光华管理学院
Guanghua School of Management

Mathematical Methods in Finance

Lecture 1: Introduction and Review of Fundamental Tools

Fall 2013

Agenda

- About the course
- Part I: Introduction
- Part II: Review of fundamental tools, e.g., probability and statistics

Course Information

Course information:

- Course title: Mathematical Methods in Finance
- Time and Place: Mondays, 2:00-5:00 PM, at Room 202, Guanghua Building $\#1$
- Instructor: LI, Chenxu (cxli@gsm.pku.edu.cn OR cxli@caa.columbia.edu)
- Office hour: immediately after class or by appointment

Teaching assistants

- JIANG, Jiajun (georgejjj@pku.edu.cn)
  LI, Chenxu (my PhD student) (lichenxu.pku@gmail.com)
  LI, Shuo (lishuo2012@pku.edu.cn)
- TA sessions: TBA

Administrative Matters

- Grading policy: homework 30%, midterm exam 30% and final exam 40%
- Homework collection: before class
- Course material:
  - Course material downloadable from: `http://course.pku.edu.cn/webapps/login/`
  - Recommended text book:
    S. E. Shreve. *Stochastic Calculus for Finance*, Volume I, II. Springer Finance. Springer-Verlag, New York, 2004
  - The book can be ordered from `http://www.amazon.cn`
    (You can probably also find a Chinese translated version; but I recommend the original English version.)
- Guest speaker presentations from academia and industry, if time allowed

Other Reference and Supplementary Material

Excellent reading material, e.g.,

- supplementary material uploaded together with lecture notes

Other reference books, e.g.,

- Thomas Mikosch, *Elementary Stochastic Calculus With Finance in View*, World Scientific, 2009
- Tomas Bjork, *Arbitrage theory in continuous-time*, Oxford University Press, 3rd Edition, 2009
- Darrel Duffie, *Dynamic Asset Pricing Theory*, Princeton University Press, 3rd Edition, 2001
- P. Glasserman, *Monte Carlo Methods in Financial Engineering*, Springer; 2003

PART I: Introduction

- Question 1: What is this course about?
- Question 2: Why this course?

What is this course about?

- An inter-discipline, preparing some sharp quantitative tools, primarily including applied probability, stochastic calculus, statistics, econometrics, optimization, differential equations, etc., for studying the financial markets and products.
- An quantitative preparation and introduction to Financial Engineering, Financial Econometrics, Mathematical Finance, Financial Mathematics, Computational Finance (with subtle difference in particular emphases).
- A young but very promising and important discipline:
  - Developing very quickly since its advent in 1980s
  - Provide deep insights into financial market
  - Provide key techniques for trading, asset and portfolio management, and risk management

Myron Scholes Predicts 'Golden Age' for Quants

Source: Risk magazine, 25 Aug 2011
Top quant sees bright future for mathematical finance as it tackles problems thrown up by the crisis.

A top quantitative analyst and economics Nobel laureate says the coming years will be a "golden age" for financial modelling as it tackles problems highlighted by the global financial crisis particularly relating to liquidity, intermediation and the role of the state.

> *Myron Scholes, emeritus professor of finance at Stanford University and co-creator of the Black-Scholes option pricing formula, says that because the fundamental assumptions of dominant pre-crisis theory have been challenged, many lines of research are now open to exploration. "I'm bullish on the future for quants," he says. "One thing about a crisis is that it shakes old opinions and you start learning new things. I hope we do - it should be a golden age for risk modelling and management."*

Example 1: Investment and Portfolio Management

- Generally speaking, for one's investment in financial market, the higher the return, the higher the risk.
- Question: How to invest in a variety of financial instruments? Or equivalently, how to select a portfolio of assets?
- No perfect answers yet, but our quantitative tools has given deep insights.
  - Mean-variance analysis and Capital Asset Pricing Model (CAPM) (**Markowitz**, Nobel Prize in Economics in 1990)
  - Minimizing the variance with the same return level to get the efficient frontier
  - In practice, this is never used; it is much better not to use it.
  - Where is the problem? The model is over-simplified and estimation of the parameters poses significant challenges when dimension is high.
  - Some advanced modeling and estimation techniques, e.g. Black-Litterman model (1990) employing Bayesian statistics

Example 1: Investment and Portfolio Management

- Real world portfolio management involves rebalancing and trading assets whenever necessary.
- Static one-period problems vs. dynamic multiperiod problems.
- Merton (1969) established the theory of dynamic consumption-portfolio optimization:
  - Merton, R. C. .Lifetime Portfolio Selection under Uncertainty: the Continuous Time Case,.Review of Economic Studies, 51, 1969: 247-257.
  - Merton, R. C. .Optimum Consumption and Portfolio Rules in a Continuous Time Model,.Journal of Economic Theory, 3, 1971: 373-413.
- Question: Tools: “stochastic process” for asset prices and the Hamilton-Jacobi-Bellman (HJB) equation
- These works greatly impact the real world applications.
- An important topic from investment banks: optimal portfolio execution (optimal strategy for buying and selling assets)

Example 2: Derivatives Valuation and Hedging

- A derivative security is a financial asset whose payoff depends on the value of some underlying variable.
- The underlying variable can be a traded asset (e.g., IBM stock), an index (e.g., Hang Seng Index), other derivatives (e.g., crude oil futures), and so on.
- An option is a derivative security that grants the buyer the right to buy or sell the underlying asset, at or before the maturity date T, for a pre-specified price K, called the strike or exercise price.
- “Right", not “obligation".
- “buy": call options; “sell": put options.
- If it can be exercised only at the maturity date, it is European style; if the exercise can happen at any time before or at the maturity date, it is American style.
  `http://finance.yahoo.com/`
- Excellent leverage and risk-management tools

Example 2: Derivatives Valuation and Hedging

- An European call option has a payoff: $(S(T) - K)^+$; whereas an European put option: $(K - S(T))^+$, where $S(t)$ denotes the underlying asset price at time t.
- Suppose one attempts to buy an European call option with underlying asset being IBM stock.
- Question: How much should he/she pay? What is the “fair" price? What does “fair" mean?
- Question: After buying the call option, the buyer will be exposed to a risk that the IBM stock declines below the strike at maturity. How to hedge the risk?

Example 2: Derivatives Valuation and Hedging

- Need models!
- Under some assumptions and the Black-Scholes-Merton model (BSM), these two questions have been solved elegantly.
- The Black-Scholes-Merton model (1973) assumes that

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t) \Longleftrightarrow \quad S(t) = S(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}.$$

- $\{W(t)\}$ a famous **stochastic process**: Brownian motion (named after the Scottish botanist Robert Brown) is the seemingly random movement of particles suspended in a fluid or the mathematical model used to describe such random movements. Widely used in physics and finance etc.
- The Black-Scholes-Merton pricing formula and delta hedging (Nobel Prize for Economics in 1997)

Example 3: Structure Products

- Shenzhen Nanshan Power Co., LTD agreed with an "Accumulator" contract with Goldman Sachs on Mar 12, 2008, with the details of the first period (Mar 3, 2008 – Dec 21, 2008) roughly given as follows:
  - If in the i^{th} month, some kind of average crude oil price $A_i \geq \$63.5/barrel$, Shenzhen Nanshan earns a profit $(\$63.5/barrel - \$62/barrel) \times 200,000\ barrel = \$300,000$.
  - If $\$63.5/barrel > A_i \geq \$62/barrel$, Shenzhen Nanshan earns a profit $(A_i - \$62/barrel) \times 200,000\ barrel$.
  - If $A_i < \$62/barrel$, Shenzhen Nanshan pays Goldman Sachs $(\$62/barrel - A_i) \times 400,000\ barrel$.

Example 3: Structure Products

- Unfortunately, since June 2008, the crude oil price declines dramatically from more than $\$140/barrel$ to around $\$40/barrel$ on Dec 30, 2008. Shenzhen Nanshan would suffer a huge loss.
- One opportunities to avoid the huge loss might be as follows.
- Even though it was speculated that the crude oil price would be likely to rise, there is still chance to decline. Shenzhen Nanshan should hedge this risk by trading in other securities, e.g., put options on securities having positive correlations with oil price.
- Upon your interest, let me know your answers as if you are Shenzhen Nanshan or a financial analyst at Goldman Sachs in the end of this semester.

Example 4: Quantitative Strategies and Trading

- The Xinhua development company is a large quasi-public developer which takes on large infrastructure development projects insider China. They completed water works, power plants, and industrial parks, etc.
- Since late 1990s, it has funded these projects by borrowing extensively from Japanese banks and insurance companies. So, their debt is Yen-dominated

Exhibit 1
Key terms for outstanding yen-denominated debt of the Xinhua Development Company

Currency:	Japanese yen
Principal:	JPY 50 billion
Remaining time to maturity now:	7 years
Coupon:	Floating, payable annually
Coupon due in year t:	125 b.p. + 1 year Libor Yen rate at end of year t-1
Prepayment:	Not allowed
Sinking fund:	No

Example 4: Quantitative Strategies and Trading

- Exposure of foreign exchange (FX) fluctuation: Receivables Yuan and futures interest expenses in Yen.
- Japanese interests are rising now!
- A direct way to eliminate Yuan/Yen risk: Borrowing in Yuan and the either buying back its own Yen debt or buying another floating rate Yen-dominated bonds as a hedge
- Borrowing in Yuan in the tightly regulated Chinese financial system is difficult; there is not Yuan-lincked swap market in which she can borrow synthetically
- A solution: Enter into a "cross-currency" interest rate swap: receive floating rate in Yen and pay a floating rate in US dollars
- The FX risk is thus reduced since Yuan is officially pegged to the US dollars
- Further refinement of this idea according to Xinhua's need.
- How to "structure" such a strategy? What financial instruments to use? How to reasonably price them?

Example 5: Development in China

- Put warrants on stocks
- Stocks index futures: trading initiated in April 2010
- Options on stock index futures: now being tested and forthcoming in the near future
- Interest rate deregulation and liberalisation: e.g., bond futures
- Credit default swap

Derivatives market in Hongkong:

- Heng Seng index futures and options and Heng Seng volatility index (Hong Kong's VIX)
  `http://www.hkex.com.hk/eng/prod/drprod/hkifo/HSIFO.htm`
  `http://www.asiaasset.com/news/Hang-Seng-VIX.aspx`

More Examples: Other Business Problems and Solutions

Application of the principle of hedging in revenue management to minimize risk

- Demand is highly correlated with financial index
- Hedging the inventory risk using market instrument

Combination with actuarial sciences: various contract involving market risk

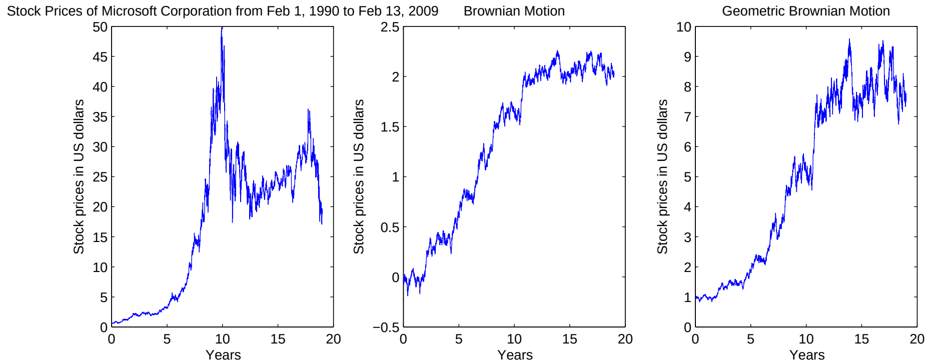
- Equity linked product
- Variable annuity

How to solve such a quantitative problem?

- Design or choose a proper model based on the statistical/econometrics properties of the asset returns data
- Solve the mathematical problem arising from modeling
- Implement the model using various suitable numerical methods
- Calibrate/fit asset prices to market trading data to find the optimal model parameters
- Use the model in trading, asset and portfolio management, and risk management, etc.

Choose a Proper Model

Empirical features of Microsoft stock prices



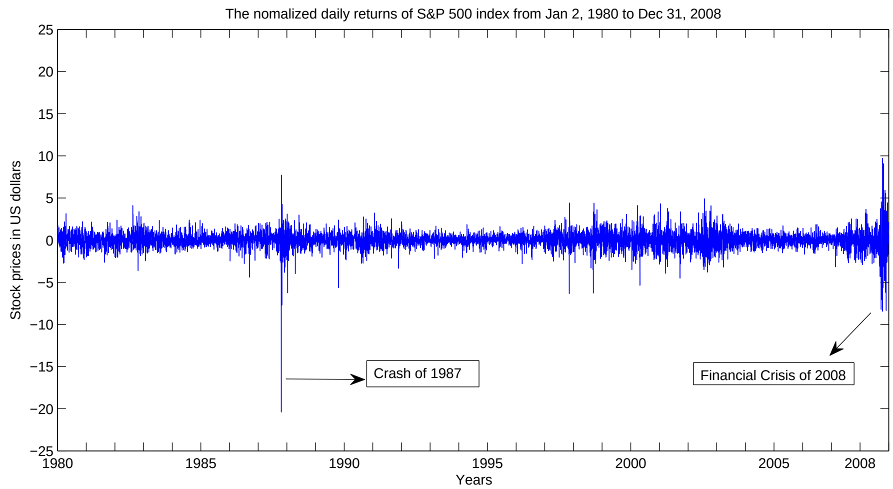
- Observations: greatly volatile sample paths
  - If we assume continuity of sample paths, the sample path seems to (1) not be that smooth, and (2) fluctuate very frequently (extreme zig-zagness).
  - Geometric Brownian motion (the Black-Scholes model) seems to be better than Brownian motion
  - It is even better to include **jumps** and **stochastic volatility**.

Limitation of the Black-Scholes Model

- The Black-Scholes model is enlightening in research and magically applied in practice, though it has several drawbacks.
- Empirical reason: the BSM model cannot capture the following statistical features:
  - Asymmetric leptokurtic feature: empirically, the asset return has a higher peak and heavier tails than that of normal distribution. Moreover, the distribution is skewed to the left.
    - Kurtosis:
$$\frac{E[(X-EX)^4]}{(VarX)^2}$$
    - Skewness:
$$\frac{E[(X-EX)^3]}{(VarX)^{3/2}}$$
  - Volatility smile: the volatility is not a constant! (the "implied volatility" curve is not constant but resembles a smile)

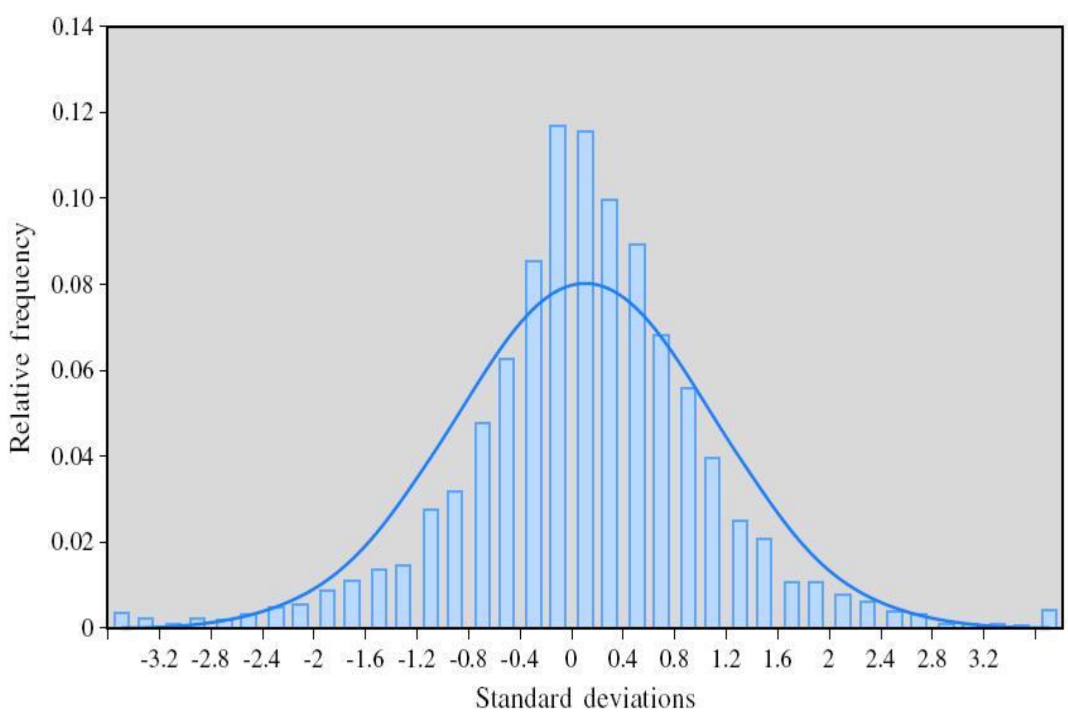
The Asymmetric Leptokurtic Feature

Normalized Daily Returns of S&P 500 index from Jan 2, 1980 to Dec 31, 2008.



- The minimum returns in 1987 and 2008 are -21.1550 and -8.4535. However, $P(Z < -21.1550) \approx 1.4 \times 10^{-107}$ and $P(Z < -8.4535) \approx 1.4 \times 10^{-17}$.

Empirical Distribution of Daily Log-returns and Best Normal Fit



Advanced Models: LV and SV

- To incorporate the features, a natural idea is to allow for change of volatility.
- More sophistic diffusion models
  - Local volatility models (e.g., Dupire (1994) as well as Derman and Kani (1994))

$$dS(t) = \mu S(t)dt + \sigma(t, S(t))S(t)dW(t)$$

  - The stochastic volatility model (e.g. Heston (1993)):

$$dS(t) = \mu S(t)dt + \sqrt{V(t)}S(t)dW_1(t)$$

$$dV(t) = \kappa(\theta - V(t))dt + \sigma_v\sqrt{V(t)}dW_2(t).$$

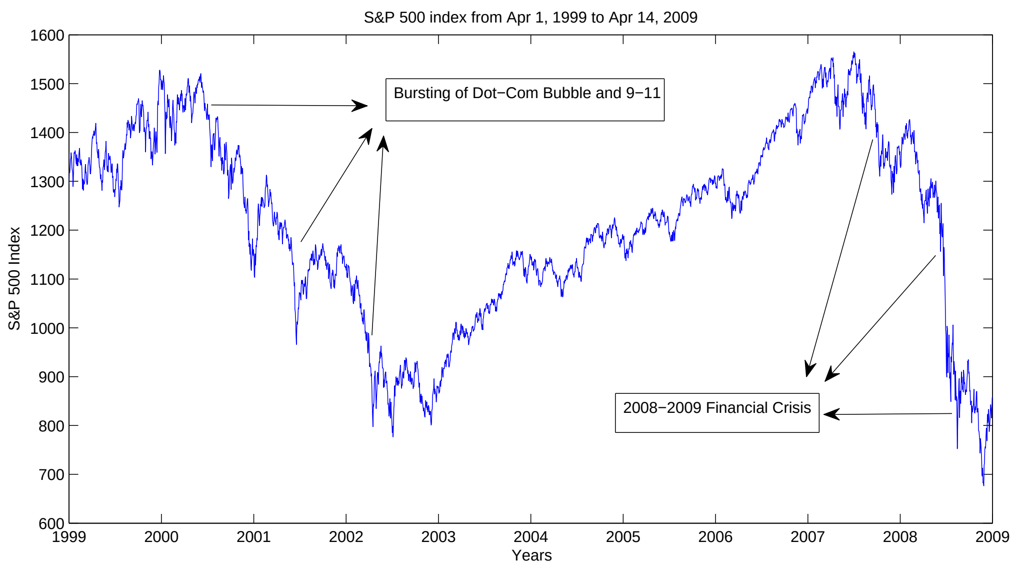
- And, even add jumps to generate market shocks

Add Jumps

- To incorporate the asymmetric leptokurtic feature, a natural idea is to allow for more extreme events.
- The jump diffusion models are proposed to capture this feature.
  - Intuitively, jumps will incur extreme events.
  - Jumps can also generate “volatility smiles”.
  - Jumps coincide with the fact that the sample path of stock price is discontinuous sometimes.
  - Jumps can reflect the instability of the financial market to some extent.
- Moreover, jump diffusion model can provide a behavioral finance interpretation of leptokurtic feature.
  - Reaction to various good or bad outside news;
  - Overreaction (heavy tails) and underreaction (high peak).

Discontinuity of Asset Prices with Jumps

Evolution of S&P500 index from Apr 1, 1999 to Apr 14, 2009.



Advanced Models: Jump-Diffusions

- A general jump-diffusion model can be expressed as $S(t) = S_0 e^{X(t)}$ where the asset return process $X(t)$ follows a jump diffusion process as follows:

$$X(t) = X(0) + \mu t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i, \quad \text{for any } t \geq 0.$$

- Different distribution types of Y_i result in different jump diffusion models.
- Merton (1976): Y_i has normal distribution.
- Kou (2002): Y_i has double exponential distribution.

Our Goal: A Tentative Syllabus

Our goal: preparation of quantitative tools for modeling financial markets. We will select from the following list of material:

- Mathematical tools: applied probability and statistics, stochastic processes, stochastic calculus, and introduction to partial differential equations
- Computational tools: binomial/multinomial lattice, Monte Carlo simulation, numerical methods for differential equations, etc.
- Tentative topics on the applications of quantitative tools: modeling and computing methods for pricing derivative securities on a wide variety of asset classes, such as equity/index, fixed-income, credit, commodity, foreign-exchange, etc.

Part II: Review of Probability Theory

- Probability Space: $(\Omega, \mathcal{F}, \mathbb{P})$.
- Ω: The set of all possible outcomes of a random experiment.
- $\mathcal{F}$: The collection of some subsets in Ω such that a probability measure can be well defined. (The collection of events)
- Formally, $\mathcal{F}$ is called *σ-algebra* satisfying following conditions:
  - (1) $\Omega \in \mathcal{F}$;
  - (2) (Closed under complementation) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$;
  - (3) (Closed under countable unions) If $A_i \in \mathcal{F}$ for all $i = 1, 2, \cdots$, then $\bigcup_{i=1}^{+\infty} A_i \in \mathcal{F}$.
- **Definition**: Probability (measure) $\mathbb{P}$: a set function on $\mathcal{F}$ such that
  - (1) $\mathbb{P}(\Omega) = 1$;
  - (2) $\mathbb{P}(\bigcup_1^\infty A_n) = \sum_1^\infty \mathbb{P}(A_n)$ if $A_i \bigcap A_j = \emptyset$ for any $i \neq j$.

Probability Measure: Example

- Probability (measure) $\mathbb{P}$ is the same as length, area, volume in the sense that they are all used to measure something in terms of real values.

- **Example (Toss of Coin)**: Consider tossing a fair coin. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ can be defined as:
  - $\Omega = \{$H (head), T (tail)$\}$;
  - $\mathcal{F} = \{\emptyset, \{H\}, \{T\}, \Omega\}$;
  - $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\{H\}) = \mathbb{P}(\{T\}) = 1/2$, and $\mathbb{P}(\Omega) = 1$.

Random Variables

- A random variable $X(\omega)$ is a function on Ω such that we always have
$$\{\omega : X(\omega) \leq x\} \in \mathcal{F}.$$
- For any interval B in the real line, we can use a potential probability measure $\mathbb{P}$ to measure the event $\{\omega : X(\omega) \in B\}$.
- $\mathcal{F}$-measurable: $X(\omega)$ is $\mathcal{F}$-measurable iff
$$\{\omega : X(\omega) \leq x\} \in \mathcal{F}$$
for any x.
- Cumulative distribution function (CDF)
$$F(x) := \mathbb{P}(\{\omega : X(\omega) \leq x\}) = \mathbb{P}(X \leq x)$$
.
- Continuous R.V.s with a probability density function (pdf) $f(x)$:
$$F(y) = \int_{-\infty}^{y} f(x)dx \; (\Longleftrightarrow \; f(y) = F'(y)).$$

Random Variables

- Discrete RVs: Bernoulli distribution with parameter p, Binomial distribution with parameter p and n, Poisson with intensity λ etc.
- Continuous RVs: Normal $\mathrm{N}(\mu, \sigma^2)$, log-normal $\log X \sim \mathrm{N}(\mu, \sigma^2)$, exponential expo$(\eta)$, etc.
- Expectation: $\mathbb{E}X$.
  - If X is a discrete RV, $\mathbb{E}X = \sum_x x\mathbb{P}(X = x)$.
  - If X is a continuous RV, $\mathbb{E}X = \int_{\mathbb{R}} xf(x)dx$.
- Variance: $Var(X) = \mathbb{E}(X - \mathbb{E}(X))^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2$
- Moment Generating Function: $M(\theta) = \mathbb{E}e^{\theta X}$
- Characteristic Function: $\widehat{F}(\theta) = \mathbb{E}e^{i\theta X}$
- How to find the higher order moments from

$$M(\theta) = \mathbb{E}e^{\theta X} = \sum_{k=0}^{\infty} \frac{\theta^k}{k!}\mathbb{E}X^k? \tag{1}$$

Random Vector: Vector of Random Variables

- $X = (X_1, X_2, ..., X_m)$ is a random vector if its components X_k are real-valued random variables.
- CDF and PDF:

$$\begin{aligned} &\mathbb{P}(X_1 \le x_1, ..., X_m \le x_m) = F(x_1, ..., x_m) \\ &= \int_{-\infty}^{x_1} ... \int_{-\infty}^{x_m} f(y_1, ..., y_m)dy_1...dy_m. \end{aligned} \tag{2}$$

- Marginal distribution/densities: distribution/density of one or several of the component of the vector. .e.g. $m = 3$

$$f_1(x_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, y_2, y_3)dy_2dy_3,$$

$$P(X_1 \le x_1) = F_1(x_1) = \int_{-\infty}^{x_1} f_1(y_1)dy_1.$$

Covariance and Correlation

- Covariance: a probabilistic relation,

$$cov(X_i, X_j) = \mathbb{E}(X_i - \mathbb{E}X_i)(X_i - \mathbb{E}X_i) = \mathbb{E}(X_iX_j) - \mathbb{E}X_i\mathbb{E}X_j.$$

- Correlation: normalized covariance,

$$corr(X_i, X_j) = \frac{cov(X_i, X_j)}{\sqrt{Var(X_i)}\sqrt{Var(X_j)}}.$$

- Expectation and variance of a linear combination:

$$E\left(\sum_{j=1}^{m} a_jX_j\right) = \sum_{j=1}^{m} a_jEX_j$$

$$Var\left(\sum_{j=1}^{m} a_jX_j\right) = \sum_{j=1}^{m} a_j^2VarX_j + 2\sum_{i=1}^{m}\sum_{j>i}^{m} a_ia_jcov(X_i,X_j) \tag{3}$$

- How about the I.I.D. case?

Independence

- For independent events:

$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} \mathbb{P}(A_i).$$

- For independent random variables,

$$\mathbb{E}[f(X)g(Y)] = \mathbb{E}f(X)\mathbb{E}g(X),$$

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B),$$

$$f_{X,Y}(x,y) = f_X(x)f_Y(y).$$

- What is the relation btw independent and uncorrelated?

A Useful Example: (Multivariate) Normal Distribution

Normal Distribution
`http://en.wikipedia.org/wiki/Normal_distribution`

Multivariate Normal Distribution
`http://en.wikipedia.org/wiki/Multivariate_normal_distribution`

- Notation, parametrization and definition
- Various properties
- Why normal?

Convergence

- Given a sequence of RVs $\{X_n : n \geq 1\}$ and another RV X, there are at least four types of convergence.
- Almost Sure Convergence $\quad X_n \xrightarrow{a.s.} X$: $\mathbb{P}(\lim_{n\to+\infty} X_n = X) = 1$;
- Convergence in Probability $\quad X_n \xrightarrow{P} X$: $\lim_{n\to+\infty} \mathbb{P}(|X_n - X| > \epsilon) = 0$ for any $\epsilon > 0$;
- Convergence in Distribution $\quad X_n \xrightarrow{D} X$: $\lim_{n\to+\infty} F_n(x) = F(x)$ for any continuous point x of $F(x)$, where F_n and F are cdf of X_n and X, respectively;
- Convergence in L^r Norm $\quad X_n \xrightarrow{L^r} X$: $\lim_{n\to+\infty} E\left(|X_n - X|^r\right) = 0$ where $r > 0$ and $E(X_n^r) < +\infty$ for any $n \geq 1$;

Law of Large Numbers and Central Limit Theorem

Let $X_1, X_2, ..., X_n$ be a sequence of i.i.d. random variables. Assume that $E|X_i| < +\infty$ and $Var(X_i) = \sigma^2 < +\infty$. Let $S_n = X_1 + X_2 + ... + X_n$. We have:

Strong Law of Large Numbers (SLLN): (A relation btw sample mean and true mean)

$$\overline{X} = \frac{S_n}{n} \xrightarrow{a.s.} \mu.$$

Center Limit Theorem (CLT): (How good is the estimation using LLN)

$$\frac{S_n - n\mu}{\sqrt{n}\sigma} \xrightarrow{D} N(0,1).$$

Supplementary Material

Suggested Reading Material (We only need to focus on the material parallel to our course slides):

- Selected Material from Shreve Vol. II: Section 1.1, 1.2, 1.3, 1.5

Suggested Exercises (Do Not Hand In; For Your Better Understanding Only)

- Shreve Vol. II: Exercise 1.5, 1.6