Midterm Exam for *Introduction to Probability and Statistics*

Note:

1. The exam time is 8:00-10:00, Nov. 8, 2011.
2. You are allowed to use calculators that only have calculation functions.
3. Everyone is allowed to have one single-sided A4-sized cheat-sheet.
4. The questions are not arranged in an easy-to-difficult order or a difficult-to-easy order.

1. (22 points) (Note: For problem 1, you do not need to evaluate the arithmetic expressions you get.)

   Suppose that there are three groups of singers. Group A contains 20 singers, Group B contains 40 singers and Group C contains 60 singers.

   (a) If we select 6 singers randomly from all 120 singers, what is the probability that at least one group is missing in the selection? (8 points)

   (b) Suppose that among the 20 singers in Group A, 8 are male and 12 are female, that among the 40 singers in Group B, 20 are male and 20 are female, and that among the 60 singers in Group A, 15 are male and 45 are female. If we first select one group randomly and then select 6 singers randomly from the chosen group, what is the probability that among the 6 selected singers, 3 are male and 3 are female? (8 points)

   (c) In part (b), if we do not know which group the singers are selected from, but we observe that among the 6 selected singers, 3 are male and 3 are female, what is the probability that the 6 singers are selected from group A? (6 points)

2. (28 points) Suppose that X is the proportion of people who prefer product A, Y is the proportion of people who prefer product B, and the joint probability density function of X and Y is given by

$$f(x,y)=\begin{cases} cx(1+y^2) & 0<x\le y<1 \\ 0 & \text{otherwise} \end{cases}$$

(1) Find the value of the constant c. (4 points)

(2) Find the marginal distribution of X. (4 points)

(3) Find the conditional distribution of X given Y. (6 points)

(4) Find $\Pr\left(X>0.25 \middle| Y=0.5\right)$. (4 points)

(5) Find $Cov\left(X,Y\right)$. (4 points)

(6) Find the distribution of W=Y-X. (6 points)

3. (12 points) Suppose that the number of customers coming to a bank in a day follows a Poisson distribution with mean $\lambda=100$, and suppose that the service time for each customer follows an exponential distribution with parameter $\beta=0.5$. What are the expectation and variance of the total service time for customers in a day?

4. (28 points) (a) Suppose that each file can be processed with a mean of 6 minutes and a standard deviation of 1.5 minutes. Suppose that each staff member works 8 hours a day. Please use the Central Limit Theorem to determine the maximum number of files that can be assigned to each staff member each day so that the probability of working over time is no larger than 0.05. (10 points)

   (b) Suppose that the time it takes staff member A to process a file follows a normal distribution with a mean of 6.5 minutes and a standard deviation of 1 minute. Suppose that the time it takes staff member B to process a file follows a normal distribution with a mean of 5.5 minutes and a standard deviation of 2.5 minutes.

   (b.1) If these two staff members process file independently, and each is assigned 10 files, what is the probability that staff member A will finish earlier than staff member B? (10 points)

   (b.2) When these two staff members work while sitting next to each other, their file processing times follow a bivariate normal distribution and the correlation between their file processing times is 0.2. If each is assigned one file in this case, what is the probability that staff member A will finish earlier than staff member B? (8 points)

5. (10 points) Suppose that $X \sim \text{Uniform}[-1,1]$ and $Y_n = X^n$. Prove that $Y_n \xrightarrow{p} 0$.

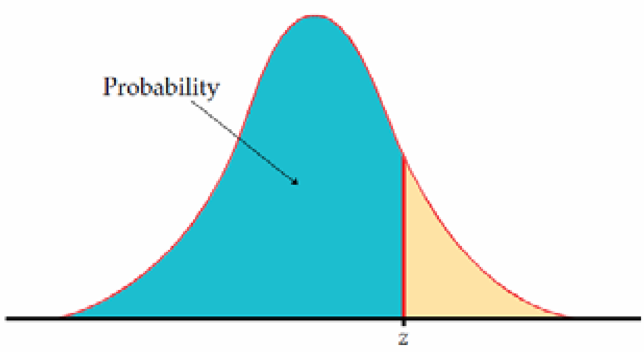

Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5	0.504	0.508	0.512	0.516	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.591	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.648	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.67	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.695	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.719	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.758	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.791	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.834	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.877	0.879	0.881	0.883
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.898	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.937	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.975	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.983	0.9834	0.9838	0.9842	0.9846	0.985	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.989
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.992	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.994	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.996	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.997	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.998	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.999	0.999
3.1	0.999	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998