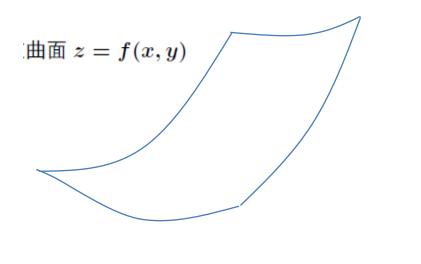
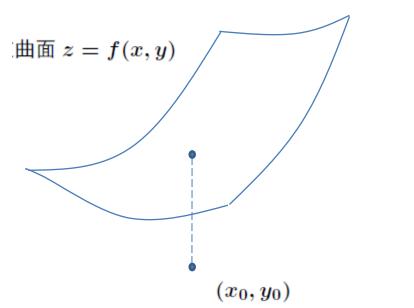
方向导数与梯度





出面 z=f(x,y) $\Delta z=f(x_0+
ho\cos\alpha,y_0+
ho\cos\beta)-f(x_0,y_0)$ $\Delta x=x-x_0=
ho\cos\alpha$

 (x_0, y_0)

(单位向量) $\vec{l} = \{\cos \alpha, \cos \beta\}$

(x,y)

 $\Delta y = y - y_0 = \rho \cos \beta$

曲面
$$z=f(x,y)$$

$$\Delta z=f(x_0+\rho\cos\alpha,y_0+\rho\cos\beta)-f(x_0,y_0)$$

$$\Delta x=x-x_0=\rho\cos\alpha$$

$$\Delta y=y-y_0=\rho\cos\beta$$
 (単位向量) $\vec{l}=\{\cos\alpha,\,\cos\beta\}$ (x_0,y_0)

如果极限

$$\lim_{\rho\to 0+0}\frac{\Delta z}{\rho}=\lim_{\rho\to 0+0}\frac{f(x_0+\rho\cos\alpha,y_0+\rho\cos\beta)-f(x_0,y_0)}{\rho}$$
存在,则称之为函数 $z=f(x,y)$ 在点 (x_0,y_0) 处在方向 \vec{l} 上方向导数记作 $\frac{\partial z}{\partial l}|_{(x_0,y_0)}$

$$\frac{\partial z}{\partial l}|_{(x_0,y_0)} = \lim_{\rho \to 0+0} \frac{f(x_0 + \rho \cos \alpha, y_0 + \rho \cos \beta) - f(x_0, y_0)}{\rho}$$

$$\Delta z = f(x_0 + \rho \cos \alpha, y_0 + \rho \cos \beta) - f(x_0, y_0)$$

$$\Delta x = x - x_0 = \rho \cos \alpha$$

$$\Delta y = y - y_0 = \rho \cos \beta$$
(单位向量) $\vec{l} = \{\cos \alpha, \cos \beta\}$

$$(x_0, y_0)$$

$$\frac{\partial z}{\partial l}|_{(x_0,y_0)} = \lim_{\rho \to 0+0} \frac{f(x_0 + \rho \cos \alpha, y_0 + \rho \cos \beta) - f(x_0, y_0)}{\rho}$$

$$\Delta z = f(x_0 + \rho \cos \alpha, y_0 + \rho \cos \beta) - f(x_0, y_0)$$

$$\Delta x = x - x_0 = \rho \cos \alpha$$

$$\Delta y = y - y_0 = \rho \cos \beta$$
(单位向量) $\vec{l} = \{\cos \alpha, \cos \beta\}$

$$(x_0, y_0)$$

 $f(x_0 + \rho \cos \alpha, y_0 + \rho \cos \beta) - f(x_0, y_0)$

(通常一点的左右侧导数相等)

 $1. \rho > 0$, 注意方向导数与左右导数(单侧导数)的区别!

$$\frac{\partial z}{\partial l}|_{(x_0,y_0)} = \lim_{\rho \to 0+0} \frac{f(x_0 + \rho \cos \alpha, y_0 + \rho \cos \beta) - f(x_0, y_0)}{\rho}$$

$$\Delta z = f(x_0 + \rho \cos \alpha, y_0 + \rho \cos \beta) - f(x_0, y_0)$$

$$\Delta x = x - x_0 = \rho \cos \alpha$$

$$\Delta y = y - y_0 = \rho \cos \beta$$
(单位向量) $\vec{l} = \{\cos \alpha, \cos \beta\}$

 (x_0, y_0)

注 2.如果 $\vec{l} = \{1,0\}$,则 $\lim_{\rho \to 0+0} \frac{f(x_0 + \rho, y_0) - f(x_0, y_0)}{\rho} = \frac{\partial f}{\partial x}(x_0, y_0)$

$$\frac{\partial z}{\partial l}|_{(x_0,y_0)} = \lim_{\rho \to 0+0} \frac{f(x_0 + \rho \cos \alpha, y_0 + \rho \cos \beta) - f(x_0, y_0)}{\rho}$$

$$\Delta z = f(x_0 + \rho \cos \alpha, y_0 + \rho \cos \beta) - f(x_0, y_0)$$

$$\Delta x = x - x_0 = \rho \cos \alpha$$

$$\Delta y = y - y_0 = \rho \cos \beta$$
(单位向量) $\vec{l} = \{\cos \alpha, \cos \beta\}$

$$(x_0, y_0)$$

$$\begin{split} \vec{l} &= \{-1,0\} \, \text{时}, \frac{\partial z}{\partial l} = \lim_{\rho \to 0+0} \frac{f(x_0 - \rho, y_0) - f(x_0, y_0)}{\rho} \\ &= \lim_{\rho \to 0+0} \frac{f(x_0 - \rho, y_0) - f(x_0, y_0)}{-\rho} = -\frac{\partial f}{\partial x}(x_0, y_0) \end{split}$$

注 2.如果 $\vec{l} = \{1,0\}$,则 $\lim_{\rho \to 0+0} \frac{f(x_0 + \rho, y_0) - f(x_0, y_0)}{\rho} = \frac{\partial f}{\partial x}(x_0, y_0)$

$$\frac{\partial z}{\partial l}|_{(x_0,y_0)} = \lim_{\rho \to 0+0} \frac{f(x_0 + \rho \cos \alpha, y_0 + \rho \cos \beta) - f(x_0, y_0)}{\rho}$$

$$\frac{\partial f}{\partial l}(x_0, y_0)$$
是沿 x 轴正向的方向导数,

$$\begin{split} \vec{l} &= \{-1, 0\} \, \text{时}, \frac{\partial z}{\partial l} = \lim_{\rho \to 0+0} \frac{f(x_0 - \rho, y_0) - f(x_0, y_0)}{\rho} \\ &= \lim_{\rho \to 0+0} \frac{f(x_0 - \rho, y_0) - f(x_0, y_0)}{-\rho} = -\frac{\partial f}{\partial x}(x_0, y_0) \end{split}$$

$$\Delta z = f(x_0 + \rho \cos \alpha, y_0 + \rho \cos \beta) - f(x_0, y_0)$$

$$\Delta x = x - x_0 = \rho \cos \alpha$$

$$\Delta y = y - y_0 = \rho \cos \beta$$
(単位向量) $\vec{l} = \{\cos \alpha, \cos \beta\}$
 (x_0, y_0)

 $\underline{f(x_0 + \rho \cos \alpha, y_0 + \rho \cos \beta) - f(x_0, y_0)}$

$$rac{\partial z}{\partial l} > 0$$
 表示 $z = f(x,y)$ 沿 $ec{l}$ 方向函数值增加.

注 3.方向导数中所说的方向是在底平面上的,切线是在空中的.

$$rac{\partial z}{\partial l} < 0$$
表示 $z = f(x,y)$ 沿 $ec{l}$ 方向函数值减少.

进一步考察,

$$\lim_{\rho \to 0+0} \frac{f(x_0 + \rho \cos \alpha, y_0 + \rho \cos \beta) - f(x_0, y_0)}{\rho}$$

$$= \lim_{\rho \to 0+0} \left[\frac{f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0)}{\rho \cos \beta} \cos \beta \right]$$

进一步考察,

$$\lim_{\rho \to 0+0} \frac{f(x_0 + \rho \cos \alpha, y_0 + \rho \cos \beta) - f(x_0, y_0)}{\rho}$$

$$\lim_{\rho \to 0+0} \frac{\frac{f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0 + \rho \cos \beta)}{\rho}}{\rho}$$

$$= \lim_{\rho \to 0+0} \left[\frac{f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0)}{\rho \cos \beta} \cos \beta \right]$$

 $=\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)}{\Delta x} \cos \alpha + \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \cos \beta$

进一步考察.

$$\lim_{\rho \to 0+0} \frac{f(x_0 + \rho \cos \alpha, y_0 + \rho \cos \beta) - f(x_0, y_0)}{\rho}$$

$$f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0 + \rho \cos \beta)$$

$$\lim_{\rho \to 0+0} \frac{\rho}{\rho}$$

$$\lim_{\rho \to 0} \left[\frac{f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0 + \rho \cos \beta)}{\cos \alpha} \cos \alpha \right]$$

 $= \frac{\partial f}{\partial x}(x_0, y_0) \cos \alpha + \frac{\partial f}{\partial y}(x_0, y_0) \cos \beta$

$$\lim_{\rho \to 0+0} \frac{\rho}{\lim_{\rho \to 0+0} \left[\frac{f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{\rho \cos \beta}{\rho \cos \alpha} \right]}$$

$$\lim_{\rho \to 0+0} \frac{\rho}{\lim_{\rho \to 0+0} \left[\frac{f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{\rho \cos \alpha}{\rho \cos \alpha}\right]}$$

$$\lim_{\rho \to 0+0} \frac{\rho}{\lim_{\rho \to 0+0} \left[\frac{f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0 + \rho \cos \beta)}{\cos \alpha + \cos \beta} \cos \alpha + \frac{1}{2}\right]}}$$

$$\frac{\cos \beta}{\cos \alpha} \cos \alpha + \frac{f(x_0, y_0)}{\cos \alpha}$$

$$\frac{1}{\rho \cos \beta} \cos \beta + \frac{f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0)}{\rho \cos \beta} \cos \beta$$

$$= \lim_{\rho \to 0+0} \left[\frac{f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0)}{\rho \cos \beta} \cos \beta \right]$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)}{\Delta x} \cos \alpha + \lim_{\substack{\Delta y \to 0}} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \cos \beta$$

进一步考察.

$$\lim_{\rho \to 0+0} \frac{f(x_0 + \rho \cos \alpha, y_0 + \rho \cos \beta) - f(x_0, y_0)}{\rho}$$
$$f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0 + \rho \cos \beta)$$

$$\lim_{\rho \to 0+0} \frac{f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0 + \rho \cos \beta)}{\rho}$$

$$= \lim_{\rho \to 0+0} \left[\frac{f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0 + \rho \cos \beta)}{\cos \beta} \cos \beta \right]$$

$$\lim_{\rho \to 0+0} \frac{\frac{f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0 + \rho \cos \beta)}{\rho}}{\rho}$$

$$\lim_{\rho \to 0+0} \left[\frac{f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos$$

$$ho \left[rac{f(x_0,y_0+
ho\coseta)-f(x_0,y_0+
ho\coseta)}{
ho\coslpha}\coslpha +
ho\sinlpha
ight]$$

$$\lim_{\rho \to 0+0} \frac{\frac{f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0 + \rho \cos \beta)}{\rho}}{\rho}$$

$$\lim_{\rho \to 0+0} \left[\frac{f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0 + \rho \cos \beta)}{\cos \alpha} \cos \alpha + \frac{1}{\rho} \cos \beta \right]$$

$$\frac{f(x_0 + \rho \cos \alpha, y_0 + \rho \cos \beta) - f(x_0, y_0)}{\rho}$$

$$\frac{f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0 + \rho \cos \beta)}{\cos \alpha + \beta}$$

$$ho$$
 $(0, y_0 +
ho\coseta) - f(x_0, y_0 +
ho\coseta) \coslpha + coslpha$

$$\frac{\rho}{\rho}$$
 $\frac{(x_0, y_0 + \rho \cos \beta)}{\cos \alpha + \frac{1}{2}}$

$$\frac{\beta}{\beta} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0)}{\rho \cos \beta} \cos \beta$$

$$= \lim_{\rho \to 0+0} \left[\frac{f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0 + \rho \cos \beta)}{\rho \cos \alpha} \cos \alpha + \frac{f(x_0, y_0 + \rho \cos \beta) - f(x_0, y_0)}{\rho \cos \beta} \cos \beta \right]$$

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)$$

$$f(x_0, y_0 + \Delta y) - f(x_0, y_0)$$

$$\frac{\rho\cos\beta}{\frac{y_0+\Delta y)}{\cos\alpha+\lim_{\Delta y\to 0}}}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}\cos\beta$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)}{\Delta x} \cos \alpha + \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \cos \beta$$

$$\Delta x \qquad \Delta y \rightarrow 0 \qquad \Delta y$$

$$= \frac{\partial f}{\partial x}(x_0, y_0) \cos \alpha + \frac{\partial f}{\partial y}(x_0, y_0) \cos \beta \qquad \text{连续}$$

如果
$$\frac{\partial f}{\partial x}$$
, $\frac{\partial f}{\partial y}$ 连续, 则 $z=f(x,y)$ 于点 (x,y) 在任意方向

$$ec{l} = \{\cos \alpha, \cos \beta\}$$
 上的导数存在且有
$$\partial f = \partial f \qquad \partial f$$

$$\frac{\partial f}{\partial l} = \frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \cos \beta$$

" $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ 连续" 可换成 " z=f(x,y) 可微"

如果 $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ 连续, 则 z = f(x,y) 于点 (x,y) 在任意方向

$$\begin{split} \vec{l} &= \{\cos\alpha,\cos\beta\} \text{ 上的导数存在且有} \\ &\frac{\partial f}{\partial l} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\cos\beta \end{split}$$

可微= 每个方向上都可导!

"
$$\frac{\partial f}{\partial x}$$
, $\frac{\partial f}{\partial y}$ 连续" 可换成 " $z=f(x,y)$ 可微"



定理 如果
$$\frac{\partial f}{\partial x}$$
, $\frac{\partial f}{\partial y}$ 连续, 则 $z=f(x,y)$ 于点 (x,y) 在任意方向

$$\vec{l} = \{\cos \alpha, \cos \beta\}$$
 上的导数存在且有

$$\frac{\partial f}{\partial l} = \frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \cos \beta$$

$$\vec{l} = \{\cos\alpha, \ \cos\beta\} \qquad \qquad \frac{\partial f}{\partial l} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\cos\beta$$

$$\vec{l} = \{\cos \alpha, \cos \beta\}$$

$$\frac{\partial f}{\partial l} = \frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \cos \beta$$

求
$$z = x^2 - y^2$$
 在点 $(1,1)$ 处沿 $\vec{l} = \{\cos \frac{\pi}{3}, \cos \frac{\pi}{6}\} = \{\frac{1}{2}, \frac{\sqrt{3}}{2}\}$ 的方向导数.

$$\vec{l} = \{\cos \alpha, \cos \beta\}$$

$$\frac{\partial f}{\partial l} = \frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \cos \beta$$

求
$$z = x^2 - y^2$$
 在点 $(1,1)$ 处沿 $\vec{l} = \{\cos \frac{\pi}{3}, \cos \frac{\pi}{6}\} = \{\frac{1}{2}, \frac{\sqrt{3}}{2}\}$ 的方向导数.

解:
$$\frac{\partial z}{\partial x} = 2x$$
, $\frac{\partial z}{\partial y} = -2y$,

$$\vec{l} = \{\cos \alpha, \cos \beta\}$$

$$\frac{\partial f}{\partial l} = \frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \cos \beta$$

例'

求
$$z = x^2 - y^2$$
 在点 $(1,1)$ 处沿 $\vec{l} = \{\cos \frac{\pi}{3}, \cos \frac{\pi}{6}\} = \{\frac{1}{2}, \frac{\sqrt{3}}{2}\}$ 的方向导数.

解:
$$\frac{\partial z}{\partial x} = 2x$$
, $\frac{\partial z}{\partial y} = -2y$, $\frac{\partial z}{\partial x}|_{(1,1)} = 2$, $\frac{\partial z}{\partial y}|_{(1,1)} = -2$,

$$\vec{l} = \{\cos \alpha, \ \cos \beta\}$$

$$\frac{\partial f}{\partial l} = \frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \cos \beta$$

求
$$z = x^2 - y^2$$
 在点 $(1,1)$ 处沿 $\vec{l} = \{\cos \frac{\pi}{3}, \cos \frac{\pi}{6}\} = \{\frac{1}{2}, \frac{\sqrt{3}}{2}\}$ 的方向导数.

$$\Re z = x^2 - y^2$$
 在点 $(1,1)$ 处沿 $l = \{\cos \frac{\pi}{3}, \cos \frac{\pi}{6}\} = \{\frac{\pi}{2}, \frac{\pi 3}{2}\}$ 的万向导致.

解:
$$\frac{\partial z}{\partial x} = 2x$$
, $\frac{\partial z}{\partial y} = -2y$,

$$\frac{\partial z}{\partial x}|_{(1,1)} = 2, \quad \frac{\partial z}{\partial y}|_{(1,1)} = -2,$$

$$\frac{\partial z}{\partial z}|_{(1,1)} = \frac{\partial z}{\partial y}|_{(1,1)} = -2,$$

$$\frac{\partial z}{\partial l}|_{(1,1)} = \frac{\partial z}{\partial x}|_{(1,1)} \cdot \frac{1}{2} + \frac{\partial z}{\partial y}|_{(1,1)} \cdot \frac{\sqrt{3}}{2} = 1 - \sqrt{3}.$$

$$\frac{\vec{l} = \{\cos \alpha, \ \cos \beta\}}{\frac{\partial f}{\partial l}} = \frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \cos \beta$$

$$\frac{\partial f}{\partial l} = \{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\} \cdot \{\cos \alpha, \cos \beta\}$$

$$\vec{l} = \{\cos \alpha, \cos \beta\}$$

$$\frac{\partial f}{\partial l} = \frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \cos \beta$$

$$\frac{\partial f}{\partial l} = \{ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \} \cdot \{ \cos \alpha, \cos \beta \}$$
 是什么矢量呢?

老家等喜线族
$$f(x, y) = c$$
 微分后得 $df(x, y) = 0$

考察等高线族,
$$f(x,y) = c$$
,微分后得 $df(x,y) = 0$, 即 $\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0 \Rightarrow \{ dx, dy \} \perp \{ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \},$ { dx , dy } 是等高线的切线方向矢量,

$$\{dx, dy\}$$
 是等高线的切线方向矢量,

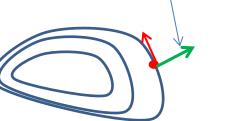


$$\frac{\vec{l} = \{\cos \alpha, \cos \beta\}}{\frac{\partial f}{\partial l} = \frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \cos \beta}$$

$$\frac{\partial f}{\partial l} = \{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\} \cdot \{\cos \alpha, \cos \beta\}$$

考察等高线族,
$$f(x,y) = c$$
, 微分后得 $df(x,y) = 0$, 即 $\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0 \Rightarrow \{ dx, dy \} \bot \{ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \},$

 $\partial x \qquad \partial y$ { dx, dy} 是等高线的切线方向矢量,



$$\{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\}|_{(x_0,y_0)}$$
 是 $z = f(x,y)$ 过 (x_0,y_0) 的等高线于此点的法线方向矢量!

$$\vec{l} = \{\cos \alpha, \cos \beta\}$$

$$\frac{\partial f}{\partial l} = \frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \cos \beta$$

向量
$$\{rac{\partial f}{\partial x},rac{\partial f}{\partial y}\}$$
 决定一切方向上的方向导数

$$\vec{l} = \{\cos \alpha, \cos \beta\}$$

$$\frac{\partial f}{\partial l} = \frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \cos \beta$$

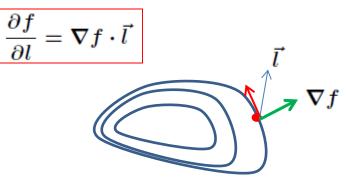
向量
$$\{rac{\partial f}{\partial x},rac{\partial f}{\partial y}\}$$
 决定一切方向上的方向导数。

称之为函数 z=f(x,y) 在 (x,y) 点的梯度,记作gradz 或gradf,即 $gradz=\{\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\}$, $gradf=\{\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\}$,也写作 ∇z , ∇f .

$$\vec{l} = \{\cos\alpha, \ \cos\beta\} \qquad \qquad \frac{\partial f}{\partial l} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\cos\beta$$

向量
$$\{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\}$$
 决定一切方向上的方向导数

称之为函数 z=f(x,y) 在 (x,y) 点的梯度,记作gradz 或gradf,即 $gradz=\{\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\}$, $gradf=\{\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\}$, 也写作 $\nabla z,\nabla f$.



底平面各点的梯度方向就是过该点的等高线的(正)法线方向。

$$\frac{\partial f}{\partial l} = \nabla f \cdot \vec{l}$$

函数在梯度方向上的方向导数为

函数在梯度万向上的万向导数为
$$\frac{\partial f}{\partial \nabla} = \left\{ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\} \cdot \frac{\left\{ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\}}{\sqrt{\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2}}$$

$$\frac{\partial f}{\partial l} = \nabla f \cdot \vec{l}$$

函数在梯度方向上的方向导数为

$$\frac{\partial f}{\partial \nabla} = \left\{ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\} \cdot \frac{\left\{ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\}}{\sqrt{\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2}}$$

$$= \sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2} = |\nabla f|$$

 ∇z 的方向是方向导数最大的方向,(自身到自身的投影的模(长度)最大!)

所以梯度方向就是函数值增加速度最大的(水平)方向