Lecture 5/3/19 Modern Portfolio Theory

Case 1: Suppose we only have two assets, 1 and 2, and that asset 2 is risk free (standard deviation = 0).

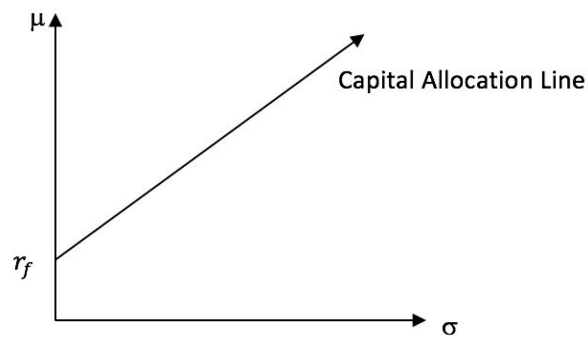
$\mu_2 = r_f$, and as well, $\sigma_2^2 = \sigma_2 = 0$. This implies that our expectation and variance of the portfolio are given by : $\mu_p = r_f + \omega_1(\mu_1 - r_f)$

$$\sigma_p^2 = \omega_1^2 \sigma_1^2 \Rightarrow \sigma_p = |\omega_1| \sigma_1$$

By substituting, $\omega_1 = \frac{\sigma_p}{\sigma_1} \Rightarrow \mu_p = r_f + \frac{\sigma_p}{\sigma_1}(\mu_1 - r_f)$

$$= r_f + \sigma_p \frac{(\mu_1 - r_f)}{\sigma_1}.$$

This form gives a representation of μ_p as a linear function of σ_p. Below is a graphical representation of this case.



Note, the slope of the capital allocation line (CAL) is the Sharpe ratio of asset 1: . $\frac{(\mu_1 - r_f)}{\sigma_1}$

Classic Problem: In order to determine where out portfolio will land on the capital allocation line we maximize the following equation with respect to ω_1 : max$_{\omega_1}$ $\mu_p - \frac{1}{2}A\sigma_p^2$, where A is a risk aversion parameter. In particular, the higher the value of A, the more the agent is risk-averse. Therefore, a higher value of A will correspond to a lower spot on Capital Allocation Line. We can view the indifference curves surrounding this equation as the set of all σ_p^2 and μ_p such that the equation $\mu_p - \frac{1}{2}A\sigma_p^2 = k$, is satisfied for some $k \in R$. Now we will solve for the maximum of the above equation with respect to ω_1 analytically.

$\max_{\omega_1} \mu_p - \frac{1}{2}A\sigma_p^2$, where $\mu_p = r_f + \omega_1(\mu_1 - r_f) \Rightarrow$

$\max_{\omega_1} r_f + \omega_1(\mu_1 - r_f) - \frac{1}{2}A\sigma_p^2$, when looking at first order conditions we see:

FOC: $(\mu_1 - r_f) - A\omega_1^* \sigma_1^2$

$\Rightarrow \quad \omega_1^* = \frac{(\mu_1 - r_f)}{A\sigma_1^2} = \frac{S_1}{A\sigma_1}$, **which is the Sharpe ratio of asset 1 over the risk aversion parameter (A) multiplied by the standard deviation of asset 1.**

Case 2: Two risky assets that are perfectly correlated ($\rho_{12} = 1$) .

$$\sigma_p^2 = \omega_1^2\sigma_1^2 + (1-\omega_1)^2\sigma_2^2 + 2\omega_1(1-\omega_1)\sigma_1\sigma_2$$

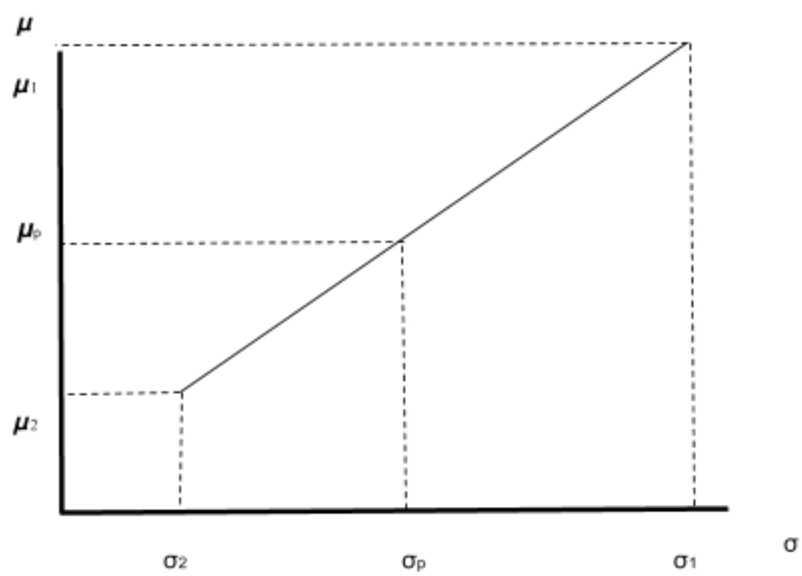
$$= (\omega_1\sigma_1 + (1-\omega_1)\sigma_2)^2$$

$$\Rightarrow \sigma_p = \omega_1\sigma_1 + (1-\omega_1)\sigma_2$$

$$\omega_1 = \frac{\sigma_p - \sigma_2}{\sigma_1 - \sigma_2}$$

$$\mu_p = \mu_2 + \left(\frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2}\right)(\sigma_p - \sigma_2)$$

$$= \mu_2 - \left(\frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2}\right)\sigma_2 + \left(\frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2}\right)\sigma_p$$



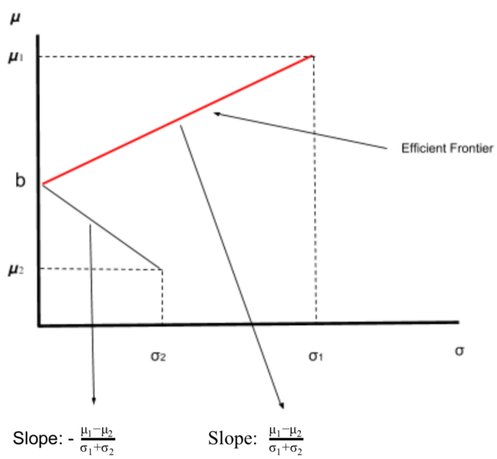
Case 3: Perfectly negatively correlated risky assets ($\rho_{12} = -1$)

σ_p^2=$w_1^2\sigma_1^2$+$(1-w_1^2)\sigma_2^2 - 2$w$_1(1-w_1)\sigma_1\sigma_2$= (w$_1\sigma_1$-$(1-w_1)\sigma_2)^2$ σ_p=$\pm$(w$_1\sigma_1$-$(1-w_1)\sigma_2$)

w_1=$(\pm\sigma_p$+$\sigma_2)/(\sigma_1$+$\sigma_2)$

μ_p=μ_2+$(\mu_1$-$\mu_2)/(\sigma_1$+$\sigma_2)$*$((\pm\sigma_p)$+$\sigma_2)$ μ_p=μ_2+$(\mu_1$-$\mu_2)/(\sigma_1$+$\sigma_2)$*$(\pm\sigma_p)\pm(\mu_1$-$\mu_2)/(\sigma_1$+$\sigma_2)$*(σ_2)

Note: Intercept:μ_2+$(\mu_1$-$\mu_2)/(\sigma_1$+$\sigma_2)$*$(\pm\sigma_p)$ Slope:$(\mu_1$-$\mu_2)/(\sigma_1$+$\sigma_2)$



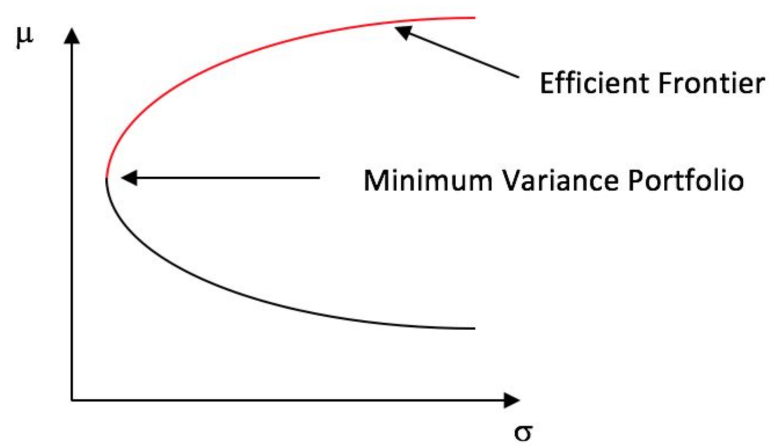
Case 4: Two imperfectly correlated risky assets ($|\rho_{12}| \leq -1$)

σ_p^2=$w_1^2\sigma_1^2$+$(1-w_1^2)\sigma_2^2 + 2$w$_1(1-w_1)\sigma_1\sigma_2\rho_{12}$ w_1=μ_2+$(\mu_p$-$\mu_2)/(\mu_1$+$\mu_2)$ =μ_p*$1/(\mu_1$+$\mu_2)$-μ_2*$1/(\mu_1$+$\mu_2)$

which implies that the variance is a quadratic function of μ_p.

Below is a graph of this scenario, with the efficient frontier highlighted in red. The minimum variance portfolio is also shown, a portfolio which can be found analytically by doing the following:

Minimize σ_p^2 with respect to ω_1, the solution to which is: $\omega_1^* = \frac{\sigma_2^2 - \sigma_1\sigma_2\rho_{12}}{\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2\rho_{12}}$.



Case 5: Many risky assets.

(i) $min_w \sigma_p^2$

s.t. $\mu = \bar{\mu}$

(ii) $max_w \mu_p$

s.t. $\sigma_p^2 = \overline{\sigma^2}$

With many Risky Assets

$$\sigma_p^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} \omega_i\omega_j \sigma_{ij} \qquad (\sigma_{ij} = Cov(r_i, r_j))$$

$$\sigma_{ij} = \sigma_i^2$$

$$min_{\omega,\omega_n} \sum_{i=1}^{N}\sum_{j=1}^{N} \omega_i\omega_j\sigma_{ij}$$

$$s.t. \sum_{i=1}^{N} \omega_i\mu_i = \bar{\mu}$$

$$\sum_{i=1}^{N} \omega_i = 1$$

$$min_\omega \ \omega^T \Sigma\omega$$

$$s.t. \ \omega^T\mu = \bar{\mu}$$

$$\omega^T e = 1$$