By definition, the moment generating function (MGT) for a random variable $X$ is written as

$$M(\theta) = Ee^{\theta X}.$$

We know that characteristic functions (CF) provide full characterizations of probability distributions, i.e., the one-to-one correspondence between CF and PDF. Since MGT and CL "look" so similar, let us believe that MGTs also provide full characterizations of probability distributions.

By Taylor expansion on the function $f(y) = e^y$ around $y = 0$, we have

$$f(y) = e^y = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(y-0)^k = \sum_{k=0}^{\infty} \frac{y^k}{k!}.$$

Now, let $y = \theta X$. We have

$$e^{\theta X} = \sum_{k=0}^{\infty} \frac{\theta^k X^k}{k!}.$$

Taking expectation on the both sides, we have

$$M(\theta) = Ee^{\theta X} = \sum_{k=0}^{\infty} \frac{\theta^k}{k!} EX^k.$$

We see that the MGT contains all "moments", i.e., $EX^k$ for all integers $k$. Since we know that moments characterize probability distributions, we are further convinced that MGT is able to characterize probability distributions.

However, why do we call this function MGT? The reason is simple, we can simply find all moments by differentiating the MGT. How? By taking the $n$th order derivatives with respect to $\theta$, we have

$$M^{(n)}(\theta) = EX^n e^{\theta X}.$$

Now, let $\theta = 0$. We obtain that

$$M^{(n)}(0) = EX^n.$$

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