夹逼原理及例题

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本段内容要点:

夹逼原理（两边夹原理）
二项式公式用于不等式放缩
系列例题，包括 $\lim\limits_{n\to\infty}\sqrt[n]{n}=1$,

$$\lim_{n\to\infty}n^kq^n=0,\ \ (k\in\mathbb{N},|q|<1).$$

定理:[夹逼原理]:

设有三个数列$\{x_n\},\{y_n\},\{z_n\}$满足:

$(a)\ x_n \leqslant y_n \leqslant z_n, n = 1, 2, \cdots,$

$(b)\ \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n = a,$

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(b) $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n = a,$

则必有$\lim\limits_{n\to\infty} y_n = a.$

例1: $x_n = \dfrac{\sin n}{n}, \ \ n \in \mathbb{N}.$

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所以 $\lim\limits_{n\to\infty} \dfrac{\sin n}{n} = 0.$

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因为$\lim\limits_{n\to\infty}\dfrac{n^2}{n^2+n\pi}=\lim\limits_{n\to\infty}\dfrac{n^2}{n^2+\pi}=1$.

故$\lim\limits_{n\to\infty} x_n = 1$.

例3: $\lim\limits_{n\to\infty}(a_1^n+a_2^n+\cdots+a_k^n)^{\frac{1}{n}}$,

其中$a_i>0,\ i=1,\cdots,k,\ k\in\mathbb{N}$是确定的数.

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即 $\lim\limits_{n\to\infty}(a_1^n+a_2^n+\cdots+a_k^n)^{\frac{1}{n}}=\max\{a_1,a_2,\cdots,a_k\}$.

例4: $x_n = \dfrac{(2n-1)!!}{(2n)!!} = \dfrac{1\cdot 3\cdot 5\cdot\ \cdots\ \cdot(2n-1)}{2\cdot 4\cdot 6\cdot\ \cdots\ \cdot(2n)}$

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解: 因为$(n+1)(n-1) = n^2 - 1 < n^2$,

故$\boxed{\frac{n-1}{n} < \frac{n}{n+1},}$

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所以$\lim\limits_{n\to\infty}\dfrac{(2n-1)!!}{(2n)!!} = 0.$

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这里使用了$n>2$时, $n-1>\dfrac{n}{2}$.

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$\boxed{C_n^k}$

事实上，

$a^n>$ $\boxed{n\text{的}k\text{次多项式}}$ $*(a-1)^k,\ n>N_0.$

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故 $0 < \sqrt[n]{n} - 1 \leqslant \frac{2}{\sqrt{n}} \to 0\ (n\to\infty).$

据夹逼原理, $\lim\limits_{n\to\infty} \sqrt[n]{n} - 1 = 0,$

即 $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1.$

$$\boxed{a^n \geqslant \frac{n^2}{4}(a-1)^2,\ (n>2).}$$

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$$\to 0\ (n\to\infty).$$

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$$\boxed{(|q|^{-1})^n \geqslant \frac{n(n-1)(n-2)}{3!}(|q|^{-1}-1)^3} \geqslant \frac{n}{6}\cdot\frac{n}{2}\cdot\frac{n}{3}\cdot(|q|^{-1}-1)^3$$

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这里使用了$n>3$时, $n-1>\frac{n}{2},\ n-2>\frac{n}{3}$.

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$$\geqslant \frac{n}{6}\cdot\frac{n}{2}\cdot\frac{n}{3}\cdot(|q|^{-1}-1)^3$$

$$= \frac{n^3}{36}(|q|^{-1}-1)^3 \qquad (n>3)$$

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例6: $\lim\limits_{n\to\infty} nq^n = 0 \quad (0<|q|<1).$

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$$|n^2q^n| = n^2|q|^n = n^2\cdot\frac{1}{(|q|^{-1})^n}$$

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推而广之,

$$\forall k\in\mathbb{N},\quad \lim_{n\to\infty} n^kq^n = 0,\ 0<|q|<1.$$

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当$a>1$时， $a^n=[1+(a-1)]^n=\sum_{k=0}^{n} C_n^k (a-1)^k$

$$=1+n(a-1)+\frac{n(n-1)}{2}(a-1)^2+\cdots+(a-1)^n$$

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本段知识要点:

夹逼原理

二项式公式用于不等式放缩

$$\lim_{n\to\infty} \sqrt[n]{n} = 1,$$

$$\lim_{n\to\infty} n^k q^n = 0,\ \ (k \in \mathbb{N}, |q| < 1).$$

$$\lim_{n\to\infty} (a_1^n + a_2^n + \cdots + a_k^n)^{\frac{1}{n}} = \max\{a_1, a_2, \cdots, a_k\}.$$