Investment

Lecture 4 & 5

Equity Invesment

- Return
- Risk
- Portfolio

Risk

- Risky return:

$$\tilde{r} = \frac{\tilde{D} + \tilde{P}_1}{P_0} - 1$$

- This notation means realized returns may take on different values (ie. D and P are risky therefore r is risky).

Risk

- Eg. Suppose a stock pays no dividends and tomorrow's price is determined on the basis of the flip of two coins. Today's price is 900 and tomorrow's price is 1000 x (# heads).
- This stock is risky.
- We don't know what tomorrow's price will be but we do know what it can be.

Risk

$\tilde{P}_1$	$\tilde{r}$	Probability
0	-100	.25
1000	11	.50
2000	122	.25

- Price can take on three possible values with different probabilities for each value.

Risk

- We also know the distribution of the stock's returns:
  - eg. What is the probability that the above stock's return is negative?

Risk

- In reality a stock's return may take on any value and its distribution of returns is subjective (ie. It's based on beliefs).
  - Eg. What does the probability distribution of Baidu returns look like?
- Risk will have to be related to the distribution of returns.

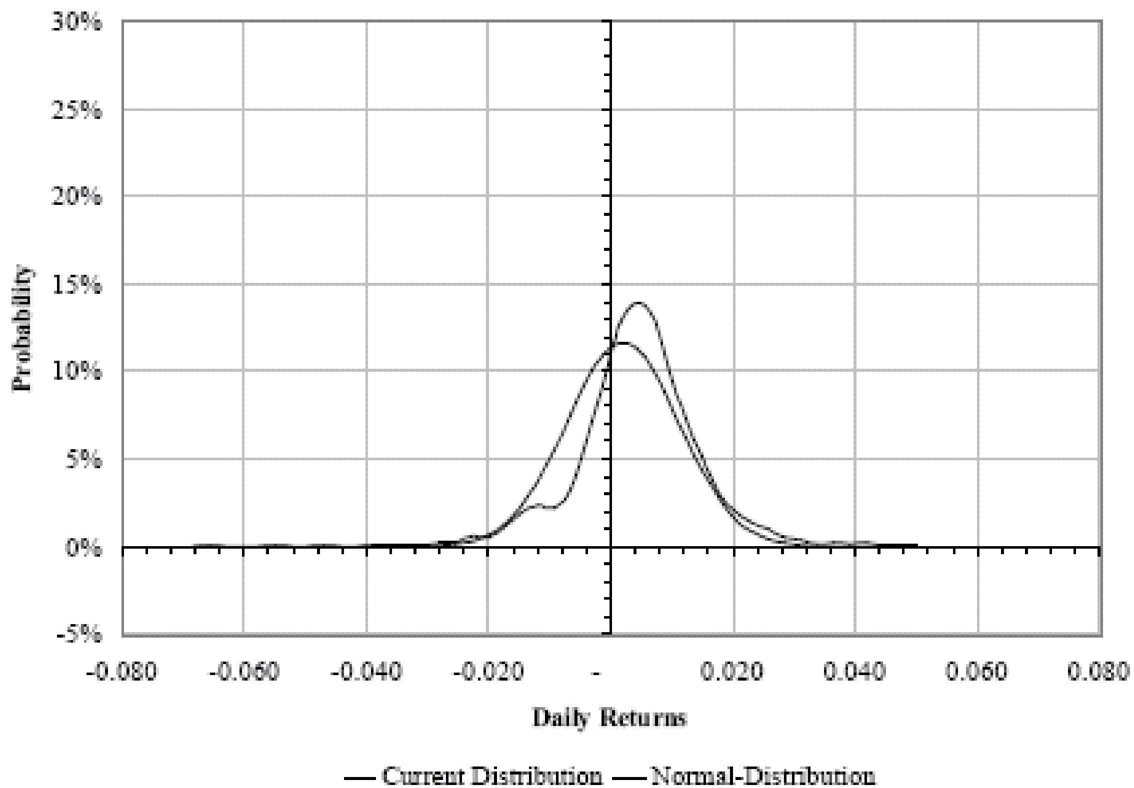


Figure 2: Return of S & P 500 Index,

Source: Bloomberg Professional.

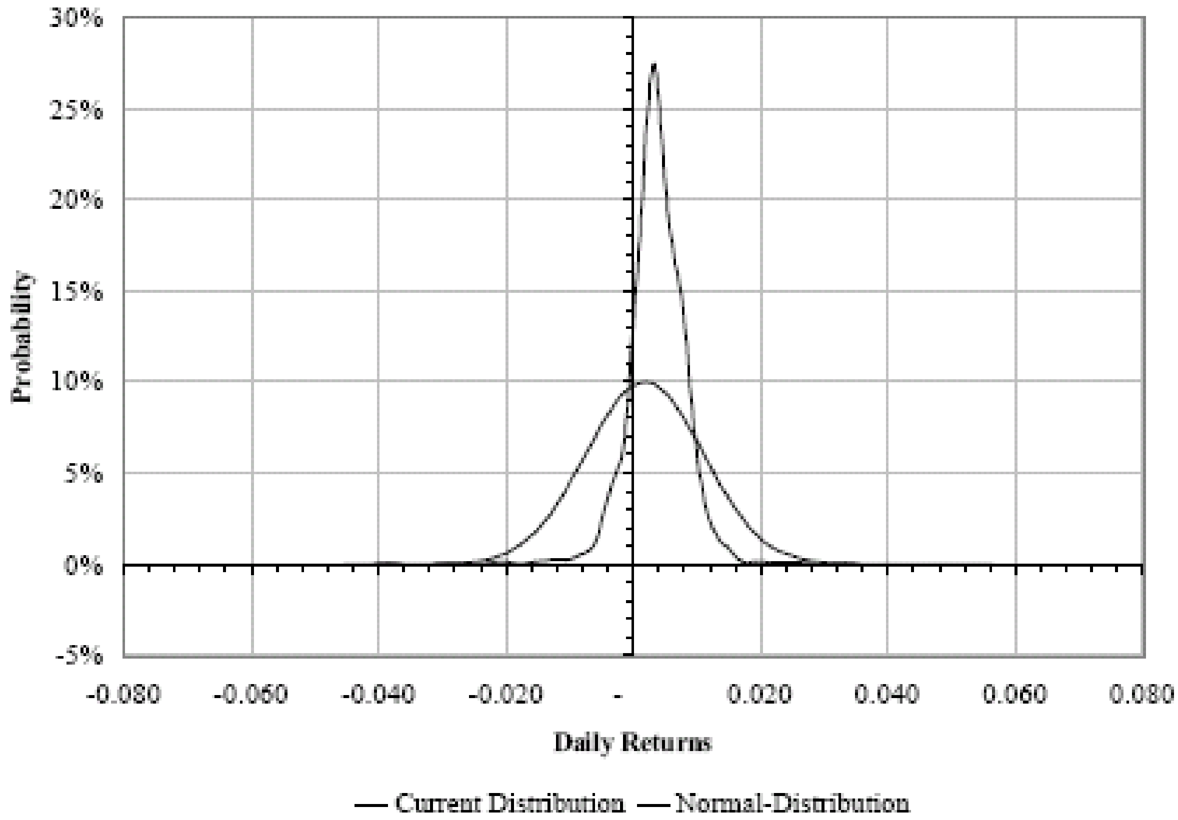


Figure 3: Return of 10 Year Treasury Bills,

Source: Bloomberg Professional.

Risk

- More risky returns demand higher expected returns:
  - Expected Stock Return = riskless rate + risk premium

- BUT - how do we measure risk?

Some Statistics

- We need to calculate expected return and variance of stock returns:

$$E(\tilde{r}) = \sum\nolimits_{\text{all } \tilde{r}} \tilde{r}\; p(\tilde{r})$$

$$Var(\tilde{r}) = \sum\nolimits_{\text{all } \tilde{r}} (\tilde{r} - E(\tilde{r}))^2\; p(\tilde{r})$$

$$SDev(\tilde{r}) = \left(Var(\tilde{r})\right)^{1/2}$$

Risk

- From previous example:

Properties of E() and Var()

a - constant (not random)

$\tilde{r}$ - random (eg. return)

$$E(a\tilde{r}) = aE(\tilde{r})$$

$$E(a + \tilde{r}) = a + E(\tilde{r})$$

$$Var(a\tilde{r}) = a^2\tilde{r}$$

$$Var(a + \tilde{r}) = Var(\tilde{r})$$

Scenario Analysis: Peace & War

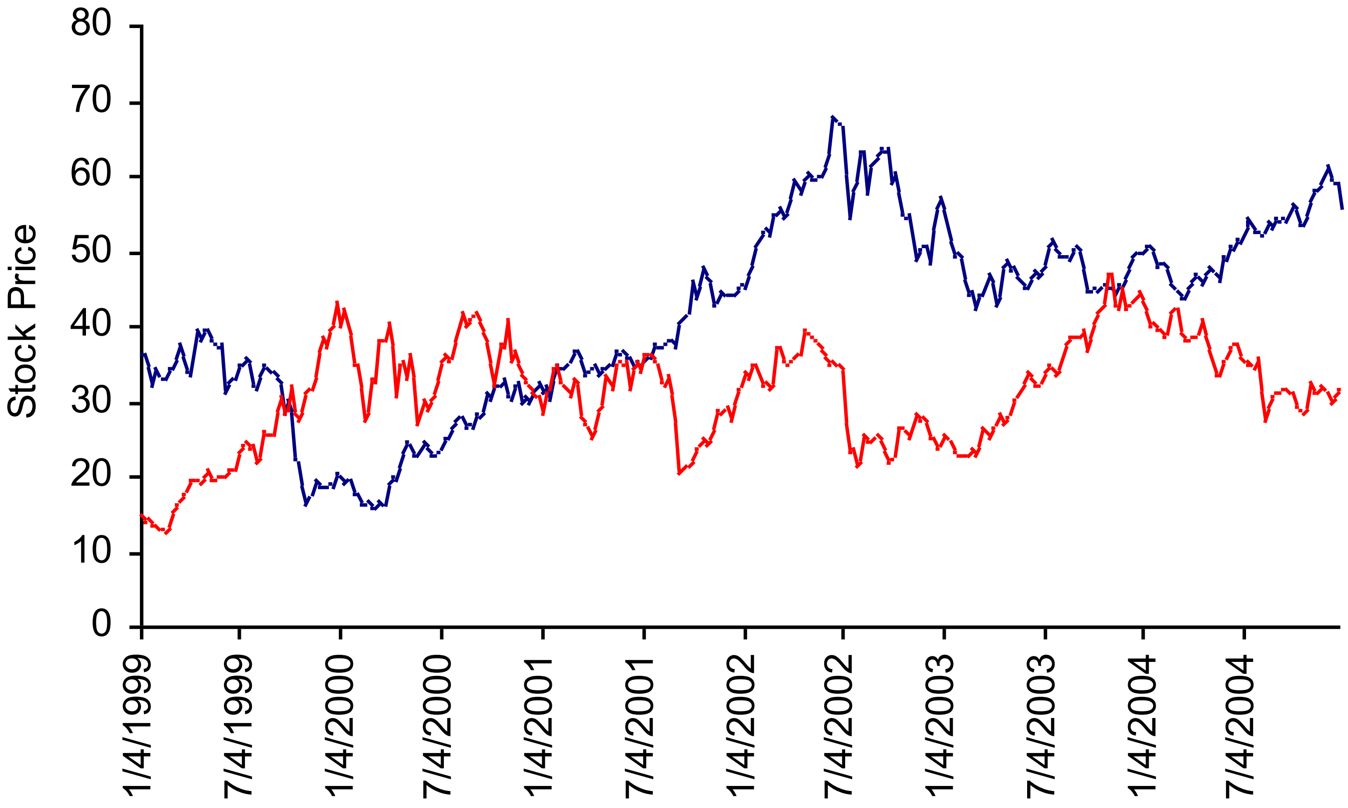
- Suppose you are a shareholder of Tiffany (TIF) and of Lockheed Martin (LKM)
- What is the return and the risk of the stocks?

State	Probability	Return TIF	Return LKM
Peace	0.75	30%	0%
War	0.25	-40%	20%

Distribution of Returns

Mean	$E(R) = \sum_{S} p_s R_s$
Variance	$VAR(R) = \sum_{S} p_s \left[R_s - E(R_s)\right]^2$
Std. Dev.	$\sigma = \sqrt{VAR(R)}$
Covariance	$COV(R_1, R_2) = \sum_{S} p_s \left[R_{1,s} - E(R_{1,s})\right]\left[R_{2,s} - E(R_{2,s})\right]$
Correlation	$\rho_{1,2} = \dfrac{COV(R_1, R_2)}{\sigma_1 \sigma_2}$

Historical Returns (1999-2004)



Distribution of Historical Returns

Arithmetic Mean	$\overline{R} = \frac{1}{T}\sum_t R_t$
Geometric Mean	$\overline{GR} = \left[(1+R_1)(1+R_2)\cdots(1+R_T)\right]^{1/T} = \left[\frac{V_T}{V_0}\right]^{1/T}$
Variance	$VAR(R) = \frac{1}{T-1}\sum_t \left[R_t - \overline{R}\right]^2$
Std. Dev.	$\overline{\sigma} = \sqrt{VAR(R)}$
Covariance	$COV(R_1, R_2) = \frac{1}{T-1}\sum_t \left[R_{t,1} - \overline{R}_1\right]\left[R_{t,2} - \overline{R}_2\right]$
Correlation	$\overline{\rho}_{1,2} = \frac{COV(R_1, R_2)}{\overline{\sigma}_1 \overline{\sigma}_2}$

Summary Statistics of Monthly Returns (1999-2004)

	Tiffany	Lockheed Martin	Standard & Poor's 500
Arith. Mean	2.12%	1.23%	0.03%
Geom. Mean	1.20%	0.78%	-0.08%
Variance	$1.86\%^2$	$0.86\%^2$	$0.21\%^2$
σ	13.65%	9.26%	4.60%
$\rho(R_i,R_{TIF})$	1		
$\rho(R_i,R_{LKM})$	0.03	1	
$\rho(R_i,R_{SPY})$	0.74	-0.11	1

A Simple Portfolio

- Eg. Suppose we combine the stock above with a t-bill one-year t-bill which is yielding 5%. What is the mean, variance and standard deviation of a portfolio where 70% of the investment is in the stock and 30% is in the t-bill?

A More Complicated Portfolio

- Eg. Suppose we combine the S&P 500 index with a t-bill one-year t-bill which is yielding 5%. What is the mean, variance and standard deviation of a portfolio where 70% of the investment is in the index and 30% is in the t-bill? The market risk premium is 5% and the annual standard deviation is 20%.

Portfolios of Risky Assets

x_i - proportion of investment in i

$\tilde{r}_i$ - random return on asset i

$$E(x_1\tilde{r}_1 + x_2\tilde{r}_2) = x_1 E(\tilde{r}_1) + x_2 E(\tilde{r}_2)$$

$$Var(x_1\tilde{r}_1 + x_2\tilde{r}_2) = x_1^2 Var(\tilde{r}_1) + x_2^2 Var(\tilde{r}_2)$$
$$+ 2x_1 x_2 Cov(\tilde{r}_1, \tilde{r}_2)$$

Properties of Covariance

$\tilde{r}_i$ - random return on asset i

a, b - constant

$$\mathrm{Cov}(\tilde{r}_1, \tilde{r}_2) = E\left((\tilde{r}_1 - \bar{r}_1)(\tilde{r}_2 - \bar{r}_2)\right)$$

$$\mathrm{Cov}(\tilde{r}_1, \tilde{r}_2) = \rho_{12}\sigma_1\sigma_2$$

$$\mathrm{Cov}(a\,\tilde{r}_1, b\tilde{r}_2) = a\,b\ \mathrm{Cov}(\tilde{r}_1, \tilde{r}_2)$$

$$\mathrm{Cov}(a\,\tilde{r}_1, \tilde{r}_1) = a\mathrm{Var}(\tilde{r}_1)$$

Portfolios of Risky Assets

- Suppose we invest \$120 in IBM stock and \$180 in Bristol-Myers stock.
  - Determine the portfolio mean. ($\mu_{BM}=21\%$, $\mu_{IBM}=17\%$)
  - Determine the portfolio standard deviation. ($\sigma_{BM}=21\%$, $\sigma_{IBM}=31\%$, $\rho=0.135$)

A Simple Method for Calculating Portfolio Covariance

	Stock 1	Stock 2	Stock 3
Stock 1	$X_1^2 \mathrm{Var}(r_1)$	$X_1 X_2 \mathrm{Cov}(r_1, r_2)$	$X_1 X_3 \mathrm{Cov}(r_1, r_3)$
Stock 2	$X_2 X_1 \mathrm{Cov}(r_2, r_1)$	$X_2^2 \mathrm{Var}(r_2)$	$X_2 X_3 \mathrm{Cov}(r_2, r_3)$
Stock 3	$X_3 X_1 \mathrm{Cov}(r_3, r_1)$	$X_3 X_2 \mathrm{Cov}(r_3, r_2)$	$X_3^2 \mathrm{Var}(r_3)$

Portfolio Variance

- Suppose portfolio P is composed of N assets, let w denote portfolio weights, r denote components' return
- Then

$$r_p = w_1 r_1 + w_2 r_2 + w_3 r_3 + \ldots + w_n r_n = \sum_{i=1}^{N} w_i r_i$$

$$E(r_p) = w_1 E(r_1) + w_2 E(r_2) + w_3 E(r_3) + \ldots + w_n E(r_n) = \sum_{i=1}^{N} w_i E(r_i)$$

Portfolio Variance

- The variance of P is:

$$\sigma_p^2 = E\left[r_p - E(r_p)\right]^2$$

$$= E\left[\sum_{i=1}^{N} w_i r_i - \sum_{i=1}^{N} w_i E(r_i)\right]^2$$

$$= E\left[\sum_{i=1}^{N} w_i (r_i - E(r_i))\right]^2$$

$$= E\left[\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j [(r_i - E(r_i)][(r_j - E(r_j)]\right]$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{i,j}$$

Example

- Calculate the expected return and variance of a portfolio with $1000 invested in each of XOM, WMT and AMZN.

Annualized Covariance Matrix (Based on last 14 returns)			
	XOM	**WMT**	**AMZN**
XOM	0.015683	-0.001051	0.002407
WMT	-0.001051	0.047663	0.023407
AMZN	0.002407	0.023407	0.130953
Expected Returns	0.06358	0.235825	0.429066
Std Dev	0.164595		
Exp Ret	0.242823		

Market Condition	Return[a]				Rainfall	Return[a]
	Asset 1	Asset 2	Asset 3	Asset 5		Asset 4
Good	15	16	1	16	Plentiful	16
Average	9	10	10	10	Average	10
Poor	3	4	19	4	Poor	4
Mean return	9	10	10	10		10
Variance	24	24	54	24		24
Standard deviation	4.9	4.9	7.35	4.90		4.9

[a]The alternative returns on each asset are assumed equally likely and, thus, each has a probability of $\frac{1}{3}$.

Condition of Market	Asset 2	Asset 3	Combination of Asset 2 (60%) and Asset 3 (40%)
Good	$1.16	$1.01	$1.10
Average	1.10	1.10	1.10
Poor	1.04	1.19	1.10

The Benefits of Diversification

- How many stocks do you need to hold a well-diversified portfolio?
  - Assume that all the assets have the same standard deviation of 60% per year.
  - Assume that the correlation of the returns between two assets is 30%.
  - Assume that you invest the same amount in the different assets.

The Benefits of Diversification

- The variance of the return of a portfolio that includes N different assets depends on the weight w and on the covariances :

$$Var(r_P) = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j Cov(r_i, r_j)$$

$$= w_1 w_1 Cov(r_1, r_1) + w_1 w_2 Cov(r_1, r_2) + \cdots + w_1 w_N Cov(r_1, r_N)$$

$$+ w_2 w_1 Cov(r_2, r_1) + w_2 w_2 Cov(r_2, r_2) + \cdots + w_2 w_N Cov(r_2, r_N)$$

$$+ \cdots$$

$$+ w_N w_1 Cov(r_N, r_1) + w_N w_2 Cov(r_N, r_2) + \cdots + w_N w_N Cov(r_N, r_N)$$

The Benefits of Diversification

- If you invest equally in N different assets, then the weights are:
  - $w = 1/N$
- The covariances simplify to:
  - $Cov(r_i, r_j) = \rho_{i,j}\, \sigma_i\, \sigma_j = 0.3*(0.6)^2 = 0.108$
  - $Cov(r_i, r_i) = Var(r_j) = \rho_{i,i}\, \sigma_i\, \sigma_i = 1*(0.6)^2 = 0.36$

$$\sigma_p^{\,2} = \sum_{j=1}^{N} w_j^{\,2}\sigma_j^2 + \sum_{j=1}^{N}\sum_{\substack{k=1\\k\neq j}}^{N} w_j w_k \sigma_{jk}$$

$$= \sum_{j=1}^{N} \frac{1}{N^2}\sigma_j^2 + \sum_{j=1}^{N}\sum_{\substack{k=1\\k\neq j}}^{N} \frac{1}{N^2}\sigma_{jk}$$

$$= \frac{N\sigma^2}{N^2} + \frac{N(N-1)}{N^2}\rho\sigma^2$$

The Benefits of Diversification

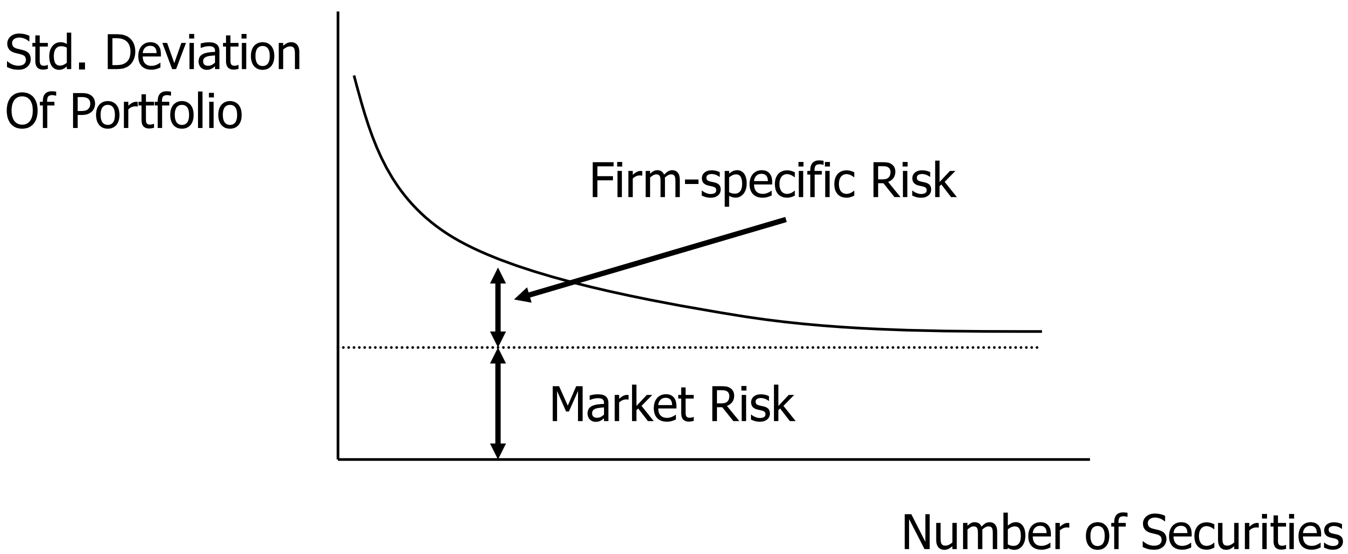
- Some tedious algebra shows that the variance of the total portfolio is:

$$Var(r_P) = \frac{1}{N}\sigma^2 + \frac{N-1}{N}\rho\sigma^2 = \frac{1}{N}0.36 + \frac{N-1}{N}0.108$$

Number of Stocks	Standard Deviation
1	60.0%
2	48.4%
5	39.8%
10	36.5%
50	33.6%
100	33.2%

Diversification

- The standard deviation of a portfolio tends to decrease as more risky assets are added to the portfolio



Diversification

- The total risk of a portfolio has two components:
  - Market Risk:
    - Risk factors common to the whole economy
    - Cannot be diversified away
  - Firm-specific Risk:
    - Risk factors that are specific to a company
    - Can be diversified away

The Benefits of Diversification in Practice (1998-2002)

Stock/ Index	Daily Standard Deviation
Yahoo	5.3%
Sun Microsystems	4.4%
Microsoft	2.7%
Ford	2.6%
Lockheed Martin	2.4%
General Electric	2.2%
Nasdaq 100	2.8%
Dow Jones Industrial Average	1.3%
Standard & Poor's 500 Index	1.4%
Wilshire 5000	1.4%

Diversification

- Only non-diversifiable risk matters.
- This means covariance or correlation (NOT variance) is what matters.
- Correlation with what?

Asset Allocation

- Asset Allocation is the portfolio choice among broad investment classes:
  - One risky asset and one risk-free asset
  - Two risky assets
  - Two risky assets and one risk-free asset
  - Many risky assets and one risk-free asset

Asset Classes

- Risky Assets
  - Stocks (S&P Comp. Index)
    - Mean Real Return: 10%
    - Standard Deviation: 20%
  - Bonds
    - Mean Real Return: 4%
    - Standard Deviation: 10%
    - Correlation with Real Stock Return: 0.2
  - T-Bills (Risk-free asset)
    - Real Return: 1%

Portfolio of Risk-Free Asset and Risky Asset

- What is the expected return and the standard deviation of a portfolio that invests:
  - w in stocks
  - $1-w$ in the risk-free asset?

Capital Allocation Line

- Expected Return of Portfolio:

$$E(r) = wE(r_s) + (1 - w)r_F$$

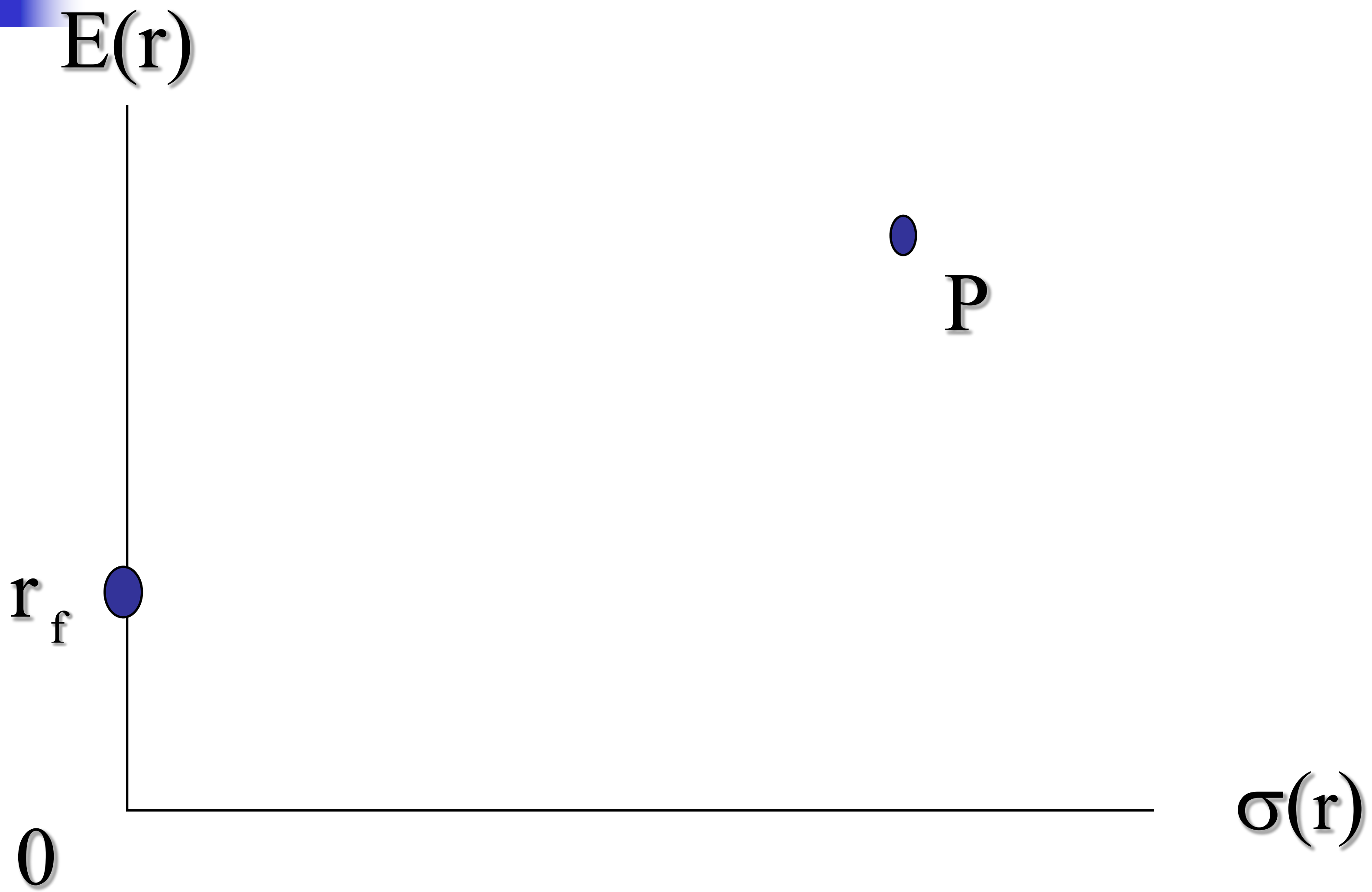
- Standard Deviation of Portfolio:

$$\sigma = w\sigma_S$$

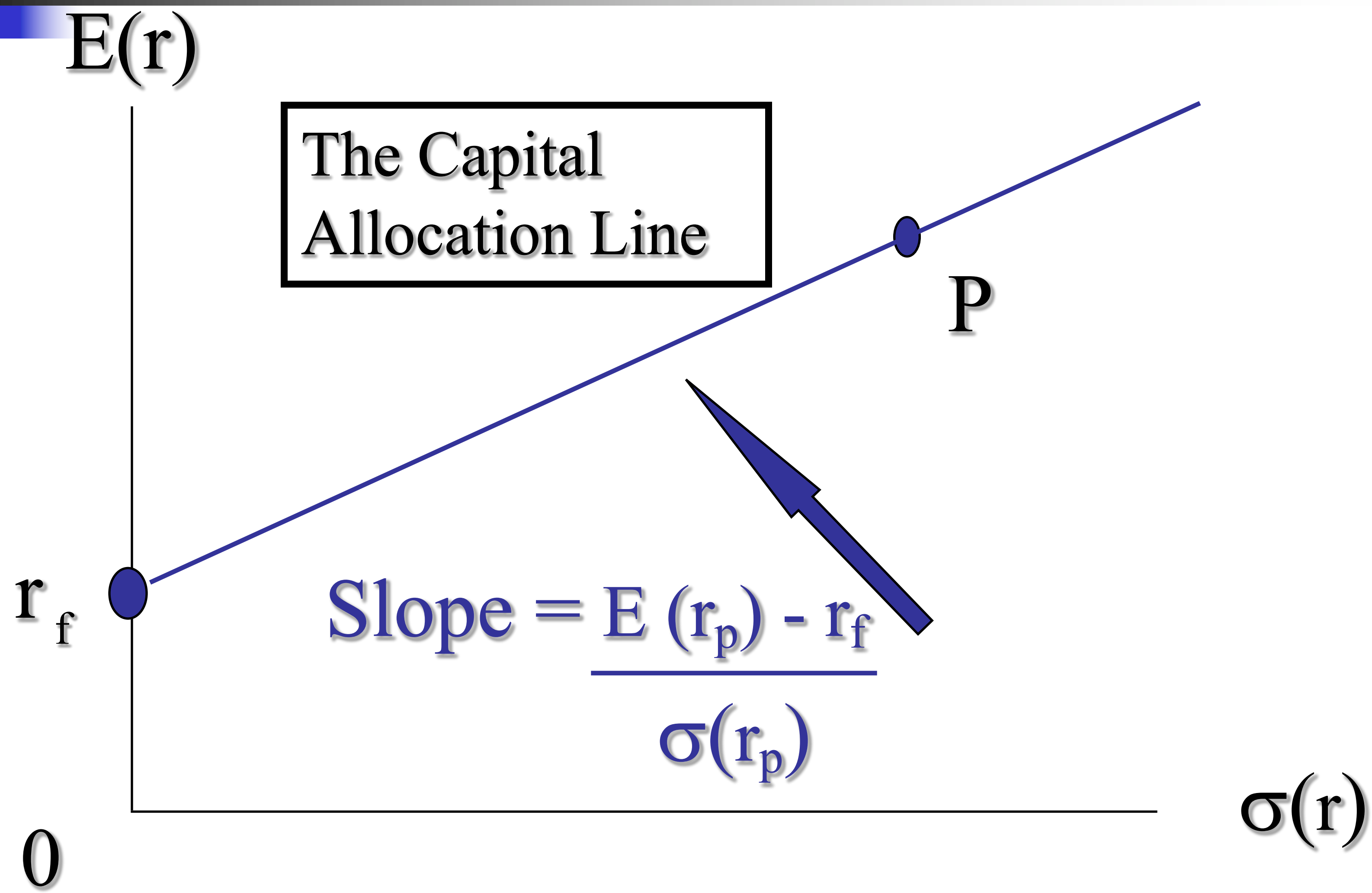
- Substituting for *w*, gives the Capital Allocation Line (CAL):

$$E(r) = r_F + \frac{E(r_S) - r_F}{\sigma_S}\sigma$$

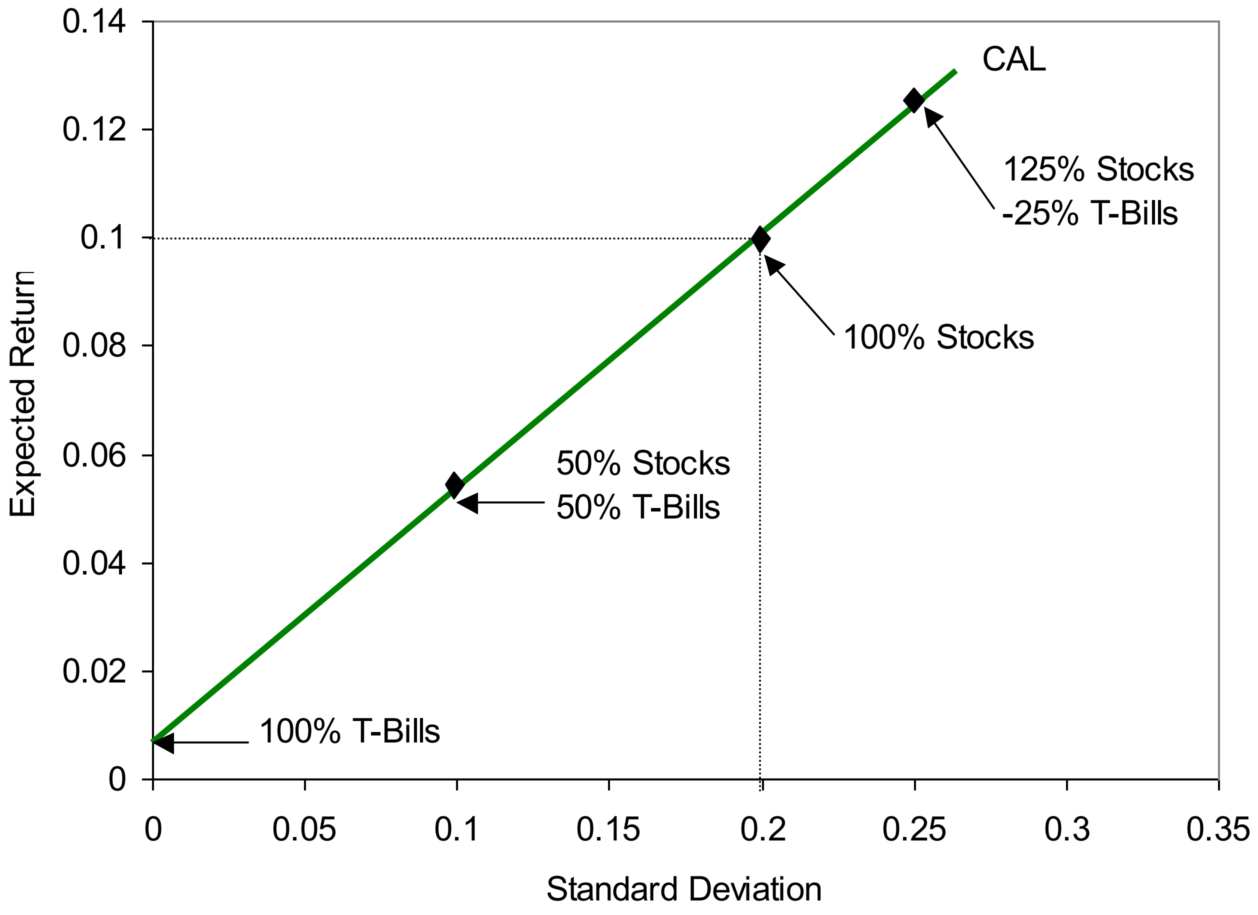
Capital Allocation Line



Capital Allocation Line



Capital Allocation Line



Capital Allocation Line

- The Capital Allocation Line shows the risk-return combinations available by changing the proportion invested in a risk-free asset and a risky asset
- The slope of the CAL is the reward-to-variability ratio

Risk Aversion

- Now the question is, which risk-return combination along the CAL do you want?
- To answer this we need to bring your preferences for risk into the picture
- We will use indifference curves to represent risk aversion
- Indifference curves represent utility functions

Risk Aversion & Utility

- Investor’s view of risk
  - Risk Averse
  - Risk Neutral
  - Risk Seeking
- Example:

Utility

- Utility
- Utility Function

$$U = E(r) - .005\, A\, \sigma^2$$

- A measures the degree of risk aversion

Risk Aversion and Value: The Sample Investment

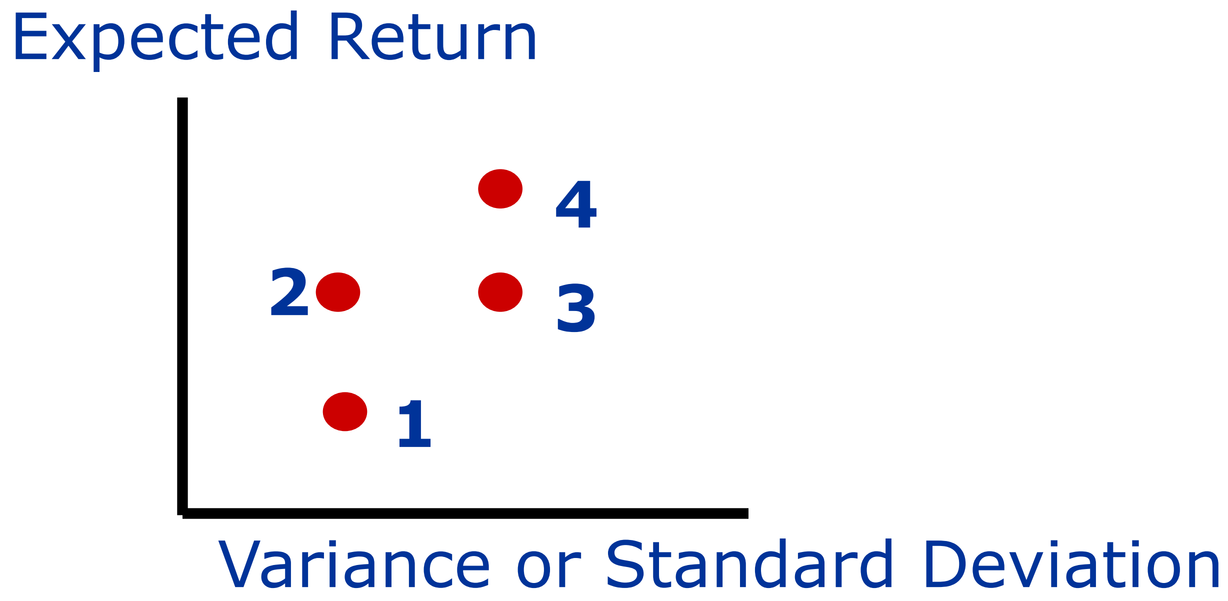
$$U = E(r) - .005\, A\, \sigma^2$$

$$= 22\% \quad - \quad .005\, A\, (34\%)^2$$

Risk Aversion	A	Utility
High	5	-6.90
↓	3	4.66
Low	1	16.22

T-bill = 5%

Dominance Principle



- 2 dominates 1; has a higher return
- 2 dominates 3; has a lower risk
- 4 dominates 3; has a higher return

Dominance Principle

A dominates B if

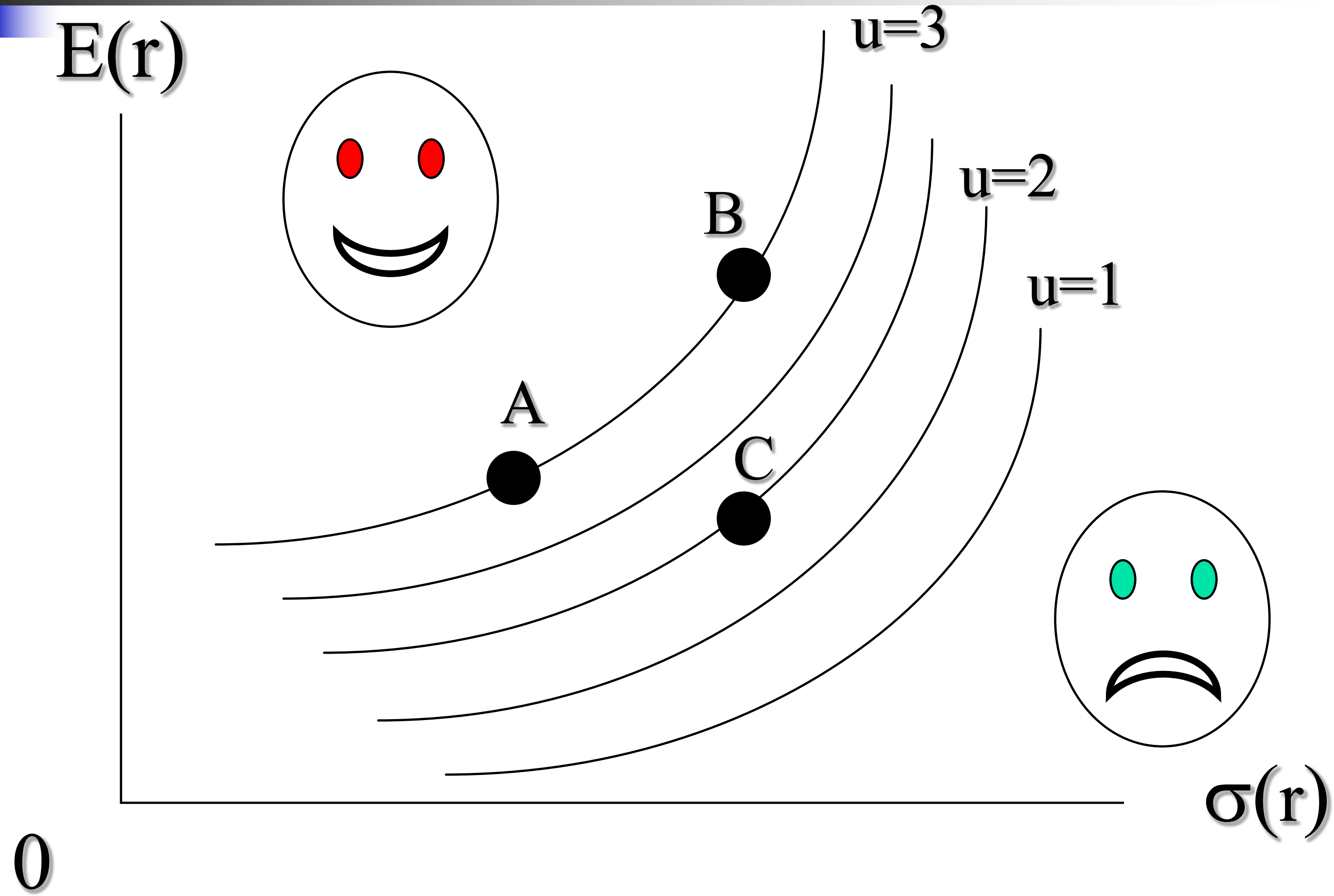
$$E(r_A) \geq E(r_B)$$

and

$$\sigma_A \leq \sigma_B$$

and at least one inequality is strict

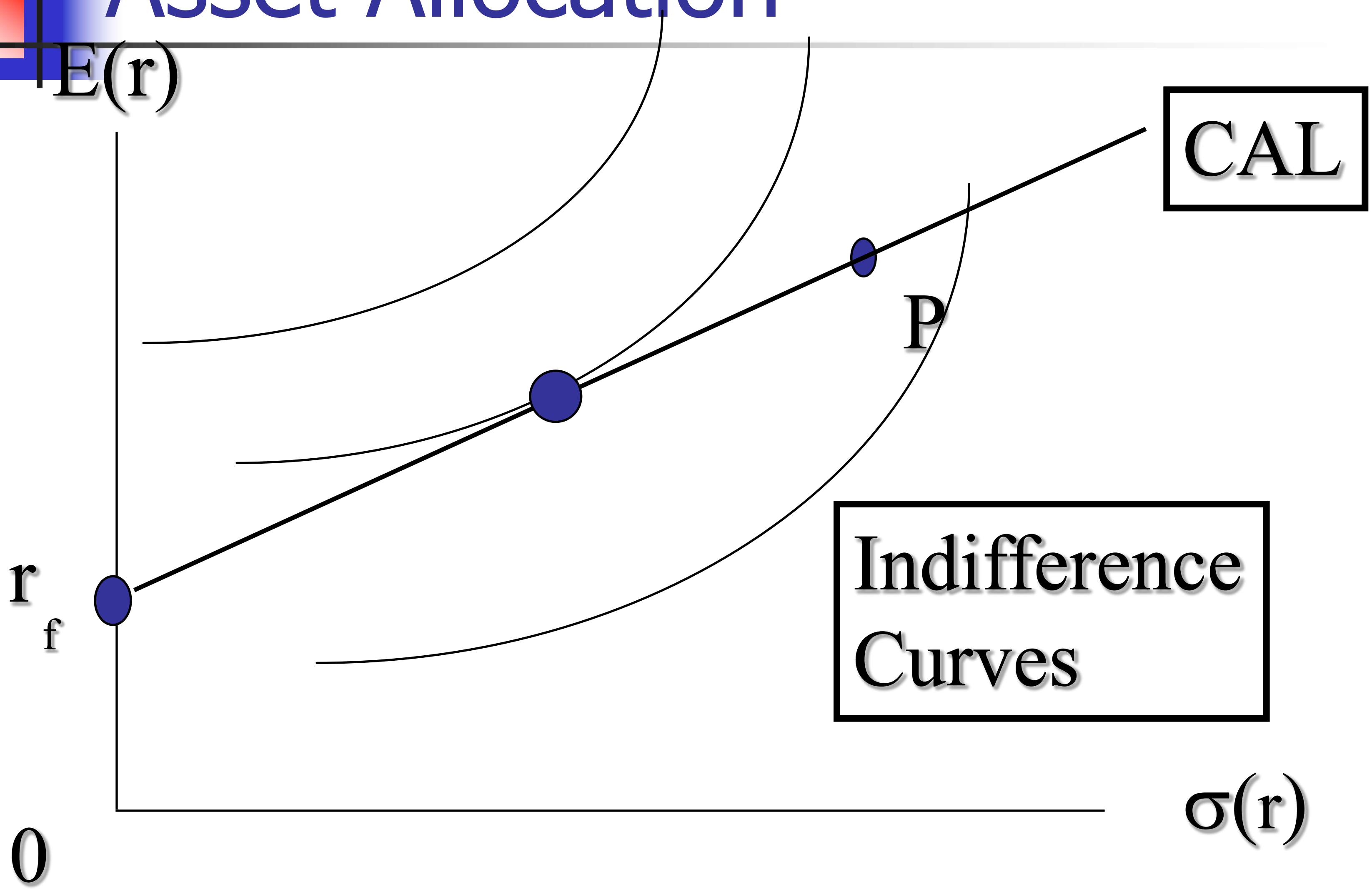
Indifference Curves



Asset Allocation

- Now we can combine the indifference curves with the capital allocation line
- If investors are maximizing their utility, they will choose the highest possible indifference curve
- The highest curve is tangent to the CAL

Asset Allocation



Utility Function

We have:

$$E(r) = wE(r_p) + (1 - w)r_f$$

So:

$$U = E(r) - 0.005A\sigma^2$$

$$= wE(r_p) + (1 - w)r_f - 0.005Aw^2\sigma_p^2$$

Optimal Holding

First-order condition to maximize U:

$$\frac{dU}{dw} = [E(r_p) - r_f] - 0.01Ay\sigma_p^2 = 0$$

Solving for w:

$$w = [E(r_p) - r_f] / 0.01A\sigma_p^2$$

An example: if

$$A = 4,$$

$$r_f = 7\%$$

$$E(r_p) = 15\%$$

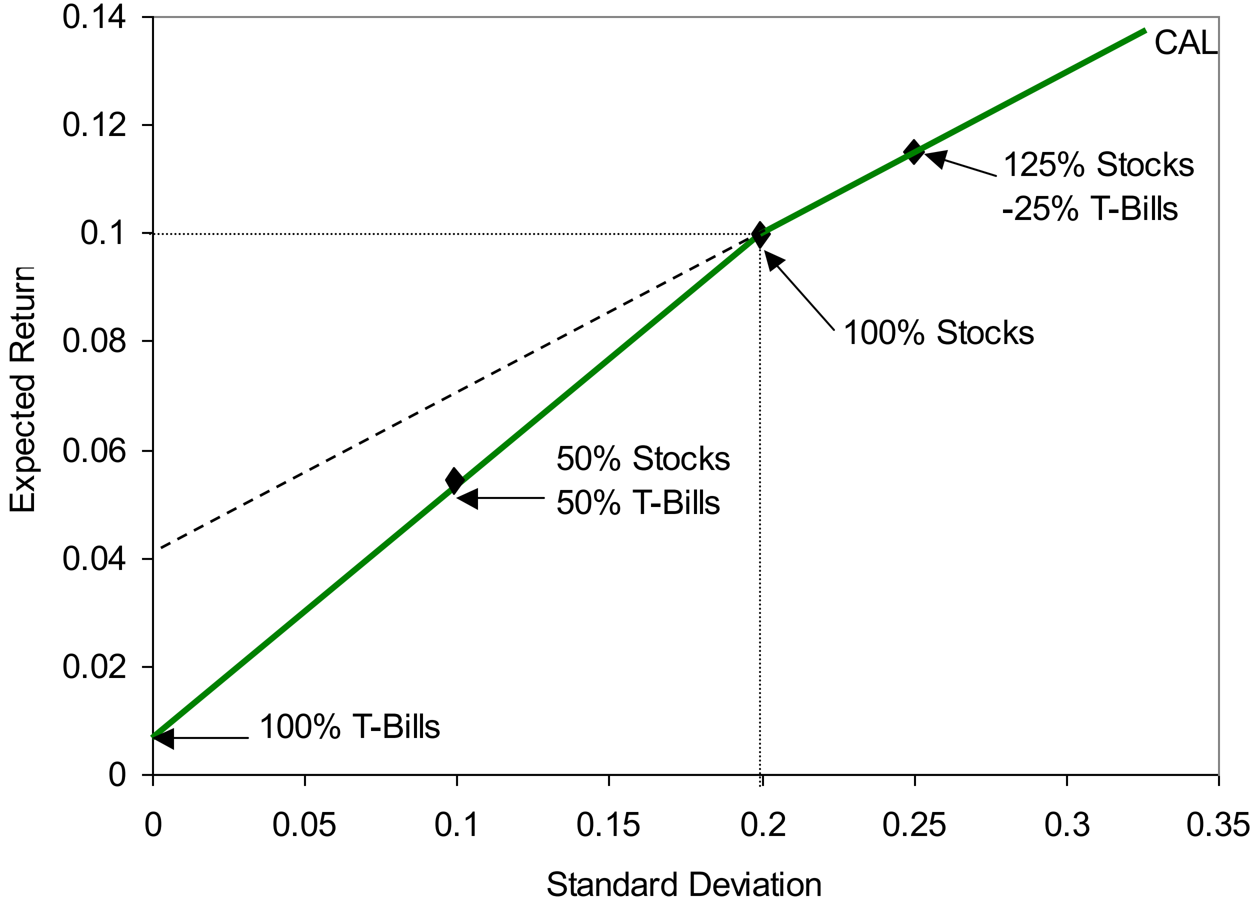
$$\sigma_p = 22\%$$

$$w = (15 - 7)/(0.01 \times 4 \times 22^2) = 0.41$$

Capital Allocation Line

- In practice, investors cannot borrow at the T-Bill rate
- How does the CAL change if the borrowing rate is 4%?

Capital Allocation Line with Higher Borrowing Rates



Portfolio Choice

- The investor chooses a point on the CAL that maximizes the utility
- The choice is determined by the risk aversion of investors
  - Risk-averse investors will invest more in the risk-free asset
  - Risk-tolerant investors will invest more in the risky asset

Asset Allocation with Two Risky Assets

- What is the expected return and the standard deviation of a portfolio that invests:
  - *w* in Stocks
  - 1-*w* in Bonds?

Risk-Return Tradeoff

- Expected Return of Portfolio:

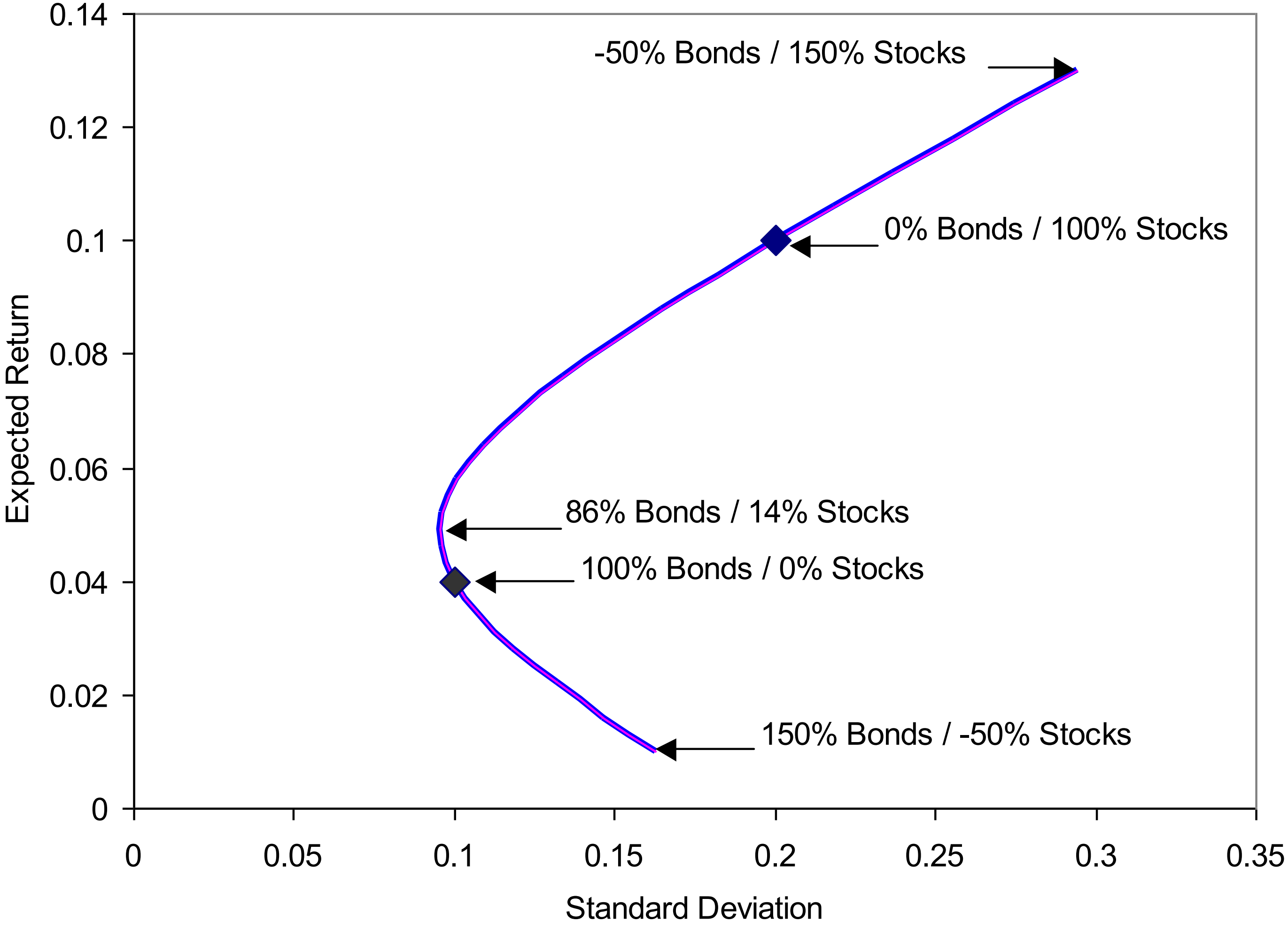
$$E(r^P) = wE(r^S) + (1-w)E(r^B)$$

- Variance of Portfolio

$$VAR(R_P) = (w_S)^2 VAR(R_S) + (1-w_S)^2 VAR(R_B)$$
$$+2(w_S)(1-w_S)COV(R_B, R_S) = (w_S)^2 VAR(R_S) + (1-w_S)^2 VAR(R_B)$$
$$+2(w_S)(1-w_S)CORR(R_B, R_S)SD(R_S)SD(R_B)$$

- Substituting for *w*, gives the Capital Investment Opportunity Set

Portfolio Frontier with Stocks and Bonds



Portfolio Frontier

- The portfolio frontier depicts the feasible portfolio choices for investors holding stocks and bonds
- The minimum variance portfolio includes 86% bonds and 14% stocks
- Portfolios below the minimum variance portfolio are inefficient
- The portfolio frontier above the minimum variance portfolio is called ‘efficient frontier’

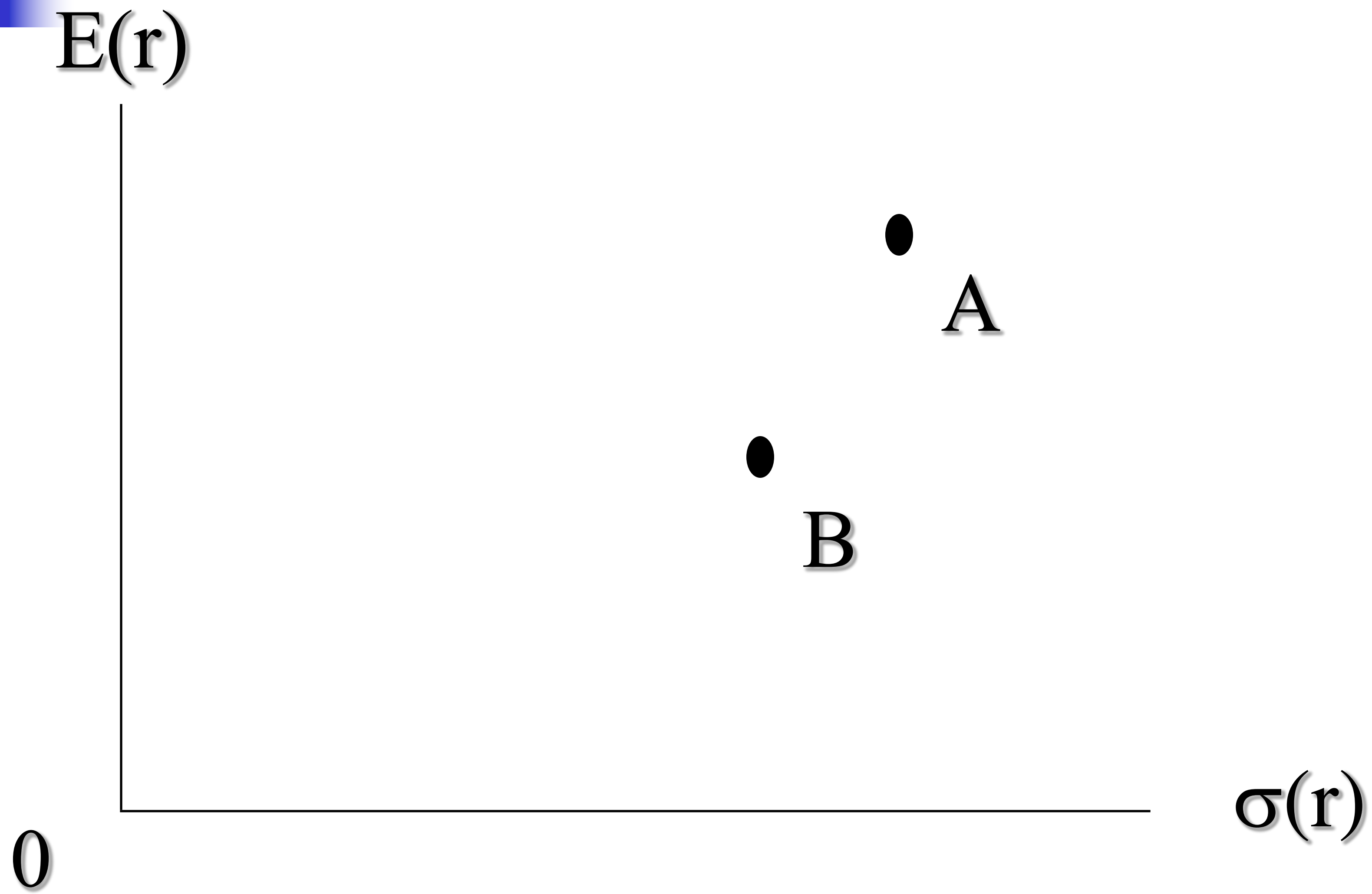
Correlation: Two Risky Assets

- To see the importance of correlation, we will look at the set of feasible portfolios under three different assumptions:
  - 1) $\rho_{AB} = 1$
  - 2) $\rho_{AB} = -1$
  - 3) $\rho_{AB} = 0$
- Then we will discuss the intermediate cases

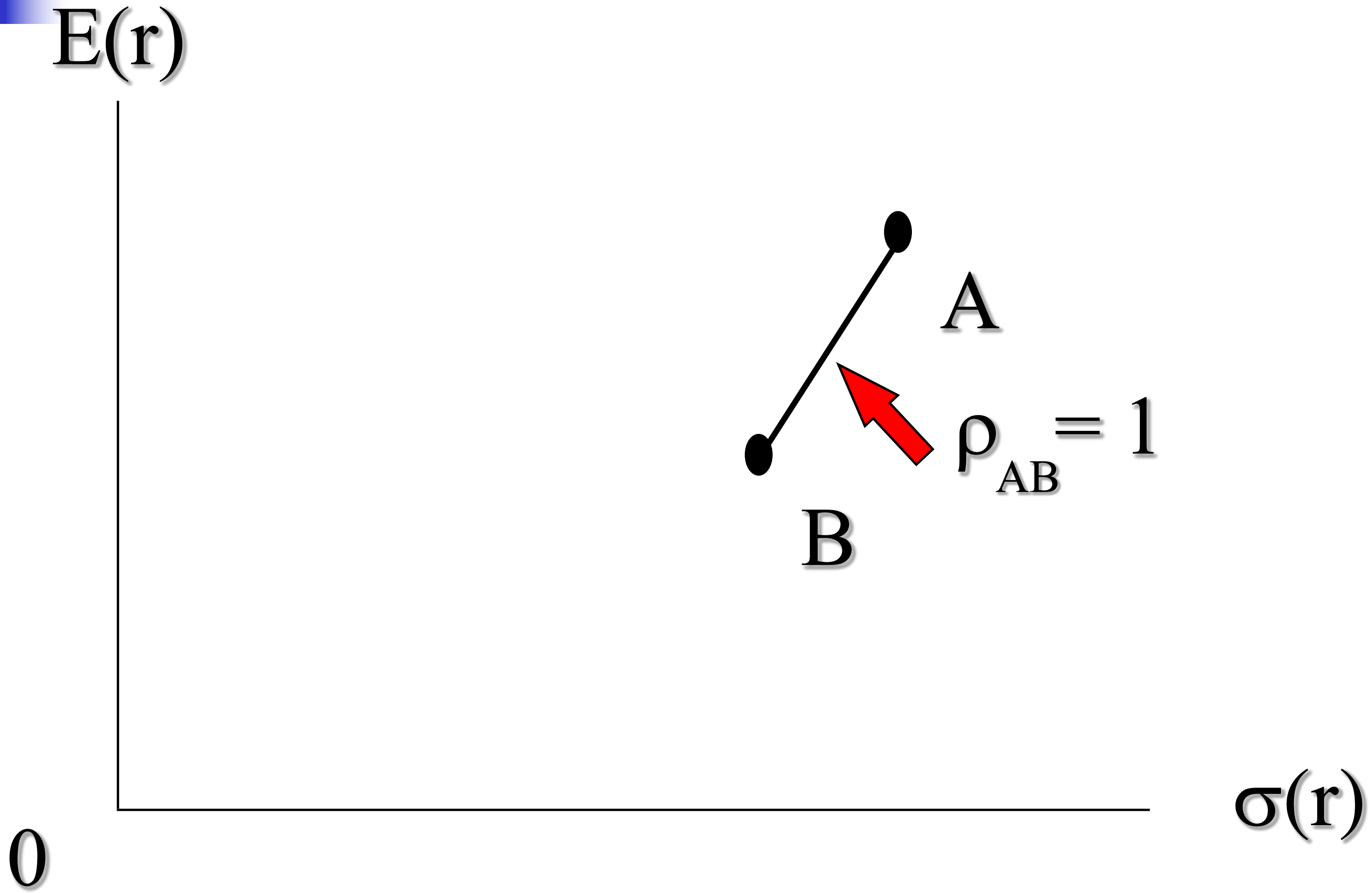
Perfect Correlation

- Case 1: $\rho_{AB} = 1$
- When $\rho_{AB} = 1$ we can simplify the variance:
- $\sigma_P^2 = (x\sigma_A + (1-x)\ \sigma_B)^2$
- The two relevant equations are therefore:
  - $E\ (r_P) = xE(r_A) + (1-x)\ E(r_B)$
  - □$\sigma_P = x\sigma_A + (1-x)\sigma_B$
- These two equations give us the set of feasible portfolios

Perfect Correlation



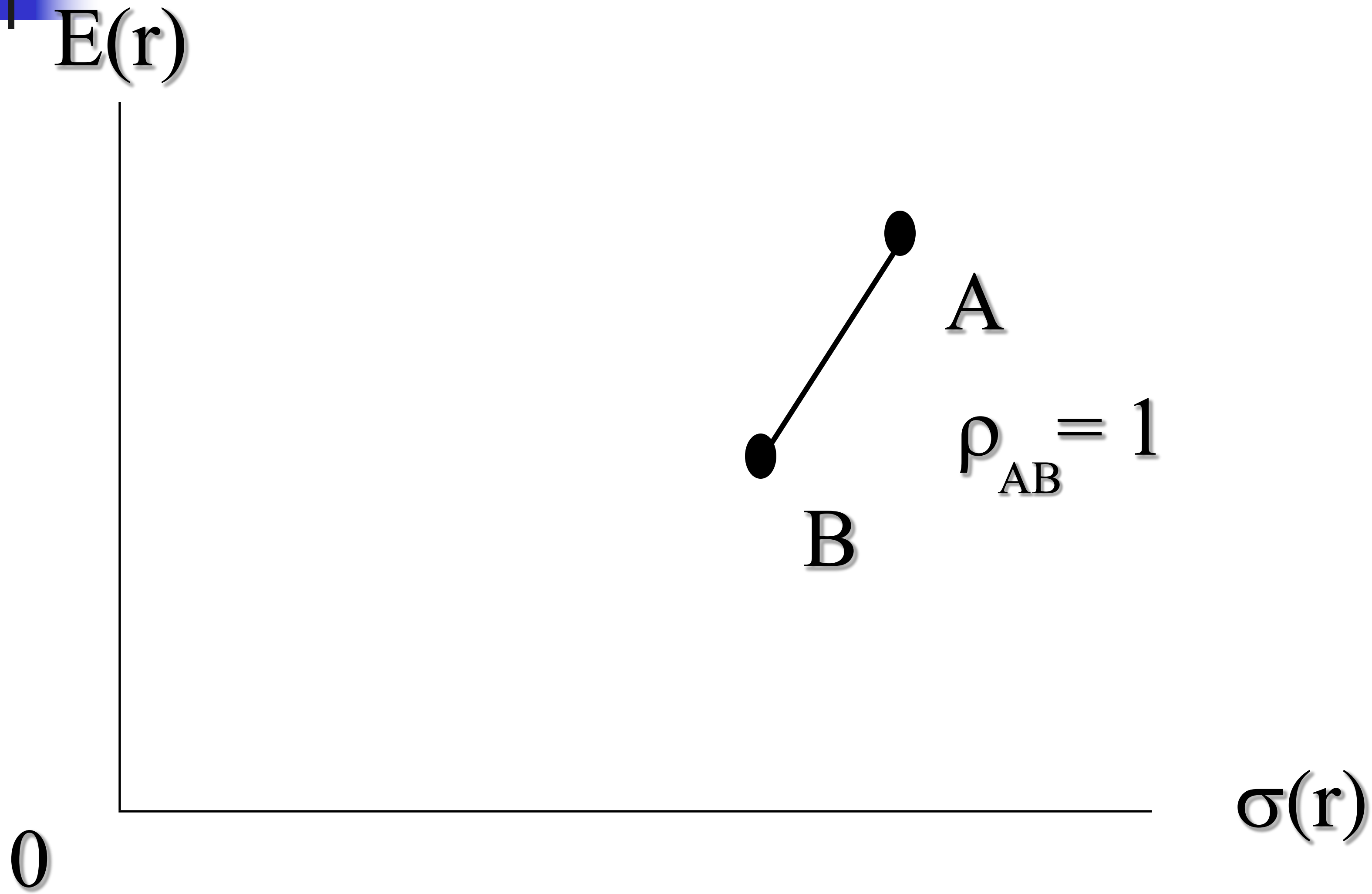
Perfect Correlation



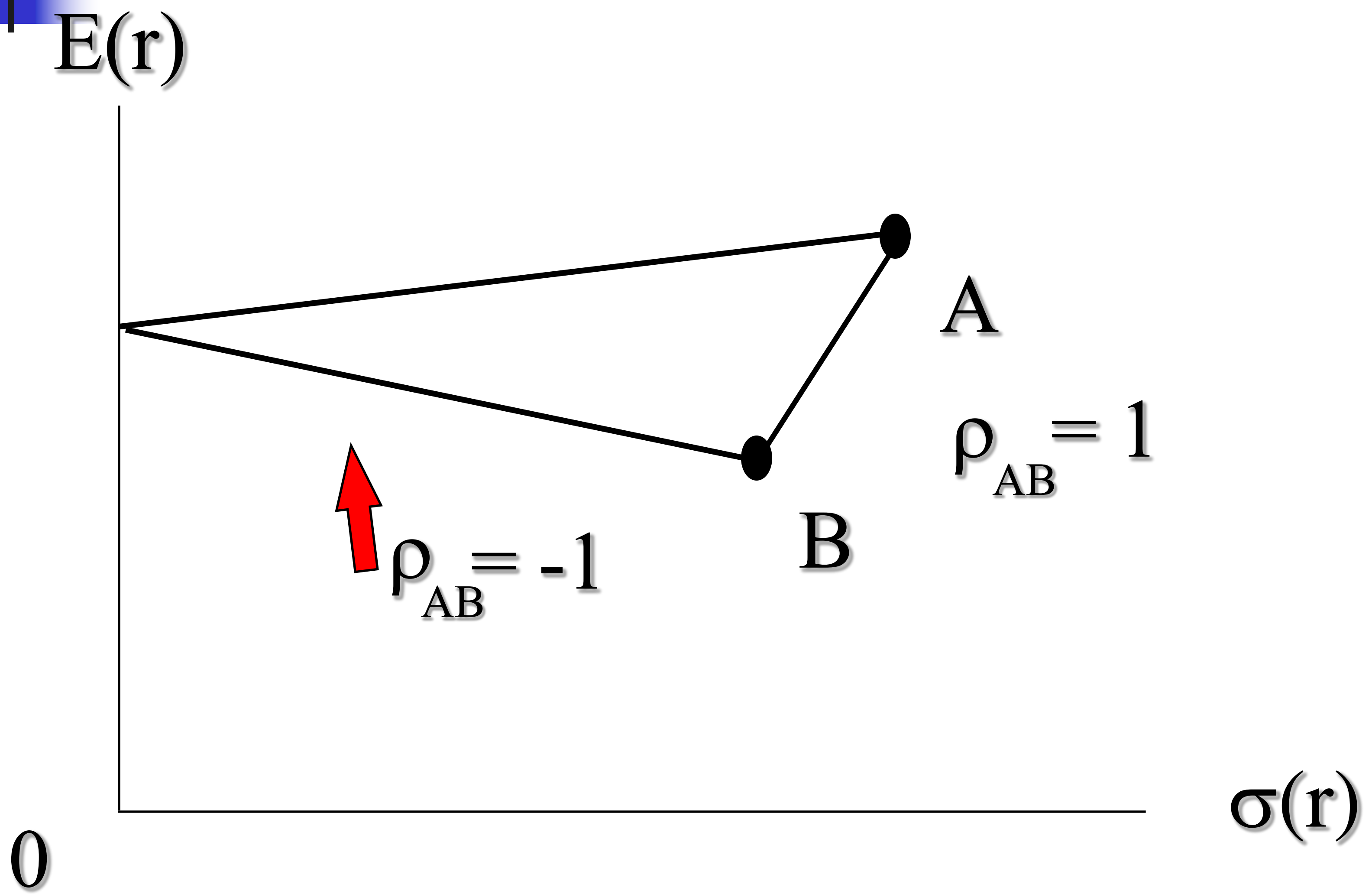
Perfect Negative Correlation

- Case 2: $\rho_{AB} = -1$
- When $\rho_{AB} = -1$ we can simplify variance:□□□□□ $\sigma_P^2 = (x\sigma_A - (1-x)\sigma_B)^2$
- The two relevant equations are therefore:
  - $E(r_P) = xE(r_A) + (1-x)\,E(r_B)$
  - □ $\sigma_P = |x\sigma_A - (1-x)\,\sigma_B|$
- These two equations give us the set of feasible portfolios

Perfect Negative Correlation



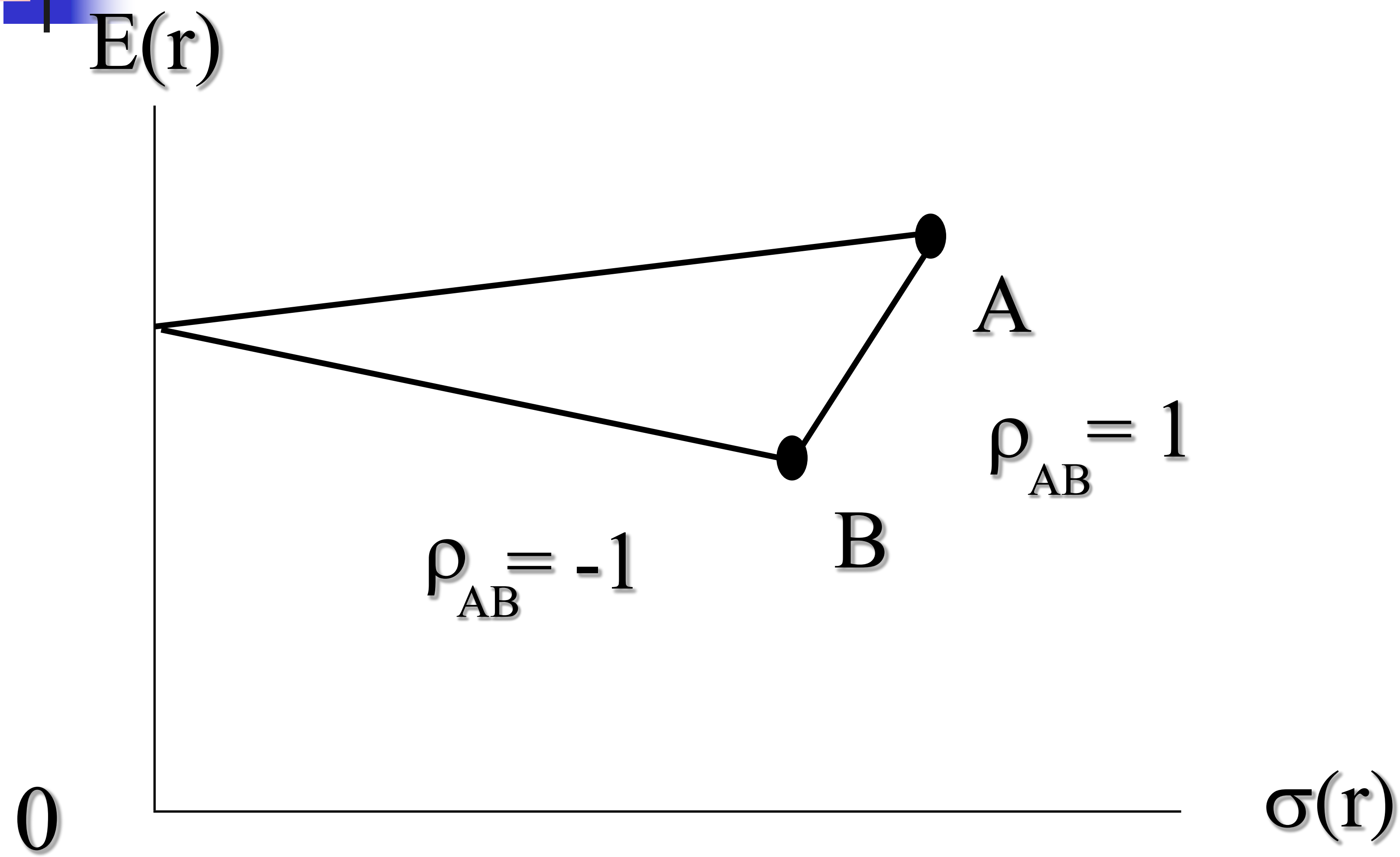
Perfect Negative Correlation



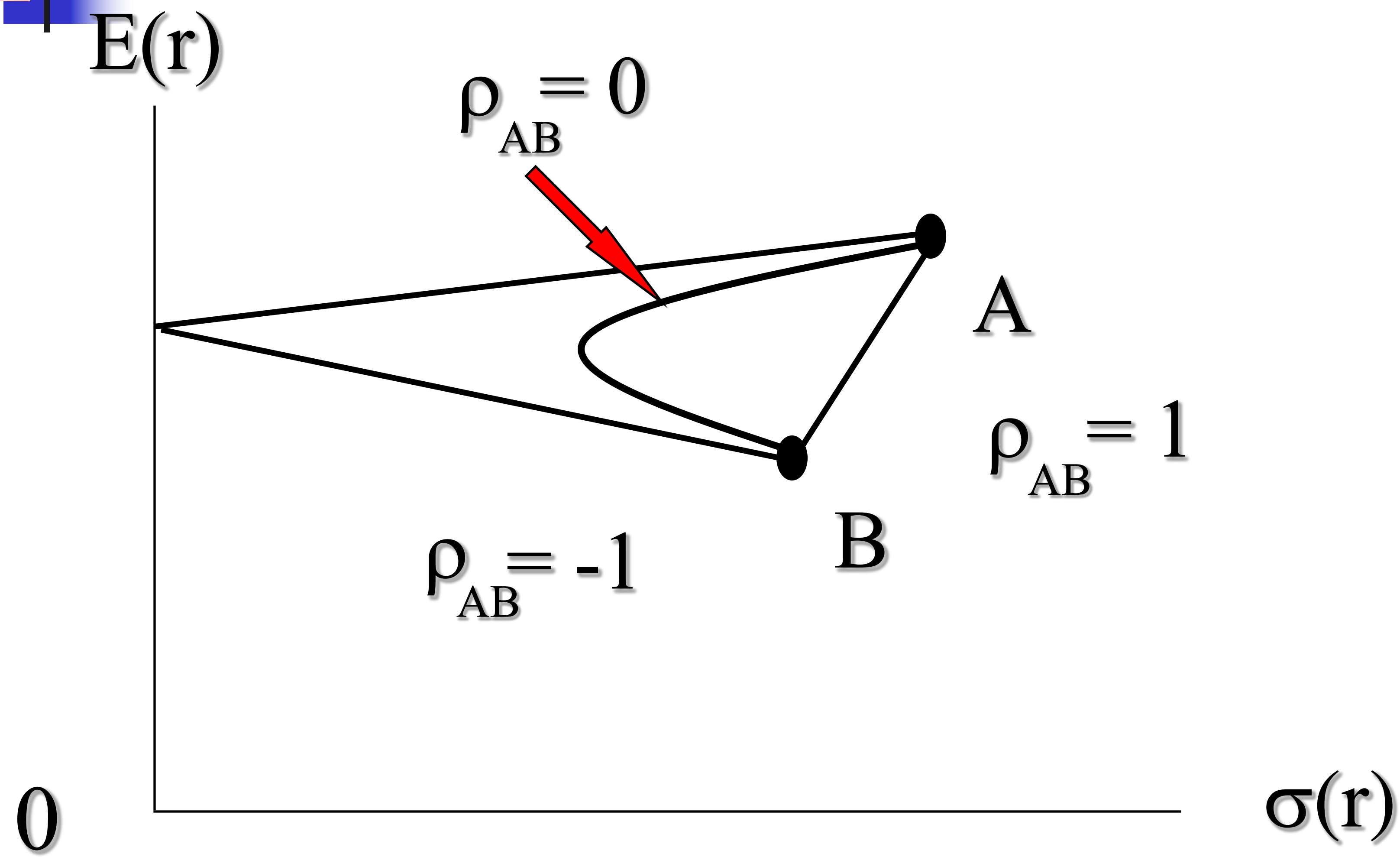
No Correlation

- Case 3: $\rho_{AB} = 0$
- When $\rho_{AB} = 0$ we cannot simplify the variance equation
- However, we can graph the set of feasible portfolios

No Correlation



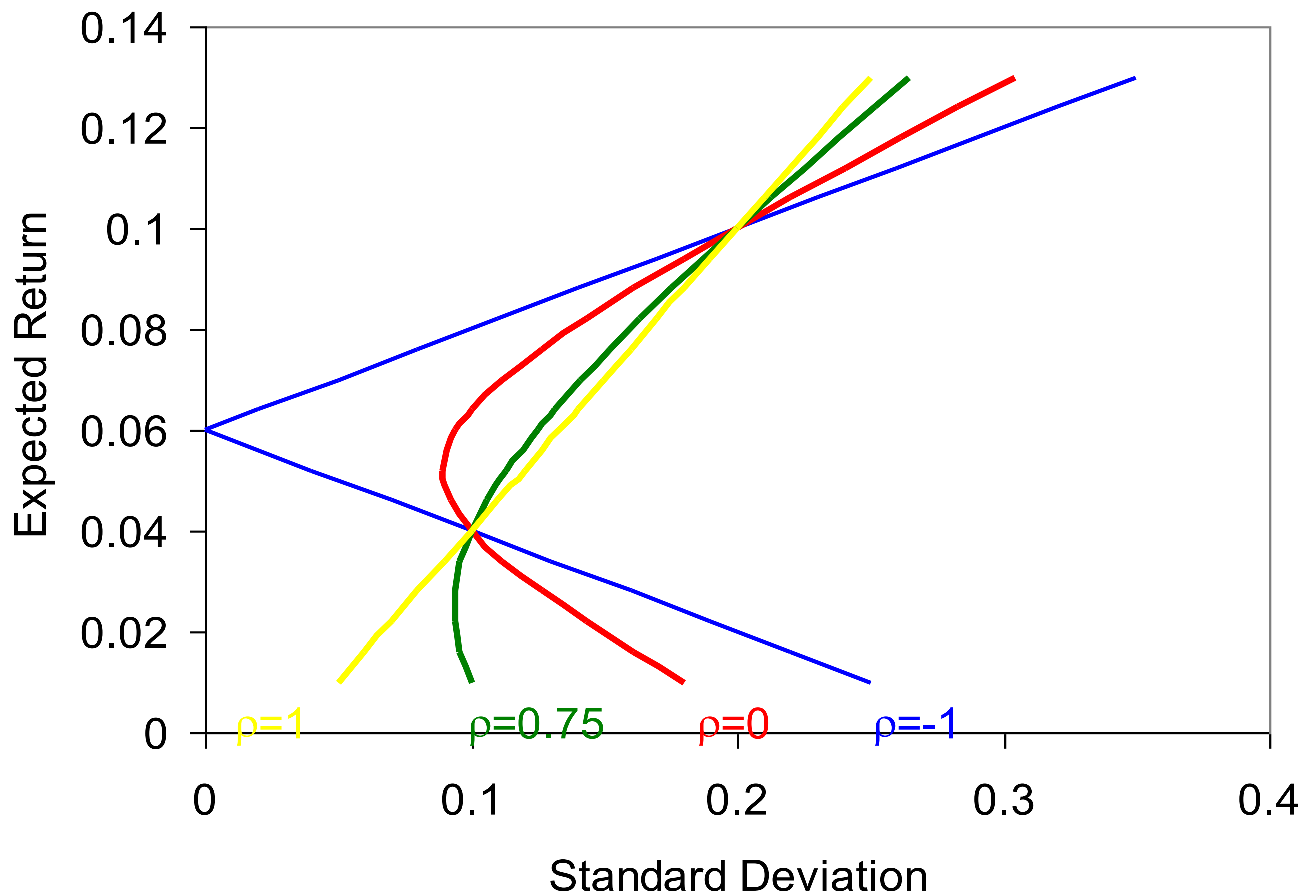
No Correlation



Assignment

- List two close-end funds in China and draw the graphic of their historical performances
- Choose two stocks in any fund above, and present their average return and risk
- Suppose you equally invested in those two stocks, present your portfolio’s average return and risk

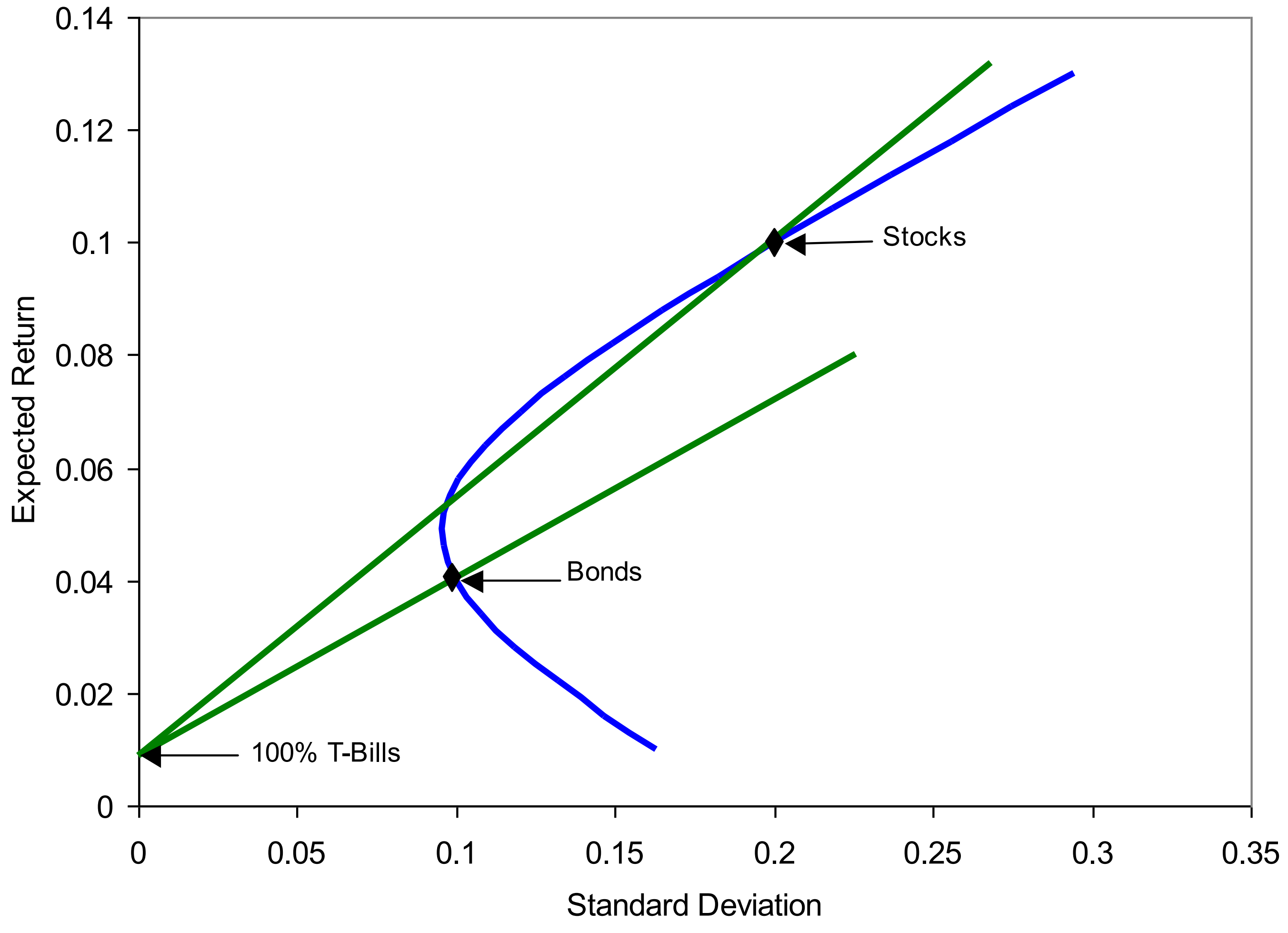
Portfolio Frontiers with Different Asset Correlations



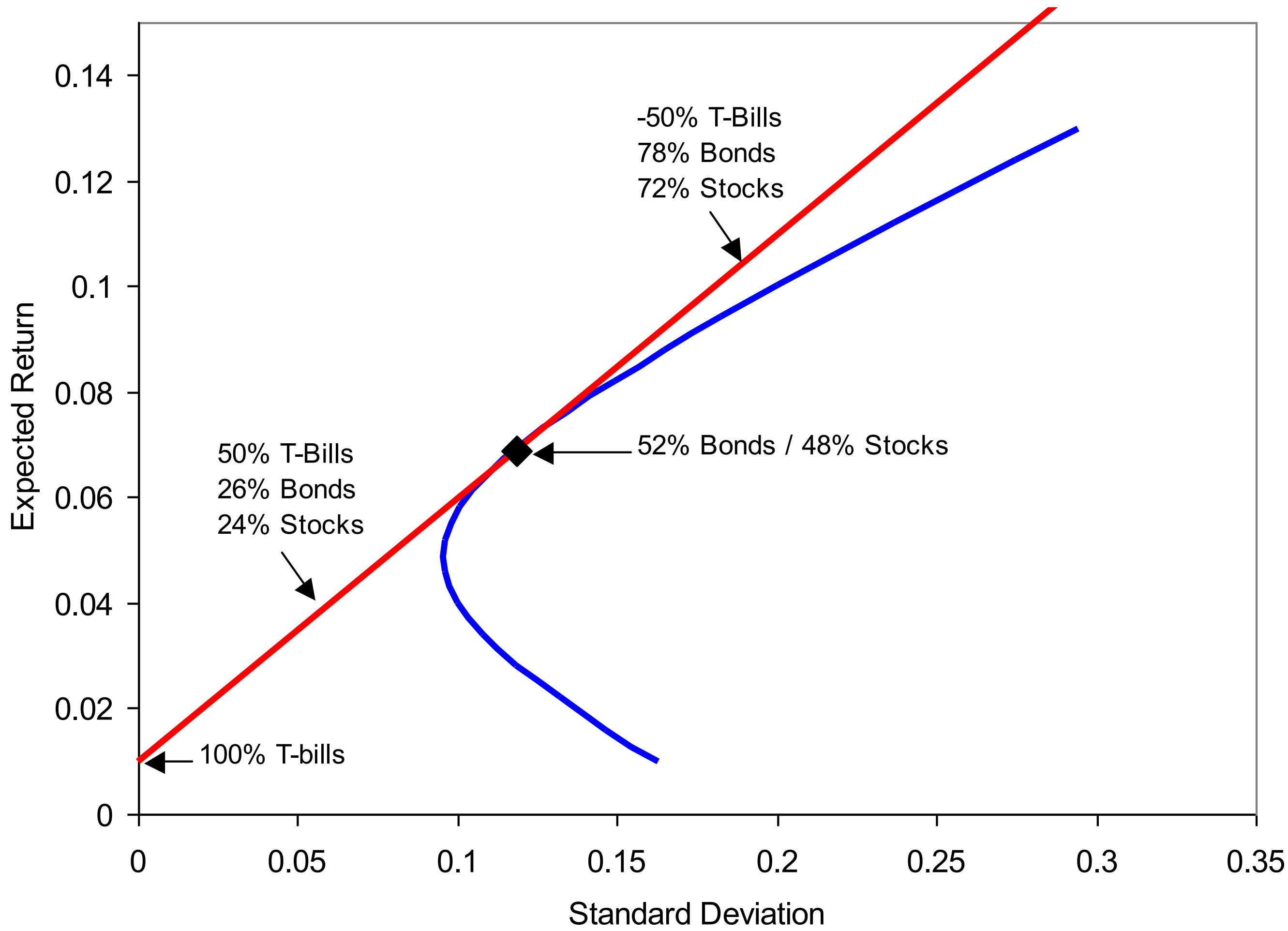
Asset Allocation with Risk-Free Asset

- Introducing a risk-free asset besides stocks and bonds improves the investment opportunities

Capital Allocation Lines using Stocks and Bonds



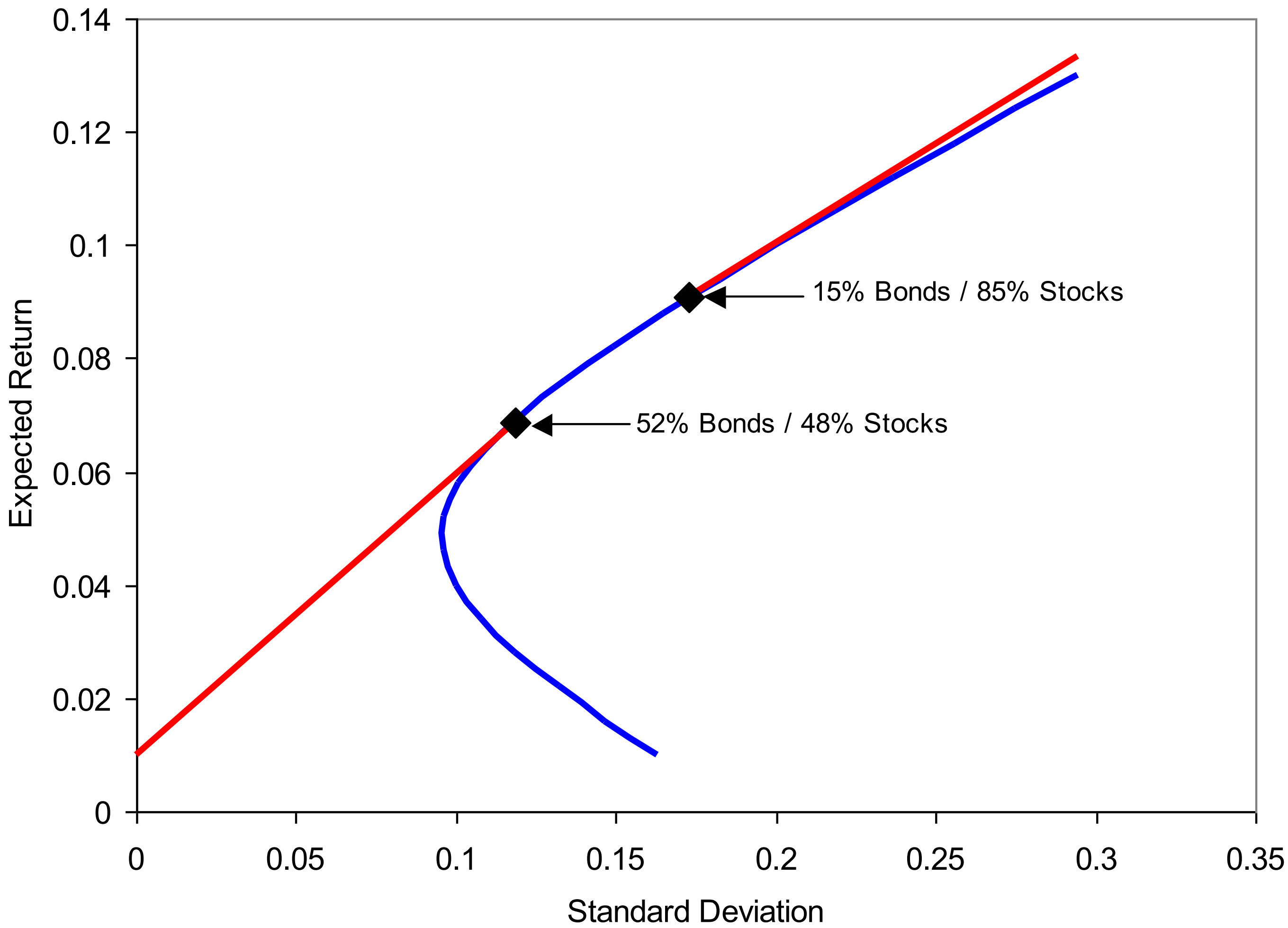
Optimal Capital Allocation Line



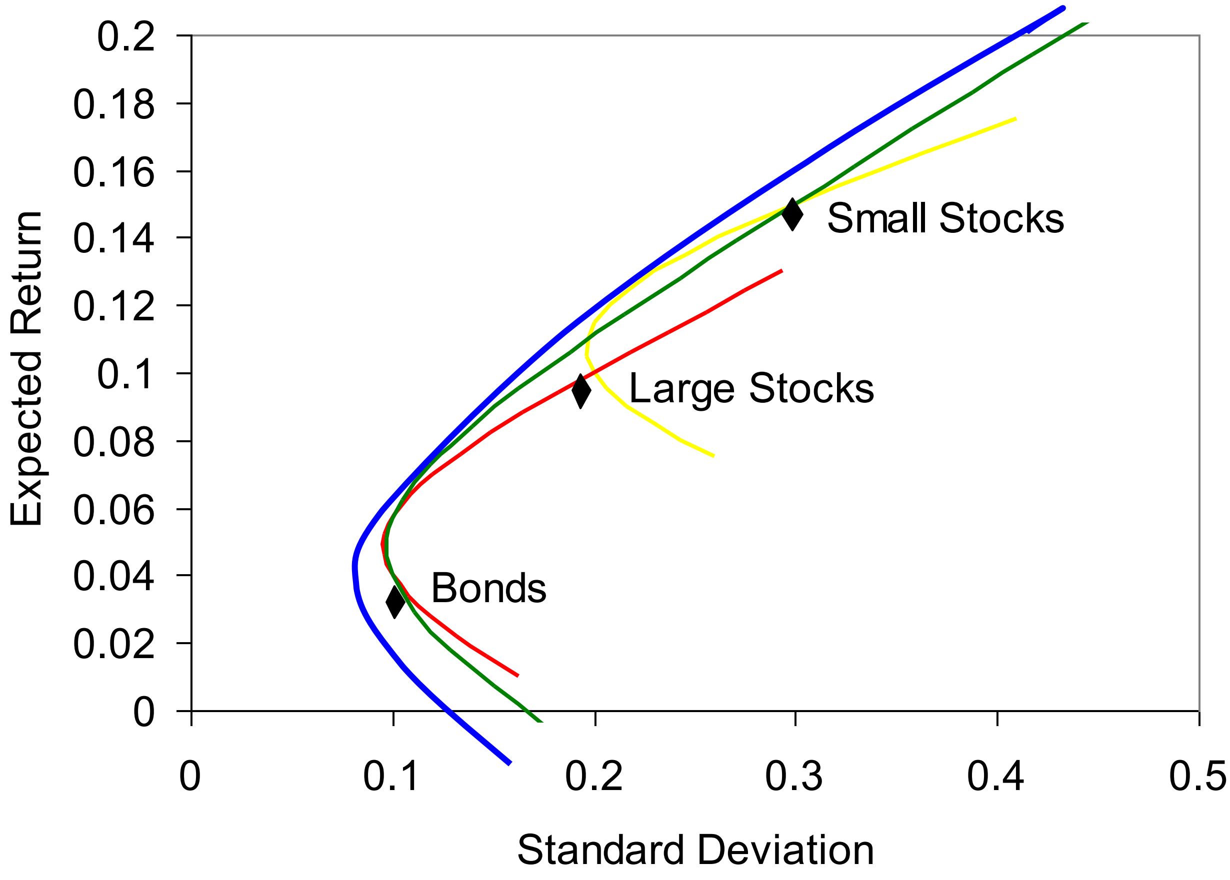
Optimal Capital Allocation Line

- The optimal capital allocation line combines T-Bills with the tangency-portfolio, which consists of 52% bonds and 48% stocks
- Separation Property
  - Determination of optimal risky portfolio (Tangency Portfolio)
  - Personal choice of best mix of tangency portfolio and risk-free asset

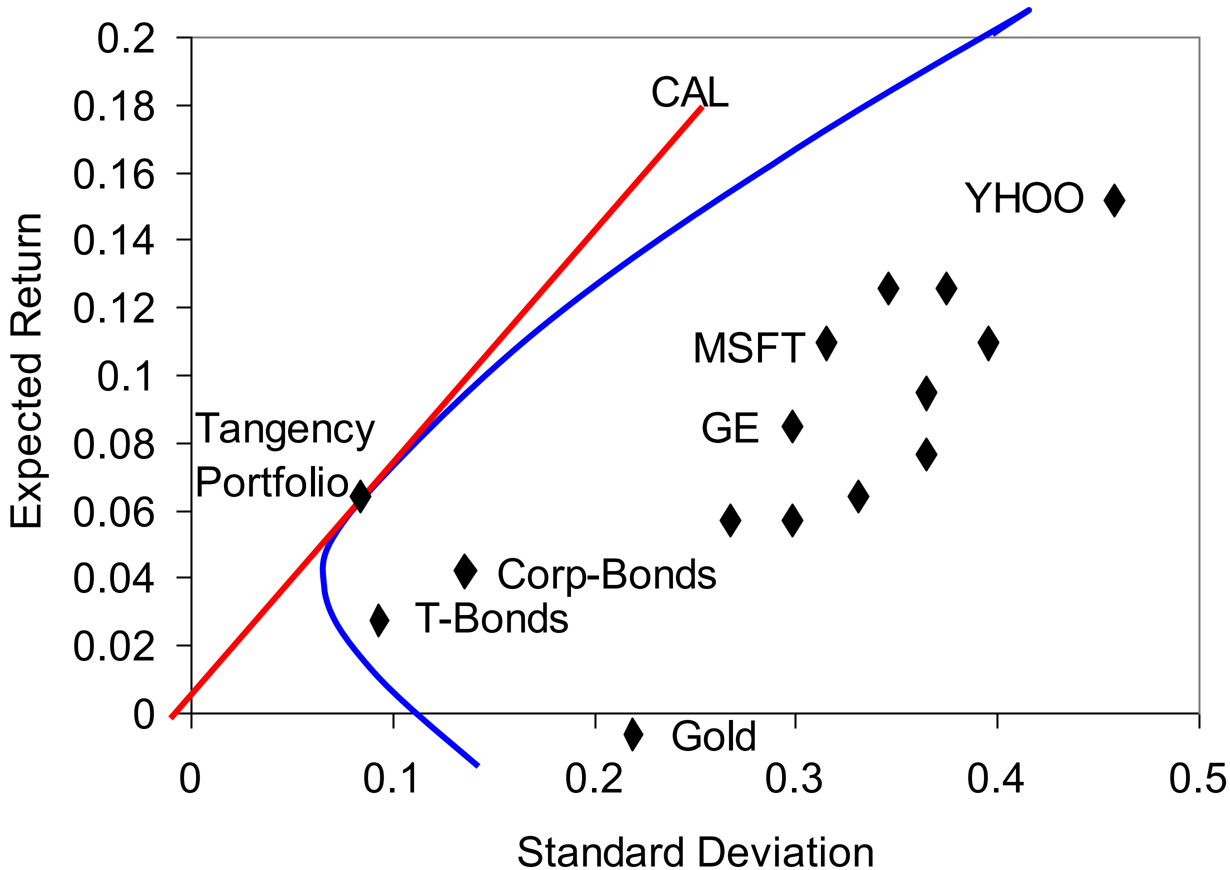
Portfolios with Different Lending and Borrowing Rates



Portfolio Frontier with Three Risky Assets



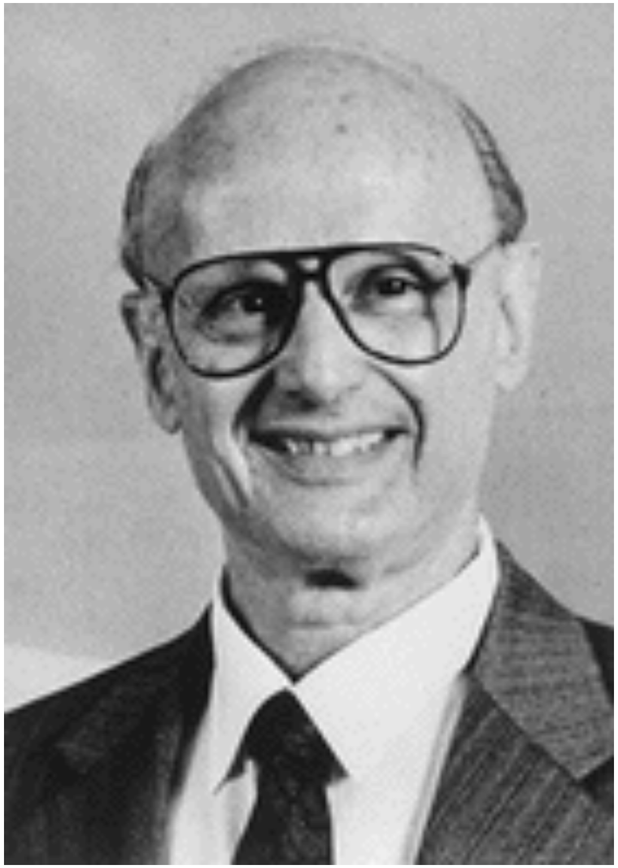
Portfolio Frontier with Many Risky Assets



Right or Wrong?

- If you are young you should be putting money into a couple of good growth stocks, maybe even into a few small stocks. Now is the time to take risks.
- If you are close to retirement, you should be putting all of your money into bonds and safe stocks, and nothing into the risky stocks.

Modern Portfolio Theory



- Harry Markowitz
- 1990 Nobel Prize
- 751,000 (473,000) in Google (name)
- 3,780,000 (1,300,000) in Google (MPT)

Portfolio Theory - The Markowitz Approach

- Assumptions
  - Investors want to **buy and hold** a portfolio of risky stocks.
  - The investors like high expected returns, don’t like high variance (equivalently standard deviation) and don’t care about other aspects of portfolio return distributions.

Return

- In this setting, portfolio return is defined by:

$$\frac{\text{End-of-period wealth - Beginning-of-period wealth}}{\text{Beginning-of-period wealth}}$$

$$\tilde{r}_p = \frac{\tilde{W}_1 - W_0}{W_0}$$

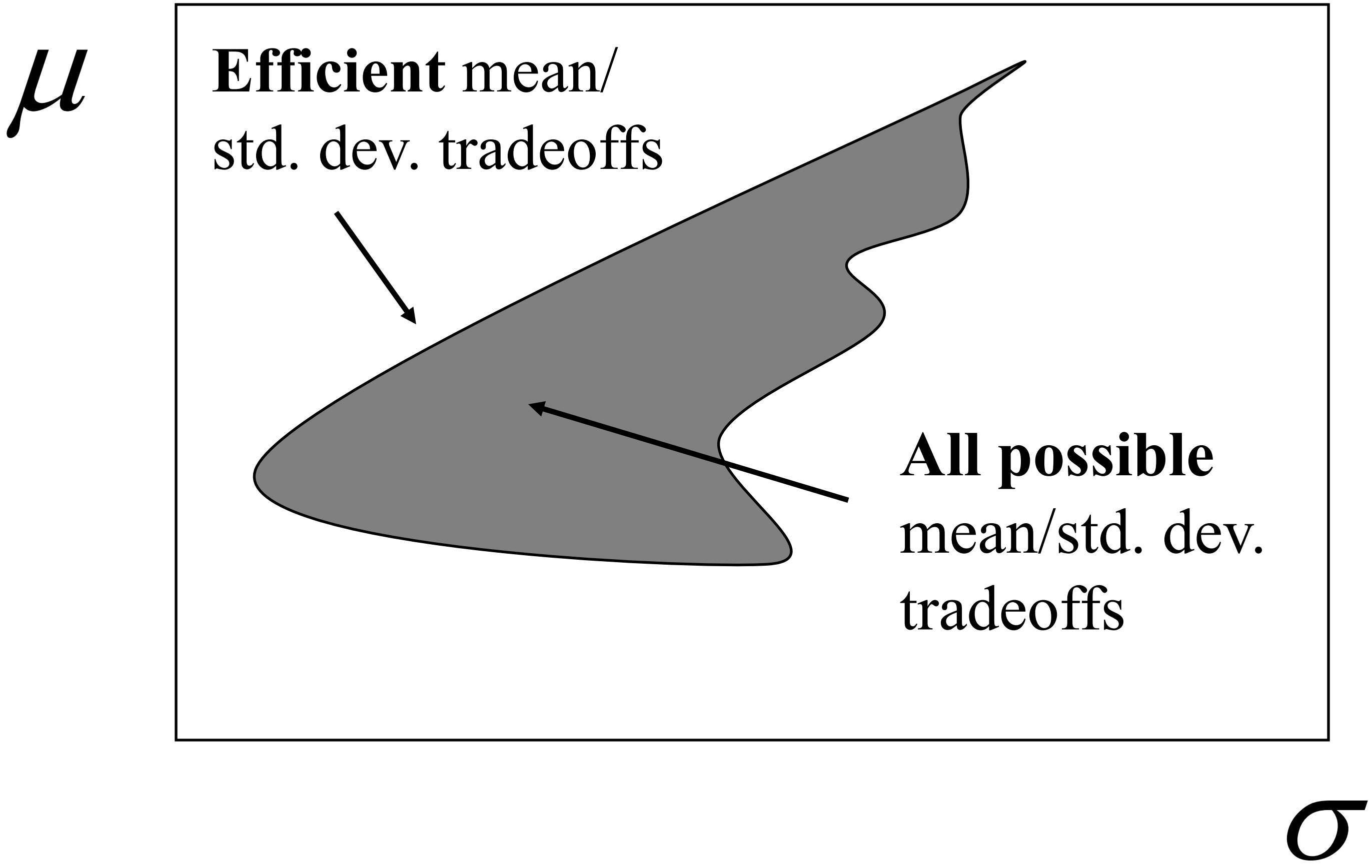
Example

- The only tricky part about applying this definition occurs when some short positions are involved:
- What is the realized portfolio return if you invested \$100 in the S&P500 last year by borrowing \$20 at 5% per year and used \$80 in cash? Assume that the \$100 invested in the S&P500 has now grown to \$110.

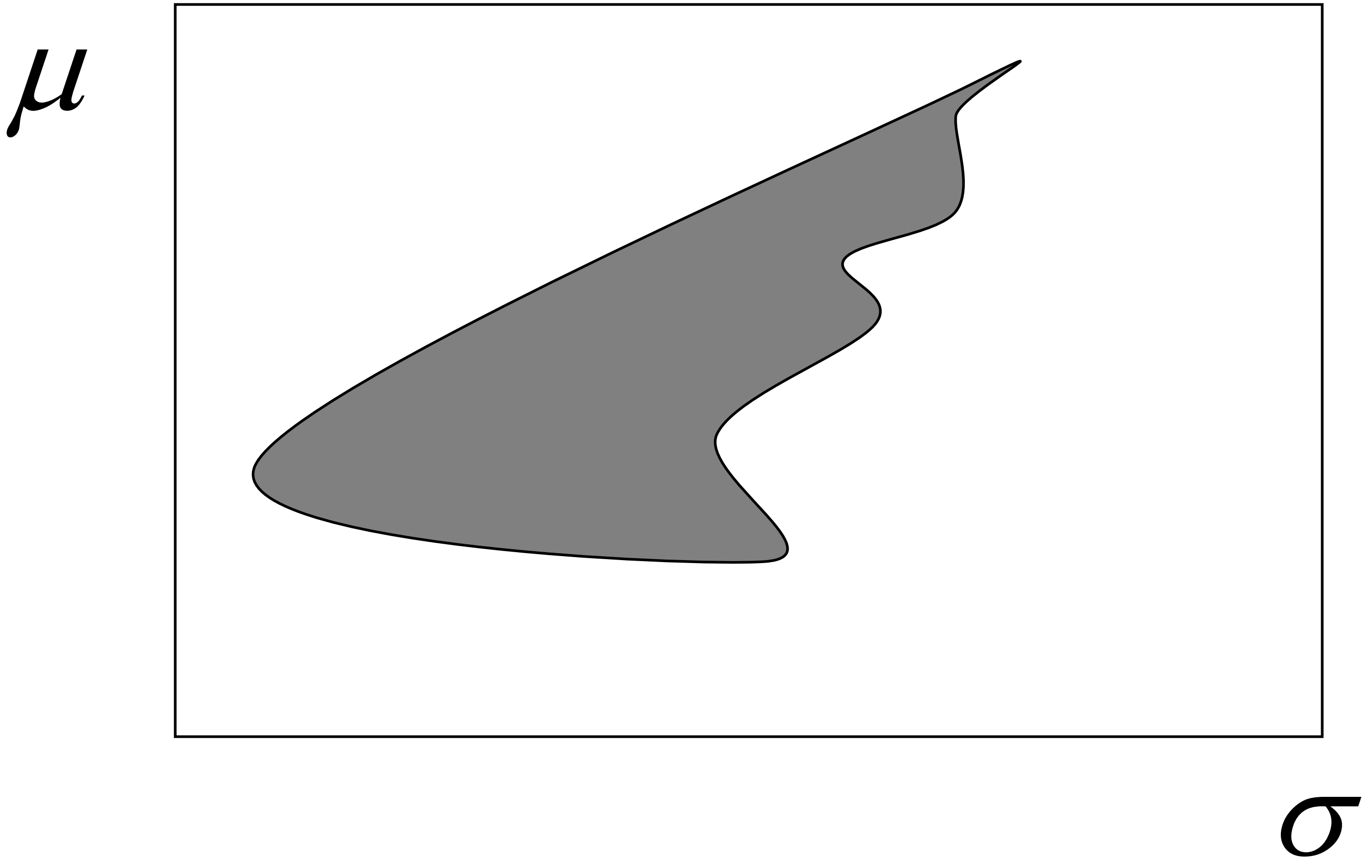
The Mean-Variance Tradeoff

- In the absence of arbitrage opportunities, the set of returns from all securities gives rise to an "efficient frontier".
- The efficient frontier bounds the tradeoffs that are available between mean and variance by trading in all securities.

Mean-Variance Analysis



Mean-Variance Analysis



Utility

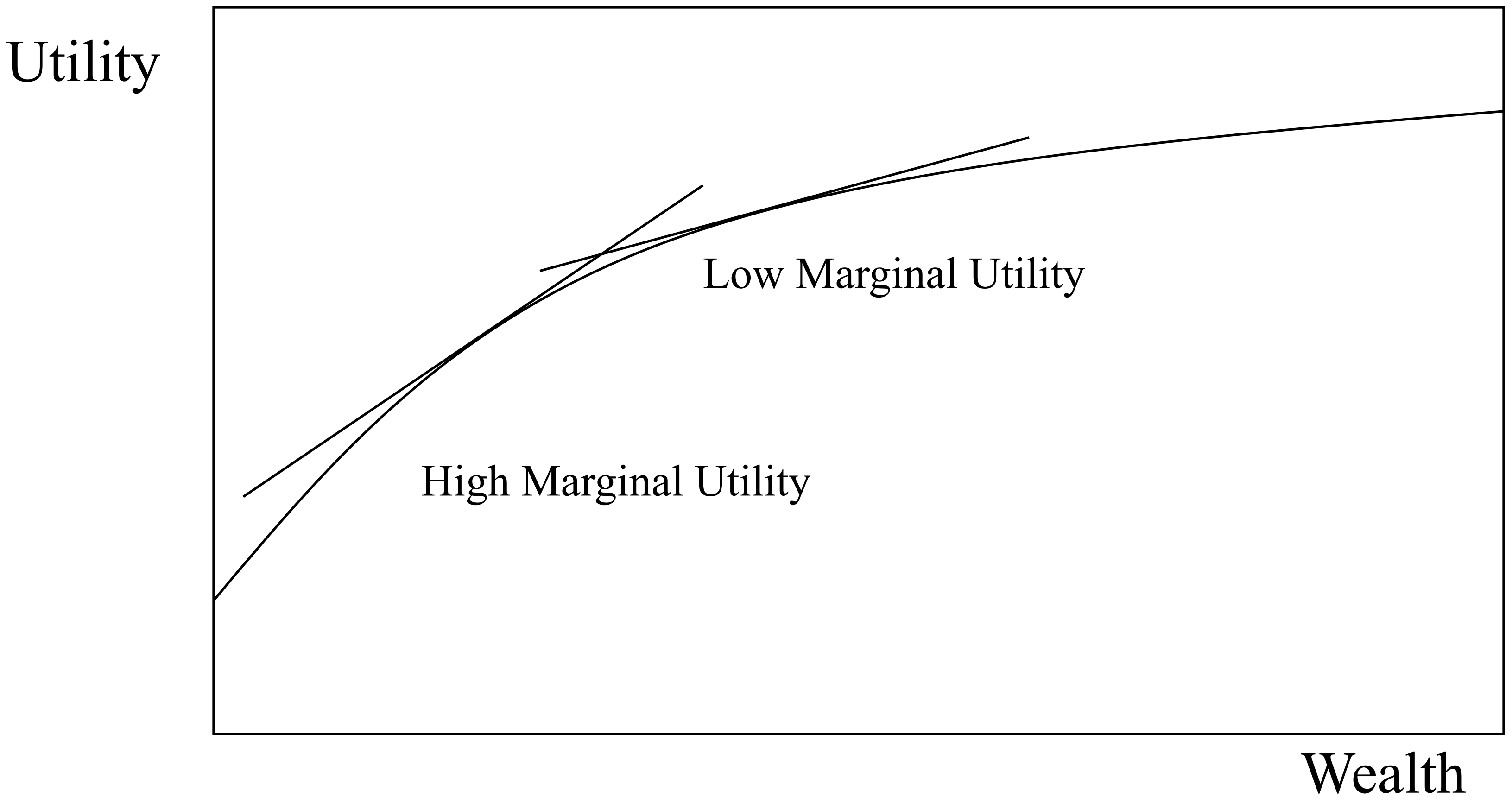
- Investors are typically thought of as being risk averse:
  - When given the choice between

    a) a riskless payment of $10 and

    b) a 50/50 chance of $20 or 0

    most people choose a).
- This is a consequence of diminishing marginal utility of wealth.

Diminishing Marginal Utility



Utility and Indifference Curves

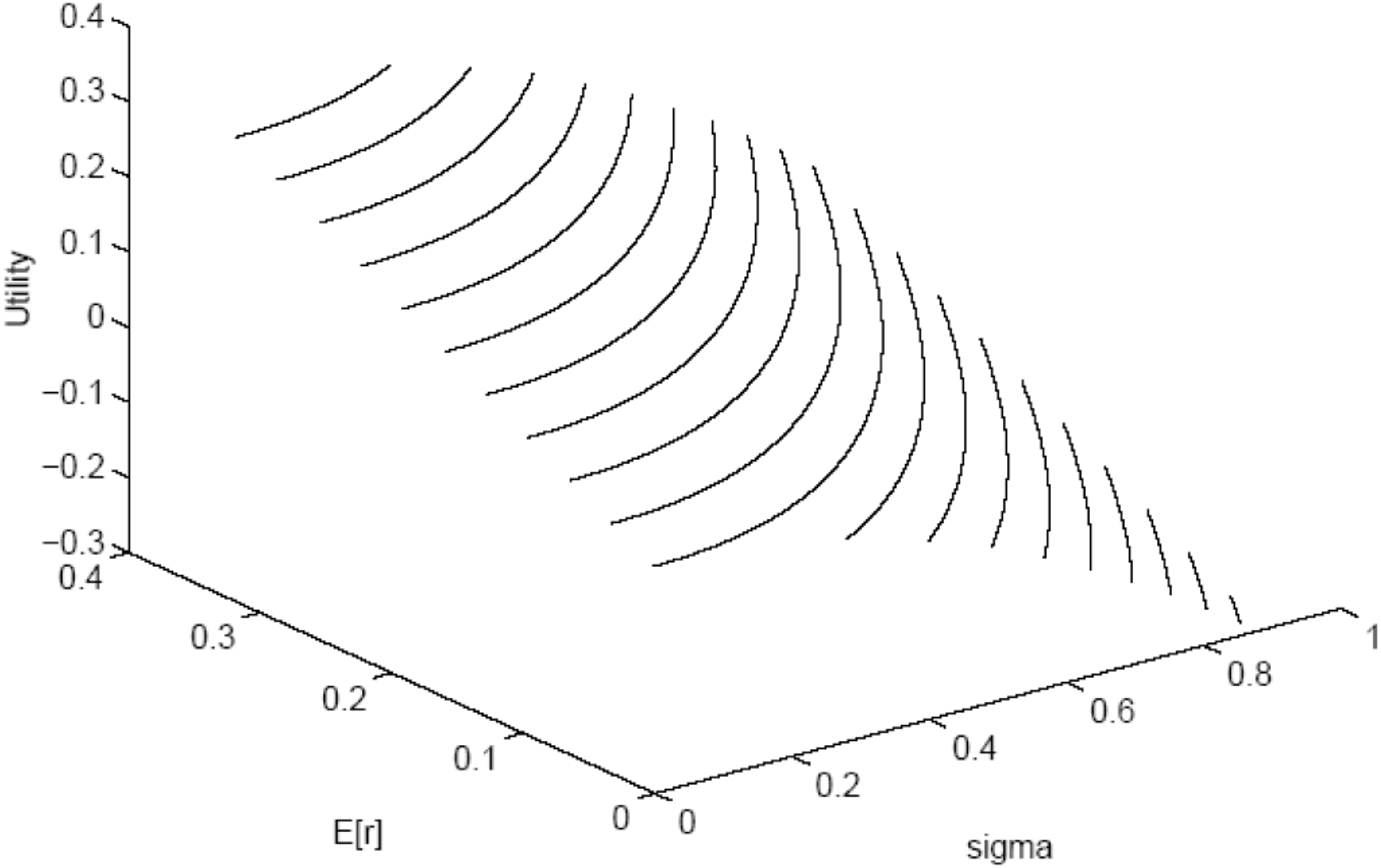
- Represent an investor's willingness to trade-off return and risk

Example (for an investor with A=4):

Exp Return (%)	St Deviation (%)	$U=E(r)-.005A\sigma^2$
10	20.0	2
15	25.5	2
20	30.0	2
25	33.9	2

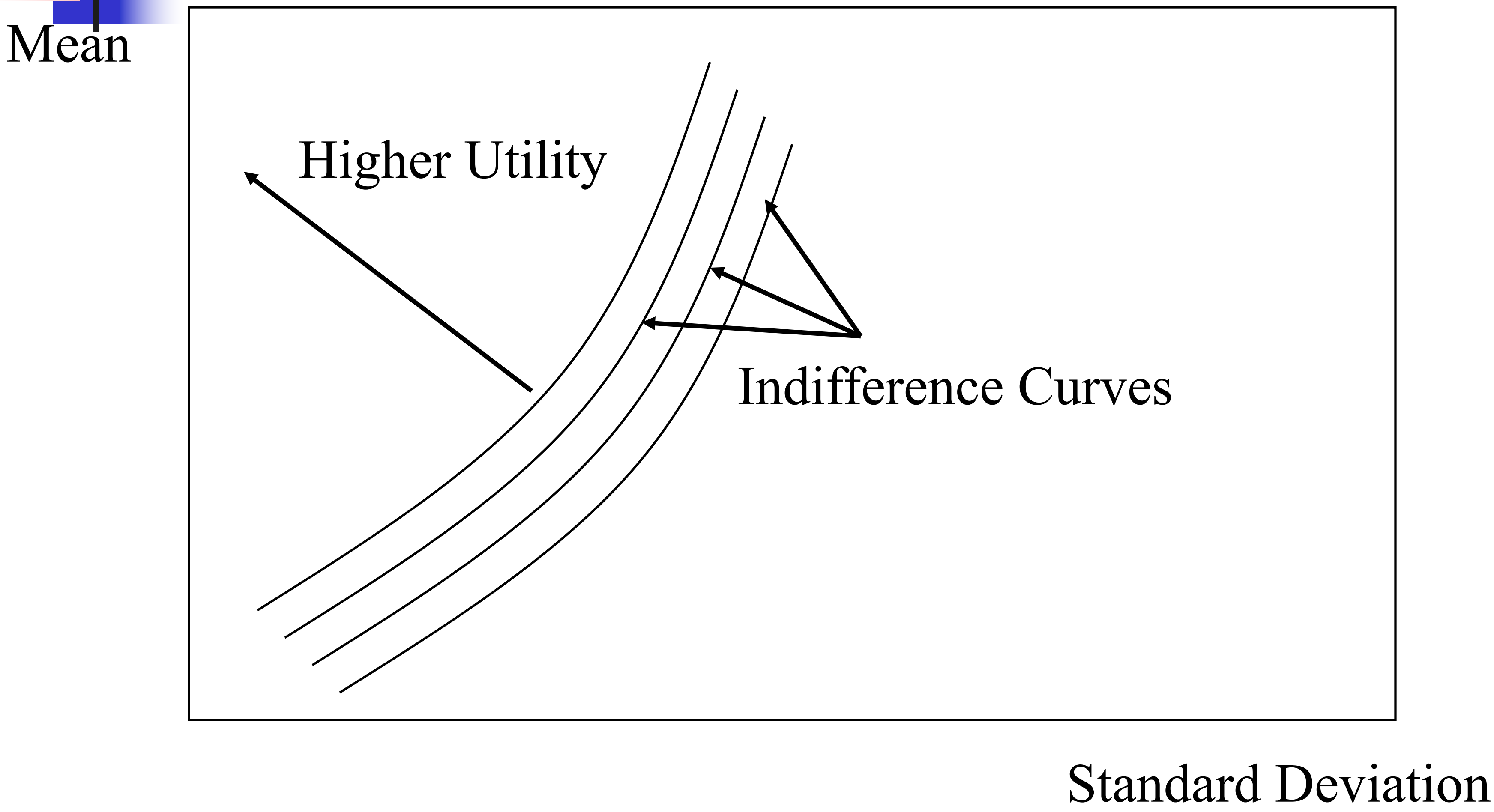
Indifference Curves

- If investors have utility functions of a special type, they only care about mean and variance.
- Indifference curves define sets of mean-standard deviation pairs that make an investor equally well off.

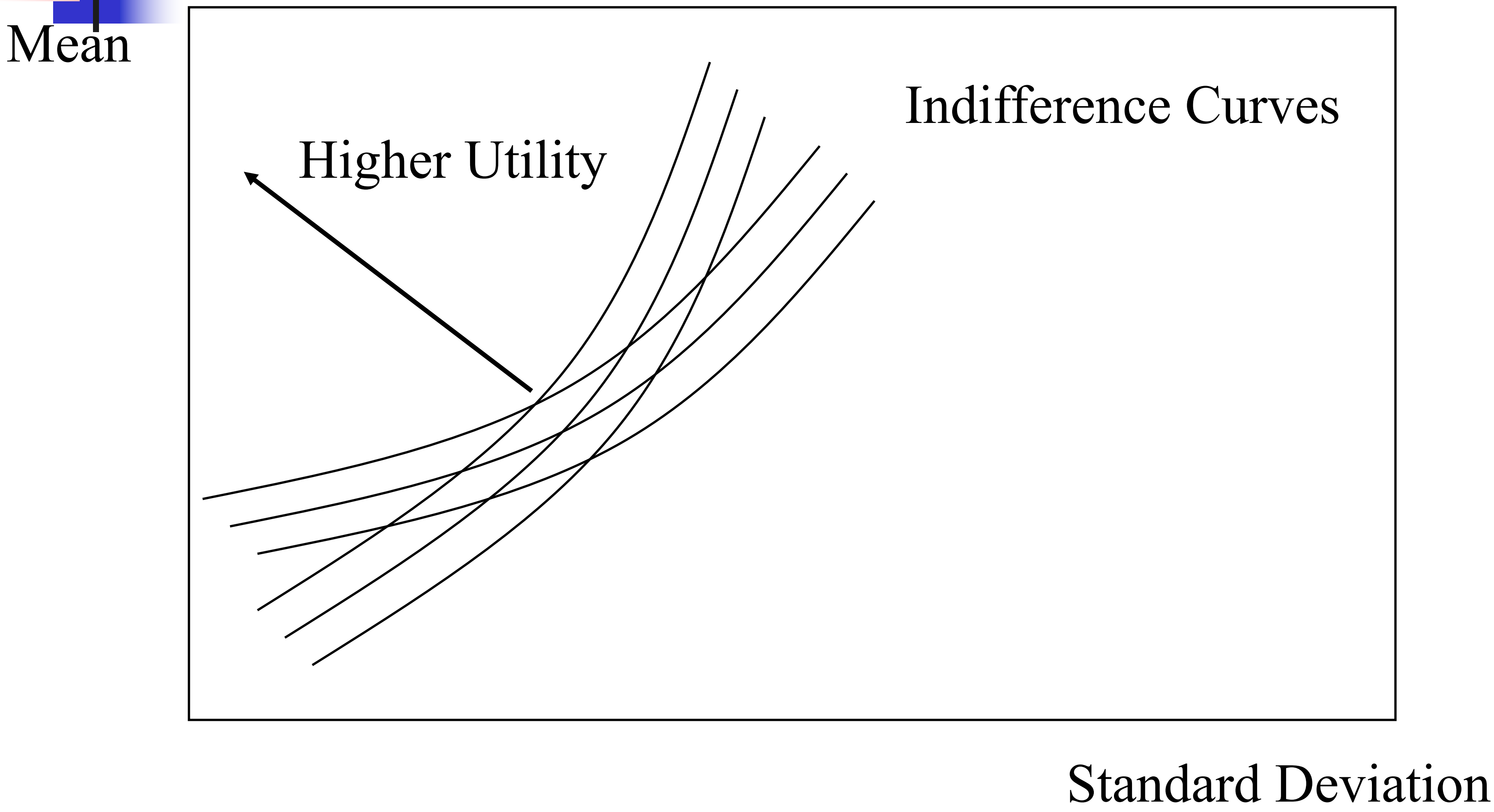


Utility
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
E[r]
0.4
0.3
0.2
0.1
0
sigma
0
0.2
0.4
0.6
0.8
1

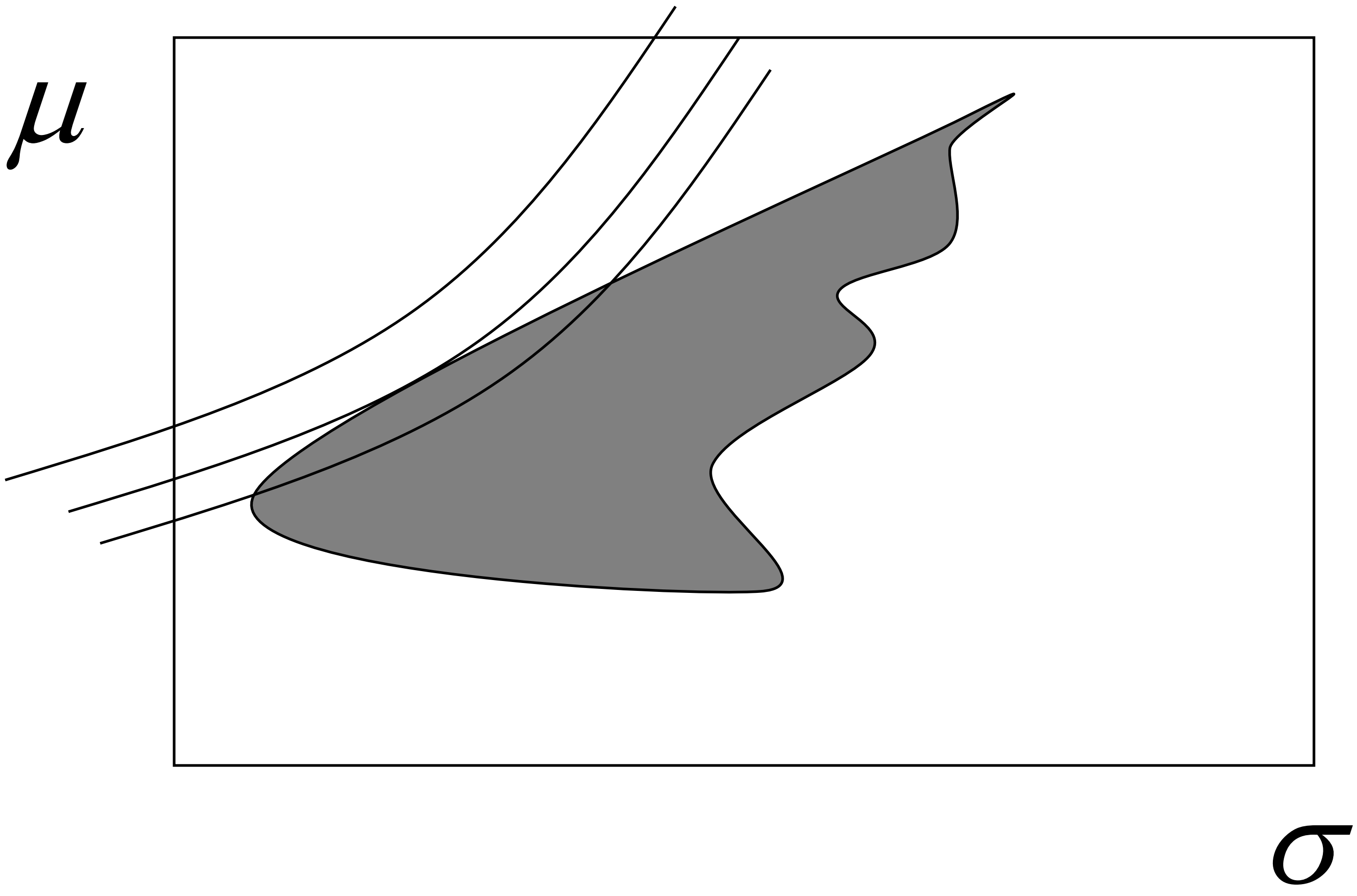
Indifference Curves



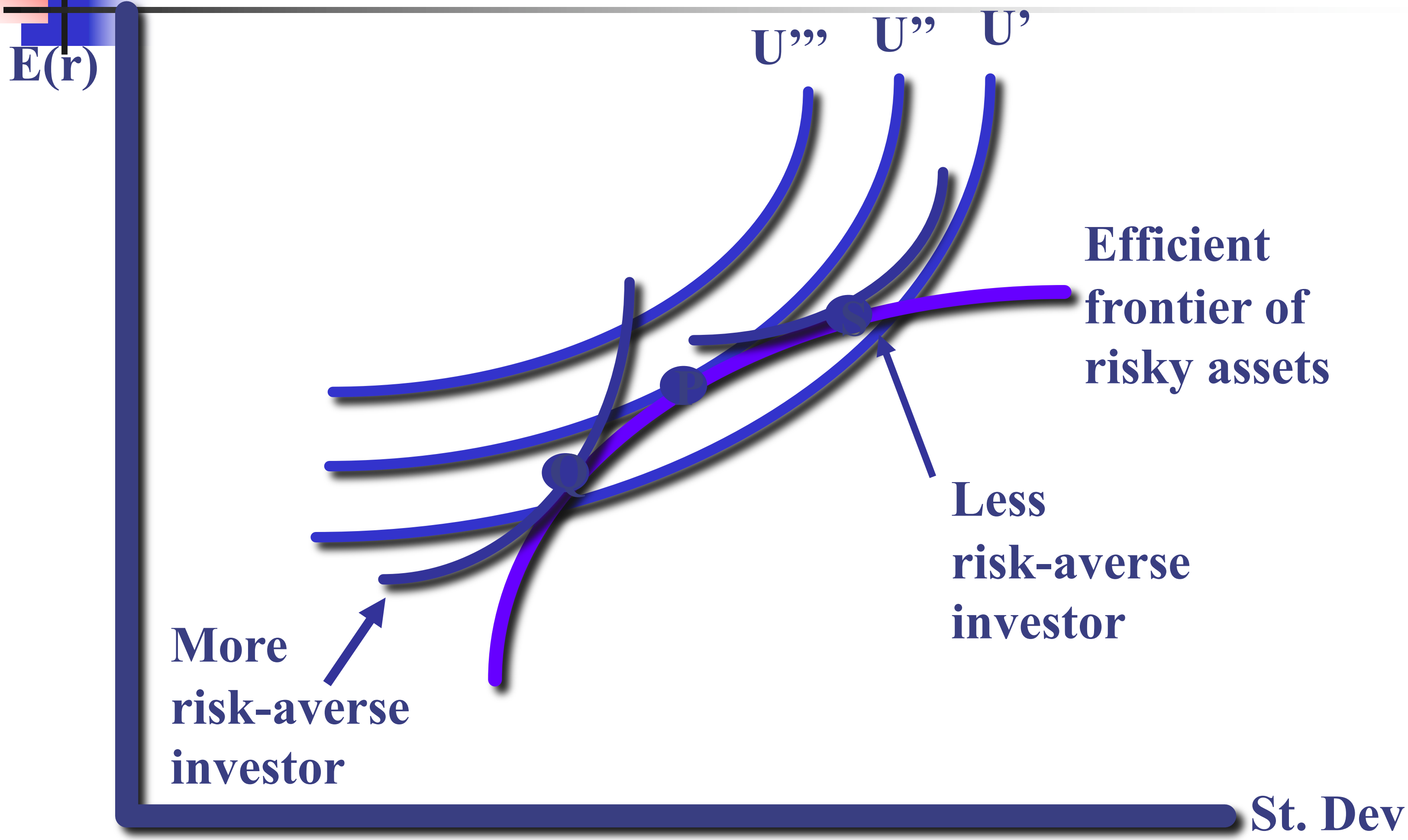
Indifference Curves



Combining Utility Theory and Mean-Variance Theory



Portfolio Selection & Risk Aversion



Mean-Variance Analysis

- Properties of MV frontier
  - Higher mean returns can only be achieved by increasing portfolio variance.
  - More risk-tolerant investors will choose higher variance portfolios but receive higher expected returns.
  - These efficient portfolios can be calculated if we know the covariance matrix.
    - There are companies who will calculate return covariances for you (e.g. BARRA)

A Practice: finding MVP

$E(r_1) = .10 \qquad \sigma \quad = .15$

$E(r_2) = .14 \qquad \sigma_2 = .20$

$\rho_{12} = .2$

$$\sigma_P^2 = w^2\sigma_1^2 + 2w(1-w)\rho_{12}\sigma_1\sigma_2 + (1-w)^2\sigma_2^2$$

The first-order condition to minimize σ_p^2:

$$\frac{\partial \sigma_P^2}{\partial w} = 2w\sigma_1^2 + 2(1-w)\rho_{12}\sigma_1\sigma_2 - 2w\rho_{12}\sigma_1\sigma_2 - 2(1-w)\sigma_2^2 = 0$$

Therefore:

$$w_{\min} = \frac{\sigma_2^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}$$

Minimum-Variance Combination: ρ = .2

$$W_1 = \frac{(.2)^2 - (.2)(.15)(.2)}{(.15)^2 + (.2)^2 - 2(.2)(.15)(.2)}$$

$$W_1 = .6733$$

$$W_2 = (1 - .6733) = .3267$$

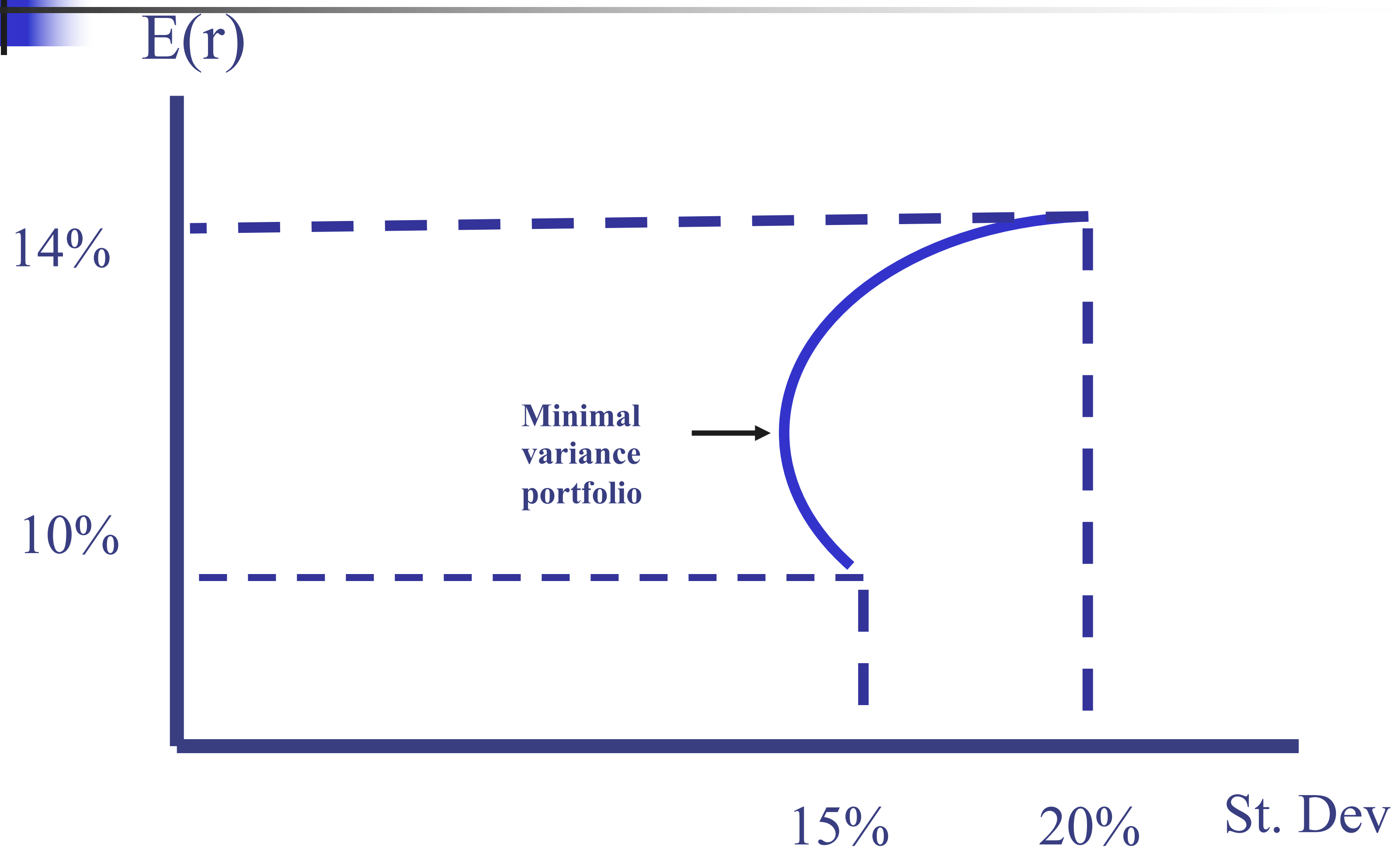
Minimum -Variance: Return and Risk with ρ = .2

$$r_p = .6733(.10) + .3267(.14) = .1131$$

$$\sigma_p = [(.6733)^2(.15)^2 + (.3267)^2(.2)^2 + 2(.6733)(.3267)(.2)(.15)(.2)]^{1/2}$$

$$\sigma_p = [.0171]^{1/2} = .1308$$

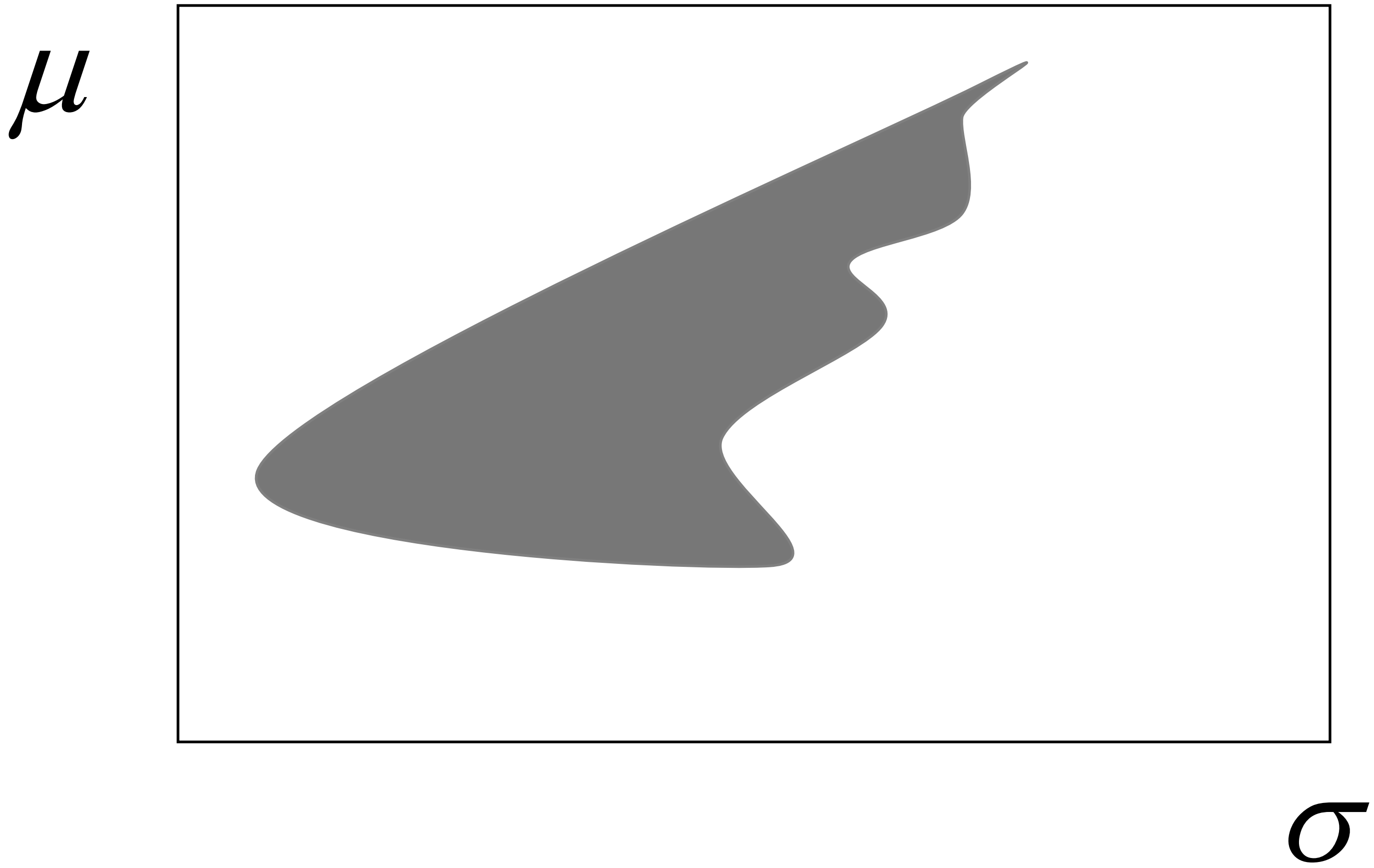
Efficient Frontier



Portfolio Separation

- An important result:
  - all frontier portfolios are a combination of the same two frontier portfolios!
- Implication:
  - If there were no information issues and investors cared only about mean and variance, only two mutual funds would be required to satisfy **all** investors stock market demands.

Mean-Variance Analysis



Model

- The quadratic program:

$$\min_{\{w\}} \frac{1}{2} w^T V w$$

$$s.t. \quad \mathrm{w}^{\mathrm{T}} e = E[\tilde{r}_p] \text{ and}$$

$$\mathrm{w}^{\mathrm{T}} 1 = 1$$

$$\min_{\{w,\lambda,\gamma\}} L = \frac{1}{2} w^T V w + \lambda(E[r_p] - w^T e) + \gamma(1 - w^T 1)$$

$$FOC:$$

$$V w_p - \lambda e - \lambda 1 = 0$$

$$E[r_p] - w^T e = 0$$

$$1 - w^T 1 = 0$$

Only risky assets

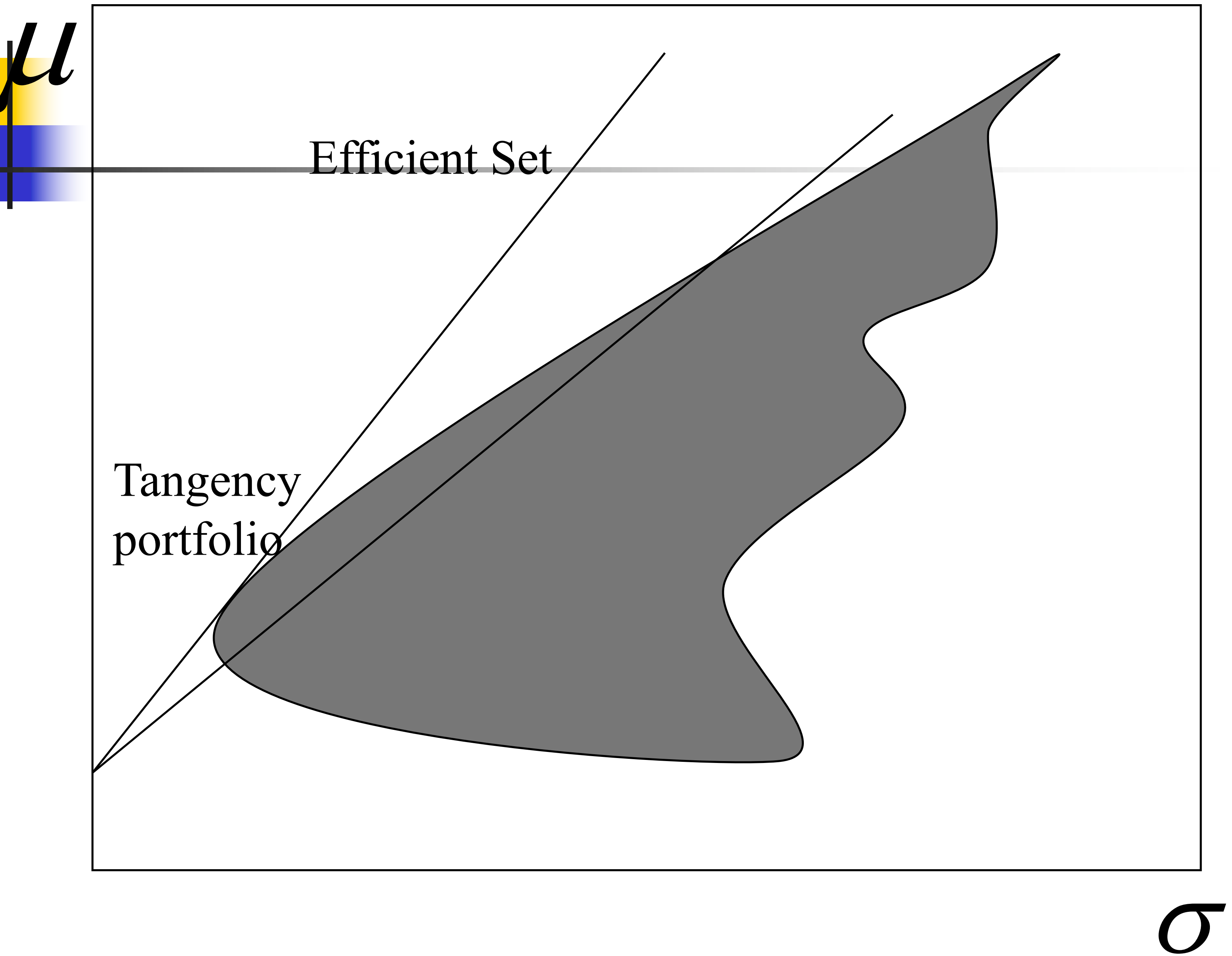
$$w_1 r_1 + w_2 r_2 + w_3 r_3 = \bar{r}$$

$$w_1 + w_2 + w_3 = 1$$

$$w_1 \sigma_{11} + w_2 \sigma_{12} + w_3 \sigma_{13} - \lambda r_1 - \gamma = 0$$

$$w_1 \sigma_{21} + w_2 \sigma_{22} + w_3 \sigma_{23} - \lambda r_2 - \gamma = 0$$

$$w_1 \sigma_{31} + w_2 \sigma_{32} + w_3 \sigma_{33} - \lambda r_3 - \gamma = 0$$



μ
Efficient Set
Tangency
portfolio
σ

Risk-free Borrowing and Lending

- You can expand the efficient set if risk-free borrowing and lending are possible.
- If you invest X in t-bills and (1-X) in an efficient portfolio the mean and variance of the portfolio are:

$$
\begin{aligned}
E(\tilde{r}_p) &= E(Xr_f + (1-X)\tilde{r}_e) \\
&= Xr_f + (1-X)E(\tilde{r}_e)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{var}(\tilde{r}_p) &= \mathrm{var}(Xr_f + (1-X)\tilde{r}_e) \\
&= (1-X)^2\,\mathrm{var}(\tilde{r}_e) \\
\sigma_p &= (1-X)\sigma_e
\end{aligned}
$$

Portfolio Separation

- If there were no information issues and investors cared only about mean and variance, only t-bills and one mutual fund (the tangency portfolio) would be required to satisfy **all** investors stock market demands.
- Investments in this tangency portfolio and t-bills dominate all other investments!

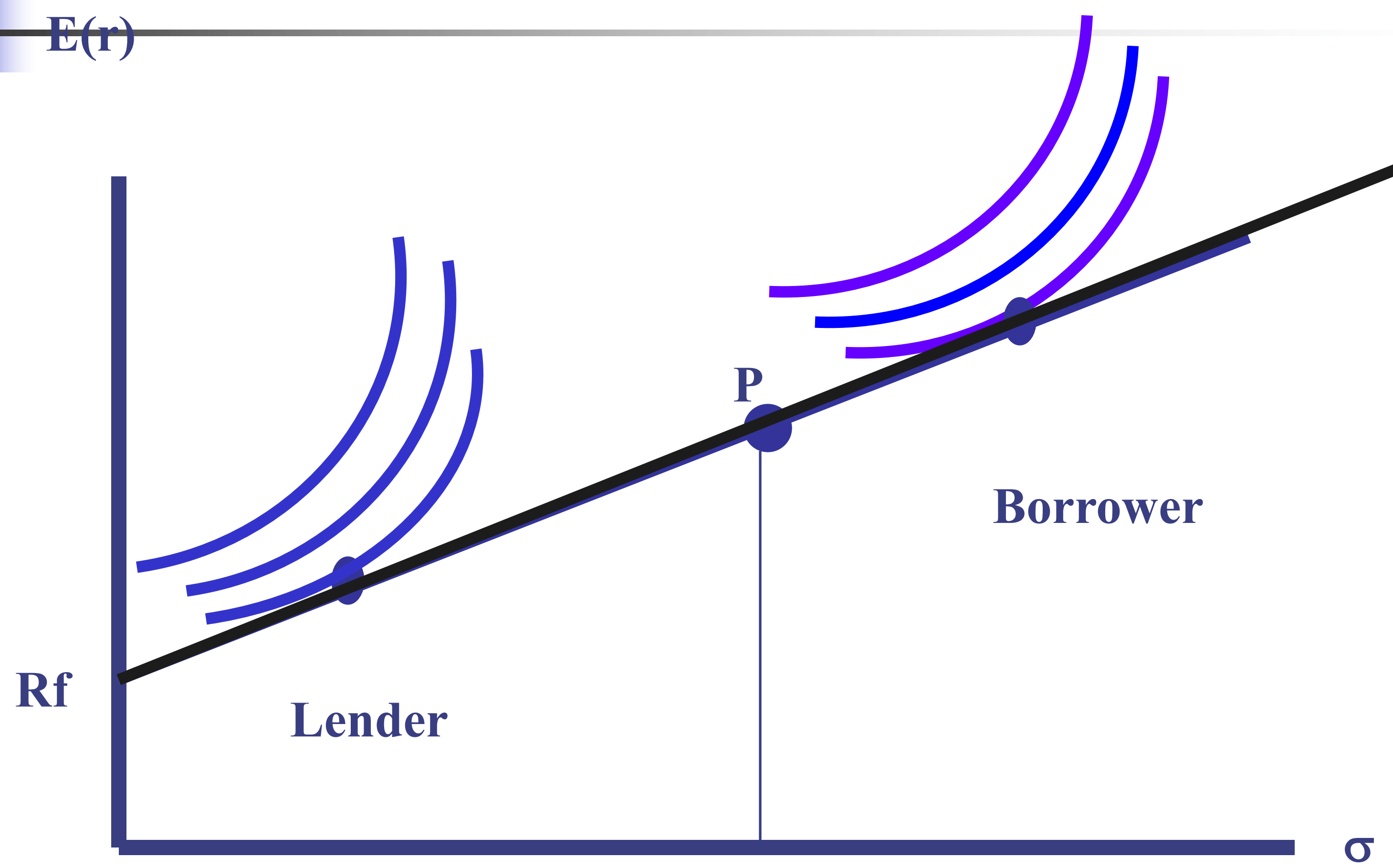
Extending Concepts to All Securities

- The optimal combinations result in lowest level of risk for a given return.
- The optimal trade-off is described as the efficient frontier.
- These portfolios are dominant.

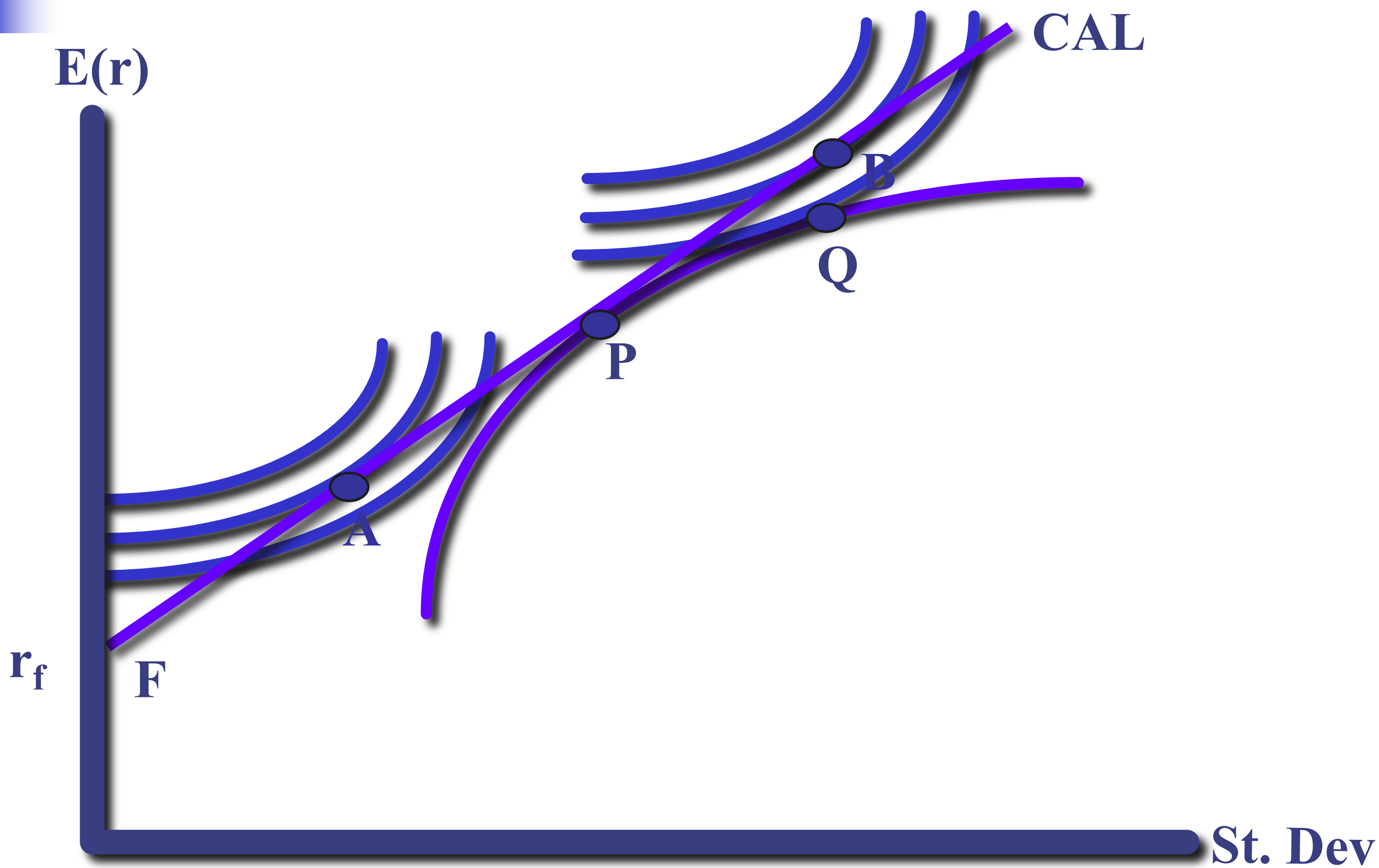
Extending to Include Riskless Asset

- The optimal combination becomes linear.
- A single combination of risky and riskless assets will dominate.

CAL between a risky asset and a risk-free asset



Borrowing and Lending: How the Risk-free Asset Originates



Finding the tangency portfolio

- Since the tangency portfolio lies on the highest CAL that touches the efficient frontier, we just have to maximize the slope of the CAL:
- Max $Slope = \dfrac{E(r_p) - r_f}{\sigma_p}$ subject to $\sum w_i = 1$
- In the case of two securities:
- $$w_1 = \frac{(r_1 - r_f)\sigma_2^2 - (r_2 - r_f)\rho_{12}\sigma_1\sigma_2}{(r_1 - r_f)\sigma_2^2 + (r_2 - r_f)\sigma_1^2 - (r_1 - r_f + r_2 - r_f)\rho_{12}\sigma_1\sigma_2}$$

Finding the tangency portfolio: generalization

- Allow short sale and riskless lending and borrowing

$$r = r_f + \left(\frac{r_p - r_f}{\sigma_p} \right) \sigma$$

$$\text{let } \theta = \frac{r_p - r_f}{\sigma_p} = \frac{\sum_{i=1}^{N} w_i (r_i - r_f)}{\left[\sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1, i \neq j}^{N} w_i w_j \sigma_{i,j} \right]^{1/2}}$$

Finding the tangency portfolio: generalization

$$\text{let } F_1(w) = \sum_{i=1}^{N} w_i (r_i - r_f)$$

$$F_2(w) = \left[\sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1, i \neq j}^{N} w_i w_j \sigma_{i,j} \right]^{-1/2}$$

- Then

$$\frac{dF_1(w)}{dw_k} = r_k - r_f$$

$$\frac{dF_2(w)}{dw_k} = -\frac{1}{2} \left(\sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1, i \neq j}^{N} w_i w_j \sigma_{i,j} \right)^{-\frac{3}{2}} \times \left(2 w_k \sigma_k^2 + 2 \sum_{j=1, i \neq j}^{N} w_j \sigma_{j,k} \right)$$

Finding the tangency portfolio: generalization

- Since $$\frac{d\theta}{dw_k} = F_1(w)\frac{dF_2(w)}{dw_k} + F_2(w)\frac{dF_1(w)}{dw_k} = 0$$
- We have:

$$-\left(\frac{\sum_{i=1}^{N} w(r_i - r_f)}{\sum_{i=1}^{N} w_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{j=1,i\neq j}^{N} w_i w_j \sigma_{i,j}}\right) \times \left(w_k\sigma_k^2 + \sum_{j=1,i\neq j}^{N} w_j\sigma_{j,k}\right) + (r_k - r_f) = 0$$

Finding the tangency portfolio: generalization

- Denote $$\lambda = \frac{\sum_{i=1}^{N} w(r_i - r_f)}{\sum_{i=1}^{N} w_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{j=1, i\neq j}^{N} w_i w_j \sigma_{i,j}} = \frac{r_p - r_f}{\sigma_p}$$
- We have:

$$-\lambda\left(w_k\sigma_k^2 + \sum_{j=1, i\neq j}^{N} w_j\sigma_{j,k} \right) + (r_k - r_f) = 0$$

$$r_k - r_f = \lambda w_k\sigma_k^2 + \sum_{j=1, i\neq j}^{N} \lambda w_j\sigma_{j,k}$$

Finding the tangency portfolio: generalization

- Let $Z_k = \lambda w_k$, we have:

$$r_1 - r_f = z_1\sigma_1^2 + z_2\sigma_{1,2} + z_3\sigma_{1,3}\ldots\ldots + z_N\sigma_{1,N}$$

$$r_2 - r_f = z_1\sigma_{1,2} + z_2\sigma_2^2 + z_3\sigma_{2,3}\ldots\ldots + z_N\sigma_{2,N}$$

.

.

.

$$r_N - r_f = z_1\sigma_{1,N} + z_2\sigma_{2,N} + z_3\sigma_{3,N}\ldots\ldots + z_N\sigma_N^2$$

Finding the tangency portfolio:example

- Given the following parameters, what is the optimal risky portfolio?

$$r_f = 5\%$$

$$r_1 = 14\%, \sigma_1 = 6\%$$

$$r_2 = 8\%, \sigma_2 = 3\%$$

$$r_3 = 20\%, \sigma_3 = 15\%$$

$$\rho_{1,2} = 0.5, \rho_{2,3} = 0.4, \rho_{1,3} = 0.2$$

Finding the tangency portfolio:example

- We have:

$$r_1 - r_f = z_1\sigma_1^2 + z_2\sigma_{1,2} + z_3\sigma_{1,3}$$

$$r_2 - r_f = z_1\sigma_{1,2} + z_2\sigma_2^2 + z_3\sigma_{2,3}$$

$$r_3 - r_f = z_1\sigma_{1,3} + z_2\sigma_{2,3} + z_3\sigma_3^2$$

$$or$$

$$9 = 36z_1 + 9z_2 + 18z_3$$

$$3 = 9z_1 + 9z_2 + 18z_3$$

$$15 = 18z_1 + 18z_2 + 225z_3$$

Finding the tangency portfolio:example

- The solution to this system of equations is: $$z_1 = \frac{14}{63}, z_2 = \frac{1}{63}, z_3 = \frac{3}{63},$$
- Scale z_k so that they add up to 1, we have: $$w_1 = \frac{14}{18}, w_2 = \frac{1}{18}, w_3 = \frac{3}{18},$$
- The expected return on such a portfolio is 14.66%, the variance is 33.83%

Optimal portfolio selection:1

- Inputs necessary to implement **The Tool**
  - identify all assets eligible for investment,
  - for all eligible assets, estimate
    - Expected return
    - Variance, *or* Standard Deviation
    - Correlations or covariances between all pairs of assets
  - identify investment restrictions for all assets.

Optimal portfolio selection:2

- **The Tool** identifies the ***Feasible Set*** or ***Investment Opportunity Set:*** set of all portfolios that can be formed by combining eligible assets. Each portfolio characterized by
  - portfolio proportion invested in each asset.
  - portfolio expected return and standard deviation.

Optimal portfolio selection: 3

- When choosing among *n* risky assets, **The Tool** yields:
  - ***Minimum variance frontier***: set of portfolios with lowest SD/Var for given level of return.
  - ***Minimum variance portfolio***: portfolio with the lowest SD/Var among all feasible portfolios.
  - **Efficient frontier**: set of portfolios with highest expected return for given level of risk (SD).

Optimal portfolio selection: 4

- When choosing among *n* risky assets *plus a Risk Free Asset*, **The Tool** yields
  - ***Minimum variance frontier:*** set of portfolios with lowest SD/Var for given level of return.
  - ***Minimum variance portfolio:*** portfolio with the lowest SD/Var among all feasible portfolios.
  - ***Efficient frontier****:* set of portfolios with highest expected return for given level of risk (SD).
  - ***Tangent/optimal risky portfolio***: efficient portfolio which combined with *rf* yields the capital allocation line with highest reward to total risk ratio, tangent to efficient set.

In Practice

- In practice quality of the output of The Tool (i.e.portfolio weights) depends on
  - the quality of the estimates of *M Var. Cov.*
  - *n*, the number of assets included.
- Estimation of mean, variance, and covariance is better for portfolio or asset classes than for individual securities
- Therefore, in practice, often use optimizer stepwise, first within each asset class and then across asset classes, or more often only at the asset allocation level.

Conclusion

- We call it mean-variance analysis because we assume that all that matters to investors is the average return and the return variance of their portfolio.
  - – This is appropriate if returns are normally distributed.

Conclusion (cont.)

- The key lessons from mean-variance analysis are:
  - You should hold the same portfolio of risky assets no matter what your tolerance for risk.
  - If you want less risk, combine this portfolio with investment in the risk-free asset.
  - If you want more risk, buy the portfolio on margin.
  - In large portfolios, covariance is important, not variance.

Conclusion (cont.)

- What is wrong with mean-variance analysis:
  - Not much! This is one of the few things in finance about which there is complete agreement.
- Caveat: remember that you have to include every asset you have in the analysis; including human capital, real estate, etc.
- Finally, Markowitz's theory does not tell us where the prices, returns, variances or covariances come from.