高阶偏导数

函数 $z=f(x,y)$ 的偏导数 $\dfrac{\partial f(x,y)}{\partial x}$，$\dfrac{\partial f(x,y)}{\partial y}$ 仍是二元函数，若它们仍可求偏导,则可求二阶、三阶等高阶偏导数.

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例

$$z = x^3 y - 3x^2 y^3$$

$$\frac{\partial z}{\partial x} = 3x^2 y - 6xy^3, \qquad \begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) &= 6xy - 6y^3, \\ \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) &= 3x^2 - 18xy^2, \end{aligned}$$

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$$\frac{\partial z}{\partial y}=x^3-9x^2y^2$$

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$$\frac{\partial z}{\partial y}=x^3-9x^2y^2\qquad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=3x^2-18xy^2,\quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=-9x^2\ .$$

引入记号:

$$\frac{\partial}{\partial x}(\frac{\partial z}{\partial x}) \triangleq \frac{\partial^2 z}{\partial x^2}$$ （z 对 x 的二阶偏导）

$$\frac{\partial}{\partial y}(\frac{\partial z}{\partial y}) \triangleq \frac{\partial^2 z}{\partial y^2}$$ （z 对 y 的二阶偏导）

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（z 对 x, y 的混合二阶偏导）

注 混合二阶偏导 $\frac{\partial^2 z}{\partial x \partial y}$ 与 $\frac{\partial^2 z}{\partial y \partial x}$ 有着求导次序的不同，

由于各种教材中记号的规定不一致，

这里我们不强调 $\frac{\partial^2 z}{\partial x \partial y}$ 到底是 $\frac{\partial}{\partial y}(\frac{\partial z}{\partial x})$ 还是 $\frac{\partial}{\partial x}(\frac{\partial z}{\partial y})$，

因为通常是 $\frac{\partial^2 z}{\partial x \partial y}$ 与 $\frac{\partial^2 z}{\partial y \partial x}$ 两个都要计算的，

而且他们两个通常都是相等的.

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通常记二阶偏导数运算符

$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ 为 Δ

即 $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

称作Laplace算子

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$$=\boxed{-(x^2+y^2+z^2)^{-\frac{3}{2}}+3x^2(x^2+y^2+z^2)^{-\frac{5}{2}}}$$

同理：

$$\frac{\partial^2 u}{\partial y^2}=\boxed{-(x^2+y^2+z^2)^{-\frac{3}{2}}+3y^2(x^2+y^2+z^2)^{-\frac{5}{2}}}$$

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通常记二阶偏导数运算符

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\text{ 为 }\Delta$$

即 $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

称作Laplace算子

$$-\Delta u=0$$

Laplace方程

两个混合二阶偏导数相同否？

定理

如果 $z=f(x,y)$ 的两个混合二阶偏导数 $\frac{\partial^2 z}{\partial x \partial y}$, $\frac{\partial^2 z}{\partial y \partial x}$ 在某区域 D 上连续则他们在此区域上相等.

证略

高阶偏导推广

三元函数$u = f(x, y, z)$的二阶偏导:

$$\frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial^2 u}{\partial z^2}, \quad \frac{\partial^2 u}{\partial x \partial y}, \quad \frac{\partial^2 u}{\partial x \partial z}, \quad \frac{\partial^2 u}{\partial y \partial z}.$$

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n元函数$z=f(x_1,\cdots,x_n)$的二阶偏导:

$$\frac{\partial^2 z}{\partial x_i\partial x_j},\quad i,j=1,\cdots,n.$$

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三元函数$u = f(x, y, z)$的二阶偏导:

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n元函数$z = f(x_1, \cdots, x_n)$的二阶偏导:

$$\frac{\partial^2 z}{\partial x_i \partial x_j}, \quad i, j = 1, \cdots, n.$$

二元函数$z = f(x, y)$的三阶偏导:

$$\frac{\partial^3 z}{\partial x^3}, \quad \frac{\partial^3 z}{\partial y^3}, \quad \frac{\partial^3 z}{\partial x^2 \partial y}, \quad \frac{\partial^3 z}{\partial x \partial y^2}.$$

二元函数$z = f(x, y)$的n阶偏导:

$$\frac{\partial^n z}{\partial x^i \partial y^j}, \quad i + j = n, \ \ i, j = 0, 1, \cdots, n.$$

二元函数$z=f(x,y)$的n阶偏导:

$$\frac{\partial^n z}{\partial x^i \partial y^j}, \quad i+j=n, \quad i,j=0,1,\cdots,n.$$

n元函数$z=f(x_1,\cdots,x_n)$的k阶偏导:

$$\frac{\partial^k z}{\partial x_1^{j_1}\cdots\partial x_n^{j_n}}, \quad j_1+\cdots+j_n=k,$$

$$j_1=0,1,\cdots,n, \quad \cdots, \quad j_n=0,1,\cdots,n.$$