Mathematical Methods in Finance

Lecture 4: Derivatives Valuation with Binomial Lattice

Fall 2013

Overview

- Introduction to option pricing: basic notions
- One-period binomial lattice for security pricing
- Multi-period binomial lattice for security pricing
- Perspectives:
  - No-arbitrage pricing
  - Replication and Hedging
  - Risk-neutral probability

What is an Option?

- An option is a derivative security that grants the buyer the right to buy or sell the underlying asset, at or before the maturity date T, for a pre-specified price K, called the strike or exercise price.
- “Right", not “obligation".
- “buy": call options; “sell": put options.
- If it can be exercised only at the maturity date, it is European style; If the exercise can happen at any time before or at the maturity date, it is American style.

- You need to pay for a “fee” (**the value/premium of this option**) in order to **obtain the “right”**.
- The options have value all the time until their maturities.

What is Hedging?

Making an investment to reduce the risk of adverse price movements in an asset.

How to hedge?

- Sell!

- Buy insurance (options, specialized contracts)

- Self-insure (correlation, replication)

Question: If you manufacture a call option (sell it to your customer), what do you need to do? Just wait after you collect the option premium until the maturity?

Why Options: A Case

The New York Times "An Airline Shrugs at Oil Prices" November 29, 2007

"The reason for Southwest's rapidly increasing advantage over other big airlines is much simpler: it loaded up years ago on hedges against higher fuel prices. And with oil trading above $90 a barrel, most of the rest of the airline industry is facing a huge run-up in costs, and Southwest is not."

"The hedges have helped keep Southwest profitable, producing gains on the hedging contracts of $455 million in 2004, $892 million in 2005 and $675 million in 2006, as well as $439 million for the first nine months of 2007, as oil prices have nearly doubled this year."

Why Options: A Case

Southwest's Profitable Bet

Percentage of each airline's fuel needs that are hedged against higher fuel prices and have been disclosed, with the price caps of their hedges.

	2007 4th quarter		2008 full years		2009		2010	
	HEDGED	PRICE CAP	HEDGED	PRICE CAP	HEDGED	PRICE CAP	HEDGED	PRICE CAP
Alaska[1]	50%	$62	32%	$64	5%	$68	0	
American[1]	40	69	14	n.d.†	0		0	
Continental[2]	30	93	10*	93	0		0	
Delta[2]	20	99	0		0		0	
JetBlue[2]	47	83	0		0		0	
Northwest[1]	50	73	10*	84	0		0	
Southwest[1]	90	51	70	51	55	51	25%	$63
United[2]	18	93	0		0		0	
US Airways[1]	56	73	15	73	0		0	

[1] Price based on crude oil.
[2] Price based on heating oil, which is more expensive.

* First quarter only. † Price not disclosed.

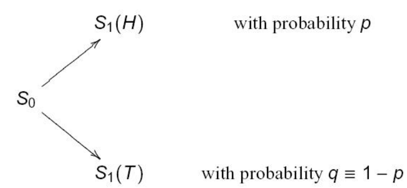
Sources: Securites and Exchange Commission filings; the companies

THE NEW YORK TIMES

Motivation

- **Question**: Consider an European call option with underlying asset being a stock. What is the fair price of this option?

- The Binomial Lattice Model is a *simplified* model for asset pricing.
  - One period binomial lattice model just considers a single period: From Time 0 to Time 1
  - Consider a stock with price per share being $S_t > 0$, $t = 0, 1$.
  - S_0 is a constant, but S_1 assumes a Bernoulli distribution.



  - Probability space $(\Omega, \mathbb{P}, \mathcal{F})$ where $\Omega = \{H, T\}$ with $\mathbb{P}(\{H\}) = p$

Motivation: Why Binomial Lattice?

- The Binomial model seems too simple to reflect the reality. Why study it?

- Reason 1: Despite the simplicity, the ideas of no arbitrage pricing (risk-neutral pricing) behind it are profound and universal.

- Reason 2: Multiperiod Binomial models converge to the continuous Black-Scholes-Merton model as the single period length goes to zero.

- Reason 3: Multiperiod Binomial models provide a powerful tool for numerical asset pricing.

One Period Binomial Lattice Model

- Consider a financial market with one stock and a money market, where the interest rate is r.
- Define $u = \frac{S_1(H)}{S_0}$ and $d = \frac{S_1(T)}{S_0}$, and assume $u > d$.
- **Definition**: A trading strategy (portfolio) is self-financing if there is no exogenous infusion or withdrawal of money; the purchase of a new asset must be financed by the sale of an old one.
- **Definition**: Arbitrage is a self-financing trading strategy that
  - (i) begins with no money ($X_0 = 0$)
  - (ii) has no probability of losing money ($P(X_1 \geq 0) = 1$) and
  - (iii) has a positive probability of making money at some future date ($P(X_1 > 0) > 0$)
- An efficient market should preclude arbitrages.
- The financial market above has no arbitrage $\Longleftrightarrow$ if $0 < d < 1 + r < u$
  - Proof of "$\Longrightarrow$": By contradiction.
  - Proof of "$\Longleftarrow$": **an excellent exercise.**

A Fair Option Price – No Arbitrage Approach

- No arbitrage pricing: It is reasonable to assign a fair price for this option such that no arbitrage is incurred, i.e. no arbitrage is created by adding the option to form a bigger market
- **Example (Two portfolios with identical values)**
  Let $S_0 = 4$, $u = 2 = \frac{1}{d}$, $r = \frac{1}{4}$, and $K = 5$. Consider two portfolios:

  Portfolio 1: A European call option with payoff $(S_1 - K)^+$

  Portfolio 2: An initial wealth $X_0 = 1.20$ which are invested into the stock and the money market in the following way: Borrow 0.8 and buy $\Delta_0 = 0.5$ shares of stock at time 0.

A Fair Option Price – No Arbitrage Approach

- At time 1, the two portfolios have the same values regardless of random outcomes.
- If the price of V_0 of this European call option is greater or less than 1.20 (the initial value of the **Portfolio 2**), there exist arbitrage opportunities.
- Assumptions of the derivation
  - Shares of stock can be subdivided for sale or purchase.
  - The interest rate for lending equals that for borrowing.
  - Purchasing price of stock equals the selling price.

Derivative Pricing: a General Backward Induction Method

- If one portfolio is a replication of another portfolio at time 1, their values at time 0 should be identical in order to preclude arbitrage.
- **Method**: To price a financial derivative security, replicate it with available financial instruments with known values.
- Consider a general derivative security with underlying asset being the stock and with payoff $V_1(H)$ or $V_1(T)$ depending on different random outcomes.
- How to get its price V_0 at time 0 with the technique of replication?

Derivatives Pricing

- Start with a wealth X_0.
- At time 0, invest X_0 into the stock and the money market by
  - buying Δ_0 shares of stock
  - borrowing or investing $X_0 - \Delta_0 S_0$ ($\Longleftarrow$ self-financing) in the money market.
- Then at time 1, we have

$$X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) \tag{1}$$

- Replication: $X_1 = V_1$ holds for both the two scenarios (H or T).
- Calculation of X_0 and Δ_0 from the following two cases of the replication equation:

$$\begin{aligned} X_1(H) = \Delta_0 S_1(H) + (1+r)(X_0 - \Delta_0 S_0) &= V_1(H) \\ X_1(T) = \Delta_0 S_1(T) + (1+r)(X_0 - \Delta_0 S_0) &= V_1(T) \end{aligned}$$

Derivatives Pricing

- Solution:

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}$$

and

$$X_0 = \frac{1}{1+r}\left[\frac{1+r-d}{u-d}V_1(H) + \frac{u-(1+r)}{u-d}V_1(T)\right].$$

- **Question**: How to intuitively explain $\Delta_0 = \frac{V_1(H)-V_1(T)}{S_1(H)-S_1(T)}$?
- **Question**: How to hedge a short position of this derivative security? How about the long position?

Derivatives Pricing

- Denote by $\tilde{p} = \frac{1+r-d}{u-d}$ and $\tilde{q} = \frac{u-(1+r)}{u-d}$ (Note: $\tilde{p} + \tilde{q} = 1$)
- $\tilde{p}$ and $\tilde{q} \Longrightarrow$ a probability measure $\mathbb{Q}$ called Risk-neutral probability with $\mathbb{Q}(H) = \tilde{p}$ and $\mathbb{Q}(H) = \tilde{q}$.
- It easy to know from algebra that

$$S_0 = \frac{1}{1+r}[\tilde{p}S_1(H) + \tilde{q}S_1(T)] \tag{2}$$

and

$$X_0 = \frac{1}{1+r}[\tilde{p}X_1(H) + \tilde{q}X_1(T)] = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)] \tag{3}$$

- No-arbitrage implies the risk-neutral pricing formula

$$V_0 = X_0 = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)] \tag{4}$$

Under the Risk-Neutral Probability $\mathbb{Q}$

- **Question**: What do (2), (3) and (4) imply?
  - Discounted stock price is a martingale under $\mathbb{Q}$;

$$S_0 = \mathbb{E}^{\mathbb{Q}}\left(\frac{S_1}{1+r}\right).$$

  - Discounted replicating porfolio is a martingale under $\mathbb{Q}$;

$$X_0 = \mathbb{E}^{\mathbb{Q}}\left(\frac{X_1}{1+r}\right).$$

  - Discounted option price is a martingale under $\mathbb{Q}$;

$$V_0 = \mathbb{E}^{\mathbb{Q}}\left(\frac{V_1}{1+r}\right).$$

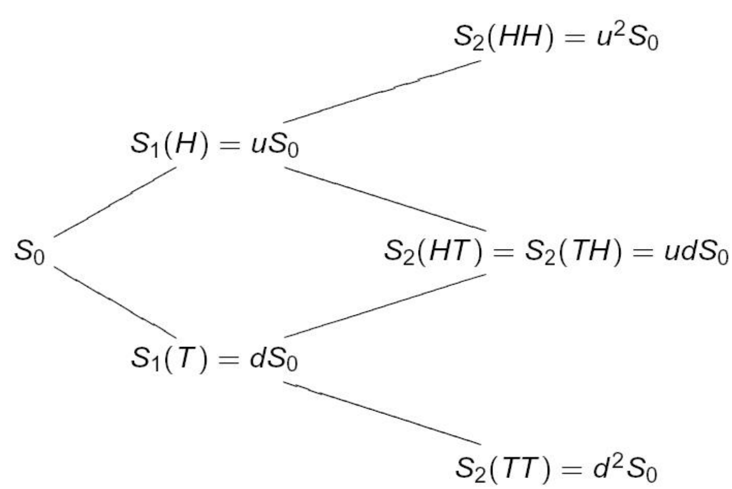
- **Question**: How to understand the “risk-neutral"?
- Untangle the **Expectation Puzzle**: Common sense told us that price could be some sort of expectation. Now, it is specified under $\mathbb{Q}$ (not $\mathbb{P}$)! $\mathbb{Q}$ is something neutral designed for derivatives valuation.

Real-World versus Risk-Neutral Probability

- **Question**: Why isn't actual probability involved?
- **A "Counter-Intuition" Example**: Consider two extreme cases of the actual probability of the stock.
  - Going-up probability: $p = 99\%$; Going-down probability: $q = 1\%$
  - Going-up probability: $p = 1\%$; Going-down probability: $q = 99\%$
- Which call option has greater value intuitively?
- Your intuitions become true as time evolves. Why?
- The risk premium is embedded in the underlying asset.

The Multiperiod Binomial Lattice Model

The Multiperiod Binomial Lattice Model: For example–two-period



An example: The Two-period Binomial Model

- Consider a two-period model with a European call option having payoff $V_2 = (S_2 - K)^+$.
- Construct a self-financing replicating portfolio, starting with a wealth X_0.
- Buy Δ_0 shares of stock and borrowing or investing $X_0 - \Delta_0 S_0$ in the money market.
- At time 1, we have $X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)$.

$$X_1(H) = \Delta_0 S_1(H) + (1+r)(X_0 - \Delta_0 S_0)$$
$$X_1(T) = \Delta_0 S_1(T) + (1+r)(X_0 - \Delta_0 S_0)$$

- At time 1, one is allowed to adjust the portfolio in a self-financing way, i.e. no extra fund or withdrawal.
  - Hold Δ_1 shares of stock
  - borrowing or investing $X_1 - \Delta_1 S_1$ in the money market.

An example: The Two-period Binomial Model

- At time 2, we have $X_2 = \Delta_1 S_2 + (1+r)(X_1 - \Delta_1 S_1)$.
- Replication: equate X_2 with V_2.

$$V_2(HH) = \Delta_1(H)S_2(HH) + (1+r)(X_1(H) - \Delta_1(H)S_1(H))$$
$$V_2(HT) = \Delta_1(H)S_2(HT) + (1+r)(X_1(H) - \Delta_1(H)S_1(H))$$
$$V_2(TH) = \Delta_1(T)S_2(TH) + (1+r)(X_1(T) - \Delta_1(T)S_1(T))$$
$$V_2(TT) = \Delta_1(T)S_2(TT) + (1+r)(X_1(T) - \Delta_1(T)S_1(T))$$

- Six equations for six variables X_0, Δ_0, $\Delta_1(H)$, $\Delta_1(T)$, $X_1(H)$, and $X_1(T)$
- $X_1(H)$ and $X_1(T)$: equal to the option prices at time 1 given the first coin toss is H and T, denoted by $V_1(H)$ and $V_1(T)$, respectively.
- X_0: equal to the option price at time 0.

An example: The Two-period Binomial Model

Viewing each sub-period as a one-period model and taking into account $\tilde{p}$ and $\tilde{q}$, we can obtain that

$$
\begin{aligned}
\Delta_1(H) &= \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} \\
X_1(H) &= \frac{1}{1+r}[\tilde{p}V_2(HH) + \tilde{q}V_2(HT)] \\
\Delta_1(T) &= \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)} \\
X_1(T) &= \frac{1}{1+r}[\tilde{p}V_2(TH) + \tilde{q}V_2(TT)] \\
\Delta_0 &= \frac{X_1(H) - X_1(T)}{S_1(H) - S_1(T)} \\
X_0 &= \frac{1}{1+r}[\tilde{p}X_1(H) + \tilde{q}X_1(T)]
\end{aligned}
$$

Generalization to Multiperiod Binomial model

The Principle of **Backward Induction**:

$$
\begin{gathered}
\text{A derivative with payoff } V_N \\
\Updownarrow (Replication) \\
\text{A portfolio with initial wealth } X_0 \text{ and self-finance strategy } \Delta_n \\
\Downarrow \\
\text{The derivative price } V_n = X_n \text{ at time } n = 0, 1, \cdots, N-1
\end{gathered}
$$

Generalization to Multiperiod Binomial Model

- Start from an initial wealth X_0, and adjust the portfolio by investing in stock and money market at each time $n = 0, 1, \cdots, N-1$
- Self-finance adjust satisfies a Wealth Equation.

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$$

- Replication

$$X_N = V_N$$

  to specify X_n and Δ_n for $n = 0, 1, \cdots, N-1$.
- The option price at time n given the fixed random outcome in the first n period: $V_n = X_n$, for $n = 0, 1, \cdots, N-1$.

Generalization to Multiperiod Binomial Model

- **Theorem**: Consider a N-period Binomial model with $0 < d < 1 + r < u$ and with $\tilde{p} = \frac{1+r-d}{u-d}$ and $\tilde{q} = \frac{u-(1+r)}{u-d}$. For $n = 0, 1, \cdots, N-1$, we have

$$X_n(\omega_1 \cdots \omega_n) = \frac{1}{1+r}[\tilde{p} X_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q} X_{n+1}(\omega_1 \cdots \omega_n T)]$$

$$\Delta_n = \frac{X_{n+1}(\omega_1\omega_2 \cdots \omega_n H) - X_{n+1}(\omega_1\omega_2 \cdots \omega_n T)}{S_{n+1}(\omega_1\omega_2 \cdots \omega_n H) - S_{n+1}(\omega_1\omega_2 \cdots \omega_n T)}$$

  Then, to preclude arbitrage, the initial value of the derivative is set as $V_0 = X_0$ and the values at times $n = 0, 1, \cdots, N$ are set as

$$V_n(\omega_1\omega_2 \cdots \omega_n) = X_n(\omega_1\omega_2 \cdots \omega_n) \quad \text{for all } \omega_1\omega_2 \cdots \omega_n$$

- Obviously, we also have

$$S_n(\omega_1 \cdots \omega_n) = \frac{1}{1+r}[\tilde{p} S_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q} S_{n+1}(\omega_1 \cdots \omega_n T)]$$

Recast: Formalizing the Binomial Lattice

Let us recast a N-period binomial lattice model. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_n\}$ can be specified as follows:

- $\Omega = \{(\omega_1\omega_2\cdots\omega_N) : \omega_i = \text{H or T for any } i = 1, 2, \cdots, N\}$: the collection of results of N coin tosses
- $\mathcal{F}$ (collection of all possible events) is a σ-algebra generated by all subsets of Ω.
- Probability measure: $\forall \omega = (\omega_1\omega_2\cdots\omega_N) \in \Omega$,

  $$\mathbb{P}(\{\omega\}) := p^{\sum_{i=1}^{N} I_{\{\omega_i=H\}}} q^{\sum_{i=1}^{N} I_{\{\omega_i=T\}}}.$$

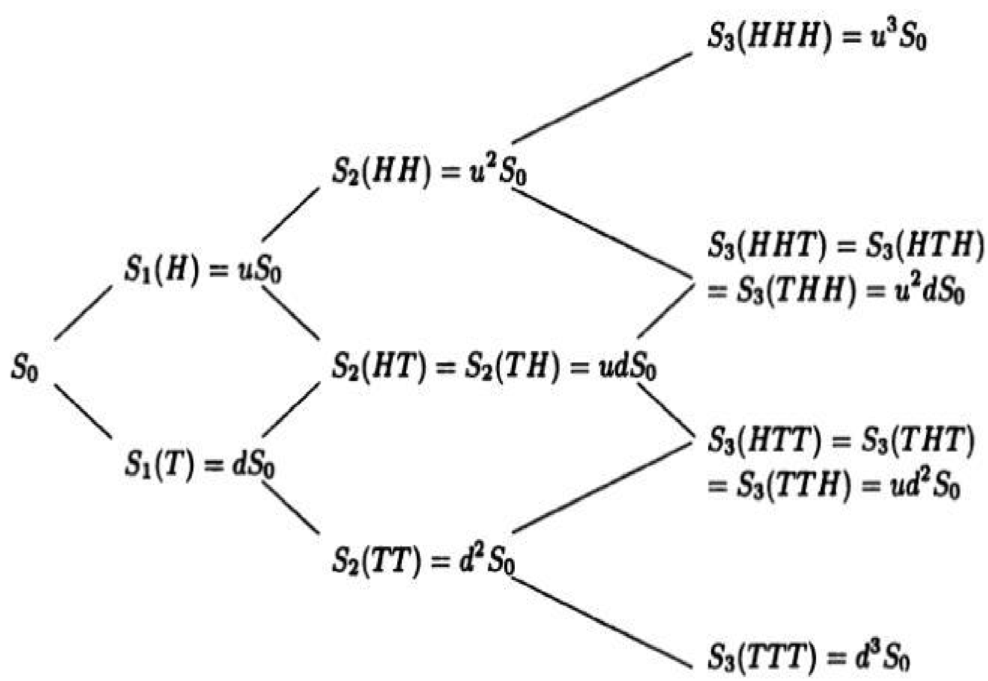
  $\forall A \in \mathcal{F}$,

  $$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\{\omega\}).$$

- Consider a filtration $\{\mathcal{F}_n\}$, $n = 1, 2, \cdots, N$. $\mathcal{F}_n$ is a σ-algebra generated by the "information" up to n. We thus have $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ and $\mathcal{F}_N = \mathcal{F}$.

Recast: Formalizing the Binomial Lattice

An illustration of the filtration: $N = 3$.



Recast: Formalizing the Binomial Lattice

An illustration of the filtration: $N = 3$. We have $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3$.

- Information up to the first day: $\mathcal{F}_1 = \{\phi, \Omega, A_H, A_T\}$. Here

$$\begin{aligned} A_H =& \{HHH, HHT, HTH, HTT\}, \\ A_T =& \{THH, THT, TTH, TTT\}. \end{aligned} \tag{5}$$

- Information up to the second day: $\mathcal{F}_2 = \{\phi, \Omega, A_H, A_T, A_{HH}, A_{HT}, A_{TH}, A_{TT}, A_{HH}^c, A_{HT}^c, A_{TH}^c, A_{TT}^c, A_{HH} \cup A_{TH}, A_{HH} \cup A_{TT}, A_{HT} \cup A_{TH}, A_{HT} \cup A_{TT}\}$. Here

$$\begin{aligned} A_{HH} =& \{HHH, HHT\}, \quad A_{HT} = \{HTH, HTT\}, \\ A_{TH} =& \{THH, THT\}, \quad A_{TT} = \{TTH, TTT\}. \end{aligned} \tag{6}$$

- Information up to the third day: $\mathcal{F}_3 =$ the set of all subsets of $\Omega = \mathcal{F}$

Martingale Approach–Risk-Neutral Measure

- Introduce the risk-neutral probability measure $\mathbb{Q}$: from $\mathbb{P}$ to $\mathbb{Q}$, we just need to replace (p, q) by $(\widetilde{p}, \widetilde{q})$.

- **Theorem**: Under the risk-neutral probability measure $\mathbb{Q}$ ($\tilde{p} = \frac{1+r-d}{u-d}$ and $\tilde{q} = \frac{u-(1+r)}{u-d}$), the discounted stock price $\{\frac{S_n}{(1+r)^n} : n = 0, 1, \cdots, N\}$ is a martingale, i.e.,

$$\frac{S_m}{(1+r)^m} = \mathbb{E}^{\mathbb{Q}} \left(\frac{S_n}{(1+r)^n} | \mathcal{F}_m \right) \quad \text{for } 0 \leq m \leq n \leq N.$$

- Again, why $\mathbb{Q}$ is called the risk-neutral probability measure?

Martingale Approach–Risk-Neutral Measure

- **Theorem**: Let X_0 be a real number and $\{\Delta_n : n = 0, 1, \cdots, N-1\}$ is an adapted process. Generate the wealth process $\{X_n : n = 0, 1, \cdots, N\}$ based on the wealth equation

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n).$$

  Then under the risk-neutral probability measure $\mathbb{Q}$, the discounted wealth process $\{\frac{X_n}{(1+r)^n} : n = 0, 1, \cdots, N\}$ is a martingale, i.e.,

$$\frac{X_m}{(1+r)^m} = \mathbb{E}^{\mathbb{Q}}\left(\frac{X_n}{(1+r)^n}|\mathcal{F}_m\right) \quad \text{for } 0 \leq m \leq n \leq N.$$

- Replication implies that the option prices $\{V_n : n = 0, 1, \cdots, N\}$ satisfy

$$\frac{V_m}{(1+r)^m} = \mathbb{E}^{\mathbb{Q}}\left(\frac{V_N}{(1+r)^N}|\mathcal{F}_m\right) \quad \text{for } 0 \leq m \leq N.$$

Risk-Neutral Pricing Formula

- Risk-Neutral Derivatives Pricing Formula:

$$V_m = \mathbb{E}^{\mathbb{Q}}\left(\frac{V_N}{(1+r)^{N-m}}|\mathcal{F}_m\right) \quad \text{for } 0 \leq m \leq N.$$

- Forward Pricing method (more convenient comparing with the backward induction method):

$$V_0 = \left(\frac{1}{1+r}\right)^N \mathbb{E}^{\mathbb{Q}} V_N = \left(\frac{1}{1+r}\right)^N \sum_{\omega \in \Omega} V_N(\omega)\mathbb{Q}(\omega).$$

- Example: pricing a call option with the two-period binomial lattice

Change of Measure

- Note that two probability measures are mentioned:
  - Actual probability measure: $\mathbb{P}$, not involved in pricing
  - Risk-neutral probability measure: $\mathbb{Q}$, involved in pricing
- What is the relationship between $\mathbb{P}$ and $\mathbb{Q}$?
- Two probability measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent on finite space $(\Omega, \mathcal{F})$ means that $\mathbb{P}(A) = 0 \iff \mathbb{Q}(A) = 0$ for any $A \in \mathcal{F}$.
- Assume $\mathbb{P}(\omega) > 0$ and $\mathbb{Q}(\omega) > 0$ for any $\omega \in \Omega$. Define Radon-Nikodym derivative of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ as

$$Z(\omega) := \frac{\mathbb{Q}(\omega)}{\mathbb{P}(\omega)}.$$

  - $\mathbb{P}(Z > 0) = 1$;
  - $\mathbb{E}^{\mathbb{P}}(Z) = 1$ (like a random density);
  - For any RV Y, we have $\mathbb{E}^{\mathbb{Q}} Y = \mathbb{E}^{\mathbb{P}}[ZY]$.

Risk-Neutral and No Arbitrage

- General definition of **Risk Neutral Probability Measure**: an equivalent probability measure under which the discounted security prices of the market are martingales.

- **Theorem**: If we can find a risk-neutral probability measure (for the whole market), the market is free of arbitrage.

- A straightforward proof by contradiction: If there exists an arbitrage, then beginning with $X_0 = 0$, we can construct a portfolio such that
  - $X_N(\omega) \geq 0$ for all $\omega \in \Omega$.
  - There exists at least one $\bar{\omega}$ such that $X_N(\omega) > 0$.

- Therefore, $\mathbb{E}^{\mathbb{Q}} X_0 = 0$ but $\mathbb{E}^{\mathbb{Q}} \left[\frac{X_N}{(1+r)^N} \right] > 0$. Contradiction!

State Price

- Consider a N-period Binomial model. Then the Radon-Nikodym derivative of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ is given by

$$Z(\omega_1\omega_2\cdots\omega_N) := \frac{\mathbb{Q}(\omega_1\omega_2\cdots\omega_N)}{\mathbb{P}(\omega_1\omega_2\cdots\omega_N)} = \left(\frac{\tilde{p}}{p}\right)^{\sum_{i=1}^N I_{\{\omega_i=H\}}} \left(\frac{\tilde{q}}{q}\right)^{\sum_{i=1}^N I_{\{\omega_i=T\}}}$$

.

- **Definition**: $\zeta(\omega) = \frac{Z(\omega)}{(1+r)^N}$ is called the state price density random variable.
- **Definition**: $\zeta(\omega)\mathbb{P}(\omega)$ is called the state price corresponding to ω.

State Price

- Arrow-Debreu Security: (a Simple Contract) with payoff $A_N^{\bar{\omega}}(\omega) = 1_{\{\omega=\bar{\omega}\}}$ for an arbitrary state $\bar{\omega} \in \Omega$ and any outcome $\omega \in \Omega$
- Its price at time 0 is

$$A_0^{\bar{\omega}} = \mathbb{E}^{\mathbb{Q}}\left(\frac{A_N^{\bar{\omega}}}{(1+r)^N}\right) = \frac{\mathbb{Q}(\bar{\omega})}{(1+r)^N} = \frac{Z(\bar{\omega})\mathbb{P}(\bar{\omega})}{(1+r)^N} = \zeta(\bar{\omega})\mathbb{P}(\bar{\omega})$$

- The state price $\zeta(\bar{\omega})\mathbb{P}(\bar{\omega})$ means the time 0 price of a contract that pays 1 at time N iff $\bar{\omega}$ happens.
  - Time value $\frac{1}{(1+r)^N}$, probability $\mathbb{P}(\bar{\omega})$, and risk adjustment $Z(\bar{\omega})$

State Price

Given a general derivative security with payoff

$$V_N(\omega) \equiv \sum_{\{\bar{\omega}\in\Omega\}} V_N(\bar{\omega}) A_N^{\bar{\omega}}(\omega),$$

we have

$$\begin{aligned} V_0 &= \mathbb{E}^{\mathbb{Q}}\left(\frac{V_N}{(1+r)^N}\right) \\ &= \mathbb{E}^{\mathbb{Q}}\left(\frac{\sum_{\{\bar{\omega}\in\Omega\}} V_N(\bar{\omega}) A_N^{\bar{\omega}}}{(1+r)^N}\right) \\ &= \sum_{\{\bar{\omega}\in\Omega\}} V_N(\bar{\omega}) \mathbb{E}^{\mathbb{Q}}\left(\frac{A_N^{\bar{\omega}}}{(1+r)^N}\right) \\ &= \sum_{\{\bar{\omega}\in\Omega\}} V_N(\bar{\omega}) \zeta(\bar{\omega}) \mathbb{P}(\bar{\omega}) = \sum_{\{\bar{\omega}\in\Omega\}} V_N(\bar{\omega}) A_0^{\bar{\omega}}. \end{aligned}$$

State Price

- **Definition**: Randon-Nykodym derivative process is defined as $Z_n = \mathbb{E}(Z|\mathcal{F}_n)$, $n = 0, 1, \cdots, N$, where $Z(\omega) := \frac{\mathbb{P}(\omega)}{\mathbb{Q}(\omega)}$.
  - Z_n is a martingale with respect to the filtration $\{\mathcal{F}_n\}$ (Doob's martingale).
  - $Z_0 = 1$ and $Z_n = Z$.
- For any $\mathcal{F}_m$-measurable RV Y and $0 \le n \le m \le N$, we have
  - $\mathbb{E}^{\mathbb{Q}} Y = \mathbb{E}(Z_m Y)$ (Proof via definition).
  - $\mathbb{E}^{\mathbb{Q}}(Y|\mathcal{F}_n) = \frac{1}{Z_n}\mathbb{E}(Z_m Y|\mathcal{F}_n)$ (Proof via definition).
- **Definition**: State price density process $\zeta_n = \frac{Z_n}{(1+r)^n}$, for $n = 0, 1, \cdots, N$
- **Theorem**: Under the actual probability $\mathbb{P}$, $\{\zeta_n V_n : n = 0, 1, \cdots, N\}$ is a martingale; in particular, we have

$$V_n = \frac{1}{\zeta_n}\mathbb{E}\left(\zeta_N V_N | \mathcal{F}_n\right).$$

Suggested Reading Material (We only need to focus on the material parallel to our course slides):

- Selected Material from Shreve Vol. I: Section 1.1, 1.2, 2.4, 3.1

Suggested Exercises (Do Not Hand In; For Your Deeper Understanding Only)

- Shreve Vol. I: Exercise 1.1, 1.2, 1.3, 1.6, 2.2, 2.10, 3.1, 3.2, 3.3, 3.4