函数的渐近线

本段内容：

函数竖直渐近线、水平渐近线和斜渐近线的概念和求法

若点M沿曲线$y = f(x)$无限远离原点时，
它与某条固定直线L之间的距离无限接近于0，
则称直线L为曲线$y = f(x)$的一条渐近线.

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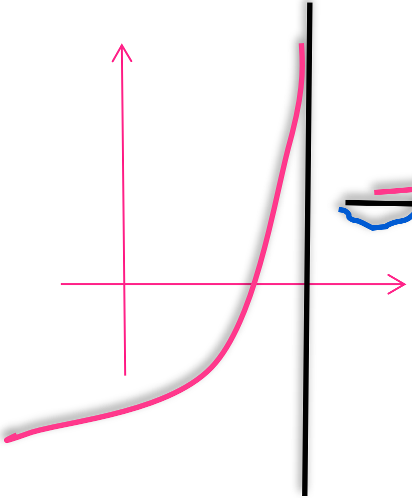
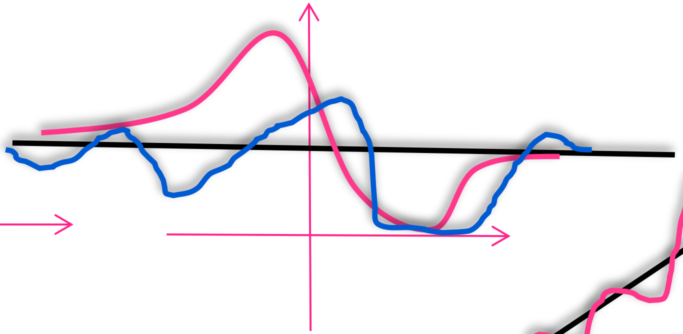
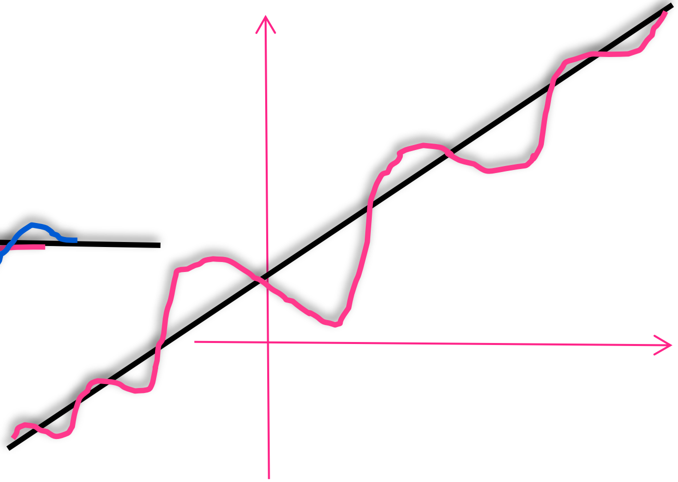
竖直渐近线



水平渐近线



斜渐近线

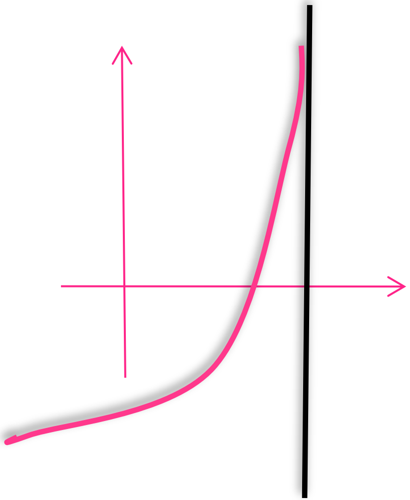


①若$\lim\limits_{x\to a} f(x)=\infty$,

或$\lim\limits_{x\to a-0} f(x)=\infty$,

或$\lim\limits_{x\to a+0} f(x)=\infty$,

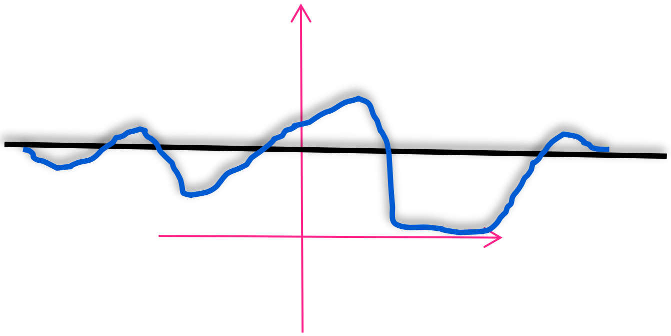
则$x=a$称为$y=f(x)$的竖直渐近线.



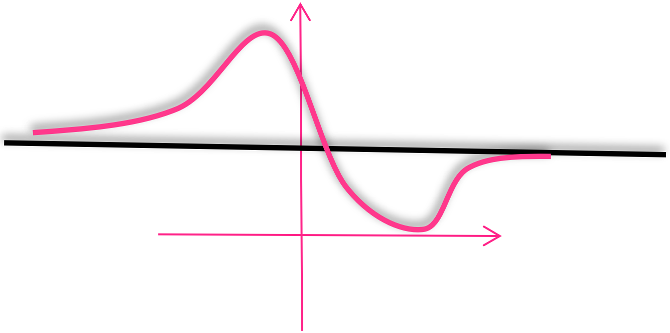
②若 $\lim\limits_{x\to\infty} f(x) = A$,

或 $\lim\limits_{x\to-\infty} f(x) = A$,

或 $\lim\limits_{x\to+\infty} f(x) = A$,



则$y = A$称为$y = f(x)$的水平渐近线.



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$$\text{则}\lim_{x\to\infty}[f(x)-(kx+b)]=0,$$

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故有:

若 $\lim\limits_{x \to +\infty} \dfrac{f(x)}{x} = k(\neq 0)$，且 $\lim\limits_{x \to +\infty} [f(x) - kx]$存在，记作$b$.

则$y = f(x)$有渐近线$y = kx + b$.

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故有:

$$\boxed{\begin{array}{l}\text{若}\lim\limits_{x\to-\infty}\dfrac{f(x)}{x}=k(\neq 0),\text{且}\lim\limits_{x\to-\infty}[f(x)-kx]\text{存在，记作}b.\\ \text{则}y=f(x)\text{有渐近线}y=kx+b\end{array}}$$

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又因为 $\lim\limits_{x\to\infty}\dfrac{y}{x}=1$,

$$\lim_{x\to\infty}(y-x)=\lim_{x\to\infty}\left[\frac{x^2}{x-1}-x\right]=\lim_{x\to\infty}\frac{x}{x-1}=1.$$

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$\therefore$有斜渐近线$y = x + 1$.

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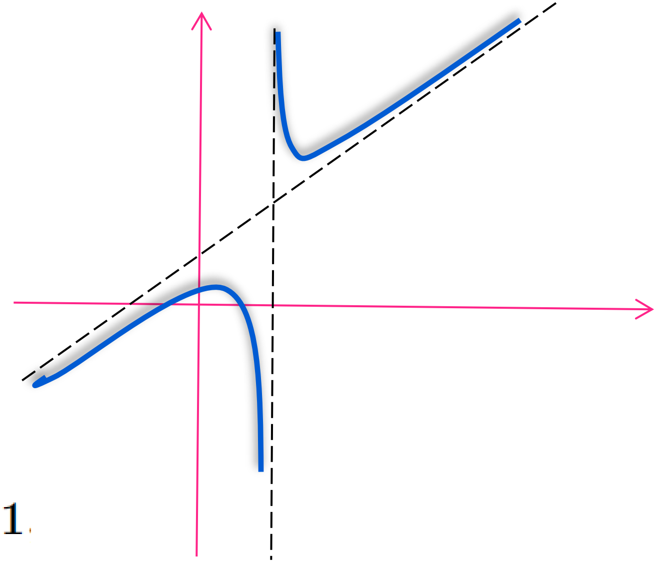

定理:

$y=f(x)$有斜渐近线$y=kx+b$的充分必要条件是下列三条至少一条成立

(1) $\lim\limits_{x\to+\infty}\dfrac{f(x)}{x}=k$且$\lim\limits_{x\to+\infty}[f(x)-kx]=b$;

(2) $\lim\limits_{x\to-\infty}\dfrac{f(x)}{x}=k$且$\lim\limits_{x\to-\infty}[f(x)-kx]=b$;

(3) $\lim\limits_{x\to\infty}\dfrac{f(x)}{x}=k$且$\lim\limits_{x\to\infty}[f(x)-kx]=b$.

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