Midterm Exam

October 17, 2016

Instructions: This is a closed book exam, but you may refer to one sheet of notes. You have 80 minutes for the exam. Answer as many questions as possible. Partial answers get partial credit. Please write legibly. *Good luck!*

Problem 1 (5 points). Determine whether or not the statement below is correct and give a *brief* (e.g., a bluebook page or less) justification for your answer.

Suppose $X_1, \ldots, X_n$ is a random sample from a continuous distribution with pdf $f(\cdot|\theta)$, where $\theta \in \Theta \subseteq \mathbb{R}^k$ is unknown. If the $\mathbb{R}^m$-valued statistic $T = T(X_1, \ldots, X_n)$ is sufficient, then $m \geq k$.

Problem 2 (45 points, each part receives equal weight). Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with mean $\sqrt{\theta}$ and pdf

$$f_X(x|\theta) = c(\theta) \exp(-\frac{1}{2}x^2) \exp[w(\theta)x],$$

where $\theta \in \Theta = [0, \infty)$ is an unknown parameter while $c(\cdot)$ and $w(\cdot)$ are some functions.

(a) Show that

$$c(\theta) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}\theta), \qquad w(\theta) = \sqrt{\theta}.$$

(b) As an estimator of θ, consider $\bar{X}^2$, where $\bar{X} = n^{-1} \sum_{i=1}^n X_i$. Give conditions (on $X_1, \ldots, X_n$) under which a method moments estimator of θ exists and equals $\bar{X}^2$. Is $\bar{X}^2$ an unbiased estimator of θ?

(c) Find the log likelihood function. Does θ admit a scalar sufficient statistic?

(d) Find the maximum likelihood estimator $\hat{\theta}_{ML}$ of θ.

(e) Find a uniform minimum variance unbiased estimator of θ.

(f) Show that the Cramér-Rao bound (on the variance of unbiased estimators of θ) is given by $4\theta/n$.

(g) Is the bound from (f) attained by the estimator from (e)?
[Hint: If $Y \sim \mathcal{N}(\mu, \sigma^2)$, then $Var(Y^2) = 4\mu^2\sigma^2 + 2\sigma^4$.]

Let $\theta_0 > 0$ be some constant and consider the one-sided testing problem

$$H_0 : \theta = \theta_0 \qquad \text{vs.} \qquad H_1 : \theta > \theta_0.$$

(h) Consider a test which rejects H_0 if (and only if) $\bar{X} > c$, where c is some constant (possibly depending on θ_0). Find c such that the test has 5% size.

[Hint: If $Z \sim \mathcal{N}(0, 1)$, then $P(Z \leq 1.645) \approx 0.95$.]

(i) Show that the test derived in (h) is uniformly most powerful (within the class of tests of the same level).