Lecture 15

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1 Content

1. CAPM: Capital Asset Pricing Model
2. Expected Return - Beta Representation
3. Security Market Line

2 Capital Asset Pricing Model

2.1 Properties

1. Market portfolio is the tangency portfolio
2. Capital Market Line connects risk-free rate to market portfolio
3. All investor portfolios lie on Capital Market Line, which is defined as
$$\mu_p - r_f = \sigma_p \frac{\tilde{r_M} - r_f}{\sigma_M} \tag{1}$$
4. For any $\tilde{r}_i$,
$$E[\tilde{r_i}] - r_f = \beta_i(E[\tilde{r_M}] - r_f) \tag{2}$$

2.2 Assumptions

1. All investors have a one-period horizon
2. All assets are tradeable in perfectly divisible amounts
3. Unlimited risk-free lending and borrowing
4. No taxes or transaction costs
5. No short sale constraints
6. All investors are mean-variance optimizers
7. Homogenous expectations: all investors have same estimation for expected returns and variance
8. Equilibrium theory model: for all assets, demand = supply

2.3 Implications

1. All investors are working with the same mean variance diagram
2. All investors hold mean variance efficient portfolios
3. **Mutual fund theorem**: All investors hold the same tangency portfolios
4. Demand = Supply implies that the tangency portfolio is the market portfolio
5. Market Portfolio is mean-variance efficient

2.4 Market Portfolio

1. First, let v_i represent the market value of asset i. As such, we know that

$$v_i = n_i P_i \tag{3}$$

where n_i represents the number of outstanding shares and p_i represents the price/share of asset i

2. If we sum the market value of all assets, we find the total market value V

$$V = \sum_{i=1}^{N} v_i \tag{4}$$

3. The market portfolio is the portfolio of all risky assets, and can be represented as a set of portfolio weights,

$$W_m = (W_{1,M}, ..., W_{n,M}) \tag{5}$$

where each $W_{i,M}$ can be defined as $\frac{v_i}{V}$

4. Let T represent the tangency portfolio. Since demand must equal supply from our CAPM assumptions,

$$\begin{aligned} V &= \text{Market value of all assets (Supply)} \\ &= \text{Total funds invested in T (Demand} \end{aligned} \tag{6}$$

5. We can now prove that the tangency portfolio is equal to the market portfolio

$$\begin{aligned} v_i &= \text{Market value of asset } i \\ &= \text{Total amount invested in asset } i \\ &= W_{i,T} * (\text{Total funds invested in T}) \\ &= W_{i,T} * V \end{aligned} \tag{7}$$

6. Manipulate $W_{i,T}$ and conclude tangency portfolio is equal to market portfolio
$$\begin{aligned} W_{i,T} &= \frac{W_{i,T} * V}{V} \\ &= \frac{v_i}{V} \\ &= W_{i,M} \end{aligned} \tag{8}$$

3 Expected Return - Beta Representation

$$\begin{aligned} E[\tilde{r_i}] - r_f &= \beta_i(E[\tilde{r_M}] - r_f) \\ \beta_i &= \frac{cov(\tilde{r_i}, \tilde{r_m})}{\sigma_m^2} \end{aligned} \tag{9}$$

1. β_i is called the "loading" and is typically estimated from a time - series regression
$$r_{i,t} - r_{f,t} = \alpha_1 + \beta_i(r_{m,t} - r_{f,t}) + \epsilon_{i,t} \tag{10}$$
$$E[\epsilon_{i,t}] = 0 \tag{11}$$

2. E[excess return] = (quantity of risk) (price of risk)

3. Let var($\tilde{r_p}$) be the starting variance
$$var(\tilde{r_p} + \theta_i(\tilde{r_i} - r_f)) = \sigma_m^2 + \sigma_i^2\sigma_i^2 + 2\theta_i cov(\tilde{r_i}, \tilde{r_m})) \tag{12}$$

4. Suppose there was a change in variance $\approx 2\theta_i$, cov($\tilde{r_i}$, $\tilde{r_p}$)

5. Replace p with n above, so there is a change in expected return θ_i (E[$\tilde{r_i}$] - r_f)

6. Intuitive proof: all assets should have same risk-adjusted return in equilibrium
$$\frac{E[\tilde{r_i}] - r_f}{cov(\tilde{r_i}, \tilde{r_m})} = \frac{E[\tilde{r_j}] - r_f}{cov(\tilde{r_j}, \tilde{r_m})} \tag{13}$$

7. What if:
$$\frac{E[\tilde{r_i}] - r_f}{cov(\tilde{r_i}, \tilde{r_m})} > \frac{E[\tilde{r_j}] - r_f}{cov(\tilde{r_j}, \tilde{r_m})} \tag{14}$$

8. Then:
$$E[r_i] = \frac{E[p_i, t+1}{p_1, t} - 1 \tag{15}$$

9. In particular:
$$\begin{aligned} \frac{E[\tilde{r_i}] - r_f}{cov(\tilde{r_i}, \tilde{r_m})} &= \frac{E[\tilde{r_j}] - r_f}{\sigma_m^2}] \\ E[\tilde{r_i}] - r_f &= (\frac{cov(\tilde{r_i}, \tilde{r_m})}{\sigma_m^2})(E[\tilde{r_i}] - r_f) \\ \beta_i &= \frac{cov(\tilde{r_i}, \tilde{r_m})}{\sigma_m^2} \end{aligned} \tag{16}$$

3.1 Formal Proof:

1. Consider 2 simulatenous shifts in asset i and asset j
2. change in risk: $2\theta_i$, cov($\tilde{r_i}$, $\tilde{r_m}$) + $2\theta_i$, cov($\tilde{r_j}$, $\tilde{r_m}$)
3. change in expected return: σ_i (E[$\tilde{r_i}$] - r_f) + σ_j (E[$\tilde{r_j}$] - r_f)

$$\sigma_j = -\sigma_i(\frac{cov(\tilde{r_i}, \tilde{r_m})}{cov(\tilde{r_j}, \tilde{r_m})} \tag{17}$$

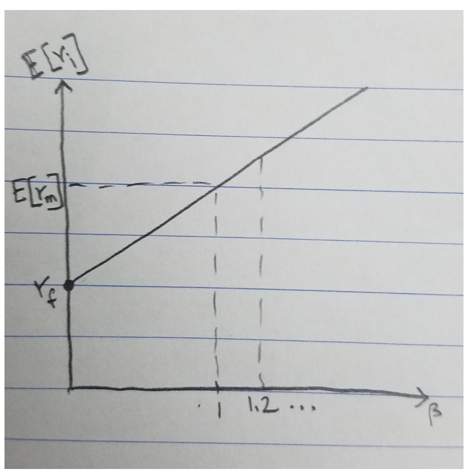
4. change in expected return:

$$\sigma_i(E[\tilde{r_i} - r_f]) - \sigma_i(\frac{cov(\tilde{r_i}, \tilde{r_m})}{cov(\tilde{r_j}, \tilde{r_m})}(E[\tilde{r_i}] - r_f) = \sigma_i cov(\tilde{r_i}, \tilde{r_m}) * [\frac{E[\tilde{r_i}] - r_f}{cov(\tilde{r_i}, \tilde{r_m})} - \frac{E[\tilde{r_j}] - r_f}{cov(\tilde{r_j}, \tilde{r_m})}] \tag{18}$$

4 Security Market Line

4.1 One Factor Model

$$E[\tilde{r_i}] - r_f = \beta_i(E[\tilde{r_M}] - r_f) \tag{19}$$



$$\sigma_i{}^2 = \beta_i{}^2 \sigma_M{}^2 + \sigma_\epsilon{}^2 \tag{20}$$

$$\sigma_{ij} = \beta_i \beta_j \sigma_M \tag{21}$$