Investments

Lecture 11

Pricing: Options

Option Pricing Models

- Two approaches:
  - Tracking Portfolio.
    - Solve explicitly for replicating portfolio.
  - Risk-neutral probabilities.
    - Adjust probabilities so that expected cashflows give option prices.
- Two settings:
  - Binomial (discrete time)
  - Black-Scholes (continuous time)

The Binomial Valuation Model:
The Tracking Portfolio Approach

- Consider the following call option:

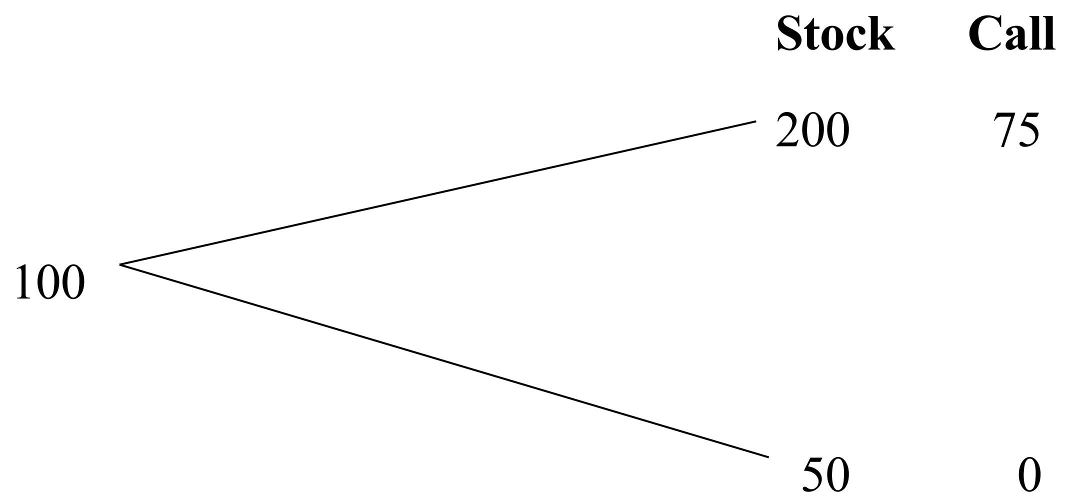
  S_0 = \$100, K =\$125, r_f = 8

  Suppose at year-end stock can take only two values:

  S_T = \$200 or \$50

  Then the payoff for call option is:

  \$75 when stock goes up and \$0 when stock goes down.



Tracking Portfolio

- Compare to the payoff from a portfolio of:

  1 share of stock + borrowing \$46.30 at 8%

  Payoffs from this position are:

  \$150 when stock goes up and \$0 when stock goes down

  Cash outlay to establish this position = \$100 - \$46.30 = \$53.70

- Payoff of levered portfolio is twice the option payoff, therefore:

  $C_0 = (1/2)\ \$53.70 = \$26.85$

Consider the following position (same data as before)

1 share of stock + 2 written calls

Net payoffs:

Stock value	\$50	\$200
- Call obligation	- 0	-\$150
Net payoff	\$50	\$50

PV of \$50 = \$50/1.08 = \$46.30

Hence:

$S_0 - 2C_0 = 46.30$; $S_0 = 100$ and C_0=\$26.85

Calculating the Hedge Ratio

Hedge ratio (H): $$H = \frac{C^{+} - C^{-}}{S^{+} - S^{-}}$$

In this case: $$H = \frac{\$75 - \$0}{\$200 - \$50} = 0.5$$

Therefore:	H = 0.5 shares	\$100	\$25
	<u>-Call obligation</u>	\$ 75	\$ 0
		\$ 25	\$25

PV of \$25 at 8% = \$25/1.08 = \$23.15

General Principle

Value of hedged position = PV of certain payoff

$$0.5S_0 - C_0 = \$23.15$$

$$\$50 - C_0 = \$23.15$$

$$C_0 = \$26.85$$

- H shares and 1 call written always results in a perfectly hedged portfolio
- Yields sure payoff at time T of
- General binomial model pricing formula:

$$HS_0 - C_0 = \frac{(HS^+ - C^+)}{(1+r_f)^T}, \text{ or}$$

$$HS_0 - C_0 = \frac{(HS^- - C^-)}{(1+r_f)^T}$$

General Principle for Dummies

1. Calculate the spread in option values
2. Calculate the spread in stock values
3. Calculate the hedge ratio H
4. Calculate the payoff of holding H share stock and writing 1 call option and confirm it is riskless
5. Calculate the PV of this payoff
6. Solve for the option value

Ex: $S^+=120$, $S^-=90$, $K=110$, $r=10\%$ -> $C=6.06$

Example: Arbitrage Profits

Given previous data except C_0=\$30

Initial cash flows:

Write 1 option	\$30
Purchase 0.5 shares	-\$50
Borrow \$20 at 8%	\$20
(Repay in 1 year)	

Cash flow in 1 year at each stock price:

Written option	\$0	-\$75
Purchase H shares	\$25	\$100
Borrow \$20 at 8%		
Repay in 1 year	-\$21.60	-\$21.60
Payoffs	\$3.40	\$3.40

For a net initial investment of \$0

Basic approach:

- If option overpriced purchase H shares, write 1 option and borrow the difference to make 0 net investment.

- If option underpriced sell H shares short, purchase 1 option and lend the difference to make 0 net investment.

The Binomial Valuation Model:
The Risk-Neutral Probability Approach

- Suppose stock price is $75 and you want to value an option with K = $90
- Suppose that r_f =2.5% and that stock price can be $60 or $100 in 1 year

Step 1:
Compute the probability "as if" investors were indifferent with respect to risk

$$\text{"Expected return"} = p \times \frac{100-75}{75} + (1-p) \times \frac{60-75}{75} = 2.5\%$$

$$p = 0.421875$$

Step 2: Use that probability to evaluate options payoff and use r_f to discount

$$p \times 10 + (1-p) \times 0 = 0.421875 \times 10 + 0.578125 \times 0 = \$4.21875$$

$$C_0 = \frac{4.21875}{1.025} = \$4.115854$$

Risk-Neutral Valuation

- To value a derivative security using the binomial model and the risk-neutral pricing approach:
  1. Calculate the risk-neutral probabilities:

     $$S_0 = \frac{pS^+ + (1-p)S^-}{R_f} \Rightarrow p = \frac{R_f - d}{u - d}$$

  2. Use these probabilities to calculate the expected payout from the instrument, and
  3. Discount the expect payout using the risk-free interest rate.

     $$C_0 = \frac{pC^+ + (1-p)C^-}{R_f}$$

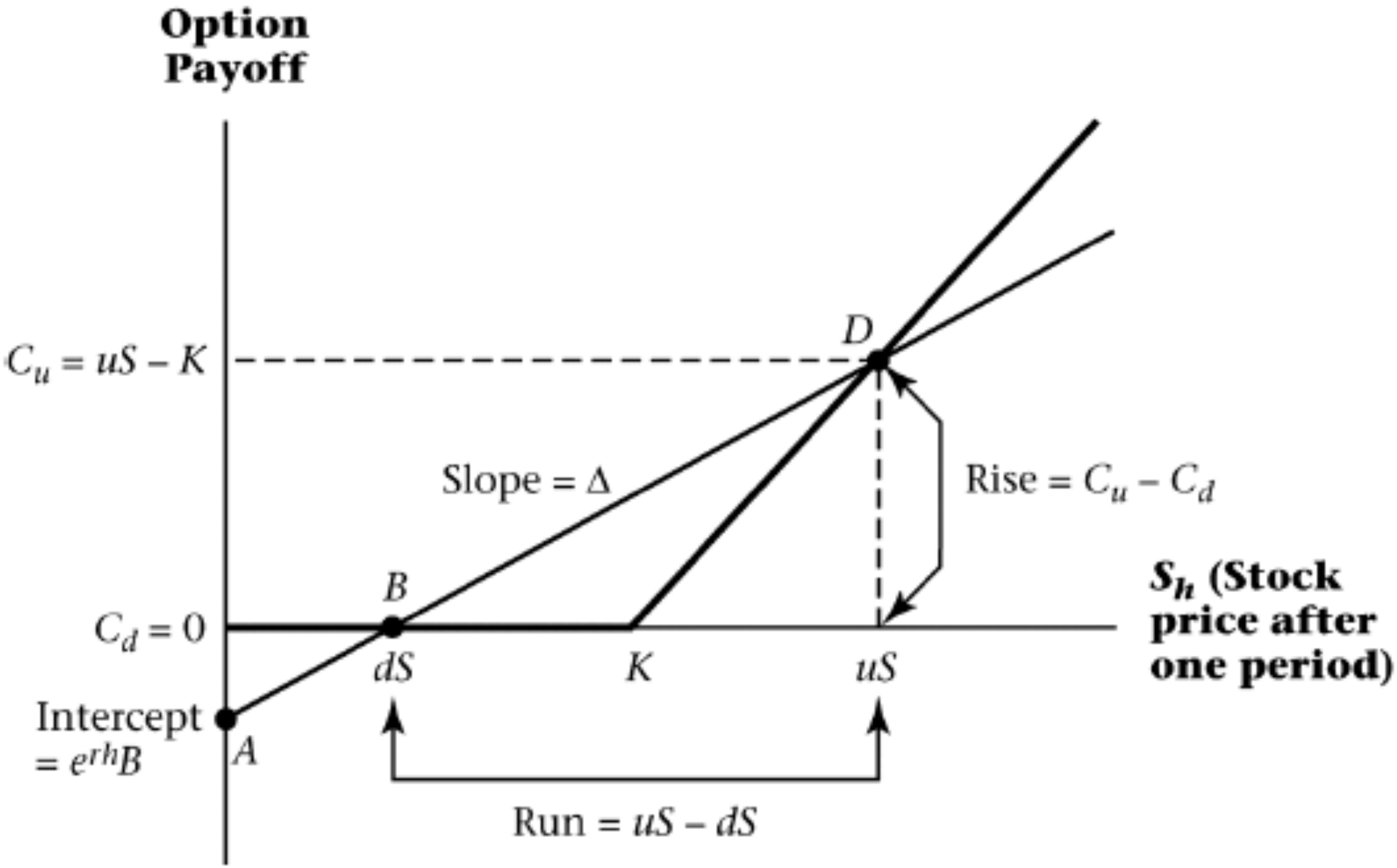
Binomial Option Pricing

- Binomial option pricing enables us to determine the price of an option, given the characteristics of the stock or other underlying asset.

- The binomial option pricing model assumes that the price of the underlying asset follows a binomial distribution—that is, the asset price in each period can move only up or down by a specified amount.

- The binomial model is often referred to as the “Cox-Ross-Rubinstein pricing model.”

A graphical interpretation of the binomial formula



Building the Binomial Model

- In order for these techniques to be useful, the binomial model must closely represent stock price dynamics.
- In order to understand the model and how to apply it, we must first understand some continuous compounding techniques.

Riskless Asset Dynamics

- Consider the behavior of the price of an investment that is growing at the continually compounding rate r_f:

$$S_T = S_0 e^{r_f T}$$

- This growth can be thought of as occurring in stages:

$$S_T = S_0 \left(e^{r_f \Delta t_1 + r_f \Delta t_2 + r_f \Delta t_1 + \ldots} \right)$$

Risky Asset Dynamics

- The same sort of model may be applied to risky asset price dynamics:

$$S_T = S_0 e^{\tilde{r}T}$$

- Here, however, the return is random.
- We will assume it is composed of a lot of small random shocks that occur over small intervals:

$$S_T = S_0\left(e^{\tilde{r}_1\Delta t_1 + \tilde{r}_2\Delta t_2 + \tilde{r}_3\Delta t_1 + \ldots}\right)$$

Risky Asset Dynamics

- Over each small interval, the random returns can be thought of as a branch of a binomial tree:

$$S \to uS = Se^{r_u \Delta t}$$

$$S \to dS = Se^{r_d \Delta t}$$

Risky Asset Dynamics

- Why do we care about these details?
- Well, if each of the binomial branches is independent of the other, stringing the branches together into a tree gives rise to stock prices whose returns are **lognormal**.
- This fact allows us to measure characteristics of the **returns** (e.g. the variance) and then use these measured variables to construct a binomial tree that describes how the stock's price will change.

The random walk model

- The idea that asset prices should follow a random walk was articulated in Samuelson (1965)
- In efficient markets, an asset price should reflect all available information. In response to new information the price is equally likely to move up or down, as with the coin flip.
- The price after a period of time is the initial price plus the cumulative up and down movements due to new information.

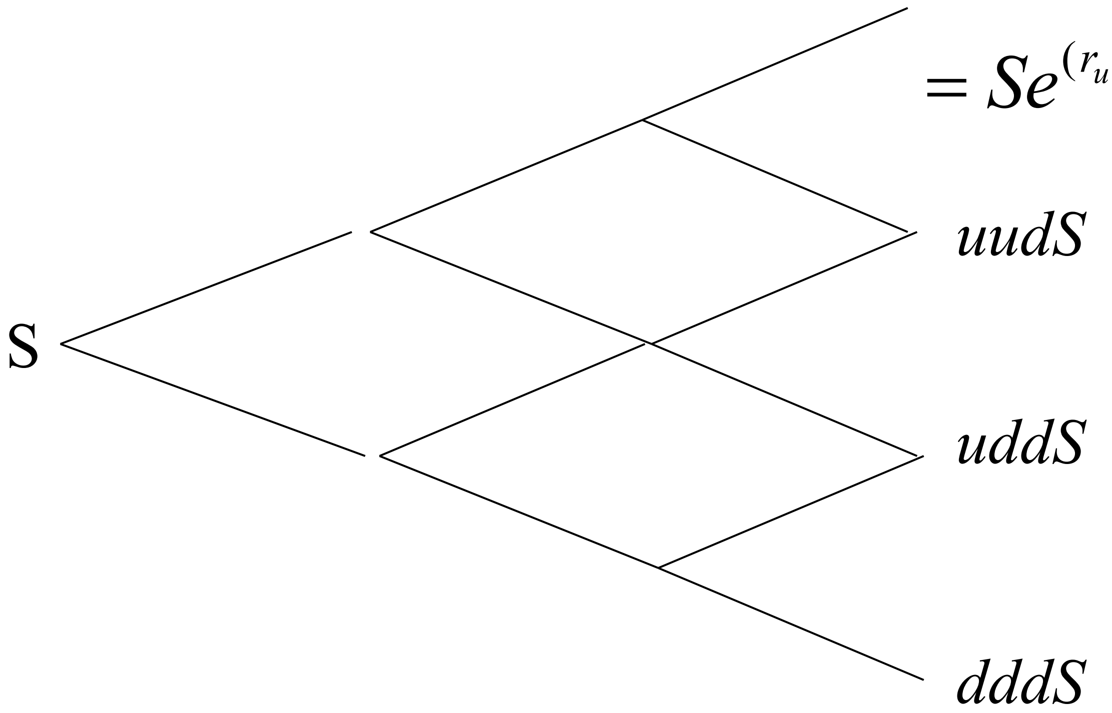
Modeling stock prices as a random walk.

- The above description of a random walk is not a satisfactory description of stock price movements. There are at least three problems with this model
  - If by chance we get enough cumulative down movements, the stock price will become negative.
  - The magnitude of the move ($1) should depend upon how quickly the coin flips occur and the level of the stock price
  - The stock, on average, should have a positive return. However, the random walk model taken literally does not permit this
- The binomial model is a variant of the random walk model that solves all of these problems.

Moving Through the Tree

$$uuuS = Se^{r_u \Delta t} e^{r_u \Delta t} e^{r_u \Delta t}$$

$$= Se^{(r_u + r_u + r_u)\Delta t}$$



Measuring Volatility

- If we know the volatility of a stock, we can start to build a binomial tree.
  - The volatility is given by the **variance of the continuously compounded return**.

$$\text{variance of each return step} = \sigma^2 \Delta t$$

  - The standard deviation of each return step is $\sigma\sqrt{\Delta t}$ (in units of %/yr).

Building the Binomial Tree

- In order to have our binomial tree accurately represent the stock's returns, we want to choose the up and down returns (*u* and *d*) so that the variance of each return is correct.
- In addition, the tree will "recombine" making computation easier.
- So, we choose $u = e^{\sigma\sqrt{\Delta t}}$ and $d = 1/u = e^{-\sigma\sqrt{\Delta t}}$

Example

- The standard deviation of returns for the S&P 500 index is 28.42% per year. The current level of the index is 881.
  - Build a 3-period binomial tree that can be used to price options with two months to maturity.
  - Use the tree to price an Aug 875 call. (current price = ?)
  - Excel!

Option Price

T	0.17		u	1.069288
N	3.00		d	0.935201
dt	0.055567		rf	0.01
sigma	0.2842			
K	875		p	0.487403
0	1	2	3	
Stock				
881	942.043	1007.316	1077.111	
	823.9125	881	942.043	
		770.5242	823.9125	
			720.5954	
Call				
47.81447	81.42347	132.8016	202.1107	
	15.90917	32.65881	67.04299	
		0	0	
			0	

Other Choices for the Returns

- There are several possibilities for *u* and *d*:

$$u = e^{r_f \Delta t + \sigma \sqrt{\Delta t}}, d = e^{r_f \Delta t - \sigma \sqrt{\Delta t}}$$

$$u = e^{(r_f - .5\sigma^2)\Delta t + \sigma \sqrt{\Delta t}}, d = e^{(r_f - .5\sigma^2)\Delta t - \sigma \sqrt{\Delta t}}$$

- With many steps, the choice is largely irrelevant. The last choice will give the closest approximation in a coarse tree.

Dividends

- Option holders do not receive dividends on the underlying stock. As a result, we must adjust the option pricing formula to account for this.
- Several assumptions can be made about the form of dividends. For now, assume dividends are paid to maintain a constant dividend yield (Div/Price = constant).

Effect of Dividends on Binomial Model

- We can account for dividends by changing the risk-neutral probabilities:

$$p = \frac{e^{(r_f - \delta)\Delta t} - d}{u - d}$$

- Notice that no changes need to be made to the binomial tree! We just reinterpret the stock prices as ex-dividend prices.

Example

- The standard deviation of returns for the S&P 500 index is 28.42% per year. The current level of the index is 881 and the current dividend yield is 1% per year.
  - Build a 3-period binomial tree that can be used to price options with two months to maturity.
  - Use the tree to price an Aug 875 call. (current price = ?)

Option Price

T	0.17		u	1.069288
N	3.00		d	0.935201
dt	0.055567		rf	0.01
sigma	0.2842		DivYld	0.01
K	875		p	0.483258
0	1	2	3	
Stock				
881	942.043	1007.316	1077.111	
	823.9125	881	942.043	
		770.5242	823.9125	
			720.5954	
Call				
47.00368	80.59487	132.242	202.1107	
	15.63971	32.38106	67.04299	
		0	0	
			0	

Put Option

- We compute put option prices using the same stock price tree and in the same way as call option prices.
- The only difference with a European put option occurs at expiration.
  - –Instead of computing the price as *max* (0, S–K), we use *max* (0, K–S).

American Option

- The value of the option if it is left "alive" (i.e., unexercised) is given by the value of holding it for another period
- The value of the option if it is exercised is given by *max* (0, S–K) if it is a call and *max* (0, K–S) if it is a put.
- For an American call, the value of the option at a node is given by

$$\max(C, \max(0, S-K))$$

American Options

- The valuation of American options proceeds as follows:
  - At each node, we check for early exercise.
  - If the value of the option is greater when exercised, we assign that value to the node. Otherwise, we assign the value of the option unexercised.
  - We work backward through the three as usual.

Properties of option prices

- American vs. European–Since an American option can be exercised at anytime, whereas a European option can only be exercised at expiration, an American option must always be at least as valuable as an otherwise identical European option

Properties of option prices (cont.)

- Call price cannot:
  - be negative
  - exceed stock price
  - be less than price implied by put-call parity using zero for put price:
- Put price cannot
  - be more than the strike price
  - be less than price implied by put-call parity using zero for put price

Properties

- Early exercise of American options
  - A non-dividend paying American call option should not be exercised early
  - That means, one would lose money be exercising early instead of selling the option
  - If there are dividends, it may be optimal to exercise early
  - It may be optimal to exercise a non-dividend paying put option early if the underlying stock price is sufficiently low

Properties

- Different strike prices ($K1 < K2 < K3$), for both European and American options
  - A call with a low strike price is at least as valuable as an otherwise identical call with higher strike price:
    - $C(K1) > C(K2)$
  - A put with a high strike price is at least as valuable as an otherwise identical put with low strike price:
    - $P(K2) > P(K1)$
  - The premium difference between otherwise identical calls with different strike prices cannot be greater than the difference in strike prices:
    - $C(K1) - C(K2) < K2 - K1$

Properties

- Different strike prices ($K1 < K2 < K3$), for both European and American options
  - The premium difference between otherwise identical puts with different strike prices cannot be greater than the difference in strike prices:
    - $P(K1) - P(K2) < K2 - K1$
  - Premiums decline at a decreasing rate for calls with progressively higher strike prices. (Convexity of option price with respect to strike price)

TABLE 9.6 Panel A shows call option premiums for which the change in the option premium ($6) exceeds the change in the strike price ($5). Panel B shows how a bear spread can be used to arbitrage these prices. By lending the bear spread proceeds, we have a zero cash flow at time 0; the cash outflow at time T is always greater than $1.

Panel A

Strike	50	55
Premium	18	12

Panel B

		Expiration or Exercise		
Transaction	**Time 0**	$S_T < 50$	$50 \le S_T \le 55$	$S_T \ge 55$
Buy 55-Strike Call	−12	0	0	$S_T - 55$
Sell 50-Strike Call	18	0	$50 - S_T$	$50 - S_T$
Total	6	0	$50 - S_T$	−5

TABLE 9.7 The example in Panel A violates the proposition that the rate of change of the option premium must decrease as the strike price rises. The rate of change from 50 to 59 is 5.1/9, while the rate of change from 59 to 65 is 3.9/6. We can arbitrage this convexity violation with an asymmetric butterfly spread. Panel B shows that we earn at least $3 plus interest at time *T*.

Panel A

Strike	50	59	65
Call Premium	14	8.9	5

Panel B

		Expiration or Exercise			
Transaction	**Time 0**	$S_T < 50$	$50 \le S_T \le 59$	$59 \le S_T \le 65$	$S_T > 65$
Buy Four 50-Strike Calls	−56	0	$4(S_T - 50)$	$4(S_T - 50)$	$4(S_T - 50)$
Sell Ten 59-Strike Calls	89	0	0	$10(59 - S_T)$	$10(59 - S_T)$
Buy Six 65-Strike Calls	−30	0	0	0	$6(S_T - 65)$
Lend $3	−3	$3e^{rT}$	$3e^{rT}$	$3e^{rT}$	$3e^{rT}$
Total	0	$3e^{rT}$	$3e^{rT} + 4(S_T - 50)$	$3e^{rT} + 6(65 - S_T)$	$3e^{rT}$