ECON 139 - Intermediate Financial Economics

Scribe Notes for Lecture 11

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1 Risk Aversion

1.1 Theorem 1

Let $a^*(w_0) = \arg\max_a E(u(\tilde{w}_1))$. Then, the theorem says that,

(a) If $R'_A(w) < 0$, then $\frac{da^*(w_0)}{dw_0} > 0$. (Declining absolute risk aversion: power utility function)

If the utility function exhibits declining absolute risk aversion, the investor will invest more dollars in the risky asset, when he/she is more wealthy.

(b) If $R'_A(w) = 0$, then $\frac{da^*(w_0)}{dw_0} = 0$. (Constant absolute risk aversion: log utility function)

If the utility function exhibits constant absolute risk aversion, the investor will invest the same amount of dollars in the risky asset, when he/she is more wealthy.

(c) If $R'_A(w) > 0$, then $\frac{da^*(w_0)}{dw_0} < 0$. (Increasing absolute risk aversion: quadratic utility function)

If the utility function exhibits increasing absolute risk aversion, the investor will invest less dollars in the risky asset, when he/she is more wealthy.

1.2 Example 1

Assume $u(w) = w - bw^2$, $b > 0$, then the absolute risk aversion coefficient,

$$R_A(w) = \frac{2b}{1 - 2bw} > 0,$$

as long as $1 - 2bw > 0$. The first derivative of the absolute risk aversion coefficient,

$$R'_A(w) = \frac{4b^2}{(1 - 2bw)^2} > 0.$$

Hence, the utility function exhibit increasing absolute risk aversion.

1.3 Theorem 2

Let $\eta = \frac{da^*(w_0)}{dw_0}\frac{w_0}{a^*(w_0)} = \frac{da^*(w_0)/a^*(w_0)}{dw_0/w_0}$ denote the elasticity of a^* with respect to w_0. Furthermore, let $a^*(w_0) = \arg\max_a E(u(\tilde{w}_1))$. Then, the theorem says that

(a) If $R'_A(w) < 0$, then $\eta > 1$.

(b) If $R'_A(w) = 0$, then $\eta = 1$.

(c) If $R'_A(w) > 0$, then $\eta < 1$.

1.4 Example 2

Assume that the investor is risk neutral. The utility function is $u(w) = c + dw$, $d > 0$. Furthermore, assume that c=0 (for simplicity). Then, the maximization problem under linear utility is

$$\begin{aligned}
&\max_a E(d(w_0(1+r_f) + a(\tilde{r} - r_f))) && \Leftrightarrow \\
&\max_a dw_0(1+r_f) + daE(\tilde{r} - r_f) && \Leftrightarrow \\
&\max_a daE(\tilde{r} - r_f) && \Leftrightarrow \\
&\max_a aE(\tilde{r} - r_f)
\end{aligned}$$

Thus, a risk neutral investor will invest as much as he/she possibly can, since a is positive and the investor maximizes expected returns.

2 Intertemporal choice

The agent solves the following utility maximization problem

$$\max_{C_0, C_1} u(C_0) + \frac{1}{1+\delta}u(C_1) \qquad s.t. \qquad C_0 + \frac{C_1}{1+r} = M_0 + \frac{M_1}{1+r}.$$

Now we rewrite the problem in terms of savings. First we assume that $M_1 = 0$. This means that the agent can only save. Let $S = M_0 - C_0$ denote savings in period 0. Then $C_0 = M_0 - S$ and $C_1 = S(1+r_f)$. Then, the maximization problem can be rewritten as

$$\max_S u(M_0 - S) + \frac{1}{1+\delta}u(S(1+r_f)).$$

The first order condition is

$$\begin{aligned}
&-u'(M_0 - S^*) + \frac{1+r_f}{1+\delta}u'(S^*(1+r_f)) = 0 && \Leftrightarrow \\
&\frac{u'(M_0 - S^*)}{u'(S^*(1+r_f))} = \frac{1+r_f}{1+\delta} && \Leftrightarrow \\
&\frac{u'(M_0 - S^*)}{u'(S^*(1+r_f))} = \frac{u'(C_0^*)}{u'(C_1^*)}
\end{aligned}$$

Case 1: $r_f = \delta$

The first order condition is

$$\begin{aligned}
u'(M_0 - S^*) &= u'(S^*(1+r_f)) && \Leftrightarrow \\
M_0 - S^* &= S^*(1+r_f) && \Leftrightarrow \\
S^* &= \frac{M_0}{2+r_f}.
\end{aligned}$$

Case 2: $r_f > \delta$

The first order condition is

$$\begin{aligned}
u'(M_0 - S^*) &> u'(S^*(1+r_f)) && \Leftrightarrow \\
M_0 - S^* &< S^*(1+r_f) && \Leftrightarrow \\
S^* &> \frac{M_0}{2+r_f}.
\end{aligned}$$

2.1 Example 1

Now assume that the investor can only invest in a risky asset. The risky asset has return $\tilde{r}$, so $\tilde{R} = 1 + \tilde{r}$. Also let $\gamma = \frac{1}{1+\delta}$. The, the maximization problem can be written as

$$\max_S E(u(M_0 - S) + \delta u(S\tilde{R}))$$

The first order condition is

$$\begin{aligned}
E(-u'(M_0 - S^*) + \gamma u'(S^*\tilde{R})\tilde{R}) = 0 && \Leftrightarrow \\
E(\gamma u'(S^*\tilde{R})\tilde{R}) = u'(M_0 - S^*)
\end{aligned}$$

Consider two possible return distributions $\tilde{R_A}$ and $\tilde{R_B}$. Let

$$\begin{aligned}
\tilde{R_B} &= \tilde{R_A} + \epsilon \\
\sigma_B^2 &= \sigma_A^2 + \sigma_\epsilon^2,
\end{aligned}$$

where $E(\epsilon) = 0$ and $E(\epsilon\tilde{R_A}) = 0$, so that $\tilde{R_B}$ is a mean-preserving spread of $\tilde{R_A}$. This condition says that the covariance between the noise term and $\tilde{R_A}$ is zero. Now let us look at two examples: 1) for quadratic utility functions and 2) in the general case.

2.1.1 Example 1.1: Quadratic Utility

Consider the utility function $u(c) = c - bc^2$, with $b > 0$ and $u'(c) = 1 - 2bc$. Then the first order condition is

$$\begin{aligned}
E(\gamma(1 - 2b(S^*\tilde{R})\tilde{R}) = 1 - 2b(M_0 - S^*) && \Leftrightarrow \\
S^* = \frac{\gamma E(\tilde{R}) + 2bM_0 - 1}{2b(1+\gamma)E(\tilde{R}^2)}.
\end{aligned}$$

It appears that if the riskiness, $E(\tilde{R})$, increases, then savings, S^*, decline.

2.1.2 Example 1.2: More General Case

Let $g(\tilde{R}) = \gamma u'(S^*\tilde{R})\tilde{R}$. Notice that

$$g'(\tilde{R}) = \gamma u''(S^*\tilde{R})S^*\tilde{R} + \gamma u'(S^*\tilde{R})$$
$$g''(\tilde{R}) = \gamma u'''(S^*\tilde{R})(S^*)^2\tilde{R} + \gamma u'(S^*\tilde{R})S^* + \gamma u'(S^*\tilde{R})S^* = \gamma u'''(S^*\tilde{R})(S^*)^2\tilde{R} + 2\gamma u'(S^*\tilde{R})S^*.$$

In this more general case, the first order condition is

$$E(g(\tilde{R})) = u'(M_0 - S^*)$$

It is apparent that

(a) if $g(\tilde{R})$ is linear, then $E(g(\tilde{R}))$ does not change as $\tilde{R}$ gets riskier. Therefore, S^* does not change as $\tilde{R}$ gets riskier.

(b) if $g(\tilde{R})$ is concave, then $E(g(\tilde{R}))$ decreases as $\tilde{R}$ gets riskier. Therefore, S^* decreases as $\tilde{R}$ gets riskier.

(c) if $g(\tilde{R})$ is convex, then upward deviations are stronger than downward deviations. Hence, $E(g(\tilde{R}))$ and S^* increases as $\tilde{R}$ gets riskier.

2.2 Theorem 1: Rothschild and Stiglitz

Let $\tilde{r_A}$ and $\tilde{r_B}$ be two return distributions, such that $\tilde{r_B}$ is a mean-preserving spread of $\tilde{r_A}$, and let s_A^* and s_B^* be the optimal savings. Then,

(a) if $R_R'(w_0) \leq 0$ and $R_R(w_0) > 1$, then $s_A^* < s_B^*$.

   Hence, as we go from the less risky to the more risky distribution, savings will increase.

(b) if $R_R'(w_0) \geq 0$ and $R_R(w_0) < 1$, then $s_A^* > s_B^*$.

   Hence, as we go from the less risky to the more risky distribution, savings will decrease.

2.3 Example 2

Let $u(w) = \frac{w^{1-\gamma)}}{1-\gamma}$, so that $u'(w) = w^{-\gamma}$ and $u''(w) = -\gamma w^{-\gamma-1}$. Then, the absolute and relative risk-aversion coefficients are

$$R_A(w) = \frac{\gamma}{w}$$
$$R_R(w) = \gamma.$$

In this case, we have constant relative risk aversion, $R_R'(w) = 0$. Using the Rothschild and Stiglitz theorem, we know that

(a) if $\gamma < 1$, then as riskiness increases, savings increases.

(b) if $\gamma > 1$, then as riskiness increases, savings decreases.

(c) if $\gamma = 1$, then as riskiness increases, savings stay the same.

3 Prudence

3.1 Definitions

The absolute prudence coefficient is defined as

$$P_A(w) = -\frac{u'''(w)}{u''(w)}.$$

The relative prudence coefficient is defined as

$$P_R(w) = -\frac{u'''(w)}{u''(w)}w.$$

3.2 Theorem 1

The theorem says that,

(a) If $P_A(w) > 2$, then savings increases, $S_A^* < S_B^*$, when riskiness increases.

(b) If $P_A(w) < 2$, then savings decreases, $S_A^* > S_B^*$, when riskiness increases.

Proof of (a):
We know from the general case example in section 2.1.2 that when $g(\tilde{R})$ is convex, $g''(\tilde{R}) > 0$, then savings increase, $S_A^* < S_B^*$, when riskiness increases. Hence,

$$\begin{aligned}
&g''(\tilde{R}) > 0 && \Leftrightarrow \\
&\gamma u'''(S^*\tilde{R})(S^*)^2\tilde{R} + 2\gamma u'(S^*\tilde{R})S^* > 0 && \Leftrightarrow \\
&-\frac{S^*\tilde{R}u'''(S^*\tilde{R})}{u''(S^*\tilde{R})} > 2,
\end{aligned}$$

which proves (a).

3.3 Example 1: Power Utility Function

Let $u(w) = \frac{w^{1-\gamma}}{1-\gamma}$, so that $u'(w) = w^{-\gamma}$, $u''(w) = -\gamma w^{-\gamma-1}$ and $u'''(w) = \gamma(1-\gamma)w^{-\gamma-2}$. Then, the relative prudence coefficient is

$$P_R(w) = -\frac{\gamma(1-\gamma)w^{-\gamma-2}}{-\gamma w^{-\gamma-1}} \cdot w = \gamma + 1.$$

I.e. when $\gamma > 1$, then savings increases, wehn riskiness increases.