Guanghua School of Management

Mathematical Methods in Finance

Lecture 3: Martingales

Fall 2013

Overview

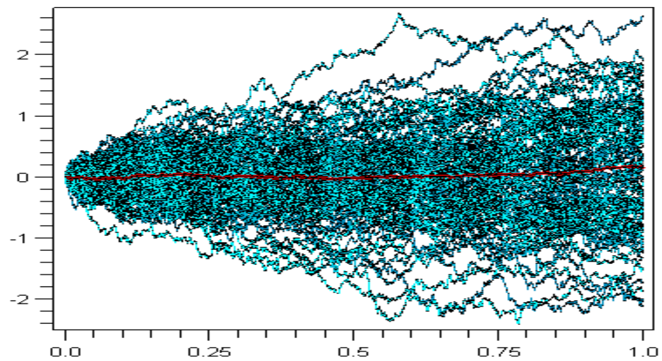
- Martingales
- Applications

Recap: Stochastic Process

- A collection of random variables:

$$\{X(t), 0 \le t \le T\}$$

- For emphasizing the dependence of both time and random events, we heuristically write $X(t, \omega)$.
- Understanding:
  - A function of two variables $(\omega, t) \in \Omega \times (0, T]$
  - Fix any $t \in (0, T]$, $X(t, \cdot)$ is a random variable
  - Fix any $\omega \in \Omega$, $X(\cdot, \omega)$ is a realization (path) of the process



Recap: Stochastic Process

- **Definition**: Let Ω be a nonempty set and T be a fixed positive number. Assume that for each $t \in (0, T]$, there exists a σ-algebra (information) $\mathcal{F}(t)$, and that for any $s \le t$, $\mathcal{F}(s) \subseteq \mathcal{F}(t)$. Then we call the collection of σ-algebra $\mathcal{F}(t)$, $0 \le t \le T$, a filtration.
- **Definition**: Let Ω be a nonempty set equipped with a filtration $\mathcal{F}(t)$, $0 \le t \le T$. Let $\{X(t); 0 \le t \le T\}$ be a collection of RVs indexed by $t \in [0, T]$. We say this collection of RVs is an adapted (continuous time) stochastic process if for each t, we have $X(t)$ is $\mathcal{F}(t)$-measurable.
- Similarly define filtration (denoted by $\mathcal{F}_n$) and Stochastic process (denoted by $\{X_n : n \in \mathbb{N}\}$) on discrete time
- Note: the state space (all possible values) of $X(t)$ (or X_n) is continuous or not is another story.

Martingales: Intuition and History

- A model of a fair game
- A stochastic process such that the conditional expected value of an observation at some time t, given all the observations up to some earlier time s, is equal to the observation at that earlier time s.
- Referred to a class of betting strategies that was popular in 18th century France
- The strategy had the gambler double his bet after every loss so that the first win would recover all previous losses plus win a profit equal to the original stake. As the gambler's wealth and available time jointly approach infinity, his probability of eventually flipping heads approaches 1, which makes the martingale betting strategy seem like a sure thing. However, the exponential growth of the bets eventually bankrupts its users.
- "Saint Petersburg Paradox"
- e.g. win at the fourth bet: $8-(1+2+4)=1$

Martingales: Formal Definition and Constant Expectation

- Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration $\mathcal{F}(t)(\subseteq \mathcal{F})$, $0 \leq t \leq T$, and an adapted stochastic process $M(t)$, $0 \leq t \leq T$.
  - If $E[M(t)|\mathcal{F}(s)] = M(s)$ for all $0 \leq s \leq t \leq T$, we say this process $M(t)$ is a martingale.
  - If $E[M(t)|\mathcal{F}(s)] \geq M(s)$ for all $0 \leq s \leq t \leq T$, we say this process $M(t)$ is a submartingale.
  - If$E[M(t)|\mathcal{F}(s)] \leq M(s)$ for all $0 \leq s \leq t \leq T$, we say this process $M(t)$ is a supermartingale.
- Discrete-time martingale M_n, $n \geq 1$, iff

$$\mathbb{E}(M_{n+1}|\mathcal{F}_n) = M_n$$

- Martingales have constant expectation over time: for any $T > t > 0$

$$\mathbb{E}M(t) = \mathbb{E}M(0)$$

Example 1: Martingale Betting Strategy (Ancient Finance)

- Betting strategy: keep doubling your bet until you eventually win
- V_n: the winnings/losses up through n trials using this strategy, $n \geq 1$
- When we win, we stop playing: $\mathbb{P}(V_{n+1} = 1 | V_n = 1) = 1$
- For first n trials resulting losses, we have $V_n = -(1 + 2 + 4 + ... + 2^{n-1}) = -(2^n - 1)$

$$\begin{aligned} &\mathbb{P}(V_{n+1} = 1 | V_n = -(2^n - 1)) = 1/2 \\ &\mathbb{P}(V_{n+1} = -(2^{n+1} - 1)) | V_n = -(2^n - 1)) = 1/2 \end{aligned} \tag{1}$$

- We have $\mathbb{E}(V_{n+1} | \mathcal{F}_n) = V_n$ for $n \geq 1$.
- On average, the winnings/losses doesn't change over time.
- So, we have $\mathbb{E}V_n = \mathbb{E}V_1$. And, $\mathbb{E}V_1 = 0$ implies that $\mathbb{E}(V_n) = 0$.

Blackjack Martingale Betting Strategy

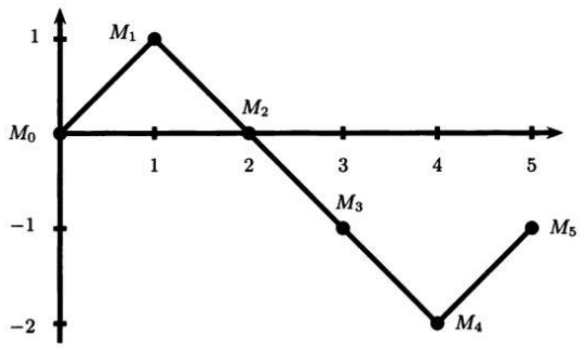
- Blackjack
- MIT Blackjack team
- Martingale betting strategies in Blackjack

Example 2: Simple Random Walk as a Martingale

A simple symmetric random walk

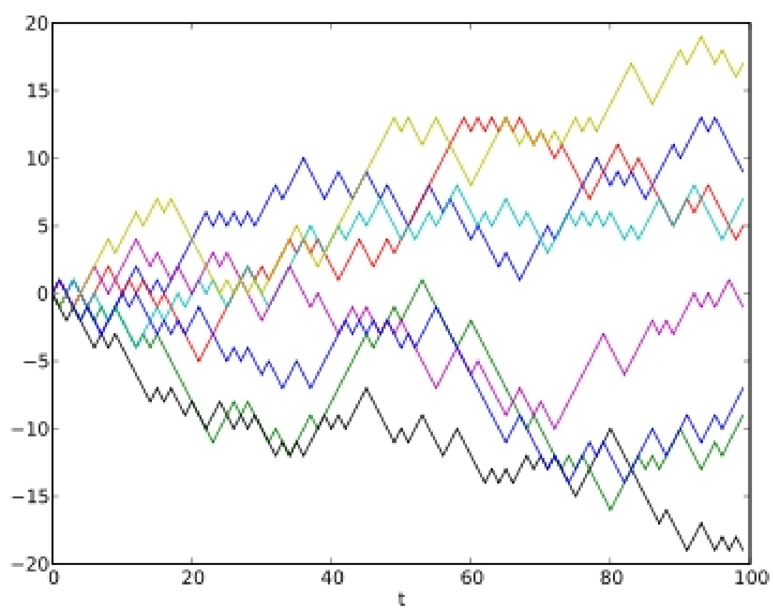
$$W_0 = 0, \quad W_n = \sum_{k=1}^{n} X_k,$$

where $X_k = 1$ or $X_k = -1$ with probability $1/2$.



Five steps of a random walk.

More Sample Paths for a Simple Random Walk



Some Properties of Simple Random Walk

- A simple symmetric random walk

$$W_0 = 0, \quad W_n = \sum_{k=1}^{n} X_k,$$

where $X_k = 1$ or $X_k = -1$ with probability $1/2$.

- A Markov process
- Transition probability:

$$\mathbb{P}(W_{n+1} = s | W_n = r) = \begin{cases} 1/2, & \text{if } s = r + 1; \\ 1/2, & \text{if } s = r - 1. \end{cases}$$

Simple Random Walk as a Martingale

- $\{W_n\}$ is martingale with respect to filtration $\mathcal{F}_n = \sigma(X_1, ..., X_n)$: just prove that

$$\mathbb{E}(W_{n+1} | \mathcal{F}_n) = W_n.$$

- $\{W_n^2 - n\}$ is also a martingale: just need to prove that

$$E(W_{n+1}^2 - (n+1) | \mathcal{F}_n) = W_n^2 - n.$$

- Generalization to asymmetric random walk (an excellent exercise): random walk with a drift (trend)

Example 3: Martingale Transform as a Discrete-time Integral

- Definition: $\{\vartheta_n\}$ is predictable iff ϑ_n is $\mathcal{F}_{n-1}$ measurable, i.e. $\{\vartheta_n\}$ is known up to the information up to $n-1$.
- "an investment decision is made according to the information previously at hand already."
- Given a martingale $\{M_n\}$, we construct

$$T_0 = 0,\ T_n = \sum_{k=1}^{n} \vartheta_k (M_k - M_{k-1})$$

- T_n is an analog to the change of wealth resulted from a dynamic trading strategy ϑ
- $\{T_n\}$ is a martingale (it is called martingale transform)
- T_n resembles an "integration" of ϑ with respect to M
- Later in this course, we will work on the stochastic integral which is a continuous time analog of this.

Recap: Poisson Process

- Model arrivals: e.g. a "jump" in financial market
- A Poisson process $\{N(t)\}$ with intensity λ:
  - N(0)=0
  - stationary and independent increment
  - $N(t+\Delta t) - N(t)$ is a Poisson R.V. with parameter $\lambda \Delta t$

$$\mathbb{P}\left(N(t+\Delta t) - N(t) = k\right) = \frac{\lambda^k \Delta t^k}{k!} e^{-\lambda \Delta t} \tag{2}$$

- Constriction from exponential R.V.s,

$$N(t) = \max\left\{ n : S_n = \sum_{i=1}^{n} \tau_i \leq t \right\},$$

where τ_i are I.I.D with exponential distribution:
$\mathbb{P}(\tau_i \leq x) = 1 - e^{-\lambda x},$ for $x \geq 0.$

Example 4: Compensated Poisson process as a Martingale

- $\{N(t) - \lambda t; t \geq 0\}$ is martingale. Why?
- We will see more as we introduce Brownian motion and stochastic integral

More Examples (Excellent Exercises)

- Levy martingale:
$$X_n = \mathbb{E}(X|\mathcal{F}_n)$$
- Product:
$$P_0 = 1,\ P_n = \mu^{-n} \prod_{j=1}^{n} X_j,$$
where X_j are I.I.D with mean $\mathbb{E}(X_j) = \mu$
- Wald martingale:
$$W_0 = 1,\ W_n = \frac{e^{\theta \sum_{j=1}^{n} X_j}}{(\phi(\theta))^n},$$
where X_j are I.I.D with moment generating function $\phi(\theta) = \mathbb{E}e^{\theta X_i}$

Some Properties: Martingale with Stopping

- Definition of **stopping times**: a random variable τ such that $\{\tau \le t\} \in \mathcal{F}(t)$ for any $0 < t < T$
- Stopping or not at time t is totally determined by the information up to that time
- Some technical properties:
  - $\sigma-$algebra $\mathcal{F}(\sigma) = \{A \in \mathcal{F} : A \cap \{\tau \le t\} \in \mathcal{F}(t)\}$
  - τ is $\mathcal{F}(\tau)$ measurable
  - $\mathcal{F}(\sigma) \subset \mathcal{F}(\tau)$ if $\sigma \le \tau$
  - $\mathcal{F}(\min(\sigma, \tau)) = \mathcal{F}(\sigma) \cap \mathcal{F}(\tau)$
- An important class of stopping times: first hitting time of a set A:

$$\tau_A = \inf\{t \ge 0 : X(t) \in A\}$$

Application: Gambler's Problem

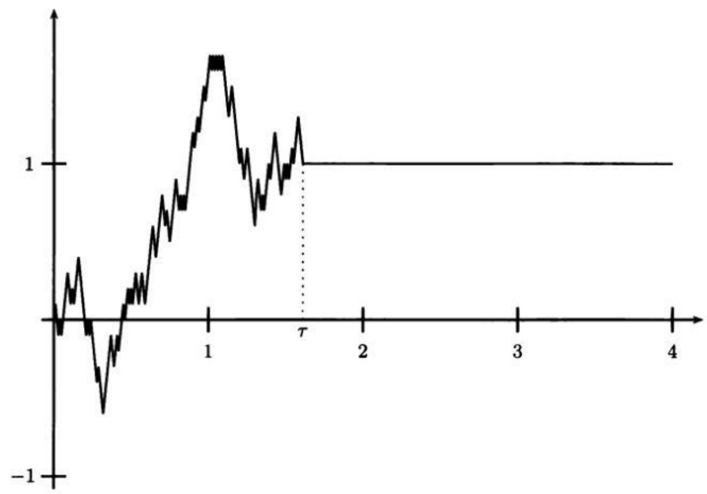
- Suppose you start with n dollars, and make a sequence of bets. For each bet, you win 1 dollar with probability $1/2$, and lose 1 dollar with probability $1/2$. You quit if either you go broke, in which case you lose, or when you reach $n+m$ dollars, what is the probability of your winning?
- The time τ at which he need to quit with a profit (or goes broke and is forced to quit) is a stopping time, i.e.

$$\tau = \min\{s \in \mathbb{N}, W_s = m \text{ or } -n\}.$$

- Previously, we can use R.W. to model and solve it by conditioning.
- Now, we solve this more systematically using "martingale"! How?

Stopped Process and Martingale

- Define: stopped process $X^{\tau}(t) = X(\min(t, \tau))$ (truncation up to a stopping time)
- Proposition: **stopped martingale is a martingale**, i.e. for martingale $\{X(t)\}$ adapted to filtration $\{\mathcal{F}(t)\}$, $\{X^{\tau}(t)\}$ is a martingale adapted to the stopped filtration $\{\mathcal{F}^{\tau}(t) = \mathcal{F}_{\min(t,\tau)}\}$



A stopped process.

Optional Sampling Theorem

- If $\{X_t\}$ is a martingale, under some technical conditions, the constant expectation property can be extended to stopping time τ, i.e.

$$\mathbb{E}X_{\tau} = \mathbb{E}X_0?$$

- Connection with betting strategies: impossible for a gambler to improve their betting strategies τ to obtain bigger expected profit!
- Interpretation of the Gambler's Ruin Problem
  The gambler's fortune over time is a martingale; and the time τ at which he need to quit with a profit (or goes broke and is forced to quit) is a stopping time, i.e.

$$\tau = \min\{s \in \mathbb{N}, W_s = m \text{ or } -n\}.$$

  So the optional sampling theorem says that (why?)

$$\mathbb{E}W_{\tau} = \mathbb{E}W_0 = 0.$$

Optional Sampling Theorem

Some technical conditions are necessary for establishing the optional sampling theorem:

$$\mathbb{E}X_\tau = \mathbb{E}X_0,$$

e.g.

- If τ is a bounded stopping time, i.e. $\mathbb{P}(\tau \leq M) = 1$,
- If the following conditions hold
  - $\mathbb{P}(\tau < \infty) = 1$
  - $\mathbb{E}|X_\tau| < \infty$
  - $\lim_{n\to\infty} \mathbb{E}(|X_n| 1\{\tau > n\}) = 0$
- other conditions...

Counter example: consider the random walk W_n and let $\tau = \min\{j \in \mathbb{N}, W_j = 1\}$, we have

$$1 = \mathbb{E}W_\tau \neq EW_0 = 0.$$

Solve Gambler's Problem

By the martingale property of the stopped martingale W_r^τ, we have

$$\mathbb{E}W_{\min\{r,\tau\}} = \mathbb{E}W_0 = 0.$$

By the fact that W_r^τ is bounded, we use the dominated convergence theorem to let $r \to \infty$ and obtain the optional sampling theorem, i.e.

$$\mathbb{E}W_\tau = \mathbb{E}W_0.$$

Therefore,

$$\begin{aligned} m\mathbb{P}(W_\tau = m) - n\mathbb{P}(W_\tau = -n) &= 0, \\ \mathbb{P}(W_\tau = m) + \mathbb{P}(W_\tau = -n) &= 1. \end{aligned} \tag{3}$$

Therefore, the probability of win is

$$\mathbb{P}(W_\tau = m) = \frac{n}{m+n}.$$

Question: Find $\mathbb{E}\tau$.

Supplementary Material

Suggested Reading Material (We only need to focus on the material parallel to our course slides):

- Shreve Vol. II: Section 2.3, 3.2.2, 3.2.3

  Or you can find equivalent material from

- Mikosch: Section 1.5

Suggested Exercises (Do Not Hand In; For Your Deeper Understanding Only)

- Shreve Vol. I: Exercise 2.4, 2.5, 2.6