Midterm Exam

October 12, 2015

Instructions: This is a closed book exam, but you may refer to one sheet of notes. You have 80 minutes for the exam. Answer as many questions as possible. Partial answers get partial credit. Please write legibly. *Good luck!*

Problem 1 (5 points). Determine whether or not the statement below is correct and give a *brief* (e.g., a bluebook page or less) justification for your answer.

Suppose $X_1, \ldots, X_n$ is a random sample from a continuous distribution with pdf $f(\cdot|\theta)$, where $\theta \in \Theta \subseteq \mathbb{R}$ is unknown. A statistic $T = T(X_1, \ldots, X_n)$ is complete if and only if

$$P_\theta[g(T) = \theta] = 1 \qquad \forall \theta \in \Theta$$

for every function $g(\cdot)$ satisfying

$$E_\theta[g(T)] = \theta \qquad \forall \theta \in \Theta.$$

Problem 2 (45 points, each part receives equal weight). Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with mean $\sqrt{\theta}$ and pdf

$$f_X(x|\theta) = \frac{x}{j(\theta)} 1[0 \leq x \leq k(\theta)],$$

where $\theta \in \Theta = \mathbb{R}_{++}$ is an unknown parameter, $j : \Theta \to \mathbb{R}_{++}$ and $k : \Theta \to \mathbb{R}_{++}$ are some functions, and $1(\cdot)$ is the indicator function.

(a) Show that

$$j(\theta) = \frac{9}{8}\theta, \qquad k(\theta) = \frac{3}{2}\sqrt{\theta}.$$

(b) Find $F_X(\cdot|\theta)$, the cdf of X.

(c) Derive a method moments estimator $\hat{\theta}_{MM}$ of θ. Is $\hat{\theta}_{MM}$ an unbiased estimator of θ?

(d) Find the likelihood function. Does θ admit a scalar sufficient statistic?

(e) Show that

$$\hat{\theta}_{ML} = \frac{4}{9}(\max\nolimits_{1 \leq i \leq n} X_i)^2$$

is the maximum likelihood estimator of θ.

(f) Show that the cdf of $\hat{\theta}_{ML}$ is given by

$$F_{ML}(x|\theta) = \begin{cases} 0 & \text{if } x < 0, \\ (x/\theta)^n & \text{if } 0 \leq x < \theta, \\ 1 & \text{if } x \geq \theta. \end{cases}$$

It can be shown that $\hat{\theta}_{ML}$ is complete.

(g) Find a uniform minimum variance unbiased estimator of θ.

Let $\theta_0 > 0$ be some constant and consider the one-sided testing problem

$$H_0 : \theta \leq \theta_0 \qquad \text{vs.} \qquad H_1 : \theta > \theta_0.$$

(h) Consider a test which rejects H_0 if (and only if) $\hat{\theta}_{ML}/\theta_0 > c$, where c is some positive constant (possibly depending on θ_0). Find c such that the test has 5% size.

(i) Find the power function of the test derived in (h).