条件极值

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求 $z = 4 - x^2 - y^2$在条件 $x + y = 1$下的极值.

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$\therefore z(\frac{1}{2})$为极大值.即所求极值为极大值,为 $z(\frac{1}{2},\frac{1}{2})=4-\frac{1}{4}-\frac{1}{4}=\frac{7}{2}$.

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则考虑曲线 $C:\begin{cases} z=f(x,y) \\ \phi(x,y)=0 \end{cases}$ 的极值问题.

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若令第一式的比例为 $-\lambda$,则有

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然后讨论此驻点是否为极值点

（Lagrange乘子法）

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$$6x^2 = S, x = \sqrt{\frac{S}{6}},$$

$\therefore$取作立方体（边长为 $\sqrt{\frac{S}{6}}$）时体积最大为 $V = (\frac{S}{6})^{\frac{3}{2}}$.

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要说明它对任意的 $\Delta x,\Delta y,\Delta z$（充分小）保持定号，

需要讨论三阶对称矩阵的正定性.

我们只根据具体问题“推断”极值问题一定有解，

从而,我们找到唯一驻点就是极值点,也就是最值点.

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在原点 $(0,0,0)$处放置一正单位电荷,

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解: 位置 (x,y,z)处负单位点电荷所受作用力大小为 $F = \dfrac{k}{x^2 + y^2 + z^2}$

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现让一负单位电荷于曲线 $\begin{cases} z = x^2 + y^2 \\ x + y + z = 1 \end{cases}$ 上,

问此负电荷位于何处受力最大,何处受力最小?

解: 位置 (x,y,z)处负单位点电荷所受作用力大小为 $F = \dfrac{k}{x^2+y^2+z^2}$

考虑 $F = x^2 + y^2 + z^2$在附加条件 $\begin{cases} z = x^2 + y^2 \\ x + y + z = 1 \end{cases}$ 下的最大最小值

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令 $G(x, y, z, \lambda, \mu) = x^2 + y^2 + z^2 + \lambda(z - x^2 - y^2) + \mu(x + y + z - 1)$,

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若 $\lambda = 1$,则 $\mu = 0$,此时 $z = -\dfrac{1}{2}$不在所考虑的范围内.

即 $x_1 = \dfrac{-1 + \sqrt{3}}{2}$, $x_2 = \dfrac{-1 - \sqrt{3}}{2}$

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$\therefore (x_1,y_1,z_1)=(\frac{-1-\sqrt{3}}{2},\frac{-1-\sqrt{3}}{2},2+\sqrt{3})$,

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$${x_1}^2+{y_1}^2+{z_1}^2=\frac{z_1}{2}+\frac{z_1}{2}+{z_1}^2=2+\sqrt{3}+4+3+4\sqrt{3}=9+5\sqrt{3},$$

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$\therefore$ 所以位于 (x_1,y_1,z_1)时作用力最小,位于 (x_2,y_2,z_2)时作用力最大

例

点到面的距离公式

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点到面的距离公式 $\Sigma: Ax+By+Cz+D=0$,面外面一点 $P_0(x_0,y_0,z_0)$,

则点 P_0到平面 Σ上任一点 $P(x,y,z)$的距离（平方）为:

$$\rho^2(x,y,z)=(x-x_0)^2+(y-y_0)^2+(z-z_0)^2$$

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$$\boxed{\therefore \rho=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}}$$