# Econ 139 Lecture 23

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## 1 Market Model

Recall $\tilde{r}_j = E(\tilde{r}_j) + \beta_j(\tilde{r}_m - E(\tilde{r}_m)) + \epsilon_j$

**Arbitrage Pricing Theory**
add two assumptions:
(1) enough individual assets to create many well-diversified portfolio
(2) investors act to eliminate arbitrage among well-diversified portfolio

Assume $E(\epsilon_j) = E(\epsilon_j \tilde{r}_m) = E(\epsilon_j \epsilon_k) = 0$
Then consider somewhat diversified portfolio of two assets and put $w$ in one asset and $(1-w)$ in the other.

We know that

$$\tilde{r}_j = E(\tilde{r}_j) + \beta_j[\tilde{r}_m - E(\tilde{r}_m)] + \epsilon_j \qquad (1)$$

$$\tilde{r}_k = E(\tilde{r}_k) + \beta_k[\tilde{r}_m - E(\tilde{r}_m)] + \epsilon_k \qquad (2)$$

$$\tilde{r}_p = w\tilde{r}_j + (1-w)\tilde{r}_k \qquad (3)$$

Then substituting $\tilde{r}_j$ and $\tilde{r}_k$ in equation (3) with equation (1) and (2) will yield

$$\tilde{r}_p = wE(\tilde{r}_j) + (1-w)(\tilde{r}_k) + w\beta_j(\tilde{r}_m - E(\tilde{r}_m)) + (1-w)\beta_k(\tilde{r}_m - E(\tilde{r}_m)) + w\epsilon_j + (1-w)\epsilon_k$$

Now consider the equation

$$\tilde{r}_p = E(\tilde{r}_p) + \beta_p(\tilde{r}_m - E(\tilde{r}_m)) + \epsilon_p$$

then observe the following equations:

$$E(\tilde{r}_p) = wE(\tilde{r}_j) + (1-w)E(\tilde{r}_k)$$

$$\beta_p = w\beta_j + (1-w)\beta_k$$

$$\epsilon_p = w\epsilon_j + (1-w)\epsilon_k$$

$$\begin{aligned}
\sigma^2_{\epsilon_p} = E(\epsilon_p^2) &= E[(w\epsilon_j + (1-w)\epsilon_k)^2] \\
&= E(w^2\epsilon_j^2 + (1-w)^2\epsilon_k^2 + 2w(1-w)\epsilon_j\epsilon_k) \\
&= w^2E(\epsilon_j^2) + (1-w)^2E(\epsilon_k^2) + 2w(1-w)E(\epsilon_j\epsilon_k) \\
&= w^2\sigma^2_{\epsilon_j} + (1-w)^2\sigma^2_{\epsilon_k}
\end{aligned}$$

Suppose $w = \frac{1}{2}, \sigma^2_{\epsilon_j} = \sigma^2_{\epsilon_k} = \sigma^2$ then we have $\frac{1}{4}\sigma^2 + \frac{1}{4}\sigma^2 = \frac{1}{2}\sigma^2$
N assets
$w_i$ weights,$\sum_{i=1}^n w_i = 0$
assume equal weighted portfolio $w_i = \frac{1}{N}$

$$\tilde{r}_p = \sum_{i=1}^n \frac{1}{N}\tilde{r}_i - \frac{1}{N}\sum_{i=1}^n \tilde{r}_i$$

$$E[\tilde{r}_p] = \frac{1}{N}\sum_{i=1}^n \tilde{r}_i E[\tilde{r}_i]$$

$$\beta_p = \frac{1}{N}\sum_{i=1}^n \beta_i$$

$$\epsilon_p = \frac{1}{N}\sum_{i=1}^n \epsilon_i$$

$$\tilde{r}_p = \frac{1}{N}\sum_{i=1}^n E[\tilde{r}_i] + \frac{1}{N}\sum_{i=1}^n \beta_p(\tilde{r}_m - E[\tilde{r}_m]) + \sum_{i=1}^N \epsilon_i$$

$$Var(\frac{1}{N^2}\sum_{i=1}^{N^2}\epsilon_i) = \frac{1}{N^2}Var(\sum_{i=1}^{N^2}\epsilon_i) = \frac{\sigma^2_\epsilon}{N} =$$

$$\frac{1}{N^2}\sum_{i=1}^N \epsilon_i = \frac{1}{N}(\frac{1}{N}\sum_{i=1}^N \sigma^2_\epsilon)$$

$$Var(\sum_{i=1}^{N} w_i \epsilon_i) = \sum_{i=1}^{N} w_i^2 \sigma_i^2 \qquad E[\epsilon_j \epsilon_k] = 0$$

$$\tilde{r_p} = E[\tilde{r_p}] + \beta_p(\tilde{r_m} - E[\tilde{r_m}]) + \epsilon_p$$

$$\tilde{r_p} = E[\tilde{r_p}] + \beta_p(\tilde{r_m} - E[\tilde{r_m}])$$

For all **well-diversified** portfolio.

## 2 Proposition1

Absence of arbitrage requires that all well-diversified portfolio with the same Bp have the same expected return.

$$\tilde{r_p^1} = E[\tilde{r_p}] + \beta_p(\tilde{r_m} - E[\tilde{r_m}])$$

$$\tilde{r_p^2} = E[\tilde{r_p}] + \delta + \beta_p(\tilde{r_m} - E[\tilde{r_m}])$$

assume $\delta < 0$

$$X(1+\tilde{r_p^1}) - X(1+\tilde{r_p^2}) = XE[\tilde{r_p}] + X\beta_p(\tilde{r_m}) - XE[\tilde{r_p}] - X\delta - X\beta_p(\tilde{r_m} - E[\tilde{r_m}])$$

left with $= X\delta > 0 (since \delta < 0)$

## 3 Proposition2

The absence of arbitrage requires the expected return of the well-diversified portfolio to satisfy:

$$E[\tilde{r_p}] = r_f + \beta_p(E[\tilde{r_m}] - r_f)$$

**Step 1:**
Consider a well diversified portfolio with $\beta\_p = 0$

$$\tilde{r_p^1} = E[\tilde{r_p}] + 0 * (\tilde{r_m} - E[\tilde{r_m}]) = r_f$$

Get constant return.

**Step 2:**
Consider a second well diversified portfolio with $beta\_p = 1$

$$\tilde{r_p^2} = E[\tilde{r_p^2}] + 1(\tilde{r_m} - E[\tilde{r_m}])$$

$$= E[\tilde{r_m}] + (\tilde{r_m} - E[\tilde{r_m}]) = \tilde{r_m}$$

For these two portfolios,

$$E[\tilde{r_p}] = r_f + \beta(E[\tilde{r_m}] - r_f)$$

$$\beta(r_f - E[\tilde{r_m}]) = r_f - E[\tilde{r_p}]$$

hold by construction.

**Step 3:**

Consider a third well-diversified portfolio,$0 \neq \beta \neq 1$,such that the expected return of the portfolio:

$$E[\tilde{r_p}] = r_f + \beta_p(\tilde{r_m} - E[\tilde{r_m}]) + \delta$$

$$= r_f + \beta(\tilde{r_m} - r_f) + \delta \qquad \delta > 0$$

$$r_p^1 = r_f$$

$$r_p^2 = E[\tilde{r_m}] + (\tilde{r_m} - E[\tilde{r_m}])$$

$$r_p^3 = \tilde{r_f} + \beta_p(E[\tilde{r_m}] - r_f) + \delta$$

**Step 4:**

Consider a fourth W-D portfolio by allocating $1 - \beta_p$ to portfolio 1 and $\beta_p$ to portfolio 2.

$$\tilde{r}_p = (1 - \beta_p)(\tilde{r}_p^1) + \beta_p(\tilde{r}_p^2)$$

$$= (1 - \beta_p)\tilde{r}_f + \beta_p(E(\tilde{r}_m)) + \beta_p(\tilde{r}_f - E(\tilde{r}_m))$$

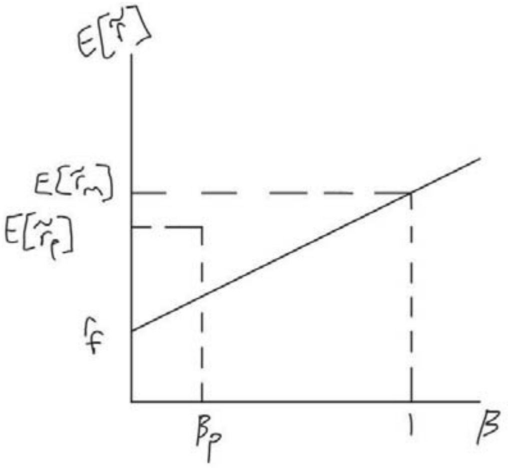
$$= \tilde{r}_f + \beta_p(E(\tilde{r}_m) - \tilde{r}_f) + \beta_p(\tilde{r}_m - E(\tilde{r}_m))$$

When $\Delta = 0$, there is no arbitrage opportunities.

$$\tilde{r}_p = \tilde{r}_f + \beta_p(E(\tilde{r}_m) - \tilde{r}_f) + \beta_p(\tilde{r}_m - E(\tilde{r}_m))$$

$$E(\tilde{r}_p)) = \tilde{r}_f + \beta_p(E(\tilde{r}_m) - \tilde{r}_f)$$

this could be applied to the multi-factor model.




# 4 CAPM Anomalies

1. Value Effect: Value stocks (high book to market ratio) have empirically high returns relative to their CAPM prediction
2. Size Effect: Small stocks have high returns relative to CAPM
3. Momentum Effect: Momentum stocks (stocks that have performed well in the last year) tend to outperform their predictions from CAPM
4. Reversal Effect: Stocks that have done well over the last 2-5 years tend to under-perform CAPM prediction