Solutions to Homework 2

§1 Selected Questions from Karatzas and Shreve [1]

1. p. 197: 3.5.9 Exercise

With $\mu > 0$ and $W_* \triangleq \inf_{t>0} W_t$, under $P^{(\mu)}$ the random variable $-W_*$ is exponentially distributed with parameter 2μ,i.e.,

$$P^{(\mu)}\left[-W_* \in db\right] = 2\mu e^{-2\mu b} db,\ \ b > 0.$$

Remark 1 *$P^{(\mu)}$ is a measure which satisfies*

$$P^{(\mu)}\left(A\right) = E\left[1_A Z_t\right];\ \ A \in \mathcal{F}_t,$$

where $Z_t \triangleq \exp\left(\mu W_t - \frac{1}{2}\mu^2 t\right)$.

Proof. Define that

$$W_t^* = \inf_{0<s\leq t} W_s, T_b = \inf\left\{t \geq 0; W_t = b\right\}.$$

Then,

$$\begin{aligned}
P^{(\mu)}(-W_* \geq b) &= \lim_{t->\infty} P^{(\mu)}(-W_t^* \geq b) \\
&= \lim_{t->\infty} P^{(\mu)}(-W_t^* \geq b) \\
&= \lim_{t->\infty} P^{(\mu)}(W_t^* \leq -b) \\
&= \lim_{t->\infty} P^{(\mu)}(T_{-b} \leq t)
\end{aligned}$$

Then, with the same method as (5.11) [[1] P196], and combining with (8.5) [[1] P96], we get

$$
\begin{aligned}
\lim_{t\to\infty} P^{(\mu)}(T_{-b} \quad \leq \quad t) &= \lim_{t\to\infty} E[1_{\{T_{-b}\leq t\}} Z_t] \\
&= \lim_{t\to\infty} E[1_{\{T_{-b}\leq t\}} E[Z_t | F^w_{t\wedge T_b}]] \\
&= \lim_{t\to\infty} E[1_{\{T_{-b}\leq t\}} Z_{t\wedge T_b}] \\
&= \lim_{t\to\infty} E[1_{\{T_{-b}\leq t\}} Z_{T_b}] \\
&= \lim_{t\to\infty} E[1_{\{T_{-b}\leq t\}} e^{-\mu b - \frac{1}{2}\mu^2 T_{-b}}] \\
&= \lim_{t\to\infty} \int_0^t e^{-\mu b - \frac{1}{2}\mu^2 s} P[T_{-b} \in ds] \\
&= \lim_{t\to\infty} \int_0^t e^{-\mu b - \frac{1}{2}\mu^2 s} (\frac{|-b|}{\sqrt{2\pi s^3}} e^{-\frac{b^2}{2s}}) ds \\
&= \int_0^\infty \frac{b}{\sqrt{2\pi s^3}} e^{-\mu b - \frac{1}{2}\mu^2 s - \frac{b^2}{2s}} ds \\
&= e^{-2\mu b} \int_0^\infty \frac{b}{\sqrt{2\pi s^3}} e^{-\frac{1}{2}(\mu\sqrt{s} - \frac{b}{\sqrt{s}})^2} ds.
\end{aligned}
$$

Take $y = \mu\sqrt{s} - \frac{b}{\sqrt{s}}$,we get

$$
\begin{aligned}
& e^{-2\mu b} \int_{-\infty}^{\infty} \frac{4\mu b}{\sqrt{2\pi}(y + \sqrt{y^2 + 4\mu b})\sqrt{y^2 + 4\mu b}} e^{-\frac{1}{2}y^2} dy \\
= \; & e^{-2\mu b} \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy - \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{y}{\sqrt{y^2 + 4\mu b}} e^{-\frac{1}{2}y^2} dy \right] \\
= \; & e^{-2\mu b} (1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{y}{\sqrt{y^2 + 4\mu b}} e^{-\frac{1}{2}y^2} dy).
\end{aligned}
$$

It is obvious that $\int_{-\infty}^{\infty} \frac{y}{\sqrt{y^2+4\mu b}} e^{-\frac{1}{2}y^2} dy = 0$,then we get

$$
P^{(\mu)}(-W_* \geq b) = e^{-2\mu b}.
$$

Thus,

$$
P^{(\mu)}(-W_* \in db) = -de^{-2\mu b} = 2\mu e^{-2\mu b} db.
$$

Remark 2 *This is a directly corollary of (5.13) [[1] P197].*

■

Proof. There is another way to get the conclusion.

Define

$$
\tau_{-x} = \inf\{t \geq 0; W_t + \mu t = -x\}.
$$

Since $\exp\left\{\lambda W_t - \frac{1}{2}\lambda^2 t\right\}$ is a martingale and for every $t > 0$,

$$E\exp\left\{\lambda W_t - \frac{1}{2}\lambda^2 t\right\} = 1.$$

So that for t big enough,

$$\begin{aligned}
1 &= E\exp\left\{\lambda W_t - \frac{1}{2}\lambda^2 t\right\} \qquad (1)\\
&= E\exp\left\{\lambda W_{t\wedge\tau_{-x}} - \frac{1}{2}\lambda^2 t\wedge\tau_{-x}\right\}\\
&= E\exp\left\{\lambda W_{t\wedge\tau_{-x}} + \lambda\mu t\wedge\tau_{-x} - \lambda\mu t\wedge\tau_{-x} - \frac{1}{2}\lambda^2 t\wedge\tau_{-x}\right\}\\
&= E\left\{\exp\left\{\lambda W_{t\wedge\tau_{-x}} + \lambda\mu t\wedge\tau_{-x} - \lambda\mu t\wedge\tau_{-x} - \frac{1}{2}\lambda^2 t\wedge\tau_{-x}\right\}|\tau_{-x}<\infty\right\}P\left(\tau_{-x}<\infty\right)\\
&\quad +E\left\{\exp\left\{\lambda W_{t\wedge\tau_{-x}} + \lambda\mu t\wedge\tau_{-x} - \lambda\mu t\wedge\tau_{-x} - \frac{1}{2}\lambda^2 t\wedge\tau_{-x}\right\}|\tau_{-x}=\infty\right\}P\left(\tau_{-x}=\infty\right)\\
&= E\left\{\exp\left\{\lambda W_{\tau_{-x}} + \lambda\mu\tau_{-x} - \lambda\mu\tau_{-x} - \frac{1}{2}\lambda^2\tau_{-x}\right\}|\tau_{-x}<\infty\right\}P\left(\tau_{-x}<\infty\right)\\
&\quad +E\left\{\exp\left\{\lambda W_t + \lambda\mu t - \lambda\mu t - \frac{1}{2}\lambda^2 t\right\}|\tau_{-x}=\infty\right\}P\left(\tau_{-x}=\infty\right).
\end{aligned}$$

Since $\tau_{-x} = \inf\left\{t \geq 0; W_t + \mu t = -x\right\}$, when $\tau_{-x} = \infty$, $W_t + \mu t > -x$ for every $t > 0$.

Let $\lambda = -2\mu$, then under the condition $\tau_{-x} = \infty$,

$$\exp\left\{\lambda W_t + \lambda\mu t - \lambda\mu t - \frac{1}{2}\lambda^2 t\right\} \leq \exp\left\{-\lambda x - \lambda\mu t - \frac{1}{2}\lambda^2 t\right\}.$$

Take $\lambda = -2\mu$ and let $t \to \infty$ in (1),

$$\begin{aligned}
1 &= E\left\{\exp\left\{\lambda W_{\tau_{-x}} + \lambda\mu\tau_{-x} - \lambda\mu\tau_{-x} - \frac{1}{2}\lambda^2\tau_{-x}\right\}|\tau_{-x}<\infty\right\}P\left(\tau_{-x}<\infty\right)\\
&= E\left\{\exp\left\{-\lambda x - \lambda\mu\tau_{-x} - \frac{1}{2}\lambda^2\tau_{-x}\right\}|\tau_{-x}<\infty\right\}P\left(\tau_{-x}<\infty\right)\\
&= E\left\{e^{-2\mu x}|\tau_{-x}<\infty\right\}P\left(\tau_{-x}<\infty\right)\\
&= e^{-2\mu x}P\left(\tau_{-x}<\infty\right).
\end{aligned}$$

So that,

$$P\left(\tau_{-x}<\infty\right) = e^{-2\mu x}, P\left(\tau_{-dx}<\infty\right) = 2\mu e^{-2\mu x}dx$$

which means,

$$P^{(\mu)}(-W_* \in db) = P\left(\tau_{-db} < \infty\right) = 2\mu e^{-2\mu b} db$$

■

2. p.283: 5.1.2 Problem

Assume that the coefficients b_i, σ_{ij} are bounded and continuous, and the $\mathbb{R}^d$-valued process X satisfies

$$X_t^{(i)} = x_i + \int_0^t b_i\left(X_s\right) ds + \sum_{j=1}^{r} \int_0^t \sigma_{ij}\left(X_s\right) dW_s^{(j)};\ 0 \le t < \infty, 1 \le i \le d. \tag{2}$$

Show that

$$E^x\left[X_t^{(i)} - x_i\right] = tb_i\left(x\right) + o\left(t\right) \tag{3}$$

$$E^x\left[\left(X_t^{(i)} - x_i\right)\left(X_t^{(k)} - x_k\right)\right] = ta_{ik}\left(x\right) + o\left(t\right)$$

as $t \downarrow 0$, for $1 \le i, k \le d$ hold for every $x \in \mathbb{R}^d$, and that

$$\lim_{t\downarrow 0} \frac{1}{t}\left[E^x f\left(X_t\right) - f\left(x\right)\right] = \left(\mathcal{A}f\right)\left(x\right);\ \forall x \in \mathbb{R}^d \tag{4}$$

hold for every $f \in C^2\left(\mathbb{R}^d\right)$ which is bounded and has bounded first- and second-order derivatives where the operator $\mathcal{A}f$ in (4) is given by

$$\left(\mathcal{A}f\right)\left(x\right) \triangleq \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d} a_{ik}\left(x\right)\frac{\partial^2 f\left(x\right)}{\partial x_i \partial x_k} + \sum_{i=1}^{d} b_i\left(x\right)\frac{\partial f\left(x\right)}{\partial x_i}$$

Remark 3 *E^x means the expectation condioning the initial value of the process being x.*

Proof. (1)

According to (2), we get

$$\begin{aligned} E^x[X_t^{(i)} - x_i] - tb_i(x) &= E^x[\int_0^t b_i(X_s)ds - \sum_{j=1}^{r}\sigma_{ij}(X_s)dW_s^{(j)} - \int_0^t b_i(x)ds] \\ &= E^x[\int_0^t (b_i(X_s) - b_i(x))ds - \sum_{j=1}^{r}\sigma_{ij}(X_s)dW_s^{(j)}]. \end{aligned}$$

Since b_i is continuous and X_t has continuous sample paths, for $\forall\varepsilon > 0, \exists t > 0$,so that $\forall 0 < s < t$, we get $|b_i(X_s) - b_i(x)| < \varepsilon$.

Thus

$$\frac{1}{t}\int_0^t |b_i(X_s)-b_i(x)|\,ds \le \frac{1}{t}\int_0^t \varepsilon ds = \varepsilon.$$

Therefore,

$$\lim_{t\to 0+}\frac{1}{t}E^x[\int_0^t (b_i(X_s)-b_i(x))ds = 0. \tag{5}$$

Meanwhile,Let $\Pi = \{0 \le t_0 < t_1 < ... < t_n \le t\}$, $\|\Pi\| = \max\{t_k - t_{k-1}\}_{1\le k\le n}$,

$$\begin{aligned}
& E^x[\int_0^t \sigma_{ij}(X_s)dW_s^{(j)}] \\
= \quad & \lim_{\|\Pi\|\to 0} E^x \sum_n \sigma_{ij}(X_{t_{n-1}})(W_{t_n}^{(j)} - W_{t_{n-1}}^{(j)}) \\
= \quad & \lim_{\|\Pi\|\to 0} E^x \left[E^x \sum_n \sigma_{ij}(X_{t_{n-1}})(W_{t_n}^{(j)} - W_{t_{n-1}}^{(j)}) \mid \mathcal{F}_{t_{m-1}}\right] \\
= \quad & \lim_{\|\Pi\|\to 0} E^x \left[\sum_n \sigma_{ij}(X_{t_{n-1}}) E^x\left[(W_{t_n}^{(j)} - W_{t_{n-1}}^{(j)}) \mid \mathcal{F}_{t_{m-1}}\right]\right] = 0.
\end{aligned}$$

Thus,

$$\lim_{t\to 0+}\frac{1}{t}E^x[\int_0^t \sigma_{ij}(X_s)dW_s^{(j)}] = 0. \tag{6}$$

Combining (5) and (6), (3) holds.

(2)

$$\begin{aligned}
& E^x[(X_t^{(i)} - x_i)(X_t^{(k)} - x_k)] \\
= \quad & E^x[(\int_0^t b_i(X_s)ds + \sum_{j=1}^r \int_0^t \sigma_{ij}(X_s)dW_s^{(j)})(\int_0^t b_k(X_s)ds + \sum_{l=1}^r \int_0^t \sigma_{kl}(X_s)dW_s^{(l)})].
\end{aligned}$$

In the same method as we did in 1) ,we get

$$E^x[(\int_0^t b_i(X_s)ds)(\sum_{l=1}^r \int_0^t \sigma_{kl}(X_s)dW_s^{(l)})] = o(t),\ E^x[(\int_0^t b_i(X_s)ds)(\int_0^t b_k(X_s)ds)] = o(t).$$

Now it is sufficient to prove

$$\sum_{j=1}^r\sum_{l=1}^r E^x[(\int_0^t \sigma_{ij}(X_s)dW_s^{(j)})(\int_0^t \sigma_{kl}(X_s)dW_s^{(l)})] = ta_{ik}(x) + o(t). \tag{7}$$

Let $\Pi = \{0 \leq t_0 < t_1 < ... < t_N \leq t\}$, $\|\Pi\| = \max\{t_k - t_{k-1}\}_{1\leq k\leq N}$.

$$
\begin{aligned}
& E^x[(\int_0^t \sigma_{ij}(X_s)dW_s^{(j)})(\int_0^t \sigma_{kl}(X_s)dW_s^{(l)})] \qquad (8)\\
= & \lim_{\|\Pi\|\to 0} E^x\left[\left(\sum_n \sigma_{ij}(X_{t_{n-1}})(W_{t_n}^{(j)} - W_{t_{n-1}}^{(j)})\right)\left(\sum_m \sigma_{kl}(X_{t_{m-1}})(W_{t_m}^{(l)} - W_{t_{m-1}}^{(l)})\right)\right]\\
= & \lim_{\|\Pi\|\to 0} \sum_n\sum_m E^x\left[\left(\sigma_{ij}(X_{t_{n-1}})\sigma_{kl}(X_{t_{m-1}})(W_{t_n}^{(j)} - W_{t_{n-1}}^{(j)})(W_{t_m}^{(l)} - W_{t_{m-1}}^{(l)})\right)\right].
\end{aligned}
$$

For those terms with $t_{n-1} < t_{m-1}$,

$$
\begin{aligned}
& E^x\left[\left(\sigma_{ij}(X_{t_{n-1}})\sigma_{kl}(X_{t_{m-1}})(W_{t_n}^{(j)} - W_{t_{n-1}}^{(j)})(W_{t_m}^{(l)} - W_{t_{m-1}}^{(l)})\right)\right] \qquad (9)\\
= & E^x\left[E^x\left[\left(\sigma_{ij}(X_{t_{n-1}})\sigma_{kl}(X_{t_{m-1}})(W_{t_n}^{(j)} - W_{t_{n-1}}^{(j)})(W_{t_m}^{(l)} - W_{t_{m-1}}^{(l)})\right)\right] \mid \mathcal{F}_{t_{m-1}}\right]\\
= & E^x\left[\left(\sigma_{ij}(X_{t_{n-1}})\sigma_{kl}(X_{t_{m-1}})(W_{t_n}^{(j)} - W_{t_{n-1}}^{(j)})E^x\left[(W_{t_m}^{(l)} - W_{t_{m-1}}^{(l)}) \mid \mathcal{F}_{t_{m-1}}\right]\right)\right]\\
= & 0.
\end{aligned}
$$

For those term with $t_{m-1} < t_{n-1}$, the process is quiet same.

For those term with $t_{n-1} = t_{m-1}$,

$$
\begin{aligned}
& E^x\left[\left(\sigma_{ij}(X_{t_{n-1}})\sigma_{kl}(X_{t_{m-1}})(W_{t_n}^{(j)} - W_{t_{n-1}}^{(j)})(W_{t_m}^{(l)} - W_{t_{m-1}}^{(l)})\right)\right] \qquad (10)\\
= & E^x\left[\left(\sigma_{ij}(X_{t_{m-1}})\sigma_{kl}(X_{t_{m-1}})(W_{t_m}^{(j)} - W_{t_{m-1}}^{(j)})(W_{t_m}^{(l)} - W_{t_{m-1}}^{(l)})\right)\right]\\
= & E^x\left[E^x\left[\left(\sigma_{ij}(X_{t_{m-1}})\sigma_{kl}(X_{t_{m-1}})(W_{t_m}^{(j)} - W_{t_{m-1}}^{(j)})(W_{t_m}^{(l)} - W_{t_{m-1}}^{(l)})\right) \mid \mathcal{F}_{t_{m-1}}\right]\right]\\
= & E^x\left[\sigma_{ij}(X_{t_{m-1}})\sigma_{kl}(X_{t_{m-1}})E^x\left[\left((W_{t_m}^{(j)} - W_{t_{m-1}}^{(j)})(W_{t_m}^{(l)} - W_{t_{m-1}}^{(l)})\right) \mid \mathcal{F}_{t_{m-1}}\right]\right].
\end{aligned}
$$

When $t_{n-1} = t_{m-1}$, $j \neq l$, since $W^{(j)}$ is independent with $W^{(l)}$, we have

$$
\begin{aligned}
& E^x\left[\left((W_{t_m}^{(j)} - W_{t_{m-1}}^{(j)})(W_{t_m}^{(l)} - W_{t_{m-1}}^{(l)})\right) \mid \mathcal{F}_{t_{m-1}}\right] \qquad (11)\\
= & E^x\left[(W_{t_m}^{(j)} - W_{t_{m-1}}^{(j)})(W_{t_m}^{(l)} - W_{t_{m-1}}^{(l)})\right]\\
= & E^x(W_{t_m}^{(j)} - W_{t_{m-1}}^{(j)})E^x(W_{t_m}^{(l)} - W_{t_{m-1}}^{(l)})\\
= & 0.
\end{aligned}
$$

When $t_{n-1} = t_{m-1}$, $j = l$,

$$
\begin{aligned}
& E^x\left[\left(\sigma_{ij}(X_{t_{m-1}})\sigma_{kj}(X_{t_{m-1}})(W_{t_m}^{(j)} - W_{t_{m-1}}^{(j)})(W_{t_m}^{(j)} - W_{t_{m-1}}^{(j)})\right)\right]\\
= & E^x\left[E^x\left[\left(\sigma_{ij}(X_{t_{m-1}})\sigma_{kj}(X_{t_{m-1}})(W_{t_m}^{(j)} - W_{t_{m-1}}^{(j)})(W_{t_m}^{(j)} - W_{t_{m-1}}^{(j)})\right) \mid \mathcal{F}_{t_{m-1}}\right]\right]\\
= & E^x\left[\sigma_{ij}(X_{t_{m-1}})\sigma_{kj}(X_{t_{m-1}})E^x(W_{t_m}^{(j)} - W_{t_{m-1}}^{(j)})^2\right]\\
= & (t_m - t_{m-1})E^x[\sigma_{ij}(X_{t_{m-1}})\sigma_{kj}(X_{t_{m-1}})].
\end{aligned}
$$

Since $\sigma_{ij}(x), \sigma_{kj}(x), X_t$ are both continuous sample paths, for $\forall \varepsilon > 0, \exists t > 0$,so that $\forall 0 < s < t, \forall 1 \leq i, j, k \leq d$, we get $|\sigma_{ij}(X_s)\sigma_{kj}(X_s) - \sigma_{ij}(x)\sigma_{kj}(x)| < \dfrac{\varepsilon}{d^2}$.

Thus

$$
\begin{aligned}
& \left| \left\{ \sum_m (t_m - t_{m-1}) E^x[\sigma_{ij}(X_{t_{m-1}})\sigma_{kj}(X_{t_{m-1}})] \right\} - t\sigma_{ij}(x)\sigma_{kj}(x) \right| \\
= & \left| \sum_m (t_m - t_{m-1}) E^x \left[\sigma_{ij}(X_{t_{m-1}})\sigma_{kj}(X_{t_{m-1}}) - \sigma_{ij}(x)\sigma_{kj}(x) \right] \right| \\
\leq & \ \varepsilon \left| \sum_m (t_m - t_{m-1}) \right| \\
= & \ \frac{\varepsilon t}{d^2}.
\end{aligned}
$$

So that

$$
\left| \left\{ \sum_i \sum_j \sum_m (t_m - t_{m-1}) E^x[\sigma_{ij}(X_{t_{m-1}})\sigma_{kj}(X_{t_{m-1}})] \right\} - t a_{ik}(x) \right| \leq \varepsilon t. \tag{12}
$$

Since (8), (9), (10), (11) and (12) hold for any Π, let $\|\Pi\| \to 0$, we can get that (7) holds, which means

$$
E^x[(X_t^{(i)} - x_i)(X_t^{(k)} - x_k)] = a_{ik}(x)t + o(t).
$$

(3)

Since $f \in C^2(R^d)$,

$$
f(X_t) - f(x) = \sum_{i=1}^d \frac{\partial}{\partial x_i} f(x)(X_t^{(i)} - x_i) + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \frac{\partial^2}{\partial x_i \partial x_j} f(x)(X_t^{(i)} - x_i)(X_t^{(j)} - x_j).
$$

Combining with the conclusion of 1) and 2), we can get

$$
\lim_{t \to 0} \frac{1}{t} E^x f(X_t) - f(x) = \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i} f(x) + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \frac{\partial^2}{\partial x_i \partial x_j} f(x) a_{ij}(x) = (\mathcal{A}f)(x).
$$

Remark 4 *We can get some intuitive sense by simply use Itô rule to $f(X_t)$, and then it is easily to remember this formula.*

■

Proof. Here is an alternative proof by applying Itô's formula to $f(X_t)$.

By Itô's formula,

$$f(X_t) - f(x) = \sum_{i=1}^{d} \int_0^t \frac{\partial f(X_s)}{\partial x_i} dX_s^{(i)} + \frac{1}{2} \sum_{i=1}^{d} \sum_{k=1}^{d} \int_0^t \frac{\partial^2 f(X_s)}{\partial x_i \partial x_k} d\langle X^{(i)}, X^{(k)} \rangle_s,$$

with

$$X_t^{(i)} = x_i + \int_0^t b_i(X_s) ds + \sum_{j=1}^{r} \int_0^t \sigma_{ij}(X_s) dW_s^{(j)},$$

and

$$\begin{aligned} \langle X^{(i)}, X^{(k)} \rangle_t &= \sum_{j=1}^{r} \int_0^t \sigma_{ij}(X_s) \sigma_{kj}(X_s) ds \\ &= \int_0^t a_{ik}(X_s) ds. \end{aligned}$$

Then

$$\begin{aligned} f(X_t) - f(x) &= \sum_{i=1}^{d} \int_0^t b_i(X_s) \frac{\partial f(X_s)}{\partial x_i} ds + \sum_{i=1}^{d} \sum_{j=1}^{r} \int_0^t \sigma_{ij}(X_s) \frac{\partial f(X_s)}{\partial x_i} dW_s^{(j)} \\ &\quad + \frac{1}{2} \sum_{i=1}^{d} \sum_{k=1}^{d} \int_0^t a_{ik}(X_s) \frac{\partial^2 f(X_s)}{\partial x_i \partial x_k} ds \\ &= \int_0^t (\mathcal{A} f)(X_s) ds + \sum_{i=1}^{d} \sum_{j=1}^{r} \int_0^t \sigma_{ij}(X_s) \frac{\partial f(X_s)}{\partial x_i} dW_s^{(j)}. \end{aligned}$$

Taking expectation on both sides of the above equation and we obtain

$$E^x f(X_t) - f(x) = E \int_0^t (\mathcal{A} f)(X_s) ds,$$

which implies

$$\lim_{t \downarrow 0} \frac{1}{t} [E^x f(X_t) - f(x)] = (\mathcal{A} f)(x),$$

since $f \in C^2(\mathbb{R}^d)$ implies $\mathcal{A} f \in C(\mathbb{R}^d)$ and X has a continous path.

If we let $f(y) = y_i$, then $(\mathcal{A} f)(y) = b_i(y)$, and we obtain

$$\lim_{t \downarrow 0} \frac{1}{t} [E^x X_t^{(i)} - x_i] = b_i(x),$$

which is equivalent to the first equation in (3).

If we let $f(y) = (y_i - x_i)(y_k - x_k)$, then $(\mathcal{A}f)(y) = a_{ik}(y) + b_i(x)(y_k - x_k) + b_k(x)(y_i - x_i)$, and we obtain

$$\lim_{t \downarrow 0} \frac{1}{t} E^x[(X_t^{(i)} - x_i)(X_t^{(k)} - x_k)] = a_{ik}(x),$$

which is equivalent to the second equation in (3). ■

3. p. 360: 5.6.15 Problem

We have an equation

$$dX_t = [A(t) X_t + a(t)] dt + \sum_{j=1}^{r} [S_j(t) X_t + \sigma_j(t)] dW_t^{(j)}, \tag{13}$$

where $W = \left\{ W_t = \left(W_t^{(1)}, \ldots, W_t^{(r)} \right), \mathcal{F}_t; 0 \leq t < \infty \right\}$ is an r-dimensional Brownian motion, and the coefficients A, a, S_j, σ_j are measurable, $\{\mathcal{F}_t\}$ −adapted, almost surely locally bounded processes. Show that the unique solution of this equation is

$$X_t = Z_t \left[X_0 + \int_0^t \frac{1}{Z_u} \{a(u) - \sum_{j=1}^{r} S_j(u) \sigma_j(u)\} du + \sum_{j=1}^{r} \int_0^t \frac{\sigma_j(u)}{Z_u} dW_u^{(j)} \right], \tag{14}$$

where we set

$$\varsigma_t \triangleq \sum_{j=1}^{r} \int_0^t S_j(u) dW_u^{(j)} - \frac{1}{2} \sum_{j=1}^{r} \int_0^t S_j^2(u) du,$$

$$Z_t \triangleq \exp\left[\int_0^t A(u) du + \varsigma_t \right].$$

In particular, the solution of the equation

$$dX_t = A(t) X_t dt + \sum_{j=1}^{r} S_j(t) X_t dW_t^{(j)}$$

is given by

$$X_t = X_0 \exp\left[\int_0^t \{A(u) - \frac{1}{2} \sum_{j=1}^{r} S_j^2(u)\} du + \sum_{j=1}^{r} \int_0^t S_j(u) dW_u^{(j)} \right]. \tag{15}$$

In the case of constant coefficients $A(t) \equiv A, S_j(t) \equiv S_j$ with $2A < \sum_{j=1}^{r} S_j^2$ in (15),show that $\lim_{t\to\infty} X_t = 0$ $a.s.$, for arbitrary initial condition X_0.

Proof. 1)

Since

$$|(A(t)x - a(t)) - (A(t)y - a(t))|^2 + \sum_{j=1}^{r} |(S_j(t)x + \sigma_j(t)) - (S_j(t)y + \sigma_j(t))|^2 \leq \left(|A(t)|^2 + \sum_{j=1}^{r} |S_j(t)|^2 \right) |x-y|^2,$$

combining with Theory (5.2.5) [[1] P287], the solution is unique.

Now it is sufficient to show that (14) satisfies (13).

Let

$$Z_t \triangleq \exp\left\{ \int_0^t A(u)du + \sum_{j=1}^{r} \int_0^t S_j(u)dW_u^{(j)} - \frac{1}{2}\sum_{j=1}^{r} \int_0^t S_j^2(u)du \right\}.$$

According to the Itô's rule,

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle M\rangle_t\,,$$

it is easy to get that

$$dZ_t = Z_t\left[A(t)dt + S_j(t)dW_t^{(j)}\right] - \frac{1}{2}\sum_{j=1}^{r} S_j^2(t)dt + Z_t\left[\frac{1}{2}\sum_{j=1}^{r} S_j^2(t)dt\right] = Z_t\left[A(t)dt + S_j(t)dW_t^{(j)}\right].$$

.

Then (13) can be simplified,

$$dX_t = (A(t)X_t + a(t))\,dt + \sum_{j=1}^{r} (S_j(t)X_t + \sigma_j(t))\,dW_t^{(j)} = \frac{X_t}{Z_t}dZ_t + a(t)dt + \sum_{j=1}^{r} \sigma_j(t)dW_t^{(j)}. \tag{16}$$

Apply Itô's rule and (16) to (14),

$$
\begin{aligned}
& d\left\{Z_t\left[X_0+\int_0^t \frac{1}{Z_u}\{a(u)-\sum_{j=1}^r S_j(u)\sigma_j(u)\}du+\int_0^t \frac{1}{Z_u}\sigma_j(u)dW_u^{(j)}\right]\right\} \\
= \quad & X_0 dZ_t+\left\{\left[a(t)-\sum_{j=1}^r S_j(t)\sigma_j(t)\right]dt+\left[\int_0^t \frac{1}{Z_u}\{a(u)-\sum_{j=1}^r S_j(u)\sigma_j(u)\}du\right]dZ_t\right\}+ \\
& \left\{\sum_{j=1}^r \sigma_j(t)dW_t^{(j)}+\left[\int_0^t \frac{1}{Z_u}\sigma_j(u)dW_u^{(j)}\right]dZ_t+\sum_{j=1}^r S_j(t)\sigma_j(t)dt\right\} \\
= \quad & \left\{X_0+\left[\int_0^t \frac{1}{Z_u}\{a(u)-\sum_{j=1}^r S_j(u)\sigma_j(u)\}du\right]+\left[\int_0^t \frac{1}{Z_u}\sigma_j(u)dW_u^{(j)}\right]\right\}dZ_t+ \\
& \left[a(t)-\sum_{j=1}^r S_j(t)\sigma_j(t)\right]dt+\sum_{j=1}^r \sigma_j(t)dW_t^{(j)}+\sum_{j=1}^r S_j(t)\sigma_j(t)dt \\
= \quad & \frac{X_t}{Z_t}dZ_t+a(t)dt+\sum_{j=1}^r \sigma_j(t)dW_t^{(j)} \\
= \quad & dX_t
\end{aligned}
$$

So (14) is the unique solution of (13).

2)

In the case of constant coefficients,

$$A(t)\equiv A, S_j(t)\equiv S_j, 2A<\sum_{j=1}^r S_j^2,$$

in (15),

$$X_t=X_0\exp\left\{\left(A-\frac{1}{2}\sum_{j=1}^r S_j^2\right)t+\sum_{j=1}^r\int_0^t S_j dW_u^{(j)}\right\}=X_0\exp\left\{\left(A-\frac{1}{2}\sum_{j=1}^r S_j^2\right)t+\sum_{j=1}^r S_j W_t^{(j)}\right\}.$$

Since $\sum_{j=1}^r S_j W_t^{(j)}$ is Brownian motion, applying the strong law of large numbers [[1] P104 Problem 9.3], we can get

$$\lim_{t\to\infty}\frac{W_t}{t}=0. \tag{17}$$

We can also uses the Law of the Iterated Logarithm [[1] P112 Theorem 9.23]

$$\limsup_{t\to\infty}\frac{W_t(w)}{\sqrt{2t\log\log t}}=1, \liminf_{t\to\infty}\frac{W_t(w)}{\sqrt{2t\log\log t}}=-1 \tag{18}$$

According to (17) or (18),

$$\lim_{t\to\infty}\frac{1}{t}\sum_{j=1}^{r}S_jW_t^{(j)}=0.$$

So

$$\begin{aligned}
&\lim_{t\to\infty}X_0\exp\left\{\left(A-\frac{1}{2}\sum_{j=1}^{r}S_j^2\right)t+\sum_{j=1}^{r}\int_0^t S_j dW_u^{(j)}\right\}\\
=&\lim_{t\to\infty}X_0\exp\left\{\left(A-\frac{1}{2}\sum_{j=1}^{r}S_j^2\right)+\frac{1}{t}\sum_{j=1}^{r}S_jW_t^{(j)}\right\}t=0
\end{aligned}$$

■

§2 Selected Questions from Shreve [2]

4. p.291: Exercise 6.8

(Kolmogorov backward equation). Considr the stochastic differential equation

$$dX(u)=\beta(u,X(u))\,du+\gamma(u,X(u))\,dW(u).$$

We assume that, just as with a geometric Brownian motion, if we begin a process at an arbitrary initial positive value $X(t)=x$ at an arbitrary initial time t and evolve it forward using this equation, its value at each time $T>t$ could be any positive number but cannot be less than or equal to zero. For $0\le t\le T$, let $p(t,T,x,y)$ be the transition density for the solution to this equation (i.e., if we solve the equation with the initial condition $X(t)=x$,then the random variable $X(T)$ has density $p(t,T,x,y)$ in the y variable).We are assuming that $p(t,T,x,y)=0$ for $0\le t\le T$ and $y\le 0$.

Show that $p(t,T;x,y)$ satisfies the *Kolmogorov backward equation*

$$-p_t(t,T,x,y)=\beta(t,x)\,p_x(t,T,x,y)+\frac{1}{2}\gamma^2(t,x)\,p_{xx}(t,T,x,y).$$

Proof. We know from the Feynman-Kac Theorem that for any borel measurable function $h(y)$,

$$g(t,x)=E^{t,x}h(X(T))=\int_0^\infty h(y)p(t,T,x,y)dy$$

satisfies

$$g_t(t,x)+\beta(t,x)g_x(t,x)+\frac{1}{2}\gamma^2(t,x)g_{xx}(t,x)=0.$$

So

$$g_t(t,x)=\frac{\partial}{\partial t}\int_0^\infty h(y)p(t,T,x,y)dy=\int_0^\infty h(y)p_t(t,T,x,y)dy.$$

$$g_x(t,x)=\frac{\partial}{\partial x}\int_0^\infty h(y)p(t,T,x,y)dy=\int_0^\infty h(y)p_x(t,T,x,y)dy.$$

$$g_{xx}(t,x)=\int_0^\infty h(y)p_{xx}(t,T,x,y)dy.$$

By these three equations, we can get

$$\int_0^\infty h(y)p_t(t,T,x,y)dy+\beta(t,x)\int_0^\infty h(y)p_x(t,T,x,y)dy+\frac{1}{2}\gamma^2(t,x)\int_0^\infty h(y)p_{xx}(t,T,x,y)dy=0.$$

Because the integration varible is y , $\beta(t,x)$ and $\gamma^2(t,x)$ is independent of it.

Thus

$$\int_0^\infty h(y)\left(p_t(t,T,x,y)dy+\beta(t,x)p_x(t,T,x,y)dy+\frac{1}{2}\gamma^2(t,x)p_{xx}(t,T,x,y)\right)dy=0.$$

Since p is the transition density of $X(t)$,it is independent of $h(y)$.

So for every $h(y)$, the equation holds. Now it is easy to see

$$p_t(t,T,x,y)dy+\beta(t,x)p_x(t,T,x,y)dy+\frac{1}{2}\gamma^2(t,x)p_{xx}(t,T,x,y)=0. \quad a.e.$$

(In fact, we can choose $h(y)=p_t(t,T,x,y)dy+\beta(t,x)p_x(t,T,x,y)dy+\frac{1}{2}\gamma^2(t,x)p_{xx}(t,T,x,y)$ for some every t,T,x.Then it is easy to see.) ■

5. p.291: Exercise 6.9

(Kolmogorov forward equation). (Also called the *Fokker-Planck equation*). We begin with the same stochastic differential equation,

$$dX(u) = \beta(u, X(u))\,du + \gamma(u, X(u))\,dW(u),$$

as in Exercise 6.8, use the same notation $p(t,T,x,y)$ for the transition density, and again assume that $p(t,T,x,y) = 0$ for $0 \le t < T$ and $y \le 0$. In this problem, we show that $p(t,T,x,y)$ satisfies the *Kolmogorov forward equation*

$$\frac{\partial}{\partial T} p(t,T,x,y) = -\frac{\partial}{\partial y}\left(\beta(t,y)\,p(t,T,x,y)\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\gamma^2(T,y)\,p(t,T,x,y)\right).$$

In contrast to the Kolmogorov backward equation, in which T and y were held constant and the variables were t and x, here t and x are held constant and the variables are y and T. The variables t and x are sometimes called the *backward variables, and* T and y are called the *forward variables.*

(i)Let b be a positive constant and let $h_b(y)$ be a function with continuous first and second derivatives such that $h_b(x) = 0$ for all $x \le 0$, $h_b'(x) = 0$ for all $x \ge b$, and $h_b(b) = h_b'(b) = 0$. Let $X(u)$ be the solution to the stochastic differential equation with initial condition $X(t) = x \in (0,b)$, and use the Itô's formula to compute $dh_b(X(u))$.

(ii)Let $0 \le t \le T$ be given, and integrate the equation you obtained in (i) from t to T. Take expectations and use the fact that $X(u)$ has density $p(t,u,x,y)$ in the $y-variable$ to obtain

$$\int_0^b h_b(y)\,p(t,T,x,y)\,dy = h_b(x) + \int_t^T \int_0^b \beta(u,y)\,p(t,u,x,y)\,h_b'(y)\,dy\,du + \frac{1}{2}\int_t^T \int_0^b \gamma^2(u,y)\,p(t,u,x,y)\,h_b''(y)\,dy. \tag{19}$$

(iii)Integrate the integrals $\int_0^b \cdots dy$ on the right-hand side of (19) by parts to abtain

$$\begin{aligned} &\int_0^b h_b(y)\,p(t,T,x,y)\,dy \\ = \;& h_b(x) - \int_t^T \int_0^b \frac{\partial}{\partial y}\left[\beta(u,y)\,p(t,u,x,y)\right] h_b(y)\,dy\,du + \frac{1}{2}\int_t^T \int_0^b \frac{\partial^2}{\partial y^2}\left[\gamma^2(u,y)\,p(t,u,x,y)\right] h_b(y)\,dydu \end{aligned} \tag{20}$$

(iv)Differentiate (20) with respect to T to obtain

$$\int_0^b h_b(y)\left[\frac{\partial}{\partial T} p(t,T,x,y) + \frac{\partial}{\partial y}\left(\beta(T,y)\,p(t,T,x,y)\right) - \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\gamma^2(T,y)\,p(t,T,x,y)\right)\right] dy = 0 \tag{21}$$

(v)Use (21) to show that there cannot be numbers $0 < y_1 < y_2$ such that

$$\frac{\partial}{\partial T}p(t,T,x,y) + \frac{\partial}{\partial y}\left(\beta(T,y)\,p(t,T,x,y)\right) - \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\gamma^2(T,y)\,p(t,T,x,y)\right) > 0 \text{ for all } y \in (y_1, y_2).$$

Similarly, there cannot be numbers $0 < y_1 < y_2$ such that

$$\frac{\partial}{\partial T}p(t,T,x,y) + \frac{\partial}{\partial y}\left(\beta(T,y)\,p(t,T,x,y)\right) - \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\gamma^2(T,y)\,p(t,T,x,y)\right) < 0 \text{ for all } y \in [y_1, y_2].$$

This is as much as you need to do for this problem. It is now obvious that if

$$\frac{\partial}{\partial T}p(t,T,x,y) + \frac{\partial}{\partial y}\left(\beta(T,y)\,p(t,T,x,y)\right) - \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\gamma^2(T,y)\,p(t,T,x,y)\right)$$

is a continuous function of y, then this expression must be zero for every $y > 0$, and hence $p(t,T,x,y)$ satisfies the Kolmogorov forward equation stated at the beginning of this problem.

Proof. (i)

We can easily find that

$$\begin{aligned}
h_b(x) = 0 \quad & x \leq 0 \ or\ x \geq b, \\
h_b'(x) = 0 \quad & x \leq 0 \ or\ x \geq b, \\
h_b''(x) = 0 \quad & x \leq 0 \ or\ x \geq b.
\end{aligned}$$

With Itô's formula,

$$\begin{aligned}
dh_b(X_u) &= h_b'(X_u)dX_u + \frac{1}{2}h_b''(X_u)dX_u dX_u \\
&= h_b'(X_u)\beta(u,X_u)du + h_b'(X_u)\gamma(u,X_u)dW_u + \frac{1}{2}h_b''(X_u)\gamma^2(u,X_u)du.
\end{aligned}$$

(ii)

Integral both side of the equation above from t to T, we have

$$h_b(X_T) - h_b(X_t) = \int_t^T h_b'(X_u)\beta(u,X_u)du + \int_t^T h_b'(X_u)\gamma(u,X_u)dW_u + \int_t^T \frac{1}{2}h_b''(X_u)\gamma^2(u,X_u)du.$$

Take expectation of both sides conditioned on t,x,

$$\begin{aligned} E^{t,x}h_b(X_T) - E^{t,x}h_b(X_t) &= E^{t,x}\int_t^T h_b'(X_u)\beta(u,X_u)du + E^{t,x}\int_t^T h_b'(X_u)\gamma(u,X_u)dW_u + \\ & E^{t,x}\int_t^T \frac{1}{2}h_b''(X_u)\gamma^2(u,X_u)du. \end{aligned}$$

The second term of the right hand side is an Itô integral, and $E^{t,x}h_b(X_t) = E[h_b(X_t) \mid X_t = x] = h_b(x)$.

$$E^{t,x}h_b(X_T) = h_b(x) + \int_t^T E^{t,x}h_b'(X_u)\beta(u,X_u)du + \int_t^T E^{t,x}\frac{1}{2}h_b''(X_u)\gamma^2(u,X_u)du.$$

$$\int_0^\infty h_b(y)p(t,T,x,y)dy = h_b(x)+\int_0^\infty\int_t^T h_b'(y)\beta(u,y)p(t,u,x,y)dudy+\frac{1}{2}\int_0^\infty\int_t^T h_b''(y)\gamma^2(u,y)p(t,u,x,y)dudy.$$

According to (22), we can ristrict the range of integration.

$$\int_0^b h_b(y)p(t,T,x,y)dy = h_b(x)+\int_t^T\int_0^b h_b'(y)\beta(u,y)p(t,u,x,y)dydu+\frac{1}{2}\int_t^T\int_0^b h_b''(y)\gamma^2(u,y)p(t,u,x,y)dydu.$$

(iii)

Thus

$$\frac{\partial}{\partial y}(h_b(y)\beta(u,y)p(t,u,x,y)) = h_b'(y)\beta(u,y)p(t,u,x,y) + h_b(y)\frac{\partial}{\partial y}\left(\beta(u,y)p(t,u,x,y)\right).$$

Integral both sides of the equation with respect to y from 0 to b,

$$h_b(y)\beta(u,y)p(t,u,x,y)\ |_0^b = \int_0^b h_b'(y)\beta(u,y)p(t,u,x,y)dy + \int_0^b h_b(y)\frac{\partial}{\partial y}\left(\beta(u,y)p(t,u,x,y)\right)dy.$$

Remend that $h_b(0) = h_b(b) = 0$,

$$\int_0^b h_b'(y)\beta(u,y)p(t,u,x,y)dy = -\int_0^b h_b(y)\frac{\partial}{\partial y}\left(\beta(u,y)p(t,u,x,y)\right)dy.$$

It is quiet same as above by do twice of partial differentiate and then integral

$$\int_0^b h_b''(y)\gamma^2(u,y)p(t,u,x,y)dy = \int_0^b h_b(y)\frac{\partial^2}{\partial y^2}(\gamma^2(u,y)p(t,u,x,y))dy.$$

Thus

$$\begin{aligned} \int_0^b h_b(y)p(t,T,x,y)dy &= h_b(x) - \int_t^T\int_0^b h_b(y)\frac{\partial}{\partial y}\left(\beta(u,y)p(t,u,x,y)\right)dydu \\ & + \int_t^T\int_0^b h_b(y)\frac{\partial^2}{\partial y^2}(\gamma^2(u,y)p(t,u,x,y))dydu. \end{aligned}$$

(iv)

Differentiate with respect to T,

$$\frac{\partial}{\partial T}\int_0^b h_b(y)p(t,T,x,y)dy = -\int_0^b h_b(y)\frac{\partial}{\partial y}\left(\beta(u,y)p(t,u,x,y)\right)dy + \int_0^b h_b(y)\frac{\partial^2}{\partial y^2}(\gamma^2(u,y)p(t,u,x,y))dy.$$

$$\int_0^b h_b(y)\left(\frac{\partial}{\partial T}p(t,T,x,y) + \frac{\partial}{\partial y}\left(\beta(u,y)p(t,u,x,y)\right) - \frac{\partial^2}{\partial y^2}(\gamma^2(u,y)p(t,u,x,y))\right)dy = 0. \tag{23}$$

For every t, T, x, if there exists a positive measurable set $A \in [0,b]$ that the () term is positive or negative, we can choose $h_b(y)$ to be a little positive in A and to be 0 in $[0,b]\backslash A$, assume () term is good enough then we can assure $h_b(y)$ to be continuous. Then, at this situation, (23) doesn't hold.

So

$$\frac{\partial}{\partial T}p(t,T,x,y) + \frac{\partial}{\partial y}\left(\beta(u,y)p(t,u,x,y)\right) - \frac{\partial^2}{\partial y^2}(\gamma^2(u,y)p(t,u,x,y)) = 0. \quad a.e.$$

■

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