Mathematical Methods in Finance

Lecture 7: Stochastic Differential Equations and Financial Applications

Fall 2013

Overview

- Stochastic Differential Equations (SDE)
- Examples in Financial Modeling

Stochastic Differential Equations (SDEs)

- **Definition:** A one-dimensional Stochastic Differential Equation (SDE) is an equation of the form

$$dX(t) = \beta(t, X(t))dt + \gamma(t, X(t))dW(t). \tag{1}$$

  - $\beta(t, x)$: drift;
  - $\gamma(t, x)$: diffusion;
  - $X(0) = x$ for $t \geq 0$ and $x \in \mathcal{R}$: the initial condition.

- Similarly define SDEs with multiple driving Brownian motions

- Similarly define multidimensional SDEs

Two types of solutions

- A strong solution is a process that solves the dynamic (2) on a given probability space (the driving Brownian motion is given as an input);

- A weak solution consists of a probability space and a process on it that solves the dynamic (2).

- strong solution $\Longrightarrow$ weak solution

SDEs

- **Existence and Uniqueness of Strong Solution**
  If there exist two constants C and D s.t. for any $t \in [0, T]$ and $x \in \mathcal{R}$,
  - $|\beta(t,x)| + |\gamma(t,x)| \le C(1 + |x|)$;
  - $|\beta(t,x) - \beta(t,y)| + |\gamma(t,x) - \gamma(t,y)| \le D|x - y|$.

  The SDE admits a unique strong solution!

- Generally speaking, a SDE is not easy to solve, but sometimes we can solve it explicitly.

- Sometimes, numerical computing (e.g. Monte Carlo simulation) are necessary!

Linear SDEs

SDE:

$$dX(t) = \beta(t, X(t))dt + \gamma(t, X(t))dW(t). \tag{2}$$

with

- Drift $\beta(t,x) = a(t) + b(t)x$;
- Diffusion $\gamma(t,x) = \gamma(t) + \sigma(t)x$; condition.

e.g. One-dimensional linear SDEs:

$$dX(t) = [a(t) + b(t)X(t)]dt + [\gamma(t) + \sigma(t)X(t)]dW(t).$$

Examples in Financial Modeling: GBM

Example 1: (Generalized) Geometric Brownian Motion for Modeling Asset Price

- $S(t)$ satisfies SDE:

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), S(0) = s_0$$

- Modeling issue: $\alpha(t)$ is instantaneous mean rate of return, and $\sigma(t)$ is volatility.
- Both $\alpha(t)$ and $\sigma(t)$ could be very general adapted stochastic processes.
- If $\alpha(t)$ and $\sigma(t)$ are both constants $\Longrightarrow$ Black-Scholes-Merton model (1973)

- Explicit solution:

$$S(t) = s_0 e^{\int_0^t \sigma(u)dW(u) + \int_0^t \left(\alpha(u) - \frac{1}{2}\sigma^2(u)\right)du}.$$

Examples in Financial Modeling: the Vasicek Model

Example 2: Vasicek Model for Interest Rate

$$dR(t) = (\alpha - \beta R(t))dt + \sigma dW(t).$$

- When $\alpha = 0$, $R(t)$ is called an Ornstein-Uhlenbeck
- Equivalently written as

$$dR(t) = \kappa(\theta - R(t))dt + \sigma dW(t).$$

  process.
- κ: mean-reverting speed
- θ: mean-reverting level

How to solve is?

If RHS does not involve $R(t)$, the integral form of $R(t)$ is ready. So our objective is to remove $R(t)$ on the RHS.

Recall from Ordinary Differential Equation

Recall ODE

$$\frac{df(x)}{dx} = -af(x) + g(x),$$

where $g(x)$ is known. We have that

$$df(x) + af(x)dx = g(x)dx,$$

and

$$e^{ax}df(x) + ae^{ax}f(x)dx = e^{ax}g(x)dx,$$

i.e.,

$$d[e^{ax}f(x)] = e^{ax}g(x)dx.$$

Therefore

$$f(x) = e^{-ax}\left[f(0) + \int_0^x e^{as}g(s)ds\right].$$

Examples in Financial Modeling: the Vasicek Model

- Similarly, multiply

$$dR(t) = (\alpha - \beta R(t))dt + \sigma dW(t)$$

by $e^{\beta t}$. Then Itô lemma applies

$$d\left[e^{\beta t}R(t)\right] = e^{\beta t}dR(t) + \beta e^{\beta t}R(t)dt = e^{\beta t}\alpha dt + e^{\beta t}\sigma dW(t)$$

- Integrating both sides yields

$$e^{\beta t}R(t) = R(0) + \frac{\alpha}{\beta}\left(e^{\beta t} - 1\right) + \int_0^t \sigma e^{\beta s}dW(s).$$

- Namely, a closed-form expression for $R(t)$ is given by

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma e^{-\beta t}\int_0^t e^{\beta s}dW(s).$$

Several Modeling Issue

- Normal Distribution:

$$R(t) \sim N\left(e^{-\beta t}R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right), \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta t}\right)\right).$$

- Disadvantage: possibility to be negative.

- Advantage: mean-reverting property.
  - β (speed of mean reversion);
  - $\lim_{t\to+\infty} ER(t) = \frac{\alpha}{\beta}$ (long-term mean level);
  - $\lim_{t\to+\infty} Var(R(t)) = \frac{\sigma^2}{2\beta}$ (long-term variance).

General Linear SDEs

Consider SDE

$$dX(t) = [a(t) + b(t)X(t)]dt + [\gamma(t) + \sigma(t)X(t)]dW(t), X(0) = X_0.$$

Apply Ito's rule to prove that

$$X(t) = Y(t)\left[X_0 + \int_0^t (a(s) - \gamma(s)\sigma(s))Y(s)^{-1}ds + \int_0^t \gamma(s)Y(s)^{-1}dW(s)\right],$$

where

$$Y(t) = \exp\left\{\int_0^t \left(b(s) - \frac{1}{2}\sigma(s)^2\right)ds + \int_0^t \sigma(s)dW(s)\right\}.$$

Question: How to find the expectation and variance of $X(t)$?

Note: Previous examples are both special cases of linear SDEs.

Examples in Financial Modeling: the CIR Model

Example 3: Cox-Ingersoll-Ross (CIR) Model for Interest Rate

$$dR(t) = (\alpha - \beta R(t))dt + \sigma\sqrt{R(t)}dW(t).$$

- So the advantage of CIR over Vasicek is its non-negativity.
- Widely used in modeling interest rate, stochastic volatility, stochastic intensity of credit default and other jumps.
- We cannot derive a closed form formula for $R(t)$.
- However, we know $R(t)$ assumes a noncentral Chi-square distribution.
- Exercise: Compute $\mathbb{E}(R(t))$ and Var$(R(t))$ via Itô formula.

Examples in Financial Modeling: Multidimensional Geometric Brownian Motion

Example 4: Multidimensional Geometric Brownian Motion Model for Multiple Correlated Asset Prices, e.g., for two correlated assets

$$\begin{aligned}\frac{dS_1(t)}{S_1(t)} &= \alpha_1 dt + \sigma_1 dW_1(t),\\ \frac{dS_2(t)}{S_2(t)} &= \alpha_2 dt + \sigma_2[\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)],\end{aligned}$$

where $\{(W_1(t), W_2(t))\}$ is a standard two-dimensional Brownian motion.

Equivalent dynamics:

$$\begin{aligned}\frac{dS_1(t)}{S_1(t)} &= \alpha_1 dt + \sigma_1 dW_1(t),\\ \frac{dS_2(t)}{S_2(t)} &= \alpha_2 dt + \sigma_2 dW_3(t).\end{aligned}$$

Example: Correlated Assets

Here $\{(W_1(t), W_3(t))\}$ is a two dimensional Brownian motion with Corr$(W_1(t), W_3(t)) = \rho$.

Apply Ito's formula to $\log S_1(t)$ and $\log S_2(t)$, we find that

$$
\begin{aligned}
S_1(t) &= S_1(0)\exp\left\{\sigma_1 W_1(t) + \left(\alpha_1 - \frac{1}{2}\sigma_1^2\right)t\right\} \\
S_2(t) &= S_2(0)\exp\left\{\sigma_2[\rho W_1(t) + \sqrt{1-\rho^2}W_2(t)] + \left(\alpha_1 - \frac{1}{2}\sigma_1^2\right)t\right\}
\end{aligned}
$$

Generalization: multidimensional linear SDEs. Even in linear specifications, not all SDEs are explicitly solvable! This is not as simple as the one-dimensional linear SDEs.

More Examples

SDE provides us a powerful tool to describe the dynamics of financial market. For example, in order to incorporate the "volatility smile", a natural idea is to allow the change of volatility.

- Local volatility models (Dupire, Derman):

$$dS(t) = \mu S(t)dt + \sigma(t, S(t))S(t)dW(t)$$

- The stochastic volatility model (e.g. Heston (1993)):

$$dS(t) = \mu S(t)dt + \sqrt{V(t)}S(t)dW_1(t)$$

$$dV(t) = \kappa(\theta - V(t))dt + \sigma_v\sqrt{V(t)}dW_2(t).$$

- In practice, we may use more advanced models according to the special necessity, e.g. adding jumps, etc.

Suggested Reading Material (We only need to focus on the material parallel to our course slides):

- Selected material from Shreve Vol. II: Examples 4.4.10, 4.4.11, Sections 6.1, 6.2
- Or equivalent material from Mikosch: 3.2, 3.3

Suggested Exercises (some of these exercises have been included in Homework Assignment $\#6$; others are for your deeper understanding)

- Shreve Vol.II: 4.5, 4.8, 6.1, 6.6