金融经济学

第四讲

张宇

4 Measuring Risk and Risk Aversion

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A Measuring Risk Aversion

B Interpreting the Measures of Risk Aversion

C Risk Premium and Certainty Equivalent

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Risk Premium and Certainty Equivalent

愿意付多少钱来avoid

Our thought experiments so far have asked about how probabilities need to be boosted in order to induce a risk-averse investor to accept an absolute or relative bet.

Let's take step away from gambling and towards investing by asking: suppose that an investor with income Y has the opportunity to buy an asset with a payoff $\tilde{Z}$ that is random and has expected value $E(\tilde{Z})$.

Risk Premium and Certainty Equivalent

If this investor is risk-averse and has vN-M expected utility, he or she will always prefer an alternative asset that pays off $E(\tilde{Z})$ for sure. Mathematically,

熟悉一下这样的写法 z是随机变量
E（Z）我直接拿到

$$u[Y + E(\tilde{Z})] \geq E[u(Y + \tilde{Z})],$$

"the utility of the expectation is greater than the expectation of utility."

Risk Premium and Certainty Equivalent

This follows from a result proven by Johan Jensen (Denmark, 1859-1925).

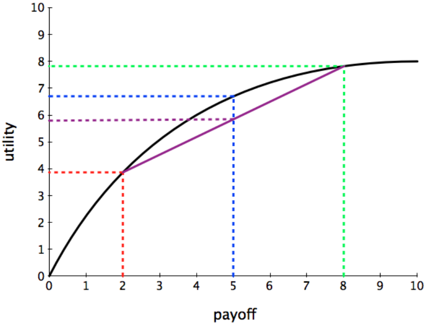
Theorem (Jensen's Inequality) Let g be a concave function and $\tilde{x}$ be a random variable. Then

$$g[E(\tilde{x})] \geq E[g(\tilde{x})].$$

因为凹

Furthermore, if g is strictly concave and the probability that $\tilde{x} \neq E(\tilde{x})$ is greater than zero, the inequality is strict.

Risk Premium and Certainty Equivalent



This graph illustrates a special case of Jensen's inequality. The result holds much more generally.

Risk Premium and Certainty Equivalent

An implication of Jensen's inequality is that the maximum riskless payoff that a risk-averse investor is willing to exchange for the asset with random payoff $\tilde{Z}$, called the certainty equivalent for that asset, will always be less than $E(\tilde{Z})$.

Since

$$u[Y + E(\tilde{Z})] \geq E[u(Y + \tilde{Z})],$$

the certainty equivalent $CE(\tilde{Z})$ defined by

$$u[Y + CE(\tilde{Z})] = E[u(Y + \tilde{Z})]$$

also satisfies $CE(\tilde{Z}) \leq E(\tilde{Z})$. 少给一点 肯定要小于期望回报 很乐意接受期望

Risk Premium and Certainty Equivalent

Since

$$u[Y + E(\tilde{Z})] \geq E[u(Y + \tilde{Z})],$$

the certainty equivalent $CE(\tilde{Z})$ defined by

$$u[Y + CE(\tilde{Z})] = E[u(Y + \tilde{Z})]$$

also satisfies $CE(\tilde{Z}) \leq E(\tilde{Z})$.

The difference between the higher expected value $E(\tilde{Z})$ and the smaller certainty equivalent $CE(\tilde{Z})$ can then be used to define the positive risk premium $\Psi(\tilde{Z})$ for the asset:

$$\Psi(\tilde{Z}) = E(\tilde{Z}) - CE(\tilde{Z}) \geq 0.$$ 会定义好几种

Risk Premium and Certainty Equivalent

The certainty equivalent and risk premium are "two sides of the same coin"

$$\Psi(\tilde{Z}) = E(\tilde{Z}) - CE(\tilde{Z})$$

$CE(\tilde{Z}) =$ lesser amount the investor is willing to accept to remain invested in the risk-free asset

$\Psi(\tilde{Z}) =$ extra amount the investor needs to take on additional risk

2w5 我愿意接受2w rish premium=5k

Risk Premium and Certainty Equivalent

Combining the definitions of the certainty equivalent $CE(\tilde{Z})$,

$$\underline{E[u(Y+\tilde{Z})] = u[Y+CE(\tilde{Z})]},$$

and the risk premium $\Psi(\tilde{Z})$,

$$CE(\tilde{Z}) = E(\tilde{Z}) - \Psi(\tilde{Z}),$$

yields

$$\underline{E[u(Y+\tilde{Z})] = u[Y+E(\tilde{Z}) - \Psi(\tilde{Z})]},$$

which we can use to link the risk premium $\Psi(\tilde{Z})$ to our measures of risk aversion.

Risk Premium and Certainty Equivalent

The risk premium can be approximated using this formula:

$$\Psi(\tilde{Z}) \approx \frac{1}{2}\sigma^2(\tilde{Z}) R_A(Y + E(\tilde{Z})),$$

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indicating that the risk premium depends directly on the coefficient of absolute risk aversion and the absolute "size of the bet" $\sigma^2(\tilde{Z})$.

(Bonus: Derive this formula using Taylor-approximation.)

Risk Premium and Certainty Equivalent

As an example, consider an investor with income $Y = 50000$ and utility function of the constant relative risk aversion form

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}$$

with $\gamma = 5$, who is considering buying an asset with random payoff $\tilde{Z}$ that equals 2000 with probability $1/2$ and 0 with probability $1/2$. For this asset

$$E(\tilde{Z}) = (1/2)2000 + (1/2)0 = 1000$$ CE不会超过1000

$$\sigma^2(\tilde{Z}) = (1/2)(2000 - 1000)^2 + (1/2)(0 - 1000)^2 = 1000^2.$$

Risk Premium and Certainty Equivalent

Our approximation formula

$$\Psi(\tilde{Z}) \approx \frac{1}{2}\sigma^2(\tilde{Z}) R_A(Y + E(\tilde{Z}))$$

indicates that

处以income+1000？

$$\Psi(\tilde{Z}) \approx \frac{1}{2}(1000)^2 \left(\frac{5}{51000}\right) = 49.02$$

since $R_A(Y) = R_R(Y)/Y$.

代入公式

Risk Premium and Certainty Equivalent

Now go back to the original formula defining the risk premium,

$$E[u(Y + \tilde{Z})] = u[Y + E(\tilde{Z}) - \Psi(\tilde{Z})],$$

and plug in the numbers to see that in this case, the exact value of $\Psi(\tilde{Z})$ must satisfy

$$\begin{aligned} &(1/2)\left(\frac{52000^{-4} - 1}{-4}\right) + (1/2)\left(\frac{50000^{-4} - 1}{-4}\right) \\ = \quad &\frac{(51000 - \Psi(\tilde{Z}))^{-4} - 1}{-4}. \end{aligned}$$

只有效用函数是二次的时候才比较准确 看懂!

Risk Premium and Certainty Equivalent

$$
\begin{aligned}
& (1/2)\left(\frac{52000^{-4}-1}{-4}\right)+(1/2)\left(\frac{50000^{-4}-1}{-4}\right) \\
= \quad & \frac{(51000-\Psi(\tilde{Z}))^{-4}-1}{-4}.
\end{aligned}
$$

$$
(1/2)52000^{-4}+(1/2)50000^{-4}=(51000-\Psi(\tilde{Z}))^{-4}
$$

$$
[(1/2)52000^{-4}+(1/2)50000^{-4}]^{-1/4}=51000-\Psi(\tilde{Z})
$$

$$
\Psi(\tilde{Z})=51000-[(1/2)52000^{-4}+(1/2)50000^{-4}]^{-1/4}
$$

$$
\Psi(\tilde{Z})=48.97.
$$

Risk Premium and Certainty Equivalent

因为风险很小 如果风险很大就不能

The approximation $\Psi(\tilde{Z}) \approx 49.02$ or the exact solution $\Psi(\tilde{Z}) = 48.97$ imply that an investor with $Y = 50000$ and constant coefficient of relative risk aversion equal to 5 will give up a riskless payoff of up to about

$$CE(\tilde{Z}) = E(\tilde{Z}) - \Psi(\tilde{Z}) \approx 1000 - 49 = 951$$

for this risky asset with expected payoff equal to 1000.

Assessing the Level of Risk Aversion

We can use similar calculations to work through thought experiments that shed light on our own levels of risk aversion.

Suppose your income is $Y = 50000$ and you have the chance to buy an asset that pays 50000 with probability $1/2$ and 0 with probability $1/2$.

This asset has $E(\tilde{Z}) = (1/2)50000 + (1/2)0 = 25000$, but what is the maximum riskless payoff you would exchange for it?

Assessing the Level of Risk Aversion

Suppose your utility function is of the constant relative risk aversion form

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}$$

and recall that the most you should pay for the asset is given by the certainty equivalent $CE(\tilde{Z})$ defined by

$$E[u(Y + \tilde{Z})] = u[Y + CE(\tilde{Z})].$$

Assessing the Level of Risk Aversion

$$E[u(Y+\tilde{Z})] = u[Y + CE(\tilde{Z})]$$

$$\begin{aligned} & (1/2)\left(\frac{100000^{1-\gamma}-1}{1-\gamma}\right) + (1/2)\left(\frac{50000^{1-\gamma}-1}{1-\gamma}\right) \\ = \quad & \frac{(50000 + CE(\tilde{Z}))^{1-\gamma}-1}{1-\gamma} \end{aligned}$$

$$CE(\tilde{Z}) = [(1/2)100000^{1-\gamma} + (1/2)50000^{1-\gamma}]^{1/(1-\gamma)} - 50000$$

Assessing the Level of Risk Aversion

Certainty equivalent for an asset that pays 50000 with probability 1/2 and 0 with probability 1/2 when income is 50000 and the coefficient of relative risk aversion is γ.

γ	$CE(\tilde{Z})$	$\Psi(\tilde{Z})$	
0	25000	0	("risk neutrality," Pascal)
1	20711	4289	(log utility, D Bernoulli)
2	16667	8333	
3	13246	11754	
4	10571	14429	
5	8566	16434	
10	3991	21009	
20	1858	23142	
50	712	24288	

市场上大家的效用函数非常不一样 想描述整个市场技术原因解不出来
找一个中间的 1要更全面 可以描述现实中想到的 2期望效用框架

The Concept of Stochastic Dominance

It is important to recognize that the coefficients of absolute and relative risk aversion, $R_A(Y)$ and $R_R(Y)$, and the certainty equivalent $CE(\tilde{Z})$ and the risk premium $\Psi(\tilde{Z})$, all help describe or summarize investors' preferences over risky cash flows.

They do not directly represent differences in market or equilibrium prices or rates of return across riskless and risky assets.

The Concept of Stochastic Dominance

Since individuals will differ in their attitudes towards risk as in their preferences over everything else, it is useful to ask whether there are properties of payoff distributions that will allow "preference-free" comparisons to be made across risky cash flows.

State-by-state dominance, as we've already seen, is one such property. But are there any others, which might be more widely applicable?

The Concept of Stochastic Dominance

state-by-state 所有情况下都 这个是说不一定永远好但是可以找到一种方法来排序

Consider two assets, with random payoffs Z_1 and Z_2:

Payoffs	10	100	1000
Probabilities for Z_1	0.40	0.60	0.00
Probabilities for Z_2	0.40	0.40	0.20

There may be no state-by-state dominance, if the payoffs $Z_1 = 10$ and $Z_2 = 100$ can occur together and the payoffs $Z_1 = 100$ and $Z_2 = 10$ can occur together.

没有state-by-state 随便选4各资产 定义为z1=100 z2=10

但是z2比z1要好 不可能有一个投资目标 你达到的概率比我高

The Concept of Stochastic Dominance

Payoffs	10	100	1000
Probabilities for Z_1	0.40	0.60	0.00
Probabilities for Z_2	0.40	0.40	0.20

Because $E(Z_1) = 64$, $\sigma(Z_1) = 44$, $E(Z_2) = 244$, and $\sigma(Z_2) = 380$, there is no mean-variance dominance either.

The Concept of Stochastic Dominance

Payoffs	10	100	1000
Probabilities for Z_1	0.40	0.60	0.00
Probabilities for Z_2	0.40	0.40	0.20

But, intuitively, asset 2 "looks" better, because its distribution takes some of the probability of a payoff of 100 and "moves" that probability to the even higher payoff of 1000.

We can make this idea more concrete by looking at the distributions of these random payoffs in a different way.

The Concept of Stochastic Dominance

In probability theory, the cumulative distribution function (cdf) for a random variable X keeps track of the probability that the realized value of X will be less than or equal to x:

$$F(x) = \text{Prob}(X \leq x).$$

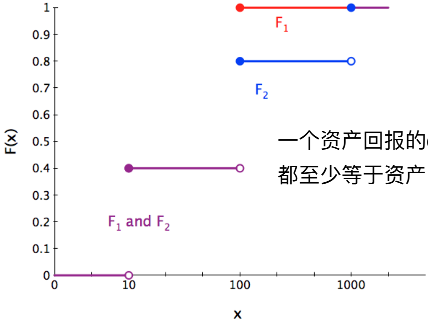
first order do
随机变量最后的realization小于某设定值

The Concept of Stochastic Dominance

Payoffs	10	100	1000
Probabilities for Z_1	0.40	0.60	0.00
Probabilities for Z_2	0.40	0.40	0.20

cdfs	$x < 10$	$10 \leq x < 100$	$100 \leq x < 1000$	$1000 \leq x$
$F_1(x)$	0.00	0.40	1.00	1.00
$F_2(x)$	0.00	0.40	0.80	1.00

The Concept of Stochastic Dominance

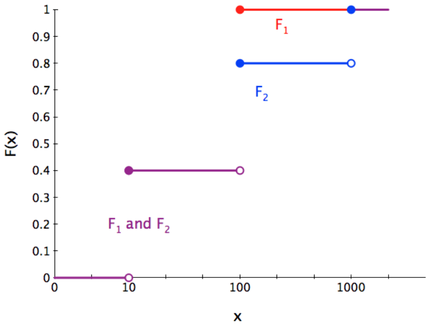


一个资产回报的cdf永远小于另一个资产，并
都至少等于资产1，并且在某些地方可以大于

Cumulative distribution functions are always nondecreasing.

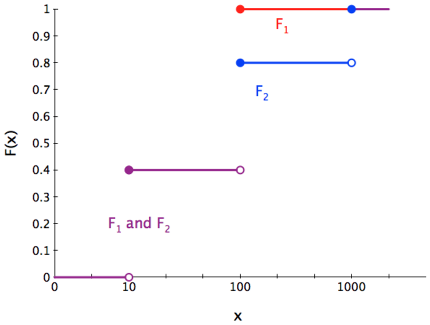
性质没什么用 唯一的用处就是出题发现不对。

The Concept of Stochastic Dominance



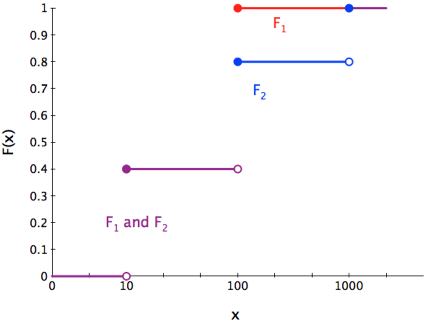
Cumulative distribution functions always satisfy $F(-\infty) = 0$, $F(\infty) = 1$ and $0 \leq F(x) \leq 1$.

The Concept of Stochastic Dominance



Cumulative distribution functions are always càdlàg ("continue à driote, limite à gauche") or RCLL "right continuous with left limits." 法语概念 右连续

The Concept of Stochastic Dominance



The fact that $F_2(x)$ always lies below $F_1(x)$ formalizes the first-order stochastic dominance of Z_2 over Z_1.

The Concept of Stochastic Dominance

cdfs	$x < 10$	$10 \leq x < 100$	$100 \leq x < 1000$	$1000 \leq x$
$F_1(x)$	0.00	0.40	1.00	1.00
$F_2(x)$	0.00	0.40	0.80	1.00

Asset 2 displays first-order stochastic dominance over asset 1 because $F_2(x) \leq F_1(x)$ for all possible values of x.

The Concept of Stochastic Dominance

1没有state-by-state那么严格 2和期望效用理论有关
资产ab 符合期望效用理论的投资者都会选

Theorem Let $F_1(x)$ and $F_2(x)$ be the cumulative distribution functions for two assets with random payoffs Z_1 and Z_2. Then

$$F_2(x) \leq F_1(x) \text{ for all } x,$$

that is, asset 2 displays first-order stochastic dominance over asset 1, if and only if

$$E[u(Z_2)] \geq E[u(Z_1)]$$

for any nondecreasing Bernoulli utility function u.

u要是单调增的 越多越好 期望效用理论只关心payoff和概率 而这里只取决于这两个
舍弃的信息是究竟是哪个state得到了这样回报 如果原先的收入Y存在风险的时候 金融行业者
金融市场而动 就会选择能对冲金融行业风险的资产

The Concept of Stochastic Dominance

没有风险+ 就没有人去买差的 好的资产价格上升 回报率下降 这种优势就会消失

First-order stochastic dominance is a weaker condition than state-by-state dominance, in that state-by-state dominance implies first-order stochastic dominance but first-order stochastic dominance does not necessarily imply state-by-state dominance.

But first-order stochastic dominance remains quite a strong condition. Since an asset that displays first-order stochastic dominance over all others will be preferred by any investor with vN-M utility who prefers higher payoffs to lower payoffs, the price of such an asset is likely to be bid up until the dominance goes away.

The Concept of Stochastic Dominance

Consider two more assets, with random payoffs Z_3 and Z_3:

state-by-state肯定没有 两个需要计算

Payoffs	1	4	5	6	8	9
Probabilities for Z_3	0.33	0.00	0.00	0.33	0.33	0.00
Probabilities for Z_4	0.00	0.25	0.50	0.00	0.00	0.25

任意设定一个投资目标 概率都要高 假设设定为5 67% 75% 设定为8 100% 75%

不同的资产有不同的表现

Asset 4 looks at least slightly better, since it always pays off at least 4 and has a non-trivial probability of 9. On the other hand, asset 3 has a higher probability of a payoff of 6 or more.

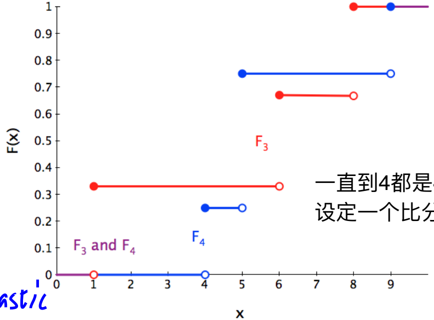
The Concept of Stochastic Dominance

Payoffs	1	4	5	6	8	9
Probabilities for Z_3	0.33	0.00	0.00	0.33	0.33	0.00
Probabilities for Z_4	0.00	0.25	0.50	0.00	0.00	0.25

cdfs	$x < 1$	$1 \leq x < 4$	$4 \leq x < 5$	$5 \leq x < 6$
$F_3(x)$	0.00	0.33	0.33	0.33
$F_4(x)$	0.00	0.00	0.25	0.75

cdfs	$6 \leq x < 8$	$8 \leq x < 9$	$9 \leq x$
$F_3(x)$	0.66	1.00	1.00
$F_4(x)$	0.75	0.75	1.00

The Concept of Stochastic Dominance



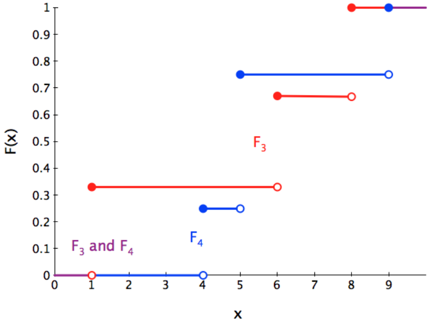
一直到4都是4领先
设定一个比分

不存在

first-order stochastic dominance

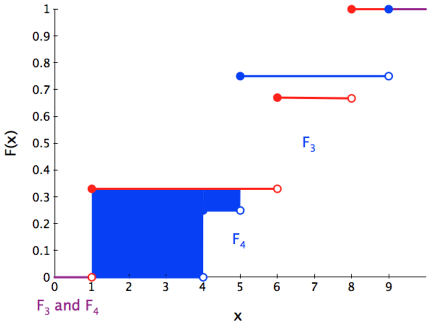
There is no first-order stochastic dominance, since $F_4(x) \leq F_3(x)$ for some values of x but $F_3(x) \leq F_4(x)$ for other values of x.

The Concept of Stochastic Dominance



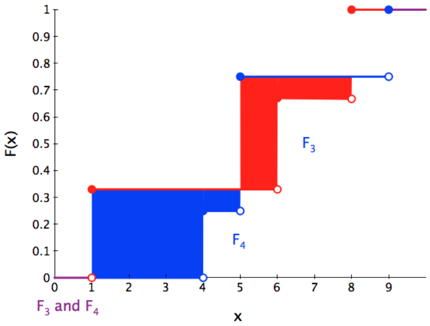
Still, as we move from left to right . . .

The Concept of Stochastic Dominance



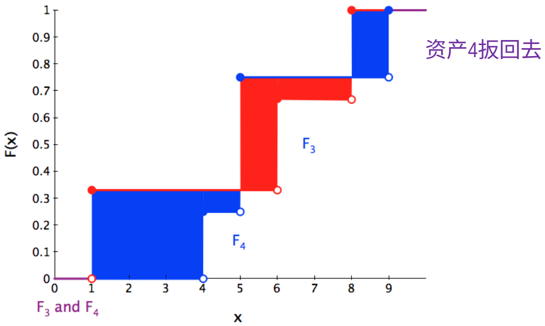
... the areas over which $F_4(x) \leq F_3(x)$ ...

The Concept of Stochastic Dominance



... seem to be consistently larger than the areas over which $F_3(x) \leq F_4(x)$.

The Concept of Stochastic Dominance



Thus, asset 4 displays second-order stochastic dominance over asset 3.

The Concept of Stochastic Dominance

cdfs	$x < 1$	$1 \leq x < 4$	$4 \leq x < 5$	$5 \leq x < 6$
$F_3(x)$	0.00	0.33	0.33	0.33
$F_4(x)$	0.00	0.00	0.25	0.75

cdfs	$6 \leq x < 8$	$8 \leq x < 9$	$9 \leq x$
$F_3(x)$	0.66	1.00	1.00
$F_4(x)$	0.75	0.75	1.00

不考!

Asset 4 displays second-order stochastic dominance over asset 4 since

没说

$$\int_{-\infty}^{\bar{x}} [F_4(x) - F_3(x)]dx \leq 0$$

for all possible values of $\bar{x}$.

总比分表 比刚刚的加起来更严格
永远不会3比4高的情况 最多只能扳到4:3 4:4

The Concept of Stochastic Dominance

Asset 4 displays second-order stochastic dominance over asset 4 since

$$\int_{-\infty}^{\bar{x}} [F_4(x) - F_3(x)]dx \leq 0$$

for all possible values of $\bar{x}$.

Note the integral in the definition runs from $-\infty$ up to $\bar{x}$, since we want to penalize assets with higher probabilities of lower payoffs (we moved from left to right in the graphs). Note, also, that the condition must hold for all values of $\bar{x}$.

The Concept of Stochastic Dominance

选择 or 判断

Theorem Let $F_3(x)$ and $F_4(x)$ be the cumulative distribution functions for two assets with random payoffs Z_3 and Z_4. Then

考!

$$\int_{-\infty}^{\bar{x}} [F_4(x) - F_3(x)]dx \leq 0 \text{ for all } \bar{x}$$

that is, asset 4 displays second-order stochastic dominance over asset 3, if and only if

$$E[u(Z_4)] \geq E[u(Z_3)]$$

for any nondecreasing and concave Bernoulli utility function u.

有直接的联系 只要满足+投资者满足期望效用理论+risk adverse就会选这个
最需要记忆的部分

The Concept of Stochastic Dominance

Second-order stochastic dominance is a weaker condition than first-order stochastic dominance, in that first-order stochastic dominance implies second-order stochastic dominance but second-order stochastic dominance does not necessarily imply first-order stochastic dominance.

But second-order stochastic dominance remains a strong condition. Since an asset that displays second-order stochastic dominance over all others will be preferred by any risk-averse investor with vN-M utility, the price of such an asset is likely to be bid up until the dominance goes away.

如果second 没有让人去买价格降低回报升高 不应该经常出现
应该能解释资产 但是不应该出现在市场上

Mean Preserving Spreads

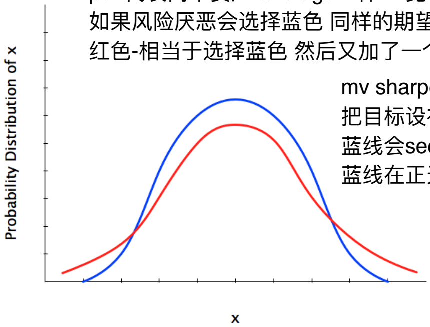
Comparisons based on state-by-state dominance, first-order stochastic dominance, and second-order stochastic dominance can reflect differences in the mean, or expected, payoff as well as in the standard deviation or variance of the payoff.

It is also useful, therefore, to consider an alternative criterion that focuses entirely on the standard deviation of a random payoff, as a measure of the riskiness of the corresponding asset, holding the mean or expected value fixed.

期望效用理论的由来 和 局限之处

Mean Preserving Spreads

pdf 代表两个资产 average一样 一宽一窄
如果风险厌恶会选择蓝色 同样的期望回报更低风险
红色-相当于选择蓝色 然后又加了一个期望回报为0但是有风险的



mv sharpe应该符合
把目标设在最右边 没有first
蓝线会second！ 只要是这种关系
蓝线在正无穷的时候才会输光

Graphically, a mean preserving spread takes probability from the center of a distribution and shifts it to the tails.

Mean Preserving Spreads

Mathematically, one way of producing a mean preserving spread is to take one random variable X_1 and define a second, X_2, by adding "noise" in the form of a third, zero-mean random variable Z:

$$X_2 = X_1 + Z,$$

where $E(Z) = 0$.

定义：x2就是x1的mean preserving spreads

Mean Preserving Spreads

As an example, suppose that

x2会更肥

$$X_1 = \begin{cases} 5 & \text{with probability } 1/2 \\ 2 & \text{with probability } 1/2 \end{cases}$$

$$Z = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

then

$$X_2 = X_1 + Z = \begin{cases} 6 & \text{with probability } 1/4 \\ 4 & \text{with probability } 1/4 \\ 3 & \text{with probability } 1/4 \\ 1 & \text{with probability } 1/4 \end{cases}$$

$E(X_1) = E(X_2) = 3.5$, but if these are random payoffs, asset 2 seems riskier.

Mean Preserving Spreads

The following theorem relates the concept of a mean preserving spread to the previous concept of second-order stochastic dominance.

两个资产期望回报相同就一定

Theorem Let X_1 and X_2, with $E(X_1) = E(X_2)$, be random payoffs on two assets. Then the following two statements are equivalent: (i) $X_2 = X_1 + Z$ for some random variable Z with $E(Z) = 0$ and (ii) asset 1 displays second-order stochastic dominance over asset 2.

Mean Preserving Spreads

Theorem Consider two assets with random payoffs Z_1 and Z_2. Asset 2 displays second-order stochastic dominance over asset 1 if and only if

$$E[u(Z_2)] \geq E[u(Z_1)]$$

for any nondecreasing and concave Bernoulli utility function u.

This theorem imply that any risk-averse investor with vN-M preferences will avoid "pure gambles," in the form of assets with payoffs that simply add more randomness to the payoff of another asset.