Econ 139 Lecture 8 Scribe Notes

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Measures of Risk Aversion

- Absolute - $r_A(x) = -\frac{u''(x)}{u'(x)}$
- Relative - $r_R(x) = -\frac{u''(x)}{u'(x)}x = r_A(x) \times x$

  $r_A(x) = -\frac{du'(x)}{dx} \cdot \frac{1}{u'(x)} = -\frac{u''(x)}{u'(x)}$

Absolute risk can be also shown in terms of growth rate per marginal wealth increase as:

$r_A(x) = -\frac{du'(x)}{dx} \cdot \frac{1}{u'(x)} = -\frac{u''(x)}{u'(x)}$

- Since the equation is negative, this can be reworded as "Rate of Decay of Marginal wealth increase by 1%

Elasticity: $\frac{df(x)}{dx} \cdot \frac{x}{f(x)} = \frac{\frac{df(x)}{f(x)}}{\frac{x}{dx}}$

- Elasticity is the loss of marginal utility per increase in percent of wealth

Absolute:

- Start with x, consider gamble paying +h w.p. π and -h w.p. 1-π.
- Want to find $\pi(x, h)$ such that we are indifferent between current wealth and current wealth plus gamble.

  $u(x) = \pi(x, h)\, u(x + h) + (1 - \pi(x, h))\, u(x - h)$

  Using Taylor's Theorem to approximate:

  $\pi(x, h) \approx \frac{1}{2} + \frac{1}{4}h\, r_A(x)$

  Probability > $\frac{1}{2}$ is required for a risk-averse investor

Relative:

- Start with x, consider gamble θ_x w.p. π and -θ_x w.p. 1-π.
- Want to find π(x, θ_x) such that we are indifferent between current wealth and current wealth plus gamble.

  $u(x) = \pi(x, \theta_x)\, u(x + \theta_x) + (1 - \pi(x, \theta_x))\, u(x - \theta_x)$

  $\pi(x, \theta_x) \approx \frac{1}{2} + \frac{1}{4}\, \theta_x r_A(x) \approx \frac{1}{2} + \frac{1}{4}\, \theta_x \cdot r_R(x)$

- Consider utility function:

  $u(x) = -\frac{1}{v}e^{-vx}$

  $r_A(x) = -\frac{-ve^{-vx}}{e^{-vx}} = v$

This is constant absolute risk aversion (CARA)
$\pi(x, h) \approx \frac{1}{2} + \frac{1}{4}hv$

- Consider utility function:
  $u(x) = \frac{x^{1-\gamma}}{1-\gamma} \quad if \;\; \gamma > 0,\; \gamma \neq 1$
  $\qquad\qquad ln(x),\;\; if\; \gamma = 1$
  $r_R(x) = -\frac{-\gamma x^{-\gamma-1}}{x^{-\gamma}} \cdot x = \gamma$

  Plug in $\pi(x, \theta_x) \approx \frac{1}{2} + \frac{1}{4}\,\theta_x \gamma$,
  $r_A(x) = \frac{\gamma}{x}$

Risk Premium
Suppose investor starts with wealth w and has the opportunity to invest in an asset with pay off X and E[X].
$u(w + E[X]) \geq E[u[w + X]]$

- Follows by Jensen's inequality

<u>Certainty Equivalent (CE)</u>
The maximum amount an investor is willing to give up in exchange for an asset.

CE satisfies
$u(w + CE[X]) = E[u[w + X]]$
So, $CE[X] \leq E[X]$ where $CE[X] = E[X]$ if the investor is risk neutral

<u>Risk Premium (RP)</u>
The difference between E[X] and CE[X]. In other words, the maximum amount to pay to make the asset risk free (like insurance).
$RP[X] = E[X] - CE[X]$

<u>**Example**</u>:
Consider:

$$u(w) = \frac{w^{1-\gamma}}{1-\gamma} \quad if \;\; \gamma > 0,\; \gamma \neq 1$$
$$ln(w)\; if \;\; \gamma = 1$$

Asset X has payoff

	Payoff	Probability of State
State 1	50000	0.5
State 2	100000	0.5

→ E(X) = 75,000
→ Assume 0 initial wealth
$u(w + CE(X)) = E[u(w + X)]$

When w = 0: $u(CE(X)) = E[u(X)]$

γ	0	1	2	5	10
CE	75000	70711	66667	58566	53991

Now suppose w = 100000. For $\gamma = 5$, CE = 66532 (which is greater than 58566 from table above, because initial wealth is higher now)

X	π_A	π_B	$F_{\pi A}$	$F_{\pi B}$
10	0.4	0.4	0.4	0.4
100	0.6	0.4	1.0	0.8
1000	0	0.2	1.0	1.0

*Recall in probability, CDF (Cumulative Distribution Function) is given by
$F(X) \; = \; P(X \leq x)$ where $F : \mathbb{R} \subset [0, 1]$

We can say that an asset $X_{\pi B}$ **<u>First Order Stochastically Dominates</u>** (FSD) asset $X_{\pi A}$ if

$$F_{\pi B}(X) \; \leq \; F_{\pi A}(X)$$

<u>Formal definition:</u>

- <u>Preamble:</u> let $F_{\pi A}$ and $F_{\pi B}$ be the CDFs for random payoffs X, with probability densities (PD) π_A and π_B respectively.
- <u>Definition and Theorem:</u> given preamble, we say that $X_{\pi B}$ FSD $X_{\pi A}$ iff

$$F_{\pi B}(X) \; \leq \; F_{\pi A}(X)$$
$$E_{\pi B}[u(X)] \; \geq E_{\pi A}[u(X)]$$

<u>Example:</u>

X	π_A	π_B	$F_{\pi A}$	$F_{\pi B}$	$\int[F_{\pi B}(t) - F_{\pi A}(t)]dx$
1	0.33	0	0.33	0	0
4	0	0.25	0.33	0.25	-0.99
5	0	0.5	0.33	0.75	-1.07
6	0.33	0	0.66	0.75	-0.65
8	0.34	0	1.0	0.75	-0.47
9	0	0.25	1.0	1.0	-0.72

We can say that an asset $X_{\pi B}$ **Second Order Stochastically Dominates** (SSD) asset $X_{\pi A}$ if

$\int[F_{\pi B}(X) - F_{\pi A}(X)]dt \leq 0 \; over \; an \; interval \; [-\infty, t]$ for any t