多元复合函数的微分法

$$z = f(u,v), \quad u = \varphi(x,y), \quad v = \psi(x,y)$$

复合成 $z = f(\varphi(x,y), \psi(x,y))$

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定理

设$z = f(u,v)$, $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial v}$ 在(u,v)点连续,

且$u = \varphi(x,y)$ $v = \psi(x,y)$的偏导数 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ 在 (x,y) 点连续

则$\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$ 在 (x,y) 点存在且连续,且

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证明：记 $\rho_1=\sqrt{(\Delta u)^2+(\Delta v)^2}$, $\rho_2=\sqrt{(\Delta x)^2+(\Delta y)^2}$,

则由于 $u=\varphi(x,y), v=\psi(x,y)$ 连续,有 $\rho_2\to 0\Rightarrow \rho_1\to 0.$

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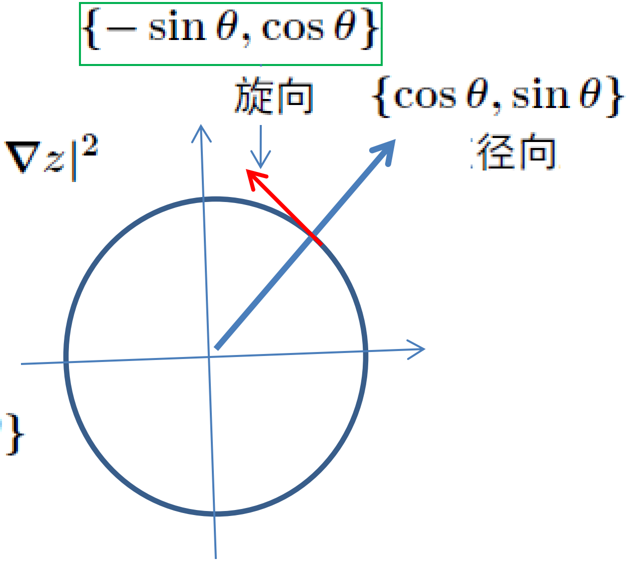

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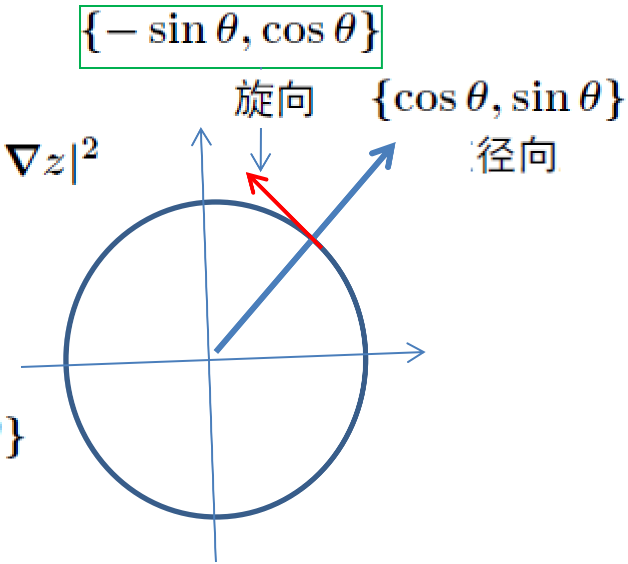

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二元函数 $z = f(x, y)$ 只与 r 有关,

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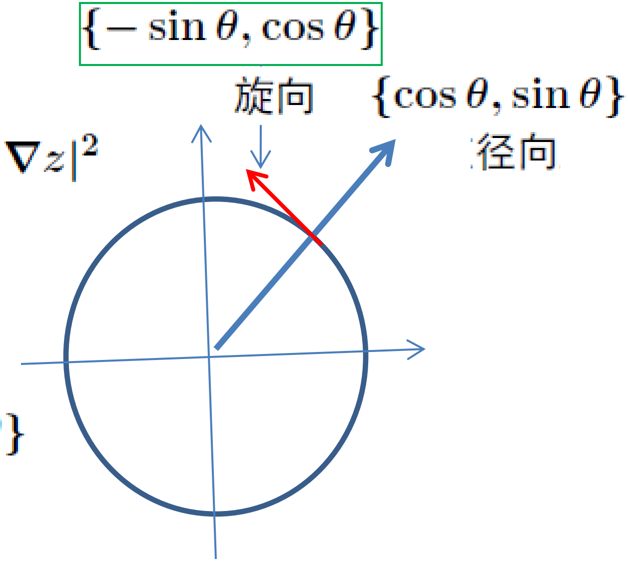

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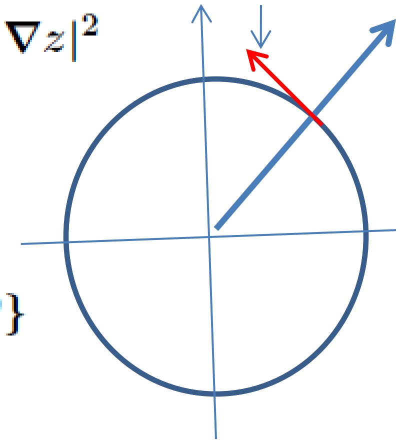
$\boxed{\{-\sin\theta, \cos\theta\}}$

旋向 $\quad \{\cos\theta, \sin\theta\}$

径向



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为辐射线族 $y = kx$

$\Rightarrow \quad z = z(\theta)$

链式法则举例：

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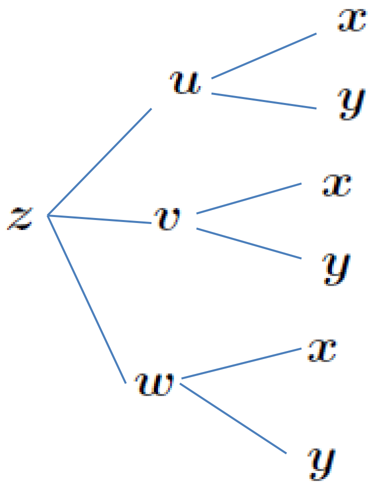

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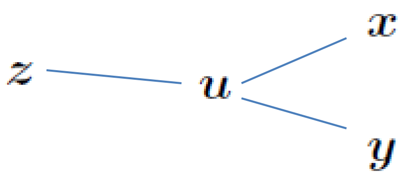

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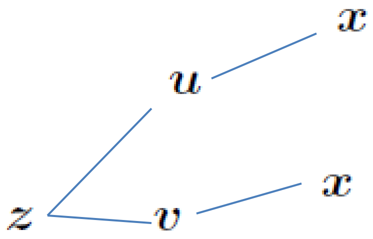

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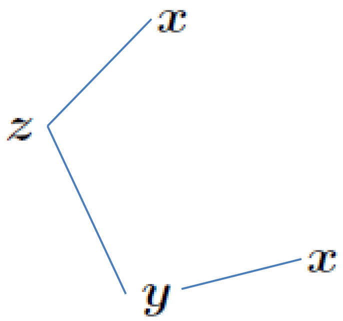

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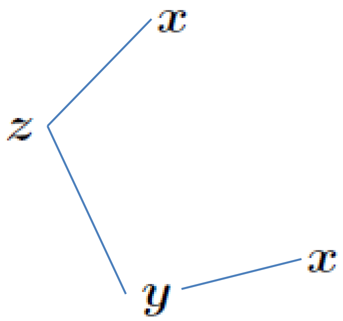

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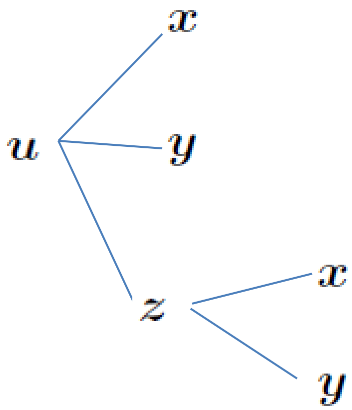

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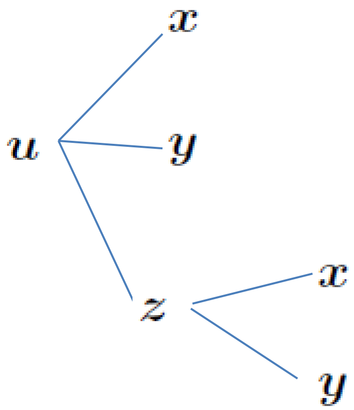

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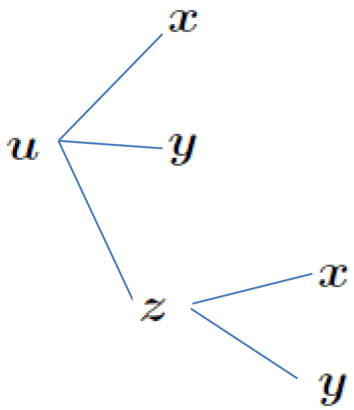

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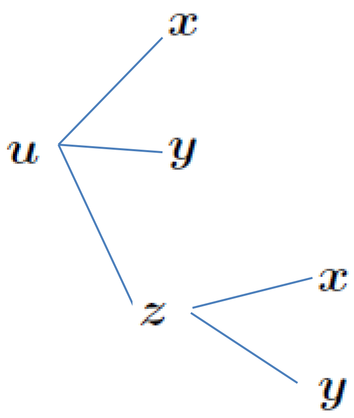

可以通过求全微分求全部偏导数

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\end{aligned}$$

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$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-y}{x^2 + y^2}, \quad \frac{\partial z}{\partial y} = \frac{x}{x^2 + y^2}.$$

例 $$z = (x^2 + y^2)^{100}$$

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