Mathematical Methods in Finance

Lecture 9: Partial Differential Equations and Monte Carlo Simulation

Fall 2013

Overview

- Motivation
- A brief introduction to PDEs
- Solution of Black-Scholes-Merton PDE
- Connect btw SDEs and PDEs: Feynman-Kac Theorem
- Change of probability measure: the Girsanov theorem
- Risk-Neutral valuation
- A brief introduction to Monte Carlo simulation

Motivation: the Black-Scholes-Merton PDE

- Consider the Black-Scholes-Merton model:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t),$$

  and suppose that the interest rate is r.
- Let $C(t) = c(t, S(t))$ be the value of a call option with maturity T with payoff $(s-K)^+$.
- $c(t,x)$ satisfies the Black-Scholes-Merton equation.

$$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x) \quad \text{for all } t \in [0,T), \tag{1}$$

  with a terminal condition $c(T,x) = (x-K)^+$.
- Question: How to solve the Black-Scholes-Merton PDE?

A Brief Introduction to Partial Differential Equation

Supplementary material (optional):

- Introduction
- Existence and uniqueness
- Classification and examples: first-order PDEs and second-order PDEs
- Initial value problems

Preparation: One-dimensional Heat Equation

Supplementary material (optional):

Selected material from 2.4 in W. Strauss' book "Partial Differential Equations: An Introduction".

Preparation: One-dimensional Heat Equation

Consider a heat equation

$$u_\tau(\tau, z) = \frac{1}{2} u_{zz}(\tau, z),$$

for all $\tau \in [0, +\infty)$ and $z \in \mathcal{R}$ with the initial condition

$$u(0, z) = f(z),$$

where $f(z)$ is a continuous and uniformly bounded function. Then the unique continuous and bounded solution to the heat equation is given by

$$u(\tau, z) = \int_{-\infty}^{+\infty} f(y) G(z, y, \tau) dy,$$

$$G(z, y, \tau) = \frac{1}{\sqrt{2\pi\tau}} \exp\left\{ -\frac{(z-y)^2}{2\tau} \right\}.$$

Back to the Black-Scholes-Merton PDE: Change of Variable

Consider the BSM PDE:

$$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x) \quad \text{for all } t \in [0,T), \tag{2}$$

Using the change of variables:

$$\begin{aligned} u &= e^{-rt}c, \\ y &= \log x, \tau = (T-t)\sigma^2, \\ z &= y + \frac{1}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)\tau, \end{aligned}$$

the Black-Scholes-Merton PDE becomes a heat equation:

$$u_\tau(\tau,z) = \frac{1}{2}u_{zz}(\tau,z),$$

with terminal condition $u(0,z) = e^{-rT}(e^z - K)^+$.

The Black-Scholes-Merton formula

- Solve the heat equation:

$$u(\tau,z) = \int_{-\infty}^{+\infty} u(0,y)G(z,y,\tau)dy,$$

$$G(z,y,\tau) = \frac{1}{\sqrt{2\pi\tau}} \exp\left\{-\frac{(z-y)^2}{2\tau}\right\}.$$

- **The Black-Scholes-Merton formula**: For any $t \in [0,T)$ and $x > 0$,

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)), \tag{3}$$

where $N(y)$ is the CDF of standard normal distribution and

$$\begin{aligned} d_+(\tau,x) &= \frac{1}{\sigma\sqrt{\tau}}\left[\log\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau\right], \\ d_-(\tau,x) &= \frac{1}{\sigma\sqrt{\tau}}\left[\log\left(\frac{x}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau\right]. \end{aligned} \tag{4}$$

Connect btw Brownian Motion and Heat Equation

Recall that the heat equation initial value problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{1}{2}\frac{\partial^2 u}{\partial x^2} &= 0, \\ u(0,x) &= f(x), \end{aligned}$$

admits solution

$$u(t,x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{2t}} f(y)dy.$$

Obviously, we can express this solution using a standard Brownian motion, i.e.

$$u(t,x) = \mathbb{E} f(W(t) + x),$$

where $\{W(t)\}$ is a standard Brownian motion.

A Heuristic Verification

An alternative heuristic verification that $\mathbb{E} f(W(t) + x)$ solves the heat equation:

The initial condition obviously holds! By Taylor expansion

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2} f''(a)(b-a)^2 + o((b-a)^2).$$

Therefore,

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \lim_{\Delta t \to 0} \frac{u(t+\Delta t, x) - u(t,x)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{\mathbb{E} f(W(t+\Delta t) + x) - \mathbb{E} f(W(t)+x)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{E}[f'(W(t)+x)(W(t+\Delta t) - W(t)) + \frac{1}{2} f''(W(t)+x) \\ &\quad (W(t+\Delta t) - W(t))^2 + o((W(t+\Delta t) - W(t))^2)] \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[0 + \frac{1}{2} \Delta t u''(t,x) + o(\Delta t) \right] \\ &= \lim_{\Delta t \to 0} 0 + \frac{1}{2} u''(t,x) + o(1) = \frac{1}{2} u''(t,x). \end{aligned}$$

A Heuristic Verification

Indeed, we have applied

$$\begin{aligned} & \mathbb{E}f'(W(t)+x)(W(t+\Delta t)-W(t)) \\ = \;& \mathbb{E}\left[\mathbb{E}\left[f'(W(t)+x)(W(t+\Delta t)-W(t))|\mathcal{F}_t\right]\right] \\ = \;& \mathbb{E}\left[f'(W(t)+x)\mathbb{E}\left[W(t+\Delta t)-W(t)|\mathcal{F}_t\right]\right]=0 \end{aligned}$$

$$\begin{aligned} & \mathbb{E}f''(W(t)+x)(W(t+\Delta t)-W(t))^2 \\ = \;& \mathbb{E}\left[\mathbb{E}\left[f''(W(t)+x)(W(t+\Delta t)-W(t))^2|\mathcal{F}_t\right]\right] \\ = \;& \mathbb{E}\left[f''(W(t)+x)\mathbb{E}\left[(W(t+\Delta t)-W(t))^2|\mathcal{F}_t\right]\right] \\ = \;& \Delta t\mathbb{E}f''(W(t)+x)=\Delta t u''(t,x) \end{aligned}$$

Multidimensional Extension

For $x=(x_1,x_2,\cdots,x_d)\in\mathbf{R}^d$ and a known function $g:\mathbf{R}^d\to\mathbf{R}$, an unknown function $u(t,x)$ satisfies that a $d-$dimensional heat equation:

$$\begin{aligned} \frac{\partial u}{\partial t}-\frac{1}{2}\Delta u &= 0, \\ u(0,x) &= g(x), \end{aligned}$$

where

$$\Delta=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_d^2}.$$

Amazingly, we have

$$u(t,x)=\mathbb{E}g(W(t)+x),$$

where $\{W(t)\}$ is a standard $d-$dimensional Brownian motion.

Connect btw Brownian Motion and Backward Heat Equation

Now, let $v(t,x) = u(T-t,x)$. Calculus yields a Backward heat equation:

$$\begin{aligned} \frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial^2 v}{\partial x^2} &= 0, \\ v(T,x) &= f(x), \end{aligned}$$

Note that,

$$v(t,x) = \mathbb{E}[f(B(T))|B(t) = x],$$

where $\{B(t)\}$ is a Brownian motion, solves the this equation!

Now, using the fact that $v(t,B(t)) = \mathbb{E}[f(B(T))|B(t)]$ is a martingale (why?), we can give a probabilistic proof! Later we will see something more general!

Connect btw Stochastic Processes and PDEs: Feynman-Kac Theorem

Question: Can we generalize the previous result on the connection btw Brownian motion and heat equation?

- Consider an SDE:

$$dX(u) = \beta(u, X(u))du + \gamma(u, X(u))dW(u). \tag{5}$$

- We assume the existence and uniqueness of its solution
- Can we employ this SDE to express the solution to some certain PDEs as conditional expectation?

Properties of Strong Solutions

- Consider a strong solution of (5) $X(t)$ and a function $h(y)$. Define

$$g(t,x) := E^{t,x}h(X(T)) \equiv E[h(X(T))|X(t)=x] \tag{6}$$

- By the Markov property (let us believe it) of $\{X(t)\}$

$$E[h(X(T))|\mathcal{F}(t)] \equiv E[h(X(T))|X(t)]. \tag{7}$$

- Note that

$$g(t,X(t)) = E^{t,X(t)}h(X(T)) \equiv E[h(X(T))|X(t)].$$

  This indicates that $g(t,X(t))$ is a martingale (Levy martingale).

Feynman-Kac Theorem

- **Feynman-Kac Theorem:** Consider the SDE (5), its strong solution $X(t)$, a function $h(y)$, and

$$g(t,x) := E^{t,x}h(X(T))(< +\infty)$$

  given by (6). Then $g(t,x)$ satisfies the PDE

$$g_t(t,x) + \beta(t,x)g_x(t,x) + \frac{1}{2}\gamma^2(t,x)g_{xx}(t,x) = 0 \tag{8}$$

  and the terminal condition

$$g(T,x) = h(x) \quad \text{for any } x. \tag{9}$$

- **Remarks**:
  - The PDE (8) does not involve $h(\cdot)$.
  - $h(\cdot)$ is only involved in the terminal condition (9).

Feynman-Kac Theorem

- **Proof:** Applying Itô lemma to the process $g(t, X(t))$ and omitting the argument $(t, X(t))$ yield

$$
\begin{aligned}
dg(t, X(t)) =& g_t dt + g_x dX + \frac{1}{2} g_{xx} dX dX \\
=& \left[g_t + \beta g_x + \frac{1}{2}\gamma^2 g_{xx} \right] dt + \gamma g_x dW.
\end{aligned} \tag{10}
$$

- Since $g(t, X(t))$ is a martingale, there is no dt term in (10), as results in the PDE (8).
- The Key point to derive a PDE is
  - (1) construct a martingale involving a Markov process $X(t)$ that solves a SDE;
  - (2) apply Itô lemma;
  - (3) Set dt term to be 0.

Feynman-Kac Theorem: A discounted version

- Consider

$$
E\left[e^{-r(T-t)} h(X(T)) | \mathcal{F}(t) \right] =: f(t, X(t)).
$$

- Question: Is there any PDE that $f(t, x)$ solves?
- First, $f(t, X(t))$ is not a martingale because

$$
\begin{aligned}
E[f(t, X(t)) | \mathcal{F}(s)] =& E[E[e^{-r(T-t)} h(X(T)) | \mathcal{F}(t)] | \mathcal{F}(s)] \\
=& E[e^{-r(T-t)} h(X(T)) | \mathcal{F}(s)],
\end{aligned} \tag{11}
$$

  where the RHS depends on t.
- However, $e^{-rt} f(t, X(t))$ is a martingale.
- Apply Itô lemma to $e^{-rt} f(t, X(t))$ yields

$$
d(e^{-rt} f(t, X(t))) = e^{-rt} \left[-rf + f_t + \beta f_x + \frac{1}{2}\gamma^2 f_{xx} \right] dt + e^{-rt} \gamma f_x dW(t) \tag{12}
$$

Feynman-Kac Theorem

- Apply Itô lemma to $e^{-rt}f(t,X(t))$ yields

$$d(e^{-rt}f(t,X(t))) = e^{-rt}\left[-rf + f_t + \beta f_x + \frac{1}{2}\gamma^2 f_{xx}\right]dt + e^{-rt}\gamma f_x dW(t) \quad (13)$$

- Setting dt term to be zero leads to a PDE

$$f_t(t,x) + \beta(t,x)f_x(t,x) + \frac{1}{2}\gamma^2(t,x)f_{xx}(t,x) = rf(t,x) \quad (14)$$

and the terminal condition

$$f(T,x) = h(x) \quad \text{for any } x. \quad (15)$$

- **Remarks:**
  - The PDE (14) does not depend on $h(\cdot)$ and solely depends on $X(t))$, the Markov process that the payoff relies on.
  - $h(\cdot)$ only affects the terminal condition (15).

Change of Measure to Risk Neutral

In our study of option pricing under binomial lattice, we have

$$\mathbb{Q}(\omega) = \mathbb{P}(\omega)Z(\omega),$$

for a Radon-Nykodim derivative Z. Under $\mathbb{Q}$, we have $\mathbb{Q}(H) = \widetilde{p}$ and $\mathbb{Q}(T) = \widetilde{q}$.

The change of measure changes the likelihood of having a head and a tail. So, it leads to the change of expected price movement.

Can we find an analogy?

The Girsanov Theorem: One-dimensional Case

Theorem. Let $W(t)$, $0 \leq t \leq T$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t); 0 \leq t \leq T$, be a filtration for this Brownian motion. Let $\Theta(t), 0 \leq t \leq T$, be an adapted process. Define

$$Z(t) = \exp\left(-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du\right) \text{ and } \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = Z(T).$$

and

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)du.$$

Assume that

$$E\int_0^T \Theta^2(u)Z^2(u)du < \infty.$$

Then $\mathbb{E}^{\mathbb{P}} Z(T) = 1$ and under the probability measure $\widetilde{\mathbb{P}}$, the process $\widetilde{W}(t)$, $0 \leq t \leq T$ is a Brownian motion.

Understanding from a Simple Example

Rather than providing a theoretical proof, we try to understand/feel the Girsanov theorem from the following simple example.

How can we use change-of-measure to move the mean of a normal random variable?

- X is a standard normal random variable on a probability space $(\Omega, \mathcal{F}, P)$, θ is a constant.
- Define
$$Z = \exp\left(-\theta X - \frac{1}{2}\theta^2\right) \text{ and } \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = Z$$
- Under the probability measure $\widetilde{\mathbb{P}}$, the random variable $Y = X + \theta$ is a standard normal.
- In particular, $\mathbb{E}^{\widetilde{\mathbb{P}}} Y = 0$, whereas $\mathbb{E}^{\mathbb{P}} Y = \mathbb{E}^{\mathbb{P}} X + \theta = \theta$.

Risk-Neutral Representation of the BSM PDE Solution

- Let
$$\Theta(t) := \frac{\mu - r}{\sigma}$$
- We have
$$W^{\mathbb{Q}}(t) = W(t) + \int_0^t \frac{\mu - r}{\sigma} du = W(t) + \frac{\mu - r}{\sigma} t$$
is a Brownian motion under $\mathbb{Q}$ satisfying $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z(T)$.
- We have
$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma S(t)dW(t) \\ &= \mu S(t)dt + \sigma S(t)\left(dW^{\mathbb{Q}}(t) - \frac{\mu - r}{\sigma}dt\right) \\ &= rS(t)dt + \sigma S(t)dW^{\mathbb{Q}}(t). \end{aligned}$$
- $\Theta(t)$: the Sharpe ratio or market price of risk.

Risk-Neutral Representation of the BSM PDE Solution

- The probability measure $\mathbb{Q}$ is called the risk-neutral (martingale) measure.
- Under $\mathbb{Q}$, we have
$$dS(u) = rS(u)du + \sigma S(u)dW^{\mathbb{Q}}(u).$$
This is a special case of the general SDE: $\beta(u, x) = rx$ and $\gamma(u, x) = \sigma x$.
- Let
$$v(t, S(t)) = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}(S(T) - K)^+ | \mathcal{F}(t)].$$
- From (14), $v(t, x)$ solves the BSM PDE:
$$v_t(t, x) + rxv_x(t, x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t, x) = rv(t, x), \qquad (16)$$
with terminal condition $v(T, x) = (x - K)^+$.

Risk-neutral Valuation

- Under $\mathbb{Q}$, $e^{-rt}S(t)$ (the discounted underlying asset price), $e^{-rt}v(t, S(t))$ (the discounted option price) and $e^{-rt}X(t)$ (the discounted replicating portfolio value) are all martingales

- Risk-Neutral Representation of the BSM PDE Solution:

$$v(t, S(t)) = \mathbb{E}^{Q}[e^{-r(T-t)}(S(T) - K)^{+}|\mathcal{F}(t)]$$

- This expresses the option price as the risk-neutral expectation of the discounted payoff.

- Using the explicit solution of $S(T)$ to derive the Black-Scholes-Merton formula.

Derivation of the BSM Formula

Derivation of the BSM Formula

An Introduction to Monte Carlo Simulation

By Monte Carlo simulation, we compute the option price based on the risk-neutral representation

$$v(0, S(0)) = \mathbb{E}^Q[e^{-rT}(S(T) - K)^+].$$

- Generate a sample $S^{(i)}(T)$ according to the lognormal distribution
- Evaluate the discounted payoff by

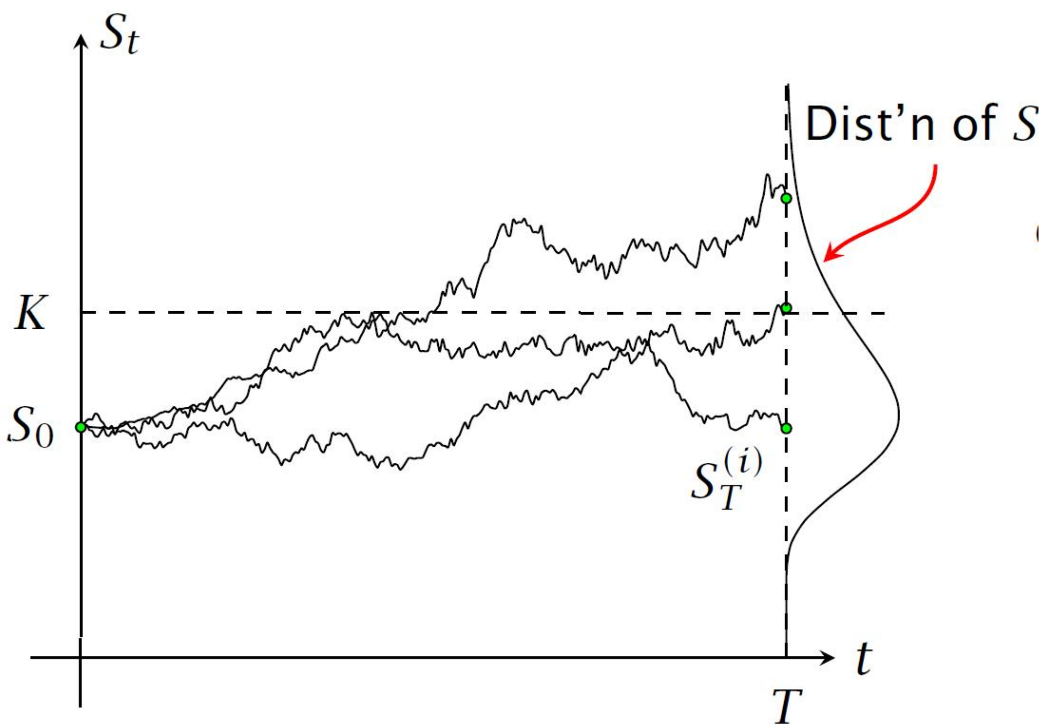
$$C^{(i)} = e^{-rT}(S^{(i)}(T) - K)^+$$

- Repeat and average $i = 1, 2, , ..., n$ simulation trials

By the law of large number,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} e^{-rT}(S^{(i)}(T) - K)^+ = \mathbb{E}^Q e^{-rT}(S(T) - K)^+ = v(0, S_0).$$

Monte Carlo Simulation



How to Sample $S(T)$

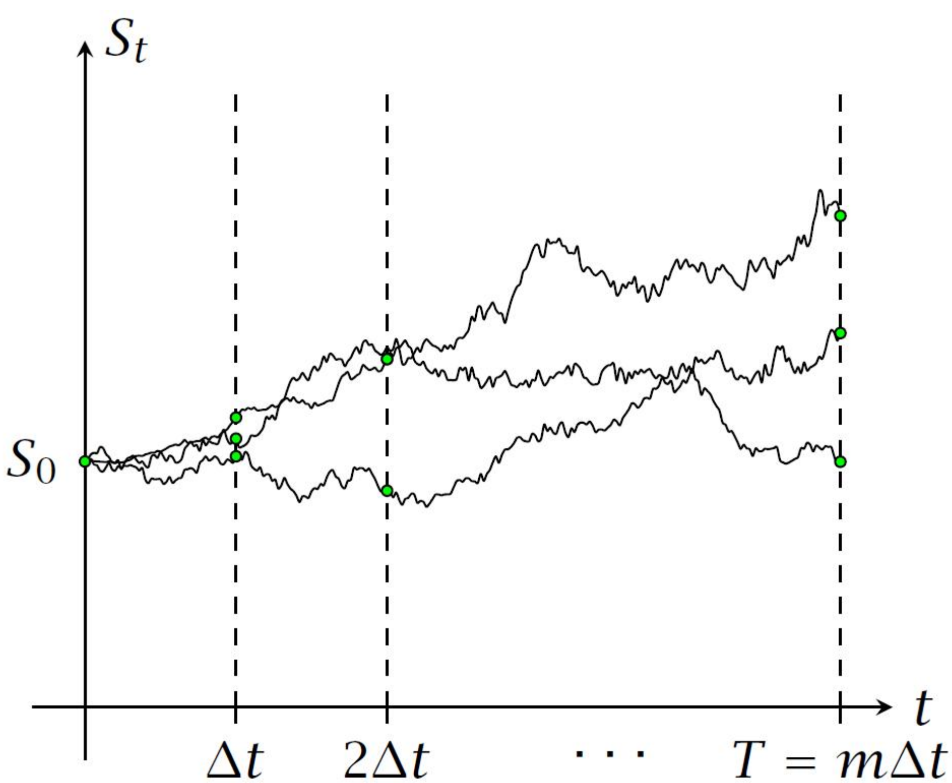
We need to perform the simulation under the risk-neutral distribution

- $$S(T) = S_0 \exp\left\{\sigma W(T) + \left(r - \frac{1}{2}\sigma^2\right) T\right\}$$
- We just need to sample
$$W(T) \sim \mathcal{N}(0, T).$$
- Further,
$$W(T) = \sqrt{T} Z,$$
where $Z \sim \mathcal{N}(0,1)$.We just need to sample a $\mathcal{N}(0,1)$ variable.
- Inverse transform method: We hope to sample a random variable with cdf $F(x)$. Computer can easily sample uniform distribution on $[0,1]$, denoted as U. We have
$$F^{-1}(U) \sim F.$$

Further Topics on Simulation

- Various sampling methods
- Estimation efficiency
- Variance reduction
- SDE Discretization (think about how do you simulate a general SDE solution?)
- etc.

Generating Sample Paths



Generating Sample Paths

For a general SDE:

$$dS(t) = \mu(t, S(t))dt + \sigma(t, S(t))dW(t),$$

- According to the exact distribution of $S((i+1)\Delta t)$ give $S(i\Delta t)$.
- In the case when this distribution is unknown, we consider the discretization approximation, e.g. Euler scheme:

  We apply the following approximation

  $$\widehat{S}(m) \approx S(m\Delta t).$$

  Here $\{\widehat{S}(i)\}$ is implemented by the following recursion:

  $$\widehat{S}(i+1) = \widehat{S}(i) + \mu(i\Delta t, \widehat{S}(i))\Delta t + \sigma(i\Delta t, \widehat{S}(i))\sqrt{\Delta t} Z_{i+1},$$

  where $Z_i \sim \mathcal{N}(0,1)$ for $i = 0, 1, ..., m-1$.

Supplementary Material

Suggested Reading Material (We only need to focus on the material parallel to our course slides):

- Selected material from Shreve Vol. II: 4.5.4, 5.2.5, 6.3, 6.4 or Mikosch 4.1
- Supplementary notes: “An Introduction to Partial Differential Equation”
- Supplementary notes: Section 2.4 in W. Strauss’ book “Partial Differential Equations: An Introduction”.

Suggested Exercises (Do Not Hand In; For Your Deeper Understanding Only)

- Shreve Vol. II: 4.10, 5.4, 5.9, 6.7, 6.8, 6.9, 6.10 (some of these are challenging questions)