Numerical Methods in Economics and Finance
Lecture 1: Basic Models

Xi Wang
Department of Finance, School of Economics
Peking University

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INTRODUCTION AND OVERVIEW

- Basic DGE and DSGE models
- Finite Horizon Version and Infinite horizon version
- Generally, I will use two ways to characterize the solutions: Euler Equation and Dynamic Programming.
- Sometime people also refer to this type of model as Ramsey model (utility maximizing problem).

BASIC SETUP

- Time is divided into intervals of unit length and extends from the current date $t = 0$ to Final horizon, if any, say $t = T$
- K_t and N_t denotes the stock of capital and flow of labor being put into production
- Only one product is available, it can be used to consume or produce capital goods.
- I call this kind of goods as final product, which can be only produced by labor and capital through a production function:

$$Y_t = F(N_t, K_t) \tag{1}$$

Assumptions

- There are several assumptions on this production function

$$Y_t = F(N_t, K_t)$$

  - $0 = F(0,0)$
  - $F'_1, F'_2 \geq 0$
  - $F''_{11}, F''_{22} \leq 0$, how about F''_{12}?
- Budget Constraint (Simplest Version)

$$Y_t \geq C_t + K_{t+1} - (1-\delta)K_t, \quad \delta \in [0,1] \tag{2}$$

- How does consumer value the consumption?

$$U(C_1, C_2, \ldots, C_T), \quad U'_i > 0, U''_{ii} \leq 0 \tag{3}$$

PROBLEM FORMULATED

- Hence the Ramsey Problem can be formulated as

$$\max_{C_1, C_2, \ldots, C_t} U(C_1, C_2, \ldots, C_T)$$
$$s.t.$$
$$C_t + K_{t+1} \leq F(N_t, K_t) + (1-\delta)K_t$$
$$C_t \geq 0$$
$$K_{t+1} \geq 0$$
$$\text{for all } t = 1, 2, \ldots, T$$
$$K_1 = K \text{ is given}$$

- Comments:
  - no uncertainty
  - All information are fully understood by?
  - $T < \infty, T = \infty$

KUHN-TUCKER THEOREM

Theorem

Let f be a concave continuous differentiable function, mapping X into R, where $X \subset R^N$ is a convex and open. For $i = 1, \ldots, K$, let $h^i : X \to R$ is a concave continuous differentiable function. Suppose there is $x_0 \in X$ with $h^i(x_0) > 0$ for all i.
Then x^* maximizes f over $D = \{x \in X | h^i(x) > 0, i = 1, \ldots, K\}$ iff there is $\lambda^* \in R^K$ s.t.

$$\frac{\partial}{\partial x_j} f(x)|_{x^*} + \sum_{i=1}^{K} \lambda_i^* \frac{\partial}{\partial x_j} h^i(x)|_{x^*} = 0, \quad i = 1, \ldots, N \tag{4}$$

$$\lambda_j^* \geq 0, \quad j = 1, \ldots, K \tag{5}$$

$$\lambda_j^* h^j(x^*) = 0 \tag{6}$$

Problem Revisited

- Here I just applied the Th to the first problem

$$
\begin{aligned}
&\frac{\partial U(C_1, C_2, \ldots, C_T)}{\partial C_t} - \lambda_t + \mu_t = 0 \\
&- \lambda_t + \lambda_{t+1} F_2(N, K_{t+1}) + \theta_{t+1} = 0 \\
&- \lambda_T + \theta_{T+1} = 0 \\
&\lambda_t (F(N, K_t) - C_t + (1-\delta) K_t - K_{t+1}) = 0 \\
&\mu_t C_t = 0 \\
&\theta_{t+1} K_{t+1} = 0
\end{aligned}
$$

- Short-cut?

Problem Simplified with previous assumptions

- $\frac{\partial U(C_1,C_2,...,C_T)}{\partial C_t} > 0$, Hence?
- What if Inada condition holds?
- Why Free-Lunch Condition?
- In the end I can reduce my array of equations into

$$K_{t+1} = F(N, K_t) + (1-\delta)K_t - C_t \tag{7}$$

$$\frac{U_{C_t}}{U_{C_{t+1}}} = F_2'(N, K_{t+1}) + 1 - \delta \tag{8}$$

- How and Why?

EXTENDED OUR PREVIOUS SETUP

- Before I move forward, what is the sequence of equations?
- number of unknown v.s. number of equations
- What if $T \to \infty$?
- Time separable Utility setup

$$U(C_1, C_2 \ldots, C_t, \ldots) = \sum_{t=1}^{\infty} \beta^{t-1} u(C_t) \tag{9}$$

- Not all sensible life-utility function are time separable, say Epstein and Zin Utility function.
- What does this mean? $u : R^+ \to R$, Ass: $u' > 0, u'' \leq 0$, $\beta < 1$

Problem Reformulated

- Infinite Deterministic Problem: Hence the Ramsey Problem can be formulated as

$$
\begin{aligned}
& \max_{C_1, C_2, \ldots, C_t} \sum_{t=1}^{\infty} \beta^{t-1} u(C_t) \\
& s.t. \\
& C_t + K_{t+1} \leq F(N_t, K_t) + (1-\delta)K_t \\
& C_t \geq 0 \\
& K_{t+1} \geq 0 \\
& \text{for all } t = 1, 2, \ldots, T \\
& K_1 = K \text{ is given}
\end{aligned}
$$

- Again? FOC?
- We need the life-time attainable utility to be finite. Do I need to impose u is bounded? Sustainable Capital.

Solution of Reformulated Problem

- Lagrangean:

$$L = \sum_{t=0}^{\infty} \beta^t [u(C_t) + \lambda_t (F(N, K_t) + (1-\delta)K_t - C_t - K_{t+1}) + \mu_t C_t + \theta_{t+1} K_{t+1}]$$

- Again? FOC?

$$\lambda_t = u'(C_t) + \mu_t \tag{10}$$
$$\lambda_t = \beta \lambda_{t+1} [(1-\delta) + F_2'(N, K_{t+1})] + \theta_{t+1} \tag{11}$$
$$\lambda_t (F(N, K_t) + (1-\delta)K_t - C_t - K_{t+1}) = 0 \tag{12}$$
$$\mu_t C_t = 0 \tag{13}$$
$$\theta_{t+1} K_{t+1} = 0 \tag{14}$$

- Inada Condition simplifies? BC simplifies?

EULER EQUATIONS OF REFORMULATED PROBLEM

- Two left:

$$u'(C_t) = \beta u'(C_{t+1})[(1-\delta) + F_2'(N, K_{t+1})] \quad (15)$$
$$F(N, K_t) + (1-\delta)K_t - C_t - K_{t+1} = 0 \quad (16)$$

- Again what is this?

$$u'(g(K_t) - K_{t+1}) = \beta u'(g(K_{t+1}) - K_{t+2})[1 - \delta + F_2'(N, K_{t+1})] \quad (17)$$

- How many boundary conditions do we need?
- K_0 and TVC $\lim_{t\to\infty} \beta^t u'(c_t)K_{t+1} = 0$ $\square$

Dynamic Programming Method

- There is a recursive nature of the infinite horizon Ramsey Problem. Given the optimal path of capital, say $\{K_{t+1}\}_{t=0}^{\infty}$

$$V(K_0) = \sum_t \beta^t u(C_t) \tag{18}$$

$$C_t = F(N, K_t) + (1-\delta)K_t - K_{t+1} \tag{19}$$

$$V(K_0) = \sum_t \beta^t u(g(K_t) - K_{t+1}) \tag{20}$$

$$= u(g(K_0) - K_1) + \beta \sum_{t=1}^{\infty} \beta^{t-1} u(g(K_t) - K_{t+1}) \tag{21}$$

$$= u(g(K_0) - K_1) + \beta V(K_1) \tag{22}$$

- The last equation is called Bellman Equation. Economics and Reinforcement Learning literature use it to update value function or policy function.

DYNAMIC PROGRAMMING METHOD II

- The Bellman Equations tells us

$$V(K) = u(C) + \beta V(K')$$
$$s.t.$$
$$g(K) \geq K' + C$$

- What is $V(K)$?
- What if we know $V(.)$?

$$L = u(C) + \beta V(K') + \lambda(g(K) - K' - C)$$
$$u'(C) = \lambda$$
$$\beta V'(K') = \lambda$$

- $\lambda =?$
- What do we get from the last FOC?
- Implicit function theorem....

Recursive Method: A Problem

- The Bellman Equations tells us

$$V(K) = u(g(K) - K') + \beta V(K')$$
$$s.t.$$
$$g(K) \geq K'$$

- But we do not know $V(.)$. How to deal with that?
- Recursive Method:

$$V^{i+1}(K) = u(g(K) - K') + \beta V^{i}(K')$$
$$s.t.$$
$$g(K) \geq K'$$
$$\lim_{i \to \infty} V^{i}(K) = V(K), \quad \text{in Sup Norm}$$

IF concave, bounded continuous condition holds. Please carefully read Ch 3 of Stokey and Lucas with Prescott (1989).

Recursive Method

- The Bellman Equations tells us

$$0 = -u'(g(K) - K') + \beta V'(K')$$
$$hence.$$
$$V'(K) = u'(g(K) - K')[g'(K) - h'(K)] + \beta V'(K')h'(K)$$
$$= u'(g(K) - K')g'(K)$$

- Divide the first equation with the last one. One will get Euler equation.

RECURSIVE METHOD V.S EULER EQUATION

- Are these two method the same?
- The Bellman Equations tells us
- Recursive Method:

$$V^{i+1}(K) = u(g(K) - K') + \beta V^i(K')$$
$$s.t.$$
$$g(K) \geq K'$$
$$\lim_{i \to \infty} V^i(K) = V(K), \quad \text{in Sup Norm}$$

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Dynamic programming and Finite Horizon Problem

- Solve backwards
- $V_{T-1}(K_{T-1}) = u(g(K_{T-1}) - K_T) + \beta V_T(k_T)$
- $V(K_T) = u(g(K_T))$, why?
- …

DYNAMICS OF THE MODEL

- How to characterize the dynamics of model? Say how does C_t, K_t evolves over time?
- Dynamic Phase:

$$K_{t+1} = g(K_t) - C_t \tag{23}$$

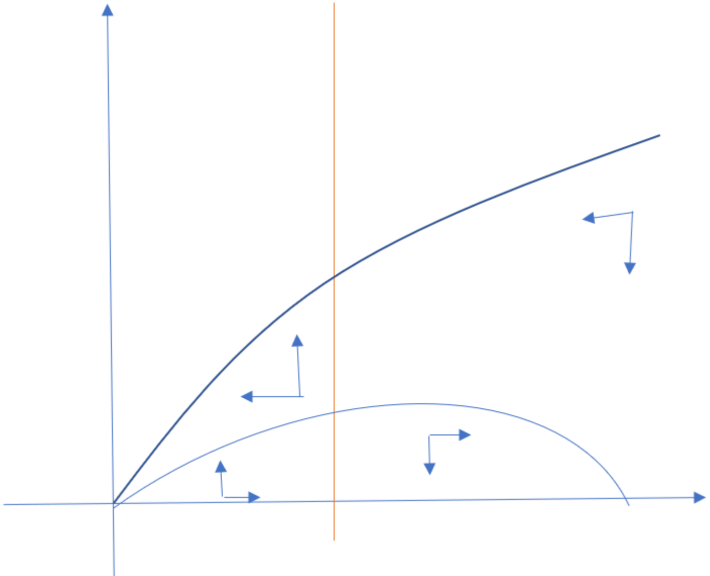
$$1 = \beta \frac{u'(C_{t+1})}{u'(C_t)} g'(K_{t+1}) \tag{24}$$

- S.S

$$K^* = g(K^*) - C^* \tag{25}$$

$$1 = \beta g'(K^*) \tag{26}$$

DYNAMICS OF THE MODEL II



HAMILTONIAN

- See lecture Notes

FURTHER READING

- Ch 1 2 3 Romer (1991)
- Ch 2, 3, 4 of Stokey and Lucas with Prescott (1989)
- Time to Build and Aggregate Fluctuations, by Finn E. Kydland and Edward C. Prescott