Lec 21 Nonlinear system — Almost linear : Linearization. [BN 4.4]

Recall last time **Warmup**

① $\frac{dy}{dt} = Ay$ ② $\frac{dy}{dt} = (A + B(t))y$

zero soln stability Q1: A sufficient condition s.t. zero soln is asym. stable?

Ans $\lim_{t\to\infty} B(t) = 0$

③ Q2.

For linear system $\frac{dy}{dt} = A(t)y$

What is the relation of zero soln stability and any soln $\phi(t)$. explain why?

Ans $\forall \varepsilon > 0$, $\exists \delta$ s.t. $|y(t_0) - \phi(t_0)| < \delta \Rightarrow |y(t) - \phi(t)| < \varepsilon$

$z(t) \triangleq y(t) - \phi(t)$ still a soln to (LE)

$\Longleftrightarrow |z(t_0) - 0| < \delta \Rightarrow |z(t) - 0| < \varepsilon$.

Nonlinear autonomous system

$F \in C^1(D)$

F cts & has its first partial derivatives in a domain $D \subset \mathbb{R}^n$.

$y' = F(y)$ $y = y^*$ is an equilibrium soln

Linearization

$z(t) = y(t) - y^*$ may no longer be a soln of (NLE)

$$\frac{dz}{dt} = \frac{dy}{dt} = F(y^* + z(t))$$

Q: properties of $g(z)$?

① cts ② $g(0) = 0$.

$$= \underbrace{F(y^*)}_{0 \text{ b/c } y^* \text{ is equilibrium}} + D_yF(y^*)z + \underbrace{g(z)}_{}$$

Higher order terms. $o(|z|)$

$$D_yF(y^*) = \left.\begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} & \cdots & \frac{\partial F_1}{\partial y_n} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} & \cdots & \frac{\partial F_2}{\partial y_n} \\ \vdots & & & \\ \frac{\partial F_n}{\partial y_1} & \frac{\partial F_n}{\partial y_2} & \cdots & \frac{\partial F_n}{\partial y_n} \end{pmatrix}_{n\times n}\right|_{y=y^*}$$

$F = (F_1, \ldots, F_n)$

$y = (y_1, \ldots, y_n)$

Const matrix!

Q: Guess what is a condition for (NLE) "behave like" its linearization?

<u>Ans</u> $\lim_{z\to 0} \frac{|g(z)|}{|z|} = 0$

$g(z) = o(|z|)$ as $z \to 0$.

"$g(z) \to 0$ faster than $|z|$"

Intuitively $\frac{dy}{dt} = Ay + B(t)y$

needs $\lim_{t\to\infty} |B(t)| = 0$

Now $\frac{dy}{dt} = Ay + g(z) \Rightarrow \frac{g(z)}{z} \to 0$.

This is exactly the def of h.o.t.

Consider a slightly more general prob. (linearization of $y' = F(t,y)$) $F(t,0)$

$$\frac{dy}{dt} = Ay + f(t,y) \qquad \text{(NL-Per)}$$

A const matrix, f cts in (t,y) in $D = \{(t,y) \mid 0 \le t < \infty, |y| < k\}$ for some const k.

$f(t,0) = 0$ and $\lim_{y\to 0} \frac{|f(t,y)|}{|y|} = 0$.

Thm Spse all e-val of A have neg real parts. $f(t,y)$, ~~$\frac{\partial f}{\partial y_j}$ $(j=1,\dots,n)$~~ are cts in (t,y) for $0 \le t < \infty$, $|y| < \tilde{k}$ where $\tilde{k} > 0$ is a const, and f is small in the sense:

$$\lim_{y\to 0} \frac{|f(t,y)|}{|y|} = 0 \quad \text{☆ unif wrt } t \text{ on } 0 \le t < \infty.$$

Then the solution $y \equiv 0$ of (NL-Per) is asym. stable.

(If $\exists$ some e-val w/ pos real part $\Rightarrow$ unstable)

Pf: WTS: "$\to 0$" & $|y(t_0)| < \delta \Rightarrow |y(t)| < \varepsilon$ "Volterra Integral Eq"

$$y(t) = e^{(t-t_0)A} y_0 + \int_{t_0}^{t} e^{(t-s)A} f(s, y(s))\,ds$$

$$|y(t)| \le K e^{-\sigma(t-t_0)} |y_0| + \int_{t_0}^{t} K e^{-\sigma(t-s)} |f(s,y(s))|\,ds$$

$\lim_{y\to 0} \frac{|f(t,y)|}{|y|} = 0 \Rightarrow \forall \eta > 0, \exists \alpha > 0$ s.t. $|y| < \alpha \Rightarrow |f(t,y)| < \eta |y|$

(naturally always take $\alpha \le \tilde{k}$)

[Skip BN Thm 4.5]

$y' = Ay + f(t,y) + h(t,y)$
$\downarrow$
$o(|y|)$

" $\times e^{\sigma t}$

$$e^{\sigma t}|y(t)| \le Ke^{\sigma t_0}|y_0| + \int_{t_0}^{t} Ke^{\sigma s}|f(s,y(s))|\,ds$$

Claim: If $|y(t_0)| < \delta \triangleq \min\{\frac{\alpha}{2K}, \frac{\alpha}{2}\} \Rightarrow |y(t)| < \alpha$ for all $0 \le t < \infty$. (will show later).

Spse claim holds, then

$$e^{\sigma t}|y(t)| \le Ke^{\sigma t_0}|y_0| + \int_{t_0}^{t} K\eta e^{\sigma s}|y(s)|\,ds \qquad (\star)$$

By Gronwall

$$\Rightarrow e^{\sigma t}|y(t)| \le Ke^{\sigma t_0}|y_0| \exp(K\eta(t-t_0))$$

$$|y(t)| \le K|y_0| e^{(-\sigma + K\eta)(t-t_0)}$$

Take η s.t. $-\sigma + K\eta < 0$. Done!

Bootstrap:

Assume $|y(t)| < \alpha$ small for some $[t_0, t^*]$

$\Rightarrow |y(t)| \le K|y_0|$ Take $|y_0| \le \min\{\frac{\alpha}{2K}, \frac{\alpha}{2}\}$

$\le \frac{\alpha}{2}$ improved result. $\square$

Pf of the claim: Bootstrap argument (Continuity argument)

pf by contradiction.

Assume $|y(t)| < \alpha$ does not hold for all $0 \le t < \infty$.

Assume $|y(t)| \le \alpha$ for $t_0 \le t \le t^*$ and $|y(t^*)| = \alpha$ for some $t^* < \infty$. (b/c y is cts)

This interval is not empty, b/c $|y(t_0)| \le \frac{\alpha}{2} < \alpha$

Consider $t \in [t_0, t^*]$, we can use $(\star)$ b/c $|y(t)| \le \alpha$ on $[t_0, t^*] \Rightarrow |f| \le \eta|y|$

we have

$$|y(t)| \le K|y_0| e^{(-\sigma + K\eta)(t-t_0)}$$

$$\le K|y_0| \le \frac{\alpha}{2}$$

It is impossible that $|y(t^*)| = \alpha$. contradiction! $\square$.

⓪ Similar result holds true for $y' = (A + B(t))y + f(t,y)$.

① Ex 2

Rmk Continuation Thm used a similar argument of pf by contradiction

If can not extend to ∂D, Then assume soln exist up to $t_0 \le t \le t^*$ where $(t^*, y(t^*)) \notin \partial D$.

Then by extension thm, must can extend soln to $t^* + \delta$, contradiction!

② Ex 2. Peano's thm. (the pf relies on this, though we did not show it in class).

"Dispersive wave"

Bootstrap / Continuity Argument

Further notes ① Ben-Artzi (Imperial College London)
② Terence Tao "Nonlinear dispersive eq"

One illustrative proposition

$f: [0,\infty) \to [0,\infty)$ cts . fix a const $C>0$. Spse the following cond. hold:

① $f(0) \le C$

② Whenever $f(t) \le 4C$, then we have in fact $f(t) \le 2C$

Then $f(t) \le 4C$ (and hence $f(t) \le 2C$) for all $t \in [0,\infty)$

Pf: Spse $f(t) \le 4C$ holds true for $0 \le t \le t^*$ where $t^* < \infty$
and $f(t^*) = 4C$

By ① $[0, t^*]$ is not empty

By ② $f(t) \le 2C$ for $t \in [0, t^*]$

contradicts w/ $f(t^*) = 4C$ $\square$.

Idea: assume what you want to prove & then to prove a strictly better version of what you assumed.

Ex Consider $\frac{dy}{dt} = \frac{y^2}{1+t^2}$ $\quad y(0) = y_0 \in \mathbb{R}$. 1D.

WTS: If $|y_0|$ is sufficient small, then the sol'n is global, i.e. $y(t)$ is defined for all $t \in \mathbb{R}$.

Furthermore, $y(t)$ is unif'ly small.

Pf: Suffices to show $y(t)$ is unif.ly small, then RHS bdd, by continuation thm, can extend.

$$f(t) \triangleq \sup_{0 \le s \le t} |y(s)| \quad \text{cts} \qquad \text{Consider } |y(t)| \text{ cts}$$

Spse $|y_0| \le \varepsilon$ small. WTS: $f(t) \le \varepsilon$ for all $t \in [0,\infty)$

Assume $|y(t)| \le 4\varepsilon$ WTS: Improved bdd for $|y(t)|$

Consider $t \ge 0$

$$|y(t)| \le |y_0| + \int_0^t \frac{|y(s)|^2}{1+s^2}\, ds \le \varepsilon + 16\varepsilon^2 \underbrace{\int_0^t \frac{1}{1+s^2}\, ds}_{\le C \text{ Const.}}$$

$\Rightarrow |y(t)| \le \varepsilon + 16 C \varepsilon^2 \le 2\varepsilon$ for ε sufficiently small.

$\Rightarrow |y(t)| \le 2\varepsilon$ for all $t \ge 0$. Similarly for $t \le 0$.

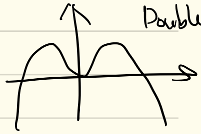
~~HW~~ (Altered from Terence Tao Notes)   V is sufficiently smooth $\in C^{\infty}(\mathbb{R})$

$$\frac{dy}{dt} = -V'(y) \qquad y(0) = y_0 \qquad V(0)=0,\ V'(0)=0,\ V''(0)>0.$$

WTS: For $|y_0|$ sufficiently small.
The unique soln exists globally.
& This soln stays bdd unif in t.

Example
$V(x) = -\frac{1}{4}x^4 + \frac{1}{2}x^2$
$V'(x) = x^3 + x$

Doublewell



Pf: Suffices to show $|y(t)|$ bdd unif in t. ⇒ RHS bdd. Then by continuation of soln. ⇒ global.

$$E(t) = \frac{(y(t))^2}{2} + V(y(t))$$

$$\frac{dE}{dt} = y(t)\,y'(t) + V'(y(t))\,y'(t) = 0$$

$$E(t) = E(0) = \frac{y_0^2}{2} + V(y_0) \quad \le \varepsilon^2$$
is suff.ly small due to smallness of $|y_0|$.

Prob Cannot conclude $y(t)$ is small from $E(t)$ is small b/c V could be negatively away from origin
(But this only occurs when y is large
⇒ Thus we need to assume y small to prove y small.

Spse $|y(t)| \le 2\varepsilon$

$$V(y) = V''(0)\,y^2 + O(|y|^3) \ \ge\ c\,y^2(t) + O(|y|^3)$$

$$E(0) \ge \frac{|y(t)|^2}{2} + c\,y^2(t) + O(|y|^3)$$

$\frac{y^2(t)}{2} \le E(0) + \frac{1}{4}y^4(t) - \frac{1}{2}y^2(t)$

$y^2(t) \le E(0) + \frac{1}{4}y^4(t) \le \varepsilon^2 + 4\varepsilon^4$ by $E(0)$ & BS asmp

$$\Rightarrow \left(\tfrac{1}{2}+c\right)|y(t)|^2 \le E(0) - O(|y|^3) \le E(0) + O(\varepsilon^3)$$

$\le 2\varepsilon^2$ for ε small enough

Choose $E(0)$ sufficiently small

$\Rightarrow y(t) \le \sqrt{2}\,\varepsilon$ improved. □

Ex 2. $y(t)$ satisfies $y'(t) = f(t,y)$, $y(t_0) = y_0$
$|y(t)| \le A + B|y(t)|^p$ $y(t)$ ets $A, B > 0$, $0 < p < 1$.   WTS: soln ∃ globally & stays bdd unif.

Pf: Spse $|y(t)| \le 4C$ const   ($y(t_0)$ certainly bdd by const)
⇒ $|y(t)| \le A + 4BC^p$ which for large enough C implies $|y(t)| \le 2C$ □.