Homework Assignment #8
Due: In class, two weeks after distribution

We follow the notation used in the lecture notes.

§1 Change of Discounted Values

Apply Itô formula to prove that the change of the discounted replicating portfolio value is only due to the change of the discounted stock price, i.e.

$$d\left[e^{-rt}X(t)\right] = \Delta(t)d\left[e^{-rt}S(t)\right].$$

§2 Verification for Black-Scholes-Merton (1973)

Follow our discussion in class to show that $c(t, S(t))$, where $c(t, x)$ solves the BSM equation, is indeed the time t value of the European call option.

§3 Understanding Option's Return

Let $\lambda = \frac{\alpha - r}{\sigma}$. This is usually called the market price of risk or the Sharpe ratio of the underlying asset. Denote by $\{\mathcal{F}(t)\}$ the filtration generated by the underlying asset price $\{S(t)\}$. Show that

$$\lim_{\Delta \to 0} \frac{1}{\Delta} \frac{E(c(t+\Delta, S(t+\Delta))|\mathcal{F}(t)) - c(t, S(t))}{c(t, S(t))} - r = \frac{\frac{\partial c}{\partial x}(t, S(t))\sigma S(t)}{c(t, S(t))}\lambda.$$

How do you interpret this result?

§4 Option Pricing with the Local Volatility Model

To remedy the unrealistic assumption on the constant volatility, B. Dupire proposed the celebrated local volatility model by generalizing the volatility as a function of the underlying stock price. Under the real world probability measure, the local volatility model is proposed as

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma(t, S(t))dW(t),$$

where μ is a positive constant representing the return of the asset; and $\sigma(t, x)$ is a "volatility" function. Please mimic the derivation of the BSM equation to derive an equation for pricing a European put option with maturity T and strike K.

§5 Exchange Options

Please note that this exercise is optional.

An exchange option holder has the right to exchange one asset for another. For example, a hedge fund manager might purchase an exchange option if she is sure that she wants to hold one of two stocks, for example the Mercedes-Benz (Daimler AG) and BMW, but is unsure of which. The option would then entitle the fund manager to hold one share of Mercedes-Benz and then switch it for BMW depending on the relative performance of the two stocks during the option's life.

Suppose two assets S_1 and S_2 follow that

$$\begin{aligned} \frac{dS_1(t)}{S_1(t)} &= \mu_1 dt + \sigma_1 dW_1(t), S_1(0) = s_1, \\ \frac{dS_2(t)}{S_2(t)} &= \mu_2 dt + \sigma_2 dW_2(t), S_2(0) = s_2, \end{aligned}$$

where μ_1, μ_2, σ_1, σ_2 and the initial asset price s_1, s_2 are all positive constants. And, the two driving standard Brownian motions are correlated in the sense that

$$dW_1(t)dW_2(t) = \rho dt$$

Suppose the interest rate is r.

Use the no-arbitrage principle to derive a partial differential equation for pricing an exchange option paying out $\max\{S_1(T) - S_2(T), 0\}$ at maturity T.