## 两个特殊极限

本段内容要点:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x = e$$

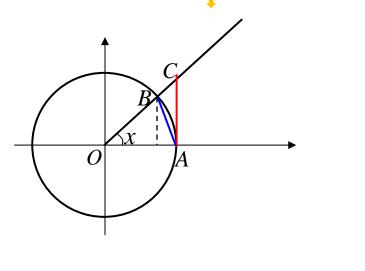
二者的若干变型及应用

$$1. \lim_{x \to 0} \frac{\sin x}{x} = 1$$

## $1. \lim_{x \to 0} \frac{\sin x}{x} = 1$

证明: 因为 $\frac{\sin x}{x}$ 是偶函数,所以左右单侧极限要么都存在并相等,要么都不存在,因而只须对 $x \to 0 + 0$ 考虑.

又因为极限过程是x o 0+0, 所以只考虑 $x \in (0, \frac{\pi}{2})$ .



 $1. \lim_{x \to 0} \frac{\sin x}{x} = 1$ 

$$O$$
 $A$ 

$$1.\lim_{x o 0}rac{\sin x}{x}=1 \ S_{ riangle OAB} \leqslant S_{AOB} \leqslant S_{ riangle AOC}$$

$$1. \lim_{x o 0} rac{\sin x}{x} = 1$$
 $S_{ riangle OAB} \leqslant S_{AOB} \leqslant S_{ riangle AOC}$ 
 $\begin{vmatrix} | & | & | \\ 1 & \sin x & \frac{1}{2}x & \frac{1}{2} \tan x \end{vmatrix}$ 

 $\Rightarrow \sin x \leqslant x \leqslant \tan x, \ x \in \left(0, \frac{\pi}{2}\right)$ 

$$\frac{1}{2}\sin x \qquad \frac{1}{2}x \qquad \frac{1}{2}\tan x$$

$$\frac{1}{2}\sin x$$
  $\frac{1}{2}\tan x$ 

$$1. \lim_{x o 0} rac{\sin x}{x} = 1$$
 $S_{ riangle OAB} \leqslant S_{AOB} \leqslant S_{ riangle AOC}$ 
 $|| \qquad || \qquad ||$ 
 $rac{1}{2} \sin x \qquad rac{1}{2} t an x$ 

从而得到:  $1 \leqslant \frac{x}{\sin x} \leqslant \frac{1}{\cos x}$ 

$$\frac{1}{2}\sin x \qquad \frac{1}{2}x \qquad \frac{1}{2}\tan x$$

$$\Rightarrow \sin x \leqslant x \leqslant \tan x, \ x \in \left(0, \frac{\pi}{-}\right)$$

$$\frac{1}{2}\sin x \qquad \frac{1}{2}x \qquad \frac{1}{2}\tan x$$

$$\Rightarrow \sin x \leqslant x \leqslant \tan x, \ x \in \left(0, \frac{\pi}{2}\right)$$

$$egin{aligned} 1. & \lim_{x o 0} rac{\sin x}{x} = 1 \ S_{ riangle OAB} \leqslant S_{AOB} \leqslant S_{ riangle AOC} \ & || & || & || \ rac{1}{2} \sin x & rac{1}{2} an x \end{aligned}$$

$$\Rightarrow \sin x \leqslant x \leqslant \tan x, \ x \in \left(0, \frac{\pi}{2}\right)$$

从而得到: 
$$1 \leqslant \frac{x}{\sin x} \leqslant \frac{1}{\cos x}$$

$$\Rightarrow \cos x \leqslant \frac{\sin x}{x} \leqslant 1, \ x \in \left(0, \frac{\pi}{2}\right)$$

令
$$x 
ightarrow 0+0$$
,则根据夹逼原理, $\lim_{x 
ightarrow 0+0} rac{\sin x}{x} = 1$ .

$$1. \lim_{x \to 0} \frac{\sin x}{x} = 1$$

例: 
$$\lim_{x \to 0} \frac{\sin 5x}{x}$$

$$1. \lim_{x \to 0} \frac{\sin x}{x} = 1$$

例: 
$$\lim_{x o 0}rac{\sin 5x}{x}$$

$$=\lim_{x\to 0}\frac{\sin 5x}{5x}\,5=5.$$

$$1. \lim_{x \to 0} \frac{\sin x}{x} = 1$$

1 例: 
$$\lim_{x o 0} rac{\sin 5x}{x}$$

$$=\lim_{x\to 0}\frac{\sin 5x}{5x}\,5=5.$$

例: 
$$\lim_{x \to \infty} x \sin \frac{1}{x}$$

1. 
$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$
 例:  $\lim_{x \to 0} \frac{\sin 5x}{x}$ 

$$x \to 0$$
  $x \to 0$   $x \to 0$   $x \to 0$ 

$$x \to 0$$
  $x$   $x \to 0$   $x$  
$$= \lim_{x \to 0} \frac{\sin 5x}{5x} \, 5 = 5.$$

$$x \rightarrow 0$$
  $x$   $x \rightarrow 0$   $x$   $\sin 5x$ 

$$x \xrightarrow{x \to 0} \frac{1}{x} = 1$$

例:  $\lim_{x \to \infty} x \sin \frac{1}{x}$   $= \lim_{x \to \infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = \lim_{t \to 0} \frac{\sin t}{t} = 1.$ 

1. 
$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$
 例:  $\lim_{x \to 0} \frac{\sin 5x}{x}$ 

$$x \to 0$$
  $x$   $x \to 0$   $x$   $= \lim_{x \to 0} \frac{\sin 5x}{5x} = 5.$ 

例:  $\lim_{x \to 0} \frac{\tan x}{x}$ 

例:  $\lim_{x \to \infty} x \sin \frac{1}{x}$   $= \lim_{x \to \infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = \lim_{t \to 0} \frac{\sin t}{t} = 1.$ 

$$x \rightarrow 0$$
  $x$   $\sin 5x$ 

$$x \to 0 \frac{1}{x}$$

$$1. \lim_{x o 0} rac{\sin x}{x} = 1$$
 例:  $\lim_{x o 0} rac{\sin 5x}{x}$ 

$$x \rightarrow 0$$
  $x$   $x \rightarrow 0$   $x$   $\sin 5x$ 

$$= \lim_{x \to 0} \frac{\sin 5x}{5x} = 5.$$

例: 
$$\lim_{x \to \infty} x \sin \frac{1}{x}$$
  $= \lim_{x \to \infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = \lim_{t \to 0} \frac{\sin t}{t} = 1.$ 

例: 
$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{x} \frac{1}{\cos x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x} = 1$$
.

$$1. \lim_{x \to 0} \frac{\sin x}{x} = 1$$

列: 
$$\lim_{x \to 0} \frac{1 - \cos x}{x^2}$$

$$1. \lim_{x \to 0} \frac{\sin x}{x} = 1$$

例: 
$$\lim_{x \to 0} \frac{1 - \cos x}{x^2}$$

$$=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}$$

$$1. \lim_{x \to 0} \frac{\sin x}{x} = 1$$

例: 
$$\lim_{x o 0} rac{1 - \cos x}{x^2}$$

$$x \to 0$$
  $x \to 0$   $x^2$ .

$$2\sin^2\frac{x}{2}$$

$$= \lim_{x \to 0} \frac{2\sin^2 \frac{x}{2}}{x^2}$$

$$=\lim_{x\to 0}\frac{1}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=\frac{1}{2}.$$

$$1. \lim_{x \to 0} \frac{\sin x}{x} = 1$$

例: 
$$\lim_{x\to 0} \frac{1-\cos x}{x^2}$$

$$2\sin^2\frac{x}{2}$$

$$=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}$$

$$=\lim_{x\to 0}\frac{1}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=\frac{1}{2}.$$

$$\lim_{x \to 0} u(x) = 0$$
,則 $\lim_{x \to 0} \frac{\sin u(x)}{\sin u(x)} = 1$ .

若
$$\lim_x u(x) = 0$$
,则 $\lim_x rac{\sin u(x)}{u(x)} = 1$ .

 $2. \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$ 

$$2.\lim_{x o \infty} \left(1 + \frac{1}{x}\right)^x = e$$
证明:

Step2: 往证 $\lim_{x o +\infty}\left(1+rac{1}{[x]}
ight)^{[x]}=e;$ 

Step3: 往证
$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e;$$
Step4: 往证 $\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$ 

$$2. \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x = e$$

证明:

Step1: 数列极限部分已证.

Step2: 往证
$$\lim_{x \to +\infty} \left(1 + \frac{1}{[x]}\right)^{[x]} = e;$$
Step3: 往证 $\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e;$ 
Step4: 往证 $\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$ 

$$2. \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x = e$$

证明: Step2: 往证
$$f(x)=\left(1+rac{1}{[x]}
ight)^{[x]}(x>1)$$
在 $x o +\infty$ 时与数列 $\left(1+rac{1}{n}
ight)^n$ 同极限.

$$s o + \infty$$
时与数列 $\left(1 + rac{1}{n}\right)^n$ 同极限.

Step2: 往证 $\lim_{x o +\infty}\left(1+rac{1}{[x]}
ight)^{[x]}=e;$ 

Step 2: 往证 
$$\lim_{x \to +\infty} \left(1 + \frac{1}{[x]}\right) = e;$$
Step 3: 往证  $\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e;$ 
Step 4: 往证  $\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$ 

$$2. \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x = e$$

$$\exists N\ s.t.\ n>N\Rightarrow \left|\left(1+rac{1}{n}
ight)\right|-e\right|从而对于任意的 $arepsilon,\ \exists X=N+1\ s.t.$$$

 $|x>X\Rightarrow \left|\left(1+rac{1}{|x|}\right)^{|x|}-e\right|<arepsilon.$ 

因为
$$\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e$$
,所以对于任意的 $\varepsilon > 0$ ,  $\exists N \ s.t. \ n > N \Rightarrow \left|\left(1 + \frac{1}{n}\right)^n - e\right| < \varepsilon$ . 从而对于任意的 $\varepsilon$ , $\exists X = N + 1 \ s.t$ .

Step 1: 往证 $\lim_{n o \infty} \left(1 + rac{1}{n}
ight)^n = e;$ Step2: 往证 $\lim_{x o +\infty} \left(1 + rac{1}{[x]}
ight)^{[x]} = e;$ 

Step3: 往证
$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e;$$
Step4: 往证 $\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$ 

$$2. \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x = e$$

 $\exists N \ s.t. \ n > N \Rightarrow \left| \left( 1 + \frac{1}{n} \right)^n - e \right| < \varepsilon.$ 从而对于任意的 $\varepsilon$ ,  $\exists X = N + 1 \ s.t.$ 

因为 $\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n=e$ , 所以对于任意的 $\varepsilon>0$ ,  $|x>X\Rightarrow \left|\left(1+rac{1}{|x|}\right)^{|x|}-e\right|<arepsilon.$ 

Step: 注证 $\lim_{n o \infty} \left(1 + \frac{1}{n}\right)^n = e;$ Step e: 注证 $\lim_{x o +\infty} \left(1 + rac{1}{|x|}
ight)^{[x]} = e;$ Step3: 往证 $\lim_{x o +\infty} \left(1 + rac{1}{x}
ight)^x = e;$ 

Step4: 往证 $\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$ 

$$2.\lim_{x o\infty}\left(1+rac{1}{x}
ight)^x=e$$
证明: Step3: 往证 $\lim_{x o+\infty}\left(1+rac{1}{x}
ight)^x=e$ 

证明: Step3: 往证 $\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$ .

Step 1: 往证 $\lim_{n o \infty} \left(1 + rac{1}{n}
ight)^n = e;$ Step e: 往证 $\lim_{x o +\infty}\left(1+rac{1}{[x]}
ight)^{[x]}=e;$ Step3: 往证 $\lim_{x o +\infty} \left(1+rac{1}{x}
ight)^x = e;$ Step4: 往证 $\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$ 

$$2. \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x = e$$

证明: Step3: 往证
$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$$
.

事实上, 
$$x \to +\infty$$
  $\left(1 + \frac{1}{x}\right) = e^{-x}$ 

证明: Step3: 往证
$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right) = e$$
. 事实上, 
$$\left(1 + \frac{1}{[x]+1}\right)^{[x]} \leqslant \left(1 + \frac{1}{x}\right)^x \leqslant \left(1 + \frac{1}{[x]}\right)^{[x]+1}$$

Step 2: 往证 
$$\lim_{x \to +\infty} \left(1 + \frac{1}{[x]}\right)^{[x]} = e;$$

Step3: 往证
$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e;$$
Step4: 往证 $\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$ 

$$2. \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x = e$$

证明: Step3: 往证
$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$$
.

事实上. Step3: 性证 
$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right) = e^{-\frac{1}{x}}$$

事实上,
$$\left(1 + \frac{1}{[x]+1}\right)^{[x]} \le \left(1 + \frac{1}{x}\right)^x \le \left(1 + \frac{1}{[x]}\right)^{[x]+1}$$

$$\overline{\prod}_{x \to +\infty} \left(1 + \frac{1}{[x]+1}\right)^{[x]}$$

$$= \lim_{x \to +\infty} \left(1 + \frac{1}{[x]+1}\right)^{[x]+1} / \left(1 + \frac{1}{[x]+1}\right) =$$

 $\overline{\mathbb{I}}\lim_{x o +\infty} \left(1 + \frac{1}{|x|+1}\right)^{[x]}$  $=\lim_{x \to +\infty} \left(1 + \frac{1}{[x]+1}\right)^{[x]+1} / \left(1 + \frac{1}{[x]+1}\right) = e^{-\frac{1}{|x|+1}}$  Step 1: 往证 $\lim_{n o \infty} \left(1 + rac{1}{n}
ight)^n = e;$ Step e: 注证 $\lim_{x o +\infty} \left(1 + rac{1}{[x]}
ight)^{[x]} = e;$ 

Step3: 往证 $\lim_{x o +\infty} \left(1 + rac{1}{x}
ight)^x = e;$ Step4: 往证  $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$ 

$$2. \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x = e$$

证明: Step 3: 往证
$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$$
.

证明: Step3: 往证
$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right) = e$$
. 事实上,  $\left(1 + \frac{1}{x}\right)^{[x]} \leqslant \left(1 + \frac{1}{x}\right)^x \leqslant \left(1 + \frac{1}{x}\right)^x$ 

$$\left(1+\frac{1}{[x]+1}\right)^{[x]}\leqslant \left(1+\frac{1}{x}\right)^x\leqslant \left(1+\frac{1}{[x]}\right)^{[x]+1}$$

$$\lim_{x \to +\infty} \left( 1 + \frac{1}{[x]} \right)^{[x]+1}$$

$$= \lim_{x \to +\infty} \left( 1 + \frac{1}{[x]} \right)^{[x]} \cdot \left( 1 + \frac{1}{[x]} \right) = e \cdot 1 = e.$$

$$= \lim_{x \to +\infty} \left(1 + \frac{1}{[x]}\right)^{[x]} \cdot \left(1 + \frac{1}{[x]}\right) = e \cdot 1 = e.$$

Step 1: 往证 $\lim_{n o \infty} \left(1 + rac{1}{n}
ight)^n = e;$ Step e: 注证 $\lim_{x o +\infty} \left(1 + rac{1}{[x]}
ight)^{[x]} = e;$ Step3: 往证 $\lim_{x o +\infty} \left(1 + rac{1}{x}
ight)^x = e;$ Step4: 往证 $\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$ 

$$2. \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x = e$$

证明: Step 3: 往证
$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$$
.

事实上, 
$$\left(1+\frac{1}{[x]+1}\right)^{[x]}\leqslant \left(1+\frac{1}{x}\right)^x\leqslant \left(1+\frac{1}{[x]}\right)^{[x]+1}$$

实上, 
$$1$$
  $1$   $1$   $1$   $1$   $1$   $1$   $1$   $1$ 

Step: 注证 $\lim_{n o \infty} \left(1 + \frac{1}{n}\right)^n = e;$ 

Step 
$$:$$
 注证 $\lim_{x o +\infty} \left(1 + rac{1}{[x]}
ight)_x^{[x]} = e;$ 

Step3: 往证 $\lim_{x o +\infty} \left(1 + rac{1}{x} \right)^x = e;$ Step4: 往证 $\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$ 

根据夹逼定理, 
$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^{\bar{x}} = e$$

$$2. \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x = e$$

证明: Step3: 往证
$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$$
.

事实上,
$$\left(1+rac{1}{x}
ight)=e.$$

实上, 
$$1 \quad {\overset{[x]}{\xrightarrow}} \quad (1 + x)$$

$$egin{aligned} eta_{\mathrm{Step 3:}} lpha egin{aligned} \dot{\Xi} \lim_{x o +\infty} \left(1+rac{1}{x}
ight) &= e. \ egin{aligned} egin{aligned} \dot{\Xi} \lim_{x o +\infty} \left(1+rac{1}{x}
ight) &= e. \end{aligned}$$

Step 1: 往证 $\lim_{n o\infty}\left(1+rac{1}{n}
ight)^n=e;$ 

Step 2: 注证
$$\lim_{x o +\infty} \left(1 + rac{1}{[x]}
ight)^{[x]} = e;$$

Step 4: 往证 $\lim_{x o +\infty} \left(1 + rac{1}{x}
ight)^x = e;$ Step 4: 往证 $\lim_{x o -\infty} \left(1 + rac{1}{x}
ight)^x = e$ 

$$+rac{1}{[x]+1}$$
  $\leqslant \left(1+rac{1}{x}
ight)$   $\leqslant \left(1+rac{1}{[x]}
ight)$  Step 4: 往证 $\lim_{x o -\infty}\left(1+rac{1}{x}
ight)^x=$ 

根据夹逼定理, 
$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$$

Step4: 往证
$$\lim_{x o -\infty} \left(1 + rac{1}{x}\right)^x = \epsilon$$

$$2. \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x = e$$

近時: Step4: 
$$\lim_{x o -\infty} \left(1 + rac{1}{x}
ight)^x$$

Step I: 往证 
$$\lim_{n o \infty} \left(1 + \frac{1}{n}\right)^n = e;$$
Step II: 往证  $\lim_{x o +\infty} \left(1 + \frac{1}{[x]}\right)^{[x]} = e;$ 

Step 3: 往证 $\lim_{x o +\infty} \left(1 + rac{1}{x}
ight)^x = e;$ Step 4: 往证 $\lim_{x o -\infty} \left(1 + rac{1}{x}
ight)^x = e$ 

Step4: 往证
$$\lim_{x o -\infty} \left(1 + rac{1}{x}
ight)^x =$$

$$2. \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$$
Step 4: 注证  $\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{t \to +\infty} \left(1 - \frac{1}{t}\right)^{-t}$ 
Step 2: 注证  $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$ ;
Step 3: 注证  $\lim_{x \to +\infty} \left(1 + \frac{1}{|x|}\right)^n = e$ ;
Step 4: 注证  $\lim_{x \to +\infty} \left(1 + \frac{1}{|x|}\right)^n = e$ ;
Step 4: 注证  $\lim_{x \to +\infty} \left(1 + \frac{1}{|x|}\right)^n = e$ ;
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Step 4: 注证  $\lim_{x \to +\infty} \left(1 + \frac{1}{|x|}\right)^n = e$ ;
Step 5: 注证  $\lim_{x \to +\infty} \left(1 + \frac{1}{|x|}\right)^n = e$ ;
Step 6: 注证  $\lim_{x \to +\infty} \left(1 + \frac{1}{|x|}\right)^n = e$ ;
Step 7: 注证  $\lim_{x \to +\infty} \left(1 + \frac{1}{|x|}\right)^n = e$ ;
Step 9: 注证  $\lim_{x \to +\infty} \left(1 + \frac{1}{|x|}\right)^n = e$ ;
Step 9: 注证  $\lim_{x \to +\infty} \left(1 + \frac{1}{|x|}\right)^n = e$ ;



$$2. \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x = e$$
证明:

Step4: 
$$\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{t \to +\infty} \left(1 - \frac{1}{t}\right)^{-t}$$

证明: 
$$\sup_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{t \to +\infty} \left(1 - \frac{1}{t}\right)^{-t}$$
 
$$\lim_{t \to +\infty} \left(\frac{t}{t-1}\right)^t = \lim_{t \to +\infty} \left(1 + \frac{1}{t-1}\right)^t$$
 Step 2: 注证  $\lim_{x \to +\infty} \left(1 + \frac{1}{[x]}\right)^{[x]} = e;$  Step 3: 注证  $\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e;$  Step 4: 注证  $\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e;$ 

$$= \lim_{t \to +\infty} \left(1 + \frac{1}{t-1}\right)^{t-1} \cdot \left(1 + \frac{1}{t-1}\right) = e \cdot 1 = e.$$

$$\lim_{t \to \infty} \left(1 + rac{1}{t-1}
ight)^{t-1} \cdot \left(1 + rac{1}{t-1}
ight) = e \cdot 1 = e$$

Step 2: 往证 
$$\lim_{x \to +\infty} \left(1 + \frac{1}{[x]}\right)^{[x]} = e$$

$$2. \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x = e$$

证明: 
$$\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{t \to +\infty} \left(1 - \frac{1}{t}\right)^{-t}$$

$$\overline{\mathbb{I}}\lim_{t \to +\infty} \left( \frac{t}{t-1} \right)^t = \lim_{t \to +\infty} \left( 1 + \frac{1}{t-1} \right)^t$$

 $= \lim_{t \to +\infty} \left(1 + \frac{1}{t-1}\right)^{t-1} \cdot \left(1 + \frac{1}{t-1}\right) = e \cdot 1 = e.$ 

Step 4: 往证 
$$\lim_{x \to +\infty} \left(1 + \frac{1}{[x]}\right)^{[x]} = e;$$
Step 4: 往证  $\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e;$ 
Step 4: 往证  $\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$ 

Step 4: 往证
$$\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$$

例:  $\lim_{x o\infty}\left(1+rac{k}{x}
ight)^x=e^k$ . 其中 $k\in\mathbb{R}$ 

•

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$$\lim_{x o\infty}\left(1+rac{k}{x}
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$$\lim_{x o\infty}\left(1+rac{a}{x}
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ight)^{bx}=e^{ab},\,orall a,b\in\mathbb{R}.$$

$$\lim_{x \to \infty} \left( 1 - \frac{1}{x} \right)^x = e^{-1}, \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^{-x} = e^{-1}$$

(推广) (1) 若 $\lim_x u(x) = +\infty \ or \ -\infty$ ,

则
$$\displaystyle \lim_x \left(1 + rac{1}{u(x)}
ight)^{u(x)} = e_;$$

(推广) (1) 若 $\lim_{x \to \infty} u(x) = +\infty \ or \ -\infty$ ,

则
$$\lim_x \left(1+rac{1}{u(x)}
ight)^{u(x)}=e;$$

(2) 若u(x)是某极限过程中的无穷大量,则在此极限过程中, $\lim_x \left(1+rac{1}{u(x)}
ight)^{u(x)}=e$ .

例:
$$\lim_{x o 0}(1+x)\overline{x}=e;$$

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$$\lim_{x \to 0} (1+x)x = 0$$

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$$\frac{1}{\ln(1+x)x} = e_1$$

若 $\lim_x u(x) = 0$ ,则 $\lim_x (1 + u(x))^{\overline{u(x)}} = e$ .

例:
$$\lim_{x o 0}(1+x)\overline{x}=e;$$

$$\lim_{x \to 0} (1+x)\overline{x} = e;$$

$$\lim_{x \to 0} (1+x)x = e;$$

若 $\lim_x u(x) = 0$ ,则 $\lim_x (1 + u(x))^{\frac{1}{u(x)}} = e$ .

例如
$$\lim_{x \to 0} (1 + \sin x)^{\frac{1}{\sin x}} = e$$

例: 
$$\lim_{x o \infty} \left(1 + rac{1}{x}
ight)^{x+10} =$$

例: 
$$\lim_{x o\infty}\left(1+rac{1}{x}
ight)^{x+10}=$$

 $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x \left(1 + \frac{1}{x}\right)^{10} = e \cdot 1 = e.$ 

$$\ell = 1 \setminus n^2$$

所以 $\exists N \ s.t. \ n > N \Rightarrow \left(1 + \frac{1}{n}\right)^n > 2.$ 

例: 
$$\lim_{n o\infty}\left(1+rac{1}{n}
ight)^{n^2}$$

事实上, 因为
$$\lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = e$$
.

例: 
$$\lim_{n o\infty}\left(1+rac{1}{n}
ight)^{n^2}$$

$$n \to \infty$$
  $\binom{1}{n}$   $\binom{1}{n}$ 

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$

例: 
$$\lim_{n o \infty} \left(1 + \frac{1}{n}\right)$$

例: 
$$\lim_{n o\infty}\left(1+rac{1}{n}
ight)$$

事实上, 因为 $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$ .

而  $\lim 2^n = +\infty$ ,

 $n \rightarrow \infty$ 

从而  $n>N\Rightarrow \left(1+rac{1}{n}
ight)^{n^2}>2^n$ ,

所以 $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n^2} = +\infty$ 

所以 $\exists N \ s.t. \ n > N \Rightarrow \left(1 + \frac{1}{n}\right)^n > 2.$ 

例: 
$$\lim_{n o\infty}\left(1+rac{1}{n}
ight)^{n^2}$$

$$\lim_{n o\infty}\left(1+rac{1}{n}
ight)^{n^2}=\lim_{n o\infty}e^n=+\infty$$

例: 
$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n^2}$$
  $= \lim_{n \to \infty} e^n = +\infty$ 

例: 
$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n^2}$$
 
$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n^2} = \lim_{n \to \infty} e^n = +\infty$$

$$\lim_{n o \infty} \left(1 + rac{1}{n}
ight)^n = \lim_{n o \infty} 1^n = 1$$

例:  $\lim_{x o 0} rac{\sin^2 lpha x - \sin^2 eta x}{x \sin x}$ 

## 例: $\lim_{x \to 0} \frac{\sin^2 \alpha x - \sin^2 \beta x}{x \sin x}$

$$x \rightarrow 0$$
  $x \sin x$ 

原式= 
$$\lim_{x \to 0} \frac{\sin^2 \alpha x}{x \sin x} - \lim_{x \to 0} \frac{\sin^2 \beta x}{x \sin x} =$$

例: 
$$\lim_{x o 0} rac{\sin^2 lpha x - \sin^2 eta x}{x \sin x}$$

$$x \rightarrow 0$$
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原式= 
$$\lim_{x \to 0} \frac{\sin^2 \alpha x}{x \sin x} - \lim_{x \to 0} \frac{\sin^2 \beta x}{x \sin x} =$$

 $\lim_{x\to 0}\frac{\sin\alpha x}{x}\lim_{x\to 0}\frac{\sin\alpha x}{\sin x}-\lim_{x\to 0}\frac{\sin\beta x}{x}\lim_{x\to 0}\frac{\sin\beta x}{\sin x}$ 

例: 
$$\lim_{x o 0} rac{\sin^2 lpha x - \sin^2 eta x}{x \sin x}$$

$$x 
ightarrow 0$$
  $x \sin x$ 

 $= \alpha^2 - \beta^2$ 

$$x\rightarrow 0$$
  $x\sin x$ 

$$x \rightarrow 0$$
  $x \sin x$ 

列: 
$$\lim_{x o 0} \overline{x \sin x}$$

$$\lim_{x o 0}rac{\sin x}{x\sin x}$$

原式=  $\lim_{x\to 0} \frac{\sin^2 \alpha x}{x \sin x} - \lim_{x\to 0} \frac{\sin^2 \beta x}{x \sin x} =$ 

$$\frac{\sin^2 \alpha x - \sin^2 \beta x}{x \sin x}$$

 $\lim_{x\to 0}\frac{\sin\alpha x}{x}\lim_{x\to 0}\frac{\sin\alpha x}{\sin x}-\lim_{x\to 0}\frac{\sin\beta x}{x}\lim_{x\to 0}\frac{\sin\beta x}{\sin x}$ 

例:  $\lim_{x o 0} rac{\cos x - \cos 3x}{x^2}$ 

例:  $\lim_{x \to 0} \frac{\cos x - \cos 3x}{x^2}$ 

$$= \lim_{x \to 0} \frac{2 \sin x \sin 2x}{x^2}$$

例: 
$$\lim_{x \to 0} \frac{\cos x - \cos 3x}{x^2}$$

$$= \lim_{x \to 0} \frac{2\sin x \sin 2x}{x^2}$$

$$= \lim_{x \to 0} \frac{2\sin x}{x} \lim_{x \to 0} \frac{\sin 2x}{x} = 4.$$

例:  $\lim_{x o\infty}\left(rac{x^2+1}{x^2-2}
ight)^{x^2}$ 

$$)$$
 $x^2$ 

例: 
$$\lim_{x o\infty}\left(rac{x^2+1}{x^2-2}
ight)^{x^2} = \lim_{x o\infty}\left(1+rac{3}{x^2-2}
ight)^{x^2}$$

例:  $\lim_{x o \infty} \left( \frac{x^2+1}{x^2-2} \right)^{x^2} = \lim_{x o \infty} \left( 1 + \frac{3}{x^2-2} \right)^{x^2}$ 

 $=\lim_{x o\infty}\left(1+rac{3}{x^2-2}
ight)^{rac{x^2-2}{3}\cdotrac{3x^2}{x^2-2}}$ 

例: 
$$\lim_{x o\infty}\left(rac{x^2+1}{x^2-2}
ight)^{x^2} = \lim_{x o\infty}\left(1+rac{3}{x^2-2}
ight)^{x^2}$$

 $=\lim_{x o\infty}\left(1+rac{3}{x^2-2}
ight)^{rac{x^2-2}{3}\cdotrac{3x^2}{x^2-2}}$ 

 $=\lim_{x o\infty}\left[\left(1+rac{3}{x^2-2}
ight)^{rac{3x^2}{3}}
ight]^{rac{3x^2}{x^2-2}}$ 

例: 
$$\lim_{x o\infty}\left(rac{x^2+1}{x^2-2}
ight)^{x^2} = \lim_{x o\infty}\left(1+rac{3}{x^2-2}
ight)^{x^2}$$

 $=\lim_{x o\infty}\left|\left(1+rac{3}{x^2-2}
ight)^{rac{x^2-2}{3}}
ight|^{\lim\limits_{x o\infty}rac{3x^2}{x^2-2}}=e^3.$ 

 $= \lim_{x \to \infty} \left( 1 + \frac{3}{x^2 - 2} \right)^{\frac{x^2 - 2}{3} \cdot \frac{3x^2}{x^2 - 2}}$ 

 $=\lim_{x o\infty}\left[\left(1+rac{3}{x^2-2}
ight)^{rac{3x^2}{3}}
ight]^{rac{3x^2}{x^2-2}}$ 

例:  $\lim_{x \to \pi} \frac{\sin x}{x - \pi}$ 

例: 
$$\lim_{x \to \pi} \frac{\sin x}{x - \pi}$$

$$=\lim_{t\to 0}\frac{-\sin t}{t}=-1$$

$$(t = x - \pi)$$

例:  $\lim_{x \to 0} \frac{\tan 2x}{x}$ 

例:  $\lim_{x \to 0} \frac{\tan 2x}{x}$ 

$$=\lim_{x\to 0}\frac{\sin 2x}{x\cos 2x}$$

例:  $\lim_{x \to 0} \frac{\tan 2x}{x}$ 

$$=\lim_{x\to 0}\frac{\sin 2x}{x\cos 2x}$$

$$=\lim_{x\to 0}\frac{\sin 2x}{x}\lim_{x\to 0}\frac{1}{\cos 2x}=2$$

例:  $\lim_{x \to 0+0} \frac{x}{\sqrt{1-\cos x}}$ 

例:  $\lim_{x \to 0+0} \frac{x}{\sqrt{1-\cos x}}$ 

$$= \lim_{x \to 0+0} \frac{x}{\sqrt{2}\sin\frac{x}{2}} = \sqrt{2}.$$

例: 
$$\lim_{x \to 0+0} \frac{x}{\sqrt{1-\cos x}}$$

$$= \lim_{x \to 0+0} \frac{x}{\sqrt{2}\sin\frac{x}{2}} = \sqrt{2}.$$

例: 
$$\lim_{x \to \infty} \frac{\sin x}{x} = 0$$

例:  $\lim_{x \to \infty} x \sin \frac{1}{3x}$ 

例:  $\lim_{x \to \infty} x \sin \frac{1}{3x}$ 

$$=\lim_{x\to\infty}\frac{\sin\frac{1}{3x}}{\frac{1}{x}}=\lim_{t\to0}\frac{\sin t}{3t}=\frac{1}{3}$$

## 本段知识要点

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

$$\lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x = e$$

注意它们的各种变型









