光华管理学院
Guanghua School of Management

Mathematical Methods in Finance

# Lecture 8: From Stochastic Calculus to Option Pricing

Fall 2013



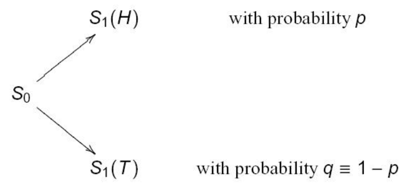
## Overview

- Brief review of the binomial lattice framework
- Derivation of The Black-Scholes-Merton (1973) framework

Note: we investigate the BSM by comparing with the binomial lattice.

## Recapitulation: Option Pricing with Binomial Lattice

- Investigated in lecture 3: How to price an option?
- The Binomial Lattice Model is a *simplified* model for asset pricing.
  - One period binomial lattice model just considers a single period: From Time $0$ to Time $1$
  - Consider a stock with price per share being $S_t > 0$, $t = 0, 1$.
  - $S_0$ is a constant, but $S_1$ assumes a Bernoulli distribution.




  - Probability space $(\Omega, \mathbb{P}, \mathcal{F})$ where $\Omega = \{H, T\}$ with $\mathbb{P}(\{H\}) = p$



## One Period Binomial Lattice Model

- Consider a financial market with one stock and a money market, where the interest rate is $r$.
- Define $u = \frac{S_1(H)}{S_0}$ and $d = \frac{S_1(T)}{S_0}$, and assume $u > d$.
- **Definition**: Arbitrage is a trading strategy that
  - (i) begins with no money,
  - (ii) has no probability of losing money, and
  - (iii) has a positive probability of making money at some future date.
- An efficient market should preclude arbitrages.
- The financial market above has no arbitrage $\Longleftrightarrow$ if $0 < d < 1 + r < u$
  - Proof of "$\Longrightarrow$": By contradiction.
  - Proof of "$\Longleftarrow$": **an excellent exercise**.

## A Fair Option Price – No Arbitrage Approach

- Consider a European call option with a payoff $(S_1 - K)^+$.
- No arbitrage pricing: It is reasonable to assign a fair price for this option such that no arbitrage is incurred, i.e. no arbitrage is created by adding the option.
- If one portfolio is a replication of another portfolio at time 1, their values at time 0 should be identical.
- **IDEA**: To price a financial derivative security, replicate it with available financial instruments with known values.
- Consider a general derivative security with underlying asset being the stock and with payoff $V_1(S_1)$ (taking value either $V_1(S_1(H))$ or $V_1(S_1(T))$) depending on different random outcomes.
- How to get its price $V_0$ at time $0$ with the technique of replication?



## Option Pricing with Binomial Lattice

- Start with a wealth $X_0$.
- At time $0$, invest $X_0$ into the stock and the money market by
  - buying $\Delta_0$ shares of stock
  - borrowing or investing $X_0 - \Delta_0 S_0$ in the money market.
- Then at time $1$, we have

$$X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) \tag{1}$$

- Replication equations for calculation of $X_0$ and $\Delta_0$

$$X_1(H) = \Delta_0 S_1(H) + (1+r)(X_0 - \Delta_0 S_0) = V_1(H)$$
$$X_1(T) = \Delta_0 S_1(T) + (1+r)(X_0 - \Delta_0 S_0) = V_1(T)$$

- $\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}$.

## Option Pricing with Binomial Lattice

- Denote $\tilde{p} = \frac{1+r-d}{u-d}$, $\tilde{q} = \frac{u-(1+r)}{u-d}$, and $\Delta_0 = \frac{V_1(H)-V_1(T)}{S_1(H)-S_1(T)}$.
- Then we have

$$X_0 = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)] \tag{2}$$

- Note that

$$S_0 = \frac{1}{1+r}[\tilde{p}S_1(H) + \tilde{q}S_1(T)] \tag{3}$$

- No arbitrage implies that

$$X_0 = V_0 = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)] \tag{4}$$



## Risk-Neutral Probability

- Risk-neutral probability measure ($\mathbb{Q}$): under which the discounted stock price (as well as the bond) is a martingale.
- In the lattice model,

$$\mathbb{Q}(\omega = H) = \tilde{p}, \quad \mathbb{Q}(\omega = T) = \tilde{q} = 1 - \tilde{p}.$$

- Alternative expressions of previous formulas:
  - Discounted stock price is a martingale under $\mathbb{Q}$;

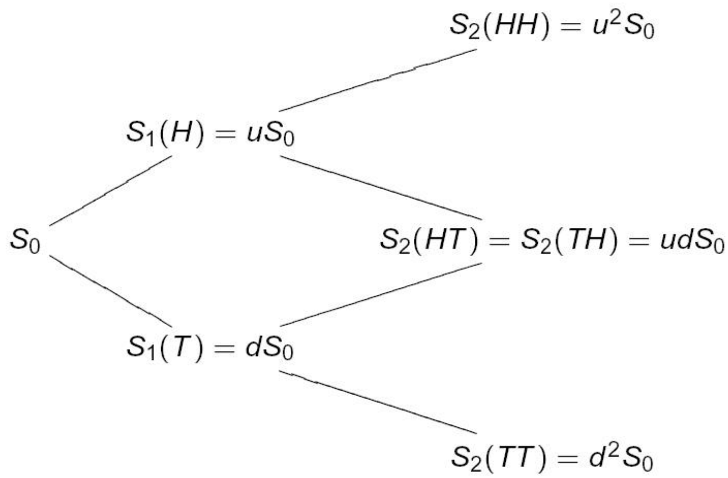
$$S_0 = \mathbb{E}^{\mathbb{Q}}\left(\frac{S_1}{1+r}\right).$$

  - Discounted option price is a martingale under $\mathbb{Q}$;

$$V_0 = \mathbb{E}^{\mathbb{Q}}\left(\frac{V_1}{1+r}\right).$$

# The Multiperiod Binomial Lattice Model

The Multiperiod Binomial Lattice Model: For example–two-period






# Generalization to multiperiod Binomial Lattice Model

The Principle of **Backward Induction**:

$$\text{A derivative with payoff } V_N$$

$$\Updownarrow \ (Replication)$$

$$\text{A portfolio with initial wealth } X_0 \text{ and self-financing strategy } \Delta_n$$

$$\Downarrow$$

$$\text{The derivative price } V_n := X_n \text{ at time } n = 0, 1, \cdots, N-1$$

## Generalization to Multiperiod Binomial Lattice Model

- Start from an initial wealth $X_0$, and adjust the portfolio by investing in stock and money market at each time $n = 0, 1, \cdots, N-1$
- Self-financing adjust satisfies a Wealth Equation.

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$$

- Replication

$$X_N = V_N$$

to specify $X_n$ and $\Delta_n$ for $n = 0, 1, \cdots, N-1$.
- The option price at time $n$ given the fixed random outcome in the first $n$ period is defined as $V_n := X_n$, for $n = 0, 1, \cdots, N-1$.



## Generalization to Multiperiod Binomial Model

- **Theorem**: Consider a N-period Binomial model with $0 < d < 1 + r < u$ and with $\tilde{p} = \frac{1+r-d}{u-d}$ and $\tilde{q} = \frac{u-(1+r)}{u-d}$. For $n = 0, 1, \cdots, N-1$, we have

$$X_n(\omega_1 \cdots \omega_n) = \frac{1}{1+r}[\tilde{p} X_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q} X_{n+1}(\omega_1 \cdots \omega_n T)]$$

$$\Delta_n = \frac{X_{n+1}(\omega_1 \omega_2 \cdots \omega_n H) - X_{n+1}(\omega_1 \omega_2 \cdots \omega_n T)}{S_{n+1}(\omega_1 \omega_2 \cdots \omega_n H) - S_{n+1}(\omega_1 \omega_2 \cdots \omega_n T)}$$

Then, to preclude arbitrage, the initial value of the derivative is set as $V_0 = X_0$ and the values at times $n = 0, 1, \cdots, N$ are set as

$$V_n(\omega_1 \omega_2 \cdots \omega_n) = X_n(\omega_1 \omega_2 \cdots \omega_n) \quad \text{for all } \omega_1 \omega_2 \cdots \omega_n$$

- Obviously, we also have

$$S_n(\omega_1 \cdots \omega_n) = \frac{1}{1+r}[\tilde{p} S_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q} S_{n+1}(\omega_1 \cdots \omega_n T)]$$

# Risk-Neutral Martingales

Under the risk-neutral probability measure $\mathbb{Q}$:

- The discounted stock price $\{\frac{S_n}{(1+r)^n} : n = 0, 1, \cdots, N\}$ is a martingale, i.e.,

$$\frac{S_m}{(1+r)^m} = \mathbb{E}^{\mathbb{Q}}\left(\frac{S_n}{(1+r)^n}|\mathcal{F}_m\right) \quad \text{for } 0 \le m \le n \le N.$$

- The discounted wealth process $\{\frac{X_n}{(1+r)^n} : n = 0, 1, \cdots, N\}$ is a martingale, i.e.,

$$\frac{X_m}{(1+r)^m} = \mathbb{E}^{\mathbb{Q}}\left(\frac{X_n}{(1+r)^n}|\mathcal{F}_m\right) \quad \text{for } 0 \le m \le n \le N.$$

- The discounted option prices process $\{\frac{V_n}{(1+r)^n} : n = 0, 1, \cdots, N\}$ is a martingale, i.e.,

$$\frac{V_m}{(1+r)^m} = \mathbb{E}^{\mathbb{Q}}\left(\frac{V_N}{(1+r)^N}|\mathcal{F}_m\right) \quad \text{for } 0 \le m \le N.$$



# Continuous-Time Finance: the Black-Scholes-Merton (1973)

Why?

- More realistic and flexible models is highly needed;
- More mathematical tools;
- etc.

Note: No model is correct! Models should get closer to the reality and capture some main features according to some certain business. Sometimes, even a “wrong” model can do something great!

# The Black-Scholes-Merton Model

- Similar as the Binomial Lattice Model, consider a simple financial market with:
  - an asset (stock) $S_t$, and
  - a money market account with a continuously compounding interest rate $r$, i.e., Investing $1$ dollar in money market becomes $e^{rt}$ at time $t$.

- We intend to price a European call option that pays $(S(T) - K)^+$ at maturity $T$.

- We propose a geometric Brownian motion model for the underlying stock:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t). \tag{5}$$



# Option Pricing with BSM

- Under this model, assume that the value of the European call option at time $t \in [0, T]$ depends only on the stock price $S(t)$ and the time to expiration $T - t$.
- Denote by $V(t) = c(t, S(t))$ the value of the European call option at time $t \in [0, T]$, where $c(t, x)$ is a deterministic function with two dummy variables $t$ and $x$.
- $V(t) = c(t, S(t))$ implies that the risk premium is embedded in the underlying asset $S(t)$!
- Moreover, assume that $c_t(t, x)$, $c_x(t, x)$, and $c_{xx}(t, x)$ exist.
- Consider hedging a short position (e.g. you are selling this option to your customer) in the option in the following way:
  - Start from $X(0) := c(0, S(0))$;
  - Construct a self-financing portfolio $X(t)$ s.t. $X(T) = c(T, S(T))$.

- In order to rule out arbitrage, we need to have

$$X(t) \equiv V(t) = c(t, S(t))$$

for all $t \in [0, T]$.

- This is equivalent to

$$dX(t) \equiv dV(t) = dc(t, S(t))$$

for any $t \in [0, T)$.





- **Step 1:** Calculate $dX(t)$.
- A self-financing strategy to reallocate the time $t$ wealth $X(t)$.
  - buy $\Delta(t)$ shares of stocks;
  - the rest $X(t) - \Delta(t)S(t)$ is invested in money market.
- Recall the self-financing condition in the Binomial lattice model:

$$X_{n+1} - X_n = \Delta_n(S_{n+1} - S_n) + r(X_n - \Delta_n S_n)$$

- By analogy, we have

$$\begin{aligned} dX(t) =& \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt \\ =& \Delta(t)\left[\alpha S(t)dt + \sigma S(t)dW(t)\right] + r(X(t) - \Delta(t)S(t))dt \\ =& rX(t)dt + \Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t) \end{aligned} \tag{6}$$

- Note! The change of the discounted replicating portfolio value is only due to the change of the discounted stock price.

$$d\left[e^{-rt}X(t)\right] = \Delta(t)d\left[e^{-rt}S(t)\right].$$

(An excellent exercise!)

# Analogy btw Discrete and Continuous Time

Change of the portfolio value $=$ Trading gain + Interest Accumulation

- Discrete-time:

$$X_n - X_0 = \sum_{i=0}^{n-1} \Delta_i (S_{i+1} - S_i) + \sum_{i=0}^{n-1} r(X_i - \Delta_i S_i).$$

- Continuous-time:

$$X(T) - X(0) = \int_0^T \Delta(u) dS(u) + \int_0^T r(X(u) - \Delta(u)S(u))du.$$



# Option Pricing with BSM

- **Step 2:** Calculate $dV(t) = d\left[c(t, S(t))\right]$.
- Apply Itô formula to $dc(t, S(t))$

$$\begin{aligned} dc(t, S(t)) = &\Big[c_t(t, S(t)) + \alpha S(t) c_x(t, S(t)) \\ &+ \frac{1}{2}\sigma^2 S^2(t) c_{xx}(t, S(t))\Big] dt + \sigma S(t) c_x(t, S(t)) dW(t) \end{aligned} \tag{7}$$

## Option Pricing with BSM

- Let
$$dX(t) \equiv dc(t,S(t)) \Longleftrightarrow X(t) \equiv c(t,S(t)).$$
We have
$$\begin{aligned} & rX(t)dt + \Delta(t)(\alpha - r)S(t)dt + \sigma\Delta(t)S(t)dW(t) \\ =& \Big[c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) + \frac{\sigma^2 S^2(t)}{2}c_{xx}(t,S(t))\Big]dt \\ & + \sigma S(t)c_x(t,S(t))dW(t) \end{aligned} \tag{8}$$
- Equate the $dW(t)$ terms $\Longrightarrow \Delta(t) = c_x(t,S(t))$.
- Equate the $dt$ terms and replace $S(t)$ by a dummy variable $x$. We can obtain a Black-Scholes-Merton equation.
$$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x) \quad \text{for all } t \in [0,T), \tag{9}$$
with a terminal condition $c(T,x) = (x-K)^+$.

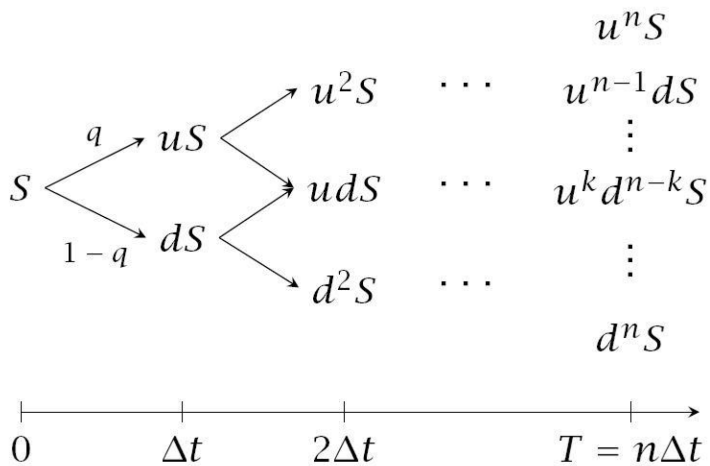


## Comparison btw Binomial and BSM

| | Binomial |
|---|---|
| Replication | $V_n = X_n$ |
| Self-Financing | $X_n - X_0 = \sum_{i=0}^{n-1} \Delta_i(S_{i+1} - S_i) + \sum_{i=0}^{n-1} r(X_i - \Delta_i S_i)$ |
| Hedging Portfolio | $\Delta_n = \frac{V_{n+1}(\omega_1\omega_2\cdots\omega_n H) - V_{n+1}(\omega_1\omega_2\cdots\omega_n T)}{S_{n+1}(\omega_1\omega_2\cdots\omega_n H) - S_{n+1}(\omega_1\omega_2\cdots\omega_n T)}$ |
| Pricing Equation | $V_n(\omega_1\cdots\omega_n) = \frac{1}{1+r}[\tilde{p}V_{n+1}(\omega_1\cdots\omega_n H) + \tilde{q}V_{n+1}(\omega_1\cdots\omega_n T)]$ |
| Terminal Condition | $V_N = (S_N - K)^+$ |

| | Black-Scholes-Merton |
|---|---|
| Replication | $V(t) = c(t,S(t)) = X_t$ |
| Self-Financing | $X(T) - X(0) = \int_0^T \Delta(u)dS(u) + \int_0^T r(X(u) - \Delta(u)S(u))du$ |
| Hedging Portfolio | $\Delta(t) = \frac{\partial c}{\partial s}(t,S(t))$ |
| Pricing Equation | $c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x)$ |
| Terminal Condition | $c(T,s) = (s-K)^+$ |

# From Binomial to BSM




Question: For what $u$ and $d$, the lattice ”converges” to BSM solution as $n \to \infty$?
Answer: Choose the Cox-Ross-Rubinstein Lattice parameters:

$$u = e^{\sigma\sqrt{\Delta t}} \quad d = \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}}.$$



# One question

- One question about the derivation of the BSM formula: How do we know the value of the European call option at time $t \in [0, T]$ depends only on the stock price $S(t)$ and the time to expiration $T - t$?
- Just now, the BSM equation is derived as a necessary condition.
- Now, let us verify it is also sufficient!
- Show that $c(t, S(t))$ is the time $t$ value of the European call option.
- It suffices to construct a self-finance portfolio with time $t$ value $c(t, S(t))$ to replicate the payoff of the European call option.
  - Start from $X(0) = c(0, S(0)) \geq 0$
  - At time $t$, buy $\Delta(t) = c_x(t, S(t))$ shares of stocks.
  - The rest $X(t) - c_x(t, S(t))S(t)$ is invested in the money market.
- An excellent exercise!

## A Puzzle about $\alpha$

- Observation: in the BSM equation, there is no $\alpha$!
- The no-arbitrage price has nothing to do with the expected return of the underlying asset. Is this counter intuitive?
- Note that $V(t) = c(t, S(t))$; thus the risk premium is embedded in $S(t)$!
- As $S(t)$ taking larger value with higher probability, $V(t) = c(t, S(t))$ is more valuable!



## From BSM to Practice

Assumptions in BSM:

- Constant volatility
- Constant interest rate
- No dividend, tax, transaction cost
- Continuously rebalancing of a perfect hedging portfolio
- etc.

Though ignoring some practical concerns, the BSM successfully opened the door to modern derivative pricing theory. BSM itself can be applied in a smart way through the notion of implied volatility; and models developed based on the idea of BSM are also moving forward the derivative business.

Suggested Reading Material (We only need to focus on the material parallel to our course slides):

- Selected material from Shreve vol. II 4.5; Or equivalent material from Mikosch 4.1 (up to the derivation of the BSM equation)

Suggested Exercises (Do Not Hand In; For Your Deeper Understanding Only)

- Shreve Vol. II: 3.8, 4.10, 4.11, 4.21.