多元函数极值

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$$F^{(n)}(t) = (\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y})^n f(x_0 + t\Delta x, y_0 + t\Delta y)$$

其中, $(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y})^n = \sum_{k=0}^{n} C_n^k (\Delta x)^k (\Delta y)^{n-k} \frac{\partial^k}{\partial x^k}\frac{\partial^{n-k}}{\partial y^{n-k}}$

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结论

若 $z=f(x,y)$于 (x_0,y_0)点取到极值,且 $\dfrac{\partial f}{\partial x}(x_0,y_0)$, $\dfrac{\partial f}{\partial y}(x_0,y_0)$存在,则

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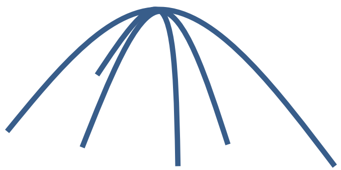

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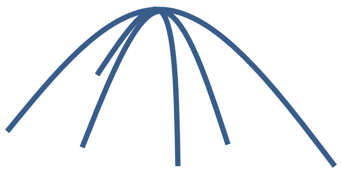

定义

使 $\frac{\partial f}{\partial x}(x_0,y_0)$, $\frac{\partial f}{\partial y}(x_0,y_0)$为零的点,称为 $z=f(x,y)$的驻点

极值可疑点

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若 $AC - B^2 > 0$,则函数值 z（除在 $(0, 0)$点外）

要么全为正（$(A > 0)$）,为开口向上的抛物面,全在 xoy平面上方;

要么全为负（$(A < 0)$）,为开口向下的抛物面,全在 xoy 平面下方;

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若 $AC - B^2 < 0$,则函数值 z对不同的 x, y有正有负,比如 $y = 0$时, $z = Ax^2$与 A同号, $x = -\frac{B}{A}y$时, $z = \frac{AC-B^2}{A}y^2$与 A异号.所以, $AC - B^2 < 0$时,函数 $z = Ax^2 + 2Bxy + Cy^2$的值(在原点附近)不具有确定的符号.

先考察二元函数 $z = Ax^2 + 2Bxy + Cy^2$的性态,其中 A, B, C为常数.

事实上,如果学过线性代数, 就知道:

$A > 0, AC - B^2 > 0$时, 二次型$Ax^2 + 2Bxy + Cy^2$正定;

$A < 0, AC - B^2 > 0$时, 二次型$Ax^2 + 2Bxy + Cy^2$负定;

$AC - B^2 < 0$时, 二次型$Ax^2 + 2Bxy + Cy^2$不定.

设 $z=f(x,y)$,在 (x_0,y_0)点,各阶偏导数连续,则有二元函数Taylor公式：

$$f(x,y)-f(x_0,y_0)=(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y})f(x_0,y_0)+\frac{1}{2!}(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y})^2 f(x_0,y_0)+\cdots$$

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考察 $\Phi(\Delta x,\Delta y)=(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y})^2 f(x_0,y_0)$在 $\forall\Delta x,\Delta y$变化时定号否.

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即 $\Phi=(\Delta x)^2\frac{\partial^2 f(x_0.y_0)}{\partial x^2}+2\Delta x\Delta y\frac{\partial^2 f(x_0.y_0)}{\partial x\partial y}+(\Delta y)^2\frac{\partial^2 f(x_0.y_0)}{\partial y^2}$

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设 $z = f(x,y)z$在某区域内有二阶连续偏导数,求出函数 $z = f(x,y)$在此区域内的驻点(x_0, y_0),在驻点处求出 $A = \dfrac{\partial^2 f}{\partial x^2}, B = \dfrac{\partial^2 f}{\partial x \partial y}, A = \dfrac{\partial^2 f}{\partial y^2}$的值,判断 $AC - B^2$是否大于0.

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设 $z = f(x,y)z$在某区域内有二阶连续偏导数,求出函数 $z = f(x,y)$在此区域内的驻点(x_0,y_0),在驻点处求出 $A = \dfrac{\partial^2 f}{\partial x^2}, B = \dfrac{\partial^2 f}{\partial x \partial y}, A = \dfrac{\partial^2 f}{\partial y^2}$的值,判断 $AC - B^2$是否大于0.

若 $AC - B^2 < 0$,则该驻点非极值点; 若 $AC - B^2 > 0$,则该驻点为极值点,此时

令 $A = \dfrac{\partial^2 f(x_0.y_0)}{\partial x^2}, B = \dfrac{\partial^2 f(x_0.y_0)}{\partial x \partial y}, C = \dfrac{\partial^2 f(x_0.y_0)}{\partial y^2}$,

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$$\begin{cases} A > 0 \Rightarrow f(x_0, y_0)\text{是极小值} \\ A < 0 \Rightarrow f(x_0, y_0)\text{是极大值} \end{cases}$$

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如 $z = x^4 + y^4 - x^2 - 2xy - y^2$有三个驻点 $(-1,-1), (0,0), (1,1)$

在 $(0,0)$处 $A = -2, B = -2, C = -2, AC - B^2 = 0$,此时无法去决定 $(0,0)$是否是极值点.

但在 $y = x$上, $z = x^4 - 4x^2 = x^2(x^2 - 4) < 0 (0 < |x| \le 1)$,在 $y = -x$上, $z = 2x^4 > 0 (0 < |x| \ll 1)$,所以 $(0,0)$ 不是极值点.

令 $A = \dfrac{\partial^2 f(x_0 . y_0)}{\partial x^2}, B = \dfrac{\partial^2 f(x_0 . y_0)}{\partial x \partial y}, C = \dfrac{\partial^2 f(x_0 . y_0)}{\partial y^2}$,

即考察 $A(\Delta x)^2 + 2B\Delta x \Delta y + C(\Delta y)^2$是否关于 $\forall \Delta x, \Delta y$定号.

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$\therefore$ 于 $(0, 0)$点, $A = 0, B = 3, C = 0, AC - B^2 = -9 < 0, \therefore (0, 0)$非极值点.

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$\because A=-6<0 \therefore (1,1)$为极大值点,极大值为 $f(1,1)=3-1-1=1.$