Lec3 How to solve. Review continued [54]

Alternative way to solve 1st order linear ODE <u>variation of parameter/consts</u>

$$\frac{dy}{dt} + a(t)y = b(t)$$

Soln for homo part $Ce^{-\int a(t)dt}$

Assume $y(t) = \underline{C(t)}\, e^{-\int a(t)dt}$

Then $C'(t)e^{-\int a(t)} + C(t)e^{-\int a(t)}\cancel{(-a(t))} + \cancel{a(t)C(t)}e^{-\int a(t)dt} = b(t)$

$$\Rightarrow C(t) = \int b(t)e^{\int a(t)dt}dt + C$$

$$\Rightarrow y(t) = e^{-\int a(t)dt}\left[\int b(t)e^{\int a(t)dt}dt + C\right]$$

same soln as we obtained from integrating factor.

Form: $y(t) = \phi(t)\left[\int \phi(t)^{-1} b(t)dt + C\right]$, where $\phi(t)$ is a soln of the homo part.

Q: Why does it even work?

Philosophy: $\frac{dy}{dt} = -a(t)y + b(t) = y\left(-a(t) + \frac{b(t)}{y}\right)$

$$\frac{dy}{y} = \left(-a(t) + \frac{b(t)}{y}\right)dt$$

Formally $y = \pm e^{-\int a(t)dt} \cdot \underbrace{e^{\int \frac{b(t)}{y(t)}dt}}_{C(t)}.$

Rmk

This method works for linear nonhomo. Eq in general, we will discuss and prove in later class.

2nd order linear ODE (homo)

$$\ddot{y} + a(t)\dot{y} + b(t)y = 0 \qquad (\star) \qquad a(t), b(t) \text{ cts in } I.$$

<u>linear superposition</u> $L[y] \triangleq \ddot{y} + a(t)\dot{y} + b(t)y$

$$L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2]$$

So clearly, if y_1 and y_2 are solns, then $c_1y_1 + c_2y_2$ is also a soln.

Q: What is the structure of soln sp?

Think of $\ddot{y}=0$ soln is $\dot{y}=C_1$, $y=C_1t+C_2$ To chose the most "representative" $y_1(t)$ and $y_2(t)$.

There exists exactly two solutions $y_1(t)$ and $y_2(t)$ of ($\star$) which are linear indep in I

linear depend def: $\exists C_1, C_2$ not both 0 s.t. $C_1y_1+C_2y_2=0$

Thm Two solns $y_1(t)$ and $y_2(t)$ of ($\star$) on I, Def $W(t) \overset{\Delta}{=} \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$ Wronskian

Then ① $y_1(t)$ and $y_2(t)$ are linear dep $\Rightarrow$ $W(t)\equiv 0$ on I.

② . . . . indep $\Rightarrow$ $W(t)\neq 0$ at each pt of I.

Thus there are only 2 possibilities: either $W(t)\equiv 0$ on I or $W(t)\neq 0$ everywhere on I.

(not discussed in class, we will revisit the pf later for linear systems.)

Pf: ① linear dep. means $\exists C_1, C_2$ not both 0 s.t.

$$C_1y_1+C_2y_2=0$$

WLOG $C_1\neq 0$, $y_1=-\frac{C_2}{C_1}y_2$ Plug in $W(t)=0$

② Assume $\exists t_0$ s.t. $W(t_0)=0$.

$\exists C_1, C_2$ not both 0 s.t. $\begin{cases} C_1y_1(t_0)+C_2y_2(t_0)=0 \\ C_1y_1'(t_0)+C_2y_2'(t_0)=0 \end{cases}$

construct $y(t)=C_1y_1(t)+C_2y_2(t)$

clearly $y(t_0)=0$ and $y'(t_0)=0$ and $y(t)$ satisfies Eq ($\star$)

So $y(t)\equiv 0$. (by uniqueness thm)

In parctice, given 2 solns $y_1(t)$ and $y_2(t)$, to determine where they are indep. ("representative enough"), we check Wronskian $W(t)=\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$ is 0 not not.

Now let us discuss specific ways to find y_1 and y_2.

2nd order scalar linear Eq. (w/ const. coefficients) linear homo

$$a\ddot{y}+b\dot{y}+cy=0$$

Try $y(t) = e^{rt}$

$ar^2 + br + c = 0$ characteristic Eq.

Three cases

Case I : $b^2 - 4ac > 0$ Distinct real roots

roots r_1 and r_2

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Case II : $b^2 = 4ac$ repeated real roots

roots $r_1 = r_2 = r$

$$y(t) = C_1 e^{r_1 t} + C_2 t e^{r_2 t}$$

Case III : $b^2 - 4ac < 0$ complex conj roots

$r_1 = u + iv$, $r_2 = u - iv$ $i = \sqrt{-1}$

$$y(t) = C_1 e^{ux} \cos vx + C_2 e^{ux} \sin vx .$$

Ex1

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 0$$

$$y = Ce^{2t} + te^{2t}$$

Ex2 $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 13y = 0$

$r = -3 \pm 2i$

$$y = Ce^{-3t}\cos(2t) + De^{-3t}\sin(2t)$$

Generally, for n-th order linear homo ODE, we have the following thm

Thm If λ is a root of multiplicity k of the characteristic Eq $P(r) \triangleq a_0 r^n + a_1 r^{n-1} + \cdots a_{n-1} r + a_n$ then the fxn $t^m e^{\lambda t}$, where $m = 0, \cdots, k-1$ are solns of

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n = 0$$ i.e. $P(D)y = 0$

Operator Calculus viewpt — A proof (optional)

Def linear op D acts on diff'ble fxn. f by

$$(Df)(x) = f'(x)$$

One can rewrite const coefficients ODE w/ D.

As a op, D is linear,

$$\begin{cases} D(u+v) = Du + Dv \\ D(cu) = cDu \end{cases} \quad (c \text{ const})$$

Lemma 1 (permutable)

$P(D) = \sum a_j D^j$, $q(D) = \sum b_k D^k$ are two linear differential op, where a_j, b_k consts.

Then

$$P(D)q(D) = q(D)P(D) = \sum a_j b_k D^{j+k}$$

Lemma 2 (Exponential Shift law) If p is a poly, λ is a const then

$$P(D)(e^{\lambda t} f) = e^{\lambda t} P(D+\lambda) f$$

Pf:

$$D(e^{\lambda t} f) = \lambda e^{\lambda t} f + e^{\lambda t} Df = e^{\lambda t}(D+\lambda) f.$$

$$\Rightarrow (D-a)(e^{\lambda t} f) = e^{\lambda t}(D+\lambda-a) f$$

$$(D-a)^2(e^{\lambda t} f) = (D-a)\left[e^{\lambda t}(D+\lambda-a) f\right]$$

$$= e^{\lambda t}(D+\lambda-a)^2 f$$

$$\cdots$$

$$(D-a)^k(e^{\lambda t} f) = e^{\lambda t}(D+\lambda-a)^k f$$

Because any poly. $P(D)$ can be written as

$$P(D) = (D-a_1)^{k_1}(D-a_2)^{k_2}\cdots(D-a_n)^{k_n}$$

$$(D-a_n)^{k_n}(e^{\lambda t} f) = e^{\lambda t}(D+\lambda-a_n)^{k_n} f$$

Apply $(D-a_{n-1})^{k_{n-1}}$ to both sides

$$(D-a_{n-1})^{k_{n-1}}(D-a_n)^{k_n}(e^{\lambda t} f) = e^{\lambda t}(D+\lambda-a_{n-1})^{k_{n-1}}(D+\lambda-a_n)^{k_n} f$$

$$\cdots$$

$$P(D) = e^{\lambda t} P(D+\lambda) f \quad \square.$$

Proof of Thm: $(D-\lambda)^k(t^r e^{\lambda t}) = e^{\lambda t} D^k t^r = 0$ by exp shift law. $r = 0, 1, \cdots, k-1$.

On the other hand $p(r)$ must contain factor $(r-\lambda)^k$

$$p(r) = (r-\lambda)^k q(r)$$

$$\Rightarrow P(D) = (D-\lambda)^k q(D) = q(D)(D-\lambda)^k \quad \Rightarrow P(D)(t^r e^{\lambda t}) = q(D)\left[(D-\lambda)^k(t^r e^{\lambda t})\right] = 0 \quad \square.$$

What about $\underset{\sim}{\text{nonhomo}}$ 2nd order linear ODE?

- undetermined coefficients / variation parameter (not discuss in class)

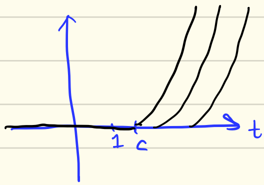
Need for theory

Exercise

1. $$\begin{cases} (y')^2 + y^2 + 1 = 0 \\ y(0) = 1 \end{cases}$$

Any solution? No soln!

2. Solve: $$\begin{cases} \frac{dy}{dt} = y^{\frac{1}{3}} \\ y(1) = 0 \end{cases}$$



Check:

$$y(x) = \begin{cases} 0 & t \leq c \\ \pm\left(\frac{2}{3}\right)^{\frac{3}{2}}(t - c)^{\frac{3}{2}} & t > c \end{cases}$$ for any $c \geq 1$ are solutions.

Q1: What is the weakest condition to ensure $\boxed{\text{local existence}}$ of IVP?

$$\frac{dy}{dt} = f(t,y), \quad y(t_0) = y_0$$

solution $y(t)$ exists on some interval containg t_0.

Q2: Under what condition the solution is $\boxed{\text{unique}}$?

To be continued next time! Theory!