Risk Premium and Certainty Equivalent

The certainty equivalent and risk premium are "two sides of the same coin"

$$\Psi(\tilde{Z}) = E(\tilde{Z}) - CE(\tilde{Z})$$

$CE(\tilde{Z}) =$ lesser amount the investor is willing to accept to remain invested in the risk-free asset

$\Psi(\tilde{Z}) =$ extra amount the investor needs to take on additional risk

Risk Premium and Certainty Equivalent

Combining the definitions of the certainty equivalent $CE(\tilde{Z})$,

$$E[u(Y + \tilde{Z})] = u[Y + CE(\tilde{Z})],$$

and the risk premium $\Psi(\tilde{Z})$,

$$CE(\tilde{Z}) = E(\tilde{Z}) - \Psi(\tilde{Z}),$$

yields

$$E[u(Y + \tilde{Z})] = u[Y + E(\tilde{Z}) - \Psi(\tilde{Z})],$$

which we can use to link the risk premium $\Psi(\tilde{Z})$ to our measures of risk aversion.

Risk Premium and Certainty Equivalent

Consider the equation defining risk premium Ψ

risk premium

$$E[u(Y + \tilde{Z})] = u[Y + \underbrace{E(\tilde{Z}) - \Psi(\tilde{Z})}_{\text{certainty equivalent}}],$$

but let

$$\begin{aligned} Y^* &= Y + E(\tilde{Z}) \\ &= \text{income plus expected payout from the risky asset} \end{aligned}$$

so that

$$E\{u[Y^* + \tilde{Z} - E(\tilde{Z})]\} = u[Y^* - \Psi(\tilde{Z})].$$

Risk Premium and Certainty Equivalent

Take a second-order Taylor approximation to $u[Y^* + \tilde{Z} - E(\tilde{Z})]$, viewing $\tilde{Z} - E(\tilde{Z})$ as the "size of the bet"

$$\begin{aligned} u[Y^* + \tilde{Z} - E(\tilde{Z})] \;&\approx\; u(Y^*) + u'(Y^*)[\tilde{Z} - E(\tilde{Z})] \\ &\quad + \frac{1}{2}u''(Y^*)[\tilde{Z} - E(\tilde{Z})]^2. \end{aligned}$$

Risk Premium and Certainty Equivalent

Now take the expected value on both sides and simplify, using the fact that Y^* is not random:

$$\begin{aligned}
E\{u[Y^* + \tilde{Z} - E(\tilde{Z})]\} &\approx E[u(Y^*)] \\
&\quad + E\{u'(Y^*)[\tilde{Z} - E(\tilde{Z})]\} \\
&\quad + E\{\frac{1}{2}u''(Y^*)[\tilde{Z} - E(\tilde{Z})]^2\} \\
&= u(Y^*) + u'(Y^*)E[\tilde{Z} - E(\tilde{Z})] \\
&\quad + \frac{1}{2}u''(Y^*)E\{[\tilde{Z} - E(\tilde{Z})]^2\}.
\end{aligned}$$

Risk Premium and Certainty Equivalent

Finally, use the fact that $E[\tilde{Z} - E(\tilde{Z})] = 0$ and the definition of the variance of $\tilde{Z}$ to simplify further:

$$\begin{aligned} E\{u[Y^* + \tilde{Z} - E(\tilde{Z})]\} &\approx u(Y^*) + u'(Y^*)E[\tilde{Z} - E(\tilde{Z})] \\ &\quad + \frac{1}{2}u''(Y^*)E\{[\tilde{Z} - E(\tilde{Z})]^2\} \\ &= u(Y^*) + \frac{1}{2}\sigma^2(\tilde{Z})u''(Y^*). \end{aligned}$$

Risk Premium and Certainty Equivalent

On the other side of our original equation, consider a first-order Taylor approximation to $u[Y^* - \Psi(\tilde{Z})]$:

$$u[Y^* - \Psi(\tilde{Z})] \approx u(Y^*) - u'(Y^*)\Psi(\tilde{Z}).$$

Risk Premium and Certainty Equivalent

Hence, the equation defining the risk premium

$$E\{u[Y^* + \tilde{Z} - E(\tilde{Z})]\} = u[Y^* - \Psi(\tilde{Z})],$$

and the approximations

$$E\{u[Y^* + \tilde{Z} - E(\tilde{Z})]\} \approx u(Y^*) + \frac{1}{2}\sigma^2(\tilde{Z})u''(Y^*)$$

$$u[Y^* - \Psi(\tilde{Z})] \approx u(Y^*) - u'(Y^*)\Psi(\tilde{Z})$$

imply

$$\frac{1}{2}\sigma^2(\tilde{Z})u''(Y^*) \approx -u'(Y^*)\Psi(\tilde{Z}).$$

Risk Premium and Certainty Equivalent

$$\frac{1}{2}\sigma^2(\tilde{Z})u''(Y^*) \approx -u'(Y^*)\Psi(\tilde{Z})$$

$$\Psi(\tilde{Z}) \approx \frac{1}{2}\sigma^2(\tilde{Z})\left[-\frac{u''(Y^*)}{u'(Y^*)}\right]$$

$$\Psi(\tilde{Z}) \approx \frac{1}{2}\sigma^2(\tilde{Z})R_A(Y^*) = \frac{1}{2}\sigma^2(\tilde{Z})R_A(Y + E(\tilde{Z})),$$

indicating that the risk premium depends directly on the coefficient of absolute risk aversion and the absolute "size of the bet" $\sigma^2(\tilde{Z})$.

Risk Premium and Certainty Equivalent

As an example, consider an investor with income $Y = 50000$ and utility function of the constant relative risk aversion form

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1 - \gamma}$$

with $\gamma = 5$, who is considering buying an asset with random payoff $\tilde{Z}$ that equals 2000 with probability $1/2$ and 0 with probability $1/2$. For this asset

$$E(\tilde{Z}) = (1/2)2000 + (1/2)0 = 1000$$

$$\sigma^2(\tilde{Z}) = (1/2)(2000 - 1000)^2 + (1/2)(0 - 1000)^2 = 1000^2.$$

Risk Premium and Certainty Equivalent

Our approximation formula

$$\Psi(\tilde{Z}) \approx \frac{1}{2}\sigma^2(\tilde{Z})R_A(Y + E(\tilde{Z}))$$

indicates that

$$\Psi(\tilde{Z}) \approx \frac{1}{2}(1000)^2 \left(\frac{5}{51000}\right) = 49.02$$

since $R_A(Y) = R_R(Y)/Y$.

Risk Premium and Certainty Equivalent

The approximation $\Psi(\tilde{Z}) \approx 49.02$ or the exact solution $\Psi(\tilde{Z}) = 48.97$ imply that an investor with $Y = 50000$ and constant coefficient of relative risk aversion equal to 5 will give up a riskless payoff of up to about

$$CE(\tilde{Z}) = E(\tilde{Z}) - \Psi(\tilde{Z}) \approx 1000 - 49 = 951$$

for this risky asset with expected payoff equal to 1000.