Numerical Methods in Economics and Finance
Lecture 2: Stochastic Models

Xi Wang
Department of Finance, School of Economics
Peking University

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Review and Overview of this lecture

- Basic DGE, Finite Horizon Version and Infinite horizon version - Deterministic Version
- Euler equation and Dynamic Programming Method
- Stochastic Model
- Models With Growth, Labor Participant Choice, and occasionally binding constraints.

STOCHASTIC ELEMENTS

- There can be uncertainty about productivity of capital, labor or your manufacturing process
- This uncertainty is beyond your control.
- Then how are we to model the objective of agent? Expected Utility Theory.
- Let me introduce one random element here, on production function:

$$Y_t = Z_t F(N_t, K_t) \tag{1}$$

- The economic agent cannot fully anticipate the future path

Stochastic Elements

- Let me introduce one random element here, on production function:
$$Y_t = Z_t F(N_t, K_t) \tag{2}$$
- Hence the economic agent cannot fully anticipate the future path

$$\max_{C_1, C_2, \dots} E[\sum_t \beta^t u(C_t)|I_0]$$
$$s.t.$$
$$K_{t+1} + C_t \leq Z_t F(N, K_t) + (1-\delta)K_t$$
$$0 \leq C_t$$
$$0 \leq K_{t+1}$$

- How many state variables do you have at the beginning of period t?

Stochastic Euler Equation

- Again Lagrangean

$$\begin{aligned} L_t =& E[\sum_t \beta^t (u(C_t) + \mu_t C_t + \theta_{t+1} K_{t+1} \\ &+ \lambda_t (Z_t F(k_t) + (1-\delta)K_t - C_t - K_{t+1})] \end{aligned}$$

- F.O.C

$$u'(C_t) + \mu_t = \lambda_t \tag{3}$$

$$E[\beta \lambda_{t+1} (Z_{t+1} F'(K_{t+1}) + 1 - \delta) + \theta_{t+1}] = \lambda_t \tag{4}$$

$$\lambda_t (Z_t F(K_t) + (1-\delta)K_t - C_t - K_{t+1}) = 0 \tag{5}$$

$$\mu_t C_t = 0 \tag{6}$$

$$\theta_{t+1} K_{t+1} = 0 \tag{7}$$

Stochastic Dynamic Programming

- Bellman equation

$$v(K, Z) = \max_{c+K' \leq (1-\delta)K+ZF(K)} u(c) \tag{8}$$

$$+ \beta E[v(Z', K')|Z] \tag{9}$$

$$E[v(Z', K')|Z] = \int v(K', Z')d\pi(Z'|Z) \tag{10}$$

- MPD.

Stochastic Euler Equation

- Simplified? Why?
- F.O.C

$$E[\beta u'(C_{t+1})(Z_{t+1}F'(K_{t+1})+1-\delta)+\theta_{t+1}] = u'(C_t) \tag{11}$$

$$(Z_tF(K_t)+(1-\delta)K_t - C_t - K_{t+1}) = 0 \tag{12}$$

$$\lim_{t\to\infty} \beta^t E[u'(C_t)K_{t+1}] = 0 \tag{13}$$

- Stochastic 2nd Order?

SDP agrees?

- Bellman equation

$$
\begin{aligned}
v(Z,K) &= u(c) + \beta E[v(Z',K')|Z] && (14)\\
&\quad + \lambda((1-\delta)K + ZF(K) - c - K') && (15)\\
u'(c) &= \lambda && (16)\\
\beta E[v_2'(Z',K')|Z] &= \lambda && (17)\\
V_2(Z,K) &= \lambda(1-\delta + ZF'(K)) && (18)
\end{aligned}
$$

- c.f. SEE

Review and Preview

- I did not take labor to be adjustable in previous slides. This one is easy, I can write

$$u = u(C_t, N_t),\ N_t \in [0, N]; \tag{19}$$

Why $N_t = N$ in my previous discussion?

  - What if there is a short jump-up of the marginal productivity of labor? Income effect, (intertemporal-) Substitution effect

- There is no growth in my model, since capital and consumption will converge to the (stochastic) s.s
- How to embed in growth? How to measure growth?
  - https://fred.stlouisfed.org/series/A939RX0Q048SBEA
  - https://fred.stlouisfed.org/series/NYGDPPCAPKDCHN

WHERE DOES GROWTH COME?

- Labor Participation; Capital Accumulation;
- Solow residual
  - Embodied Tech and disembodied

$$Y_t = A_t F(\theta_t N_t, \eta_t K_t) \tag{20}$$

  - Normal Assumption on F and Constant Return to Scale $\lambda F(x, y) = F(\lambda x, \lambda y)$
  - Balanced growth path: output per working hour grows constantly, share of saving in output is constant. `https://fred.stlouisfed.org/series/A072RC1Q156SBEA`
  - Does balance growth path impose any assumption on $F(.,.)$?

SETUP

- Production function $Y_t = F(A_t N_t, K_t)$
- There is a growth assumption as in Solow model $\dot{A}_t / A_t = a$; Where is fluctuation?
- Z_t is a covariance stationary process. $Y_t = Z_t F(A_t N_t, K_t)$ or $Y_t = F(A_t Z_t N_t, K_t)$
- If $Z_t = 1$, emm?
- Then $Y_t = A_t F(N_t, \frac{K_t}{A_t})$, what did you learn from Solow Model?
- How to model Z_t and A_t? Linear trend and fluctuation decomposition.
- $Z' = Ze^{\epsilon_t},\ \epsilon_t \sim N(0, \sigma^2)$
- I will nest AZ

SETUP II

- Problem becomes

$$\max_{c_t, N_t} E[\sum_t \beta^t u(C_t, 1 - N_t)]$$

$$s.t.$$

$$C_t + K' \leq F(AN_t, K_t) + (1 - \delta)K_t$$

$$0 \leq C_t$$

$$0 \leq K_{t+1}$$

$$N_t \in [0, 1]$$

- F.O.C

$$\lambda_t = u_c(C_t, 1 - N_t)$$

$$\lambda_t A_t F(A_t N_t, K_t) = u_l(C_t, 1 - N_t)$$

$$\lambda_t = \beta \lambda_{t+1}[1 - \delta + F_2(A_{t+1} N_{t+1}, K_{t+1})]$$

SETUP II

- Problem Turns to be

$$u_l(C_t, 1-N_t) = u_c(C_t, 1-N_t) A_t F_1(A_t N_t, K_t)$$
$$u_c(C_t, 1-N_t) = \beta\, u_c(C_{t+1}, 1-N_{t+1})[1-\delta + F_2(A_{t+1}N_{t+1}, K_{t+1})]$$

- $F_2(x, y) = F_2(\lambda x, \lambda y)$
- What does this tell us about consumption growth? $\frac{K_{t+1}}{K_t} = \frac{Y_t}{K_t} - \frac{C_t}{K_t} + 1 - \delta$
- I will generally use, $u(c, 1-N) = C^{1-\eta} v(1-N)$ *or* $\ln(C) + v(1-N)$
- How to solve it?

Euler Equation

- Euler Equation turns to be

$$
\begin{aligned}
\frac{C_t^{-\eta} v(1-N_t)}{C_{t+1}^{-\eta} v(1-N_{t+1})} &= \beta(1-\delta+F_2(N_{t+1}, K_{t+1}/A_{t+1})) \\
&= \frac{(C_t/(aA_t))^{-\eta} v(1-N_t)}{(C_{t+1}/A_{t+1})^{-\eta} v(1-N_{t+1})} \\
a\frac{K_{t+1}}{A_{t+1}} &= F(N_t, K_t/A_t) + (1-\delta)\frac{K_t}{A_t} - C_t/A_t \\
\frac{v'(1-N_t)}{(1-\eta)v(1-N_t)} \frac{C_t}{A_t} &= F_1(N_t, K_t/A_t)
\end{aligned}
$$

- How to solve it?
- Or one can rewrite the problem with new variables.

Decentralized Economy

- What I write down is called the representative agent's problem
- Where does the "general" comes from? Decentralization
- Firms
  - Large number of identical firms, all uses labor and capital to produce with same production function
- Household
  - Sell their labor, rent out capital and own the firm
  - Choose how much labor and capital to rent out
  - How much to consume

DECENTRALIZATION: FIRMS

- What if firms' decision?

$$\max_{N_t, K_t} F(A_t N_t, K_t) - w_t N_t - r_t K_t \tag{21}$$

- Demand?
  - What is there exogenous state variable? What will they choose?

$$w_t = A_t F_1(A_t N_t, K_t) \tag{22}$$

$$r_t = F_2(A_t N_t, K_t) \tag{23}$$

  - $Y_t = \sum_i Y^i = N_t F(A_t, K_t/N_t)$
  - Which Assumption does this aggregation result rest on?
  - BTW what is the profit of the firm?

DECENTRALIZATION: HOUSEHOLD

- A continuum of households of mass 1, $h \in [0,1]$, they are endowed 1 unite of labor and some capital.

$$\max_{C_t, K'} \sum_t \beta^t u(C_t, 1-N_t)$$

$$s.t.$$

$$K' + C_t \leq w_t N_t + (r_t + 1 - \delta)K_t$$

- F.O.C?
  - What is there exogenous state variable? What will they choose? AGAIN

$$\lambda_t = u_1(C_t, 1-N_t) \tag{24}$$

$$w_t \lambda_t = u_2(C_t, 1-N_t) \tag{25}$$

$$\lambda_t = \beta \lambda_{t+1}[1 - \delta + r_{t+1}] \tag{26}$$

$$C_t + K' = (r_t + 1 - \delta)F_2 + w_t N_t + \Pi_t \tag{27}$$

  - Representative Euler Equation? B.C?

MORE ON MARKOV MODEL

- Ok, what you learn about Markovian Chain may be different with what we assumed. What We assume is that stationary Markovian process, which is easy to get as a limit state of a Markov Chain.
- It is a stationary distribution of z, k, we will see more numerical examples on this
- Markov Transition Matrix with ergodic assumption
  - Given there are N states, we have initial draw of π_0, a distribution on N states and transition Matrix

$$M = \begin{pmatrix} p_{11} & p_{12} & \dots & p_{1N} \\ p_{21} & p_{22} & \dots & p_{2N} \\ \dots & & & \\ p_{N1} & p_{N2} & \dots & p_{NN} \end{pmatrix} \tag{28}$$

  - what is the distribution of states on period 1? $\pi_1' = \pi_0' M$
  - Do we have what is $\lim_{t\to\infty} \pi_0 M^t$?, will this converge?
  - Or what is the stationary distribution? $\pi_{t+1}' = \pi_t' M$, $\pi'(I - M) = 0, \; or \; (I - M')\pi = 0$, eigenvectors of M'

Problem

- We would like to solve

$$\max_{c_t,N_t} E[\sum_t \beta^t \frac{C_t^{1-\eta}(1-N_t)^{\theta(1-\eta)}}{1-\eta}]$$

$$s.t.$$

$$C_t + K' \leq Z_t(AN_t)^{1-\alpha}K_t^{\alpha} + (1-\delta)K_t$$

$$A_{t+1} = aA_t$$

$$\ln(Z_{t+1}) = \rho \ln(Z_t) + \epsilon_t,\ \epsilon_t \sim N(0,\sigma^2)$$

$$0 \leq C_t$$

$$0 \leq K_{t+1}$$

$$N_t \in [0,1]$$

PARAMETER TO CHOOSE

- The parameters will be

Parameter	Value
ρ	.9
β	0.994
a	1.005
α	0.33
δ	0.05
N	.4
σ	0.007
η	2

- Solve the model under these assumptions.

Values I would like to have

- AFTER solving…
- Average of gdp, consumption, investment growth, their variance and covariance
- Auto correlations
- Impulse response functions

RESTRICTIONS

- What if you would like to have :(1)a constant growth of variables (2) a constant share of savings in output.
- Implication

$$\frac{K_{t+1}}{K_t} = \frac{Y_t - C_t}{K_t} + 1 - \delta = \frac{Y_t}{K_t} s_t + 1 - \delta \tag{29}$$

- If capital grows constantly, then it grows as fast as? Consumption Grows?
- Given production function $Y_t = F(A_0 a^t N_t, B_0 b^t K_t) = B_t K_t F(\frac{A_t N_t}{B_t K_t}, 1)$

$$\frac{Y_{t+1}}{Y_t} = b \frac{K_{t+1}}{K_t} \frac{F(X_{t+1})}{F(X_t)}, \; X_t = \frac{A_t N_t}{B_t K_t} \tag{30}$$

Restrictions II

- Implies?
- Case 1: $b = 1, X_t = constant$, hence K_t grows like A_t since?
- Case 2: If we further impose that X_t grows constantly means $x_t = \theta^t x_0, and\ \frac{F(x_0\theta^{t+1})}{F(x_0\theta^t)} = constant$

$$F'(x_0\theta^{t+1})\theta^{t+1}F(x_0\theta^t) = F(x_0\theta^{t+1})F'(x_0\theta^t)\theta^t \tag{31}$$

$$\frac{F'(x_0\theta^{t+1})\theta^{t+1}}{F(x_0\theta^{t+1})} = \frac{F'(x_0\theta^t)\theta^t}{F(x_0\theta^t)} \tag{32}$$

$$constant = \frac{F'(x)x}{F(x)} \tag{33}$$

- Then Production function become $(A_tN_t)^\alpha(B_tK_t)^{1-\alpha}$, you can sort variables

RESTRICTIONS III

- Given our form of production function, Euler Equation

$$\beta\frac{u_1(C_{t+1},1-N_{t+1})}{u_1(C_t,1-N_t)} = F_2(A_{t+1}N_{t+1},K_{t+1})+1-\delta \quad (34)$$

$$= F_2(N_{t+1},K_{t+1}/A_{t+1})+1-\delta \quad (35)$$

$$C_t = A_t[F(N_t,K_t/A_t)+(1-\delta)K_t/A_t+aK_{t+1}/A_{t+1}] \quad (36)$$

- Then we have $\frac{u_1(C_0a^{t+1},1-N)}{u_1(C_0a^t,1-N)} = contant$
  Similar to our previous derivation

$$\frac{u_{11}(C,1-N)C}{u_1(C,1-N)} = constant \quad (37)$$

$$C^{-\gamma}v(1-N) = u_1(C,1-N) \quad (38)$$

$$C^{1-\gamma}v(1-N)+\eta(1-N) = u(C,1-N) \quad (39)$$

Restrictions IV

- What if $\gamma = 1$?
- Given our form of production function, Euler Equation

$$\beta \frac{u_2(C_t, 1-N_t)}{u_1(C_t, 1-N_t)} = A_t F_1(A_t N_t, K_t) \tag{40}$$

$$= A_t F_1(N_t, K_t/A_t) \tag{41}$$

- RHS is growing constantly

$$\frac{C^{-\gamma} v(1-N)}{C^{1-\gamma} v'(1-N) + \eta'(1-N)} \tag{42}$$

- For $\ln(C)$:

$$\frac{C^{-1} v(1-N)}{\ln(C) v'(1-N) + \eta'(1-N)} \tag{43}$$

Further Reading

- Growth Theory and After, Robert Solow, http://www.depfe.unam.mx/doctorado/teorias-crecimiento-desarrollo/solow_1988.pdf
  Summarize growth facts of advanced industrial economies.
- Time to Build and Aggregate Fluctuations, by Finn E. Kydland and Edward C. Prescott