# 5 Risk Aversion and Investment Decisions

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# Risk Aversion and Portfolio Allocation

Let's now put our framework of decision-making under uncertainty to use.

Consider a risk-averse investor with vN-M expected utility who divides his or her initial wealth $Y_0$ into an amount $a$ allocated to a risky asset – say, the stock market – and an amount $Y_0 - a$ allocated to a safe asset – say, a bank account or a government bond.

# Risk Aversion and Portfolio Allocation

$Y_0 =$ initial wealth

$a =$ amount allocated to stocks

$\tilde{r} =$ random return on stocks

$r_f =$ risk-free return

$\tilde{Y}_1 =$ terminal wealth

$$\begin{aligned}\tilde{Y}_1 &= (1 + r_f)(Y_0 - a) + a(1 + \tilde{r}) \\ &= Y_0(1 + r_f) + a(\tilde{r} - r_f)\end{aligned}$$

# Risk Aversion and Portfolio Allocation

The investor chooses $a$ to maximize expected utility:

$$\max_a E\left[u(\tilde{Y}_1)\right] = \max_a E\{u[Y_0(1+r_f) + a(\tilde{r} - r_f)]\}$$

If the investor is risk-averse, u is concave. 只需要F.O.C就好

Then the first-order condition is both a necessary and sufficient condition for the value $a^*$ of $a$ that solves this unconstrained optimization problem.

# Risk Aversion and Portfolio Allocation

The investor's problem is

$$\max_a E\{u[Y_0(1+r_f)+a(\tilde{r}-r_f)]\}$$

The first-order condition is

$$E\{u'[Y_0(1+r_f)+a^*(\tilde{r}-r_f)](\tilde{r}-r_f)\}=0.$$

Note: we are allowing the investor to sell stocks short $(a^*<0)$ or to buy stocks on margin $(a^*>Y_0)$ if he or she desires.

# Risk Aversion and Portfolio Allocation

Theorem If the Bernoulli utility function $u$ is increasing and concave, then

$$a^* > 0 \text{ if and only if } E(\tilde{r}) > r_f$$

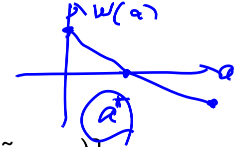
$$a^* = 0 \text{ if and only if } E(\tilde{r}) = r_f$$

$$a^* < 0 \text{ if and only if } E(\tilde{r}) < r_f$$

Thus, a risk-averse investor will always allocate at least some funds to the stock market if the expected return on stocks exceeds the risk-free rate.

# Risk Aversion and Portfolio Allocation

(Bonus) To prove the theorem, consider for example the case $E(\tilde{r}) > rf$.




Let

$$W(a) = E\{u'[Y_0(1+r_f) + a(\tilde{r} - r_f)](\tilde{r} - r_f)\},$$

You can verify these three properties of W(a) below:

(a) $W(0)>0$

(b) $W'(a) \cancel{>0}$, for all a $W'(a)<0$

(c) exist $K>0$ s.t. $W(a)<0$, for all $a>K$

$W(+\infty)<0$

It follows then $a^* > 0$.

# Risk Aversion and Portfolio Allocation

Danthine and Donaldson (3rd ed., p.41) report that in the United States, 1889-2010, average real (inflation-adjusted) returns on stocks and risk-free bonds are

$$E(\tilde{r}) = 0.075 \text{ (7.5 percent per year)}$$

$$r_f = 0.011 \text{ (1.1 percent per year)}$$

The equity risk premium of $E(\tilde{r}) - r_f = 0.064$ (6.4 percent) is not only positive, it is huge. The implication of the theory is that all investors, even the most risk averse, should have some money invested in the stock market.

# Portfolios, Risk Aversion, and Wealth

As an example, suppose

CRRA

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma} \text{ implies } u'(Y) = Y^{-\gamma} = \frac{1}{Y^{\gamma}}.$$

and then assume stock returns can either be good or bad

$$\tilde{r} = \begin{cases} r_G & \text{with probability } \pi \\ r_B & \text{with probability } 1-\pi \end{cases}$$

where $r_G > r_f > r_B$ defines the "good" and "bad" states and

$$\pi r_G + (1-\pi) r_B > r_f,$$

so that $E(\tilde{r}) > r_f$ and the investor will choose $a^* > 0$.

# Portfolios, Risk Aversion, and Wealth

With CRRA (constant relative risk aversion) utility and two states for $\tilde{r}$, the problem

$$\max_a E\{u[Y_0(1+r_f)+a(\tilde{r}-r_f)]\}$$

specializes to

$$\begin{aligned}\max_a \quad & \pi\left\{\frac{[Y_0(1+r_f)+a(r_G-r_f)]^{1-\gamma}-1}{1-\gamma}\right\}\\ & +(1-\pi)\left\{\frac{[Y_0(1+r_f)+a(r_B-r_f)]^{1-\gamma}-1}{1-\gamma}\right\}\end{aligned}$$

# Portfolios, Risk Aversion, and Wealth

The problem

$$\max_{a} \quad \pi \left\{ \frac{[Y_0(1+r_f)+a(r_G-r_f)]^{1-\gamma}-1}{1-\gamma} \right\} + (1-\pi) \left\{ \frac{[Y_0(1+r_f)+a(r_B-r_f)]^{1-\gamma}-1}{1-\gamma} \right\}$$

has first-order condition

$$\underbrace{\frac{\pi(r_G-r_f)}{[Y_0(1+r_f)+a^*(r_G-r_f)]^{\gamma}}}_{>0} + \underbrace{\frac{(1-\pi)(r_B-r_f)}{[Y_0(1+r_f)+a^*(r_B-r_f)]^{\gamma}}}_{<0} = 0.$$

# Portfolios, Risk Aversion, and Wealth

$$\frac{\pi(r_G - r_f)}{[Y_0(1+r_f) + a^*(r_G - r_f)]^\gamma} + \frac{(1-\pi)(r_B - r_f)}{[Y_0(1+r_f) + a^*(r_B - r_f)]^\gamma} = 0$$

$$\begin{aligned} & \pi(r_G - r_f)[Y_0(1+r_f) + a^*(r_B - r_f)]^\gamma \\ = \ & (1-\pi)(r_f - r_B)[Y_0(1+r_f) + a^*(r_G - r_f)]^\gamma \end{aligned}$$

$$\begin{aligned} & [\pi(r_G - r_f)]^{1/\gamma}[Y_0(1+r_f) + a^*(r_B - r_f)] \\ = \ & [(1-\pi)(r_f - r_B)]^{1/\gamma}[Y_0(1+r_f) + a^*(r_G - r_f)] \end{aligned}$$

# Portfolios, Risk Aversion, and Wealth

$$\frac{a^*}{Y_0} = \frac{(1+r_f)\{[\pi(r_G - r_f)]^{1/\gamma} - [(1-\pi)(r_f - r_B)]^{1/\gamma}]\}}{(r_G - r_f)[(1-\pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B)[\pi(r_G - r_f)]^{1/\gamma}}$$

| $\gamma$ | $r_f$ | $r_G$ | $r_B$ | $\pi$ | $E(\tilde{r})$ | $a^*/Y_0$ |
|---|---|---|---|---|---|---|
| 0.5 | 0.011 | 0.40 | −0.25 | 0.50 | 0.075 | 1.27 |
| 1 | 0.011 | 0.40 | −0.25 | 0.50 | 0.075 | 0.64 |
| 2 | 0.011 | 0.40 | −0.25 | 0.50 | 0.075 | 0.32 |
| 3 | 0.011 | 0.40 | −0.25 | 0.50 | 0.075 | 0.21 |
| 5 | 0.011 | 0.40 | −0.25 | 0.50 | 0.075 | 0.13 |
| 10 | 0.011 | 0.40 | −0.25 | 0.50 | 0.075 | 0.06 |

为什么其实没买那么多股票？

Answer:

①交易成本高

②住房中占据的比例

(human capital)

Result 1: Higher coefficients of relative risk aversion are associated with smaller values of $a^*$.

# Portfolios, Risk Aversion, and Wealth

$$\frac{a^*}{Y_0} = \frac{(1+r_f)\{[\pi(r_G - r_f)]^{1/\gamma} - [(1-\pi)(r_f - r_B)]^{1/\gamma}]\}}{(r_G - r_f)[(1-\pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B)[\pi(r_G - r_f)]^{1/\gamma}}$$

Result 2: with constant relative risk aversion, a* rises proportionally with wealth.

Two additional results, one related to ARA, another related to RRA, tell us more about the relationship between a* and wealth.

# Portfolios, Risk Aversion, and Wealth

Theorem (ARA and the absolute level of risky investment) Let $a^*(Y_0)$ be optimal amount of wealth allocated to stocks in the investor's problem below

$$\max_a E\{u[Y_0(1 + r_f) + a(\tilde{r} - r_f)]\}.$$

If $u(Y)$ is such that

一般认为 ARA
不会递增的

(a) $R'_A(Y) < 0$ then $\frac{da^*(Y_0)}{dY_0} > 0$

(b) $R'_A(Y) = 0$ then $\frac{da^*(Y_0)}{dY_0} = 0$

(c) $R'_A(Y) > 0$ then $\frac{da^*(Y_0)}{dY_0} < 0$

This result relates changes in absolute risk aversion to the absolute amount of wealth allocated to stocks.

# Portfolios, Risk Aversion, and Wealth

Part (a)

$$R_A'(Y) < 0 \text{ then } \frac{da^*(Y_0)}{dY_0} > 0$$

describes the "normal" case where absolute risk aversion falls as wealth rises.

In this case, wealthier individuals allocate more wealth to stocks.

# Portfolios, Risk Aversion, and Wealth

Part (b)

$$R_A'(Y) = 0 \text{ then } \frac{da^*(Y_0)}{dY_0} = 0$$

means that investors with constant absolute risk aversion

$$u(Y) = -\frac{1}{\nu} e^{-\nu Y}$$

allocate a constant amount of wealth to stocks.

This may seem surprising, but it reflects that fact that absolute risk aversion describes preferences over bets of a given size... so a CARA investor finds a bet of the ideal size and sticks with it, even when wealth increases.

# Portfolios, Risk Aversion, and Wealth

Part (c)

$$R_A'(Y) > 0 \text{ then } \frac{da^*(Y_0)}{dY_0} < 0$$

describes the case where absolute risk aversion rises as wealth rises.

The implication that wealthier individuals allocate less wealth to stocks makes this case (increasing absolute risk aversion) seem less plausible.

# Portfolios, Risk Aversion, and Wealth

Consistent with our results with CRRA utility, the next theorem relates changes in relative risk aversion to changes in the proportion of wealth allocated to stocks.

Define the elasticity of the function $a^*(Y_0)$ as

$$\eta = \frac{d \ln a^*(Y_0)}{d \ln Y_0} = \frac{Y_0}{a^*(Y_0)} \frac{da^*(Y_0)}{dY_0} \qquad \frac{\% \Delta a^*(Y_0)}{\% \Delta Y_0}$$

The elasticity measures the percentage change in $a^*$ brought about by a percentage-point change in $Y_0$.

# Portfolios, Risk Aversion, and Wealth

Theorem (RRA and the share of risky investment) Let $a^*(Y_0)$ be be optimal amount of wealth allocated to stocks in the investor's problem below

$$\max_a E\{u[Y_0(1+r_f)+a(\tilde{r}-r_f)]\}.$$

If $u(Y)$ is such that

(a) $R_R'(Y) < 0$ then $\eta > 1$

(b) $R_R'(Y) = 0$ then $\eta = 1$

(c) $R_R'(Y) > 0$ then $\eta < 1$

The theorem confirms what we know about CRRA utility: it implies that $a^*$ rises proportionally with $Y_0$ (risky asset share stays constant)

# Portfolios, Risk Aversion, and Wealth

With CRRA utility:

$$\frac{a^*}{Y_0} = K$$

where

$$K = \frac{(1+r_f)\{[\pi(r_G - r_f)]^{1/\gamma} - [(1-\pi)(r_f - r_B)]^{1/\gamma}]\}}{(r_G - r_f)[(1-\pi)(r_f - r_B)]^{1/\gamma} + (r_f - r_B)[\pi(r_G - r_f)]^{1/\gamma}}.$$

Hence

$$\ln(a^*(Y_0)) = \ln(K) + \ln(Y_0)$$

and

$$\eta = \frac{d \ln a^*(Y_0)}{d \ln Y_0} = 1.$$

# Portfolios, Risk Aversion, and Wealth

Theorem Let $a^*(Y_0)$ be the solution to

$$\max_a E\{u[Y_0(1+r_f)+a(\tilde{r}-r_f)]\}.$$

If $u(Y)$ is such that

(a) $R_R'(Y) < 0$ then $\eta > 1$
(b) $R_R'(Y) = 0$ then $\eta = 1$
(c) $R_R'(Y) > 0$ then $\eta < 1$

But this theorem on RRA and wealth allocated to the risk asset extends the results to the cases of decreasing and increasing relative risk aversion.