Solutions to General Linear SDEs

We discussed in class the solution to Vasicek model for interest rates:

$$dR(t) = (\alpha(t) - \beta R(t))dt + \sigma dW(t).$$

Suppose by multiplying $R(t)$ with a suitable $f(t)$ and applying Itô lemma, we can eliminate the random factor in drift (i.e. $R(t)$) so as to integrate. It turns out one suitable $f(t)$ is $e^{\beta t}$.

We can integrate into closed form of $R(t)$:

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + e^{-\beta t}\int_0^t \sigma e^{\beta s} dW(s)$$

Now consider the General Linear SDE:

$$dX(t) = [a(t) + b(t)X(t)]dt + [\gamma(t) + \sigma(t)X(t)]dW(t). \tag{1}$$

Note that this situation, aside from drift, diffusion is also random with $X(t)$. So multiplying function should be dependent on both t and $W(t)$.

Suppose $f(t)$ satisfies the following SDE:

$$df(t) = \alpha(t)f(t)dt + \beta(t)f(t)dW(t).$$

Then we have:

$$\begin{aligned}
d[f(t)X(t)] &= f(t)dX(t) + X(t)df(t) + df(t)dX(t) \quad (It\hat{o}'s \quad product) \\
&= [a(t)f(t) + b(t)f(t)X(t)]dt + [\gamma(t)f(t) + \sigma(t)f(t)X(t)]dW(t) + X(t)\alpha(t)f(t)dt \\
&\quad +X(t)\beta(t)f(t)dW(t) + \beta(t)f(t)[\gamma(t) + \sigma(t)X(t)]dt \\
&= [P(t) + Q(t)X(t)]dt + [R(t) + S(t)X(t)]dW
\end{aligned}$$

Using the same trick as in Vasicek model in order to integrate, we should have:

$$Q(t) = [b(t) + \beta(t)\sigma(t) + \alpha(t)]f(t) = 0$$
$$S(t) = [\sigma(t) + \beta(t)]f(t) = 0.$$

Hence, we have:

$$\beta(t) = -\sigma(t), \quad \alpha(t) = \sigma(t)^2 - b(t)$$

$$df(t) = (\sigma(t)^2 - b(t))f(t)dt - \sigma(t)f(t)dW(t). \tag{2}$$

Solving the above SDE (2)(using Itô lemma to $log f(t)$), and integrating $f(t)X(t)$, we have the final result:

$$f(t) = f(0) \quad exp\{\int_0^t (\frac{1}{2}\sigma(s)^2 - b(s))ds - \int_0^t \sigma(s)dW(s)\}.$$

or in slides' notations:

$$Y(t) = Y(0) \quad exp\{\int_0^t (b(s) - \frac{1}{2}\sigma(s)^2)ds + \int_0^t \sigma(s)dW(s)\} \tag{3}$$

(there's a little mistake on slides)

$$X(t) = Y(t)[X(0) + \int_0^t (a(s) - \gamma(s)\sigma(s))Y(s)^{-1}ds + \int_0^t \gamma(s)Y(s)^{-1}dW(s)].$$

When we let $a(t) = \alpha, b(t) = -\beta, \gamma(t) = \sigma, \sigma(t) = 0$ in (3), we can return back to $Y(t) = e^{-\beta t}$, which is the case in Vasicek model. $\square$