闭区间上连续函数性质

本段内容要点:

闭区间上的连续函数有界
闭区间上的连续存在最大最小值
连续函数具有介值性
函数的零点定理
相关例题

引理: 闭区间上的连续函数必有界.

(有限覆盖)

定理: (最值定理)

闭区间上的连续函数必有(必能达到)最大值最小值.即

$$f(x) \in C_{[a,b]} \Rightarrow \begin{cases} \exists x_1 \in [a,b]\ s.t.\ \ f(x_1) = \max\limits_{x\in[a,b]} f(x), \\ \exists x_2 \in [a,b]\ s.t.\ \ f(x_2) = \min\limits_{x\in[a,b]} f(x). \end{cases}$$

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上面两个结论对开区间则未必!

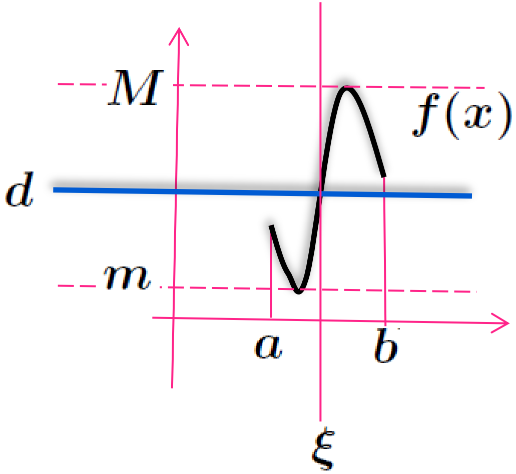
(1) $f(x) = \dfrac{1}{x}$在$(0,1)$上连续但无界.

(2) $f(x) = x^2$在$(0,1)$上连续有界但无最大最小值.

定理: (中介值定理)

设$f(x) \in C_{[a,b]}$, $m = \min\limits_{x\in[a,b]} f(x)$, $M = \max\limits_{x\in[a,b]} f(x)$.

则对$\forall d \in [m, M]$, $\exists \xi \in [a, b]\ s.t.\ f(\xi) = d$;



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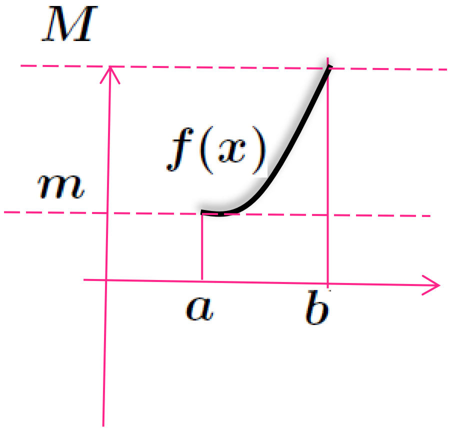

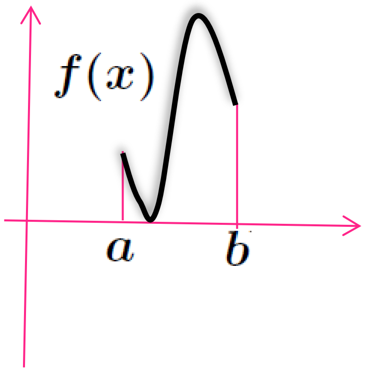

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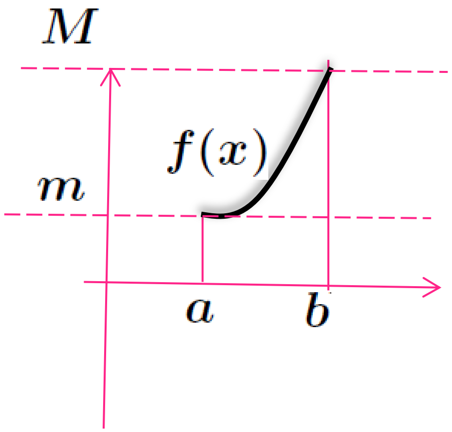

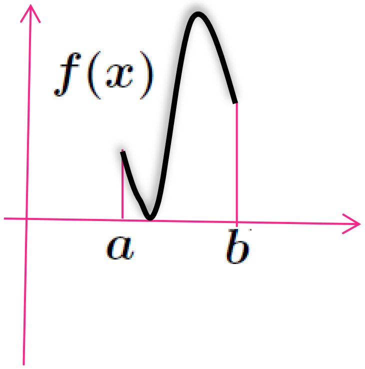

定理: (中介值定理) 设$f(x) \in C_{[a,b]}$, $f(a) \neq f(b)$(不妨设$f(a) < f(b)$). 则对于任意的$c \in (f(a), f(b))$, $\exists \xi \in (a, b)$ $s.t.$ $f(\xi) = c$.

推论:(零点定理) 设$f(x) \in C_{[a,b]}$, $f(a)f(b) < 0$, 则一定$\exists \xi \in (a, b)$ $s.t.$ $f(\xi) = 0$.

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令$f(x) = a_0x^{2n+1} + a_1x^{2n} + \cdots + a_{2n}x + a_{2n+1}$
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因为$\lim\limits_{x\to+\infty} f(x) = +\infty$, 故$\exists x_2 > 0$ $s.t.$ $f(x_2) > 0$.
因为$\lim\limits_{x\to-\infty} f(x) = -\infty$, 故$\exists x_1 < 0$ $s.t.$ $f(x_1) < 0$.

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在$[x_1, x_2]$区间上对$f(x)$使用零点定理,
则得到$\xi \in (x_1, x_2)\ s.t.\ f(\xi) = 0$.

定理:(代数基本定理):

任何n次实系数多项式, 都存在n个根(零点)
(包括复根, 重根按重数计算个数).
若有复根, 则均是成对(共轭)出现.

定义:

若定义在(a,b)上的函数$f(x)$满足:
$\exists L > 0 \; s.t.$
$|f(x_1) - f(x_2)| \leqslant L|x_1 - x_2|, \; \forall x_1, x_2 \in (a,b)$.
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如$f(x)=|x|$是Lipschitz连续的,
$|f(x_1)-f(x_2)|=||x_1|-|x_2||\leqslant|x_1-x_2|.$
但不可导.

例: 设$f(x)\in C_{[0,1]}$, $0\leqslant f(x)\leqslant 1, \forall x\in[0,1]$,
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由于$g(0) = f(0) \geqslant 0$; $g(1) = f(1) - 1 \leqslant 0$.
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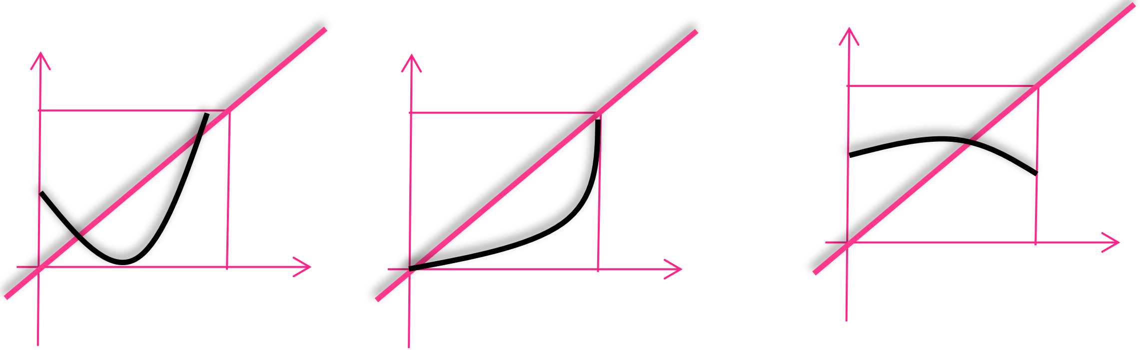

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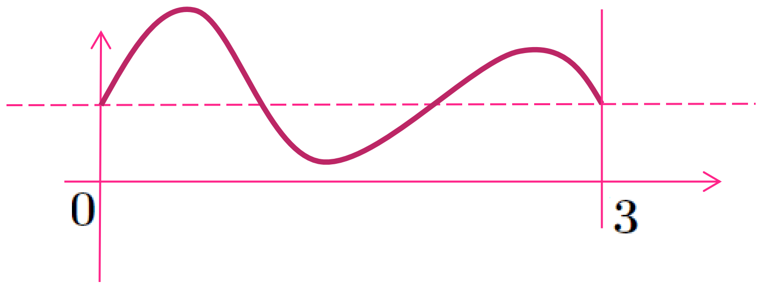

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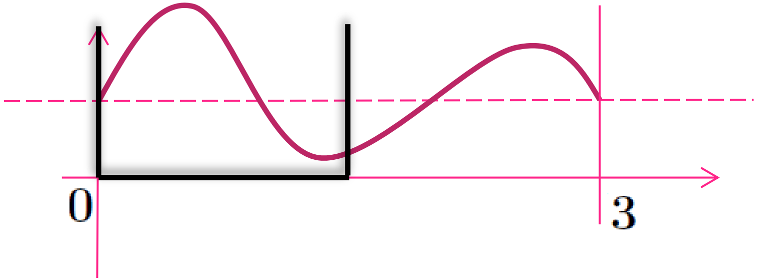

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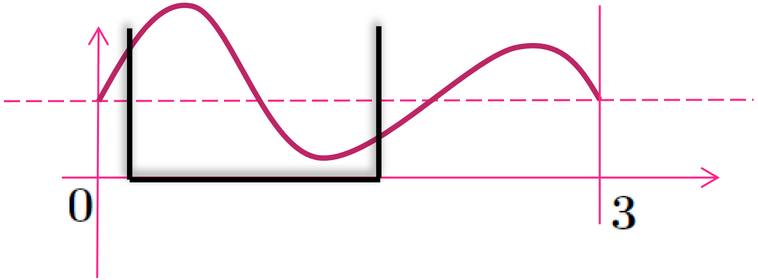

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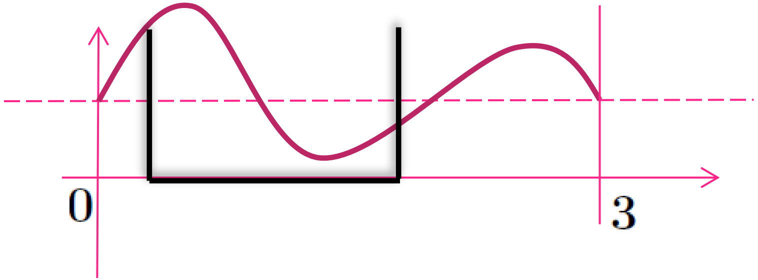

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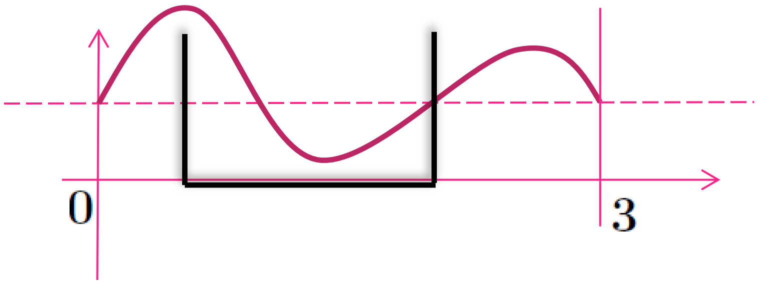

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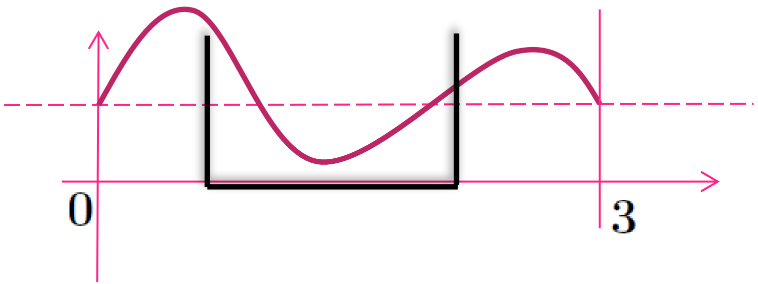

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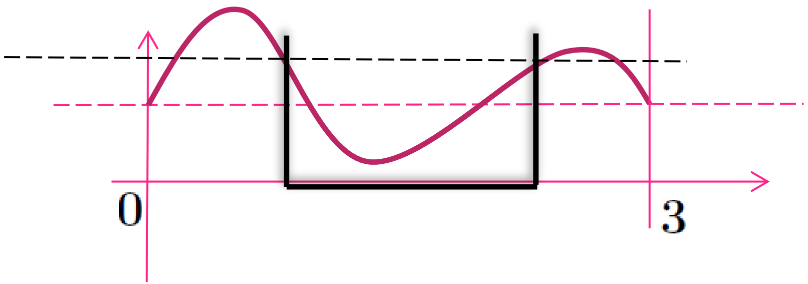

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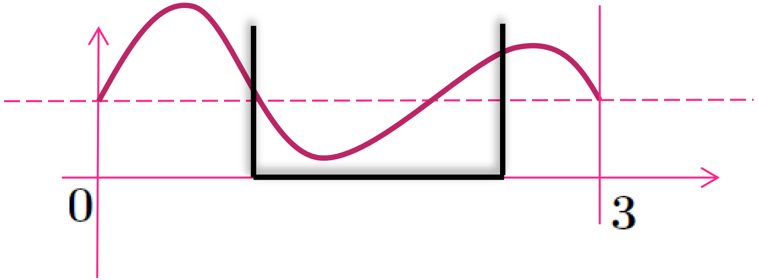

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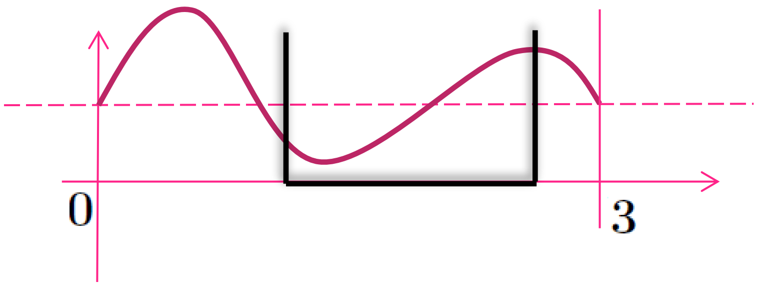

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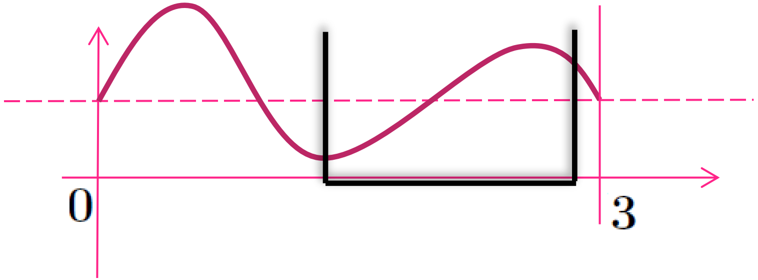

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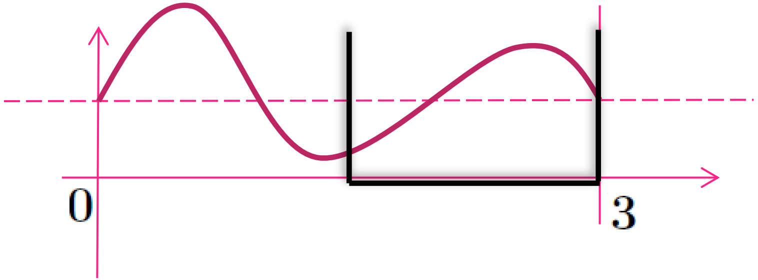

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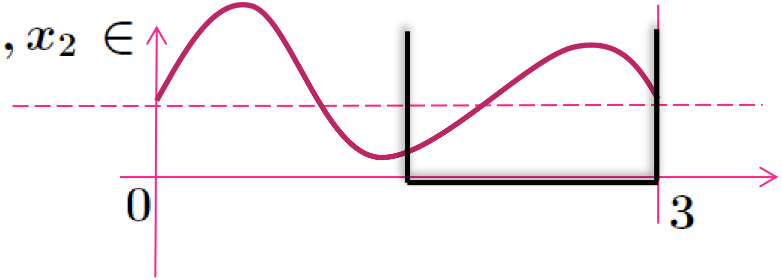

证明:

令$g(x) = f(x + \frac{3}{2}) - f(x)$, $x \in [0, \frac{3}{2}]$.

则$g(x) \in C_{[0,\frac{3}{2}]}$. $\boxed{\text{往证: } \exists \xi \in [0, \frac{3}{2}]\ s.t.\ g(\xi) = 0.}$

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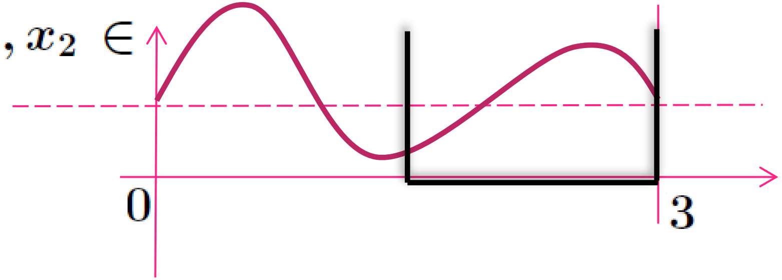

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因为 $g(0)=f(\dfrac{3}{2})-f(0)$,

$g(\dfrac{3}{2})=f(3)-f(\dfrac{3}{2})=f(0)-f(\dfrac{3}{2})$. 故$g(0)g(\dfrac{3}{2})\leqslant 0$.

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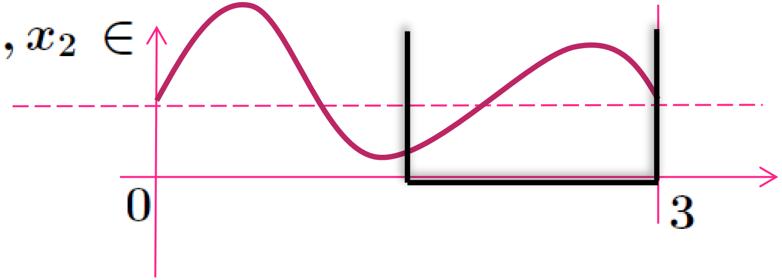

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所以根据零点定理, $\exists \xi \in [0, \frac{3}{2}]$ $s.t.$ $g(\xi) = 0$.

即$\exists x_1 = \xi, x_2 = \xi + \frac{3}{2}$ $s.t.$ $f(x_1) = f(x_2)$.

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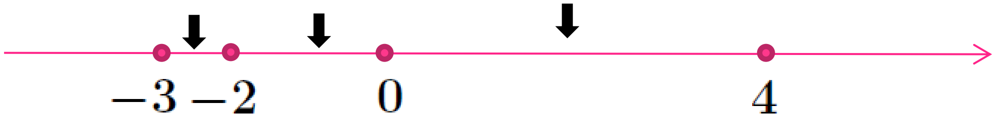
令$f(x) = x^3 - 9x - 1$,则计算显示

$f(-3) = -1 < 0,\ f(-2) = 9 > 0,$

$f(0) = -1 < 0,\ f(4) = 27 > 0.$



例: 求证方程$x^3 - 9x - 1 = 0$恰有三个实根.

证明: 据代数基本定理,方程最多有三个实根.现证方程确实存在三个实根.

令$f(x) = x^3 - 9x - 1$, 则计算显示

$f(-3) = -1 < 0,\ f(-2) = 9 > 0,$

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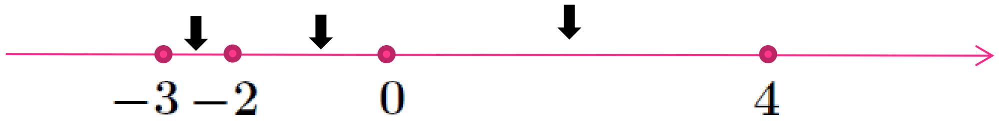

所以

$\exists \xi_1 \in (-3, -2)\ s.t.\ f(\xi_1) = 0,$

$\exists \xi_2 \in (-2, 0)\ s.t.\ f(\xi_2) = 0,$

$\exists \xi_3 \in (0, 4)\ s.t.\ f(\xi_3) = 0.$

所以, 原方程恰有三个实根.

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求证$f(x)$在$[a,+\infty)$有界.

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$\exists X>a\ s.t.\ x>X\Rightarrow|f(x)-A|<1$.

即$|f(x)|<A+1, x\in(X,+\infty)$.

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根据闭区间上连续函数的最值定理,

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则$|f(x)|\leqslant M, x\in[a,+\infty)$. $f(x)$的有界性得证.

本段知识要点:

闭区间上的连续函数有界
闭区间上的连续存在最值
连续函数具有介值性
连续函数的零点定理