Econ 240A, Fall 2018
Problem Set 1

Due date: Wednesday, Sept. 5

Review of axioms of probability, random variables, distribution functions, cdf, pdf, pms, conditional probability, marginal and joint distribution, independence, transformation, expectation, preview of Markov inequality.

1. **Conditional probability**

   (a) Prove each of the following statements assuming that any conditioning event has positive probability.

   i. If $P(B) = 1$, then $P(A \cap B) = P(A)$ and $P(A|B) = P(A)$ for any event A.
   Hint: $P(A) = P(A \cap B) + P(A \cap B^c)$.

   ii. If $A \subset B$, then $P(B|A) = 1$ and $P(A|B) = P(A)/P(B)$.

   iii. If $A \cap B = \emptyset$, then $P(A|A \cup B) = \frac{P(A)}{P(A)+P(B)}$.

   iv. $P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C)$.

   (b) Suppose $(\Omega, \mathcal{F}, P)$ is a probability space. (Remember the notation: Ω is a sample set; $\mathcal{F}$ is a σ-algebra on Ω; P is a probability function with domain $\mathcal{F}$.) $B \in \mathcal{F}$ is an event with $P(B) > 0$. Prove $P(\cdot|B) : \mathcal{F} \to \mathbb{R}$ is a probability function by verifying it satisfies the axioms of probability.

2. **Boole's inequality and Bonferroni's method**

   (a) Suppose $(\Omega, \mathcal{F}, P)$ is a probability space. $A_n \in \mathcal{F},\ n = 1, 2, \ldots$, is a sequence of events. Show that

   i. If $A_n \uparrow A$, then $P(A_n) \uparrow P(A)$. Here $A_n \uparrow A$ means $A_n \subset A_{n+1},\ \forall n \in \mathbb{N}$ and $A = \cup_{n=1}^{\infty} A_n$; $P(A_n) \uparrow P(A)$ means $P(A_n) \leq P(A_{n+1}),\ \forall n \in \mathbb{N}$ and $P(A) = \lim_{n\to\infty} P(A_n)$.
   Hint: Consider sets $B_n = A_n \backslash A_{n-1}$. This is the disjointification trick.

   ii. $P(\cup_{n=1}^{\infty} A_n) \leq \sum_{i=1}^{\infty} P(A_i)$ for any sets $A_1, A_2, \ldots$. This is called Boole's inequality.
   Hint: Consider sets $S_n = \cup_{i=1}^{n} A_n$ and use part i.

   (b) Let $p_i,\ i = 1, \ldots, n$ be random variables with marginal distributions $p_i \sim \mathrm{U}[0,1],\ i = 1, \ldots, n$. $\mathrm{U}[0,1]$ is the uniform distribution on $[0,1]$. Suppose $\alpha \in (0,1)$ is a constant.

   i. Show that $P(\min_{1\leq i\leq n} p_i \leq \alpha/n) \leq \alpha$ by applying Boole's inequality.

   ii. Now assume $p_1, \ldots, p_n$ are jointly independent. Calculate $P(\min_{1\leq i\leq n} p_i \leq \alpha/n)$ and its limit as n approaches infinity.

   iii. If $\alpha = 0.05$, compare numerically the upper bound α from Boole's inequality to the limit of the exact value you get from part ii.

3. **cdf, pdf, and transformations**

   (a) Suppose X is a random variable with cdf $F(x) = \frac{e^x}{1+e^x}$. Find its density f.

(b) Suppose X is a random varible with pdf $f(x) = cx(1-x)$ for $0 < x < 1$ and $f(x) = 0$ otherwise, where c is a constant. Find c.

(c) Suppose X has pdf $f_X(x;\theta) = \frac{1}{k(\theta)} x \mathbb{1}(0 \leq x \leq \theta)$, where $\theta \in (0, \infty)$ is an unknown constant, $k()$ is an unknown function.

i. Show $k(\theta) = \theta^2/2$.

ii. Find $F_X(x;\theta)$, the cdf of X.

iii. Let $Y = X^2$. Find $F_Y(\cdot;\theta)$, the cdf of Y, and $f_Y(\cdot;\theta)$, the pdf of Y. What is the distribution of Y?

4. Markov's inequality

Let X be a random variable with $X \geq 0$. Suppose $b > 0$, where b is a constant.

(a) Draw the functions $g_1(x) = \mathbb{1}(x \geq b)$ and $g_2(x) = \frac{x}{b}$, $x \in \mathbb{R}$ in a coordinate plane.

(b) Illustrate the inequality between random variables $\mathbb{1}(X \geq b) \leq \frac{X}{b}$ by the diagram.

(c) Show that $P(X \geq b) \leq \frac{\mathbb{E}X}{b}$.