Module 4: Interest Rate Model and Derivatives

Interest Rate Models

Roadmap

- Discussion on Interest Rate Models
- One Factor Models
- Two & Three Factor Models

The Goal

- Our goal is to model distribution of interest rates.
- We may want to do this in order to price interest rate derivatives such as:
  - Call or Put features in Corporate Bonds
  - Prepayment option on Home Mortgages
  - Caps and Floors in Adjustable Rate Mortgages
  - Interest Rate Swaps
  - Options of Interest Rate Swaps
  - Options on Bonds

One Factor Models

- A general single factor model for short term interest rates $r_t = r(t)$ is:
  - For discrete-time models

$$\Delta r_{k+1} = \mu(t, r_k)\Delta t + \sigma(t, r_k) \cdot z\sqrt{\Delta t}$$
$$z \sim N(0,1)$$

  - This can be described in continuous-time using Brownian Motion

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dZ_t$$

One Factor Models

- Note that we can always talk about rates in terms of bond prices.

$$r_k = \frac{\Delta P}{P} = \frac{P_{k+1} - P_k}{P_k}$$

- This allows us to talk about rates between two periods – what we normally call forward rates.
- Term structure models really give us a distribution of forward rates.
- We need to be careful here – because term structure model has to give us the same bond prices as we get if we use forward rates to price.

One Factor Models

- What features would we want in a model of interest rates?
  - Rates are positive and don't get "too big".
  - Mean Reversion
    - If rates are high, downward changes are more likely.
    - If rates are low, upward changes are more likely.
  - Treasury bonds must be priced correctly.

Vasicek Model

- Changes in rates can be written as:

$$dr = \alpha(\mu - r_t)dt + \sigma dZ$$

$$\Delta r = \alpha(\mu - r)\Delta t + \sigma\sqrt{\Delta t} \cdot z$$

- Where

r is current rate

Δr is projected change in rate

α indicates how fast rates revert to mean

μ long term mean spot rate

Δt is change in time, in years

σ is standard deviation (annualized volatility)

$z \sim N(0,1)$

Vasicek Model

- Good news
  - Easy to implement.
  - Includes mean reversion.
  - Model leads to a nice formula for pricing bonds.
- Bad news
  - Model can give negative yields.
  - Not a “no arbitrage model”.
    - It may not price all treasuries well.
  - Assumes constant volatility
    - We observe that volatility is higher when rates are high in practice.

Vasicek Model

- This leads to a pricing formula for zero-coupon bonds

$$B(T)=e^{\left(\frac{1}{\alpha}\left(1-e^{-\alpha T}\right)(R-r)-TR-\frac{\sigma^2}{4\alpha^3}\left(1-e^{-\alpha T}\right)^2\right)}$$

$B(T)$ Price of zero coupon bond maturing at time T

T time until maturity

R Long term bond rate

- Coefficients have to be fit to past data.
  - Choose coefficients to minimize error with yield curve, e.g., GMM estimation.

Cox-Ingersoll-Ross (CIR) Model

- CIR model is often written as

$$dr = a(b - r_t)dt + \sigma\sqrt{r}dz$$
$$\Delta r = a(b - r)\Delta t + \sigma\sqrt{r}\left(Z\sqrt{\Delta t}\right)$$

  - Where *b* is long term average rate and *a* is rate of mean reversion.

Cox-Ingersoll-Ross (CIR) Model

- CIR model is hard to implement.
- Discrete time version allows interest rates to become negative.
  - This makes it impossible to implement, because we can't take square root of a negative number.
- Good news:
  - On continuous time model, rate never goes negative.
  - Because volatility is scaled by square root of r, volatility is higher for higher interest rates.
  - There is a nice formula for pricing bonds and options on bonds.

Cox-Ingersoll-Ross (CIR) Model

- This leads to a pricing formula for bonds

$$B(t) = A(t)e^{-C(t)r}$$

where

$$C(t) = \frac{2\left(e^{\gamma t} - 1\right)}{(\gamma + a)\left(e^{\gamma t} - 1\right) + 2\gamma}$$

$$A(t) = \left(\frac{2\gamma \cdot e^{\frac{(a+\gamma)t}{2}}}{(\gamma + a)\left(e^{\gamma t} - 1\right) + 2\gamma}\right)^{\frac{2ab}{\sigma^2}}$$

$$\gamma = \sqrt{a^2 + 2\sigma^2}$$

Cox-Ingersoll-Ross (CIR) Model

- CIR also provide formulas for pricing European puts and calls on zero coupon bonds.
- In practice, coefficients can be fit so that bond prices match yield curve.
  - CIR does a decent job pricing bonds.
  - Options are not priced well.

One Factor Models

- Following table shows models that follow general form:

$$\Delta r = (\mu_1 + \mu_2(\Delta t, r_k))\Delta t + (\sigma_1 + \sigma_2(\Delta t, r_k))^{\alpha}(Z\sqrt{\Delta t})$$

Model	**μ_1**	**μ_2**	**σ_1**	**$\sigma 2$**	**α**
Brennan-Schwartz (1980)	•	•		•	1.0
Cox-Ingersoll-Ross (1985)	•	•		•	0.5
Dothan (1978)				•	1.0
Merton (1973)	•		•		1.0
Pearson-Sun (1994)	•	•	•	•	0.5
Vasicek (1977)	•	•	•		1.0

Salomon Model

- First interest rate model used on Wall Street was developed in the early 1980s by Mark Gordon and Michael Waldman of Salomon Brothers.

Salomon Model

- The basic model is

$$dr = a(t) r \cdot dt + \sigma \cdot r \cdot dz$$

- Taking natural log of this, and using Itô’s Lemma and some algebra, we get:

$$d(\ln[r]) = a(t) dt + \sigma \cdot dz$$

$$\Delta \ln[r] \approx a(t) \Delta t + \sigma \cdot Z \sqrt{\Delta t}$$

$$r_{n+\Delta t} \approx r_n e^{a(t)\Delta t + \sigma \cdot Z \sqrt{\Delta t}}$$

$a(t)$ is simply the forward rate at the time t for the period Δt

  - You don’t need to know how to do this.

Salomon Model

- With these models we can think of the rate as having two pieces – one that is random, and one that is not random.
- For the Salomon Model:

$$\Delta \ln[r] = a_n \Delta t + \sigma Z \sqrt{\Delta t}$$

$$\text{where } Z \sim \mathrm{N}(0,1)$$

So

$$r_{\Delta t} \approx r_0 e^{a_1 \Delta t + \sigma \cdot Z \sqrt{\Delta t}} = r_0 e^{a_1 \Delta t} e^{\sigma \cdot Z \sqrt{\Delta t}}$$

If we set the level of uncertainty to zero $(Z = 0)$ then:

$$r_{\Delta t} \approx r_0 e^{a_1 \Delta t}$$

Salomon Model

- This means that a_n is determined by change in forward rates.
- In general, non-random piece accounts for two things: change in forward rates and speed at which rates revert to mean.

Salomon Model

- Note, that Salomon Model was set up as a binomial tree that was wrapped around forward rates.
- It was built to ensure that there was no arbitrage (Treasuries were all priced correctly).
- Path Dependent Options were priced using a Monte Carlo simulation.
- Non-Path Dependent Options were priced using backwards induction.

Extensions

- Many two and three factor models have been created to extend these models
  - Factors are usually short rate, slope of yield curve and curvature of yield curve.
  - These models do a better job of pricing bonds.
  - They still have problems pricing options
    - Note options priced are usually caps, floors, swaptions, etc. These are usually portfolios of options, so pricing errors compound.

Calibrating Models

- Models are calibrated by minimizing tracking error between model and data.
- For example, for CIR:
  - Use long-term data to get volatility.
  - Guess at a and b.
  - Price zeros using B(t) and actual prices.
  - Calibrate a and b to minimize tracking error.

Interest Rate Derivatives

Interest Rate Derivatives

- The following securities can be considered interest rate derivatives
  - Forward contracts on fixed income instruments.
  - Futures contracts on fixed income instruments.
  - Options on these contracts.
  - Interest Rate Swaps.
  - Swaptions (options on swaps).
- Today, we will concentrate more on what these securities are. In the late classes, we will return to talk more about pricing, risks, using them for hedging, etc.

Forward Contracts

- A Forward Contract is:
  - An OTC Agreement between two parties.
  - Contract between a buyer and a seller.
  - Seller delivers a predetermined amount of a good at a predetermined date (Settlement or Delivery Date).
  - Buyer takes delivery on that day, and pays a predetermined price.

Forward Contracts

- No money changes hands until delivery – Price predetermined.
- Two sided default risk
  - Seller could default, and not deliver.
  - Buyer could default, refusing delivery (not paying).
- Agreements are tailored to participants – therefore, not liquid.

Example

- When-issued trading of bonds is like a forward contract.
  - Securities trade on a when issued basis when they have been announced, but not yet issued. The transaction is settled only after the security has been issued.
  - Investment bank agrees to deliver the security when it is issued.
  - Investor agrees on a yield.
  - Both parties are exposed to interest rate risk.

Example

- You enter into an agreement with Goldman Sachs to buy a one-year bond with a coupon rate of 10% in three months.
- The bank guarantees a price.
- In three months, you deliver money, Goldman Sachs delivers the bond.

Pricing Forward Contracts

- Two ways to get goods at settlement:
  - Forward Contract
    - Enter contract now for 1-yr 10% bond.
    - Take delivery at settlement.
    - Make payment at settlement.
  - Replicate contract
    - Borrow money now.
    - Purchase 15 month, 10% bond.
    - Store bond for three months.
    - Pay back loan on settlement date with cash and coupon.

Forward Contracts

- Liquidity
  - Forward contracts can’t be sold easily.
  - Participants may “get out” of a contract by entering a contract with someone else
    - Investor sells contract to Citibank.
    - Goldman sells contract to Morgan Stanley.
    - Note – this increases counterparty risk.

Futures Contracts

- A Futures Contract is like a forward contract , but
  - Traded on exchanges.
  - Standard contracts make them more liquid.
  - Counterparty is *always* the exchange -- so safer
  - Margin requirements.
  - Participants rarely take delivery.
  - Marked-to-market
    - Daily cash settlement.
    - Keeps contracts standard.

Futures Contracts

- Contracts specify:
  - Commodity to be delivered.
  - Quantity.
  - Date of delivery.
  - Place of delivery.
  - Minimum price tick and daily limits.
  - Margin requirements.

Example

- Eurodollar Futures (traded on CME)
  - Long 1 contract = receive, at settlement, $250,000 less interest payable on a three-month Eurodollar deposit of $1 million.
  - Prices are quoted as (100 - interest rate, in percent, on a three-month Eurodollar deposit). Therefore current value of one contract is $2,500 times quoted price. A price of 95 corresponds to a 5% interest rate.
  - Result: get/pay $25 for each b.p. difference when marked-to-market.

Example

- There are 120 different futures contracts, corresponding to delivery in each month of the next 10 years.
- Contract is cash-settled with exchange based on British Bankers Association Interest Settlement Rate for three-month dollar deposits as of contract's last trading day.
- Minimum price tick is 0.01 (one basis point of three-month interest rate), or $25/contract.
- No maximum daily price move.

Mechanics

- Trades only executed by members of an exchange.
- Brokers own chairs, act as intermediaries
- “Locals” act as market-makers.
- Clearinghouse is *always* the counterparty. This minimizes risk when someone defaults.
- Margins are required. Daily settlement.

Margin

- Initial Margin
  - Cash Deposit to cover daily mark-to-market changes.
  - Usually greater than most one-day changes.
- Daily profits added to margin account.
- Daily losses taken from margin account.
- Maintenance margin: Investor must restore margin when balance falls below maintenance level.

Treasury Bond Futures

- Contract quoted in terms of a fictitious 6%, 30 year bond with a face value of $100,000.
- Trades on CME.
- Quotation: Price of bond – Percent of Par (e.g., 139-31 – 10/26/2011 price for Dec 11 contract).
- Tick Size: 1/64 ($15.625 per contract).
- Daily price limit – 3%.
- Contract Months – March, June, September, December.

Treasury Bond Futures

- Trading Ends – 7th business day before last business day of month.
- Seller delivers
  - $100,000 face value, 15-30 years.
  - Price depends on bond delivered (conversion factor used).
  - Seller determines which bond.
  - Seller determines when in month to deliver.
- Buyer pays price plus accrued.

Treasury Bond Futures

- Quality Option – seller uses ‘cheapest to deliver’ (explanation later).
- Timing option – seller can pick date in month to deliver.
- Wildcard option: Seller can give intent to deliver several hours after market closes.

Bond Factor

- Note that at a yield of 6%, a 6% coupon bond has a value equal to par amount plus any accrued interest.
- The factor for a 6% bond is 1. You may deliver $100,000 worth of it.
- For a bond with a different coupon, find price of bond at 6%.
- For coupons greater than 6%, the value will be greater than par, so factor > 1.
  - Coupons < 6% → value < one, so factor < 1.

Cheapest to Deliver

- As yields change, value of bonds that could be delivered change.
- Since they have different durations, they change at different rates.
- Since they have different factors, relative cost of these bonds could change.
- Since investor with short position in futures contract has ability to switch between deliverable securities, he has an option.

Cheapest to Deliver

- Many analysts use current cheapest-to-deliver when doing their analysis.
  - Technically, this is not correct – as the cheapest to deliver can change through time.
  - What they are doing is analogous to using Duration-to-Worst and Yield-to-Worst for callable bonds. It ignores the probability that the worst case can change.
- In practice, you can use a model of interest rates to price all securities at delivery date, and determine, empirically, which is best to deliver.
  - This method is completely deterministic, since the universe of deliverable bonds is finite.

Options on Futures

- Options can be bought on futures contracts:
  - Contract size: One futures contract.
  - Tick size: 1/64 of a point.
  - Contract months: Next three months plus four quarterly expiration contracts (Mar, June, Sep, Dec).
  - Strike Price: Integral values close to price.

Treasury Note Futures

- T-Note Futures – CBOT
  - Deliverable: 6.5-10 year maturity.
  - Less volume than T-Bond.
- 5 yr-Note Futures – CBOT
  - Deliverable: 4.25-5.25 year maturity.

Treasury Bill Futures (CME)

- Seller delivers $1 mil face value 90-day T-Bills
- Price Paid = $1 mil (1-Rate(days/360))
- Minimum Tick – 0.5 bp ($12.5)
- Daily limit – None
- Contract Months – March, June, September, December – plus two months in next quarter
- Delivery – Issue date of T-bill
- One bp. change: $25

Euro CD (CME)

- Same as T-Bill Futures except:
  - $1mil Euro CD is underlying asset.
  - Trading ends Second London Business Day preceding third Wednesday of contract month.
  - Cash Settlement (no physical delivery).

Interest Rate Swaps

- Swaps: An OTC Derivative Contract between two parties.
- Contract specifies:
  - Notional Principal.
  - Rate (or price).
  - Payment Frequency.
  - One party pays floating rate.
  - The other party pays fixed rate.
  - Only net change gets paid – principal is never paid.
  - Fixed rate chosen so price is zero at start.

Interest Rate Swaps

- Example:
  - $10mil Notional Principal.
  - Semiannual interest payments for five years.
  - One party pays LIBOR.
  - Other party pays fixed rate (close to 5-year Treasury rate).
- Q: How does this differ from 10 forward contracts with expirations of 0.5, 1, 1.5, 2, . . ., 10 Years?
- A: It doesn't.

Interest Rate Swaps

- What is the point?
- Consider:
  - Issue 5-year floating rate debt.
  - Issue 5-year fixed rate debt and enter into 5-year swap where you get fixed and pay floating.
  - What is the difference?
- Original swaps were between US companies who had access to fixed rate market and European companies who had access to floating rate market.

Interest Rate Swaps

- How can banks hedge with Swaps?
- Consider a bank with:
  - Short term (floating rate) liabilities.
  - Long-term (fixed rate) assets.
- The bank can hedge its interest rate risk with a swap where they pay fixed and receive floating.

Interest Rate Swaps

- "Normal Swap Agreement"
  - Pay Floating based on LIBOR at previous payment (e.g., first payment based on current LIBOR).
- Counterparty risk
  - What is lost if one party defaults?
  - No principal is lost.
  - Just difference between fixed and floating on future payment is lost.

Futures on Swaps

- The CBOT offers contracts on:
  - 30-year Interest Rate Swap.
  - 10-year Interest Rate Swap.
  - 5-year Interest Rate Swap.
- They also offer options on these contracts.

Futures on Swaps

- Contract Specifications for Futures Contract on 30-year Interest Rare Swaps
  - Notional value is $100,000.
  - Buyer will get contract where he pays 6% fixed (semiannual payments) and receive floating (based on 3-month LIBOR).
  - Tick Size: 1/64 of a point (longer term) and 1/128 (short term).
  - Contract months: Next three months plus four quarterly expiration contracts (Mar, June, Sep, Dec).

Futures on Swaps

- Final settlement price is determined by:

$$100{,}000 * \left(\frac{6\%}{r} + \left(1 - \frac{6\%}{r}\right)\left(1 + \frac{r}{2}\right)^{-20} \right)$$

- Where *r* is the ISDA Benchmark rate for a 10-year swap on the last day of trading.

Swaptions

- A Swaption is an option on a swap.
- Receiver Swaption
  - The owner has the right to enter into an IRS where he receives fixed and pays floating.
- Payer Swaption
  - The owner has the right to enter into an IRS where he pays fixed and receives floating.
- Very liquid market.

Caps and Floors

- A Cap is made up of a series of payments that are made (or received) if a rate exceeds the Cap.
  - Payment is the interest paid on a notional amount of the difference in interest at actual rate and cap rate – when actual rate exceeds cap.
  - Each payment is called a “caplet” – so a Cap can be though of as a portfolio of caplets.
- Floors work just the opposite.
- Caps and Floors are sold at a positive price.
- Note that an Interest Rate Swap is equivalent to being long a cap and short a floor at the same rate.