Math 123 Lec 1 Course Policy, Philosophy, Terminology

Textbook: Qualitative theory of ODE Brauer - Nohel

Tentative Outline:

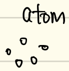
① Terms ; Review : Tricks & Techniques; need for theory. [BN1.5]
② Theory of ODE: existence, unique, ··· [BN 3]
③ linear system (Phase sp analysis) [BN2]
④ Stability of linear / almost linear system [BN4]
⑤ Lyapunov Method. [BN5]
⑥ Applications / further topics.
e.g. flocking, numerical methods, Control theory.

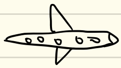
Philosophy / Motivation

What is DE?

- Philosophical Answer: Human being's interpretation of nature.
- Mathematical Answer: Eq involving derivatives.

Ex:





Schröding Eq. Newton's 2nd law Navier - Stokes Eq / Euler Eq (Fluid Eqs)

$F = ma = m\ddot{x}$

$m\ddot{x} = -mg$

free fall

Why ODE?

- phylosophical: To understand nature & this universe!
- specifically the mathematical theory / tools we learn here is a buiding block (or related w./):
  - Math majored: PDE, geometry
  - Applied math: biological modeling, dynamical systems, etc
    design algorithms for PDE/ODE (preserve properties)
  - Computer Sciences: Data gradient descent
    gradient flow
  - engineering : control theory
  - finance/business: SDE

  ...

Terminology

Q: What is an ODE?

Ans: An equation for a fxn of one var. that involves "ordinary" derivatives of the function.

For us: Implicit form

$$F(t, y(t), y'(t), \cdots, y^{(n)}(t)) = 0$$

Explicit / Standard form

$$y^{(n)}(t) = f(t, y(t), y'(t), \cdots, y^{(n-1)}(t))$$

Def: Dependent var, independent var, order.

Autonomous

Independent var does not "appear" explicitly.

Rmk All scalar ODEs of n-th order can be rewritten as first order ODEs with dependent var being a vector of n dims.

Ex

	dep. var	indep. var	order	auto.	linear
① $\frac{d^2x}{dt^2}+\omega^2 x=0$	x	t	2	✓	✓
② $\frac{d^2x}{dt^2}-x\frac{dx}{dt}-x+x^3=\sin(\omega t)$	x	t	2	X	X
③ $\ddot{x}(t)-(1-x^2(t))\dot{x}(t)+x(t)=0$	x	t	2	✓	X
④ $\frac{d^3f}{d\eta^3}+f\frac{d^2f}{d\eta^2}+\pi\left[1-\left(\frac{d^2f}{d\eta^2}\right)^2\right]=0$	f	η	3	✓	X
⑤ $y''' = (y''+x)^{\sqrt{x}}$	y	x	3	X	X

Ex Change n-th order ODE to vector form.

① $\begin{cases}\dot{x}=u\\ \dot{u}=-\omega^2 x\end{cases}$ $\qquad \begin{pmatrix}\dot{x}\\ \dot{u}\end{pmatrix}=\begin{pmatrix}u\\ -\omega^2 x\end{pmatrix}$

④ $\dot{f}=u$
$\dot{u}=v$
$\dot{v}=-fv-\pi(1-v^2)$ $\qquad \begin{pmatrix}\dot{f}\\ \dot{u}\\ \dot{v}\end{pmatrix}=\begin{pmatrix}u\\ v\\ -fv-\pi(1-v^2)\end{pmatrix}$

⑤ $\begin{cases}\dot{y}=u\\ \dot{u}=v\\ \dot{v}=(v+x)^{\sqrt{x}}\end{cases}$ $\qquad \begin{pmatrix}\dot{y}\\ \dot{u}\\ \dot{v}\end{pmatrix}=\begin{pmatrix}u\\ v\\ (v+x)^{\sqrt{x}}\end{pmatrix}$

So WLOG, we consider systems of first-order ODEs

$$\dot{\vec{y}}(t)=\vec{f}(t,\vec{y}(t)) \qquad \text{where } \underbrace{\vec{y}=(y_1,\cdots,y_n)}_{\in\mathbb{R}^n},\ \vec{f}=(f_1,\cdots,f_n)$$

Then in vector form, autonomous reads

$$\dot{\vec{y}}(t) = f(\vec{y}(t)).$$

Linear

if f (or F) is a linear function of the dependent variable (and its derivatives).

In vector form, one can write

$$\dot{\vec{y}}(t) = A(t)\,y(t) + g(t) \qquad (*)$$

Homogeneous

If $g(t)=0$, we call $(*)$ a homogeneous linear ODE.
If $g(t)\neq 0$, we call $(*)$ a nonhomogeneous linear ODE.

Solution

Assume $f: D \subseteq \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$

① $\phi(t)$ is called a solution of

$$y^{(n)} = f(t, y(t), y'(t), \cdots, y^{(n-1)}(t))$$

on interval $I = (t_1, t_2)$, if $\phi(t)$ satisfies

① $\phi(t), \phi'(t), \cdots, \phi^{(n-1)}(t)$ exists. for $\forall t \in I$
② $(t, \phi(t), \cdots, \phi^{(n-1)}(t)) \in D$, for $\forall t \in I$.
} so that RHS makes sense!

③ $\phi^{(n)}(t) = f(t, \phi(t), \phi'(t), \cdots, \phi^{(n-1)}(t))$ $\rightarrow$ Eq is satisfied.

② Similarly for the vector form. Assume $f_i : D \subseteq \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$, $i=1,\cdots,n$.

$\phi(t) = (\phi_1(t), \cdots, \phi_n(t))$ is a solution of

$$\begin{cases} \dot{y}_1 = f_1(t, y_1, \cdots, y_n) \\ \vdots \\ \dot{y}_n = f_1(t, y_1, \cdots, y_n) \end{cases}$$

on interval $I = (t_1, t_2)$, if

① $\phi_1'(t), \cdots, \phi_n'(t)$ exists for $\forall t \in I$

② $(t, \phi_1'(t), \cdots, \phi_n'(t)) \in D$ for $\forall t \in I$

③ Eq is satisfied, i.e.

$$\begin{cases} \phi_1'(t) = f_1(t, \phi_1(t), \cdots, \phi_n(t)) \\ \vdots \\ \phi_n'(t) = f_n(t, \phi_1(t), \cdots, \phi_n(t)) \end{cases}$$

Initial value problem (IVP)

initial condition is given. (also called Cauchy problem)

- For system:

$$\begin{cases} \dfrac{d\vec{y}}{dt} = \vec{f}(t, \vec{y}) \\ \vec{y}(0) = \vec{y}_0 \end{cases}$$

- For scalar n-th order:

$$\begin{cases} y^{(n)}(t) = f(t, y(t), y'(t), \cdots, y^{(n-1)}(t)) \\ y(0) = y_0 \\ y'(0) = y_1 \\ \vdots \\ y^{(n-1)}(0) = y_{n-1} \end{cases}$$

Rmk: How many derivatives is involved, then how many initial conditions in need.

Why? Think about the example of "Di jump from building"

$\ddot{x} = -g$ <u>2nd order</u>

$\Rightarrow \dot{x} = -gt + C_1$

$x = -\frac{1}{2}gt^2 + C_1 t + C_2$.

So two condition is needed to determine a specific solution.

we can have $x(0) = x_0$, $x'(0) = v_0$ initial position and velocity.