6 Modern Portfolio Theory

A Generalizing the Portfolio Problem

B Justifying Mean-Variance Utility

C The Gains From Diversification

D The Efficient Frontier

E A Separation Theorem

F Strengths and Shortcomings of MPT

1 Generalizing the Portfolio Problem

Generalizing the Portfolio Problem

We can elaborate on our previous portfolio problem

$$\max_a E\{u[Y_0(1 + r_f) + a(\tilde{r} - r_f)]\}$$

by allowing the investor to allocate funds to $N > 1$ risky assets.

$\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_N$	risky (random) returns
$a_1, a_2, \ldots, a_N$	amounts allocated at the risky assets
$w_i = a_i / Y_0$	share of initial wealth allocated to each risky asset (portfolio weights)

Generalizing the Portfolio Problem

$$
\begin{aligned}
\tilde{Y}_1 &= \text{random terminal wealth} \\
&= (1 + r_f)\left(Y_0 - \sum_{i=1}^{N} a_i\right) + \sum_{i=1}^{N} a_i(1 + \tilde{r}_i) \\
&= (1 + r_f)Y_0 + \sum_{i=1}^{N} a_i(\tilde{r}_i - r_f) \\
&= (1 + r_f)Y_0 + \sum_{i=1}^{N} w_i Y_0(\tilde{r}_i - r_f)
\end{aligned}
$$

Generalizing the Portfolio Problem

With

$$\tilde{Y}_1 = (1 + r_f)Y_0 + \sum_{i=1}^{N} w_i Y_0(\tilde{r}_i - r_f),$$

the generalized problem can be stated as

$$\max_{w_1, w_2, \ldots, w_N} E\left\{u\left[Y_0(1 + r_f) + \sum_{i=1}^{N} w_i Y_0(\tilde{r}_i - r_f)\right]\right\}$$

Generalizing the Portfolio Problem

Modern Portfolio Theory examines the solution to this extended problem assuming that investors have mean-variance utility, that is, assuming that investors' preferences can be represented by a trade-off between the mean (expected value) and variance (or standard deviation) of terminal wealth.

MPT was developed by Harry Markowitz (US, b.1927, Nobel Prize 1990) in the early 1950s, the classic paper being his article "Portfolio Selection," *Journal of Finance* Vol.7 (March 1952): pp.77-91.

Generalizing the Portfolio Problem

The mean-variance utility hypothesis seemed natural at the time the MPT first appeared, and it retains some intuitive appeal today. But viewed in the context of more recent developments in financial economics, particularly the development of vN-M expected utility theory, it now looks a bit peculiar.

A first question for us, therefore, is: Under what conditions will investors have preferences over the means and variances of asset returns?

Generalizing the Portfolio Problem

Under what conditions will investors have preferences over the means and variances of asset returns?

1. Portfolio risks are small, so that a quadratic approximation to a general Bernoulli utility function is accurate (quadratic utility approximation), or
2. The Bernoulli utility function is quadratic, so that the approximation in (1) is always exact (quadratic utility function), or
3. Asset returns are normally distributed, so that terminal wealth is normally distributed as well (we will see under return normality only mean and variance matter in the preference, but not necessarily quadratic)

2 Justifying Mean-Variance Utility

Justifying Mean-Variance Utility

If we start by assuming an investor has preferences over terminal wealth $\tilde{Y}_1$ described by a vN-M expected utility function

$$E[u(\tilde{Y}_1)]$$

we can write

$$\tilde{Y}_1 = E(\tilde{Y}_1) + [\tilde{Y}_1 - E(\tilde{Y}_1)]$$

and interpret the portfolio problem as a trade-off between the expected payoff

$$E(\tilde{Y}_1)$$

and the size of the "bet"

$$[\tilde{Y}_1 - E(\tilde{Y}_1)]$$

Justifying Mean-Variance Utility

With this interpretation in mind, consider a second-order Taylor approximation of the Bernoulli utility function u once the outcome $[\tilde{Y}_1 - E(\tilde{Y}_1)]$ of the bet is known:

$$\begin{aligned} u(\tilde{Y}_1) &\approx u[E(\tilde{Y}_1)] + u'[E(\tilde{Y}_1)][\tilde{Y}_1 - E(\tilde{Y}_1)] \\ &\quad + \frac{1}{2}u''[E(\tilde{Y}_1)][\tilde{Y}_1 - E(\tilde{Y}_1)]^2 \end{aligned}$$

Justifying Mean-Variance Utility

$$\begin{aligned} u(\tilde{Y}_1) &\approx u[E(\tilde{Y}_1)] + u'[E(\tilde{Y}_1)][\tilde{Y}_1 - E(\tilde{Y}_1)] \\ &\quad + \frac{1}{2}u''[E(\tilde{Y}_1)][\tilde{Y}_1 - E(\tilde{Y}_1)]^2 \end{aligned}$$

Now go back to the beginning of the period, before the outcome of the bet is known, and take expected values to obtain

$$E[u(\tilde{Y}_1)] \approx u[E(\tilde{Y}_1)] + \frac{1}{2}u''[E(\tilde{Y}_1)]\sigma^2(\tilde{Y}_1)$$

since

$$E[\tilde{Y}_1 - E(\tilde{Y}_1)] = 0 \text{ and } E\{[\tilde{Y}_1 - E(\tilde{Y}_1)]^2\} = \sigma^2(\tilde{Y}_1)$$

Justifying Mean-Variance Utility

$$E[u(\tilde{Y}_1)] \approx u[E(\tilde{Y}_1)] + \frac{1}{2}u''[E(\tilde{Y}_1)]\sigma^2(\tilde{Y}_1)$$

The right-hand side of this expression is in the desired form: if u is increasing, it rewards higher mean returns and if u is concave, it penalizes higher variance in returns.

So one possible justification for mean-variance utility is to assume that the size of the portfolio bet $\tilde{Y}_1 - E(\tilde{Y}_1)$ is small enough to make this Taylor approximation a good one.

But is it safe to assume that portfolio bets are small?

Justifying Mean-Variance Utility

A second possibility is to assume that the Bernoulli utility function is quadratic, with

$$u(Y) = a + bY_1 + cY_1^2,$$

with $b > 0$ and $c < 0$. Then

$$u'(Y_1) = b + 2cY_1 \text{ and } u''(Y_1) = 2c.$$

In this case, the second-order (quadratic) Taylor approximation holds exactly, even for large bets.

Justifying Mean-Variance Utility

Note, however, that for a quadratic utility function

$$R_A(Y_1) = -\frac{u''(Y_1)}{u'(Y_1)} = -\frac{2c}{b + 2cY_1}$$

which is increasing in Y_1.

Hence, quadratic utility has the undesirable implication that the amount of wealth allocated to risky investments declines when wealth increases.

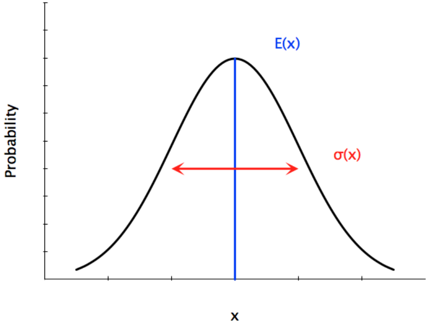
Justifying Mean-Variance Utility

Fortunately, there is a result from probability theory: if $\tilde{Y}_1$ is normally distributed with mean $\mu_Y = E(\tilde{Y}_1)$ and standard deviation $\sigma_{Y_1} = \{E[\tilde{Y}_1 - E(\tilde{Y}_1)]^2\}^{1/2}$ then the expectation of any function of $\tilde{Y}_1$ can be written as a function of μ_Y and σ_Y.

Hence, in particular, there exists a function v such that

$$E[u(\tilde{Y}_1)] = v(\mu_Y, \sigma_Y)$$

Justifying Mean-Variance Utility



The result follows from a more basic property of the normal distribution: its location and shape is described completely by its mean and variance.

Justifying Mean-Variance Utility

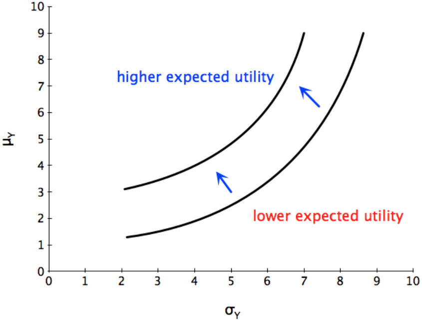
If $\tilde{Y}_1$ is normally distributed, there exists a function v such that

$$E[u(\tilde{Y}_1)] = v(\mu_Y, \sigma_Y).$$

Moreover, if $\tilde{Y}_1$ is normally distributed and

1. u is increasing, then v is increasing in μ_Y
2. u is concave, then v is decreasing in σ_Y
3. u is concave, then indifference curves defined over μ_Y and σ_Y are convex (like quadratic utility, the indifference curve is convex - will be useful in the MPT)

Justifying Mean-Variance Utility



Since μ_Y is a "good" and σ_Y is a "bad," indifference curves slope up. And if u is concave, these indifference curves will be convex (like in the case of quadratic utility).

Justifying Mean-Variance Utility

Problems with the normality assumption:

1. Returns on assets like options are highly non-normal.
2. Departures from normality, including skewness (asymmetry) and excess kurtosis ("fat tails") can be detected in returns for both individual stocks and stock indices.

Justifying Mean-Variance Utility

The mean-variance utility hypothesis is intuitively appealing and can be justified with reference to vN-M expected utility theory under various additional assumptions.

Still, it's important to recognize its limitations: you probably wouldn't want to use it to design sophisticated investment strategies that involve very large risks or make use of options and you probably wouldn't want to use it to study how portfolio strategies or risk-taking behavior changes with wealth.

3 The Gains From Diversification

The Gains From Diversification

One of the most important lessons that we can take from modern portfolio theory involves the gains from diversification.

To see where these gains come from, consider forming a portfolio from two risky assets:

$$\tilde{r}_1,\ \tilde{r}_2 = \text{random returns}$$

$$\mu_1,\ \mu_2 = \text{expected returns}$$

$$\sigma_1,\ \sigma_2 = \text{standard deviations}$$

Assume $\mu_1 > \mu_2$ and $\sigma_1 > \sigma_2$ to create a trade-off between expected return and risk.

The Gains From Diversification

If w is the fraction of initial wealth allocated to asset 1 and $1 - w$ is the fraction of initial wealth allocated to asset 2, the random return $\tilde{r}_P$ on the portfolio is

$$\tilde{r}_P = w\tilde{r}_1 + (1 - w)\tilde{r}_2$$

and the expected return μ_P on the portfolio is

$$\begin{aligned}
\mu_P &= E[w\tilde{r}_1 + (1 - w)\tilde{r}_2] \\
&= wE(\tilde{r}_1) + (1 - w)E(\tilde{r}_2) \\
&= w\mu_1 + (1 - w)\mu_2
\end{aligned}$$

The Gains From Diversification

$$\mu_P = w\mu_1 + (1-w)\mu_2$$

The expected return on the portfolio is a weighted average of the expected returns on the individual assets.

Since $\mu_1 > \mu_2$, μ_P can range from μ_2 up to μ_1 as w increases from zero to one. Even higher (or lower) expected returns are possible if short selling is allowed.

The Gains From Diversification

But now let's calculate the variance of the random portfolio return

$$\tilde{r}_P = w\tilde{r}_1 + (1-w)\tilde{r}_2$$

$$\begin{aligned}
\sigma_P^2 &= E[(\tilde{r}_P - \mu_P)^2] \\
&= E\{[w\tilde{r}_1 + (1-w)\tilde{r}_2 - w\mu_1 - (1-w)\mu_2]^2\} \\
&= E\{[w(\tilde{r}_1 - \mu_1) + (1-w)(\tilde{r}_2 - \mu_2)]^2\} \\
&= E[w^2(\tilde{r}_1 - \mu_1)^2 + (1-w)^2(\tilde{r}_2 - \mu_2)^2 \\
&\quad + 2w(1-w)(\tilde{r}_1 - \mu_1)(\tilde{r}_2 - \mu_2)]
\end{aligned}$$

The Gains From Diversification

$$\begin{aligned}\sigma_P^2 &= E[w^2(\tilde{r}_1-\mu_1)^2+(1-w)^2(\tilde{r}_2-\mu_2)^2 \\ &\quad + 2w(1-w)(\tilde{r}_1-\mu_1)(\tilde{r}_2-\mu_2)]\end{aligned}$$

$$\begin{aligned}\sigma_P^2 &= w^2E[(\tilde{r}_1-\mu_1)^2]+(1-w)^2E[(\tilde{r}_2-\mu_2)^2] \\ &\quad + 2w(1-w)E[(\tilde{r}_1-\mu_1)(\tilde{r}_2-\mu_2)]\end{aligned}$$

The Gains From Diversification

In probability theory, the covariance between two random variables X_1 and X_2 is defined as

$$\sigma(X_1, X_2) = E\{[X_1 - E(X_1)][X_2 - E(X_2)]\}$$

and the correlation between X_1 and X_2 is defined as

$$\rho(X_1, X_2) = \frac{\sigma(X_1, X_2)}{\sigma(X_1)\sigma(X_2)}$$

The Gains From Diversification

The covariance

$$\sigma(X_1, X_2) = E\{[X_1 - E(X_1)][X_2 - E(X_2)]\}$$

is positive if

$$X_1 - E(X_1) \text{ and } X_2 - E(X_2)$$

tend to have the same sign, negative

$$X_1 - E(X_1) \text{ and } X_2 - E(X_2)$$

tend to have opposite signs, and zero if

$$X_1 - E(X_1) \text{ and } X_2 - E(X_2)$$

show no tendency to have the same or opposite signs.

The Gains From Diversification

Mathematically, therefore, the covariance

$$\sigma(X_1, X_2) = E\{[X_1 - E(X_1)][X_2 - E(X_2)]\}$$

measures the extent to which the two random variables tend to move together.

Economically, buying two assets with returns that are imperfectly, and especially, negatively correlated is like buying insurance: one return will be high when the other is low and vice versa, reducing the overall risk of the portfolio.

The Gains From Diversification

The correlation

$$\rho(X_1, X_2) = \frac{\sigma(X_1, X_2)}{\sigma(X_1)\sigma(X_2)}$$

has the same sign as the covariance, and is therefore also a measure of co-movement.

But "scaling" the covariance by the two standard deviations makes the correlation range between -1 and 1:

$$-1 \leq \rho(X_1, X_2) \leq 1$$

The Gains From Diversification

Hence

$$\begin{aligned}\sigma_P^2 &= w^2E[(\tilde{r}_1-\mu_1)^2]+(1-w)^2E[(\tilde{r}_2-\mu_2)^2] \\ &\quad + 2w(1-w)E[(\tilde{r}_1-\mu_1)(\tilde{r}_2-\mu_2)]\end{aligned}$$

implies

$$\begin{aligned}\sigma_P^2 &= w^2\sigma_1^2+(1-w)^2\sigma_2^2+2w(1-w)\sigma_{12} \\ &= w^2\sigma_1^2+(1-w)^2\sigma_2^2+2w(1-w)\sigma_1\sigma_2\rho_{12}\end{aligned}$$

where

$\sigma_{12} =$ the covariance between $\tilde{r}_1$ and $\tilde{r}_2$

$\rho_{12} =$ the correlation between $\tilde{r}_1$ and $\tilde{r}_2$

The Gains From Diversification

This is the source of the gains from diversification: the expected portfolio return

$$\mu_P = w\mu_1 + (1 - w)\mu_2$$

is a weighted average of the expected returns on the individual asset returns, but the standard deviation of the portfolio return

$$\sigma_P = [w^2\sigma_1^2 + (1 - w)^2\sigma_2^2 + 2w(1 - w)\sigma_1\sigma_2\rho_{12}]^{1/2}$$

is not a weighted average of the standard deviations of the returns on the individual assets and can be reduced by choosing a mix of assets ($0 < w < 1$) when ρ_{12} is less than one and, especially, when ρ_{12} is negative.

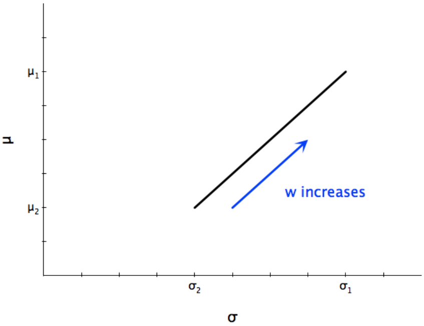
The Gains From Diversification

To see more specifically how this works, start with the case where $\rho_{12} = 1$ so that the individual asset returns are perfectly correlated. This is the one case in which there are no gains from diversification. With $\rho_{12} = 1$,

$$
\begin{aligned}
\sigma_P &= [w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2\rho_{12}]^{1/2} \\
&= [w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2]^{1/2} \\
&= \{[w\sigma_1 + (1-w)\sigma_2]^2\}^{1/2} \\
&= |w\sigma_1 + (1-w)\sigma_2|.
\end{aligned}
$$

In this special case, the standard deviation of the return on the portfolio is a weighted average of the standard deviations of the returns on the individual assets.

The Gains From Diversification



When $\rho_{12} = 1$, so that individual asset returns are perfectly correlated, there are no gains from diversification.

The Gains From Diversification

Next, let's consider the opposite extreme, in which $\rho_{12} = -1$ so that the individual asset returns are perfectly, but negatively, correlated:

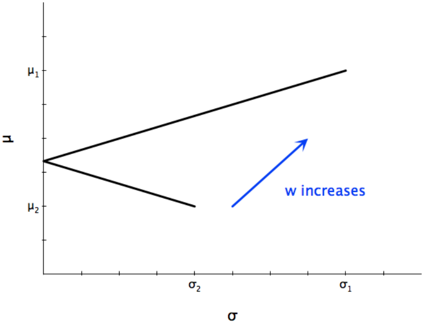
$$\begin{aligned}
\sigma_P &= [w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2\rho_{12}]^{1/2} \\
&= [w^2\sigma_1^2 + (1-w)^2\sigma_2^2 - 2w(1-w)\sigma_1\sigma_2]^{1/2} \\
&= \{[w\sigma_1 - (1-w)\sigma_2]^2\}^{1/2} \\
&= |w\sigma_1 - (1-w)\sigma_2|.
\end{aligned}$$

In this special case, the setting

$$w = \frac{\sigma_2}{\sigma_1 + \sigma_2}$$

creates a "synthetic" risk free portfolio!

The Gains From Diversification



When $\rho_{12} = -1$, so that individual asset returns are perfectly, but negatively correlated, risk can be eliminated via diversification.

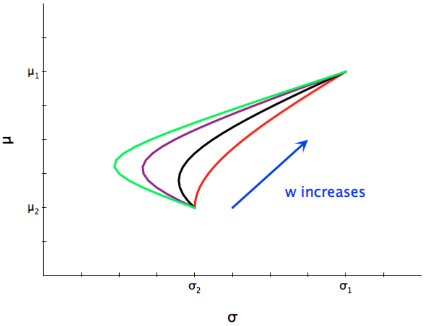
The Gains From Diversification

$$\mu_P = w\mu_1 + (1-w)\mu_2$$

$$\sigma_P = [w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2\rho_{12}]^{1/2}$$

In all intermediate cases, there will still be gains from diversification. These gains will become stronger as ρ_{12} declines from 1 to -1.

The Gains From Diversification



As ρ_{12} decreases from 0.5 to 0 to -0.5 to -0.75, the gains from diversification strengthen.

4 The Efficient Frontier

The Efficient Frontier

$$\mu_P = w\mu_1 + (1-w)\mu_2$$

$$\sigma_P = [w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2\rho_{12}]^{1/2}$$

In the case with two risky assets, the choice of w simultaneously determines μ_P and σ_P. But with more than two risky assets, the portfolio problem takes on an added dimension, since then we can ask: how can we select $w_1, w_2, \ldots, w_N$ to minimize σ_P for any given choice of μ_P?

The Efficient Frontier

Consider two portfolios, A and B, with expected returns μ_A and μ_B and standard deviations σ_A and σ_B.

Recall that portfolio A is said to exhibit mean-variance dominance over portfolio B if either

$$\mu_A > \mu_B \text{ and } \sigma_A \leq \sigma_B$$

or

$$\mu_A \geq \mu_B \text{ and } \sigma_A < \sigma_B$$

The Efficient Frontier

Hence, choosing portfolio shares to minimize variance for a given mean will allow us to characterize the efficient frontier: the set of all portfolios that are not mean-variance dominated by any other portfolio.

This is a useful intermediate step in modern portfolio theory, since investors with mean-variance utility will only choose portfolios on the efficient frontier.

The Efficient Frontier

With three assets, for example, an investor can choose

$$w_1 = \text{share of initial wealth allocated to asset 1}$$

$$w_2 = \text{share of initial wealth allocated to asset 2}$$

$$1 - w_1 - w_2 = \text{share of wealth allocated to asset 3}$$

The Efficient Frontier

Given the choices of w_1 and w_2:

$$\tilde{r}_P = w_1\tilde{r}_1 + w_2\tilde{r}_2 + (1 - w_1 - w_2)\tilde{r}_3$$

$$\mu_P = w_1\mu_1 + w_2\mu_2 + (1 - w_1 - w_2)\mu_3$$

$$\begin{aligned}\sigma_P^2 \quad = \quad & w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + (1 - w_1 - w_2)^2\sigma_3^2 \\ & + 2w_1w_2\sigma_1\sigma_2\rho_{12} \\ & + 2w_1(1 - w_1 - w_2)\sigma_1\sigma_3\rho_{13} \\ & + 2w_2(1 - w_1 - w_2)\sigma_2\sigma_3\rho_{23}\end{aligned}$$

The Efficient Frontier

Our problem is to solve

$$\min_{w_1, w_2} \sigma_P^2 \text{ subject to } \mu_P = \bar{\mu}$$

for a given value of $\bar{\mu}$.

But since we are more used to solving constrained maximization problems, consider the reformulated, but equivalent, problem:

$$\max_{w_1, w_2} -\sigma_P^2 \text{ subject to } \mu_P = \bar{\mu}$$

The solutions to this for different values of $\bar{\mu}$ will define the minimum variance frontier and later the efficient frontier.

The Efficient Frontier

Set up the Lagrangian, using the expressions for σ_P and μ_P derived previously:

$$\begin{aligned} L &= -w_1^2\sigma_1^2 - w_2^2\sigma_2^2 - (1-w_1-w_2)^2\sigma_3^2 \\ &\quad - 2w_1w_2\sigma_1\sigma_2\rho_{12} \\ &\quad - 2w_1(1-w_1-w_2)\sigma_1\sigma_3\rho_{13} \\ &\quad - 2w_2(1-w_1-w_2)\sigma_2\sigma_3\rho_{23} \\ &\quad + \lambda[w_1\mu_1 + w_2\mu_2 + (1-w_1-w_2)\mu_3 - \bar{\mu}] \end{aligned}$$

The Efficient Frontier

$$
\begin{aligned}
L \;=\; & -w_1^2\sigma_1^2 - w_2^2\sigma_2^2 - (1 - w_1 - w_2)^2\sigma_3^2 \\
& - 2w_1 w_2 \sigma_1\sigma_2\rho_{12} \\
& - 2w_1(1 - w_1 - w_2)\sigma_1\sigma_3\rho_{13} \\
& - 2w_2(1 - w_1 - w_2)\sigma_2\sigma_3\rho_{23} \\
& + \lambda[w_1\mu_1 + w_2\mu_2 + (1 - w_1 - w_2)\mu_3 - \bar{\mu}]
\end{aligned}
$$

First-order condition for w_1: (Slow and steady...)

$$
\begin{aligned}
0 \;=\; & -2w_1^*\sigma_1^2 + 2(1 - w_1^* - w_2^*)\sigma_3^2 - 2w_2^*\sigma_1\sigma_2\rho_{12} \\
& - 2(1 - w_1^* - w_2^*)\sigma_1\sigma_3\rho_{13} + 2w_1^*\sigma_1\sigma_3\rho_{13} \\
& + 2w_2^*\sigma_2\sigma_3\rho_{23} + \lambda^*\mu_1 - \lambda^*\mu_3
\end{aligned}
$$

The Efficient Frontier

$$\begin{aligned} L &= -w_1^2\sigma_1^2 - w_2^2\sigma_2^2 - (1-w_1-w_2)^2\sigma_3^2 \\ &\quad - 2w_1w_2\sigma_1\sigma_2\rho_{12} \\ &\quad - 2w_1(1-w_1-w_2)\sigma_1\sigma_3\rho_{13} \\ &\quad - 2w_2(1-w_1-w_2)\sigma_2\sigma_3\rho_{23} \\ &\quad + \lambda[w_1\mu_1 + w_2\mu_2 + (1-w_1-w_2)\mu_3 - \bar{\mu}] \end{aligned}$$

First-order condition for w_2: (Slow and steady...)

$$\begin{aligned} 0 &= -2w_2^*\sigma_2^2 + 2(1-w_1^*-w_2^*)\sigma_3^2 - 2w_1^*\sigma_1\sigma_2\rho_{12} \\ &\quad + 2w_1^*\sigma_1\sigma_3\rho_{13} - 2(1-w_1^*-w_2^*)\sigma_2\sigma_3\rho_{23} \\ &\quad + 2w_2^*\sigma_2\sigma_3\rho_{23} + \lambda^*\mu_2 - \lambda^*\mu_3 \end{aligned}$$

The Efficient Frontier

The two first-order conditions and the constraint

$$\begin{aligned}
0 &= -2w_1^*\sigma_1^2 + 2(1-w_1^*-w_2^*)\sigma_3^2 - 2w_2^*\sigma_1\sigma_2\rho_{12} \\
&\quad - 2(1-w_1^*-w_2^*)\sigma_1\sigma_3\rho_{13} + 2w_1^*\sigma_1\sigma_3\rho_{13} \\
&\quad + 2w_2^*\sigma_2\sigma_3\rho_{23} + \lambda^*\mu_1 - \lambda^*\mu_3
\end{aligned}$$

$$\begin{aligned}
0 &= -2w_2^*\sigma_2^2 + 2(1-w_1^*-w_2^*)\sigma_3^2 - 2w_1^*\sigma_1\sigma_2\rho_{12} \\
&\quad + 2w_1^*\sigma_1\sigma_3\rho_{13} - 2(1-w_1^*-w_2^*)\sigma_2\sigma_3\rho_{23} \\
&\quad + 2w_2^*\sigma_2\sigma_3\rho_{23} + \lambda^*\mu_2 - \lambda^*\mu_3
\end{aligned}$$

$$w_1^*\mu_1 + w_2^*\mu_2 + (1-w_1^*-w_2^*)\mu_3 = \bar{\mu}$$

form a system of three equations in the three unknowns: w_1^*, w_2^*, and λ^*.

The Efficient Frontier

Moreover, the equations are linear in the unknowns w_1^*, w_2^*, and λ^*:

$$\begin{aligned}
0 = -2w_1^* \quad \sigma_1^2 + 2(1 - w_1^* - w_2^*)\sigma_3^2 - 2w_2^*\sigma_1\sigma_2\rho_{12} \\
- 2(1 - w_1^* - w_2^*)\sigma_1\sigma_3\rho_{13} + 2w_1^*\sigma_1\sigma_3\rho_{13} \\
+ 2w_2^*\sigma_2\sigma_3\rho_{23} + \lambda^*\mu_1 - \lambda^*\mu_3
\end{aligned}$$

$$\begin{aligned}
0 = -2w_2^* \quad \sigma_2^2 + 2(1 - w_1^* - w_2^*)\sigma_3^2 - 2w_1^*\sigma_1\sigma_2\rho_{12} \\
+ 2w_1^*\sigma_1\sigma_3\rho_{13} - 2(1 - w_1^* - w_2^*)\sigma_2\sigma_3\rho_{23} \\
+ 2w_2^*\sigma_2\sigma_3\rho_{23} + \lambda^*\mu_2 - \lambda^*\mu_3
\end{aligned}$$

$$w_1^*\mu_1 + w_2^*\mu_2 + (1 - w_1^* - w_2^*)\mu_3 = \bar{\mu}$$

Given specific values for μ_1, μ_2, μ_3, σ_1, σ_2, σ_3, ρ_{12}, ρ_{13}, ρ_{23}, and $\bar{\mu}$ they can be solved quite easily.

The Efficient Frontier

In linear algebra, a vector is just a column of numbers. With $N \geq 3$ assets, you can organize the portfolio shares and expected returns into a vectors:

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix} \text{ and } \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{bmatrix}$$

where

$$w_1 + w_2 + \ldots + w_N = 1$$

Also in linear algebra, the transpose of a vector just reorganizes the column as a row; for example:

$$w' = \begin{bmatrix} w_1 & w_2 & \ldots & w_N \end{bmatrix}$$

The Efficient Frontier

Meanwhile, the variances and covariances can be organized into a matrix – a collection of rows and columns:

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} & \dots & \sigma_1\sigma_N\rho_{1N} \\ \sigma_1\sigma_2\rho_{12} & \sigma_2^2 & \dots & \sigma_2\sigma_N\rho_{2N} \\ \vdots & \vdots & \dots & \vdots \\ \sigma_1\sigma_N\rho_{1N} & \sigma_2\sigma_N\rho_{2N} & \dots & \sigma_N^2 \end{bmatrix}$$

The Efficient Frontier

Using the rules from linear algebra for multiplying vectors and matrices, the expected return on any portfolio with shares in the vector w is

$$\mu' w$$

and the variance of the random return on the portfolio is

$$w' \Sigma w.$$

Hence, the problem of minimizing the variance for a given mean can be written compactly as

$$\max_{w} -w'\Sigma w \text{ subject to } \mu' w = \bar{\mu} \text{ and } \ell' w = 1$$

where ℓ is a vector of N ones.

The Efficient Frontier

$$\max_{w} -w'\Sigma w \text{ subject to } \mu' w = \bar{\mu} \text{ and } \ell' w = 1$$

Problems of this form are called quadratic programming problems and can be solved very quickly on a computer even when the number of assets N is large.

We can also add more constraints, such as $w_i \geq 0$, ruling out short sales.

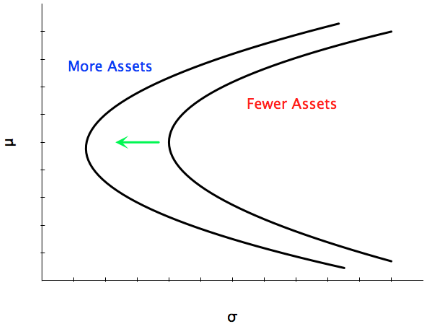
The Efficient Frontier

Going back to the case with three assets, once the optimal shares w_1^* and w_2^* have been found, the minimized standard deviation can be computed using the general formula

$$\begin{aligned}\sigma_P^2 \;=\;& w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + (1 - w_1 - w_2)^2\sigma_3^2 \\ &+ 2w_1 w_2 \sigma_1 \sigma_2 \rho_{12} \\ &+ 2w_1(1 - w_1 - w_2)\sigma_1\sigma_3\rho_{13} \\ &+ 2w_2(1 - w_1 - w_2)\sigma_2\sigma_3\rho_{23}\end{aligned}$$

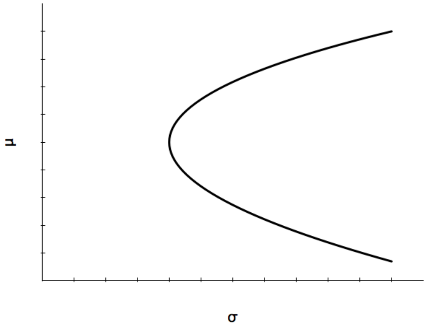
Doing this for various values of $\bar{\mu}$ allows us to trace out the minimum variance frontier.

The Efficient Frontier



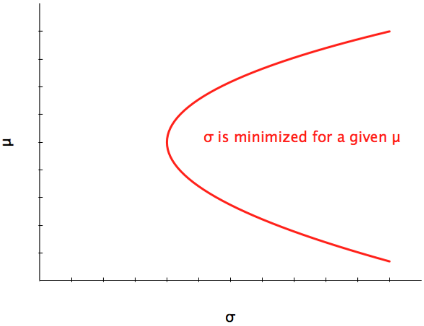
Adding assets shifts the minimum variance frontier to the left, as opportunities for diversification are enhanced.

The Efficient Frontier



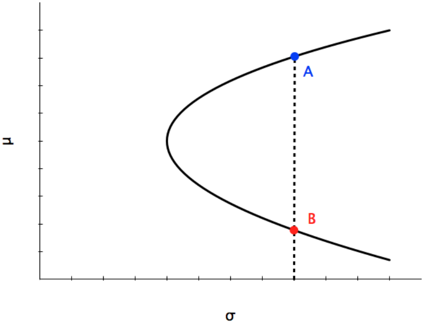
However, the minimum variance frontier retains its sideways parabolic shape.

The Efficient Frontier



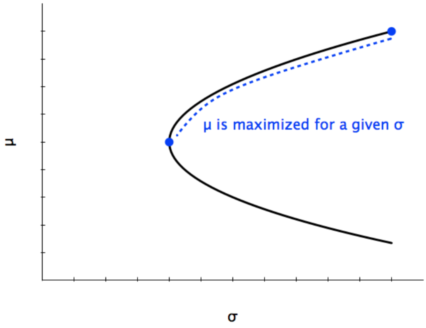
The minimum variance frontier traces out the minimized variance or standard deviation for each required mean return.

The Efficient Frontier



But portfolio A exhibits mean-variance dominance over portfolio B, since it offers a higher expected return with the same standard deviation.

The Efficient Frontier



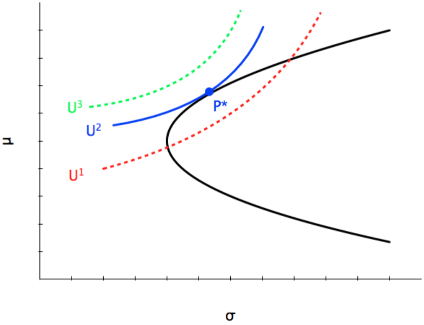
Hence, the efficient frontier extends only along the top arm of the minimum variance frontier.

The Efficient Frontier

Recall that any of the following assumptions imply that indifference curves in this $\sigma - \mu$ diagram slope upward and are convex:

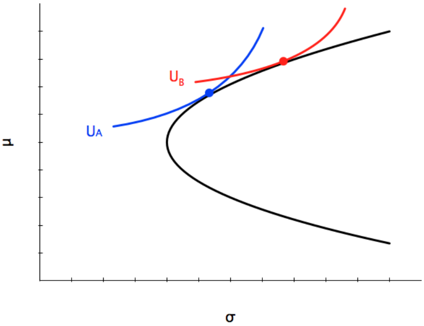
1. Risks are small enough to justify a second-order Taylor approximation to any increasing and concave Bernoulli utility function within the vN-M expected utility framework
2. Investors have vN-M expected utility with quadratic Bernoulli utility functions
3. Asset returns are normally distributed and investors have vN-M expected utility with increasing and concave Bernoulli utility functions

The Efficient Frontier



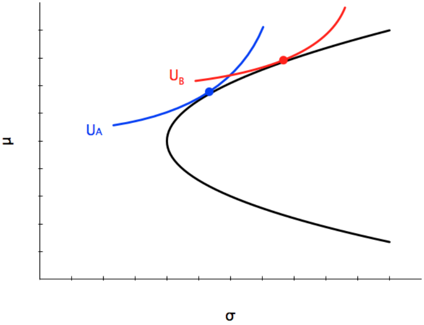
Portfolios along U^1 are suboptimal. Portfolios along U^3 are infeasible. Portfolio P^*, located where U^2 is tangent to the efficient frontier, is optimal.

The Efficient Frontier



Investor B is less risk averse than investor A. But both choose portfolios along the efficient frontier.

The Efficient Frontier



Thus, the mean-variance utility hypothesis built into Modern Portfolio Theory implies that all investors choose optimal portfolios along the efficient frontier.

Summary of MPT up to this slide

Step 1: Given expected return μ and covariance matrix Σ, solve out the minimum variance frontier and then the efficient frontier.

(Excel can do three stocks, use Python/Matlab/R/etc for general cases)

Step 2: Given the solved efficient frontier and a given utlity function, find the tangency point (optimal portfolio) between the efficient frontier and the indifference curve.

5 Separation Theorem

Separation Theorem

So far, however, our analysis has assumed that there are only risky assets. An additional, quite striking, result emerges when we add a risk free asset to the mix.

This implication was first noted by James Tobin (US, 1918-2002, Nobel Prize 1981) in his paper "Liquidity Preference as Behavior Towards Risk," *Review of Economic Studies* Vol.25 (February 1958): pp.65-86.

Separation Theorem

Consider, therefore, the larger portfolio formed when an investor allocates the fraction w of his or her initial wealth to a risky asset or to a smaller portfolio of risky assets and the remaining fraction $1 - w$ to a risk free asset with return r_f.

Separation Theorem

If the risky part of this portfolio has random return $\tilde{r}$, expected return $\mu_r = E(\tilde{r})$, and variance $\sigma_r^2 = E[(\tilde{r} - \mu_r)^2]$ then the larger portfolio has random return $\tilde{r_P} = w\tilde{r} + (1 - w)r_f$ with expected return

$$\mu_P = E[w\tilde{r} + (1 - w)r_f] = w\mu_r + (1 - w)r_f$$

and variance

$$\begin{aligned} \sigma_P^2 &= E[(\tilde{r_P} - \mu_P)^2] \\ &= E\{[w\tilde{r} + (1 - w)r_f - w\mu_r - (1 - w)r_f]^2\} \\ &= E\{[w(\tilde{r} - \mu_r)]^2\} = w^2\sigma_r^2. \end{aligned}$$

Separation Theorem

The expression for the portfolio's variance

$$\sigma_P^2 = w^2 \sigma_r^2$$

implies

$$\sigma_P = w\sigma_r$$

and hence

$$w = \frac{\sigma_P}{\sigma_r}.$$

Hence, with σ_r given, a larger share of wealth w allocated to risky assets is associated with a higher standard deviation σ_P for the larger portfolio.

Separation Theorem

But the expression for the portfolio's expected return

$$\mu_P = w\mu_r + (1 - w)r_f$$

indicates that so long as $\mu_r > r_f$, a higher value of w will yield a higher expected return as well.

What is the trade-off between risk σ_P and expected return μ_P of the mix of risky and riskless assets?

Separation Theorem

To see, substitute

$$w = \frac{\sigma_P}{\sigma_r}$$

into

$$\mu_P = w\mu_r + (1 - w)r_f$$

to obtain

$$\begin{aligned}\mu_P &= \left(\frac{\sigma_P}{\sigma_r}\right)\mu_r + \left(1 - \frac{\sigma_P}{\sigma_r}\right) r_f \\ &= r_f + \left(\frac{\mu_r - r_f}{\sigma_r}\right)\sigma_P\end{aligned}$$

Separation Theorem

The expression

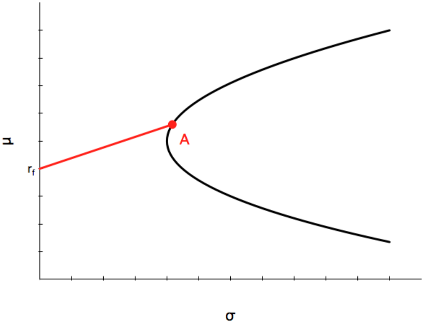
$$\mu_P = r_f + \left(\frac{\mu_r - r_f}{\sigma_r}\right)\sigma_P$$

shows that for portfolios of risky and riskless assets:

1. The relationship between σ_P and μ_P is linear.
2. The slope of the linear relationship is given by the Sharpe ratio, defined here as the "expected excess return" offered by the risky components of the portfolio divided by the standard deviation of the return on that risky component:

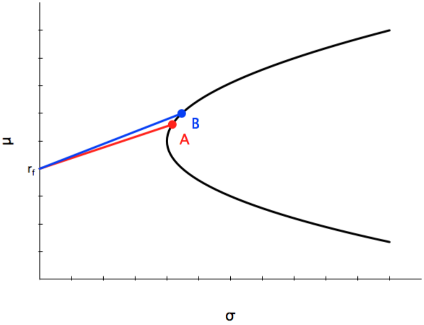
$$\frac{\mu_r - r_f}{\sigma_r}.$$

Separation Theorem



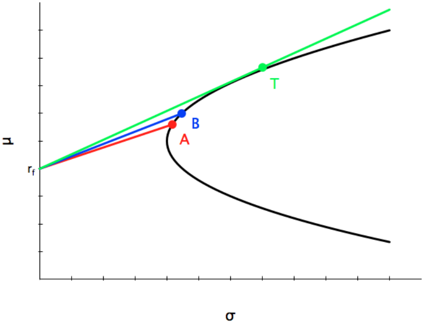
Hence, any investor can combine the risk free asset with risky portfolio A to achieve a combination of expected return and standard deviation along the red line.

Separation Theorem



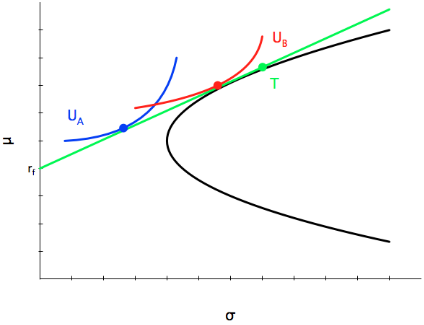
However, any investor with mean-variance utility will prefer some combination of the risk free asset and risky portfolio B to all combinations of the risk free asset and risky portfolio A.

Separation Theorem



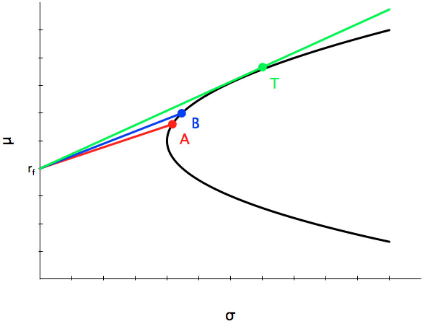
And all investors with mean-variance utility will prefer some combination of the risk free asset and risky portfolio T to any other portfolio.

Separation Theorem



Investor B is less risk averse than investor A. But both choose some combination of the "tangency portfolio" T and the risk free asset.

Separation Theorem



Note that the tangency portfolio T can be identified as the portfolio along the efficient frontier of risky assets that has the highest Sharpe ratio.

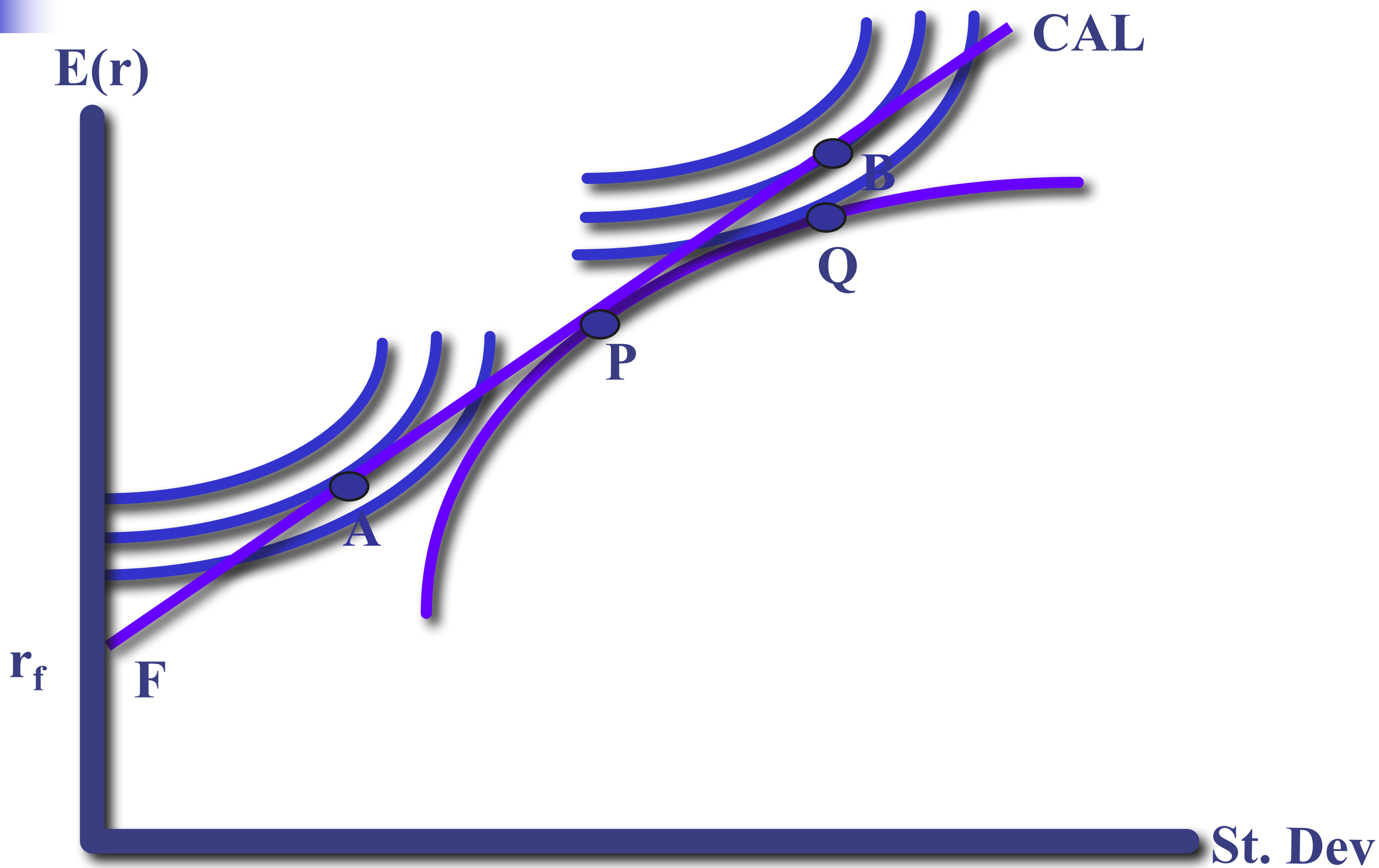
Separation Theorem

This is the two-fund theorem or separation theorem implied by Modern Portfolio Theory.

Equity mutual fund managers can all focus on building the unique portfolio that lies along the efficient frontier of risky assets and has the highest Sharpe ratio.

Each individual investor can then tailor his or her own portfolio by choosing the combination of the riskless assets (the riskless fund) and the risky mutual fund that best suits his or her own aversion to risk.

Borrowing and Lending: How the Risk-free Asset Originates



6 Strengths and Shortcomings of MPT

Strengths and Shortcomings of MPT

We've already considered one shortcoming of the MPT: its mean-variance utility hypothesis must rest on one of two more basic assumptions.

Either utility must be quadratic or asset returns must be normal.

Strengths and Shortcomings of MPT

A second problem involves the estimation or "calibration" of the model's parameters.

With N risky assets, the vector μ of expected returns contains N elements and the matrix Σ of variances and covariances contains $N(N+1)/2$ unique elements. When $N = 100$, for example, there are $100 + (100 \times 101)/2 = 5150$ parameters to estimate!

And to use data from the past to estimate these parameters, one has to assume that past averages and correlations are a reliable guide to the future.

Strengths and Shortcomings of MPT

On the other hand, the MPT teaches us a very important lesson about how individual assets with imperfectly, and especially negatively, correlated returns can be combined into a diversified portfolio to reduce risk.

And the MPT's separation theorem suggests that a retirement savings plan that allows participants to choose between a money market mutual fund and a well-diversified equity fund is fully optimal under certain circumstances and perhaps close enough to optimal more generally.

Strengths and Shortcomings of MPT

Finally, our first equilibrium model of asset pricing, the Capital Asset Pricing Model, builds directly on the foundations provided by Modern Portfolio Theory.