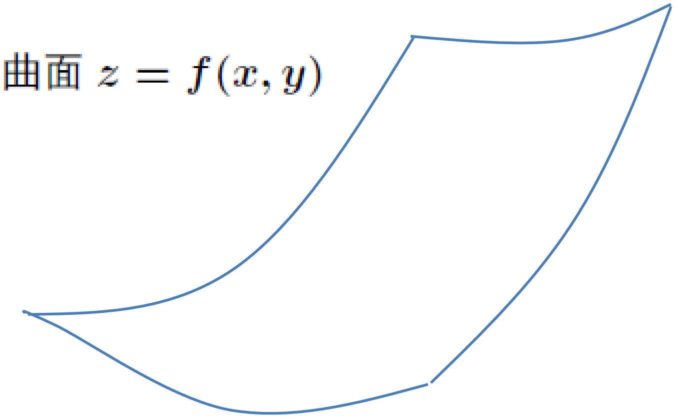
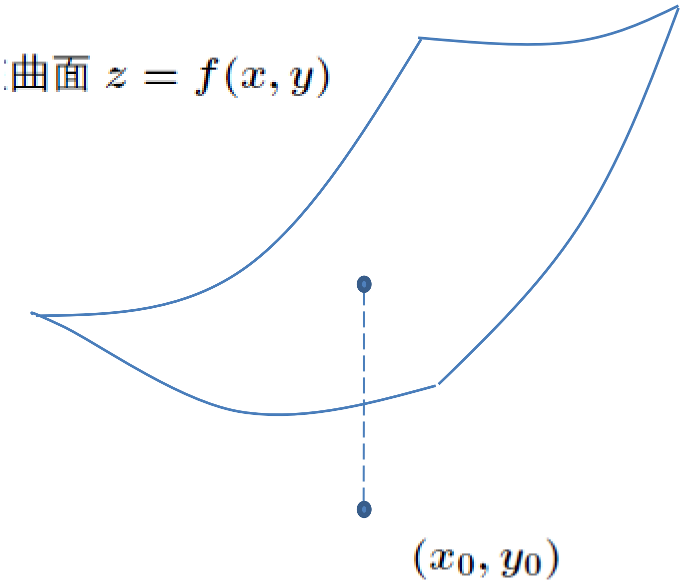
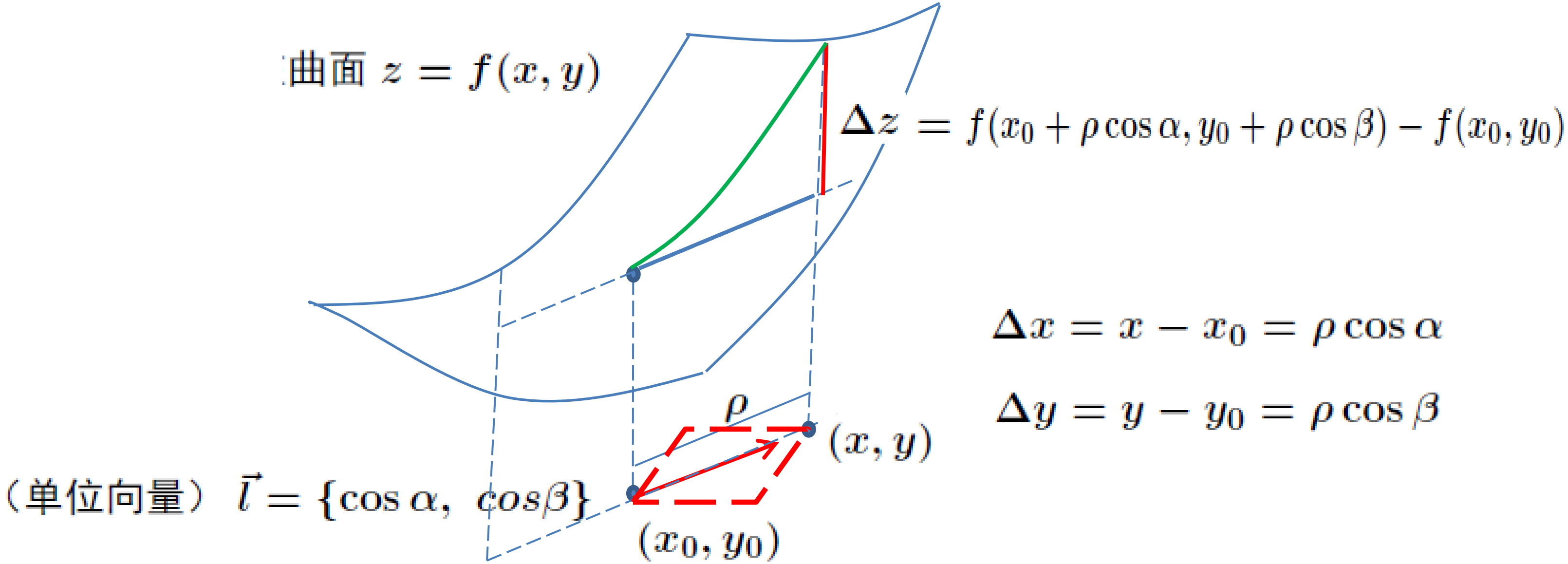
方向导数与梯度

曲面 $z = f(x,y)$

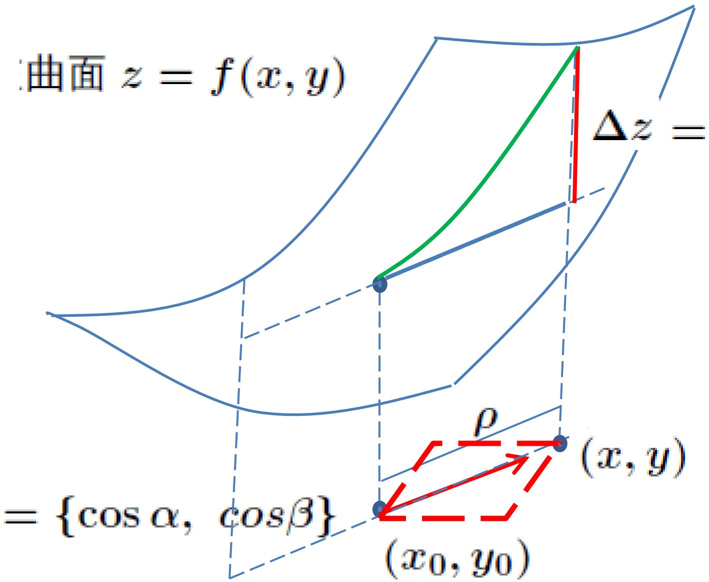



曲面 $z = f(x, y)$
(x_0, y_0)



$$\Delta x = x - x_0 = \rho\cos\alpha$$

$$\Delta y = y - y_0 = \rho\cos\beta$$



$$\Delta z = f(x_0+\rho\cos\alpha, y_0+\rho\cos\beta) - f(x_0,y_0)$$

（单位向量） $\vec{l} = \{\cos\alpha,\ \cos\beta\}$

$$\Delta x = x - x_0 = \rho\cos\alpha$$

$$\Delta y = y - y_0 = \rho\cos\beta$$

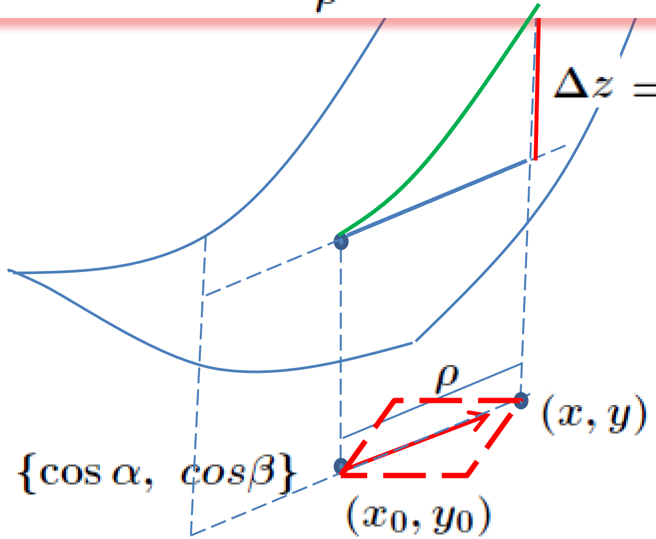
如果极限

$$\lim_{\rho\to 0+0}\frac{\Delta z}{\rho} = \lim_{\rho\to 0+0}\frac{f(x_0+\rho\cos\alpha, y_0+\rho\cos\beta) - f(x_0,y_0)}{\rho}$$

存在,则称之为函数 $z = f(x,y)$ 在点 (x_0,y_0) 处在方向 $\vec{l}$ 上方向导数

记作 $\left.\dfrac{\partial z}{\partial l}\right|_{(x_0,y_0)}$

$$\frac{\partial z}{\partial l}\Big|_{(x_0,y_0)} = \lim_{\rho\to 0+0} \frac{f(x_0+\rho\cos\alpha, y_0+\rho\cos\beta) - f(x_0,y_0)}{\rho}$$



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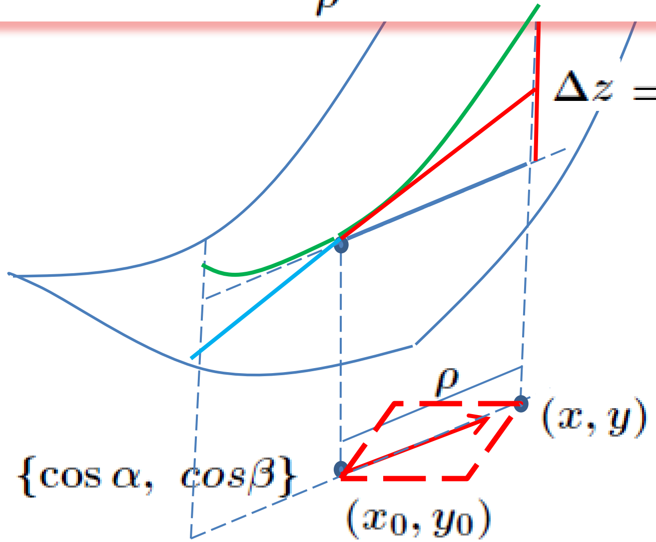


（单位向量） $\vec{l} = \{\cos\alpha,\ \cos\beta\}$

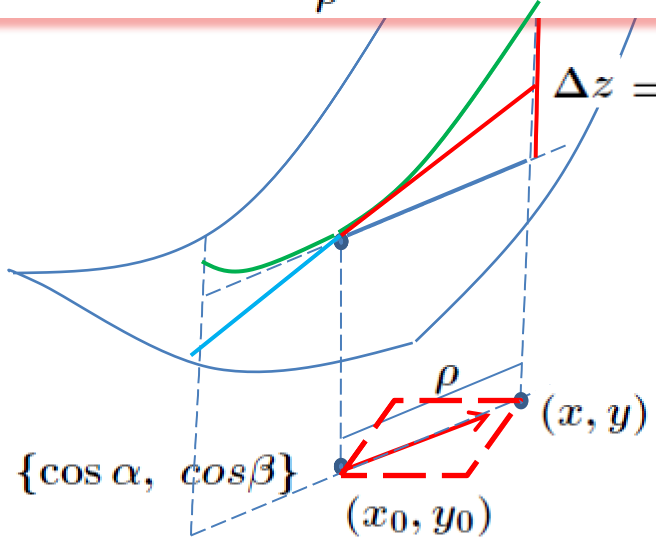
$$\Delta x = x - x_0 = \rho\cos\alpha$$

$$\Delta y = y - y_0 = \rho\cos\beta$$

注　1. $\rho > 0$, 注意方向导数与左右导数（单侧导数）的区别!

（通常一点的左右侧导数相等）

$$\frac{\partial z}{\partial l}\Big|_{(x_0,y_0)} = \lim_{\rho\to 0+0} \frac{f(x_0+\rho\cos\alpha, y_0+\rho\cos\beta) - f(x_0,y_0)}{\rho}$$



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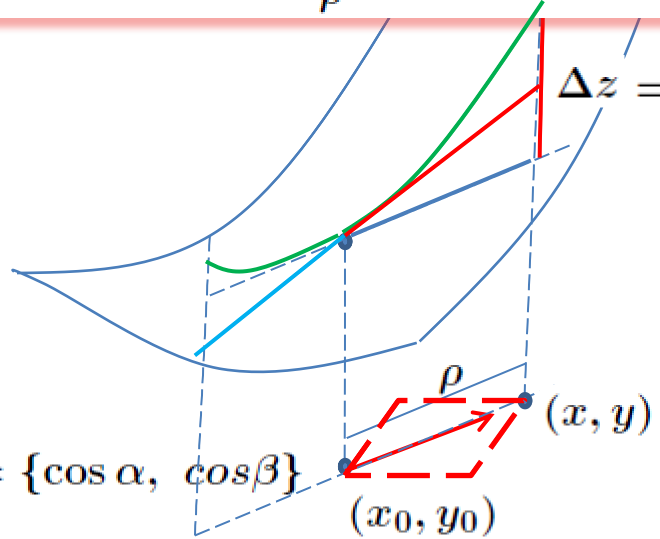
注 2.如果$\vec{l} = \{1, 0\}$, 则 $\lim\limits_{\rho\to 0+0} \frac{f(x_0+\rho,y_0)-f(x_0,y_0)}{\rho} = \frac{\partial f}{\partial x}(x_0, y_0)$

$$\boxed{\frac{\partial z}{\partial l}\Big|_{(x_0,y_0)} = \lim_{\rho\to 0+0}\frac{f(x_0+\rho\cos\alpha, y_0+\rho\cos\beta)-f(x_0,y_0)}{\rho}}$$



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$\vec{l} = \{-1, 0\}$ 时，$\frac{\partial z}{\partial l} = \lim\limits_{\rho\to 0+0}\frac{f(x_0-\rho,y_0)-f(x_0,y_0)}{\rho}$

$$= \lim_{\rho\to 0+0}\frac{f(x_0-\rho,y_0)-f(x_0,y_0)}{-\rho} = -\frac{\partial f}{\partial x}(x_0, y_0)$$

$$\boxed{\frac{\partial z}{\partial l}\Big|_{(x_0,y_0)} = \lim_{\rho\to 0+0}\frac{f(x_0+\rho\cos\alpha, y_0+\rho\cos\beta)-f(x_0,y_0)}{\rho}}$$

$\frac{\partial f}{\partial x}(x_0, y_0)$ 是沿 x 轴正向的方向导数,

$-\frac{\partial f}{\partial x}(x_0, y_0)$ 是沿 x 轴负向的方向导数($\vec{l} = \{1, 0\}$),

$\frac{\partial f}{\partial y}(x_0, y_0)$ 是沿 y 轴正向的方向导数,

$-\frac{\partial f}{\partial y}(x_0, y_0)$ 是沿 y 轴负向的方向导数($\vec{l} = \{-1, 0\}$).

注 2.如果$\vec{l} = \{1, 0\}$, 则 $\lim\limits_{\rho\to 0+0}\frac{f(x_0+\rho,y_0)-f(x_0,y_0)}{\rho} = \frac{\partial f}{\partial x}(x_0, y_0)$

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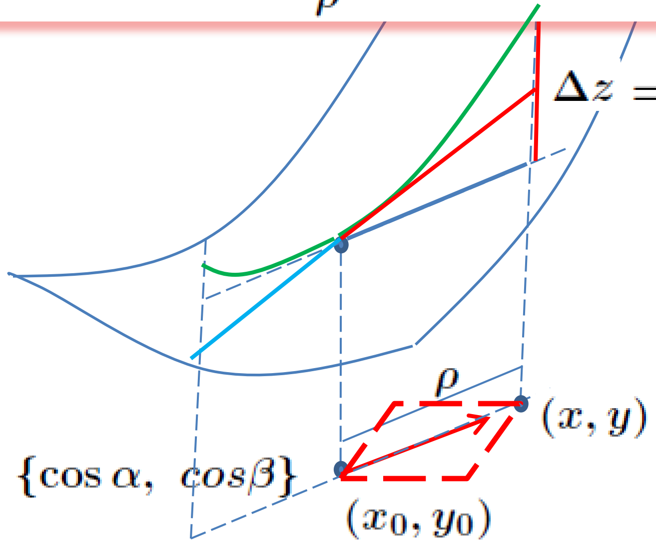

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注 3.方向导数中所说的方向是在底平面上的,切线是在空中的.

$\dfrac{\partial z}{\partial l} > 0$ 表示$z = f(x,y)$ 沿 $\vec{l}$ 方向函数值增加.

$\dfrac{\partial z}{\partial l} < 0$表示 $z = f(x,y)$ 沿 $\vec{l}$ 方向函数值减少.

进一步考察，

$$\lim_{\rho \to 0+0} \frac{f(x_0 + \rho\cos\alpha, y_0 + \rho\cos\beta) - f(x_0, y_0)}{\rho}$$

$$= \lim_{\rho \to 0+0} [\frac{f(x_0, y_0 + \rho\cos\beta) - f(x_0, y_0 + \rho\cos\beta)}{\rho\cos\alpha}\cos\alpha + \frac{f(x_0, y_0 + \rho\cos\beta) - f(x_0, y_0)}{\rho\cos\beta}\cos\beta]$$

进一步考察,

$$\lim_{\rho \to 0+0} \frac{f(x_0+\rho\cos\alpha, y_0+\rho\cos\beta)-f(x_0,y_0)}{\rho}$$

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$$=\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x_0+\Delta x, y_0+\Delta y)-f(x_0,y_0+\Delta y)}{\Delta x}\cos\alpha+\lim_{\Delta y \to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}\cos\beta$$

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连续

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连续

定理 如果 $\dfrac{\partial f}{\partial x},\dfrac{\partial f}{\partial y}$ 连续，则 $z=f(x,y)$ 于点 (x,y) 在任意方向 $\vec{l}=\{\cos\alpha,\cos\beta\}$ 上的导数存在且有

$$\frac{\partial f}{\partial l}=\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\cos\beta$$

$$\boxed{\text{“}\ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\ \text{连续” 可换成 “}\ z = f(x, y)\ \text{可微”}}$$

$$\Downarrow$$

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$$\frac{\partial f}{\partial l} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\cos\beta$$

$\boxed{\text{可微}= \text{每个方向上都可导!}}$

“$\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ 连续” 可换成 “$z = f(x, y)$ 可微”

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例'

求 $z = x^2 - y^2$ 在点 $(1,1)$ 处沿 $\vec{l} = \{\cos\frac{\pi}{3}, \cos\frac{\pi}{6}\} = \{\frac{1}{2}, \frac{\sqrt{3}}{2}\}$ 的方向导数.

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解: $$\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = -2y,$$

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$$\frac{\partial z}{\partial l}|_{(1,1)} = \frac{\partial z}{\partial x}|_{(1,1)} \cdot \frac{1}{2} + \frac{\partial z}{\partial y}|_{(1,1)} \cdot \frac{\sqrt{3}}{2} = 1 - \sqrt{3}.$$

$\vec{l} = \{\cos\alpha,\ cos\beta\}$ $$\frac{\partial f}{\partial l} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\cos\beta$$

$$\frac{\partial f}{\partial l} = \{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\} \cdot \{\cos\alpha, \cos\beta\}$$

$\vec{l} = \{\cos\alpha,\ cos\beta\}$ $\qquad$ $\dfrac{\partial f}{\partial l} = \dfrac{\partial f}{\partial x}\cos\alpha + \dfrac{\partial f}{\partial y}\cos\beta$

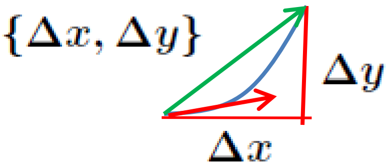
$$\frac{\partial f}{\partial l} = \{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\} \cdot \{\cos\alpha, \cos\beta\}$$

是什么矢量呢？

考察等高线族，$f(x,y) = c$,微分后得 $\mathrm{d}f(x,y) = 0$,

即 $\dfrac{\partial f}{\partial x}\mathrm{d}x + \dfrac{\partial f}{\partial y}\mathrm{d}y = 0 \Rightarrow \{\mathrm{d}x,\ \mathrm{d}y\} \perp \{\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}\}$,

$\{\mathrm{d}x,\ \mathrm{d}y\}$ 是等高线的切线方向矢量，



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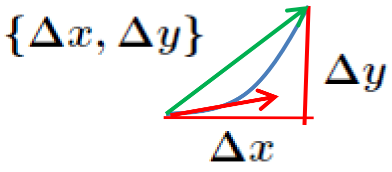

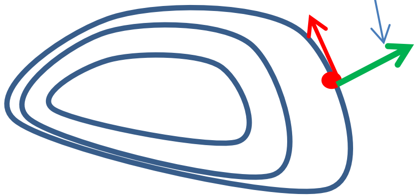

$\{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\}|_{(x_0,y_0)}$ 是 $z = f(x,y)$ 过 (x_0, y_0) 的等高线于此点的法线方向矢量!

$\vec{l} = \{\cos\alpha,\ cos\beta\}$ $$\frac{\partial f}{\partial l} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\cos\beta$$

向量 $\{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\}$ 决定一切方向上的方向导数

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向量 $\{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\}$ 决定一切方向上的方向导数

称之为函数 $z = f(x, y)$ 在 (x, y) 点的梯度,记作$gradz$ 或$gradf$,

即 $gradz = \{\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\}$, $gradf = \{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\}$,

也写作 $\nabla z, \nabla f$.

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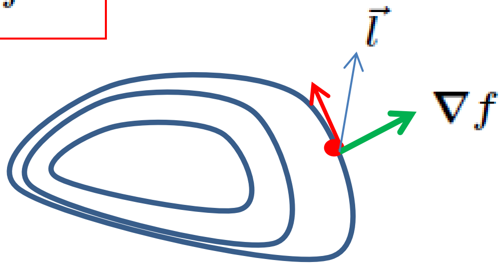
$$\boxed{\frac{\partial f}{\partial l} = \nabla f \cdot \vec{l}}$$



底平面各点的梯度方向就是过该点的等高线的（正）法线方向.

$$\boxed{\frac{\partial f}{\partial l} = \nabla f \cdot \vec{l}}$$

函数在梯度方向上的方向导数为

$$\frac{\partial f}{\partial \nabla} = \{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\} \cdot \frac{\{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\}}{\sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}}$$

$$\boxed{\frac{\partial f}{\partial l}=\nabla f\cdot\vec{l}}$$

函数在梯度方向上的方向导数为

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$$=\sqrt{(\frac{\partial f}{\partial x})^2+(\frac{\partial f}{\partial y})^2}=|\nabla f|$$

∇z 的方向是方向导数最大的方向，（自身到自身的投影的模（长度）最大！）

所以梯度方向就是函数值增加速度最大的（水平）方向