Econ 139 Lecture Note 22

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1 First Fundamental Theorem of Asset Pricing

Definition: a probability measure π_θ^{RN} is said to be a risk-neutral measure if:
(i) $\pi_\theta^{RN} = 0$, $\forall\theta$
(ii)$\sum \pi_\theta^{RN} = 1$
(iii)$X_i(0) = E_{\pi^{RN}}[\frac{X_i(\theta,1)}{1+r_f}] = \sum \pi_\theta^{RN}[\frac{X_i(\theta,1)}{1+r_f}]$ $\forall i = 1, ..., M$.

2 Examples :

Example1

	Bond	Stock
State 1	1.1	5
State 2	1.1	7
P	1	4

$$\pi_1^{RN}\frac{3}{1.1} + \pi_2^{RN}\frac{7}{1.1} = 4$$

$$\pi_1^{RN} + \pi_2^{RN} = 1$$
$$\pi_1^{RN} = 0.65$$
$$\pi_2^{RN} = 0.35$$

Example2

	Bond	Stock1	Stock2
State 1	1.1	3	1
State 2	1.1	2	4
State 3	1.1	1	6
	1	2	3

$$\pi_1^{RN}\frac{3}{1.1} + \pi_2^{RN}\frac{2}{1.1} + \pi_3^{RN}\frac{1}{1.1} = 2$$

$$\pi_1^{RN}\frac{1}{1.1} + \pi_2^{RN}\frac{4}{1.1} + \pi_3^{RN}\frac{6}{1.1} = 3$$
$$\pi_1^{RN} + \pi_2^{RN} + \pi_3^{RN} = 1$$

$$\pi_1^{RN} = 0.3 \qquad \pi_2^{RN} = 0.6 \qquad \pi_3^{RN} = 0.1$$

$$\begin{vmatrix} 1.1 & 3 & 1 \\ 1.1 & 2 & 4 \\ 1.1 & 1 & 6 \end{vmatrix} = X$$

$$p = \begin{vmatrix} 1 & 2 & 3 \end{vmatrix}^T \qquad q = \begin{vmatrix} q_1 & q_2 & q_3 \end{vmatrix}^T$$

$$q = (x^T)^-1p$$

$$\sum q_i = 0.9$$

$$\sum q_i = \frac{1}{1+r_f}$$

$$\pi_\theta^{RN} = q_\theta^{RN}(1+r_f)$$

3 Incomplete Market :

	Bond	Stock
State 1	1.1	1
State 2	1.1	2
State 3	1.1	3
	1	2

$$\pi_1^{RN} + \pi_2^{RN} + \pi_3^{RN} = 1$$

$$\pi_1^{RN}\frac{1}{1.1} + \pi_2^{RN}\frac{2}{1.1} + \pi_3^{RN}\frac{3}{1.1} = 2$$

$$\pi_2^{RN} = 0.8 - 2\pi_1^{RN}(> 0)$$

$$\pi_3^{RN} = 0.2 + 2\pi_1^{RN}(> 0)$$

$$\pi_1^{RN}(> 0)$$

$$0 < \pi_1^{RN} < 4$$

$$\{(\pi_1^{RN}, \pi_2^{RN}, \pi_2^{RN}) \in (\lambda, 0.8 - 0.2\lambda, 0.2 + \lambda) : 0 < \lambda < 0.4\}$$

4 Arbitrage Opportunity :

	Bond	Stock1	Stock2
State 1	1.1	2	4
State 2	1.1	3	5
State 3	1.1	1	2.5
	1	2	3

$$\begin{vmatrix}1.1\\1.1\\1.1\end{vmatrix} + \begin{vmatrix}2\\3\\1\end{vmatrix} << \begin{vmatrix}4\\5\\2.5\end{vmatrix}$$

Proposition: suppose set of fundamental securities is arbitrage free,
then for any portfolio:

$V_0 = \frac{1}{1+r_f} E_{\pi^{RN}}[V(\theta,\ 1)]$ for any RN measure.
$V_0 = n_b X_b(0) + \sum n_i X_i(0)$
$V(\theta,\ 1) = n_b X_b(1) + \sum n_i X_i(1)$

Proof:
$$\sum \pi_\theta^{RN} \frac{n_b X_b(1) + \sum n_i X_i(\theta, 1)}{1+r_f} = n_b \sum_{n=\theta}^{N} \frac{X_b(1)}{1+r_f} \pi_\theta^{RN} + \sum_{n=\theta}^{N} n_1 \sum_{n=\theta}^{N} \pi_\theta^{RN} \frac{x_1(\theta_1, 1)}{1+r_f}$$

Proposition: The market is complete if and only if there exist a unique RN probability measure.
Proof:
Suppose the market is incomplete we get more than 1 RN measure.
Suppose there are two distinct RN measures and market is complete.
Since the market is incomplete. We must be able to construct a portfolio such that:

$$V(0) > 0 \quad and \quad V(\theta, 1) = 0 \quad if \quad \theta \neq k$$

$$V(0) > 0 \quad and \quad V(\theta, 1) = 1 \quad if \quad \theta = k$$

$$V(0) = E_\pi^{RN}[\frac{V(\theta, 1}{1+r_f}]$$

$$\pi^{RN}, \hat{\pi}^{RN} : \pi^{RN} \neq \hat{\pi}^{RN}$$

$$E_\pi^{RN}[\frac{V(\theta, 1)}{1+r_f}] = \frac{\pi_k^{RN} * 1 + (1 - \pi_k^{RN}) * 0}{1+r_f}$$

$$= \frac{\pi_k^{RN}}{1+r_f} \neq \frac{\hat{\pi}_k^{RN}}{1+r_f} = E_{\hat{\pi}^{RN}}[\frac{V(\theta,1)}{1+r_f}] = V(0)$$

$$V(0) = E_{\hat{\pi}^{RN}}[\frac{V(\theta,1)}{1+r_f}]$$

$$V(0) = E_{\pi^{RN}}[\frac{V(\theta,1)}{1+r_f}]$$

5 Arbitrage Pricing Theory:

Market Model

$$\hat{j} = \alpha_j + \beta_j(\hat{r}_M \;-\; E[\hat{r}_M]) + \epsilon_j$$

$$where: \quad E[\tilde{\epsilon_j}] = 0$$

$$cov(\hat{r}_M, \tilde{\epsilon}_j) = 0$$

$$cov(\hat{\epsilon}_j\hat{\epsilon}_k) = 0 \quad for \quad j \neq k$$

decomposes $\hat{\epsilon}_j$ into two orthogonal components:

$$\beta_j(\hat{r}_M \;-\; E[\hat{r}_M])$$

$$\tilde{\epsilon_j}: \; purely\; idiosyncratic\; component$$

Typically, we estimate $\alpha_j\; and\; \beta_j$ from a time-series regression

$$\beta_j = \frac{cov(\hat{r}_j, \hat{r}_M \;-\; E[\hat{r}_M])}{var(\hat{r}_M \;-\; E[\hat{r}_M])} = \frac{cov(\hat{r}_j, \hat{r}_M)}{var(\hat{r}_M)}$$

$$E[\hat{r}_j] = \alpha_i + \beta_j(\hat{r}_M - E[\hat{r}_M])]$$

$$\alpha_j = E[\hat{r}_j]$$

Hence. we can write:

$$\hat{r}_j = E[\hat{r}_j] = \beta_j(\hat{r}_M - E[\hat{r}_M])] = \epsilon_j$$

$$\sigma_j^2 = E[(\hat{r}_j - E[\hat{r}_j])^2] = Bi^2 * \sigma_M^2 + \sigma_{\epsilon_j}^2$$

$$Cov(\hat{r}_j, \hat{r}_k) = E[(r_j - E[r_j]) * (\hat{r}_k - E[\hat{r}_k])] = B_jB_k\sigma_M^2$$

6 MPT: to estimate cov matrix

$$N\,variances,\ \frac{N(N-1)}{2}\,covariances$$

$$N = 100\ :\ 50,50\,parameters$$

MM: N betas, 1 market variance, N idiosyncratic variances

$$B = (B_1, ..., B_N)^T$$

$$\sigma_M^2 = marketvariance$$

$$\sigma_\epsilon^2 = (\sigma_\epsilon^2, ..., \sigma_{\epsilon M}^2)$$

$$\sum_{(n*n)} = \sigma_M^2(BB^T) + diag(\sigma_\epsilon^2)$$

$$diag(\begin{vmatrix} a \\ b \\ c \end{vmatrix}) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$