Homework Assignment #2
Due: In class, two weeks after distribution

§1 Bernoulli Trials and Conditional Expectation

Let $X_1, X_2, \ldots$ be independent identically distributed random variables with Bernoulli distribution, i.e.

$$\mathbb{P}(X_i = 1) = p, \quad \mathbb{P}(X_i = 0) = 1 - p,$$

where $p \in (0, 1)$. Let $S_k = \sum_{i=1}^{k} X_i$. For $i \leq m < n$, Evaluate $E(X_i|S_n)$ and $E(S_m|S_n)$. And, give an intuitive probabilistic interpretation of the two results you just obtained. In other words, what does this result describe?

§2 Conditional Variance Formula

Prove that, for any random variable X and Y,

$$\mathrm{Var}(X) = \mathrm{Var}[\mathbb{E}(X|Y)] + \mathbb{E}[\mathrm{Var}(X|Y)],$$

where $\mathrm{Var}(X|Y) := E((X - E(X|Y))^2 \,|Y)$.

§3 Conditional Distribution from Normal Vectors

Let (X, Y) be a bivariate normal variable with correlation ρ. Denote $\mathrm{Var}X = \sigma_X^2$ and $\mathrm{Var}Y = \sigma_Y^2$. Find the conditional distribution of X given Y.

§4 A Characterization of the Transition Probability of Markov Processes

Suppose $\{X_n\}$ is a discrete-time Markov process (Markov chain) taking values in a discrete set E. Prove that for any $0 < m < k$, we have

$$\mathbb{P}(X_{n+k} = a|X_n = b) = \sum_{c \in E} \mathbb{P}(X_{n+k} = a|X_{n+m} = c)\mathbb{P}(X_{n+m} = c|X_n = b),$$

for any arbitrary integer n and $a, b \in E$.

Can you write an analogous result for a continous-time Markov process taking values in a continuous set (e.g., the whole set of real numbers)?

§5 Poisson Process and Conditional Expectation

Suppose that $N(t)$ is a Poisson process with parameter $\lambda = 1$. Find $E(N(1)|N(2))$ and $E(N(2)|N(1))$.

§6 Roll a Dice Again

You can roll a dice no more than three times. You can stop immediately after the first roll, or immediately after the second, or you can wait for the third. I will pay you the same number of dollars as there are dots on the single upturned face on your last roll (roll number three unless you stop sooner? What is your rolling strategy? If you were running this game, how much should you charge players for repeated plays of this game?

§7 Poisson Process: Conditional Distribution of Arrival Times

For a Poisson process $\{N(t)\}$, given $N(T) = n$ for some $T > 0$, find the joint distribution of the arrival times $S_1, S_2, \cdots, S_n$.