Econ 240A, Fall 2018
Problem Set 3

Due date: Wednesday, Sept. 26

Review of exponential family, sufficient statistics, minimal sufficiency, completeness, Rao-Blackwell theorem, Cramér lower bound, UMVU estimation.
Note: Problems start with a star, *, are optional and don't count for grade. Of course, you can feel free to write them up if you want.

1. **Sufficient statistic of Gamma random sample**
   Suppose $X_i \overset{\text{iid}}{\sim} \text{Gamma}(\alpha, \beta),\ i = 1, \ldots, n$ with pdf $f_{\alpha,\beta}(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp\left\{-\frac{x}{\beta}\right\} \mathbb{1}\{x \geq 0\}$ where α, β are unknown. (More precisely, the model here is $\mathcal{P} = \{\Pi_{i=1}^n \text{Gamma}(\alpha, \beta) : \alpha, \beta > 0\}$.) Find a two-dimensional complete sufficient statistic for α, β.

2. **Poisson random sample**
   Suppose $X_i \overset{\text{iid}}{\sim} \text{Poisson}(\lambda)$ and $i = 1, \ldots, n$ with $\lambda > 0$ unknown.

   (a) Show that $\sum_{i=1}^n X_i$ is a sufficient statistic for λ.

   (b) Find a method of moments estimator, $\hat{\lambda}_{MM}$, and maximum likelihood estimator, $\hat{\lambda}_{ML}$, for λ.

   (c) Show that $\hat{\lambda}_{MM}$ is an unbiased estimator of λ.

   (d) Find the Cramér-Rao lower bound on variance for any unbiased estimator of λ.

   (e) Is $\hat{\lambda}_{MM}$ UMVU?

3. **Uniform location family**
   Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{U}[\theta - \frac{1}{2}, \theta + \frac{1}{2}]$, with $\theta \in \mathbb{R}$ unknown.

   (a) Find a two-dimensional minimal sufficient statistic and show it is minimal.

   (b) Show that the minimal sufficient statistic is not complete.

   (c) Suppose we want to estimate θ under the squared error loss $L(\theta, d) = (\theta - d)^2$. The sample mean $\bar{X}$ seems to be a reasonable estimator of θ. However, we can improve upon it by Rao-Blackwellizing it. Find this new estimator $\delta(X_1, \ldots, X_n)$.

   (d) * Compare the MSE of these two estimators. What happens to the ratio $\text{MSE}_\theta(\bar{X}) / \text{MSE}_\theta(\delta)$ as $n \to \infty$?

   (e) * Is δ UMVU?

4. **Minimal sufficiency of the likelihood ratio**
   Suppose $\mathcal{P} = \{p_\theta : \theta \in \Theta\}$ is a family of densities (defined with respect to a common measure μ on $\mathcal{X}$). Assume $\Theta = \{\theta_0, \theta_1, \ldots, \theta_m\}$ is a finite set, and there exists some $\theta \in \Theta$ such that $p_\theta(x) > 0,\ \forall x \in \mathcal{X}$ (without loss of generality we assume it is θ_0).

(a) Show that the *likelihood function*, defined as

$$T(X) = (p_\theta(X))_{\theta\in\Theta} = (p_{\theta_0}(X), \ldots, p_{\theta_m}(X))$$

is sufficient.

(b) Prove that the *likelihood ratio*, defined as

$$T(X) = \left(\frac{p_\theta(X)}{p_{\theta_0}(X)}\right)_{\theta\in\Theta}$$

is minimal sufficient.

(c) Show that the *likelihood function* is *not*, in general, minimal sufficient.