Homework Assignment #7
Due: In class, two weeks after distribution

§1 The Geometric Mean-Reversion Process

The geometric mean-reversion process is specified as

$$dS(t) = \kappa S(t)[\theta - \log S(t)]dt + \sigma S(t)dW(t), \quad S(0) = s_0.$$

1. Use Ito's formula to prove that $X(t) = \log S(t)$ satisfies a linear SDE covered in class, and thus solve $S(t)$ explicitly;
2. What is the distribution of $S(t)$?
3. Find $ES(t)$ and $\text{Var}S(t)$?

§2 A Double Mean-reversion Model

The price of an asset or an index follows $A(t) = \exp(X_1(t))$ where

$$\begin{aligned} dX_1(t) &= \kappa_1(X_2(t) - X_1(t))dt + \sigma_1 dW_1(t),\ X_1(0) = x_1, \\ dX_2(t) &= \kappa_2(\theta_2 - X_2(t))dt + \sigma_2\left[\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)\right],\ X_2(0) = x_2. \end{aligned}$$

I ever used this double mean-reverting process for modeling stochastic volatility and thus pricing options on VIX (the CBOE volatility index).

1. Suppose $\kappa_1 > \kappa_2$. Based on your intuition, could you immediately draw a picture to illustrate a pair of trajectories of X_1 and X_2?
2. Solve $(X_1(t), X_2(t))$ explicitly.
3. What is the joint distribution of $(X_1(t), X_2(t))$?
4. Suppose a buy-side econometrician hopes to estimate the model based on some discretely monitored data (the time series of (X_1, X_2)). To perform maximum-likelihood estimation, she needs the transition density of the process $\{(X_1(t), X_2(t))\}$. Please find it explicitly and briefly discribe how the estimation can be done, i.e., find

$$P(X_1(t+\Delta) \in dx_1, X_2(t+\Delta) \in dx_2 | X_1(t) = x_{10}, X_2(t) = x_{20}).$$

§3 Correlated Assets

Suppose stock prices $S_1(t), \ldots, S_n(t)$ satisfy the SDEs

$$dS_i(t) = \mu_i S_i(t)dt + \sigma_i S_i(t)dW_i(t),$$

where $S_i(0)$ are constants; and the $W_i(t)$ are correlated Brownian motions with correlation coefficients

$$\rho_{ij} = \text{corr}(W_i(t), W_j(t)).$$

1. Solve $S_1(t), \ldots, S_n(t)$ explicitly;
2. Use a standard $n-$dimensional Brownian motion to equivalently represent the dynamics of $S_1(t), \ldots, S_n(t)$;
3. Find
$$\lim_{\Delta t \to 0} \text{corr}\left(S_i(\Delta t) - S_i(0), S_j(\Delta t) - S_j(0)\right) = ?$$
This implies that the correlations can be estimated (at least theoretically) from the time-series data of the asset prices.

§4 A Bonus Question: Econometric Analysis for Continuous-time Models

Let us take the simple Vasicek model as an example. Consider

$$dX(t) = \kappa(\theta - X(t))dt + \sigma dW(t).$$

Follow the idea proposed in Question 2 on "Finite Dimensional Distribution of a Brownian Motion" of Exercise 5 to develop a procedure to perform maximum-likelihood estimation for the parameters. Note that you need to find the transition density of this model. Then, construct the (log-)likelihood function. For those of you who are interested in being a buy-side quant or technical trader, this is very helpful. And, for those for you interested in theoretical study of asset pricing, this adds your skills. Free free to talk to me if you are interested in something like this.