光华管理学院
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Mathematical Methods in Finance

Lecture 9: Introduction to Partial Differential Equations and their Applications in Financial Modeling

An Introduction to Partial Dfferential Equations

In mathematics, **partial differential equations** (**PDE**) are a type of differential equation, i.e., a relation involving an unknown function (or functions) of several independent variables and their partial derivatives with respect to those variables. Originally, partial differential equations are used to formulate, and thus aid the solution of, problems involving functions of several variables; such as the propagation of sound or heat, electrostatics, electrodynamics, fluid flow, and elasticity. Just as ordinary differential equations often model dynamical systems, partial differential equations often model multidimensional systems.

1. Introduction

A partial differential equation (PDE) for the function $u(x_1,\ldots x_n)$ is of the form

$$F(x_1,\cdots x_n,u,\frac{\partial}{\partial x_1}u,\cdots\frac{\partial}{\partial x_n}u,\frac{\partial^2}{\partial x_1\partial x_1}u,\frac{\partial^2}{\partial x_1\partial x_2}u,\cdots)=0$$

F is a linear function of u and its derivatives if, by replacing u with $v+w$, F can be written as

$F(v+w)= F(v) + F(w)$, *and* if, by replacing u with ku, F can be written as $k\cdot F(u)$

If F is a linear function of u and its derivatives, then the PDE is linear. Common examples of linear PDEs include the heat equation, the wave equation and Laplace's equation.

A relatively simple PDE is

$$\frac{\partial}{\partial x}u(x,y) = 0\,.$$

This relation implies that the function $u(x,y)$ is independent of x. Hence the general solution of this equation is

$$u(x,y) = f(y),$$

where f is an arbitrary function of y. The analogous ordinary differential equation is

$$\frac{du(x)}{dx} = 0$$

which has the solution

$$u(x) = c,$$

where c is any constant value (independent of x). These two examples illustrate that general solutions of ordinary differential equations (ODEs) involve arbitrary constants, but solutions of PDEs involve arbitrary functions. A solution of a PDE is generally not unique; additional conditions must generally be specified on the boundary of the region where the solution is defined. For instance, in the simple example above, the function $f(y)$ can be determined if u is specified on the line $x = 0$.

Existence and uniqueness

Although the issue of the existence and uniqueness of solutions of ordinary differential equations has a very satisfactory answer, that is far from the case for partial differential equations.

Notation

In PDEs, it is common to denote partial derivatives using subscripts. That is:

$$u_x = \frac{\partial u}{\partial x}$$

$$u_{xy} = \frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right).$$

Especially in (mathematical) physics, one often prefers the use of del (which in cartesian coordinates is written $\nabla = (\partial_x, \partial_y, \partial_z)$) for spatial derivatives and a dot $\dot{u}$for time derivatives. For example, the wave equation (described below) can be written as

$\ddot{u} = c^2\nabla^2 u$(physics notation),

or

$\ddot{u} = c^2\Delta u$(math notation), where Δ is the Laplace operator. This often leads to misunderstandings in regards of the Δ-(delta)operator.

2 First-Order Partial Differential Equations

A first-order partial differential equation with n independent variables has the general form

$$F\left(x_1, x_2, \ldots, x_n, w, \frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}, \ldots, \frac{\partial w}{\partial x_n}\right) = 0,$$

where $w = w(x_1, x_2, \ldots, x_n)$is the unknown function and $F(\ldots)$is a given function.

General form of first-order quasilinear PDE

A *first-order quasilinear partial differential equation with two independent variables* has the general form

$$f(x, y, w)\frac{\partial w}{\partial x} + g(x, y, w)\frac{\partial w}{\partial y} = h(x, y, w).$$

Such equations are encountered in various applications (continuum mechanics, gas dynamics, hydrodynamics, heat and mass transfer, wave theory, acoustics, multiphase flows, chemical engineering, etc.).

If the functions f, g, and h are independent of the unknown w, then equation is called *linear*.

3 Second Order PDEs

3.1 Classification of equations of second order

Some linear, second-order partial differential equations can be classified as parabolic, hyperbolic or elliptic.

Assuming $u_{xy} = u_{yx}$, the general second-order PDE in two independent variables has the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + \cdots = 0,$$

where the coefficients A, B, C etc. may depend upon x and y. If $A^2 + B^2 + C^2 > 0$ over a region of the xy plane, the PDE is second-order in that region. This form is analogous to the equation for a conic section:

$$Ax^2 + 2Bxy + Cy^2 + \cdots = 0.$$

More precisely, replacing ∂_x by X, and likewise for other variables (formally this is done by a Fourier transform), converts a constant-coefficient PDE into a polynomial of the same degree, with the top degree (a homogeneous polynomial, here a quadratic form) being most significant for the classification.

Just as one classifies conic sections and quadratic forms into parabolic, hyperbolic, and elliptic based on the discriminant $(2B)^2 - 4AC$, the same can be done for a second-order PDE at a given point.

However, the discriminant in a PDE is given by $B^2 - AC$, due to the convention of the xy term being $2B$ rather than B; formally, the discriminant (of the associated quadratic form) is $(2B)^2 - 4AC = 4(B^2 - AC)$, with the factor of 4 dropped for simplicity.

1. $B^2 - AC < 0$: **elliptic PDEs**
2. $B^2 - AC = 0$: **parabolic PDEs**

At every point can be transformed into a form analogous to the heat equation by a change of independent variables. Solutions smooth out as the transformed time variable increases.

3. $B^2 - AC > 0$: **hyperbolic equations**.

If there are n independent variables x_1, x_2 , ..., x_n, a general linear partial differential equation of second order has the form

$$Lu = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}\frac{\partial^2 u}{\partial x_i \partial x_j} \quad \text{plus lower order terms} = 0.$$

The classification depends upon the signature of the eigenvalues of the coefficient matrix.

1. **Elliptic**: The eigenvalues are all positive or all negative.
2. **Parabolic** : The eigenvalues are all positive or all negative, save one that is zero.
3. **Hyperbolic**: There is only one negative eigenvalue and all the rest are positive, or there is only one positive eigenvalue and all the rest are negative.

3.2 Heat equation (a parabolic equation)

1. The simplest example of a *parabolic equation* is the *heat equation (also called diffusion equation)*

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = 0,$$

where the variables t and x play the role of time and a spatial coordinate, respectively. Note that equation contains only one highest derivative term. It describes one-dimensional unsteady thermal processes in quiescent media or solids with constant thermal diffusivity. The heat equation has infinitely many particular solutions (this fact is common to many PDEs); in particular, it admits solutions

$$\begin{aligned}
w(x,t) &= A(x^2+2t)+B,\\
w(x,t) &= A\exp(\mu^2 t \pm \mu x)+B,\\
w(x,t) &= A\frac{1}{\sqrt{t}}\exp\left(-\frac{x^2}{4t}\right)+B,\\
w(x,t) &= A\exp(-\mu^2 t)\cos(\mu x + B)+C,\\
w(x,t) &= A\exp(-\mu x)\cos(\mu x - 2\mu^2 t + B)+C,
\end{aligned}$$

where A, B, C, and μ are arbitrary constants.

3.3 Wave equation (a hyperbolic equation)

The simplest example of a *hyperbolic equation* is the *wave equation*

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = 0,$$

where the variables t and x play the role of time and the spatial coordinate, respectively. Note that the highest derivative terms in equation differ in sign. This equation is also known as the *equation of vibration of a string*. It is often encountered in elasticity, aerodynamics, acoustics, and electrodynamics. The general solution of the wave equation is

$$w = \varphi(x+t) + \psi(x-t),$$

where $\varphi(x)$ and $\psi(x)$ are arbitrary twice continuously differentiable functions. This solution has the physical interpretation of two *traveling waves* of arbitrary shape that propagate to the right and to the left along the x -axis with a constant speed equal to 1.

3.4 Laplace equation (an elliptic equation)

The simplest example of an *elliptic equation* is the *Laplace equation*

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0,$$

where x and y play the role of the spatial coordinates. Note that the highest derivative terms in equation have like signs. The Laplace equation is often written briefly as $\Delta w = 0$, where Δ is the Laplace operator. The Laplace equation is often encountered in heat and mass transfer theory, fluid mechanics, elasticity, electrostatics, and other areas of mechanics and physics. For example, in heat and mass transfer theory, this equation describes steady-state temperature distribution in the absence of heat sources and sinks in the domain under study.

4 Initial value problem

In mathematics, in the field of differential equations, an **initial value problem** is a partial differential equation together with specified value, called the **initial condition**, of the unknown function at a given point in the domain of the solution. In physics, financial modeling or other sciences, modeling a system frequently amounts to solving an initial value problem; in this context, the differential equation is an evolution equation specifying how, given initial conditions, the system will evolve with time. In this course, we will focus on the **initial value problem of heat equation.**

Reference: (see attached lecture notes) Section 2.4 from W. Strauss's book on Partial Differential Equations