两个特殊极限

本段内容要点:

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x}=1}$$

$$\boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e}$$

二者的若干变型及应用

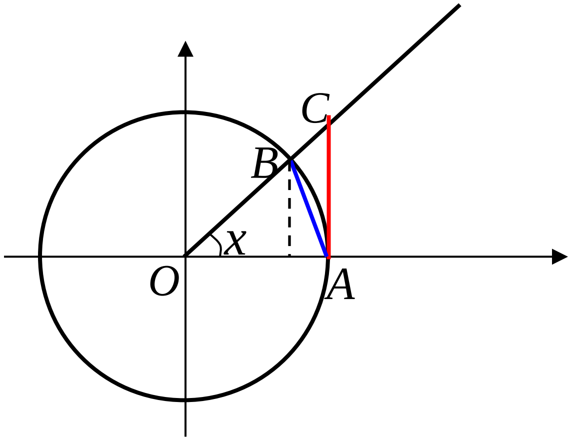
1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$

1. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$

证明: 因为$\dfrac{\sin x}{x}$是偶函数, 所以左右单侧极限要么都存在并相等, 要么都不存在, 因而只须对$x\to 0+0$考虑.

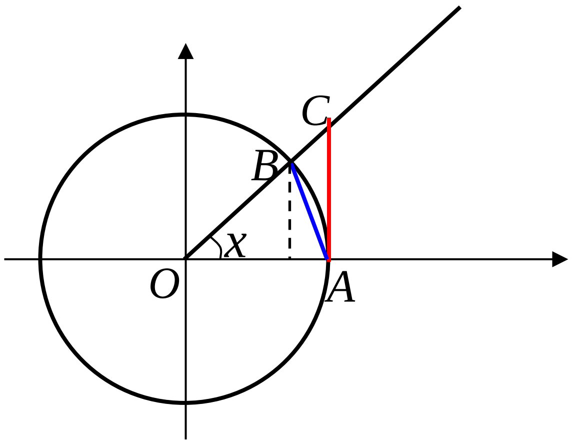
又因为极限过程是$x\to 0+0$, 所以只考虑$x\in(0,\dfrac{\pi}{2})$.

1. $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$



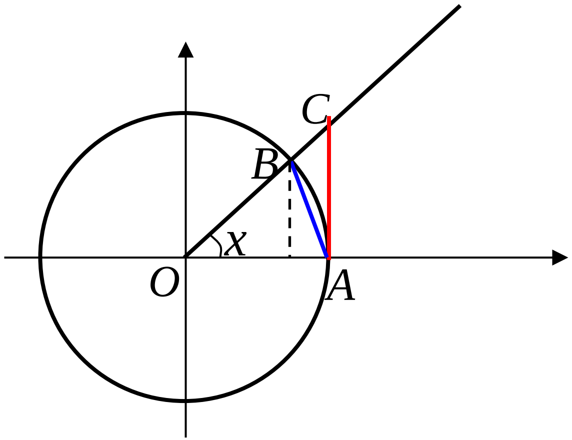
1. $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

$$S_{\triangle OAB} \leqslant S_{AOB} \leqslant S_{\triangle AOC}$$



1. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$

$$
\begin{array}{ccccc}
S_{\triangle OAB} & \leqslant & S_{AOB} & \leqslant & S_{\triangle AOC} \\
\| & & \| & & \| \\
\dfrac{1}{2}\sin x & & \dfrac{1}{2}x & & \dfrac{1}{2}\tan x
\end{array}
$$



$$1.\ \lim_{x\to 0}\frac{\sin x}{x}=1$$

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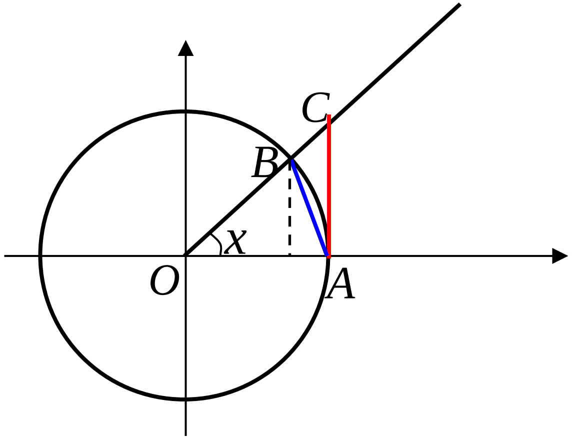
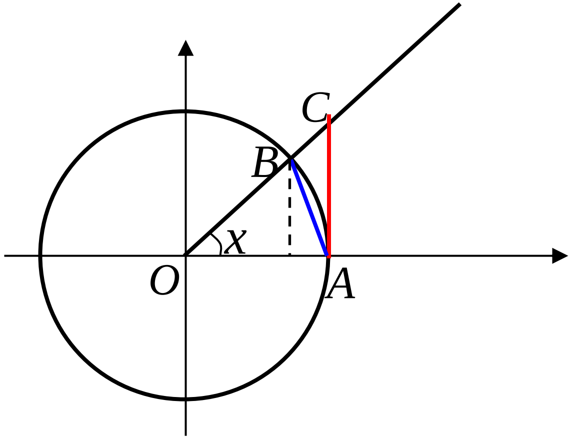


$$\Rightarrow\ \sin x \leqslant x \leqslant \tan x,\ x\in\left(0,\frac{\pi}{2}\right)$$

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$



$$
\begin{array}{ccccc}
S_{\triangle OAB} & \leqslant & S_{AOB} & \leqslant & S_{\triangle AOC} \\
\| & & \| & & \| \\
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\end{array}
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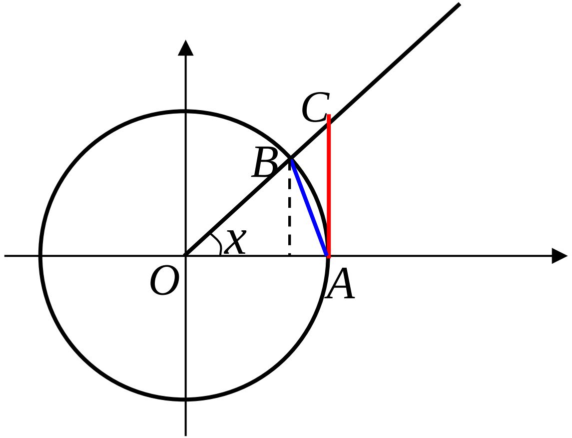
$$\Rightarrow \ \sin x \leqslant x \leqslant \tan x, \ x \in \left(0, \frac{\pi}{2}\right)$$

从而得到: $1 \leqslant \dfrac{x}{\sin x} \leqslant \dfrac{1}{\cos x}$

$$1.\ \lim_{x\to 0}\frac{\sin x}{x}=1$$



$$\begin{array}{ccccc} S_{\triangle OAB} & \leqslant & S_{AOB} & \leqslant & S_{\triangle AOC} \\ \| & & \| & & \| \\ \frac{1}{2}\sin x & & \frac{1}{2}x & & \frac{1}{2}\tan x \end{array}$$

$$\Rightarrow\ \sin x \leqslant x \leqslant \tan x,\ x\in\left(0,\frac{\pi}{2}\right)$$

从而得到: $1 \leqslant \dfrac{x}{\sin x} \leqslant \dfrac{1}{\cos x}$

$$\Rightarrow\ \cos x \leqslant \frac{\sin x}{x} \leqslant 1,\ x\in\left(0,\frac{\pi}{2}\right)$$

令$x \to 0+0$,则根据夹逼原理, $\lim\limits_{x\to 0+0}\dfrac{\sin x}{x}=1$.

$$\boxed{1.\ \lim_{x\to 0}\frac{\sin x}{x}=1}$$

例: $\lim_{x\to 0}\frac{\sin 5x}{x}$

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$$=\lim_{x\to 0}\frac{\sin 5x}{5x}5=5.$$

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例: $\lim\limits_{x\to\infty} x\sin\frac{1}{x}$

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$$=\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}=\lim_{t\to 0}\frac{\sin t}{t}=1.$$

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$$=\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}=\lim_{t\to 0}\frac{\sin t}{t}=1.$$

例: $\lim\limits_{x\to 0}\dfrac{\tan x}{x}=\lim\limits_{x\to 0}\dfrac{\sin x}{x}\dfrac{1}{\cos x}=\lim\limits_{x\to 0}\dfrac{\sin x}{x}\cdot\lim\limits_{x\to 0}\dfrac{1}{\cos x}=1.$

$$\boxed{1.\ \lim_{x \to 0} \frac{\sin x}{x} = 1}$$

例: $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$.

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例: $\lim_{x\to 0}\frac{1-\cos x}{x^2}$

$$=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}$$

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例: $\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

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$$=\lim_{x\to 0}\frac{1}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=\frac{1}{2}.$$

$$\boxed{1.\ \lim_{x\to 0}\frac{\sin x}{x}=1}$$

推广:

$$\boxed{\text{若}\lim_{x} u(x)=0,\ \text{则}\lim_{x}\frac{\sin u(x)}{u(x)}=1.}$$

例: $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$

$$=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}$$

$$=\lim_{x\to 0}\frac{1}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=\frac{1}{2}.$$

2. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e$

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证明:

Step1: 往证 $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e$;

Step2: 往证 $\lim\limits_{x\to+\infty}\left(1+\frac{1}{[x]}\right)^{[x]}=e$;

Step3: 往证 $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x}=e$;

Step4: 往证 $\lim\limits_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x}=e$

2. $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$

证明: Step1: 数列极限部分已证.

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$$2.\ \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e$$

证明: Step2: 往证 $f(x)=\left(1+\frac{1}{[x]}\right)^{[x]}\ (x>1)$

在 $x\to+\infty$ 时与数列 $\left(1+\frac{1}{n}\right)^{n}$ 同极限.

Step1: 往证 $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e$;

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在 $x\to+\infty$ 时与数列 $\left(1+\frac{1}{n}\right)^n$ 同极限.

因为 $\lim\limits_{n\to+\infty}\left(1+\frac{1}{n}\right)^n=e$, 所以对于任意的 $\varepsilon>0$,

$\exists N\ s.t.\ n>N\Rightarrow\left|\left(1+\frac{1}{n}\right)^n-e\right|<\varepsilon$.

从而对于任意的 ε, $\exists X=N+1\ s.t.$

$$x>X\Rightarrow\left|\left(1+\frac{1}{[x]}\right)^{[x]}-e\right|<\varepsilon.$$

$$2.\ \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e$$

Step1: 往证 $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{n}=e$;

Step2: 往证 $\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{[x]}\right)^{[x]}=e$;

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证明: Step3: 往证 $\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{x}\right)^x = e$.

Step1: 往证 $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = e$;

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2. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x}=e$

证明: Step3: 往证 $\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{x}\right)^{x}=e.$

事实上,

$$\left(1+\frac{1}{[x]+1}\right)^{[x]}\leqslant\left(1+\frac{1}{x}\right)^{x}\leqslant\left(1+\frac{1}{[x]}\right)^{[x]+1}$$

Step1: 往证 $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{n}=e$;

Step2: 往证 $\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{[x]}\right)^{[x]}=e$;

Step3: 往证 $\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{x}\right)^{x}=e$;

Step4: 往证 $\lim\limits_{x\to-\infty}\left(1+\dfrac{1}{x}\right)^{x}=e$

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Step3: 往证 $\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{x}\right)^{x}=e$;

Step4: 往证 $\lim\limits_{x\to-\infty}\left(1+\dfrac{1}{x}\right)^{x}=e$

证明: Step3: 往证 $\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{x}\right)^{x}=e$.

事实上,

$$\left(1+\frac{1}{[x]+1}\right)^{[x]}\leqslant\left(1+\frac{1}{x}\right)^{x}\leqslant\left(1+\frac{1}{[x]}\right)^{[x]+1}$$

而 $\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{[x]+1}\right)^{[x]}$

$$=\lim_{x\to+\infty}\left(1+\frac{1}{[x]+1}\right)^{[x]+1}\Big/\left(1+\frac{1}{[x]+1}\right)=e$$

$$2.\ \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e$$

证明: Step3: 往证 $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x}=e.$

事实上,

$$\left(1+\frac{1}{[x]+1}\right)^{[x]}\leqslant\left(1+\frac{1}{x}\right)^{x}\leqslant\left(1+\frac{1}{[x]}\right)^{[x]+1}$$

$$\lim_{x\to+\infty}\left(1+\frac{1}{[x]}\right)^{[x]+1}$$

$$=\lim_{x\to+\infty}\left(1+\frac{1}{[x]}\right)^{[x]}\cdot\left(1+\frac{1}{[x]}\right)=e\cdot 1=e.$$

Step1: 往证 $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e;$

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证明: Step3: 往证 $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x}=e.$

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$$\left(1+\frac{1}{[x]+1}\right)^{[x]}\leqslant\left(1+\frac{1}{x}\right)^{x}\leqslant\left(1+\frac{1}{[x]}\right)^{[x]+1}$$

根据夹逼定理, $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x}=e$

Step1: 往证 $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e$;

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2. $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e$

证明: Step3: 往证 $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x}=e$.

事实上,

$$\left(1+\frac{1}{[x]+1}\right)^{[x]}\leqslant\left(1+\frac{1}{x}\right)^{x}\leqslant\left(1+\frac{1}{[x]}\right)^{[x]+1}$$

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$$2.\ \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

证明:

$$\text{Step4: } \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x$$

$$\text{Step1: 往证} \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e;$$

$$\text{Step2: 往证} \lim_{x\to+\infty}\left(1+\frac{1}{[x]}\right)^{[x]} = e;$$

$$\text{Step3: 往证} \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e;$$

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$$2.\ \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

证明:

Step4: $\lim\limits_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = \lim\limits_{t\to+\infty}\left(1-\frac{1}{t}\right)^{-t}$

Step1: 往证 $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$;

Step2: 往证 $\lim\limits_{x\to+\infty}\left(1+\frac{1}{[x]}\right)^{[x]} = e$;

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证明:

Step4: $\displaystyle\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x=\lim_{t\to+\infty}\left(1-\frac{1}{t}\right)^{-t}$

$$\text{而}\lim_{t\to+\infty}\left(\frac{t}{t-1}\right)^t=\lim_{t\to+\infty}\left(1+\frac{1}{t-1}\right)^t$$

$$=\lim_{t\to+\infty}\left(1+\frac{1}{t-1}\right)^{t-1}\cdot\left(1+\frac{1}{t-1}\right)=e\cdot 1=e.$$

Step1: 往证 $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$;

Step2: 往证 $\displaystyle\lim_{x\to+\infty}\left(1+\frac{1}{[x]}\right)^{[x]}=e$;

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证明:

Step4: $\lim\limits_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x}=\lim\limits_{t\to+\infty}\left(1-\frac{1}{t}\right)^{-t}$

而 $\lim\limits_{t\to+\infty}\left(\frac{t}{t-1}\right)^{t}=\lim\limits_{t\to+\infty}\left(1+\frac{1}{t-1}\right)^{t}$

$$=\lim_{t\to+\infty}\left(1+\frac{1}{t-1}\right)^{t-1}\cdot\left(1+\frac{1}{t-1}\right)=e\cdot 1=e.$$

Step1: 往证 $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e$;

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例: $\lim_{x\to\infty}\left(1+\frac{\boldsymbol{k}}{\boldsymbol{x}}\right)^{x}=\boldsymbol{e}^{k}$. 其中$\boldsymbol{k}\in\mathbb{R}$

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例: $\lim\limits_{x\to\infty}\left(1+\dfrac{\boldsymbol{a}}{\boldsymbol{x}}\right)^{bx}=\boldsymbol{e}^{ab},\ \forall\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}$.

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例: $\lim\limits_{x\to\infty}\left(1+\dfrac{a}{x}\right)^{bx}=e^{ab},\ \forall a,b\in\mathbb{R}$.

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^{x}=e^{-1},\ \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{-x}=e^{-1}$$

(推广) (1) 若$\lim\limits_{x} u(x) = +\infty \ or \ -\infty$,

$$\text{则}\lim_{x}\left(1+\frac{1}{u(x)}\right)^{u(x)} = e;$$

(推广) (1) 若$\lim\limits_{x} u(x) = +\infty\ or\ -\infty$,

$$\text{则}\lim_{x}\left(1+\frac{1}{u(x)}\right)^{u(x)} = e;$$

(2) 若$u(x)$是某极限过程中的无穷大量, 则在此极限过程中, $\lim\limits_{x}\left(1+\dfrac{1}{u(x)}\right)^{u(x)} = e$.

例:

$$\lim_{x \to 0}(1+x)^{\frac{1}{x}} = e;$$

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$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e;$$

若$\lim\limits_{x} u(x)=0$, 则$\lim\limits_{x}(1+u(x))^{\frac{1}{u(x)}}=e$.

例:

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e;$$

$$若\lim_{x} u(x) = 0,\ 则\lim_{x}(1+u(x))^{\frac{1}{u(x)}} = e.$$

$$例如\lim_{x\to 0}(1+\sin x)^{\frac{1}{\sin x}} = e$$

例: $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x+10} =$

例: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x+10}=$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\left(1+\frac{1}{x}\right)^{10}=e\cdot 1=e.$$

例: $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{n^2}$

事实上, 因为 $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{n}=e$.

所以$\exists N\ s.t.\ n>N \Rightarrow \left(1+\dfrac{1}{n}\right)^{n}>2$.

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从而 $n>N\Rightarrow\left(1+\frac{1}{n}\right)^{n^2}>2^n,$

而 $\lim\limits_{n\to\infty}2^n=+\infty,$

所以 $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n^2}=+\infty$

例: $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n^2}$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n^2}=\lim_{n\to\infty}e^n=+\infty$$

例: $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{n^2}$

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n^2} = \lim_{n\to\infty} e^n = +\infty}$$

例: $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n^2}$

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n^2} \neq \lim_{n\to\infty} e^n = +\infty}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = \lim_{n\to\infty} 1^n = 1 \qquad ?$$

例: $\lim\limits_{x \to 0} \dfrac{\sin^2 \alpha x - \sin^2 \beta x}{x \sin x}$

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原式$= \lim\limits_{x \to 0} \dfrac{\sin^2 \alpha x}{x \sin x} - \lim\limits_{x \to 0} \dfrac{\sin^2 \beta x}{x \sin x} =$

例: $\lim\limits_{x\to 0}\dfrac{\sin^2\alpha x-\sin^2\beta x}{x\sin x}$

原式$=\lim\limits_{x\to 0}\dfrac{\sin^2\alpha x}{x\sin x}-\lim\limits_{x\to 0}\dfrac{\sin^2\beta x}{x\sin x}=$

$$\lim_{x\to 0}\frac{\sin\alpha x}{x}\lim_{x\to 0}\frac{\sin\alpha x}{\sin x}-\lim_{x\to 0}\frac{\sin\beta x}{x}\lim_{x\to 0}\frac{\sin\beta x}{\sin x}$$

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$$原式=\lim_{x\to 0}\frac{\sin^2\alpha x}{x\sin x}-\lim_{x\to 0}\frac{\sin^2\beta x}{x\sin x}=$$

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$$=\alpha^2-\beta^2$$

例: $\lim\limits_{x\to0}\dfrac{\cos x-\cos 3x}{x^2}$

例: $\lim\limits_{x \to 0} \dfrac{\cos x - \cos 3x}{x^2}$

$$= \lim_{x \to 0} \frac{2 \sin x \sin 2x}{x^2}$$

例: $\lim\limits_{x\to 0}\dfrac{\cos x-\cos 3x}{x^2}$

$$=\lim_{x\to 0}\frac{2\sin x\sin 2x}{x^2}$$

$$=\lim_{x\to 0}\frac{2\sin x}{x}\lim_{x\to 0}\frac{\sin 2x}{x}=4.$$

例: $\lim\limits_{x\to\infty}\left(\dfrac{x^2+1}{x^2-2}\right)^{x^2}$

例: $\lim_{x\to\infty}\left(\frac{x^2+1}{x^2-2}\right)^{x^2} \quad = \lim_{x\to\infty}\left(1+\frac{3}{x^2-2}\right)^{x^2}$

例: $\lim\limits_{x\to\infty}\left(\frac{x^2+1}{x^2-2}\right)^{x^2} = \lim\limits_{x\to\infty}\left(1+\frac{3}{x^2-2}\right)^{x^2}$

$$= \lim_{x\to\infty}\left(1+\frac{3}{x^2-2}\right)^{\frac{x^2-2}{3}\cdot\frac{3x^2}{x^2-2}}$$

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$$= \lim_{x\to\infty}\left[\left(1+\frac{3}{x^2-2}\right)^{\frac{x^2-2}{3}}\right]^{\frac{3x^2}{x^2-2}}$$

例: $\lim_{x\to\infty}\left(\frac{x^2+1}{x^2-2}\right)^{x^2} = \lim_{x\to\infty}\left(1+\frac{3}{x^2-2}\right)^{x^2}$

$$= \lim_{x\to\infty}\left(1+\frac{3}{x^2-2}\right)^{\frac{x^2-2}{3}\cdot\frac{3x^2}{x^2-2}}$$

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$$= \lim_{x\to\infty}\left[\left(1+\frac{3}{x^2-2}\right)^{\frac{x^2-2}{3}}\right]^{\lim\limits_{x\to\infty}\frac{3x^2}{x^2-2}} = e^3.$$

例: $\lim_{x \to \pi} \frac{\sin x}{\boldsymbol{x - \pi}}$

例: $\lim_{x \to \pi} \frac{\sin x}{x - \pi}$

$$= \lim_{t \to 0} \frac{-\sin t}{t} = -1$$

$(t = x - \pi)$

例: $\lim\limits_{x \to 0} \dfrac{\tan 2x}{x}$

例: $\lim_{x\to 0}\dfrac{\tan 2x}{x}$

$$= \lim_{x\to 0}\frac{\sin 2x}{x\cos 2x}$$

例: $\lim\limits_{x\to 0}\dfrac{\tan 2x}{x}$

$$=\lim_{x\to 0}\frac{\sin 2x}{x\cos 2x}$$

$$=\lim_{x\to 0}\frac{\sin 2x}{x}\lim_{x\to 0}\frac{1}{\cos 2x}=2$$

例: $\lim\limits_{x \to 0+0} \dfrac{x}{\sqrt{1-\cos x}}$

例: $\lim\limits_{x\to 0+0}\dfrac{x}{\sqrt{1-\cos x}}$

$$=\lim_{x\to 0+0}\frac{x}{\sqrt{2}\sin\dfrac{x}{2}}=\sqrt{2}.$$

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$$=\lim_{x\to 0+0}\frac{x}{\sqrt{2}\sin\dfrac{x}{2}}=\sqrt{2}.$$

例: $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}=0$

例: $\lim\limits_{x\to\infty} x\sin\frac{1}{3x}$

例: $\lim\limits_{x \to \infty} x \sin \dfrac{1}{3x}$

$$= \lim_{x \to \infty} \frac{\sin \dfrac{1}{3x}}{\dfrac{1}{x}} = \lim_{t \to 0} \frac{\sin t}{3t} = \frac{1}{3}$$

本段知识要点

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$$

注意它们的各种变型