ECON 139 Lecture 16 Scribe Notes

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Lecture Outline:

1. Pricing with CAPM
2. Efficient Set Mathematics
3. Proposition

Pricing with CAPM

$\tilde{X} = \tilde{P}_{x,t+1} + d_{t+1}$

1. $P_x = \frac{E(\tilde{x})}{1+r_f+s}$ where $E(\tilde{x})$ is the expected payoff and s is the spread

$E(\tilde{r}) = \frac{E(\tilde{x})}{P_x} - 1 \Rightarrow \frac{E(\tilde{x})}{P_x} = 1 + E(\tilde{r})$

According to CAPM: $E(\tilde{r}) = r_f + \beta(E(r_m) - r_f)$

$\frac{E(\tilde{x})}{P_x} = 1 + E(\tilde{r}) = 1 + r_f + \beta(E(r_m) - r_f)$

$P_x = \frac{E(\tilde{x})}{1+r_f+\beta(E(r_m)-r_f)}$ spread = $\beta(E(r_m) - r_f)$

2. How to change the expected payoff to get rid of the spread?

Adjusted expected payoff: $P_x = \frac{E(\tilde{x})+\pi}{1+r_f}$

$\beta = \frac{cov(\tilde{r},\tilde{r}_m)}{var(\tilde{r}_m)}$ $r = \frac{\tilde{x}}{P_x} - 1$

$\frac{E(\tilde{x})}{P_x} = 1 + r_f + \beta\left(E(r_m) - r_f\right) = 1 + r_f + \frac{cov(\tilde{r},\tilde{r}_m)}{var(\tilde{r}_m)} * \left(E(r_m) - r_f\right)$

$= 1 + r_f + \frac{cov\left(\frac{\tilde{x}}{P_x},\tilde{r}_m\right)}{var(\tilde{r}_m)} * \left(E(r_m) - r_f\right) = 1 + r_f + \frac{\frac{1}{P_x}cov(\tilde{x},\tilde{r}_m)}{var(\tilde{r}_m)} * \left(E(r_m) - r_f\right)$

$E(\tilde{x}) = P_x\left(1 + r_f\right) + \frac{cov(\tilde{x},\tilde{r}_m)}{var(\tilde{r}_m)} * \left(E(r_m) - r_f\right)$

$P_x\left(1 + r_f\right) = E(\tilde{x}) - \frac{cov(\tilde{x},\tilde{r}_m)}{var(\tilde{r}_m)} * \left(E(r_m) - r_f\right)$

$P_x = \frac{E(\tilde{x}) - \frac{cov(\tilde{x},\tilde{r}_m)}{var(\tilde{r}_m)} * \left(E(r_m)-r_f\right)}{(1+r_f)}$

So, the adjusted expected payoff now is $E(\tilde{x}) - \frac{cov(\tilde{x},\tilde{r}_m)}{var(\tilde{r}_m)} * \left(E(r_m) - r_f\right)$

Efficient Set Mathematics

$\Sigma^{-1}\Sigma = \mathrm{I}$ Σ^{-1}: N×N Inverse of asset return covariance matrix

Σ: N×N Covariance matrix of asset returns

I: N×N Identity matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}$$

$\mathrm{W}^T\Sigma\mathrm{W} = w_1\sigma_1^2 + w_2\sigma_2^2 + 2w_1w_2\sigma_{1,2}$

$\boldsymbol{\mathrm{W}^T\mu = \mu_p}$

$\mathrm{W}^T e = 1$

W: N×1 Asset weights that define a portfolio

W^T: 1×N Transpose of asset weight vector

μ: N×1 Vector of asset expected returns

e: N×1 Vector of ones

e^T: 1×N Transpose of ones' vector

$\underset{w}{\mathrm{Min}}\, \boldsymbol{\mathrm{W}^T\Sigma\mathrm{W}} = w_1\sigma_1^2 + w_2\sigma_2^2 + 2w_1w_2\sigma_{1,2}$

s.t $\boldsymbol{\mathrm{W}^T\mu = \mu_p}$ and $\boldsymbol{\mathrm{W}^T e = 1}$

Definition:

Define portfolio P as being the minimum-variance portfolio among all portfolios with expected return μ_p.

$\boldsymbol{W_p = \underset{w}{\mathrm{argmin}} \frac{1}{2} W^T\Sigma W}$

s.t $\boldsymbol{\mathrm{W}^T\mu = \mu_p}$ and $\boldsymbol{\mathrm{W}^T e = 1}$

Lagrangian Multiplier: $\mathcal{L}(w,\lambda,\delta) = \frac{1}{2}W^T\Sigma W - \lambda\left(W^T\mu - \mu_p\right) - \delta(W^T e - 1)$

$\frac{\partial L}{\partial \mathrm{W}} = \Sigma W_p - \lambda\mu - \delta e = 0 \Rightarrow \Sigma W_p = \lambda\mu + \delta e$

$\Rightarrow \Sigma^{-1}\Sigma W_p = \Sigma^{-1}(\lambda\mu + \delta e) \Rightarrow W_p = \Sigma^{-1}(\lambda\mu + \delta e) = \lambda\Sigma^{-1}\mu + \delta\Sigma^{-1}e$

Since transpose of a scalar is the scalar itself, we can get

$W_p^T\mu = \mu_p \Rightarrow \left(W_p^T\mu\right)^T = \mu^T W_p = \mu_p$

$W_p^T e = 1 \Rightarrow \left(W_p^T e\right)^T = e^T W_p = 1$

According to the FOC:

$\mu^T W_p = \mu^T\lambda\Sigma^{-1}\mu + \mu^T\delta\Sigma^{-1}e = \lambda\mu^T\Sigma^{-1}\mu + \delta\mu^T\Sigma^{-1}e = \mu_p$

$e^T W_p = e^T\lambda\Sigma^{-1}\mu + e^T\delta\Sigma^{-1}e = \lambda e^T\Sigma^{-1}\mu + \delta e^T\Sigma^{-1}e = 1$

Consider $\boldsymbol{\mu^T\Sigma^{-1}e}$ (equals to $\boldsymbol{e^T\Sigma^{-1}\mu}$) as a scalar A, $\boldsymbol{\mu^T\Sigma^{-1}\mu}$ as a scalar B, $\boldsymbol{e^T\Sigma^{-1}\mu}$ as a scalar C, we can get

$\lambda\mathrm{B} + \delta\mathrm{A} = \mu_p$

$\lambda\mathrm{A} + \delta\mathrm{C} = 1$

Let $D = BC - \mathrm{A}^2$

$\lambda = \frac{\mu_p C - A}{BC - \mathrm{A}^2} = \frac{\mu_p C - A}{D}$

$$\delta = \frac{B-\mu_p A}{BC-A^2} = \frac{B-\mu_p A}{D}$$

$$W_p = \lambda\Sigma^{-1}\mu + \delta\Sigma^{-1}e = \frac{\mu_p C-A}{D}\Sigma^{-1}\mu + \frac{B-\mu_p A}{D}\Sigma^{-1}e$$

$$= \frac{\mu_p}{D}(C\Sigma^{-1}\mu - A\Sigma^{-1}e) + \frac{1}{D}(B\Sigma^{-1}e - A\Sigma^{-1}\mu)$$

Let $h = \frac{C\Sigma^{-1}\mu - A\Sigma^{-1}e}{D}$, $g = \frac{B\Sigma^{-1}e - A\Sigma^{-1}\mu}{D}$ where h and g are both N×1 matrix

$$\boldsymbol{W_p = h\mu_p + g}$$

$$\sigma_p^2 = W_p^T \Sigma W$$

Propositions and Proofs

▪ **Proposition 1**:

Entire set of MV (Mean-Variance) frontier portfolios can be generated by g and $g+h$.

Proof:

Let g be an arbitrary frontier portfolio with expected return μ_q and $\alpha = 1-\mu_q$, then

$$\alpha g + (1-\alpha)(g+h) = (1-\mu_q)g + \mu_q(g+h) = g + h\mu_q = W_q$$

▪ **Proposition 2**:

Entire set of MV frontier portfolios can be generated by affine combinations of ay two distinct frontier portfolios.

Proof:

Let P_1 and P_2 be two distinct frontier portfolios ($\mu_{P_1} \neq \mu_{P_2}$) and let q be any arbitrary frontier portfolio, then exists α such that $\mu_q = \alpha\mu_{P_1} + (1-\alpha)\mu_{P_2}$

WTS: $$W_q = \alpha W_{P_1} + (1-\alpha)W_{P_2} = \alpha(g + h\mu_{P_1}) + (1-\alpha)(g + h\mu_{P_2})$$
$$= g + h(\alpha\mu_{p_1} + (1-\alpha)\mu_{p_2})$$
$$= g + h\mu_q = W_q$$

▪ **Proposition 3**:

For the global MV portfolio, we have

$$W_{MV} = \frac{1}{C}\Sigma^{-1}e \qquad \mu_{MV} = \frac{A}{C} \qquad \sigma_{MV}^2 = \frac{1}{C}$$

Proof:

$$\underset{W}{Min}\frac{1}{2}W^T\Sigma W$$
$$s.t.\, W^T e = 1$$
$$\mathcal{L}(\mathrm{W},\eta) = \frac{1}{2}W^T\Sigma W - \eta(W^T e - 1)$$

FOC:

$$\frac{\partial\mathcal{L}}{\partial W} = \Sigma W_{MV} - \eta e = 0 \quad \Rightarrow \quad \Sigma W_{MV} = \eta e$$

$\Rightarrow$ $\Sigma^{-1}\Sigma W_{MV} = \Sigma^{-1}\eta e$ $\Rightarrow$ $W_{MV} = \eta\Sigma^{-1}e$

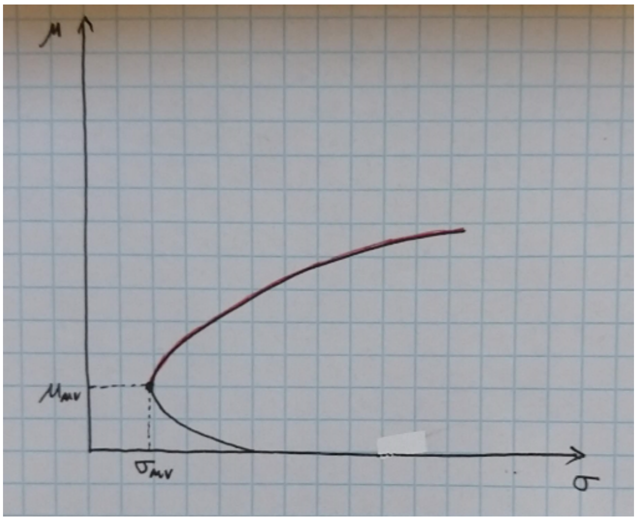
$\Rightarrow$ $e^T W_{MV} = e^T\eta\Sigma^{-1}e = 1$ $\Rightarrow$ $\eta = \frac{1}{C}$

Expected return of the MV portfolio:

$$\mu_{MV} = W_{MV}^T\mu = \left(\frac{1}{C}\Sigma^{-1}e\right)^T \mu = \frac{1}{C}e^T\Sigma^{-1}\mu = \frac{A}{C}$$

Standard deviation of the MV portfolio:

$$\sigma_{MV}^2 = W_{MV}^T \sum W_{MV} = \frac{1}{C}e^T\Sigma^{-1}\Sigma\Sigma^{-1}e\frac{1}{C} = \frac{1}{C}e^T\Sigma^{-1}e\frac{1}{C} = \frac{1}{C} * C * \frac{1}{C} = \frac{1}{C}$$



▪ **Proposition 4**:

Any convex combination of MV frontier portfolios is also a MV frontier portfolio

<u>**Proof**</u>:

▫ Let $W_1, \dots, W_k$ be the weight vector of k portfolios with expected returns $\mu_1, \dots, \mu_k$

▫ Let $\alpha_1, \dots, \alpha_k$ be scalars, and $\sum_{i=1}^k \alpha_i = 1, \quad \alpha_i \geq 0$

▫ $\sum_{i=1}^k \alpha_i W_i = \sum_{i=1}^k \alpha_i(g + h\mu_i) = \sum_{i=1}^k \alpha_i\, g + \sum_{i=1}^k \alpha_i\, h\mu_i$

$= g\sum_{i=1}^k \alpha_i + h\sum_{i=1}^k \alpha_i\,\mu_i = g + h(\sum_{i=1}^k \alpha_i\mu_i)$

<u>Corollary:</u>

The set of efficient portfolios is a convex set

▫ Convex set:

If X is a convex set, then for any $x, y \in X$, we also have $\alpha x + (1-\alpha)y \in X$

<u>**Proof:**</u>

$\mu_1, \dots, \mu_k \geq \frac{A}{C}$ $\Rightarrow$ $\sum_{i=1}^K \alpha_i\mu_i \geq \frac{A}{C}$

▪ **Proposition 5**:

Let p and r be any two MV frontier portfolios.

Then the covariance of the returns of p and r is

$$cov(\widetilde{r_p}, \widetilde{r_r}) = \frac{C}{D}\left(\mu_p - \frac{A}{C}\right)\left(\mu_r - \frac{A}{C}\right) + \frac{1}{C}$$

<u>Proof:</u>

$$\begin{aligned} cov(\widetilde{r_p}, \widetilde{r_r}) &= W_p^T \Sigma W_r = W_p^T \Sigma(\mathrm{g} + \mathrm{h}\mu_r) \\ &= W_p^T (\tfrac{1}{D}(Be - Au) + \tfrac{1}{D}(Cu - Ae)\mu_r) \\ &= \frac{C}{D}\left(\mu_p - \frac{A}{C}\right)\left(\mu_r - \frac{A}{C}\right) + \frac{1}{C} \end{aligned}$$