Scribe Notes for Lecture 19

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1 Arrow-Debreu Pricing

- Recall from our numerical example:

$$w^1 = (10, (1, 2))$$
$$w^2 = (5, (4, 6))$$
$$\pi_1 = \frac{1}{3}, \quad \pi_2 = \frac{2}{3}$$

  - Agents have same utility function:

$$U_0(c) = \frac{1}{2}c, \quad U(c) = \ln(c)$$

  - Agents have same time discount factor: $\delta^1 = \delta^2 = 0.9$,

    Then,

$$\max_{c^i} \frac{1}{2}c_0^i + 0.9\left(\frac{1}{2}\ln\left(c_1^i\right) + \frac{2}{3}\ln\left(c_2^i\right)\right)$$

$$\text{s.t.} \quad c_0^i + q_1 c_1^i + q_2 c_2^i = w_0^i + q_1 w_1^i + q_2 w_2^i$$

- Solution:

$$q_1 = 0.24, \quad q_2 = 0.3$$
$$c^1 = (9.04, (2.5, 4))$$
$$c^2 = (5.96, (2.5, 4))$$

- Social Planner Problem:

$$\max_{c^1, c^2} u^1\left(c^1\right) + u^2\left(c^2\right)$$
$$\text{s.t.} \quad c_0^1 + c_0^2 = 15 \quad c_1^1 + c_1^2 = 5 \quad c_2^1 + c_2^2 = 8$$
$$\frac{u_0^1}{u_0^2} = \frac{u_1^1}{u_1^2} = \frac{u_2^1}{u_2^2} = \lambda$$
$$\frac{u_0^1}{u_1^1} = \frac{u_0^2}{u_1^2}, \frac{u_0^1}{u_2^1} = \frac{u_0^2}{u_2^2}, \frac{u_1^1}{u_2^1} = \frac{u_1^2}{u_2^2}$$
$$\frac{\frac{1}{2}}{0.9 * \frac{1}{3} * \frac{1}{C_1^1}} = \frac{\frac{1}{2}}{0.9 * \frac{1}{3} * \frac{1}{c_1^2}}$$

- The consumption is linear:

$$\frac{u_0^1}{u_0^2} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

  - $\lambda = 1$, interior solution: all states are positive

$$c_0^1, c_1^1, c_2^1, c_0^2, c_1^2, c_2^2 > 0$$

  - $\lambda \neq 1$, boundary solution
  - Solve in a different way:

$$c^1 = (9.04, (2.5, 4))$$

$$c^2 = (5.96, (2.5, 4))$$

$$q_1 = 0.24, \quad q_2 = 0.3$$

- Market clearing condition:

$$Z_1^1 + Z_1^2 = 0$$

$$Z_2^1 + Z_2^2 = 0$$

$$c_0^1 + q_1 c_1^1 + q_2 c_2^1 = 10 + q_1 + 2q_2$$

$$c_0^1 + q_1 \underbrace{\left(c_1^1 - 1\right)}_{z_1'} + q_2 \underbrace{\left(c_2^1 - 2\right)}_{z_2^{\frac{1}{2}}} = 10$$

$$c_0^1 + q_1 z_1^1 + q_2 z_2^1 = 10$$

$$c_0^1 = 10 - q_1 z_1^1 - q_2 z_2^1$$

  - For agent 1:

$$\max_{Z_1^1, Z_2^1} \frac{1}{2} \left(10 - q_1 z_1^1 - q_2 z_2^1\right) + 0.9 \left[\frac{1}{3} \ln\left(1 + z_1^1\right) + \frac{2}{3} \ln\left(2 + z_2^1\right)\right]$$

  - For agent 2:

$$\max_{Z_1^2, Z_2^2} \frac{1}{2} \left(5 - q_1 z_1^2 - q_2 z_2^2\right) + 0.9 \left[\frac{1}{3} \ln\left(4 + z_1^2\right) + \frac{2}{3} \ln\left(6 + z_2^2\right)\right]$$

  - FOC:

    * Agent 1:

$$Z_1^1: -\tfrac{1}{2}q_1 + 0.9 * \tfrac{1}{3} * \tfrac{1}{1+z_1^1} = 0$$
$$Z_2^1: -\tfrac{1}{2}q_2 + 0.9 * \tfrac{1}{3} * \tfrac{1}{1+z_2^1} = 0$$
$$q_1 = \tfrac{0.6}{1+z_1^1}, \quad q_2 = \tfrac{1.2}{2+Z_2^1}$$

* Agent 2:

$$q_1 = \frac{0.6}{4+Z_1^2}, \quad q_2 = \frac{1.2}{6+Z_2^2}$$

With market clear condition:

$$Z_1^1 + Z_1^2 = 0$$
$$Z_2^1 + Z_2^2 = 0$$

$$\begin{aligned}
q_1 &= \frac{0.6}{1+Z_1^1} = \frac{0.6}{4+Z_1^2}, \quad Z_1^1 = 3 + Z_1^2, \quad Z_1^1 = 1.5, \quad Z_1^2 = -1.5 \\
q_2 &= \frac{1.2}{2+Z_2^1} = \frac{1.2}{6+Z_2^2} \quad , \quad Z_2^1 = 4 + z_2^2, \quad Z_2^1 = 2, \quad Z_2^2 = -2 \\
q_1 &= \frac{0.6}{1+1.5} = 0.24 \\
q_2 &= \frac{1.2}{2+2} = 0.3
\end{aligned} \tag{1}$$

2 CCAPM

- Key assumption: all agents are identical in terms of preferences and endowments, and assume U_0 and U are the same. Now consider the following:

$$\max u(c_0) + \delta \sum_{\theta=1}^{N} \pi_\theta u(c_\theta) = \max u(c_0) + \delta E_\pi[u(c_\theta)]$$

where $c = (c_0, (c_1, \ldots, c_N))$ and $w = (w_0, (w_1, \ldots, w_N))$

- Introduce single security x with payoffs in every possible state θ, and then agent wants to choose amount a to invest in x. Giving us the following:

$$\max_a u(c_0) + \delta E_\pi[u(c_\theta)]$$

where $c_0 = w_0 - p_x a$ and $c_\theta = w_\theta + aX_\theta$

Hence, we can rewrite the previous equation as:

$$\max_a u(w_0 - p_x a) + \delta E_\pi[u(w_\theta + aX_\theta)]$$

Now consider the FOC:

$$-p_x u'(w_0 - p_x a^*) + \delta E_\pi[u(w_\theta + aX_\theta)] = 0$$

$p_x = E_\pi\left(\frac{\delta u'(c_\theta)}{u'(c_0)} X_\theta\right) \Rightarrow p_x u'(c_0) = E_\pi[\delta u'(c_\theta) X_\theta$

This says that at the optimum, the following must be equal 1. $p_x u'(c_0)$: loss in utility if agent buys another unit of x 2. $E_\pi[\delta u'(c_\theta) X_\theta$: gain in discounted expected utility of extra payoff at $t = 1$

We can also write CCAPM pricing equation as:

$P_x = E_\pi[m_\theta x_\theta]$ where $m_\theta = \frac{\delta u'(c_\theta}{u'(c_0)}$

- Special Cases

  Price of gross returns $1 + r_\theta = \frac{X_\theta}{p_x}$ with $p_x = E_\pi[m_\theta x_\theta]$ then if we divide this by 1, we get:

$$E_\pi\left[m_\theta \left(\frac{X_\theta}{p_x}\right)\right] = E_\pi[m_\theta(1 + r_f)] = 1$$

  which hold for risk-free asset $E_\pi[m_\theta] = \frac{1}{1+r_f}$

- Risk-Neutral Valuation

  Recall that with AD securities, $m_\theta = \frac{q_\theta}{\pi_\theta}$ where $q_\theta = \frac{\pi_\theta \delta u'(c_\theta)}{u'(c_0)}$ then

$$\begin{aligned} E_\pi[m_\theta] &= \sum_{\theta=1}^{N} \pi_\theta \left(\frac{q_\theta}{\pi_\theta \theta} \right) \\ &= \sum_{\theta=1}^{N} q_\theta = \frac{1}{1+r_f} \end{aligned}$$

  Then $p_x = E_\pi[m_\theta X_\theta] = \sum_{\theta=1}^{N} \pi_\theta \left(\frac{q_\theta}{pi_\theta}\right) X_\theta = \sum_{\theta=1}^{N} q_\theta X_\theta$

  Multiplying and diving $\sum_{\theta=1}^{N} q_\theta$ will give us:

$$p_x = \left(\sum_{\theta=1}^{N} q_\theta\right)\left(\frac{\sum_{\theta=1}^{N} q_\theta X_\theta}{\sum_{\theta=1}^{N} q_\theta}\right) = \frac{1}{1+r_f}\left(\frac{\sum_{\theta=1}^{N} q_\theta X_\theta}{\sum_{\theta=1}^{N} q_\theta}\right) = \frac{1}{1+r_f}\sum_{\theta=1}^{N} \pi_\theta^{RN} X_\theta = \frac{1}{1+r_f} E_\pi^{RN}[X_\theta]$$

  Find a probability π^{RN} under the measure we can price the asset by discounting the expectation by risk-free rate

State	AD_1	AD_2	AD_3	X
State1	1	0	0	2
State2	0	1	0	1
State3	0	0	1	4
Price	q_1	q_2	q_4	$2q_1 + q_2 + 4q_3$

- Back to CCAPM: We know $E_\pi[m_\theta(1+r_\theta)] = 1$ then recall the formula for covariance of two random variables:

$$Cov(X,Y) = E(XY) - E(X)E(Y) \Rightarrow E(XY) = Cov(X,Y) + E(X)E(Y)$$

Then,

$$E_\pi[m_\theta(1+r_\theta)] = E_\pi[m_\theta]E_\pi[1+r_\theta] + Cov(m_\theta, 1+r_\theta) = E_\pi[m_\theta]E_\pi[1+r_\theta] + Cov(m_\theta, r_\theta) = 1$$

$$E_\pi[1+r_f] + (1+r_f)Cov(m_\theta, r_\theta) = 1 + r_f$$

$$E_\pi[r_\theta] - r_f = -(1+r_f)Cov(m_\theta, r_\theta)$$

$$E_\pi[r_\theta] = r_f - (1+r_f)Cov(m_\theta, r_\theta)$$

Risk adjustment: $-(1+r_f)Cov(m_\theta, r_\theta) = -(1+r_f)\frac{\delta}{u'(C_0)}Cov(u'(C_\theta), r_\theta)v$