ECON 139: Intermediate Financial Economics

Lecture 27 (last lecture) Scribe Notes

Yuqian Yu, Gina Wang, Jialin Dai, Tianyao Deng

May 2, 2019

Model of stock price movements: $dS = \mu S dt + \sigma S dz \quad where$

$$dz = \tilde{\varepsilon}(t)\sqrt{dt}, \tilde{\varepsilon}(t) \sim N(0,1)$$

By Ito's Lemma, a function G of S and t follows

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial G}{\partial S}\sigma S dz$$

Forward price: $F_t = S_t e^{r(T-t)}, \;\; where\ t \in [0, T]$

To apply Ito's Lemma:

$$\frac{\partial F}{\partial S} = e^{r(T-t)}\ , \;\; \frac{\partial^2 F}{\partial S^2} = 0\ , \;\; \frac{\partial F}{\partial t} = -rS_t e^{r(T-t)}$$

$$dF = \left(e^{r(T-t)}\mu S - rSe^{r(T-t)}\right)dt + e^{r(T-t)}\sigma S dz$$

Since $S = e^{-r(T-t)}F$

We get:

$$dF = \left(e^{r(T-t)}\mu e^{-r(T-t)}F - re^{-r(T-t)}Fe^{r(T-t)}\right)dt + e^{r(T-t)}\sigma e^{-r(T-t)}F dz$$

$$= (\mu F - rF)dt + \sigma F dz$$

$$= (\mu - r)F dt + \sigma F dz$$

Example: logarithm of stock price

$$G = lnS$$

We know that $\frac{\partial G}{\partial S} = \frac{1}{S}$, $\frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}$, $\frac{\partial G}{\partial t} = 0$

$$dG = \left(\frac{1}{S}\mu S - \frac{1}{2}\frac{1}{S^2}\sigma^2 S^2\right) dt + \frac{1}{S}\sigma S dz$$
$$= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dz$$

- Implies that change in G between time 0 and time T

$\sim N\left((\mu - \frac{\sigma^2}{2})*T, \sigma^2 T\right)$

at time = 0, G = ln S
at time = T, G = ln St

ln St- ln So ~ N $(\mu - \frac{\sigma^2}{2})T, \sigma^2 T)$

The log normal price distribution will be ln St ~ N (ln So + $(\mu - \frac{\sigma^2}{2})T, \sigma^2 T))$

- Distribution of CTS compounded return of stock

$$St = So * e^{\eta T}$$

$$\ln St - \ln So = \ln \left(\frac{St}{So}\right) = \ln \left(\frac{So e^{\eta T}}{So}\right) = e^{\eta T}$$

The expected return over some time T is ηT ~ N ($(\mu - \frac{\sigma^2}{2})*T, \sigma^2 T)$

The rate of return is all normally distributed as η ~ N $(\mu - \frac{\sigma^2}{2}, \frac{\sigma 2}{T})$

- Supposed we have the following:

  Case 1:
  Annual returns:
  Assume $\mu = 0.15$

	Year 1	Year 2	Year 3	Year 4	Year 5
R (%)	0.15	0.05	0.2	0.1	0.25
After a year	1.15	1.05	1.2	1.1	1.25
Five Years	2.01				

  Total after five years is 1.15+1.05+1.2+1.1+1.25 = 1.99 compared to 2.01, the difference is due to volatility correction.

  Case 2:

  Annual returns:
  Assume $\mu = 0.15$

	Year 1	Year 2	Year 3	Year 4	Year 5
R (%)	0.15	0.35	-0.05	0.05	0.25
After a year	1.15	1.35	0.95	1.05	1.25
Five Years	2.01				

  Total after five years is 1.15+1.35+0.95+1.05+1.25 = 1.94 compared to 2.01, the difference is due to volatility correction.

Black-Scholes differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf$$

Assumptions:
1.Stock follows $dS = \mu S dt + \sigma S dz$
2.Short-selling permitted
3.No transaction costs or taxes
4.No dividends
5.No arbitrage opportunities
6.Security trading continuous
7.Risk-free rate constant and same for all maturities

$$dS = \mu S dt + \sigma S dz$$
$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)$$

Consider a portfolio that short 1 share of derivative and long $\frac{\partial f}{\partial S}$ share of stock, and let V be the value of this portfolio.

$$V = \frac{\partial f}{\partial S}S - f$$

$$\begin{aligned} dV &= \frac{\partial f}{\partial S}dS - df \\ &= \frac{\partial f}{\partial S}\mu S dt + \frac{\partial f}{\partial S}\sigma S dz - \frac{\partial f}{\partial S}\mu S dt - \frac{\partial f}{\partial t}dt - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 dt - \frac{\partial f}{\partial S}\sigma S dz \\ &= -\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt \end{aligned}$$

$$\frac{dV}{V} = rdt$$
$$dV = rVdt$$

$$rV\,dt = -\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) dt$$

$$V = \frac{\partial f}{\partial S} \cdot S - f$$

$$r\left(\frac{\partial f}{\partial S} \cdot S - f\right) = -\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)$$

$$\frac{\partial f}{\partial S} \cdot r\,S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = r\,f$$

European call: $f = max\,(S_T - K,\ 0)$ when t = T
European put: $f = max\,(K - S_T,\ 0)$ when t = T

- Example: forward contract

$$f_t = S_t - K\,e^{-r\,(T-t)} \qquad t \in [0,\ T]$$

$$\frac{\partial f}{\partial t} = -r\,K\;e^{-r\,(T-t)}$$

$$\frac{\partial f}{\partial S} = 1$$

$$\frac{\partial^2 f}{\partial S^2} = 0$$

$$r\,S - r\,K\;e^{-r\,(T-t)} = r\,f$$

$$r\,S - r\,K\;e^{-r\,(T-t)} = r\,S - r\,K\,e^{-r\,(T-t)}$$