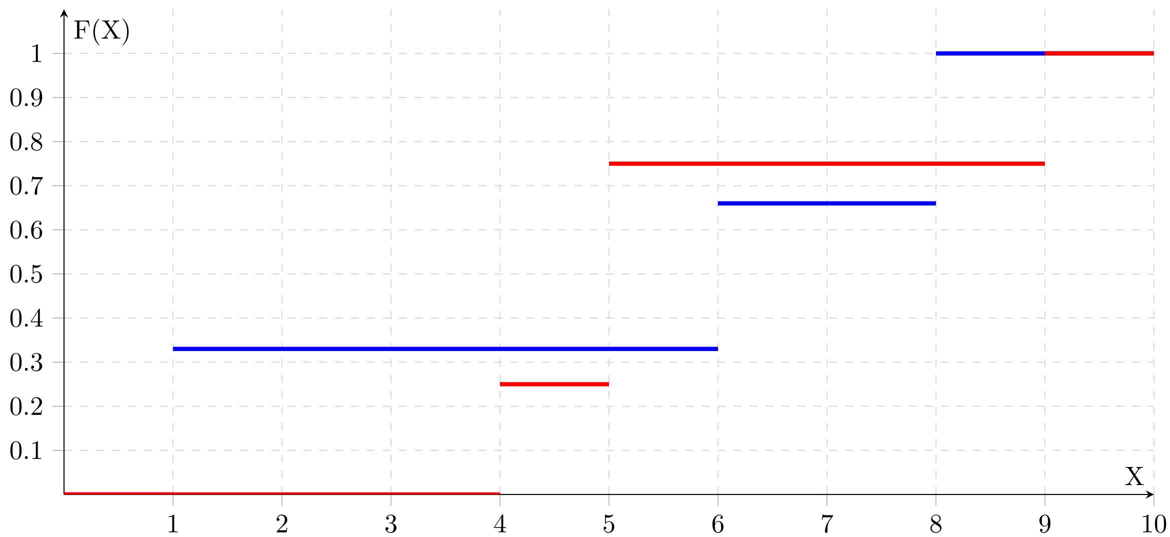
Lecture 9

Econ139

February 19, 2019

1 Second Order Stochastic Dominance(SSD)

$\tilde{X}$	π_A	π_B	$F_{\pi A}$	$F_{\pi B}$	$\int F_{\pi A}(t) - F_{\pi B}(t)$
1	0.33	0	0.33	0	0
4	0	0.25	0.33	0.25	-0.99
5	0	0.5	0.33	0.75	-1.07
6	0.33	0	0.66	0.75	-0.65
8	0.34	0	1	0.75	-0.47
9	0	0.25	1	1	-0.72



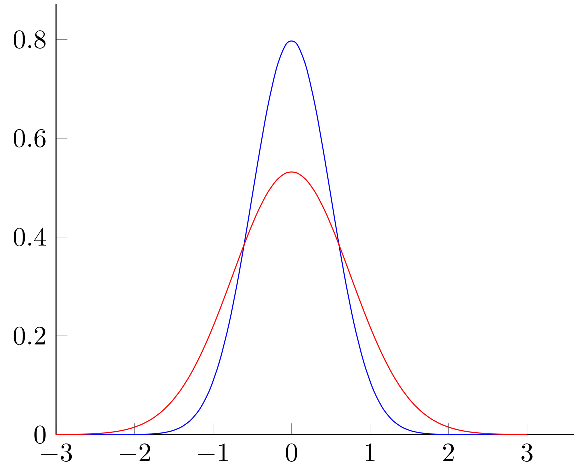
Blue = Fπ_A $\quad$ *Red* $= F\pi_B$

Definition: given preamble, we say that $\tilde{X}_{\pi A}$ SSD $\tilde{X}_{\pi B}$ iff $\int_{-\infty}^{t} F_{\pi B}(Z) - F_{\pi A}(Z)dz \leq 0 \forall t$

- Asset B SSD because it is negative at every point.

Theorem: Given preamble, then $\tilde{X}_{\pi B}$ SSD $\tilde{X}_{\pi A}$ iff $E_{\pi B}[U(\tilde{X})] > E_{\pi A}[U(\tilde{X})]$ for all increasing and concave U.

2 Mean Preserving Spread



Blue=Y Red=X

Y is a mean preserving spread of X

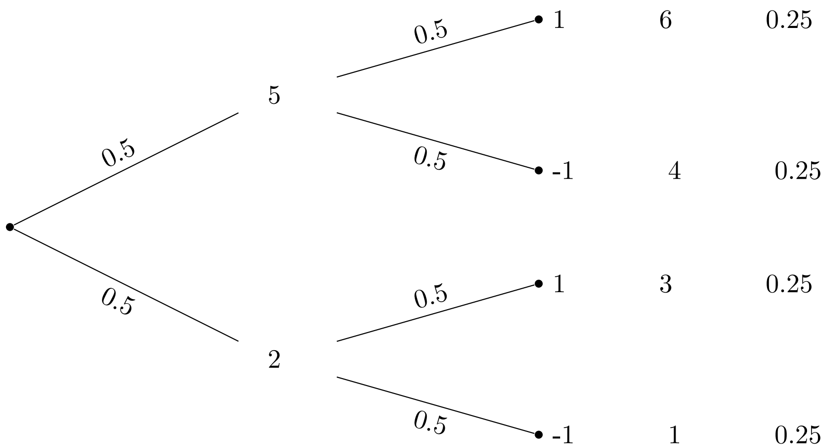
$$Y = X + Z \text{ where } E[Z] = 0$$

$$E[Y] = E[X]$$

$$\theta_y^2 > \theta_x^2$$

$\tilde{X}$	π_Z	π_A	$F_{\pi B}$
-1	0.5	0	0
1	0.5	0	0.25
2	0	0.5	0
3	0	0	0.25
4	0	0	0.25
5	0	0.5	0
6	0	0	0.25

$$E_{\pi A}[\tilde{X}] = 3.5,\ E_{\pi B}[\tilde{X}] = 3.5,\ \theta_{\pi A}^2 = 2.25,\ \theta_{\pi B}^2 = 3.25,\ E_{\pi Z}[\tilde{X}] = 0$$



Definition: Given preamble, we say that $\tilde{X}_{\pi B}$ is a mean preserving spread (MPS) of $\tilde{X}_{\pi A}$ if there exists a PDF π_Z over $\tilde{X}$ such that $E_{\pi Z}[\tilde{X}] = 0$ and $\tilde{X}_{\pi B} = \tilde{X}_{\pi A} + \tilde{X}_{\pi Z}$

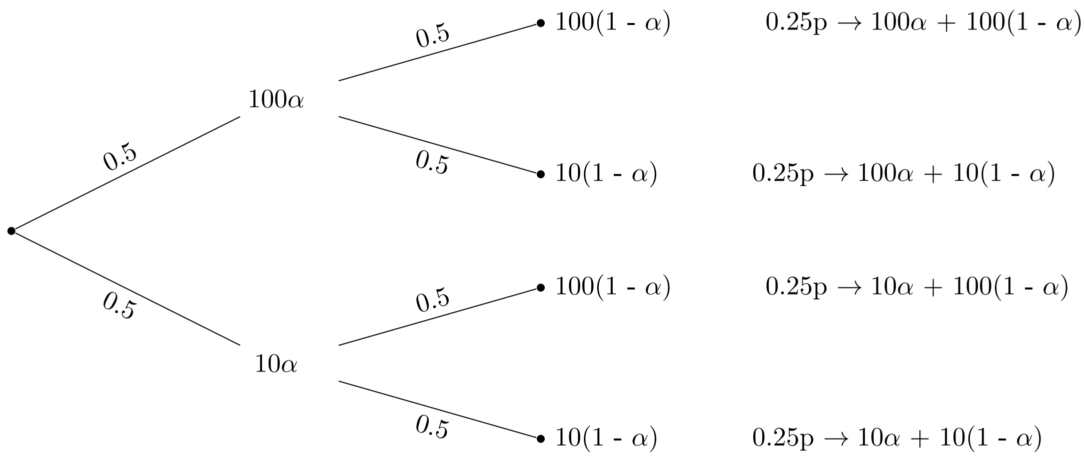
Theorem: Given preamble, if $E_{\pi B}[\tilde{X}] = E_{\pi A}[\tilde{X}]$, the following statements are equivalent:
(i) $\tilde{X}_{\pi A}$ SSD $\tilde{X}_{\pi B}$
(ii) $\tilde{X}_{\pi B}$ is a MPS of $\tilde{X}_{\pi A}$

3 Diversification

Consider: A set of n lotteries given $\tilde{x}_1$, ..., $\tilde{x}_n$ that are assumed to be independent and identically distributed.
feasible strategy: Characterized by a vector, A = (α_1, ..., α_n), where α_i is one's share in lottery i where $\sum_{i\in 1}^{n} \alpha_i = 1$. This gives a random payoff $\tilde{y} = \sum_{i\in 1}^{n} \alpha_i \tilde{x}_i$

Proposition: The random payoff $\tilde{Y}_D$ generated by the "perfect" diversification strategy D = $(\frac{1}{n}, ..., \frac{1}{n})$ SSD the random payoff generated by any other strategy.

Example: Two lotteries: $\tilde{x}_i$ = (10, 100) with π_i = (0.5, 0.5) and i = 1,2



With perfect diversification: D = ($\frac{1}{2}$, $\frac{1}{2}$)

w/ prob.	payoff
0.25	100
0.25	55
0.25	55
0.25	10

$E_[\tilde{Y}_D] = 55; \theta^2(\tilde{Y}_D) = 101.25,$

With different strategy: A = ($\frac{1}{3}$, $\frac{2}{3}$)

w/ prob.	payoff
0.25	100
0.25	40
0.25	70
0.25	10

$E_[\tilde{Y}_D] = 55; \theta^2(\tilde{Y}_D) = 102.5,$

We see the second strategy has the same mean but a higher variance. Thus, the first strategy of perfect diversification SSD the second strategy.

4 Investment in Risky Asset II

Say we have the following:

- individual:
  - initial wealth w_0
  - utility function u_1 with $u' > 0$, $u'' < 0$
- two possible investments:
  - (I) bond that pays $(1 + r_f)$ for every dollar invested
  - (II) risky asset with uncertain return $\tilde{r}$
- let a represent the number of dollars invested in the risky asset
- future wealth: $\tilde{w}_1 = (w_0 - a)(1 + r_f) + a(1 + \tilde{r}) = w_0(1 + r_f) + a(\tilde{r} - r_f)$

We will maximize expected utility:

$$\max_a \; E(u(w_1^2))$$
$$\max_a \; E[u(w_0(1 + r_f) + a(\tilde{r} - r_f))]$$

We will take the FOC (derivative with respect to a):

$$E[u'(w_0(1 + r_f) + a * (\tilde{r} - r_f))(\tilde{r} - r_f)] = 0$$

Theorem: Assume $u' > 0$ and $u'' < 0$ and a* $= argmax_a E[u(\tilde{w}_1)]$. Then:

$$\text{(I) For } a* > 0 \leftrightarrow E[\tilde{r}] > r_f$$
$$\text{(II) For } a* = 0 \leftrightarrow E[\tilde{r}] = r_f$$
$$\text{(III) For } a* < 0 \leftrightarrow E[\tilde{r}] < r_f$$

Following this theorem, we can make assumptions about our expected return per investment.

Proof:
Let $g(a) = E[u(w_0(1 + r_f) + a(\tilde{r} - r_f))]$
Then $g'(a) = E[u'(w_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)]$ and $g''(a) = E[u''(w_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)]$
Observe since $u'' < 0, g''(a) < 0$ implies $g'(a)$ is strictly decreasing in a

$$g'(0) = E[u'(w_0(1 + r_f))(\tilde{r} - r_f)]$$
$$= u'(w_0(1 + r_f))E[\tilde{r} - r_f]$$
$$= u'(w_0(1 + r_f))(E[\tilde{r}] - r_f)$$

We know the first term is positive, so this implies $g'(0)$ has the same sign as $E[\tilde{r}] - r_f$

Summary:
(I) $g'(a)$ is strictly decreasing
(II) $g'(0)$ has the sign of $E[\tilde{r}] - r_f$
(III) $g'(a*) = 0$ because of FOC

$$\text{Now } a* = 0 \textit{ iff } E[\tilde{r}] = r_f$$

To prove forward $\rightarrow$:

$$g'(a*) = 0 \text{ and } a* = 0$$
$$\rightarrow g'(0) = 0 \text{ since } g'(0) \text{ has the same sign as } E[\tilde{r}] - r_f$$
$$\rightarrow E[\tilde{r}] - r_f = 0$$
$$\rightarrow E[\tilde{r}] = r_f$$

To prove backwards $\leftarrow$:

$$\begin{gathered}\text{Suppose } E[\tilde{r}] = r_f \\ E[\tilde{r}] - r_f = 0 \\ \rightarrow g'(0) = 0 \text{ since } g'(a) \text{ is strictly decreasing} \\ \rightarrow g'(a*) = 0 \text{ from FOC} \\ \rightarrow a* = 0\end{gathered}$$