ECON 139 Lecture 12

Edward Xu, Yihuan Yao

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Joint Saving-Portfolio Problem

- Let s be amount saved and a be amount invested in risky asset at t=0, then

$$c_0 = M_0 - s$$

$$\widehat{c_1} = (s-a)\left(1+r_f\right) + a(1+\hat{r}) = s\left(1+r_f\right) + a(\hat{r} - r_f)$$

$$\max_{a,s} u(c_0) + \delta E[u(\widehat{c_1}(s,a))]$$

First Order Condition:

WRT s $\quad -u'(M_0 - s^*) + \delta E\left[u'\left(c_1(s^*, a^*)\right)\left(1+r_f\right)\right] = 0$

WRT a $\quad \delta E\left[u'\left(c_1(s^*, a^*)\right)\left(\hat{r} - r_f\right)\right] = 0$

- "Stochastic" Euler Equation

$$u'(c_0^*) = \delta\left(1+r_f\right)E\left[u'\left(\widehat{c_1^*}\right)\right]$$

Where $\widehat{c_1}^* = \widehat{c_1}(s^*, a^*)$

Without uncertainty,

$$u'(c_0^*) = \delta\left(1+r_f\right)u'\left(\widehat{c_1^*}\right)$$

- Euler Equation and Power Utility
  - Let $\quad u(w) = \frac{w^{1-\beta}}{1-\beta}, \beta > 0, \beta \neq 1$

$$u'(w) = w^{-\beta}$$

$$c_0^{-\beta} = \delta(1 + r_f)c_1^{-\beta}$$

$$\frac{c_0^{-\beta}}{c_1^{-\beta}} = \delta(1 + r_f)$$

$$\frac{c_1^{\beta}}{c_0^{\beta}} = \delta(1 + r_f)$$

$$\frac{c_1}{c_0} = [\delta(1 + r_f)]^{\frac{1}{\beta}}$$

$$\ln\left(\frac{c_1}{c_0}\right) = \frac{1}{\beta}\ln(\delta) + \frac{1}{\beta}ln(1 + r_f)$$

where $\ln\left(\frac{c_1}{c_0}\right)$ is the consumption growth rate.

– Elasticity of intertemporal substitution with respect to savings rate

$$\frac{d\ln(\frac{c_1}{c_0})}{d\ln(1 + r_f)} = \frac{1}{\beta}$$

- $\max_a E[u(w_0(1 + r_f) + a(\hat{r} - r_f))]$

  If we allow investor to invest in N>1 risky assets, then the maximization problem becomes

$$\max_{a_1,\dots,a_N} E[u(w_0(1 + r_f) + \sum_{i=1}^{N} a_i(\hat{r_i} - r_f))]$$

Let $w_i = \frac{a_i}{w_0}$,then

$$\max_{w_1,\ldots,w_N} E[u(w_0\left(1+r_f\right)+\sum_{i=1}^{N} w_i w_0(\widehat{r_i}-r_f))]$$

$$\max_{w_1,\ldots,w_N} E[u(w_0(\left(1+r_f\right)+\sum_{i=1}^{N} w_i(\widehat{r_i}-r_f)))]$$

$$\max_{w_1,\ldots,w_N} E[u(w_0(1+\widehat{r_p}))]$$

where $\widehat{r_p} = r_f\left(1-\sum_{i=1}^{N} w_i\right)+\sum_{i=1}^{N} w_i\widehat{r_i}$

- Modern Portfolio Theory (MPT)

$$\max_{a} E[u(w_0(1+r_f)+a^*(\tilde{r}-r_f))]$$

Open to possibly many risky assets (N>1),

$$\max_{a_1,\dots,a_N} E[u(w_0(1+r_f)+\sum_{i=1}^{N} a_i(\tilde{r}_i-r_f))]$$

Since $w_1 = \frac{a_i}{w_0}$,

$$\max_{w_1,\dots,w_N} E[u(w_0(1+r_f)+\sum_{i=1}^{N} w_0 w_i(\tilde{r}_i-r_f))]$$

$$\Rightarrow \max_{w_1,\dots,w_N} E[u(w_0\left((1+r_f)+\sum_{i=1}^{N} w_i(\tilde{r}_i-r_f)\right))]$$

Let net portfolio return be: $\tilde{r_p} = r_f + \sum_{i=1}^{N} w_i(\tilde{r}_i - r_f)$

$$\max_{w_1,\dots,w_N} E[u(w_0(1+\tilde{r}_p))]$$

Possible ways to justify mean-variance utility is an expected utility framework:

1) Portfolio bets are small.
2) Investors have quadratic utility.
3) Asset returns are normally distributed. (terminal wealth is normally distributed)

1) Portfolio bets are small

Suppose investors have VNM expected utility:

$$\max u(\tilde{w})$$

Wealth is represented by:

$$\tilde{w}_1 = E[\tilde{w}_1] + (\tilde{w}_1 - E[\tilde{w}_1])$$

Look at second order of Taylor approximation of Bernoulli utility function, once the outcome is known:

$$u(\tilde{w}_1) \approx u(E(\tilde{w}_1)) + u'[E(\tilde{w}_1)][(\tilde{w}_1 - E(\tilde{w}_1)] + \frac{1}{2}u''[E(\tilde{w}_1)][\ \tilde{w}_1 - E(\tilde{w}_1)]^2$$

$$E[u(\tilde{\boldsymbol{w}}_{\mathbf{1}})] \approx u\big(E(\tilde{\boldsymbol{w}}_{\mathbf{1}})\big) + \frac{1}{2}u^{"}[E(\tilde{\boldsymbol{w}}_{\mathbf{1}})]{\sigma_{w_1}}^2$$

2) Investors have quadratic utility

$$u(w) = aw - bw^2, \text{ a,b>0}$$

$$u'(w) = a - 2bw, u"(w) = -2b$$

$$R_A = -\frac{u"(w)}{u'(w)} = \frac{2b}{a-2bw}$$

$$R'_A = \frac{4b^2}{(a-2bw)^2} > 0$$

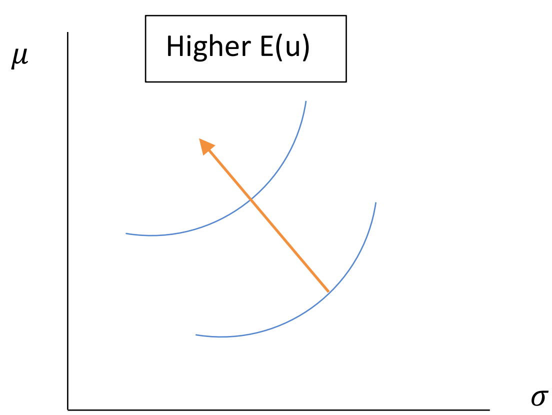
3) Asset returns are normally distributed

If $\tilde{\boldsymbol{w}}_{\mathbf{1}}$ is normally distributed with mean $\mu = E(\tilde{\boldsymbol{w}}_{\mathbf{1}})$ and variance $\sigma^2 = E(\tilde{\boldsymbol{w}}_{\mathbf{1}} - E(\tilde{\boldsymbol{w}}_{\mathbf{1}}))^2$, then the expectation of any function of $\tilde{\boldsymbol{w}}_{\mathbf{1}}$ can be written as a function of μ and σ.

Recall: if $\tilde{w}_1 \sim N(\mu, \sigma^2)$, $\tilde{z} = \frac{\tilde{w}_1 - \mu}{\sigma} \sim N(0,1)$

Therefore, $\tilde{w}_1 = \mu + \sigma z$, and $E[u(\tilde{w}_1)] = g(\mu + \sigma z)$

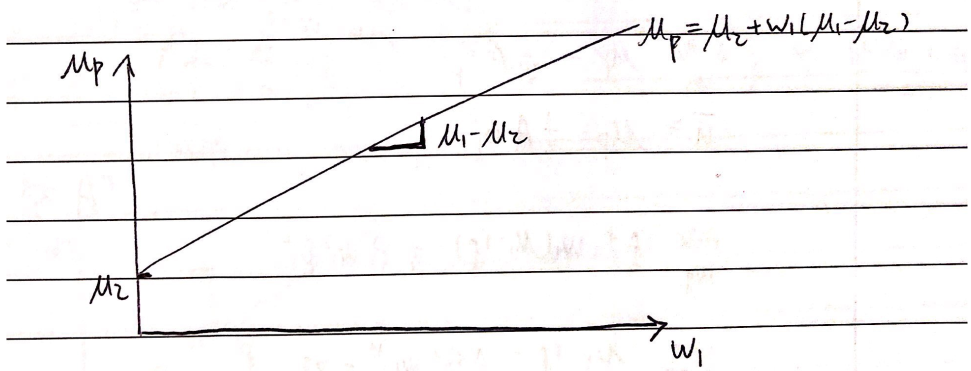
i) If μ is increasing, then g is increasing in μ.
ii) If μ is concave, then g is decreasing in σ.
iii) If μ is increasing and concave, then the indifference curves μ and σ are convex.



Gains from Diversification

- Consider two risky assets
  - $\tilde{r}_1$, $\tilde{r}_2$: risky returns
  - μ_1, μ_2: expected returns
  - σ_1^2, σ_2^2: variance of returns
- Assume $\mu_1 > \mu_2$ and $\sigma_1^2 > \sigma_2^2$, the possible portfolio expected returns are

$$\mu_p = w_1 \mu_1 + w_2 \mu_2 = w_1 \mu_1 + (1-w_1)\mu_2 = \mu_2 + w_1(\mu_1 - \mu_2)$$



$$\sigma_p^2 = Var(w_1\tilde{r}_1 + (1-w_1)\tilde{r}_2)$$
$$= w_1^2 \sigma_1^2 + (1-w_1)^2 \sigma_2^2 + 2w_1(1-w_1)cov(\tilde{r}_1, \tilde{r}_2),$$

where $cov(r_1, r_2)$ can be calculated as:

$$\rho_{12} = cov(\tilde{r}_1, \tilde{r}_2)/(\sigma_1 \sigma_2)$$
$$cov(\tilde{r}_1, \tilde{r}_2) = \sigma_1 \sigma_2 \rho_{12} = w_1^2 \sigma_1^2 + (1-w_1)^2 \sigma_2^2 + 2w_1(1-w_1)\sigma_1 \sigma_2 \rho_{12}$$