{\tilde{i}} \sigma_{\tilde{m}}})(\frac{B_j \sigma_{\tilde{m}}^2}{\sigma_{\tilde{j}} \sigma_{\tilde{m}}}) \tag{26}$$

$$= corr(\tilde{r_i}, \tilde{r_m}) * corr(\tilde{r_j}, \tilde{r_m}) \tag{27}$$

3.2 Single-Index Model, summarized:

So, for $N = 100$ we must calculate 100 of α_1, 100 of B_i, 100 of $\sigma_{\epsilon_i}^2$, 1 μ_m, and 1 σ_m^2, for a grand total of 302 parameters to be calculated. This compares to a total of 5,150 parameters if using the "covariance" method (because there are over 4900 covariances to calculate etc.)

4 Diversification:

$$\tilde{r_i} - r_f = \alpha_i + B_i(\tilde{r_m} - r_f) + \tilde{\epsilon_i} \tag{28}$$

$$\sum_{i=1}^{N} = (\frac{1}{N})\tilde{r_i} = \tilde{r_P}; \frac{1}{N}\sum \tilde{r_i} - r_f = \frac{1}{N}\sum_i \alpha_i + \frac{1}{N}\sum_i B_i(\tilde{r_m} - r_f) + \frac{1}{N}\sum_i \tilde{\epsilon_i} \tag{29}$$

$$\tilde{r_P} - r_f = \alpha_P + B_P(\tilde{r_m} - r_f) + \epsilon_P \tag{30}$$

Where P denotes the (diversified) portfolio. Continued:

$$\sigma_P^2 = B_P^2 \sigma_m^2 + \sigma_{\tilde{\epsilon_i}}^2 \tag{31}$$

$$\sigma_{\epsilon_P}^2 = var(\frac{1}{N}\sum_i \epsilon_i) = \frac{1}{N^2}\sum_i \sigma_{\epsilon_i}^2 = \frac{1}{N}(\frac{1}{N}\sum_i \sigma_{\epsilon_i}^2) = \frac{\sigma_{\epsilon_i}^2}{N} \tag{32}$$

Which approaches 0 as N approaches infinity. Also, we can do the arithmetic above because we assume $cov_{i,j} = 0$

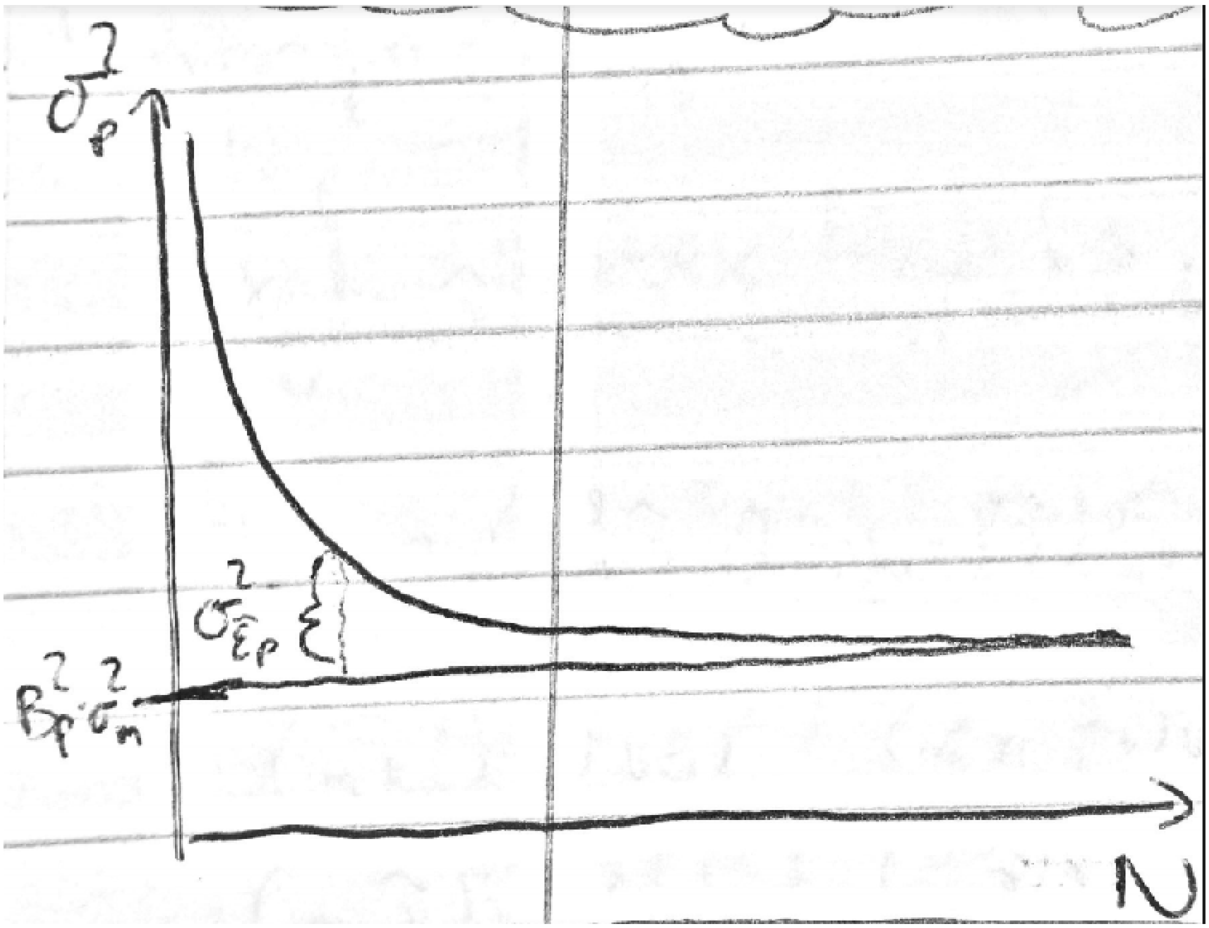
Explained: we can eliminate idiosyncratic variance in a portfolio with N = infinity assets, however, systematic variance remains.

/subsectionConnection to Single-Index Model: Thus, the Single-Index model for a large, diversified portfolio is:

$$\tilde{r_P} - r_f = \alpha_P + B_P(\tilde{r_m} - r_f) \tag{33}$$

Note that the Capital Asset Pricing Model (CAPM) shows:

$$E(\tilde{r_P} - r_f) = B_P E(\tilde{r_m} - r_f) \tag{34}$$


σ_p^2
$\sigma_{\hat{\varepsilon}_p}^2$
$\beta_p^2 \sigma_m^2$
N