Problem Set 1
Econ 139, Fall 2019
Due in class on Th September 19. No late Problem Sets accepted, sorry!

This problem set is focused on general equilibrium, arbitrage, and choice under uncertainty. General rules for problem sets: show your work, write down the steps that you use to get a solution (no credit for right solutions without explanation), write legibly. If you cannot solve a problem fully, write down a partial solution. We give partial credit for partial solutions that are correct. Do not forget to write your name on the problem set!

Problem 1. General Equilibrium. Consider the case of pure exchange with two consumers. Both consumers have Cobb-Douglas preferences, but with different parameters. Consumer 1 has utility function $u(x_1^1, x_2^1) = (x_1^1)^\alpha (x_2^1)^{1-\alpha}$. Consumer 2 has utility function $u(x_1^2, x_2^2) = (x_1^2)^\beta (x_2^2)^{1-\beta}$. The endowment of good j owned by consumer i is ω_j^i. The price of good 1 is p_1, while the price of good 2 is normalized to 1 without loss of generality.

1. Only for point 1, assume $\omega_1^1 = 1, \omega_1^2 = 3, \omega_2^1 = 3, \omega_2^2 = 1$. (that is, total endowment of each good is 4). Assume further $\alpha = 1/2$, $\beta = 1/2$. Draw the Pareto set and the contract curve for this economy in an Edgeworth box. (you do not need to give the exact solutions, only a graphical representation) What is the set of points that could be the outcome under barter in this economy?

2. For each consumer, compute the utility maximization problem. Solve for x_j^{i*} for $j = 1, 2$ and $i = 1, 2$ as a function of the price p_1 and of the endowments.

3. We now solve analytically for the general equilibrium. Require that the total sum of the demands for good 1 equals the total sum of the endowments, that is, that $x_1^{1*} + x_1^{2*} = \omega_1^1 + \omega_1^2$. Solve for the general equilibrium price p_1^*.

4. What are the comparative statics of p_1^* with respect to the endowment of good 1, that is, with respect to ω_1^i for $i = 1, 2$? What about with respect to the endowment of the other good? Does this make sense? What are the comparative statics of p_1^* with respect to the taste for good 1, that is, with respect to α and β? Does this make sense?

5. Now require the same general equilibrium condition in market 2. Solve for p_1^* again, and check that this solution is the same as the one you found in the point above. In other words, you found a property that is called Walras' Law. In an economy with n markets, if $n - 1$ markets are in equilibrium, the nth market will be in equilibrium as well.

Problem 2. Investment in Risky Asset. We consider here a standard problem of investment in risky assets, similar to the one that we covered in class. The agent can invest in bonds or stocks. Bonds have a return $r > 0$. (in class we asumed $r = 0$) Stocks have a stochastic return, $r_+ > r$ with probability p, and $r_- < r$ with probability $1 - p$. In expectations, the stocks outperform bonds, that is, $pr_+ + (1-p)\, r_- > r$. The agent has income w and utility function u, with $u'(x) > 0$ and $u''(x) < 0$ for all x. The agents wants to decide the optimal share α of his wealth to invest in stocks. The agent maximizes

$$\begin{aligned} \max_{\alpha} \quad & (1-p)\, u\left(w\left[(1-\alpha)(1+r) + \alpha(1+r_-)\right]\right) + \\ & \quad pu\left(w\left[(1-\alpha)(1+r) + \alpha(1+r_+)\right]\right) \\ s.t. \quad & \alpha \in [0,1] \end{aligned}$$

or, after some simplification,

$$\begin{aligned} \max_{\alpha} \quad & (1-p)\, u\left(w\left[1+r+\alpha(r_- - r)\right]\right) + pu\left(w\left[1+r+\alpha(r_+ - r)\right]\right) \\ s.t. \quad & \alpha \in [0,1] \end{aligned}$$

1. Assume an interior solution (i.e., $\alpha^* \in (0,1)$) and write down the first order conditions for this problem with respect to α.

2. Write down the second order condition. Is it satisfied?

3. Use the first order conditions to derive the comparative statics of α^* with respect to w. Use the implicit function theorem to write down $\partial\alpha^*/\partial w$. (this is a long expression – sorry!)

4. What is the sign of the denominator? You have checked this already. Where?

5. Argue that, given your answer to point 4, the sign of $\partial\alpha^*/\partial w$ is given by the sign of the numerator. Simplify the numerator using the first order conditions. Once you do this simplification, you should get the following expression for the numerator:

$$\begin{aligned} & (1-p)\, w\,(r_- - r)\left[1+r+\alpha(r_- - r)\right] u''\left(w\left[1+r+\alpha(r_- - r)\right]\right) + \\ & pw\,(r_+ - r)\left[1+r+\alpha(r_+ - r)\right] u''\left(w\left[1+r+\alpha(r_+ - r)\right]\right). \end{aligned}$$

6. Now, let me do one piece of the argument for you. We are interested in the sign of the expression from point 5, since it coincides with the sign of $\partial\alpha^*/\partial w$. We can rewrite it

as

$$(1-p)(r_- - r)\, u'(w[1+r+\alpha(r_- - r)]) \left\{ \frac{u''(w[1+r+\alpha(r_- - r)])}{u'(w[1+r+\alpha(r_- - r)])} w[1+r+\alpha(r_- - r)] \right\}$$
$$+\; p(r_+ - r)\, u'(w[1+r+\alpha(r_+ - r)]) \left\{ \frac{u''(w[1+r+\alpha(r_+ - r)])}{u'(w[1+r+\alpha(r_+ - r)])} w[1+r+\alpha(r_+ - r)] \right\}.$$

All we did was to multiply and divide by $u'(w[1+r+\alpha(r_- - r)])$ in the first half of the expression and by $u'(w[1+r+\alpha(r_+ - r)])$ in the second half.

It turns out that for a power utility function $u(c) = \frac{c^{1-\rho}}{1-\rho}$ the two expressions in curly brackets are both equal to $-\rho$. Using this nice result, rewrite the expression above by substituting the two expressions in curly brackets with $-\rho$.

7. Consider the simplified expression from point 6 where you substituted $-\rho$ for the curly brackets. Argue, using the first order conditions, that the resulting expression is in fact equal to zero! Now, if you go back and look at the steps of this exercise, you will realize that you have proven the following important result: with power utility function, the ratio of wealth invested in stocks (α) is independent of wealth w, i.e., $\partial\alpha/\partial w = 0$. Therefore, the model predicts that individuals earning \$20,000 should invest the same *fraction* of their earnings in stocks as individuals earning \$100,000.

Problem 3. Arbitrage and the Law of One Price. Each of the following situations is an arbitrage opportunity. For each situation, explain the source of the arbitrage opportunity and how you would trade to exploit it.

1.

State	Asset 1	Asset 2
State 1	1	5
State 2	-0.6	-3
Price	0.5	4

2.

State	Asset 1	Asset 2
State 1	1.5	4.5
State 2	2	5.5
Price	1	3

3.

State	Asset 1	Asset 2	Asset 3
State 1	1	0	4
State 2	0	1	7
Price	0.6	0.3	4.2

Problem 4. Choice Under Uncertainty (short problems).

1. Under certainty, any increasing monotone transformation of a utility function is also a utility function representing the same preferences. Under uncertainty, we must restrict this statement to linear transformations if we are to keep the same preference representation. Give a mathematical as well as an economic interpretation for this.

   Check this with the following example. Assume an initial utility function attributes the following values to three possible outcomes:

$$\begin{array}{cc} B & u(B) = 100 \\ M & u(M) = 10 \\ P & u(P) = 50 \end{array}$$

   (a) Check that with this initial utility function, the lottery $L = (B, M, 0.5)$ is strictly preferred to P, that is
$$L = (B, M, 0.5) \succ P.$$

   (b) Suppose we have the following transformations:
$$\begin{aligned} f(x) &= a + bx, \quad a \geq 0, b > 0, \\ g(x) &= \ln(x). \end{aligned}$$
   Check that under f, $L \succ P$, and under g, $P \succ L$.

2. Show that the lotteries (x, z, π) and $(x, y, \pi + (1-\pi)\tau)$ are equivalent when $z = (x, y, \tau)$.

3. Consider a two-date economy and an agent with utility function over consumption:
$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$
   in each period (where $\gamma > 0, \gamma \neq 1$). Define the intertemporal utility function as $U(c_0, c_1) = u(c_0) + u(c_1)$. Show that the agent will always prefer a smooth consumption stream to a more variable one with the same mean, that is
$$U(\bar{c}, \bar{c}) > U(c_0, c_1), \quad \text{if } \bar{c} = \frac{c_0 + c_1}{2}.$$