Econ 139 – Midterm Suggested Solutions
Spring 2018

Problem 1. True or false.
Are the following statements true or false? Explain your answer as concisely as possible. You will be graded on your explanation.

1. Consider a European put and a European call option on ABC stock with the same strike price X and expiration date T. Suppose that due to a change in market conditions, the put price increases, but the price of ABC stock and the riskfree rate r_f are unchanged. Then the call price must decrease. (6 points)

   False. By put-call parity we have:

   $$C + \frac{X}{(1+r_f)^T} = P + S,$$

   where C is the call price, P is the put price, and S is the stock price. Hence, if the put price increases, while r_f and S remain unchanged, the call price must also increase.

2. All Walrasian equilibria are Pareto efficient. (7 points)

   True. This is the first fundamental theorem of welfare economics.

3. Recall that in our intertemporal consumption problem without uncertainty, we found that the optimal consumption vector (c_0^*, c_1^*), where c_0^* represents current consumption and c_1^* represents future consumption, must satsify the first-order condition:

   $$\frac{u'(c_0^*)}{u'(c_1^*)} = \gamma(1+r),$$

   where $0 < \gamma \leq 1$ is a time discount factor and $r \geq 0$ is the saving rate. Suppose two investors have the following utility functions over consumption:

   $$u_1(c) = \frac{c^{1-\alpha}}{1-\alpha}, \qquad u_2(c) = \frac{c^{1-\beta}}{1-\beta},$$

   where $\alpha > \beta > 0$ and $\alpha, \beta \neq 1$.

   If the saving rate r increases, the consumption growth rate for investor 1 (with utility function u_1) will increase more than the consumption growth rate for investor 2 (with utility function u_2). (7 points)

False. We can write the first-order condition for investor 1 as

$$\left(\frac{c_1^*}{c_0^*}\right)^\alpha = \gamma(1+r).$$

Then we have

$$\frac{c_1^*}{c_0^*} = [\gamma(1+r)]^{1/\alpha}.$$

Taking logs gives

$$\ln(c_1^*/c_0^*) = \frac{1}{\alpha}\ln(\gamma) + \frac{1}{\alpha}\ln(1+r).$$

The derivative of $\ln(c_1^*/c_0^*)$, which is the consumption growth rate, with respect to $\ln(1+r)$ is then

$$\frac{d\ln(c_1^*/c_0^*)}{d\ln(1+r)} = \frac{1}{\alpha}.$$

Hence, the increase in the consumption growth rate due to an increase in the saving rate is inversely proportional to α for investor 1 and by a similar argument, inversely proportional to β for investor 2. Since $\alpha > \beta$, when r increases the increase in the consumption growth rate for investor 1 will be *less than* the increase for investor 2.

Problem 2. Stochastic dominance and expected utility. (20 points, 5 points each)

Consider risky assets A, B, C, and D, with probability distributions π_A, π_B, π_C, and π_D, respectively, over the payoffs $\tilde{X}$:

$\tilde{X}$	π_A	π_B	π_C	π_D
-1	0	0	0	0
1	0	0	1/6	1/9
2	1/3	1/7	0	1/9
3	0	1/7	1/6	1/9
5	0	1/7	1/6	1/9
6	1/3	1/7	0	1/9
7	0	1/7	1/6	1/9
9	0	1/7	1/6	1/9
10	1/3	1/7	0	1/9
11	0	0	1/6	1/9
$\mathbb{E}_\pi[\tilde{X}]$	6	6	6	6
$\mathbb{E}_\pi[\ln(\tilde{X})]$	1.60	1.66	1.54	1.56
$\sigma^2_\pi(\tilde{X})$	10.67	7.43	11.67	11.33

Note that none of these assets first-order stochastically dominates (FSD) another.

1. Rank these assets in terms of mean-variance dominance.

   Since $\sigma^2_{\pi_B}(\tilde{X}) < \sigma^2_{\pi_A}(\tilde{X}) < \sigma^2_{\pi_D}(\tilde{X}) < \sigma^2_{\pi_C}(\tilde{X})$ and $\mathbb{E}_{\pi_A}[\tilde{X}] = \mathbb{E}_{\pi_B}[\tilde{X}] = \mathbb{E}_{\pi_C}[\tilde{X}] = \mathbb{E}_{\pi_D}[\tilde{X}]$ we have

   $$B \succ A \succ D \succ C.$$

2. Can you find a probablity distribution over $\tilde{X}$, call it π_Z, such that:

   $$\tilde{X}_{\pi_C} = \tilde{X}_{\pi_A} + \tilde{X}_{\pi_Z}$$

   and $\mathbb{E}_{\pi_Z}[\tilde{X}] = 0$? Does asset A second-order stochastically dominate asset C? Justify your answer (do not appeal to the cumulative distribution functions of these assets).

   Let $\tilde{X}_{\pi_Z} = -1$ with probability 0.5 and $\tilde{X}_{\pi_Z} = 1$ with probability 0.5. Then asset C is a mean preserving spread of asset A. Since $\mathbb{E}_{\pi_A}[\tilde{X}] = \mathbb{E}_{\pi_C}[\tilde{X}]$ and asset C is a mean preserving spread of asset A, by theorem this is equivalent to asset A second-order stochastically dominating asset C.

3. Rank these assets for an investor whose preferences are represented by a von Neumann-Morgenstern expected utility function, with a Bernoulli utility function $u(W) = \ln(W)$.

   Since $\mathbb{E}_{\pi_B}[\ln(\tilde{X})] > \mathbb{E}_{\pi_A}[\ln(\tilde{X})] > \mathbb{E}_{\pi_D}[\ln(\tilde{X})] > \mathbb{E}_{\pi_C}[\ln(\tilde{X})]$ we have

   $$B \succ A \succ D \succ C.$$

   This is the same as our ordering by the mean-variance criterion.

4. Without doing any calculations, explain your relative ranking of assets B and D in terms of the shape of the Bernoulli utility function in point 4. What would the ranking of assets B and D be if the utility function instead was $u(W) = W^2$?

   One way to think of this is in terms of deviations from the expected value, where we can think of positive deviations as gains and negative deviations as losses. When the utility function is concave, losses cause a larger decrease in utility than the increase in utility from the the same size gains. Looking at asset D versus asset B, the largest possible gain is greater with asset D, but so is the largest possible loss. Hence, the expected utility of asset D is lower. If the utility function is instead convex, then the opposite is true. Since $u(W) = W^2$ is convex, under this utility function the expected utility of asset D will be higher than the expected utility of asset B.

Problem 3. Expected utility, risk aversion, CE, and RP. (30 points, 5 points each)

Consider an investor with initial wealth equal to $W_0 = 10$, who must choose between a stock and a bond with the following payoffs:

State	Stock	Bond	π
State 1	4	2	0.5
State 2	1	2	0.5
Price	1	1	

where π is the probability of each state occurring. Assume that this investor has a von Neumann-Morgenstern expected utility function, with a Bernoulli utility function of the constant relative risk aversion (CRRA) form:

$$u(W) = \frac{W^{1-\gamma} - 1}{1-\gamma}.$$

1. Assume the investor will purchase one share of either the stock or the bond, but not both, which asset will they prefer if they have the risk aversion coefficient $\gamma = 1/2$?

   If the investor buys the stock their final wealth will be $10 - 1 + 1 = 10$ with probability 0.5 and $10 - 1 + 4 = 13$ with probability 0.5. If the investor buys the bond their final wealth will be $10 - 1 + 2 = 11$ with probability 1. Hence, the expected utility with the stock is given by

   $$\mathbb{E}[u(W)] = 0.5\left(\frac{10^{1/2} - 1}{1/2}\right) + 0.5\left(\frac{13^{1/2} - 1}{1/2}\right) = 4.77,$$

   and the utility with the bond is given by

   $$\mathbb{E}[u(W)] = \left(\frac{11^{1/2} - 1}{1/2}\right) = 4.63.$$

   Therefore, the investor prefers the stock.

2. Assume the investor will purchase one share of either the stock or the bond, but not both, which asset will they prefer if they instead have the risk aversion coefficient $\gamma = 2$?

   In this case, the expected utility with the stock is given by

   $$\mathbb{E}[u(W)] = 0.5\left(\frac{10^{-1} - 1}{-1}\right) + 0.5\left(\frac{13^{-1} - 1}{-1}\right) = 0.9115,$$

and the utility with the bond is given by

$$\mathbb{E}[u(W)] = \left(\frac{11^{-1} - 1}{-1}\right) = 0.9091.$$

Therefore, the investor still prefers the stock.

3. With the risk aversion coefficient $\gamma = 1/2$, which asset will the investor purchase if their Bernoulli utility function is instead of the form:

$$v(W) = 4 + \frac{3(W^{1-\gamma} - 1)}{1 - \gamma}.$$

You could do the calculations again, or realize that v is an affine transformation of u, so that the preferences remain the same. Hence, the investor also prefers the stock under this utility function.

4. Under the original utility function $u(W)$ with $\gamma = 1/2$, what is the investor's certainty equivalent (CE) for the stock?

The certainty equivalent must satisfy

$$\left(\frac{(10 + CE)^{1/2} - 1}{1/2}\right) = 0.5\left(\frac{10^{1/2} - 1}{1/2}\right) + 0.5\left(\frac{13^{1/2} - 1}{1/2}\right).$$

Simplifying gives

$$CE = [0.5 \cdot 10^{1/2} + 0.5 \cdot 13^{1/2}]^2 - 10 = 1.45.$$

5. Under the original utility function $u(W)$ with $\gamma = 2$, what is the investor's CE for the stock? Did the CE increase or decrease relative to point 4? Explain.

The certainty equivalent must satisfy

$$\left(\frac{(10 + CE)^{-1} - 1}{-1}\right) = 0.5\left(\frac{10^{-1} - 1}{-1}\right) + 0.5\left(\frac{13^{-1} - 1}{-1}\right).$$

Simplifying gives

$$CE = [0.5 \cdot 10^{-1} + 0.5 \cdot 13^{-1}]^{-1} - 10 = 1.30.$$

Yes, the certainty equivalent has decreased. A higher value for γ indicates a more risk-averse investor. Hence, they will accept a smaller amount for sure, in lieu of buying the stock.

6. What is the investor's risk premium (RP) for the stock when $\gamma = 1/2$? When $\gamma = 2$? Is the RP higher when $\gamma = 1/2$ or when $\gamma = 2$? Explain.

   The risk premium is the difference between the expected net payoff of the stock and the certainty equivalent of the stock. For $\gamma = 1/2$ we have

   $$RP = 1.50 - 1.45 = 0.05,$$

   and for $\gamma = 2$ we have

   $$RP = 1.50 - 1.30 = 0.20,$$

   The risk premium is higher for $\gamma = 2$. A higher value for γ indicates a more risk-averse investor. Hence, they will be willing to pay more to eliminate the uncertainty in the payoff of the stock.

Problem 4. Portfolio allocation and mean-var utility. (30 points, 5 points each)

Consider an economy with a risk-free asset with return r_f and two risky assets, one with random return $\tilde{r}_1$, expected value $\mathbb{E}[\tilde{r}_1] = \mu_1$, and standard deviation σ_1, and the second with random return $\tilde{r}_2$, expected value $\mathbb{E}[\tilde{r}_2] = \mu_2$, and standard deviation σ_2. Let ρ_{12} denote the correlation between the two random returns on the two risky assets. Suppose that an investor forms a portfolio of these three assets by allocating the share w_1 of his or her wealth to risky asset 1, with with random return $\tilde{r}_1$, and allocating share w_2 to risky asset 2, with random return $\tilde{r}_2$, and the remaining share $1 - w_1 - w_2$ to the risk-free asset.

1. Write down a formula for the expected return μ_P of this portfolio.

$$\mu_P = (1 - w_! - w_2)r_f + w_1\mu_1 + w_2\mu_2.$$

2. Write down a formula for the variance σ_P^2 of this portfolio, in terms of w_1, w_2, σ_1, σ_2, and ρ_{12}.

$$\sigma_P^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{12}.$$

3. Suppose that the investor has a preferences defined directly over the mean and variance of the return of their portfolio, as described by the utility function:

$$U(\mu_P, \sigma_P^2) = \mu_P - \frac{A}{2}\sigma_P^2,$$

where A is a parameter that measures the investor's degree of risk aversion. Write down the first-order conditions for the investor's optimal choices w_1^* and w_2^* for the shares allocated to each of the two risky assets.

We are maximizing the following function over w_1 and w_2:

$$(1 - w_! - w_2)r_f + w_1\mu_1 + w_2\mu_2 - \frac{A}{2}(w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{12}).$$

The first order conditions are given by:

$$\begin{aligned} w_1 &: (\mu_1 - r_f) - A(w_1^*\sigma_1^2 + w_2^*\sigma_1\sigma_2\rho_{12}) = 0 \\ w_2 &: (\mu_2 - r_f) - A(w_2^*\sigma_2^2 + w_1^*\sigma_1\sigma_2\rho_{12}) = 0 \end{aligned}$$

4. Sum the first order conditions to show that w_1^* and w_2^* must satisfy the following equation:

$$S_1 + S_2 - A(1 + \rho_{12})(w_1^* \sigma_1 + w_2^* \sigma_2) = 0,$$

where S_1 and S_2 are the Sharpe ratios of assets 1 and 2, i.e.,

$$S_1 = \frac{\mu_1 - r_f}{\sigma_1}, \quad S_2 = \frac{\mu_2 - r_f}{\sigma_2}.$$

Dividing the FOC with respect to w_1 by σ_1 and the FOC with respect to w_2 by σ_2 gives

$$\begin{aligned} \frac{\mu_1 - r_f}{\sigma_1} - A(w_1^* \sigma_1 + w_2^* \sigma_2 \rho_{12}) &= 0 \\ \frac{\mu_2 - r_f}{\sigma_2} - A(w_2^* \sigma_2 + w_1^* \sigma_1 \rho_{12}) &= 0 \end{aligned}$$

or

$$\begin{aligned} S_1 - A(w_1^* \sigma_1 + w_2^* \sigma_2 \rho_{12}) &= 0 \\ S_2 - A(w_2^* \sigma_2 + w_1^* \sigma_1 \rho_{12}) &= 0. \end{aligned}$$

Then adding gives

$$S_1 + S_2 - A(w_1^* \sigma_1 + w_2^* \sigma_2 + (w_1^* \sigma_2 + w_2^* \sigma_2)\rho_{12}) = 0,$$

which finally yields

$$S_1 + S_2 - A(1 + \rho_{12})(w_1^* \sigma_1 + w_2^* \sigma_2) = 0,$$

as required.

5. Show that we can write:
$$\frac{\partial w_2^*}{\partial \rho_{12}} = -\frac{(w_1^*\sigma_1 + w_2^*\sigma_2)}{(1+\rho_{12})\sigma_2}.$$
Assume $-1 < \rho_{12} < 1$. What will happen to w_2^* if the correlation between the two risky assets, ρ_{12}, increases? Will w_2^* increase, decrease, or it depends? If it depends, what does it depend on? [Hint: there were no restrictions placed on the weights in the utility maximization problem, i.e., they could be positive or negative.]

   Letting
$$f(w_2^*, \rho_{12}) = S_1 + S_2 - A(1+\rho_{12})(w_1^*\sigma_1 + w_2^*\sigma_2) = 0,$$
we can apply the implicit function theorem to get
$$\frac{\partial w_2^*}{\partial \rho_{12}} = -\frac{\partial f/\partial \rho_{12}}{\partial f/\partial w_2^*}.$$
Applying this to our equation gives
$$\begin{aligned}\frac{\partial w_2^*}{\partial \rho_{12}} &= -\frac{-A(w_1^*\sigma_1 + w_2^*\sigma_2)}{-A(1+\rho_{12})} \\ &= -\frac{(w_1^*\sigma_1 + w_2^*\sigma_2)}{(1+\rho_{12})},\end{aligned}$$
as required. Since $-1 < \rho_{12} < 1$, the denominator is positive, hence the sign of the derivative depends on the sign of the numerator. If the numerator is positive, w_2^* will decrease if ρ_{12} increases, and if the numerator is negative, w_2^* will increase if ρ_{12} increases. The latter case makes sense if you think of the investor as being (net) short risky assets, in which case increasing w_2^* would correspond to having a smaller (net) short position.

6. Assume $w_1^*, w_2^* > 0$. Argue that if the correlation between the two risky assets, ρ_{12}, increases $w_1^* + w_2^*$ will decrease.

   If ρ_{12} increases then a combination of the risky assets becomes more risky, hence any risk averse investor will allocate fewer funds to the risky portion (represented by $w_1^* + w_2^*$) of their overall portfolio.