隐函数的微分法

定义

如果在某平面区域内 $F(x, y) = 0$ 可以确定 y 为 x 的函数（或者 x 为 y 的函数），则称 $F(x, y) = 0$ 在该区域内为**隐函数**.

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$$\Rightarrow \frac{\partial z}{\partial x}=-\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial z},\quad \frac{\partial z}{\partial y}=\frac{\partial F}{\partial y}\Big/\frac{\partial F}{\partial z}.$$

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同理可计算$\dfrac{\partial x}{\partial y}$, $\dfrac{\partial x}{\partial z}$ 和$\dfrac{\partial y}{\partial x}$, $\dfrac{\partial y}{\partial z}$

例

设 $e^{-xy}-2z-e^{z}=0$ 确定 z 为 x,y 的函数,求 $\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.

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$$\mathrm{d}z=\frac{e^{-xy}(x\,\mathrm{d}y+y\,\mathrm{d}x)}{e^{z}-2}$$

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$$\therefore\frac{\partial z}{\partial x}=\frac{ye^{-xy}}{e^{z}-2},\quad\frac{\partial z}{\partial y}=\frac{xe^{-xy}}{e^{z}-2}.$$

（不用死记隐函数的导数公式!）只要求全微分

例 设 $F(x,y,z)=0$ 中,任一变量可以看作是另两个的函数,则有 $\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial y}\dfrac{\partial y}{\partial z}=-1$

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$$\mathrm{d}F(x,y,z)=0,\Rightarrow F_x'\,\mathrm{d}x+F_y'\,\mathrm{d}y+F_z'\,\mathrm{d}z=0,$$

$$\begin{aligned}
\mathrm{d}x&=-\frac{F_y'}{F_x'}\mathrm{d}y-\frac{F_z'}{F_x'}\mathrm{d}z\\
\mathrm{d}y&=-\frac{F_x'}{F_y'}\mathrm{d}x-\frac{F_z'}{F_y'}\mathrm{d}z\\
\mathrm{d}z&=-\frac{F_x'}{F_z'}\mathrm{d}x-\frac{F_y'}{F_z'}\mathrm{d}y
\end{aligned}
\quad\Rightarrow\quad
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\frac{\partial x}{\partial y}&=-\frac{F_y'}{F_x'}, & \frac{\partial x}{\partial z}&=-\frac{F_z'}{F_x'}\\
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另法:

$$F(x,y,z)=0 \Rightarrow \begin{array}{l} F'_x+F'_z\dfrac{\partial z}{\partial x}=0 \Rightarrow \dfrac{\partial z}{\partial x}=-\dfrac{F'_x}{F'_z} \\ F'_y+F'_z\dfrac{\partial z}{\partial y}=0 \Rightarrow \dfrac{\partial z}{\partial y}=-\dfrac{F'_y}{F'_z} \end{array}$$

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$$\Rightarrow \begin{aligned} F'_x+F'_y\frac{\partial y}{\partial x}=0 \Rightarrow \frac{\partial y}{\partial x}=-\frac{F'_x}{F'_y} \\ F'_z+F'_y\frac{\partial y}{\partial z}=0 \Rightarrow \frac{\partial y}{\partial z}=-\frac{F'_z}{F'_y} \end{aligned}$$

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$$\Rightarrow \begin{aligned} F'_x\frac{\partial x}{\partial y}+F'_y=0 \Rightarrow \frac{\partial x}{\partial y}=-\frac{F'_y}{F'_x} \\ F'_x\frac{\partial x}{\partial z}+F'_z=0 \Rightarrow \frac{\partial x}{\partial z}=-\frac{F'_z}{F'_x} \end{aligned}$$

推广：$\boldsymbol{F(x,y,z,t)=0}$, 若任一变量可以看作是另三个的函数,

$$\text{则 } \frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial t}\frac{\partial t}{\partial x}=1.$$

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$$\therefore\ \frac{\partial z}{\partial x}=\frac{F_1'}{F_2'},\quad \frac{\partial z}{\partial y}=\frac{F_2'-F_1'}{F_2'}.$$

(三) $\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$ 决定 $\begin{cases} y=y(x) \\ z=z(x) \end{cases}$

(两面定一线),空间曲线方程 求 $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, $\dfrac{\mathrm{d}z}{\mathrm{d}x}$.

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$$\Rightarrow \mathrm{d}y=\frac{\begin{vmatrix} -F'_x\,\mathrm{d}x & F'_z \\ -G'_x\,\mathrm{d}x & G'_z \end{vmatrix}}{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}}=-\frac{\begin{vmatrix} F'_x & F'_z \\ G'_x & G'_z \end{vmatrix}}{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}}\mathrm{d}x,$$

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$$\boxed{\text{引入记号: } \frac{D(F,G)}{D(x,z)} = \begin{vmatrix} F'_x & F'_z \\ G'_x & G'_z \end{vmatrix}, \text{ 则}}$$

Jacobi 行列式

$$\Rightarrow \mathrm{d}y = \frac{\begin{vmatrix} -F'_x\,\mathrm{d}x & F'_z \\ -G'_x\,\mathrm{d}x & G'_z \end{vmatrix}}{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}} = -\frac{\begin{vmatrix} F'_x & F'_z \\ G'_x & G'_z \end{vmatrix}}{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}}\mathrm{d}x, \quad \mathrm{d}z = -\frac{\begin{vmatrix} F'_y & F'_x \\ G'_y & G'_x \end{vmatrix}}{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}}\mathrm{d}x$$

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$$\boxed{\frac{dy}{dx} = -\frac{D(F,G)}{D(x,z)} \Big/ \frac{D(F,G)}{D(y,z)}, \quad \frac{dz}{dx} = -\frac{D(F,G)}{D(y,x)} \Big/ \frac{D(F,G)}{D(y,z)}.}$$

$$\Rightarrow \mathrm{d}y = \frac{\begin{vmatrix} -F'_x\,\mathrm{d}x & F'_z \\ -G'_x\,\mathrm{d}x & G'_z \end{vmatrix}}{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}} = -\frac{\begin{vmatrix} F'_x & F'_z \\ G'_x & G'_z \end{vmatrix}}{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}}\mathrm{d}x, \quad \mathrm{d}z = -\frac{\begin{vmatrix} F'_y & F'_x \\ G'_y & G'_x \end{vmatrix}}{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}}\mathrm{d}x$$

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根据实际情况可以灵活应用,不必记公式!

例 $\begin{cases} x^2+y^2+z^2=a^2 \\ x+y+z=0 \end{cases}$ (一个大圆) ,求 $\dfrac{\mathrm{d}y}{\mathrm{d}x}, \dfrac{\mathrm{d}z}{\mathrm{d}x}$.

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$$\begin{cases} x\,\mathrm{d}x+y\,\mathrm{d}y+z\,\mathrm{d}z=0 \\ \mathrm{d}x+\mathrm{d}y+\mathrm{d}z=0 \end{cases}$$

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