Lec 12 const coefficients linear cont.d e^{tA}.

Different cases:

① diagonal $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

② Jordan block $\begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & & \lambda \end{pmatrix}$

③ Diagonalizable. sufficient: (distinct e-vals) $A = TDT^{-1}$

④ General $A = TJT^{-1}$.

$$e^{tA} = e^{tTJT^{-1}} = T e^{tJ} T^{-1}$$

no need to compute inv.

Part 1 finish ex. of distinct roots

In particular, cplx e-vals

$$\begin{cases} Av = \lambda v. \\ A\bar{v} = \bar{\lambda}\bar{v} \end{cases}$$

A real

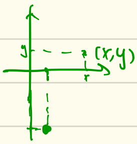
Basic ex. $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\lambda = i$ $\begin{pmatrix} 1 \\ i \end{pmatrix}$

$\lambda = -i$ $\begin{pmatrix} 1 \\ -i \end{pmatrix}$

Thm λ is a cplx e-val of real matrix A.
if v is a corresponding cplx e-vec
then $\bar{\lambda}$ is also an e-val.
$A\bar{v} = \bar{\lambda}\bar{v}$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$$



270° rotation

$A^2 = -I$, $A^3 = -A$, $A^4 = I$, ...

$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ counter-clockwise

$A(-\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ clockwise.

consists of only I and A.

$$\Rightarrow e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \cdots$$

$$= I + tA - \frac{t^2}{2!} - \frac{t^3}{3!}(-A) + \frac{t^4}{4!} + \cdots$$

$$= \left(I - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots\right) I + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right) A$$

$$= \cos t\, I + \sin t\, A$$

$$= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

A clockwise rotation of angle t.

Slightly General ex.

$$A = \begin{pmatrix} u & w \\ -w & u \end{pmatrix}$$

$\lambda = u \pm i w$ $\qquad \lambda = u + iw \quad \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\lambda = u - iw \quad \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$$e^{tA} = e^{t(uI + wA_0)}$$

$$= e^{tu} e^{twA_0} = e^{tu} \begin{pmatrix} \cos wt & \sin wt \\ -\sin wt & \cos wt \end{pmatrix} = \begin{pmatrix} e^{tu} \cos wt & e^{tu} \sin wt \\ -e^{tu} \sin wt & e^{tu} \cos wt \end{pmatrix}$$

So what?

"not-quite-diagonal"

A has l real e-vals, k pairs of cplx e-vals. $u_k \pm i w_k$. WLOG $w_k > 0$.

$\downarrow$ e-val. $\qquad \downarrow$

$v_1, \cdots, v_l \qquad w_k, \overline{w_k} \qquad w_k = a_k + i b_k$

$$V = [v_1 \ v_2 \ \cdots \ v_l \ a_1 \ b_1 \ \cdots \ a_k \ b_k]$$

Thm $AV = VD$ where D is not-quite-diag.

$$D = \begin{bmatrix} \lambda_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \lambda_l & & & & & \\ & & & u_1 & w_1 & & & \\ & & & -w_1 & u_1 & & & \\ & & & & & \ddots & & \\ & & & & & & u_k & w_k \\ & & & & & & -w_k & u_k \end{bmatrix}$$

Pf: $AV = VD$

$Av_i = \lambda_i v_i$ first l

Q: What is Aa_i ?

Ab_i

$A w_i = \lambda_i w_i$

$A(a_i + ib_i) = (u_i + iw_i)(a_i + ib_i)$

$= (u_i a_i - w_i b_i) + i(w_i a_i + u_i b_i)$

$\Rightarrow \begin{cases} Aa_i = u_i a_i - w_i b_i \\ Ab_i = w_i a_i + u_i b_i \end{cases}$

$A[a_i \ b_i] = [a_i \ b_i] \begin{bmatrix} u_i & w_i \\ -w_i & u_i \end{bmatrix}$

$\Rightarrow AV = [Av_1 \cdots Av_l \quad Aa_1 \ Ab_1 \ \cdots \ Aa_k \ Ab_k]$

$= [\lambda_1 v_1 \cdots \lambda_l v_l \quad \alpha_1 a_1 - w b_1 \ \cdots \ ]$

$= [v_1 \cdots v_l \quad a_1 b_1 \cdots a_k b_k] D = VD$ $\square$.

More on Jordan Canonical form