The maximum principle also proves the stability, but with a different way to measure nearness. Consider two solutions of (3) in a rectangle. We then have $w \equiv u_1 - u_2 = 0$ on the lateral sides of the rectangle and $w = \phi_1 - \phi_2$ on the bottom. The maximum principle asserts that throughout the rectangle

$$u_1(x, t) - u_2(x, t) \leq \max|\phi_1 - \phi_2|.$$

The "minimum" principle says that

$$u_1(x, t) - u_2(x, t) \geq -\min|\phi_1 - \phi_2|.$$

Therefore,

$$\max_{0 \leq x \leq l} |u_1(x, t) - u_2(x, t)| \leq \max_{0 \leq x \leq l} |\phi_1(x) - \phi_2(x)|, \tag{6}$$

valid for all $t > 0$. Equation (6) is in the same spirit as (5), but with a quite different method of measuring the nearness of functions. It is called stability in the "uniform" sense.

EXERCISES

1. Consider the solution $1 - x^2 - 2kt$ of the diffusion equation. Find the locations of its maximum and its minimum in the closed rectangle $\{0 \leq x \leq 1, 0 \leq t \leq T\}$.
2. Consider a solution of the diffusion equation $u_t = u_{xx}$ in $\{0 \leq x \leq l, 0 \leq t < \infty\}$.
   (a) Let $M(T)$ = the maximum of $u(x, t)$ in the rectangle $\{0 \leq x \leq l, 0 \leq t \leq T\}$. Does $M(T)$ increase or decrease as a function of T?
   (b) Let $m(T)$ = the minimum of $u(x, t)$ in the rectangle $\{0 \leq x \leq l, 0 \leq t \leq T\}$. Does $m(T)$ increase or decrease as a function of T?
3. Consider the diffusion equation $u_t = u_{xx}$ in the interval $(0, 1)$ with $u(0, t) = u(1, t) = 0$ and $u(x, 0) = 1 - x^2$. Note that this initial function does not satisfy the boundary condition at the left end, but that the solution will satisfy it for all $t > 0$.
   (a) Show that $u(x, t) > 0$ at all interior points $0 < x < 1, 0 < t < \infty$.
   (b) For each $t > 0$, let $\mu(t)$ = the maximum of $u(x, t)$ over $0 \leq x \leq 1$. Show that $\mu(t)$ is a decreasing (i.e., nonincreasing) function of t. (*Hint:* Let the maximum occur at the point $X(t)$, so that $\mu(t) = u(X(t), t)$. Differentiate $\mu(t)$, assuming that $X(t)$ is differentiable.)
   (c) Draw a rough sketch of what you think the solution looks like (u versus x) at a few times. (If you have appropriate software available, compute it.)
4. Consider the diffusion equation $u_t = u_{xx}$ in $\{0 < x < 1, 0 < t < \infty\}$ with $u(0, t) = u(1, t) = 0$ and $u(x, 0) = 4x(1 - x)$.
   (a) Show that $0 < u(x, t) < 1$ for all $t > 0$ and $0 < x < 1$.
   (b) Show that $u(x, t) = u(1 - x, t)$ for all $t \geq 0$ and $0 \leq x \leq 1$.
   (c) Use the energy method to show that $\int_0^1 u^2\, dx$ is a strictly decreasing function of t.
5. The purpose of this exercise is to show that the maximum principle is not true for the equation $u_t = xu_{xx}$, which has a variable coefficient.
   (a) Verify that $u = -2xt - x^2$ is a solution. Find the location of its maximum in the rectangle $\{-2 \leq x \leq 2, 0 \leq t \leq 1\}$.
   (b) Where precisely does our proof of the maximum principle break down for this equation?
6. Prove the comparison principle for the diffusion equation: If u and v are two solutions, and if $u \leq v$ for $t = 0$, for $x = 0$, and for $x = l$, then $u \leq v$ for $0 \leq t < \infty, 0 \leq x \leq l$.
7. (a) More generally, if $u_t - ku_{xx} = f$, $v_t - kv_{xx} = g$, $f \leq g$, and $u \leq v$ at $x = 0$, $x = l$ and $t = 0$, prove that $u \leq v$ for $0 \leq x \leq l, 0 \leq t$.
   (b) If $v_t - v_{xx} \geq \sin x$ for $0 \leq x \leq \pi$, $0 < t < \infty$, and if $v(0, t) \geq 0$, $v(\pi, t) \geq 0$ and $v(x, 0) \geq \sin x$, use part (a) to show that $v(x, t) \geq (1 - e^{-t}) \sin x$.
8. Consider the diffusion equation on $(0, l)$ with the Robin boundary conditions $u_x(0, t) - a_0 u(0, t) = 0$ and $u_x(l, t) + a_l u(l, t) = 0$. If $a_0 > 0$ and $a_l > 0$, use the energy method to show that the endpoints contribute to the decrease of $\int_0^l u^2(x, t)\, dx$. (This is interpreted to mean that part of the "energy" is lost at the boundary, so we call the boundary conditions "*radiating*" or "*dissipative.*")

2.4 DIFFUSION ON THE WHOLE LINE

Our purpose in this section is to solve the problem

$$\boxed{\begin{aligned} &u_t = ku_{xx} \qquad (-\infty < x < \infty, 0 < t < \infty) \\ &u(x, 0) = \phi(x). \end{aligned}} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

As with the wave equation, the problem on the infinite line has a certain "purity," which makes it easier to solve than the finite-interval problem. (The effects of boundaries will be discussed in the next several chapters.) Also as with the wave equation, we will end up with an explicit formula. But it will be derived by a method *very different* from the methods used before. (The characteristics for the diffusion equation are just the lines t = constant and play no major role in the analysis.) Because the solution of (1) is not easy to derive, we first set the stage by making some general comments.

Our method is to solve it for a *particular* $\phi(x)$ and then build the general solution from this particular one. We'll use five basic *invariance properties* of the diffusion equation (1).

(a) The *translate* $u(x - y, t)$ of any solution $u(x, t)$ is another solution, for any fixed y.

(b) Any *derivative* (u_x or u_t or u_{xx}, etc.) of a solution is again a solution.

(c) A *linear combination* of solutions of (1) is again a solution of (1). (This is just linearity.)

(d) An *integral* of solutions is again a solution. Thus if $S(x, t)$ is a solution of (1), then so is $S(x-y, t)$ and so is

$$v(x, t) = \int_{-\infty}^{\infty} S(x-y, t) g(y)\, dy$$

for any function $g(y)$, as long as this improper integral converges appropriately. (We'll worry about convergence later.) In fact, (d) is just a limiting form of (c).

(e) If $u(x, t)$ is a solution of (1), so is the *dilated* function $u(\sqrt{a}\, x, at)$, for any $a > 0$. Prove this by the chain rule: Let $v(x, t) = u(\sqrt{a}\, x, at)$. Then $v_t = [\partial(at)/\partial t] u_t = a u_t$ and $v_x = [\partial(\sqrt{a}\, x)/\partial x] u_x = \sqrt{a}\, u_x$ and $v_{xx} = \sqrt{a} \cdot \sqrt{a}\, u_{xx} = a u_{xx}$.

Our goal is to find a particular solution of (1) and then to construct all the other solutions using property (d). The particular solution we will look for is the one, denoted $Q(x, t)$, which satisfies the *special initial condition*

$$\boxed{Q(x, 0) = 1 \quad \text{for } x > 0 \qquad Q(x, 0) = 0 \quad \text{for } x < 0.} \tag{3}$$

The reason for this choice is that this initial condition does not change under dilation. We'll find Q in three steps.

Step 1 We'll look for $Q(x, t)$ of the special form

$$Q(x, t) = g(p) \qquad \text{where } p = \frac{x}{\sqrt{4kt}} \tag{4}$$

and g is a function of only one variable (to be determined). (The $\sqrt{4k}$ factor is included only to simplify a later formula.)

Why do we expect Q to have this special form? Because property (e) says that the equation (1) doesn't "see" the dilation $x \to \sqrt{a}\, x$, $t \to at$. Clearly, (3) doesn't change at all under the dilation. So $Q(x, t)$, which is defined by the conditions (1) and (3), ought not see the dilation either. How could that happen? In only one way: if Q depends on x and t solely through the combination $x/\sqrt{t}$. For the dilation takes $x/\sqrt{t}$ into $\sqrt{a}x/\sqrt{at} = x/\sqrt{t}$. Thus let $p = x/\sqrt{4kt}$ and look for Q which satisfies (1) and (3) and has the form (4).

Step 2 Using (4), we convert (1) into an ODE for g by use of the chain rule:

$$Q_t = \frac{dg}{dp}\frac{\partial p}{\partial t} = -\frac{1}{2t}\frac{x}{\sqrt{4kt}} g'(p)$$

$$Q_x = \frac{dg}{dp}\frac{\partial p}{\partial x} = \frac{1}{\sqrt{4kt}} g'(p)$$

$$Q_{xx} = \frac{dQ_x}{dp}\frac{\partial p}{\partial x} = \frac{1}{4kt} g''(p)$$

$$0 = Q_t - kQ_{xx} = \frac{1}{t}\left[-\frac{1}{2} p g'(p) - \frac{1}{4} g''(p)\right].$$

Thus

$$g'' + 2pg' = 0.$$

This ODE is easily solved using the integrating factor $\exp \int 2p\, dp = \exp(p^2)$. We get $g'(p) = c_1 \exp(-p^2)$ and

$$Q(x, t) = g(p) = c_1 \int e^{-p^2}\, dp + c_2$$

Step 3 We find a completely explicit formula for Q. We've just shown that

$$Q(x, t) = c_1 \int_0^{x/\sqrt{4kt}} e^{-p^2}\, dp + c_2.$$

This formula is valid only for $t > 0$. Now use (3), expressed as a limit as follows.

$$\text{If } x > 0,\ 1 = \lim_{t \searrow 0} Q = c_1 \int_0^{+\infty} e^{-p^2}\, dp + c_2 = c_1 \frac{\sqrt{\pi}}{2} + c_2.$$

$$\text{If } x < 0,\ 0 = \lim_{t \searrow 0} Q = c_1 \int_0^{-\infty} e^{-p^2}\, dp + c_2 = -c_1 \frac{\sqrt{\pi}}{2} + c_2.$$

This determines the coefficients $c_1 = 1/\sqrt{\pi}$ and $c_2 = \frac{1}{2}$. Therefore, Q is the function

$$\boxed{Q(x, t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-p^2}\, dp} \tag{5}$$

for $t > 0$. Notice that it does indeed satisfy (1), (3), and (4).

Step 4 Having found Q, we now *define* $S = \partial Q/\partial x$. (The explicit formula for S will be written below.) By property (b), S is also a solution of (1). Given any function ϕ, we also *define*

$$\boxed{u(x, t) = \int_{-\infty}^{\infty} S(x-y, t)\phi(y)\, dy \qquad \text{for } t > 0.} \tag{6}$$

By property (d), u is another solution of (1). *We claim that u is the unique* *solution of* (1), (2). To verify the validity of (2), we write

$$
\begin{aligned}
u(x,t) &= \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y,t)\phi(y)\,dy \\
&= -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}[Q(x-y,t)]\phi(y)\,dy \\
&= +\int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)\,dy \;-\; Q(x-y,t)\phi(y)\Big|_{y=-\infty}^{y=+\infty}
\end{aligned}
$$

upon integrating by parts. We assume these limits vanish. In particular, let's temporarily assume that $\phi(y)$ itself equals zero for $|y|$ large. Therefore,

$$
\begin{aligned}
u(x,0) &= \int_{-\infty}^{\infty} Q(x-y,0)\phi'(y)\,dy \qquad ? \; ! \; ! \\
&= \int_{-\infty}^{x} \phi'(y)\,dy = \phi\Big|_{-\infty}^{x} = \phi(x)
\end{aligned}
$$

because of the initial condition for Q and the assumption that $\phi(-\infty) = 0$. This is the initial condition (2). We conclude that (6) is our solution formula, where

$$\boxed{S = \frac{\partial Q}{\partial x} = \frac{1}{2\sqrt{\pi k t}}\, e^{-x^2/4kt} \qquad \text{for } t > 0.} \tag{7}$$

That is,

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\,\phi(y)\,dy.} \tag{8}$$

$S(x, t)$ is known as the *source function, Green's function, fundamental solution, gaussian,* or *propagator* of the diffusion equation, or simply the *diffusion kernel.* It gives the solution of (1), (2) with any initial datum ϕ. The formula only gives the solution for $t > 0$. When $t = 0$ it makes no sense. □

The *source function* $S(x, t)$ is defined for all real x and for all $t > 0$. $S(x, t)$ is positive and is even in x [$S(-x, t) = S(x, t)$]. It looks like Figure 1 for various values of t. For large t, it is very spread out. For small t it is a very tall thin spike (a "*delta function*") of height $(4\pi kt)^{-1/2}$. The area under its graph is

$$\int_{-\infty}^{\infty} S(x,t)\,dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-q^2}\,dq = 1$$

by substituting $q = x/\sqrt{4kt}$, $dq = (dx)/\sqrt{4kt}$ (see Exercise 7). Now look more carefully at the sketch of $S(x, t)$ for a very small t. If we cut out the tall spike, the rest of $S(x, t)$ is very small. Thus

$$\max_{|x|>\delta} S(x,t) \to 0 \qquad \text{as} \quad t \to 0 \tag{9}$$

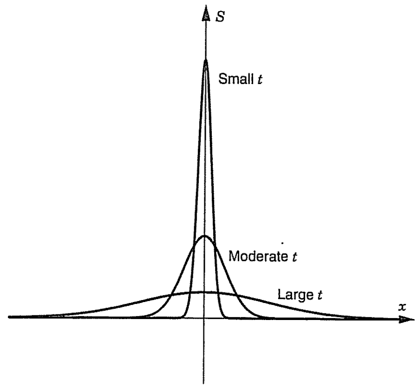

Figure 1

Notice that the value of the solution $u(x, t)$ is a kind of weighted *average* of the initial values around the point x. Indeed, letting $z = x - y$ in the integral (6), we can write

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy \simeq \sum_i S(x-y_i,t)\phi(y_i)\,\Delta y_i$$

approximately. This is the average of the solutions $S(x - y_i, t)$ with the weights $\phi(y_i)$. For very small t, the source function is a spike so that the formula exaggerates the values of ϕ near x. For *any* $t > 0$ the solution is a spread-out version of the initial values at $t = 0$.

Here's the physical interpretation. Consider diffusion. $S(x - y, t)$ represents the result of a unit mass (say, 1 gram) of substance located at time zero exactly at the position y which is diffusing (spreading out) as time advances. For any initial distribution of concentration, the amount of substance initially in the interval Δy spreads out in time and contributes approximately the term $S(x - y_i, t)\phi(y_i)\,\Delta y_i$. All these contributions are added up to get the whole distribution of matter. Now consider heat flow. $S(x - y, t)$ represents the result of a "hot spot" at y at time 0. The hot spot is cooling off and spreading its heat along the rod.

Another physical interpretation is brownian motion, where particles move randomly in space. For simplicity, we assume that the motion is one-dimensional; that is, the particles move along a tube. Then the probability that a particle which begins at position x ends up in the interval (a, b) at time t is precisely $\int_a^b S(x - y, t)\,dy$ for some constant k. In other words, if we let $u(x, t)$ be the probability density (probability per unit length) and if the initial probability density is $\phi(x)$, then the probability at all later times is given by formula (6). That is, $u(x, t)$ satisfies the diffusion equation.

It is usually impossible to evaluate the integral (8) completely in terms of elementary functions. Answers to particular problems, that is, to particular initial data $\phi(x)$, are sometimes expressible in terms of the *error function* of statistics,

$$\mathscr{E}\mathrm{rf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2}\, dp. \tag{10}$$

Notice that $\mathscr{E}\mathrm{rf}(0) = 0$. By Exercise 6, $\lim_{x\to+\infty} \mathscr{E}\mathrm{rf}(x) = 1$.

Example 1.

From (5) we can write $Q(x, t)$ in terms of $\mathscr{E}\mathrm{rf}$ as

$$Q(x, t) = \frac{1}{2} + \frac{1}{2}\mathscr{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right). \qquad \square$$

Example 2.

Solve the diffusion equation with the initial condition $u(x, 0) = e^{-x}$. To do so, we simply plug this into the general formula (8):

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\, e^{-y}\, dy.$$

This is one of the few fortunate examples that can be integrated. The exponent is

$$-\frac{x^2 - 2xy + y^2 + 4kty}{4kt}.$$

Completing the square in the y variable, it is

$$-\frac{(y + 2kt - x)^2}{4kt} + kt - x.$$

We let $p = (y + 2kt - x)/\sqrt{4kt}$ so that $dp = dy/\sqrt{4kt}$. Then

$$u(x, t) = e^{kt-x} \int_{-\infty}^{\infty} e^{-p^2} \frac{dp}{\sqrt{\pi}} = e^{kt-x}. \qquad \square$$

EXERCISES

1. Solve the diffusion equation with the initial condition

$$\phi(x) = 1 \quad \text{for } |x| < l \qquad \text{and} \qquad \phi(x) = 0 \quad \text{for } |x| > l.$$

Write your answer in terms of $\mathscr{E}\mathrm{rf}(x)$.

2. Do the same for $\phi(x) = 1$ for $x > 0$ and $\phi(x) = 3$ for $x < 0$.

3. Use (8) to solve the diffusion equation if $\phi(x) = e^{3x}$. (You may also use Exercises 6 and 7 below.)
4. Solve the diffusion equation if $\phi(x) = e^{-x}$ for $x > 0$ and $\phi(x) = 0$ for $x < 0$.
5. Prove properties (a) to (e) of the diffusion equation in Section 2.4.
6. Compute $\int_0^\infty e^{-x^2}\, dx$. (*Hint:* This is a function that *cannot* be integrated by formula. So use the following trick. Transform the double integral $\int_0^\infty e^{-x^2}\, dx \cdot \int_0^\infty e^{-y^2}\, dy$ into polar coordinates and you'll end up with a function that can be integrated easily.)
7. Use Exercise 6 to show that $\int_{-\infty}^{\infty} e^{-p^2}\, dp = \sqrt{\pi}$. Then substitute $p = x/\sqrt{4kt}$ to show that

$$\int_{-\infty}^{\infty} S(x, t)\, dx = 1.$$

8. Show that for any fixed $\delta > 0$ (no matter how small),

$$\max_{\delta \le |x| < \infty} S(x, t) \to 0 \qquad \text{as } t \to 0.$$

[This means that the tail of $S(x, t)$ is "uniformly small".]

9. Solve the diffusion equation $u_t = ku_{xx}$ with the initial condition $u(x, 0) = x^2$ by the following special method. First show that u_{xxx} satisfies the diffusion equation with *zero* initial condition. Therefore, by uniqueness, $u_{xxx} \equiv 0$. Integrating this result thrice, obtain $u(x, t) = A(t)x^2 + B(t)x + C(t)$. Finally, it's easy to solve for A, B, and C by plugging into the original problem.
10. (a) Solve Exercise 9 using the general formula discussed in the text. This expresses $u(x, t)$ as a certain integral. Substitute $p = (x - y)/\sqrt{4kt}$ in this integral.
    (b) Since the solution is unique, the resulting formula must agree with the answer to Exercise 9. Deduce the value of

$$\int_{-\infty}^{\infty} p^2 e^{-p^2}\, dp.$$

11. (a) Consider the diffusion equation on the whole line with the usual initial condition $u(x, 0) = \phi(x)$. If $\phi(x)$ is an *odd* function, show that the solution $u(x, t)$ is also an *odd* function of x. (*Hint:* Consider $u(-x, t) + u(x, t)$ and use the uniqueness.)
    (b) Show that the same is true if "odd" is replaced by "even."
    (c) Show that the analogous statements are true for the wave equation.
12. The purpose of this exercise is to calculate $Q(x, t)$ approximately for large t. Recall that $Q(x, t)$ is the temperature of an infinite rod that is initially at temperature 1 for $x > 0$, and 0 for $x < 0$.
    (a) Express $Q(x, t)$ in terms of $\mathscr{E}\mathrm{rf}$.
    (b) Find the Taylor series of $\mathscr{E}\mathrm{rf}(x)$ around $x = 0$. (*Hint:* Expand e^z, substitute $z = -y^2$, and integrate term by term.)

(c) Use the first two nonzero terms in this Taylor expansion to find an approximate formula for $Q(x, t)$.
(d) *Why* is this formula a good approximation for x fixed and t large?

13. Prove from first principles that $Q(x, t)$ *must* have the form (4), as follows.
(a) Assuming uniqueness show that $Q(x, t) = Q(\sqrt{a}\, x, at)$. This identity is valid for all $a > 0$, all $t > 0$, and all x.
(b) Choose $a = 1/(4kt)$.

14. Let $\phi(x)$ be a continuous function such that $|\phi(x)| \le Ce^{ax^2}$. Show that formula (8) for the solution of the diffusion equation makes sense for $0 < t < 1/(4ak)$, but not necessarily for larger t.

15. Prove the uniqueness of the diffusion problem with Neumann boundary conditions:

$$u_t - ku_{xx} = f(x, t) \quad \text{for } 0 < x < l,\ t > 0 \qquad u(x, 0) = \phi(x)$$
$$u_x(0, t) = g(t) \qquad u_x(l, t) = h(t)$$

by the energy method.

16. Solve the diffusion equation with constant dissipation:

$$u_t - ku_{xx} + bu = 0 \qquad \text{for } -\infty < x < \infty \qquad \text{with } u(x, 0) = \phi(x),$$

where $b > 0$ is a constant. (*Hint:* Make the change of variables $u(x, t) = e^{-bt}v(x, t)$.)

17. Solve the diffusion equation with variable dissipation:

$$u_t - ku_{xx} + bt^2u = 0 \qquad \text{for } -\infty < x < \infty \qquad \text{with } u(x, 0) = \phi(x),$$

where $b > 0$ is a constant. (*Hint:* The solutions of the ODE $w_t + bt^2w = 0$ are $Ce^{-bt^3/3}$. So make the change of variables $u(x, t) = e^{-bt^3/3}v(x, t)$ and derive an equation for v.)

18. Solve the heat equation with convection:

$$u_t - ku_{xx} + Vu_x = 0 \quad \text{for } -\infty < x < \infty \qquad \text{with } u(x, 0) = \phi(x),$$

where V is a constant. (*Hint:* Go to a moving frame of reference by substituting $y = x - Vt$.)

19. (a) Show that $S_2(x, y, t) = S(x, t)S(y, t)$ satisfies the diffusion equation $S_t = k(S_{xx} + S_{yy})$.
(b) Deduce that $S_2(x, y, t)$ is the source function for two-dimensional diffusions.

2.5 COMPARISON OF WAVES AND DIFFUSIONS

We have seen that the basic property of waves is that information gets transported in both directions at a finite speed. The basic property of diffusions is that the initial disturbance gets spread out in a smooth fashion and gradually

disappears. The fundamental properties of these two equations can be summarized in the following table.

Property	Waves	Diffusions
(i) Speed of propagation?	Finite ($\le c$)	Infinite
(ii) Singularities for $t > 0$?	Transported along characteristics (speed $= c$)	Lost immediately
(iii) Well-posed for $t > 0$?	Yes	Yes (at least for bounded solutions)
(iv) Well-posed for $t < 0$?	Yes	No
(v) Maximum principle	No	Yes
(vi) Behavior as $t \to +\infty$?	Energy is constant so does not decay	Decays to zero (if ϕ integrable)
(vii) Information	Transported	Lost gradually

For the wave equation we have seen most of these properties already. That there is no maximum principle is easy to see. Generally speaking, the wave equation just moves information along the characteristic lines. In more than one dimension we'll see that it spreads information in expanding circles or spheres.

For the diffusion equation we discuss property (ii), that singularities are immediately lost, in Section 3.5. The solution is differentiable to all orders even if the initial data are not. Properties (iii), (v), and (vi) have been shown already. The fact that information is gradually lost [property (vii)] is clear from the graph of a typical solution, for instance, from $S(x, t)$.

As for property (i) for the diffusion equation, notice from formula (2.4.8) that the value of $u(x, t)$ depends on the values of the initial datum $\phi(y)$ for *all* y, where $-\infty < y < \infty$. Conversely, the value of ϕ at a point x_0 has an *immediate effect everywhere* (for $t > 0$), even though most of its effect is only for a short time near x_0. Therefore, the *speed of propagation is infinite.* This is in stark contrast to the wave equation (and all hyperbolic equations).

As for (iv), there are several ways to see that *the diffusion equation is not well-posed for* $t < 0$ ("backward in time"). One way is the following. Let

$$u_n(x, t) = \frac{1}{n} \sin nx\, e^{-n^2kt}. \tag{1}$$

You can check that this satisfies the diffusion equation for all x, t. Also, $u_n(x, 0) = n^{-1} \sin nx \to 0$ uniformly as $n \to \infty$. But consider any $t < 0$, say $t = -1$. Then $u_n(x, -1) = n^{-1} \sin nx\, e^{+kn^2} \to \pm\infty$ uniformly as $n \to \infty$ except for a few x. Thus u_n is close to the zero solution at time $t = 0$ but not at time $t = -1$. This violates the stability, in the uniform sense at least.

Another way is to let $u(x, t) = S(x, t + 1)$. This is a solution of the diffusion equation $u_t = ku_{xx}$ for $t > -1$, $-\infty < x < \infty$. But $u(0, t) \to \infty$ as $t \searrow -1$, as we saw above. So we cannot solve backwards in time with the perfectly nice-looking initial data $(4\pi k)^{-1}e^{-x^2/4}$.