# 偏导数

$$z = f(x, y), (x, y) \in D$$

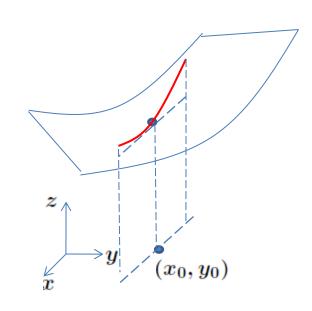
$$\forall (x_0,y_0) \in D, z=f(x,y_0)$$
 是  $y=y_0$  平面上的关于  $x$  的一元函数

即空间曲线 
$$\begin{cases} z = f(x,y) \\ y = y_0 \end{cases}$$

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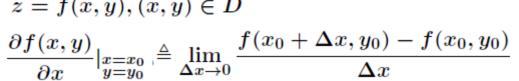


 $df(x, y_0)$ 





$$y) \in L$$



$$\in D$$



$$\in D$$

$$\in D$$

$$\in D$$



$$\in D$$



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$$z = f(x, y), (x, y) \in D$$

$$\in D$$







- $\frac{\partial f(x,y)}{\partial x}\Big|_{\substack{y=x_0\\y=y_0}} \stackrel{\triangle}{=} \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) f(x_0, y_0)}{\Delta x}$

 $z = f(x_0, y)$ 

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$$z=f(x,y),(x,y)\in D$$

$$\in D$$

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 $\partial f(x,y)$ 

 $\frac{df(x,y_0)}{dx}$ 

$$z = f(x, y), (x, y) \in D$$

$$\frac{\partial f(x,y)}{\partial x}\Big|_{\substack{x=x_0\\y=y_0}} \triangleq \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \quad (if \quad \exists)$$

$$\frac{df(x,y_0)}{dx}\Big|_{x=x_0}$$

$$z = f(x_0,y)$$

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$$\frac{\partial f}{\partial x}|_{(x_0,y_0)}, \quad \frac{\partial f(x_0,y_0)}{\partial x},$$





- $f'_x(x_0, y_0), \quad \frac{\partial z}{\partial x}|_{(x_0, y_0)}, \quad z'_x(x_0, y_0)$

$$z = f(x, y), (x, y) \in D$$

$$\frac{\partial f(x,y)}{\partial x}|_{x=x_0} \triangleq \lim_{x \to \infty} \frac{f(x_0 + x_0)}{x}$$

$$\frac{\partial f(x,y)}{\partial x}\Big|_{\substack{x=x_0\\y=y_0}} \triangleq \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \quad (if \quad \exists)$$

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$$f'_x(x_0,y_0), \quad \frac{\partial z}{\partial x}|_{(x_0,y_0)}, \quad z'_x(x_0,y_0)$$

$$\frac{\partial f(x_0,y_0)}{\partial y}$$

$$\frac{(x_0, y)}{dy}|_{y=y_0}$$

$$\frac{(x, y)}{|x=x_0|}|_{x=x_0} \triangleq \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

$$z = f(x, y), (x, y) \in D$$

$$\partial f(x,y)$$
 .

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 $f'_x(x_0, y_0), \quad \frac{\partial z}{\partial x}|_{(x_0, y_0)}, \quad z'_x(x_0, y_0)$ 

$$(y) \in \mathcal{A}$$

$$y) \in D$$

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$$\in D$$

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- $\frac{\partial f(x,y)}{\partial x}\Big|_{\substack{y=x_0\\y=y_0}} \triangleq \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) f(x_0, y_0)}{\Delta x} \qquad (if \quad \exists)$

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 $f'_y(x_0, y_0)$   $f'_y(x_0, y_0)$ ,  $\frac{\partial z}{\partial x}|_{(x_0, y_0)}, z'_x(x_0, y_0)$ 

在一般点 (x,y) 处同样定义

$$\frac{\partial f(x,y)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x,y)}{\Delta x}$$

$$\frac{\partial f(x,y)}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x,y + \Delta y) - f(x,y)}{\Delta y}$$

$$z = x^2 + xy + y^2$$

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则 
$$\frac{\partial z}{\partial x} = 2x + y$$
 ,  $\frac{\partial z}{\partial y} = x + 2y$  ,

$$\frac{\partial z}{\partial x}|_{(1,1)} = 3, \quad \frac{\partial z}{\partial y}|_{(0,1)} = 2.$$

 $z = \arctan \frac{y}{x}$ 

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$$u=\sqrt{x^2+y^2+z^2}$$
 (三元函数)

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$$\frac{\partial u}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \,.$$

注'  $\frac{\partial}{\partial x}$  和  $\frac{\partial}{\partial y}$  分别是一个整体运算符号,

不像一元微商  $\frac{dy}{dx}$  中的 d 是一个独立的运算符号(微分运算) 有时也写成  $\partial_x$ ,  $\partial_y$  等.

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 (如果两个导数都存在的话!)

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但对一个三元隐函数 F(x,y,z)=0 而言,

$$\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} = -1 \text{ (如果三个偏导数都存在的话!}}$$

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世界一下三九陽函数 
$$F(x,y,z)=0$$
 明旨,  $\partial x \quad \partial y \quad \partial z$  \_ 1 (加里三介億

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$$rac{\partial x}{\partial y}\cdotrac{\partial y}{\partial z}\cdotrac{\partial z}{\partial x}=-1$$
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$$x=\pm\sqrt{1-y^2-z^2},\quad \frac{\partial x}{\partial y}=\mp\frac{y}{\sqrt{1-y^2-z^2}}=-\frac{y}{x};$$

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十二九陽函数 
$$F(x,y)=0$$
 而言, $\dfrac{dy}{dx}\cdot\dfrac{dx}{dy}=1$  (如果两个导数都存在的话!) 个三元隐函数  $F(x,y,z)=0$  而言, $\partial x\quad \partial y\quad \partial z$ 

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所以  $\frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} = -1$ .

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在恒压下,单位体积的物体在单位温度增加量下导致的体积改变量

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物体的压缩系数:

$$\beta = -\lim_{\Delta P \to 0} \frac{1}{V} \cdot \frac{V(P + \Delta P, T) - V(P, T)}{\Delta P} = -\frac{1}{V} \cdot \frac{\partial V(P, T)}{\partial P}$$

在恒温下,单位体积的物体在单位压强增加量下导致的体积改变量(减小量)