Economy 139 Lecture 26 Scribe Notes

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Suppose we have a sequence of random variables $\tilde{x}_1, \tilde{x}_2 \ldots$ that iid $(w|E[|\tilde{x}_i|] < \infty)$. Consider $S_n = S_0 + \sum_{i=1}^{n} \tilde{x}_i$. The process $\{S_n, n \geq 1\}$ is called a <u>random walk</u>.

Simple Random Walk

Suppose we start with W_0. Make a series of bets on the flip of a coin
$P_r(\tilde{x}_i = 1) = P_r(\tilde{x}_i = -1) = \frac{1}{2}$.

$$
\begin{aligned}
\tilde{W}_N &= W_0 + \tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_N \\
&= W_0 + \sum_{i=1}^{N} \tilde{x}_i \\
\tilde{W}_N - W_0 &= \sum_{i=1}^{N} \tilde{x}_i \\
E[\tilde{W}_N - W_0] &= E\Big[\sum_{i=1}^{N} \tilde{x}_i\Big] = \sum_{i=1}^{N} E[\tilde{x}_i] = 0
\end{aligned}
$$

each increment $\tilde{x}_i$.

$$
\begin{aligned}
E[\tilde{x}_i] &= 0 \\
Var(\tilde{x}_i) &= E[\tilde{x}_i^2] - (E[\tilde{x}_i])^2 = \frac{1}{2} = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 1 \\
Var(\tilde{W}_N - W_0) &= Var\Big(\sum_{i=1}^{N} \tilde{x}_i\Big) = \sum_{i=1}^{N} Var(\tilde{x}_i) + \underbrace{\sum_{i=1}^{N}\sum_{j=1}^{N} Cov(\tilde{x}_i, \tilde{x}_j)}_{0} \\
&= N
\end{aligned}
$$

Another Random Walk

Suppose we start with zero and at the end of each time interval Δt we receive,

$$
\begin{aligned}
\tilde{\epsilon}_{(t_{i+1})}\sqrt{\Delta t} & \qquad\qquad \tilde{\epsilon}_{(t_{i+1})} \sim N(0,1) \\
\tilde{x}(t_{i+1}) &= \tilde{x}(t_i) + \tilde{\epsilon}(t_{i+1})\sqrt{\Delta t} \\
\tilde{x}(t_{i+1}) - \tilde{x}(t_i) &= \tilde{\epsilon}(t_{i+1})\sqrt{\Delta t} \\
E[\tilde{x}(t_{i+1}) - \tilde{x}(t_i)] &= E\Big[\tilde{\epsilon}(t_{i+1})\sqrt{\Delta t}\Big] = 0 \\
Var(\tilde{x}(t_{i+1}) - \tilde{x}(t_i)) &= Var\Big(\tilde{\epsilon}(t_{i+1})\sqrt{\Delta t}\Big) \\
&= \Delta t \, Var(\tilde{\epsilon}(t_{i+1})) \\
&= \Delta t
\end{aligned}
$$

$$
\begin{aligned}
N \cdot \Delta t &= T \\
\tilde{x}(T_N) &= \sum_{i=0}^{N-1} \tilde{\epsilon}(t_{i+1)\sqrt{\Delta t}} \\
E[\tilde{x}(T_N)] &= E[\sum_{i=0}^{N-1} \tilde{\epsilon}(t_{i+1})\sqrt{\Delta t}] = 0 \\
Var(\tilde{x}(T_N)) = Var\big(\sum_{i=0}^{N-1} \tilde{\epsilon}(t_{i+1})\sqrt{\Delta t}\big) &= \sum_{i=0}^{N-1} Var(\tilde{\epsilon}(t_{i+1})\sqrt{\Delta t}) + \Delta t \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} Cov(\tilde{\epsilon}(t_{i+1})\tilde{\epsilon}(t_{j+1})) \\
&= \sum_{i=0}^{N-1} \Delta t = N\Delta t = T_N
\end{aligned}
$$

As we take $\Delta t \to 0$ we have the Brownian Motion Winner Process, and

$$\tilde{x}(T_N) \sim N(0, T_N)$$

Brownian Motion Caka Wiener Process

Let $dz = \tilde{\epsilon}(t)\sqrt{dt}, \tilde{\epsilon}_t \sim N(0,1)$.

$$
\begin{aligned}
E[dz] &= E[\tilde{\epsilon}_t\sqrt{dt}] = 0 \\
Var(dz) &= Var(\tilde{\epsilon}_{(t)}\sqrt{dt}) = dtVar(\tilde{\epsilon}_{(t)}) = dt \\
dz &\sim N(0, dt)
\end{aligned}
$$

Assume $Cov(\tilde{\epsilon}(t)\tilde{\epsilon}(s)) = 0, \forall t \neq s$. At anytime $t > 0$

$$
\begin{aligned}
E[\tilde{z}(t)] &= E\left[\int_0^t dz\right] = \int_0^t E[dz] = 0 \\
Var(\tilde{z}(t)) &= Var\left(\int_0^t dz\right) = \int_0^t Var(\tilde{\epsilon}_{(u)}\sqrt{du}) \\
&= \int_0^t du = t
\end{aligned}
$$

further for $t > s > 0$,

$$
\begin{aligned}
Var(\tilde{z}(t) - \tilde{z}(s)) &= \int_s^t dz = \int_s^t Var(\tilde{\epsilon}(u)\sqrt{du}) \\
&= \int_s^t du = t - s
\end{aligned}
$$

Formal Definition: A stochastic process $\tilde{z}(t)$ is called a standard Brownian motion if the following properties are satisfied:

(i) $z(0) = 0$

(ii) For any $t_1 < t_2$, we have $\tilde{z}(t_2) - \tilde{z}(t_1) \sim N(0, t_2 - t_1)$.

(iii) For any $t_1 < t_2 < t_3 < t_4$, we have $\tilde{z}(t_4) - \tilde{z}(t_3)$ is independent of $\tilde{z}(t_2) - \tilde{z}(t_1)$

Brownian Motion with Drift

There are two types:

(i) $dx = udt + dz$, $dz = \tilde{\epsilon}(t)\sqrt{dt}$, $dx \sim N(udt, dt)$

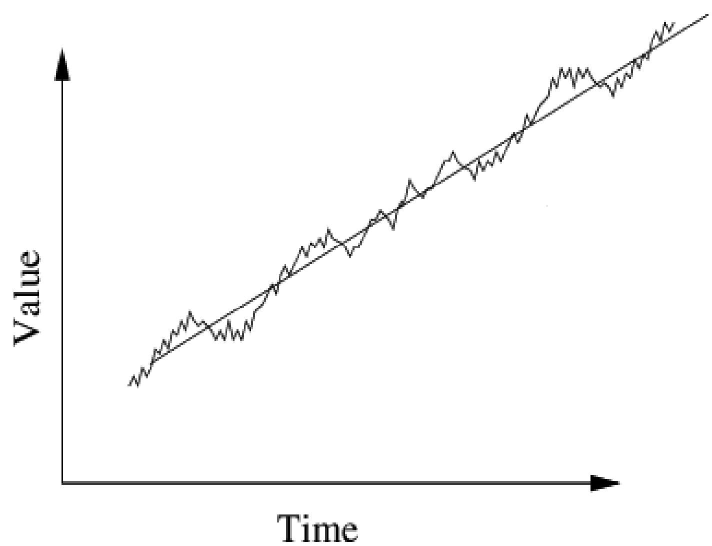

Figure 1: Brownian Motion with Drift

(ii) $dx = udt + \sigma dz$, $dx \sim N(udt, \sigma^2 dt)$

Generalized Brownian Motion (Generalized Wiener Process)

$dx = a(x,t)dt + b(x,t)dz$

Ito Process

$$dx = a(x,t)dt + b(x,t)dz$$
$$E[dx] = a(x,t)dt$$
$$Var(dx) = b(x,t)^2 dt$$

Most common stochastic process used to model stock returns (in cts time)

$$\frac{ds}{s} = udt + \sigma dz$$
$$ds = usdt + \sigma sdz$$

Called geometric Brownian Motion.

$$\frac{ds}{s} \sim N(udt, \sigma^2 dt) \Rightarrow ds \sim N\big((us)dt, (\sigma^2 s^2)dt\big)$$

Part Ⅲ:

Suppose $\sigma^2=0$,then the differential equation has solution:

$$\frac{dS}{S}=\mu dt$$

Let $S=S_0$at $t=0$

$$S=S_0e^{\mu t}$$

$$\Rightarrow \frac{dS}{dt}=\mu S_0e^{\mu t}$$

$$\Rightarrow dS=\mu S_0e^{\mu t}dt$$

$$\Rightarrow \frac{dS}{S}=\frac{\mu S_0e^{\mu t}}{S_0e^{\mu t}}dt=\mu dt$$

Ito's lemma

Suppose we have a variable X that follows an Ito process:

$$dX=a(X,t)\,dt+b(X,t)\,dZ$$

Given a function G of X and t, what stochastic process does G follow?

By Ito's lemma:

$$dG=\left[\frac{\partial G}{\partial X}a(X,t)+\frac{\partial G}{\partial t}+\frac{1}{2}\frac{\partial^2 G}{\partial X^2}b(X,t)^2\right]dt+\frac{\partial G}{\partial X}b(X,t)\,dZ$$

Variance $\left[\frac{\partial G}{\partial X}b(X,t)\right]^2dt$

$eg.\ dS=\mu Sdt+\sigma SdZ$

What is the stochastic process followed by a function G of S and t?

$$dG=\left(\frac{\partial G}{\partial S}\mu S+\frac{\partial G}{\partial t}+\frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2S^2\right)dt+\frac{\partial G}{\partial S}\sigma SdZ$$

eg. forward contracts

--at maturity, forward pays off S_T-K (where K is the delivery price)

--for $0<t<T$, by risk-neutral valuation

$$f_t = e^{-r(T-t)} E_{RN}\left[\left(S_T - K\right)\right]$$

$$\Rightarrow = e^{-r(T-t)} E_{RN}\left[S_T\right] - e^{-r(T-t)} K$$

since $S_t = e^{-r(T-t)} E_{RN}\left[S_T\right]$,

$$E_{RN}\left[S_T\right] = S_t e^{r(T-t)}$$

$so\ \ f_t = e^{-r(T-t)} S_t e^{r(T-t)} - e^{-r(T-t)} K$

$let\ F_t = K,\ such\ that\, f_z = 0$

$$0 = S_t - e^{-r(T-t)} F_t$$

$$F_t = S_t e^{r(T-t)}$$