多元复合函数的微分法

$$z = f(u, v), \quad u = \varphi(x, y), \quad v = \psi(x, y)$$

复合成 $z = f(\varphi(x, y), \psi(x, y))$

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定理

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的偏导数 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ 在 (x, y) 点连续

则 $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ 在 (x,y) 点存在且连续,且

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$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial x} \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial y} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial y} \frac{\partial v}{\partial y}.$$

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$$\begin{split} z &= f(u,v), \quad u = \varphi(x,y), \quad v = \psi(x,y) \text{ $[]$ \mathbb{Z} $] \mathbb{Z} $] = f(\varphi(x,y),\psi(x,y)) \\ \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}. \end{split}$$

 $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}.$

证明: 记
$$\rho_1 = \sqrt{(\Delta u)^2 + (\Delta v)^2}, \quad \rho_2 = \sqrt{(\Delta x)^2 + (\Delta y)^2},$$

则由于 $u = \varphi(x, y), v = \psi(x, y)$ 连续,有 $\rho_2 \to 0 \Rightarrow \rho_1 \to 0.$

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$$z = f(u,v) \ \text{可微}, \ \Delta z = \frac{\partial z}{\partial u} \Delta u + \frac{\partial z}{\partial v} \Delta v + o(\rho_1), \quad (\rho_1 \to 0) \ ,$$

$$\because z = f(u,v) \ \text{可微}, \ \triangle z = \frac{\partial z}{\partial u} \Delta u + \frac{\partial z}{\partial v} \Delta v + o(\rho_1), \quad (\rho_1 \to 0) \ ,$$

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$$\therefore u = \varphi(x,y) 可微, \therefore \Delta u = \frac{\partial u}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y + o(\rho_2), \quad (\rho_2 \to 0) ,$$

$$\rho(x,y)$$
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$$\psi(x,y)$$
 可微, $\Delta v = \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y + o(\rho_2), \quad (\rho_2 \to 0),$

$$\psi(x,y) \ \overline{\exists} \ \partial x, \therefore \Delta v = \frac{\partial v}{\partial x} \Delta x + \frac{\partial y}{\partial y} \Delta y + o(\rho_2), \quad (\rho_2 \to 0),$$

$$\because v = \psi(x,y) \ \overline{\exists} \ \partial_x \cdot \Delta v = \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y + o(\rho_2), \quad (\rho_2 \to 0) \ ,$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}.$$

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$$\rho_1 = \sqrt{(\Delta u)^2 + (\Delta v)^2}, \quad \rho_2 = \sqrt{(\Delta x)^2 + (\Delta y)^2},$$

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 $\phi(x,y)$ 可微, $\Delta v = \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial x} \Delta y + o(\rho_2), \quad (\rho_2 \to 0),$

$$\Delta z = (\frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}) \Delta x + (\frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}) \Delta y$$

$$+\frac{\partial z}{\partial u} o(\rho_2) + \frac{\partial z}{\partial v} o(\rho_2) + o(\rho_1), \quad (\rho_1 \to 0, \rho_2 \to 0)$$

$$z = f(u, v), \quad u = \varphi(x, y), \quad v = \psi(x, y) \text{ $\underline{\beta}$ $\underline{\beta}$ $\underline{\alpha}$ \underline{z} = $f(\varphi(x, y), \psi(x, y))$}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}.$$

$$\overline{\partial x} = \overline{\partial u} \, \overline{\partial x} + \overline{\partial v} \, \overline{\partial x}, \quad \overline{\partial y} = \overline{\partial u} \, \overline{\partial y} + \overline{\partial v} \, \overline{\partial y}.$$

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$$\rho_1 = \sqrt{(\Delta u)^2 + (\Delta v)^2}, \quad \rho_2 = \sqrt{(\Delta x)^2 + (\Delta y)^2},$$

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 可微, $\Delta u = \frac{\partial x}{\partial x} \Delta x + \frac{\partial y}{\partial y} \Delta y + o(\rho_2), \quad (\rho_2 \to 0),$
 $v = \psi(x,y)$ 可微, $\Delta v = \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial x} \Delta y + o(\rho_2), \quad (\rho_2 \to 0),$

$$\Delta z = (\frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}) \Delta x + (\frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}) \Delta y + \frac{\partial z}{\partial v} o(\rho_2) + \frac{\partial z}{\partial v} o(\rho_2) + o(\rho_1), \quad (\rho_1 \to 0, \rho_2 \to 0)$$

$$+\frac{\partial z}{\partial u} o(\rho_2) + \frac{\partial z}{\partial v} o(\rho_2) + o(\rho_1), \quad (\rho_1 \to 0, \rho_2 \to 0)$$

往证
$$\lim_{\rho_2 \to 0} \frac{\frac{\partial z}{\partial u} \ o(\rho_2) + \frac{\partial z}{\partial v} \ o(\rho_2) + o(\rho_1)}{\rho_2} = 0$$
,

$$\frac{z}{z}\frac{\partial u}{\partial z} + \frac{\partial z}{z}\frac{\partial v}{\partial z}, \quad \frac{\partial z}{\partial z} = \frac{\partial z}{z}\frac{\partial u}{\partial z} + \frac{\partial z}{z}\frac{\partial v}{\partial z}.$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}.$$

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$$z = f(u,v) \; \exists \, \exists \, (u,y), \quad \forall (u,y) \; \exists \, \exists \, (n,y) \; \exists \, (n,$$

$$z : z = f(u, v) \ \text{Tilde}(z), \ \Delta z = \frac{\partial z}{\partial u} \Delta u + \frac{\partial z}{\partial v} \Delta v + o(\rho_1), \ (\rho_1 \to 0),$$

$$\because u = \varphi(x,y) \, \overline{\eta} \, \mathring{\mathbb{Q}}, \, \therefore \, \Delta u = \frac{\partial u}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y + o(\rho_2), \quad (\rho_2 \to 0) \,\,,$$

$$\psi(x,y) \; \exists \exists x \; \exists x \; \exists y \;$$

$$\Delta z = (\frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}) \Delta x + (\frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}) \Delta y + \frac{\partial z}{\partial u} o(\rho_2) + \frac{\partial z}{\partial v} o(\rho_2) + o(\rho_1), \quad (\rho_1 \to 0, \rho_2 \to 0)$$

往证
$$\lim_{\rho_2 \to 0} \frac{\frac{\partial z}{\partial u} \ o(\rho_2) + \frac{\partial z}{\partial v} \ o(\rho_2) + o(\rho_1)}{\rho_2} = 0$$
,即要证 $\lim_{\rho_2 \to 0} \frac{o(\rho_1)}{\rho_2} = 0$.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}.$$

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$$0 \le |\frac{o(\rho_1)}{\rho_2}| = |\frac{o(\rho_1)}{\rho_1}||\frac{\rho_1}{\rho_2}| \le C|\frac{o(\rho_1)}{\rho_1}| \implies$$
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$$0 \le |\frac{o(\rho_1)}{\rho_2}| = |\frac{o(\rho_1)}{\rho_1}| \frac{\rho_1}{\rho_2}| \le C \frac{o(\rho_1)}{\rho_1}| \implies \text{DWET } \lim_{\rho_2 \to 0} \frac{o(\rho_1)}{\rho_2} = 0.$$

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$$\rho_1 = \sqrt{(\Delta u)^2 + (\Delta v)^2}, \quad |\Delta x| = \sqrt{(\Delta u)^2 + (\Delta v)^2},$$

$$|\Delta y| = \sqrt{(\Delta u)^2 + (\Delta v)^2}, \quad |\Delta y| = \sqrt{(\Delta u)^2 + (\Delta v)^2},$$

$$2\frac{\rho_{1}}{\rho_{2}} = \sqrt{\frac{(\Delta u)^{2} + (\Delta v)^{2}}{(\Delta x)^{2}}} \; \frac{|\Delta x|}{\sqrt{(\Delta x)^{2} + (\Delta y)^{2}}} + \sqrt{\frac{(\Delta u)^{2} + (\Delta v)^{2}}{(\Delta y)^{2}}} \; \frac{|\Delta y|}{\sqrt{(\Delta x)^{2} + (\Delta y)^{2}}}$$

$$0 \le |\frac{o(\rho_1)}{\rho_2}| = |\frac{o(\rho_1)}{\rho_1}|\frac{\rho_1}{\rho_2}| \le C \frac{o(\rho_1)}{\rho_1}|$$
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$$z = f(u, v), \quad u = \varphi(x, y), \quad v = \psi(x, y) \text{ $\underline{\beta}$ $\underline{\beta}$ $\underline{\alpha}$ \underline{z} = $f(\varphi(x, y), \psi(x, y))$}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}.$$

$$\rho_1 = \sqrt{(\Delta u)^2 + (\Delta v)^2}, \quad \rho_2 = \sqrt{(\Delta x)^2 + (\Delta y)^2},$$

$$2\frac{\rho_1}{\rho_2} = \sqrt{\frac{(\Delta u)^2 + (\Delta v)^2}{(\Delta x)^2}} \frac{|\Delta x|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} + \sqrt{\frac{(\Delta u)^2 + (\Delta v)^2}{(\Delta y)^2}} \frac{|\Delta y|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

$$\frac{1}{\Delta x} = \sqrt{\frac{(\Delta u)^2 + (\Delta v)^2}{(\Delta x)^2}} \frac{|\Delta x|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} + \sqrt{\frac{(\Delta u)^2 + (\Delta v)^2}{(\Delta y)^2}} \frac{|\Delta y|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

$$\leq \sqrt{\frac{(\Delta u)^2 + (\Delta v)^2}{(\Delta x)^2 + (\Delta y)^2}} + \sqrt{\frac{(\Delta u)^2 + (\Delta v)^2}{(\Delta y)^2}} + \sqrt{\frac{(\Delta u)^2 + (\Delta v)^2}{(\Delta y)^2}} + \sqrt{\frac{(\Delta u)^2 + (\Delta v)^2}{(\Delta y)^2}} \leq C (\overline{\eta}, \overline{\eta})$$

$$o(\rho_1)$$
, $o(\rho_1)$, ρ_1 , $o(\rho_1)$, $o(\rho_1)$

$$0 \le |\frac{o(\rho_1)}{\rho_2}| = |\frac{o(\rho_1)}{\rho_1}| \frac{\rho_1}{\rho_2}| \le C \frac{o(\rho_1)}{\rho_1}| \implies 即要证 \lim_{\rho_2 \to 0} \frac{o(\rho_1)}{\rho_2} = 0.$$

$$z = f(u, v), \quad u = \varphi(x, y), \quad v = \psi(x, y) \text{ 复合成 } z = f(\varphi(x, y), \psi(x, y))$$

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链式法则

$$z = f(u, v), \quad u = \varphi(x, y), \quad v = \psi(x, y) \text{ 复合成 } z = f(\varphi(x, y), \psi(x, y))$$

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例
$$z = (x^2 + y^2)^{xy}$$

 $z = f(u, v), \quad u = \varphi(x, y), \quad v = \psi(x, y)$ 复合成 $z = f(\varphi(x, y), \psi(x, y))$ $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}.$ 链式法则

$$z = (x^2 + y^2)^{xy}$$

例

复合函数为
$$z = u^v$$
, $u = x^2 + y^2$, $v = xy$

 $z = f(u, v), \quad u = \varphi(x, y), \quad v = \psi(x, y) \text{ 复合成 } z = f(\varphi(x, y), \psi(x, y))$ $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}.$ [链式法则]

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 $z = (x^2 + y^2)^{xy}$

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

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例
$$z=(x^2+y^2)^{xy}$$
 复合函数为 $z=u^v, \quad u=x^2+y^2, \quad v=xy$

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = vu^{v-1} \cdot 2x + u^v \ln u \cdot y$$
$$= 2x^2 y (x^2 + y^2)^{xy-1} + y (x^2 + y^2)^{xy} \ln(x^2 + y^2)$$

$$z = f(u, v), \quad u = \varphi(x, y), \quad v = \psi(x, y) \text{ 复合成 } z = f(\varphi(x, y), \psi(x, y))$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}.$$
[链式法则]

$$z = (x^2 + y^2)^{xy}$$

复合函数为
$$z = u^v$$
, $u = x^2 + y^2$, $v = xy$

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = vu^{v-1} \cdot 2x + u^v \ln u \cdot y$$
$$= 2x^2 y (x^2 + y^2)^{xy-1} + y (x^2 + y^2)^{xy} \ln(x^2 + y^2)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

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$$=2xy^{2}(x^{2}+y^{2})^{xy-1}+x(x^{2}+y^{2})^{xy}\ln(x^{2}+y^{2})$$

 $\text{ for } z=f(x,y), \quad x=r\cos\theta, \quad y=r\sin\theta$

例 $z = f(x, y), \quad x = r \cos \theta, \quad y = r \sin \theta$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r}$$

∂z _	$\partial z \ \partial x$	$\partial z \ \partial y$	$-\cos \theta \partial z$	t sin 0

 $\frac{\partial}{\partial r} = \frac{\partial}{\partial x} \frac{\partial}{\partial r} + \frac{\partial}{\partial y} \frac{\partial}{\partial r} = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y},$

列	z = f(x, y),	$x = r \cos \theta,$	$y = r\sin\theta$

$$\frac{\partial z}{\partial z} = \frac{\partial z}{\partial z} \frac{\partial x}{\partial z} + \frac{\partial z}{\partial z} \frac{\partial y}{\partial z} = \cos \theta \frac{\partial z}{\partial z} + \sin \theta$$

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$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta}$$

列	z = f(x, y),	$x = r \cos \theta,$	$y = r \sin \theta$

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 $\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta} = -r \sin \theta \frac{\partial z}{\partial x} + r \cos \theta \frac{\partial z}{\partial y}$

$$r\sin\theta$$

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$$z = f(x, y), \quad x = r \cos \theta, \quad y = r \sin \theta$$

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并且
$$\partial z = 1$$
 $\partial z = \partial z = \partial z$

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = |\nabla z|^2$$

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$$z = f(x, y), \quad x = r \cos \theta, \quad y = r \sin \theta$$

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$$\frac{\partial \theta}{\partial \theta} = \frac{\partial x}{\partial x} \frac{\partial \theta}{\partial \theta} + \frac{\partial y}{\partial y} \frac{\partial \theta}{\partial \theta} = -r \sin \theta \frac{\partial x}{\partial x} + r \cos \theta \frac{\partial y}{\partial y}$$
$$\{-\sin \theta, \cos \theta\}$$

$$\partial z$$
 \vec{r} \vec{r}

$$\frac{\partial z}{\partial z} = \nabla z \cdot \frac{\vec{r}}{|\vec{z}|} = \nabla z \cdot \{\cos \theta, \sin \theta\}$$

$$\frac{\partial z}{\partial r} = \nabla z \cdot \frac{\vec{r}}{|\vec{r}|} = \nabla z \cdot \{\cos \theta, \sin \theta\}$$

$$\lim_{\Delta \theta \to 0} \frac{\Delta z}{r \Delta \theta} = \frac{1}{r} \frac{\partial z}{\partial \theta} = \nabla z \cdot \{-\sin \theta, \cos \theta\}$$

例 $z = f(x, y), \quad x = r \cos \theta, \quad y = r \sin \theta$

 $\frac{\partial z}{\partial \theta} = 0 \Rightarrow \nabla z$ 处处上旋向

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} = \cos \theta \frac{\partial z}{\partial x} + \sin \theta \frac{\partial z}{\partial y},$$

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 $\{-\sin\theta,\cos\theta\}$

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$$\frac{\partial z}{\partial \theta} = 0 \Rightarrow \nabla z$$
处处上旋向 $\Rightarrow z = f(x, y)$ 的等高线族为同心圆, 二元函数 $z = f(x, y)$ 只与 r 有关, 即 $z = z(r)$;

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta} = -r \sin \theta \frac{\partial z}{\partial x} + r \cos \theta \frac{\partial z}{\partial y}$$

并且
$$\frac{\{-\sin\theta,\cos\theta\}\}}{\hbar \Box}$$
 旋向 $\{\cos\theta,\sin\theta\}$
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abla z | 2 $\frac{\partial z}{\partial r} = 0 \Rightarrow \nabla z \perp \vec{r}$ $\Rightarrow z = f(x, y) \text{ 的等高线族}$ $\Rightarrow x = z(\theta)$

 $\{\cos\theta,\sin\theta\}$

 $\{-\sin\theta,\cos\theta\}$

链式法则举例:

1.
$$z = f(u, v, w), \quad u = \varphi(x, y), \quad v = \psi(x, y), \quad w = \chi(x, y)$$

链式法则举例:

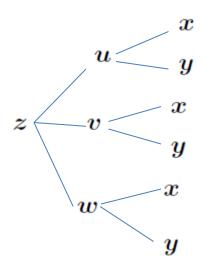
1. $z = f(u, v, w), \quad u = \varphi(x, y), \quad v = \psi(x, y), \quad w = \chi(x, y)$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial z}{\partial w} \cdot \frac{\partial w}{\partial x}$$
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链式法则举例:

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 $2. z = f(u), \quad u = \varphi(x, y)$

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$$z$$
 — u $\stackrel{x}{<}_y$

$$2. z = f(u), \quad u = \varphi(x, y)$$

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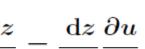
例
$$u=f(r), \quad r=\sqrt{x^2+y^2+z^2}$$

$$\frac{\partial z}{\partial x} = \frac{\mathrm{d}z}{\mathrm{d}u} \frac{\partial u}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\mathrm{d}z}{\mathrm{d}u} \frac{\partial u}{\partial y}.$$

例
$$u=f(r), \quad r=\sqrt{x^2+y^2+z^2}$$

$$f(r), \quad r = \sqrt{x^2 + y^2} +$$

$$rac{\partial u}{\partial y} = rac{\mathrm{d}u}{\mathrm{d}r} rac{\partial r}{\partial y} = f'(r) rac{y}{r}$$
 $rac{\partial u}{\partial z} = rac{\mathrm{d}u}{\mathrm{d}r} rac{\partial r}{\partial z} = f'(r) rac{z}{r}$

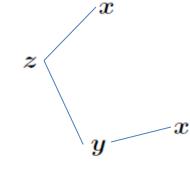


3. $z = f(u, v), u = \varphi(x), v = \psi(x)$

3. $z = f(u, v), \quad u = \varphi(x), \quad v = \psi(x)$

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial f}{\partial u} \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial f}{\partial v} \frac{\mathrm{d}v}{\mathrm{d}x}$$

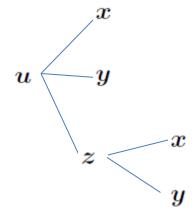
4. $z = f(x, y), \quad y = \varphi(x)$



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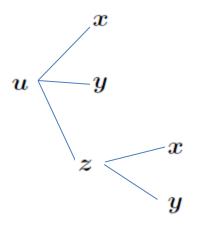
$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \varphi'(x)$$

5. $u = f(x, y, z), \quad z = \varphi(x, y)$



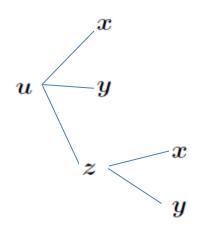
5. $u = f(x, y, z), \quad z = \varphi(x, y)$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial y},} \boxed{?}$$



5. $u = f(x, y, z), \quad z = \varphi(x, y)$

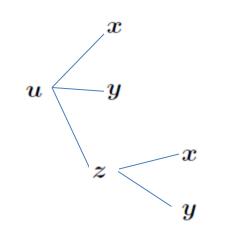
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial y},$$
$$f'_{x}$$
$$f'_{y}$$



5.
$$u = f(x, y, z), \quad z = \varphi(x, y)$$

$$rac{\partial u}{\partial x} = rac{\partial u}{\partial x} + rac{\partial u}{\partial z} rac{\partial z}{\partial x}, \quad rac{\partial u}{\partial y} = rac{\partial u}{\partial y} + rac{\partial u}{\partial z} rac{\partial z}{\partial y}, \ f'_y$$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial y}$$



可以通过求全微分求全部偏导数
$$z=f(u,v),\quad u=\varphi(x,y),\quad v=\psi(x,y)$$

$$\mathrm{d}z=f_u'\,\mathrm{d}u+f_v'\,\mathrm{d}v$$

 $dz = f_u' du + f_v' dv$

$$z=f(u,v),\;\;u=arphi(x,y),\;\;v=\psi(x,y)$$

 $= f'_u[\varphi'_x dx + \varphi'_y dy] + f'_v[\psi'_x dx + \psi'_y dy]$

可以通过來全微分來全部偏导数
$$z=f(u,v),\;\;u=arphi(x,y),\;\;v=\psi(x,y)$$

 $dz = f'_u du + f'_v dv$

 $= f'_{u}[\varphi'_{x} dx + \varphi'_{u} dy] + f'_{v}[\psi'_{x} dx + \psi'_{u} dy]$

 $= (f'_u \varphi'_x + f'_v \psi'_x) \, dx + (f'_u \varphi'_u + f'_v \psi'_u) \, dy$

可以通过水主阀刀水主部拥守致
$$z=f(u,v),\;\;u=arphi(x,y),\;\;v=\psi(x,y)$$

 $dz = f'_u du + f'_v dv$

 $= f'_{u}[\varphi'_{x} dx + \varphi'_{u} dy] + f'_{v}[\psi'_{x} dx + \psi'_{u} dy]$

 $= (f'_{u}\varphi'_{x} + f'_{v}\psi'_{x}) dx + (f'_{u}\varphi'_{y} + f'_{v}\psi'_{y}) dy$

 $\frac{\partial z}{\partial x} = f'_u \varphi'_x + f'_v \psi'_x, \quad \frac{\partial z}{\partial y} = f'_u \varphi'_y + f'_v \psi'_y.$

例
$$z = \arctan \frac{y}{x}$$

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$$\mathrm{d}z = \frac{\mathrm{d}(\frac{y}{x})}{1 + (\frac{y}{x})^2}$$

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$$z = \arctan \frac{y}{x}$$

$$dz = \frac{d(\frac{y}{x})}{1 + (\frac{y}{x})^2} = \frac{\frac{x \, dy - y \, dx}{x^2}}{\frac{x^2 + y^2}{x^2}}$$

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$$z = \arctan \frac{y}{x}$$

$$dz = \frac{d(\frac{y}{x})}{1 + (\frac{y}{x})^2} = \frac{\frac{x \, dy - y \, dx}{x^2}}{\frac{x^2 + y^2}{x^2}}$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-y}{x^2 + y^2}, \quad \frac{\partial z}{\partial y} = \frac{x}{x^2 + y^2}.$$

例
$$z = (x^2 + y^2)^{100}$$

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$$dz = 100(x^2 + y^2)^{99} d(x^2 + y^2)$$
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例
$$z = (x^2 + y^2)^{100}$$

$$dz = 100(x^2 + y^2)^{99} d(x^2 + y^2)$$
$$= 200(x^2 + y^2)^{99} (x dy + y dx)$$

$$\Rightarrow \frac{\partial z}{\partial x} = 200(x^2 + y^2)^{99}, \quad \frac{\partial z}{\partial y} = 200y(x^2 + y^2)^{99}.$$

$$u = f(r), \quad r = \sqrt{x^2 + y^2 + z^2}$$

$$u = f(r), \quad r = \sqrt{x^2 + y^2 + z^2}$$

$$du = f'(r) dr = f'(r) \frac{x dx + y dy + z dz}{r}$$

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$$du = f'(r) dr = f'(r) \frac{x dx + y dy + z dz}{r}$$

$$\frac{\partial u}{\partial x} = \frac{x}{r}f'(r), \quad \frac{\partial u}{\partial y} = \frac{y}{r}f'(r), \quad \frac{\partial u}{\partial z} = \frac{z}{r}f'(r).$$