**Econ 240A Econometrics** **Fall 2018**

# Problem Set 4 Solutions

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## 1. Normal testing: one-sided

(a) Write the density in the canonical form of the exponential family

$$\begin{aligned} f_\theta(x) &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i-\theta)^2}{2\sigma^2}\right) \\ &= \prod_{i=1}^n \left\{ \exp\left(\frac{\theta}{\sigma^2}x_i - \frac{\theta^2}{2\sigma^2}\right) \exp\left(-\frac{1}{2\sigma^2}x_i^2\right) \frac{1}{\sqrt{2\pi\sigma^2}} \right\} \\ &= \exp\left(\frac{\theta}{\sigma^2}\sum_{i=1}^n x_i - \frac{n\theta^2}{2\sigma^2}\right) \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2\right) \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}}. \end{aligned}$$

$T(X) = \sum_{i=1}^n X_i$ is the sufficient statistic. For any $0 \le \theta_1 < \theta_2$,

$$\frac{f_{\theta_2}(x)}{f_{\theta_1}(x)} = \exp\left(\frac{\theta_2-\theta_1}{\sigma^2}\sum_{i=1}^n x_i - \frac{n(\theta_2^2-\theta_1^2)}{2\sigma^2}\right)$$

is nondecreasing in $T(x) = \sum_{i=1}^n x_i$. This means the the family of distribution $\{f_\theta : \theta \ge 0\}$ has monotone likelihood ratio w.r.t. $T(x) = \sum_{i=1}^n x_i$. The UMP level-$\alpha$ test is therefore

$$\varphi(x) = \begin{cases} 1 & T(x) > c_\alpha \\ 0 & T(x) \le c_\alpha \end{cases}.$$

The threshold $c_\alpha$ satistifies

$$\begin{aligned} &\alpha = \beta(0) = P_0(T(X) > c_\alpha) = 1 - \Phi(\frac{1}{\sqrt{n}\sigma}c_\alpha) \qquad \Big(T(X) \sim \mathrm{N}(n\theta, n\sigma^2)\Big) \\ \Longrightarrow\ & c_\alpha = \sqrt{n}\sigma\Phi^{-1}(1-\alpha) = \sqrt{n}\sigma z_\alpha \qquad (z_\alpha := \Phi^{-1}(1-\alpha)). \end{aligned}$$

Hence, the rejection region is

$$R = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i > \sqrt{n}\sigma z_\alpha\}.$$

Another way to write it is

$$R = \{x \in \mathbb{R}^n : \frac{\bar{x}}{\sigma/\sqrt{n}} > z_\alpha\}.$$

(b) The power function of this test is

$$\begin{aligned}\beta_n(\theta) = P_\theta\left(X \in R\right) = P_\theta\left(\frac{\bar{X}}{\sigma/\sqrt{n}} > z_\alpha\right) &= P_\theta\left(\frac{\bar{X}-\theta}{\sigma/\sqrt{n}} > z_\alpha - \frac{\theta}{\sigma/\sqrt{n}}\right)\\
&= P_\theta\left(Z > z_\alpha - \frac{\theta}{\sigma/\sqrt{n}}\right) \qquad \left(\text{here } Z := \frac{\bar{X}-\theta}{\sigma/\sqrt{n}} \sim \mathrm{N}(0,1)\right)\\
&= 1 - \Phi\left(z_\alpha - \frac{\theta}{\sigma/\sqrt{n}}\right).\end{aligned}$$

$\beta_n(\theta)$ is increasing in $n$ and decreasing in $\sigma^2$.

(c)

$$\begin{aligned}\gamma(\theta) &= \lim_{n\to\infty} \beta_n(\theta)\\
&= \lim_{n\to\infty} 1 - \Phi\left(z_\alpha - \frac{\theta}{\sigma/\sqrt{n}}\right)\\
&= \begin{cases} \alpha & \theta = 0 \\ 1 & \theta > 0 \end{cases}.\end{aligned}$$

The limit of type I error is $\alpha$; the limit of type II error is 0. See Figure 1.

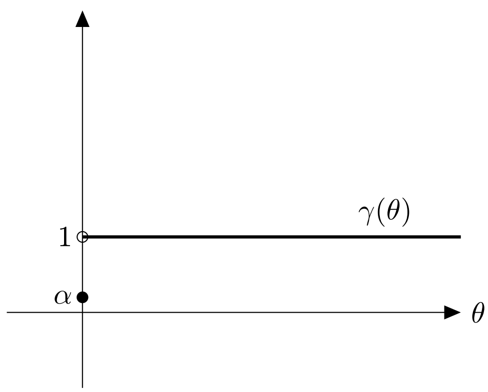


Figure 1: The limit of power $\gamma(\theta) = \lim_{n\to\infty} \beta_n(\theta)$

(d) To make $\beta_n(1) \geq 0.95$,

$$\begin{aligned}0.95 \leq \beta_n(1) &= 1 - \Phi\left(z_{0.05} - \frac{\sqrt{n}\cdot 1}{\sigma}\right)\\
\Longrightarrow \Phi\left(z_{0.05} - \frac{\sqrt{n}}{\sigma}\right) &\leq 0.05\\
\Longrightarrow \sqrt{n} &\geq \sigma\left(z_{0.05} - \Phi^{-1}(0.05)\right)\\
\Longrightarrow n &\geq \sigma^2\left(z_{0.05} - (-z_{0.05})\right)^2\\
\Longrightarrow n^* &= \left\lceil 4 \cdot z_{0.05}^2 \sigma^2 \right\rceil\\
\Longrightarrow n^* &\approx \left\lceil 10.82217 \cdot \sigma^2 \right\rceil.\end{aligned}$$

$n^*$ is roughly linear in $\sigma^2$.

(e) Denote the 1-$\alpha$ confidence set dual to the family of tests by $S(X)$.

$$\begin{aligned} S(X) &= \{\theta \geq 0 : \varphi_\theta(X) = 0\} \\ &= \left\{\theta \geq 0 : \frac{\bar{X} - \theta}{\sigma/\sqrt{n}} \leq \Phi^{-1}(1-\alpha) = z_\alpha\right\} \\ &= \left\{\theta \geq 0 : \theta \geq \bar{X} - \frac{\sigma}{\sqrt{n}} z_\alpha\right\} \\ &= \left[\max\left\{0, \bar{X} - \frac{\sigma}{\sqrt{n}} z_\alpha\right\}, \infty\right). \end{aligned}$$

**2. * Normal testing: two-sided**

**3. Normal Testing: nuisance parameter**

(a) Let's calculate the power function

$$\begin{aligned} \beta(\theta_0) &= P_{\theta_0}\left(\frac{|\bar{X} - \theta_0|}{\sqrt{S^2}/\sqrt{n}} > t_{n-1,\frac{\alpha}{2}}\right) \\ &= P_{\theta_0}\left(\frac{\frac{|\bar{X}-\theta_0|}{\sigma/\sqrt{n}}}{\frac{\sqrt{S^2}/\sqrt{n}}{\sigma/\sqrt{n}}} > t_{n-1,\frac{\alpha}{2}}\right) \\ &= P_{\theta_0}\left(\left|\frac{\frac{\bar{X}-\theta_0}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{\sigma^2}\frac{1}{n-1}}}\right| > t_{n-1,\frac{\alpha}{2}}\right) \\ &= P_{\theta_0}\left(\left|\frac{Z}{\sqrt{Y/(n-1)}}\right| > t_{n-1,\frac{\alpha}{2}}\right) \\ Z &= \frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} \sim \mathrm{N}(0,1),\ Y = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1},\ Z \perp\!\!\!\perp Y \end{aligned}$$

$$\frac{Z}{\sqrt{Y/(n-1)}} \sim t_{n-1}.$$

We use the property of normal distribution (Theorem 5.3.1 of Casella and Berger) and the definition of $t$ distribution in the derivation. Hence,

$$\begin{aligned} \beta(\theta_0) &= P_{\theta_0}\left(\left|\frac{Z}{\sqrt{Y/(n-1)}}\right| > t_{n-1,\frac{\alpha}{2}}\right) = \frac{\alpha}{2} + \frac{\alpha}{2} \\ &= \alpha. \end{aligned}$$

(b) *

**4. Likelihood ratio test**

(a) **Method 1**

Write the density in the canonical form of an exponential family

$$p_\theta(x) = \exp\{-\theta x - (-\log\theta)\}\,\mathbf{1}(x > 0).$$

This family has monotone likelihood ratio w.r.t. $-x$. Hence, the UMP level-$\alpha$ test for this one-sided testing problem (which is also the likelihood ratio test) is

$$\varphi_\alpha(x) = \begin{cases} 1 & -x > c_\alpha \\ 0 & -x \le c_\alpha \end{cases}.$$

The threshold $c_\alpha$ satisfies

$$\begin{aligned} & \alpha = \beta(1) = P_1(-X > c_\alpha) = P_1(X < -c_\alpha) = 1 - e^{c_\alpha} \\ \implies & c_\alpha = \log(1-\alpha). \end{aligned}$$

The test is

$$\varphi_\alpha(x) = \begin{cases} 1 & -x > \log(1-\alpha) \\ 0 & -x \le \log(1-\alpha) \end{cases}.$$

**Method 2**
As before, we consider the test statistic

$$\begin{aligned} -\log\lambda(X) &= \sup_{\theta\in[1,\infty)} \log L_X(\theta) - \sup_{\theta=1} \log L_X(\theta) \\ &= \begin{cases} 0 & X \ge 1 \\ -1 - \log X + X & X < 1 \end{cases} \\ -\log\lambda(X) > c' &\iff 0 < X < c \\ c' > 0,\ 0 < c < 1 & \text{ are some constant.} \end{aligned}$$

The equivalence holds because $-1-\log x + x$ is increasing in $x$.The rejection region of the likelihood ratio test is therefore $R = \{x : 0 < x < c\}$. For any $\alpha \le 1 - e^{-1}$,

$$\begin{aligned} & \alpha = \beta(1) = P_1(X < c) = 1 - e^{-c} \\ \implies & c = -\log(1-\alpha) \end{aligned}$$

The level-$\alpha$ likelihood ratio test for $\alpha \le 1 - e^{-1}$ is hence

$$\varphi_\alpha(x) = \begin{cases} 1 & x < -\log(1-\alpha) \\ 0 & x \ge -\log(1-\alpha) \end{cases}.$$

**5. * p-value**