光华管理学院
Guanghua School of Management

Mathematical Methods in Finance

Lecture 6: Stochastic Calculus

Fall 2013

Overview

- Stochastic integral
- Ito's formulae
- Examples

Motivation

- Consider trading in an asset with unit price $W(t)$ (unrealistic, just for simplicity).
  - A partition $\Pi = \{t_0, t_1, \cdots, t_n\}$ s.t. $0 = t_0 < t_1 < \cdots < t_n = T$.
  - In the time period $[t_j, t_{j+1})$, hold Δ_j (adapted process) shares of this asset.
  - Note that the time period is left closed but right open.
- The gain process $I(t)$ at time $t \in [t_k, t_{k+1})$ is given by

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)]. \quad (1)$$

- Question: When $||\Pi|| := \max_{1 \le j \le n}(t_{j+1} - t_j)$ goes to zero, how to define the associated limiting gain process?
  - A kind of limit of summation (1) as $||\Pi|| \to 0$.
  - A kind of integral written as $\int_0^t \Delta(t)dW(t)$ (recall the definition of Riemann integral in ordinary calculus).

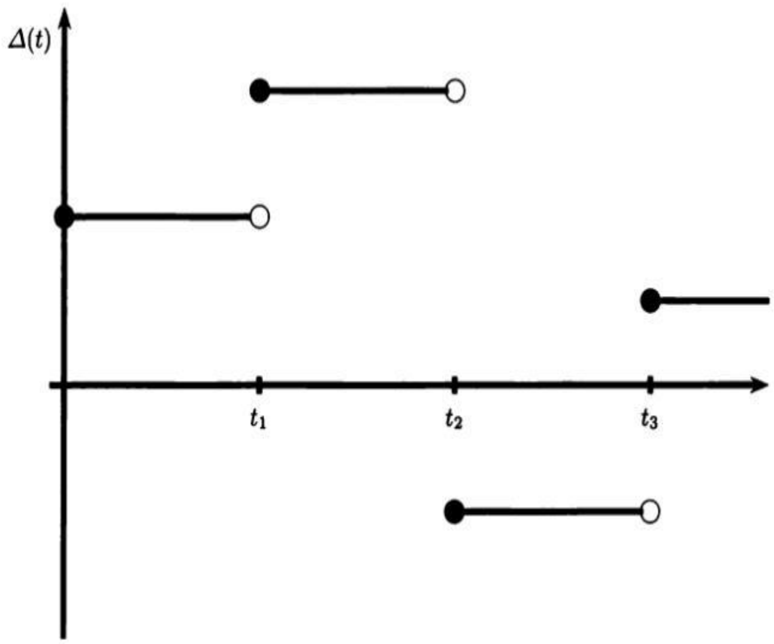
Motivation

- However, it is more complicated.
  - What is the definition of the related limit? In what sense?
  - It is not “traditional” (professionally speaking, Lebesgue or Riemann) integral because $W(t)$ is non-differentiable. It doesn’t make sense that

$$\int_0^t \Delta(t)dW(t) = \int_0^t \Delta(t)W'(t)dt.$$

- First, define the integral for a simple process $\Delta(t)$, which is
  - adapted (the investment decisions are made based on the available information up to that time) and $E\int_0^t \Delta^2(u)du < +\infty$
  - equals $\Delta(t_j)$ in the time period $[t_j, t_{j+1})$ for any $j = 0, 1, \cdots, n-1$. (see a graph next page)
- Then the Itô integral at time $t \in [t_k, t_{k+1}]$ is defined to be (1).

Simple Process



A path of a simple process.

Construction of Itô integral

- Properties of Itô integral $I(t)$ for simple processes $\Delta(t)$.
  - (1) $I(t)$ is $\mathcal{F}_t$-measurable; **Linearity**;
  - (2) $I(t)$ is a **martingale**;
  - (3) **(Itô isometry)** $EI^2(t) = E\int_0^t \Delta^2(u)du$;
  - (4) **(Quadratic variation)** $[I, I](t) = \int_0^t \Delta^2(u)du$.
- Another way to express quadratic variation

$$dI(t)dI(t) = \Delta^2(t)dW(t)dW(t) = \Delta^2(t)dt.$$

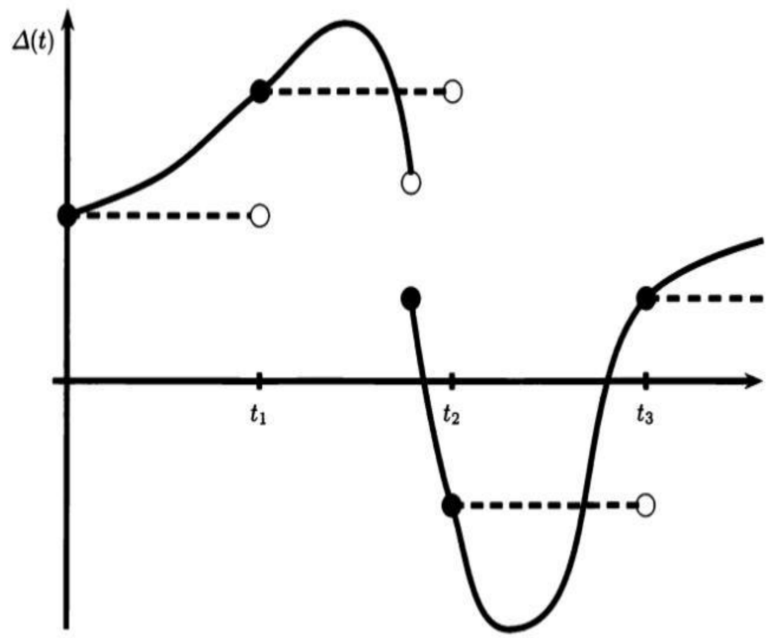
- Another way to express Itô integral

$$dI(t) = \Delta(t)dW(t)$$

  (Differential Form).
- Second, let us construct Itô integral for a general adapted process $\Delta(t)$ that can be approximated by simple processes in some sense.

Construction of Itô integral

Approximate a general adapted process $\Delta(t)$ by simple processes.



Approximating a continuously varying integrand.

Construction of Itô integral

- We can find a sequence of simple processes Δ_n which approximate Δ.
- Note that $\int_0^T \Delta_n(t)dW(t)$ has already been well defined.
- It is natural to define the $\int_0^T \Delta(t)dW(t)$ to be a limit of

$$I_n(t) := \int_0^T \Delta^n(t)dW(t).$$

- Question (1): How do we know a limit exists? What do we mean by “limit"?
- Question (2): Is the limit unique?

Construction of Itô integral

- **Answer**: for an adapted process $\Delta(t) \in L^2[0,T]$, we can define the related Itô integral

$$\int_0^T \Delta(t)dW(t) := \lim_{n\to+\infty}\int_0^T \Delta_n(t)dW(t),$$

where $\{\Delta_n(t) \in L^2[0,T] : n = 0, 1, \cdots\}$ are a sequence of simple processes and the "limit" is unique in some sense.

- The so called "sense" is in $L^2(T)$ (square integrable).

- The meaning of $\int_0^T \Delta(t)dW(t)$: the gain process by holding $\Delta(t)$ shares of asset $W(t)$.

Properties of Itô integral

- **(Continuity)** $I(t)$ is continuous in t;

- **(Adaptivity)** $I(t)$ is $\mathcal{F}_t$-measurable;

- **(Linearity)** If $I(t) = \int_0^t \Delta(u)dW(u)$ and $J(t) = \int_0^t \Gamma(u)dW(u)$, then $I(t) \pm J(t) = \int_0^t(\Delta(u) \pm \Gamma(u))dW(u)$ and $cJ(t) = \int_0^t c\Gamma(u)dW(u)$ for any constant c;

- **(Martingale)** $I(t)$ is a martingale;

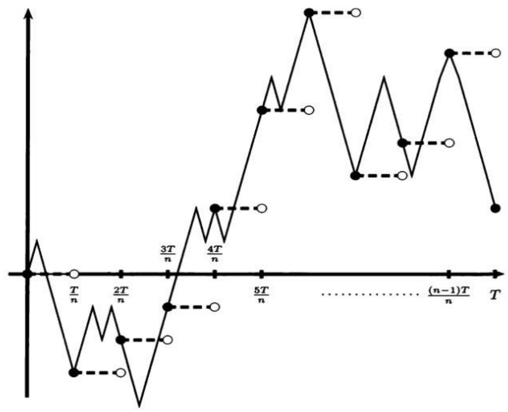
- **(Itô Isometry)** $EI^2(t) = E\int_0^t \Delta^2(u)du$;

- **(Quadratic variation)** $(I, I)(t) = \int_0^t \Delta^2(u)du$.

An Example

- **An Example**: Compute $\int_0^T W(t)dW(t)$.
- Select one particular sequence of simple processes as follows:

$$\Delta_n(t) = \sum_{j=0}^{n-1} W\left(\frac{jT}{n}\right) I_{\{t\in[\frac{jT}{n},\frac{(j+1)T}{n})\}}.$$



Simple process approximating Brownian motion.

An Example

- Next, compute

$$\int_0^T \Delta_n(t)dW(t) := \sum_{j=0}^{n-1} W\left(\frac{jT}{n}\right)\left[W\left(\frac{(j+1)T}{n}\right) - W\left(\frac{jT}{n}\right)\right].$$

- By algebra, we have

$$\begin{aligned}
&2\sum_{j=0}^{n-1} W\left(\frac{jT}{n}\right)\left[W\left(\frac{(j+1)T}{n}\right) - W\left(\frac{jT}{n}\right)\right] \\
=&\sum_{j=0}^{n-1}\left[W^2\left(\frac{(j+1)T}{n}\right) - W^2\left(\frac{jT}{n}\right)\right] - \sum_{j=0}^{n-1}\left[W\left(\frac{(j+1)T}{n}\right) - W\left(\frac{jT}{n}\right)\right]^2 \qquad (2) \\
=&W^2(T) - \sum_{j=0}^{n-1}\left[W\left(\frac{(j+1)T}{n}\right) - W\left(\frac{jT}{n}\right)\right]^2.
\end{aligned}$$

An Example

- Therefore,

$$\int_0^T \Delta_n(t)dW(t) = \frac{1}{2}W^2(T) - \frac{1}{2}\sum_{j=0}^{n-1}\left[W\left(\frac{(j+1)T}{n}\right) - W\left(\frac{jT}{n}\right)\right]^2.$$

- Recall the definition of quadratic variation, we have that

$$\sum_{j=0}^{n-1}\left[W\left(\frac{(j+1)T}{n}\right) - W\left(\frac{jT}{n}\right)\right]^2 \to T.$$

- So

$$\int_0^T W(t)dW(t) = \frac{1}{2}W^2(T) - \frac{1}{2}T.$$

Itô Formula

- Ordinary integral: if $W(t)$ were differentiable, then we have the chain rule

$$df(W(t)) = f'(W(t))dW(t) = f'(W(t))W'(t)dt$$

(differential form) and

$$f(W(T)) - f(W(0)) = \int_0^T f'(W(t))dW(t) = \int_0^T f'(W(t))W'(t)dt$$

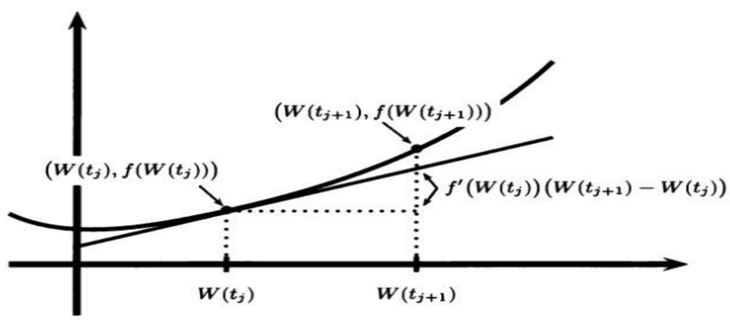
(integral form).

- Itô integral: however, $W(t)$ is non-differentiable. Then
  - $df(W(t)) = f'(W(t))dW(t)$ is incorrect;
  - $f(W(T)) - f(W(0)) = \int_0^T f'(W(t))dW(t)$ is incorrect, either.
- Question: What is the counterpart of the chain rule for Itô integral?
- We seek to derive a corresponding integral form of $f(W(T)) - f(W(0)) =?$

Itô Formula: A Heuristic Derivation

Note that

$$\begin{aligned}
&f(W(T)) - f(W(0)) \\
&= \sum_{j=0}^{n-1} [f(W(t_{j+1})) - f(W(t_j))] \\
&= \sum_{j=0}^{n-1} f'(W(t_j))\,[W(t_{j+1}) - W(t_j)] + \sum_{j=0}^{n-1} \frac{1}{2} f''(W(t_j))\,[W(t_{j+1}) - W(t_j)]^2 \\
&\quad + \text{higher order smaller error.}
\end{aligned} \tag{3}$$



Taylor approximation to $f(W(t_{j+1})) - f(W(t_j))$.

Itô Formula: A Heuristic Derivation

- Roughly speaking, if $W(t)$ were differentiable, the second term of the RHS goes to 0 as $||\Pi||$ goes to 0;
- If $W(t)$ is non-differentiable, the second term of the RHS roughly goes to $\int_0^T \frac{1}{2} f''(W(t))dt$ as $||\Pi||$ goes to 0 due to finite quadractic variation.
- Higher order small errors vanish
- **Theorem** (An Easiest Version of Itô's Formula)

$$f(W(T)) - f(W(0)) = \int_0^T f'(W(t))dW(t) + \int_0^T \frac{1}{2} f''(W(t))dt.$$

- More general versions...

Itô Formula

- **Theorem (Itô Formula for Brownian Motion)** Let $f(t,x)$ be a function for which $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are well defined and continuous. Then

$$f(T,W(T)) = f(0,W(0)) + \int_0^T f_t(t,W(t))dt + \int_0^T f_x(t,W(t))dW(t) + \frac{1}{2}\int_0^T f_{xx}(t,W(t))dt. \tag{4}$$

- Differential form:

$$df(t,W(t)) = f_t(t,W(t))dt + f_x(t,W(t))dW(t) + \frac{1}{2}f_{xx}(t,W(t))dt.$$

- **Example**: apply this theorem to $f(x) = \frac{x^2}{2}$.

Itô Formula for Itô Processes

- Motivation: we need more realistic models!
- **Definition:** An **Itô process** is a stochastic process of the form

$$X(t) = X(0) + \int_0^t \Delta(u)dW(u) + \int_0^t \Theta(u)du,$$

where $X(0)$ is non-random, $\Delta(u)$ and $\Theta(u)$ are adapted, and $E\int_0^t \Delta^2(u)du < +\infty$ and $\int_0^t |\Theta(u)|du < +\infty$ for any t.
- Differential form:

$$dX(t) = \Delta(t)dW(t) + \Theta(t)dt.$$

- **Proposition**: The quadratic variation of the Itô process $X(t)$ is

$$[X,X](t) = \int_0^t \Delta^2(u)du.$$

Itô Formula for Itô Processes

- **Definition: The integral of an adapted process** $\Gamma(t)$ w.r.t. an Itô process $X(t)$, with $E\int_0^t \Gamma^2(u)\Delta^2(u)du < +\infty$ and $\int_0^t |\Gamma(u)\Theta(u)|du < +\infty$ for any t,

$$\int_0^t \Gamma(u)dX(u) = \int_0^t \Gamma(u)\Delta(u)dW(u) + \int_0^t \Gamma(u)\Theta(u)du.$$

- Itô process is employed to describe a **gain process**!
- Differential form of the Itô Formula for an Itô process $X(t)$.
$df(t, X(t))$
$= f_t(t, X(t))dt + f_x(t, X(t))dX(t) + \frac{1}{2}f_{xx}(t, X(t))d[X, X](t)$
$= f_t(t, X(t))dt + f_x(t, X(t))\Delta(t)dW(t) + f_x(t, X(t))\Theta(t)dt + \frac{1}{2}f_{xx}(t, X(t))\Delta^2(t)dt.$

Applications of Itô Formula

Example: Generalized Geometric Brownian Motion for Modeling Stock Process

Consider $S(t) := S(0)e^{X(t)}$, where

$$dX(t) = \sigma(t)dW(t) + \left(\alpha(t) - \frac{1}{2}\sigma(t)^2\right)dt, \quad X(0) = 0$$

- $S(t)$ satisfies the following stochastic differential equation

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t) \Longleftrightarrow \frac{dS(t)}{S(t)} = \alpha(t)dt + \sigma(t)dW(t).$$

- Modeling issue: $\alpha(t)$ is the instantaneous mean rate of return, and $\sigma(t)$ is the volatility.
- When $\alpha = 0$, we get a martingale

$$S(t) = S(0)\exp\left\{\int_0^t \sigma(s)dW(s) - \frac{1}{2}\int_0^t \sigma(s)^2 ds\right\} = S(0) + \int_0^t \sigma(s)S(s)dW(s).$$

- Generalization of the exponential martingale

Applications of Itô Formula: Itô integral of a Deterministic Integrand

- What is the distribution of $I(t) = \int_0^t \Delta(s)dW(s)$, where $\Delta(s)$ is non-random (deterministic!) function of time
- Claim: $I(t)$ is normally distributed $I(t) \sim N\left(0, \int_0^t \Delta(s)^2 ds\right)$.
- We just need to prove that

$$\mathbb{E}e^{uI(t)} = \exp\left\{\frac{1}{2}u^2 \int_0^t \Delta(s)^2 ds\right\}, \text{ for all } u \in \mathbf{R}.$$

- Indeed, we observe a fact that the moment generating function of $I(t)$ satisfies

$$\exp\left\{\int_0^t u\Delta(s)dW(s) - \frac{1}{2}\int_0^t (u\Delta(s))^2 ds\right\}$$

is a martingale ($\Leftarrow$ generalized geometric Brownian motion with $\alpha = 0$ and $\sigma(s) = u\Delta(s)$)

Applications of Itô Formula: Characterizing a Brownian motion

- Recall that $W(t)$ satisfies the following three conditions
  - (1) a martingale with $M(0) = 0$;
  - (2) with continuous paths;
  - (3) with quadratic variation $[W, W](t) = t$.
- Surprisingly, conditions (1), (2) and (3) are sufficient to characterize a BM.

Theorem 4.6.4 (Lévy Theorem): Let $M(t)$ be a martingale relative to a filtration $\mathcal{F}(t)$, $t \geq 0$. Assume that $M(0) = 0$, $M(t)$ has continuous paths, and $[M, M](t) = t$ for all $t \geq 0$. Then $M(t)$ is a BM.
A rough proof: Consider a function $f(t, x)$ with partial derivatives f_t, f_x, and f_{xx} continuous. We use the following formula (an Ito formula with respect to martingales):

$$df(t, M(t)) = f_t(t, M(t))dt + f_x(t, M(t))dM(t) + \frac{1}{2}f_{xx}(t, M(t))[M, M](t).$$

Characterizing a Brownian motion

The integral form:
$f(t,M(t)) = f(0,M(0)) + \int_0^t \left[f_t(s,M(s)) + \frac{1}{2}f_{xx}(s,M(s))\right] ds + \int_0^t f_x(s,M(s))dM(s)$. Taking expectations leads to

$$Ef(t,M(t)) = f(0,M(0)) + E\int_0^t \left[f_t(s,M(s)) + \frac{1}{2}f_{xx}(s,M(s))\right] ds.$$

Select $f(t,x) = e^{ux-\frac{1}{2}u^2t}$. we can verify that

$$f_t(t,x) + \frac{1}{2}f_{xx}(t,x) = 0.$$

Therefore $Ee^{uM(t)-\frac{1}{2}u^2t} = 1$, i.e., $Ee^{uM(t)} = e^{-\frac{1}{2}u^2t}$. Thus, $M(t)$ has a normal distribution $N(0,t)$. $\square$
Note: The Lévy Theorem can be extended to the multi-dimensional case.

Multivariable Stochastic Calculus

- Recall: A d-dimensional Brownian motion is a process $W(t) = (W_1(t), W_2(t), \cdots, W_d(t))$ such that
  - Each $W_i(t)$ is a one-dimensional BM;
  - $W_i(t)$ and $W_j(t)$ are independent for any $i \neq j$;
  - Independent increments.
- Some Properties:
  - $[W_i, W_i](t) = t$;
  - $[W_i, W_j](t) = 0$ if $i \neq j$, i.e.,

$$\lim_{||\Pi||\to 0} E\left\{\left(\sum_{k=0}^{n-1}[W_i(t_{k+1}) - W_i(t_k)][W_j(t_{k+1}) - W_j(t_k)]\right)^2\right\} = 0.$$

Multivariable Stochastic Calculus

- Without loss of generality, consider a two-dimensional BM $(W_1(t), W_2(t))$.
- Consider two Itô processes
  $X(t) = X(0) + \int_0^t \Theta_1(u)du + \int_0^t \sigma_{11}(u)dW_1(u) + \int_0^t \sigma_{12}(u)dW_2(u)$
  $Y(t) = Y(0) + \int_0^t \Theta_2(u)du + \int_0^t \sigma_{21}(u)dW_1(u) + \int_0^t \sigma_{22}(u)dW_2(u)$
- The corresponding differential forms
  $dX(t) = \Theta_1(t)dt + \sigma_{11}(t)dW_1(t) + \sigma_{12}(t)dW_2(t)$
  $dY(t) = \Theta_2(t)dt + \sigma_{21}(t)dW_1(t) + \sigma_{22}(t)dW_2(t)$
- Quadratic and cross variations:

$$\begin{aligned}
dX(t)dX(t) &= (\sigma_{11}^2(t) + \sigma_{12}^2(t))dt,\\
dY(t)dY(t) &= (\sigma_{21}^2(t) + \sigma_{22}^2(t))dt,\\
dX(t)dY(t) &= (\sigma_{11}(t)\sigma_{21}(t) + \sigma_{12}(t)\sigma_{22}(t))dt.
\end{aligned}$$

Multivariable Stochastic Calculus

- **Theorem 4.6.2 (Two-dimensional Itô formula**) Let $f(t,x,y)$ be a function with partial derivatives f_t, f_x, f_y, f_{xx}, f_{xy}, and f_{yy} well defined and continuous. Consider two Itô processes $X(t)$ and $Y(t)$. Then we have

$$\begin{aligned}
&df(t, X(t), Y(t)) \qquad (5)\\
=&f_t(t,X(t),Y(t))dt + f_x(t,X(t),Y(t))dX(t) + f_y(t,X(t),Y(t))dY(t)\\
&+\frac{1}{2}f_{xx}(t,X(t),Y(t))dX(t)dX(t) + f_{xy}(t,X(t),Y(t))dX(t)dY(t)\\
&+\frac{1}{2}f_{yy}(t,X(t),Y(t))dY(t)dY(t)
\end{aligned}$$

- (**Itô product formula**).

$$d[X(t)Y(t)] = X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t).$$

Example: Correlated Stock Prices

- Two asstes with price:

$$\begin{aligned} S_1(t) &= S_1(0)\exp\left\{\sigma_1 W_1(t) + (\alpha_1 - \frac{1}{2}\sigma_1^2)t\right\} \\ S_2(t) &= S_2(0)\exp\left\{\sigma_2[\rho W_1(t) + \sqrt{1-\rho^2}W_2(t)] + (\alpha_2 - \frac{1}{2}\sigma_2^2)t\right\} \end{aligned}$$

  where $W_1(t)$ and $W_2(t)$ are two independent Brownian motions.
- Use Ito's formula, we prove that

$$\begin{aligned} \frac{dS_1(t)}{S_1(t)} &= \alpha_1 dt + \sigma_1 dW_1(t), \\ \frac{dS_2(t)}{S_2(t)} &= \alpha_2 dt + \sigma_2[\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)] \end{aligned}$$

Example: Correlated Stock Prices

For $S_2(t)$, we let $X(t) = W_1(t),\ Y(t) = W_2(t)$ and use 2-dimensional Ito's formula to find

$$dS_2(t) = df(t, X(t), Y(t)) = ?,$$

where

$$f(t,x,y) = S_2(0)\exp\left\{\sigma_2[\rho x + \sqrt{1-\rho^2}y] + (\alpha_2 - \frac{1}{2}\sigma_2^2)t\right\}.$$

Example: Correlated Stock Prices

- Denote $W_3(t) = \rho W_1(t) + \sqrt{1-\rho^2} W_2(t)$. Obviously, $W_3(t)$ is a standard Brownian motion.
- Apply Itô product formula to prove that Corr$(W_1(t), W_3(t)) = \rho$
- Thus, we may spell the joint dynamics as

$$\begin{aligned} \frac{dS_1(t)}{S_1(t)} &= \alpha_1 dt + \sigma_1 dW_1(t), \\ \frac{dS_2(t)}{S_2(t)} &= \alpha_2 dt + \sigma_2 dW_3(t) \end{aligned}$$

- The log-return satisfies that

$$\text{Corr}\left(\log \frac{S_1(t)}{S_1(0)}, \log \frac{S_2(t)}{S_2(0)}\right) = \rho$$

  i.e. the correlation btw Brownian motions is exactly that for the returns.

Supplementary Material

Suggested Reading Material (We only need to focus on the material parallel to our course slides):

- Selected material from Shreve Vol. II 4.1-4.4, 4.6
- Or equivalent material from Mikosch: Chapter 2

Suggested Exercises (some of these exercises have been included in Homework Assignment $\#5$; others are for your deeper understanding)

- Shreve Vol.II: Exercise 4.3, 4.5, 4.6, 4.7, 4.13, 4.15, 4.16, 4.17, 4.19