Cauchy收敛准则

Cauchy收敛准则.

本段内容要点:

Cauchy数列的定义

Cauchy收敛准则

数列收敛的意义

定义: 对数列$\{x_n\}$, 如果

$$\forall \varepsilon > \mathbf{0},\ \exists \boldsymbol{N} \in \mathbb{N},\ \ \boldsymbol{s.t.}\ \ |x_n - x_m| < \varepsilon,\ \ \boldsymbol{n}, \boldsymbol{m} > \boldsymbol{N},$$

则称$\{x_n\}$为一个Cauchy列.

定义: 对数列$\{x_n\}$, 如果

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ s.t.\ \ |x_n - x_m| < \varepsilon,\ \ n, m > N,$$

则称$\{x_n\}$为一个Cauchy列.

另一种描述

对数列$\{x_n\}$, 如果

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ s.t.$$
$$|x_{n+p} - x_n| < \varepsilon,\ \ \forall n > N,\ \forall p \in \mathbb{N},$$

则称$\{x_n\}$为一个Cauchy列.

定理[Cauchy收敛准则]

$\boxed{\{x_n\}\text{收敛} \Leftrightarrow \{x_n\}\text{是一个Cauchy列.}}$

定理[Cauchy收敛准则]

$\boxed{\{x_n\}\text{收敛} \Leftrightarrow \{x_n\}\text{是一个Cauchy列.}}$

证明:

(1)必要性. 设 $\lim\limits_{n\to\infty} x_n = a$, 往证$\{x_n\}$Cauchy.

定理[Cauchy收敛准则]

$\boxed{\{x_n\}\text{收敛} \Leftrightarrow \{x_n\}\text{是一个Cauchy列.}}$

证明:

(1)必要性. 设 $\lim\limits_{n\to\infty} x_n = a$, 往证$\{x_n\}$Cauchy.

即往证: $\forall\varepsilon > 0,\ \exists N\ \ s.t.\ |x_n - x_m| < \varepsilon,\ \ n, m > N.$

$$|x_n - x_m| \leqslant |x_n - a| + |x_m - a|$$

定理[Cauchy收敛准则]

$\boxed{\{x_n\}\text{收敛} \Leftrightarrow \{x_n\}\text{是一个Cauchy列.}}$

证明:

(1)必要性. 设 $\lim_{n\to\infty} x_n = a$, 往证$\{x_n\}$Cauchy.

即往证: $\forall \varepsilon > 0,\ \exists N\ \ s.t.\ |x_n - x_m| < \varepsilon,\ \ n, m > N.$

现在, 对给定的$\varepsilon > 0$, 因为 $\lim_{n\to\infty} x_n = a$,

所以就$\dfrac{\varepsilon}{2}$来说, $\quad \exists N\ \ s.t.\ |x_n - a| < \dfrac{\varepsilon}{2},\ \ n > N.$

定理[Cauchy收敛准则]

$\boxed{\{x_n\}\text{收敛} \Leftrightarrow \{x_n\}\text{是一个Cauchy列.}}$

证明:

(1)必要性. 设 $\lim\limits_{n\to\infty} x_n = a$, 往证$\{x_n\}$Cauchy.

即往证: $\forall \varepsilon > 0, \ \exists N \ \ s.t. \ |x_n - x_m| < \varepsilon, \ \ n, m > N.$

现在, 对给定的$\varepsilon > 0$, 因为 $\lim\limits_{n\to\infty} x_n = a$,

所以就$\dfrac{\varepsilon}{2}$来说, $\exists N \ \ s.t. \ |x_n - a| < \dfrac{\varepsilon}{2}, \ \ n > N.$

从而,$n, m > N$时,

$$|x_n - x_m| \leqslant |x_n - a| + |x_m - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

所以$\{x_n\}$Cauchy.

定理[Cauchy收敛准则]

$\boxed{\{x_n\}\text{收敛} \Leftrightarrow \{x_n\}\text{是一个Cauchy列.}}$

(2)充分性. 设$\{x_n\}$Cauchy, 往证$\{x_n\}$收敛.

其思路是:

$\{x_n\}$Cauchy $\Rightarrow$ $\{x_n\}$有界 $\Rightarrow$ $\{x_n\}$有聚点.

然后证$\{x_n\}$的聚点唯一, 其即为数列的极限.

由于用到聚点原理, 超出本课程范围, 此段证明从略.

例1：数列$x_n = \sum_{k=1}^{n} \frac{1}{k}$，$n = 1, 2, \cdots$发散.

例1：数列$x_n = \sum_{k=1}^{n} \frac{1}{k}$, $n = 1, 2, \cdots$发散.

因为$|x_{2n} - x_n| = \sum_{k=n+1}^{2n} \frac{1}{k}$

例1：数列$x_n = \sum_{k=1}^{n} \frac{1}{k}$, $n = 1, 2, \cdots$发散.

$$\text{因为}|x_{2n} - x_n| = \sum_{k=n+1}^{2n} \frac{1}{k}$$

$$> \sum_{k=n+1}^{2n} \frac{1}{2n} = \frac{n}{2n} = \frac{1}{2}, \ \forall n \in \mathbb{N}.$$

例1：数列$x_n = \sum_{k=1}^{n} \frac{1}{k}$, $n = 1, 2, \cdots$ 发散.

因为$|x_{2n} - x_n| = \sum_{k=n+1}^{2n} \frac{1}{k}$

$$> \sum_{k=n+1}^{2n} \frac{1}{2n} = \frac{n}{2n} = \frac{1}{2}, \ \forall n \in \mathbb{N}.$$

所以, 对于$\varepsilon_0 = \frac{1}{4}$, 不可能存在$N$使得,

任何$n, m > N$时, $|x_n - x_m| < \varepsilon_0$总成立.

例1：数列$x_n = \sum_{k=1}^{n} \frac{1}{k}$, $n = 1, 2, \cdots$发散.

$$\text{因为}|x_{2n} - x_n| = \sum_{k=n+1}^{2n} \frac{1}{k}$$

$$> \sum_{k=n+1}^{2n} \frac{1}{2n} = \frac{n}{2n} = \frac{1}{2}, \ \forall n \in \mathbb{N}.$$

所以, 对于$\varepsilon_0 = \frac{1}{4}$, 不可能存在$N$使得,

任何$n, m > N$时, $|x_n - x_m| < \varepsilon_0$总成立.

所以数列发散.

例2：数列$x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$收敛.

例2：数列$x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$收敛.

$$|x_{n+p} - x_n| = \frac{1}{(n+1)^2} + \cdots + \frac{1}{(n+p)^2}$$

例2：数列$x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$收敛.

$$|x_{n+p} - x_n| = \frac{1}{(n+1)^2} + \cdots + \frac{1}{(n+p)^2}$$

$$< \frac{1}{n(n+1)} + \cdots + \frac{1}{(n+p-1)(n+p)}$$

例2：数列$x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$收敛.

$$
\begin{aligned}
|x_{n+p} - x_n| &= \frac{1}{(n+1)^2} + \cdots + \frac{1}{(n+p)^2} \\
&< \frac{1}{n(n+1)} + \cdots + \frac{1}{(n+p-1)(n+p)} \\
&= \left(\frac{1}{n} - \frac{1}{n+1}\right) + \cdots + \left(\frac{1}{n+p-1} - \frac{1}{n+p}\right) \\
&= \frac{1}{n} - \frac{1}{n+p} < \frac{1}{n}
\end{aligned}
$$

例2：数列$x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$收敛.

$$
\begin{aligned}
|x_{n+p} - x_n| &= \frac{1}{(n+1)^2} + \cdots + \frac{1}{(n+p)^2} \\
&< \frac{1}{n(n+1)} + \cdots + \frac{1}{(n+p-1)(n+p)} \\
&= (\frac{1}{n} - \frac{1}{n+1}) + \cdots + (\frac{1}{n+p-1} - \frac{1}{n+p}) \\
&= \frac{1}{n} - \frac{1}{n+p} < \frac{1}{n}
\end{aligned}
$$

要想$|x_{n+p} - x_n| < \varepsilon$, 只需$n > \frac{1}{\varepsilon}$.

例2：数列$x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$收敛.

$$
\begin{aligned}
|x_{n+p} - x_n| &= \frac{1}{(n+1)^2} + \cdots + \frac{1}{(n+p)^2} \\
&< \frac{1}{n(n+1)} + \cdots + \frac{1}{(n+p-1)(n+p)} \\
&= \left(\frac{1}{n} - \frac{1}{n+1}\right) + \cdots + \left(\frac{1}{n+p-1} - \frac{1}{n+p}\right) \\
&= \frac{1}{n} - \frac{1}{n+p} < \frac{1}{n}
\end{aligned}
$$

要想$|x_{n+p} - x_n| < \varepsilon$, 只需$n > \frac{1}{\varepsilon}$.

所以, $\forall \varepsilon > 0,\ \exists N = [\frac{1}{\varepsilon}]\ s.t.$

$|x_{n+p} - x_n| < \varepsilon,\ \ \forall n > N, \forall p \in \mathbb{N}$

本段知识要点:

Cauchy列就是收敛列

Cauchy性不管是正用还是反用, 都是强有力的工具.