Lec15 - Asymptotic behavior of soln of linear ODE w/ const coeff.

Warm up: (Write it on board first and give them some time to do it)

$$\frac{dy}{dt} = Ay + g(t) \quad , \quad y(t_0) = y_0 \qquad (*)$$

where $A \in M_{n\times n}(\mathbb{R})$ is a const matrix indep of t.
Write down the variation of constant formula to this problem.

<u>Ans</u> Recall the variation of const formula for $\frac{dy}{dt} = A(t)y + g(t)$

Given $\Phi(t)$ as the fundamental matrix.

$$y(t) = \Phi(t)\Phi^{-1}(t_0)y_0 + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)\, g(s)\, ds.$$

In our case now, $\Phi(t) = e^{tA}$

$$\Rightarrow y(t) = \underbrace{e^{(t-t_0)A}} y_0 + \int_{t_0}^{t} e^{(t-s)A} g(s)\, ds$$

Q1: Why $e^{tA}e^{-t_0 A} = e^{(t-t_0)A}$? (because tA and $t_0 A$ commute)

$\Phi(t)\Phi^{-1}(s) = e^{(t-s)A}$ in this case.

Q2: You may wonder from earlier $\Phi(t)$ might not be e^{tA}, but $e^{tA}V = VJ$ where V are generalized e-vecs, and J is the Jordan form, or generally is the formula still true w/ $e^{tA}C_{n\times n}$? (C nonsingular)

Ans: Yes! $\Phi(t)\Phi^{-1}(s) = e^{tA}CC^{-1}e^{-sA} = e^{(t-s)A}$

In previous 3 lectures, we've discussed how to find fundamental matrix

Today: Consider non-homo Eq $(*)$. in many cases $\int_{t_0}^{t} e^{(t-s)A}g(s)\,ds$ is difficult to compute analytically. Then <u>can we say anything qualitatively</u> as $t \to \infty$.

To do that, we first give a bound of e^{tA}.

Lemma [B-N Thm 2.10] textbook — estimate of e^{tA}

If $\lambda_1, \cdots, \lambda_k$ are the distinct e-vals of A, where λ_j has multiplicity n_j and $n_1 + \cdots + n_k = n$. Then for any $\rho > \max_{j=1,\cdots,k} \{\mathrm{Re}\,\lambda_j\}$,

$\exists$ a const $K > 0$ s.t.

$$|e^{tA}| \le K e^{\rho t} \qquad (0 \le t < \infty)$$

Understanding (a simple case)

(i) $A = \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix}$ $\quad |e^{tA}| = \left| \begin{matrix} e^{\lambda t} & & \\ & \ddots & \\ & & e^{\lambda t} \end{matrix} \right| = n|e^{\lambda t}| = n e^{\mathrm{Re}\lambda\, t}$

matrix norm for this course is $\sum_{ij} |a_{ij}|$

(ii) $A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ $\quad |e^{tA}| = \left| \begin{matrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{matrix} \right| = \sum_{j=1}^n |e^{\lambda_j t}| = \sum_{j=1}^n |e^{\mathrm{Re}\lambda_j t}| = \sum_{j=1}^n e^{\mathrm{Re}\lambda_j t}$

$\le n e^{\rho t}$.

Pf: For any A, $\exists\, T$ nonsingular s.t. $TJT^{-1} = A$

J: Jordan canonical form

$$|e^{tA}| = |T e^{tJ} T^{-1}|$$
$$\le |T||e^{tJ}||T^{-1}| \quad \text{(by properties of matrix norm)}$$
$$= \underbrace{|T||T^{-1}|}_{\text{const.}} \left| \begin{matrix} e^{tJ_1} & & & \\ & e^{tJ_2} & & \\ & & \ddots & \\ & & & e^{tJ_s} \end{matrix} \right|$$

Here s not necessarily $= k$, b/c $s = \#$ of Jordan block $= \#$ of indep eig-vec not $\#$ of distinct e-val

$$\underset{p\times p}{e^{tJ_i}} = \begin{pmatrix} 1 & t & \cdots & \frac{t^{p-1}}{(p-1)!} \\ & 1 & \ddots & \vdots \\ & & \ddots & t \\ & & & 1 \end{pmatrix} e^{\underbrace{\lambda[J_i]}_{\text{e-val correspond to } J_i} t}$$

$$\Rightarrow |e^{tJ_i}| \overset{\text{sum all entries } |\cdot|}{=} P_i(t)|e^{\lambda[J_i]t}| = P_i(t)\, e^{\mathrm{Re}\lambda[J_i]\, t} \le P_i(t)\, e^{\max_{j=1,\cdots,k}\{\mathrm{Re}\lambda_j\}\, t}$$

$P_i(t)$: some poly of t w/ all positive coefficients

Note that for any $\rho - \max_j\{Re\lambda_j\} > 0$

we can take $C > 0$ s.t.

$$|P_i(t)| \leq C e^{(\rho - \max_j\{Re\lambda_j\})t} \quad ☆$$

where C is indep of i.

Hence $|e^{tJ_i}| \leq C e^{(\rho - \max_j\{Re\lambda_j\})t} e^{\max_j\{Re\lambda_j\}t} \leq C e^{\rho t}$

Thus,

$$|e^{tA}| \leq |T||T^{-1}| \sum_{i=1}^{s} |e^{tJ_i}|$$

$$\leq \underbrace{|T||T^{-1}| s C}_{=K} e^{\rho t} \qquad \square.$$

Rmk ① choice of ρ

Q1: Can ρ be chosen as "=" ?

$$\rho = \max_{j=1,\cdots,k}\{Re\lambda_j\}$$

Ans: In general, not. (Why?)

b/c in general, the repeated e-vals could give rise to polynomial Growth in t ($P_i(t)$) on the top of e-val propagation --- (☆)

Then Q2: Is there any special case that we can do better? choose "=".

Ans: Yes! In what case? The idea is that we want to suppress poly growth.

Where does it arise? When geometric multiplicity ≠ Algebraic multiplicity

We must use generalized eigvec, and hence poly-growth.

Therefore, in the lucky case, where

the evals achieve the $\max_j\{Re\lambda_j\}$ are all simple, in the sense that

its geometric multiplicity = Algebraic multiplicity.

② This states: solns of linear homo ODE behave at most exponential growth.

An immediate consequence of Lemma:

Cor If all e-vals of A have real part negative, then every soln $\phi(t)$ of

$$y' = Ay$$

approaches 0 as $t \to \infty$.

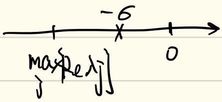
More precisely, $\exists$ const $\tilde{k} > 0$, $\sigma > 0$ s.t.

$$|\phi(t)| \le \tilde{k} e^{-\sigma t} \qquad (0 \le t < \infty)$$

pf: $y(t) = \Phi(t) \vec{c}$

column vector

$$|y(t)| \le |\Phi(t)||\vec{c}| \le \underbrace{k|\vec{c}|}_{\tilde{k}} e^{\rho t}$$ where ρ is any number $> \max_j \{\operatorname{Re} \lambda_j\}$

simply choose 

Some $-\sigma$ s.t. $\max_j \{\operatorname{Re} \lambda_j\} < -\sigma < 0$. $\square$

Now consider non-homo case.

Thm Spse that in the linear non-homogeneous system $y' = Ay + g(t)$ **(NH)**
the fxn $g(t)$ grows no faster than an exponential fxn, that is,

$\exists$ real consts $M > 0$, $T > 0$ s.t.

$$|g(t)| \le M e^{at} \quad (t \ge T)$$

Then every soln ϕ of Eq (NH) grows no faster than an exponential fxn, that is,

$\exists$ real consts $k > 0$, b s.t.

$$|\phi(t)| \le k e^{bt} \quad (t \ge T)$$

some column vector depends on initial data

Pf: $\phi(t) = e^{tA}\vec{c} + \int_0^t e^{(t-s)A} g(s)\,ds.$

$$= e^{tA}\vec{c} + \int_0^T e^{(t-s)A} g(s)\,ds + \int_T^t e^{(t-s)A} g(s)\,ds$$

$$|\phi(t)| \le |e^{tA}||\vec{c}| + \int_0^T |e^{(t-s)A}||g(s)|\,ds + \int_T^t |e^{(t-s)A}||g(s)|\,ds$$

by Lemma & condition of g

$$\le k|\vec{c}|\, e^{\rho t} + \int_0^T k e^{\rho(t-s)} \underbrace{\sup_{0\le t\le T}|g(s)|}_{\text{const} \triangleq k_0} ds + \int_T^t k e^{\rho(t-s)} M e^{as}\,ds$$

So infact only w/ exponential growth mode w/ const ρ or a.
why?

can give them sometime to compute the integrals.

$$= k_1 e^{\rho t} + \frac{k k_0}{\rho}\left(e^{\rho t} - e^{\rho(t-T)}\right) + \frac{kM}{\rho(a-\rho)}\left[e^{at} - e^{\rho(t-T)} e^{aT}\right]$$

$$\le \bar{K}\, e^{\max\{a,\rho\}t} \qquad (t \ge T)$$

$\square$.

Rmk ① Key idea of the pf is the split the integral into two parts.
the finite part, g is bdd by const; the tail part, use exponential tail condition.

② Connection with Gronwall Ineq (Why not use Gronwall?)

(optional) You may find wonder that to study an estimate of soln to (NH) Eq. Why don't we just use Gronwall, and there we do not even need to estimate e^{tA} first. Isn't that easier? I mean: just write (NH) Eq in integral form

$$|y(t)| \le |y_0| + \int_{t_0}^t |A||y(s)|\,ds + \int_{t_0}^t |g(s)|\,ds$$

and apply Gronwall. We can NOT do this! Recall we need $f(t) \le \underset{\text{const.}}{K} + \int g(s) f(s)\,ds$

Here $\int_{t_0}^t |g(s)|\,ds$ grows exp (by condition in thm), not bdd by a const.

③ $\phi'(t)$ satisfies a similar Ineq, i.e.

$$|\phi'(t)| = |A\phi(t) + g(t)|$$

$$\leq A\tilde{k} e^{\max\{\rho, a\}t} + Me^{at}$$

$$\leq \tilde{c}\, e^{\max\{\rho, a\}t}$$

Cor If $\mathrm{Re}\,\lambda_j < 0 \quad \forall j$. and $a < 0$ (the non-homo term also decays exp.)

Then

$$\begin{cases} \lim_{t\to\infty} y(t) = 0 \\ \lim_{t\to\infty} y'(t) = 0 \end{cases}$$