Finite-Dimensional Distribution of One-Dimensional Brownian Motion

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Suppose W is a one-dimensional Brownian motion. For $0 < t_1 < t_2 < \cdots < t_n \leq T$, the joint density of $(W(t_1), W(t_2), \ldots, W(t_n))$ is the product of n transition densities from $W(t_{i-1})$ to $W(t_i)$, $i = 1, 2, \ldots, n$:

$$p(w_1, w_2, w_3, \ldots, w_n) = p(w_n|w_{n-1})p(w_{n-1}|w_{n-2}) \ldots p(w_2|w_1)p(w_1), \tag{1}$$

where

$$P(W(t_i) \in dw_i | W(t_{i-1}) = w_{i-1}) = p(w_i|w_{i-1})dw_i.$$

Note that this is true for any one-dimensional Markov process.

For continuous random vectors, we have

$$p(\mathbf{x}, \mathbf{y}) = p_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y})p_{\mathbf{Y}}(\mathbf{y}), \tag{2}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are m and n-dimensional random vectors, $m = 1, 2, \ldots,$ $n = 1, 2, \ldots$. In particular, when $m = n = 1$, we have

$$p(x, y) = p_{X|Y}(x|y)p_Y(y).$$

In physics notations, we have

$$P(\mathbf{X} \in \mathbf{dx}, \mathbf{Y} \in \mathbf{dy}) = P(\mathbf{X} \in \mathbf{dx}|\mathbf{Y} = \mathbf{y})P(\mathbf{Y} \in \mathbf{dy}), \tag{3}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are m and n-dimensional random vectors, $m = 1, 2, \ldots,$ $n = 1, 2, \ldots,$ $\mathbf{dx} = (dx_1, dx_2, \ldots, dx_m)'$, $\mathbf{dy} = (dy_1, dy_2, \ldots, dy_n)'$.

Indeed, it is very easy to verify that (2) and (3) are equivalent. This is simply because

$$P(\mathbf{X} \in \mathbf{dx}, \mathbf{Y} \in \mathbf{dy}) = P(\mathbf{X} \in \mathbf{dx}|\mathbf{Y} = \mathbf{y})P(\mathbf{Y} \in \mathbf{dy})$$

is equivalent to

$$p(\mathbf{x}, \mathbf{y})dx_1dx_2 \ldots dx_m dy_1 dy_2 \ldots dy_n = p_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y})dx_1dx_2 \ldots dx_m \cdot p_{\mathbf{Y}}(\mathbf{y})dy_1dy_2 \ldots dy_n$$

and further equivalent to

$$p(\mathbf{x}, \mathbf{y}) = p_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y})p_{\mathbf{Y}}(\mathbf{y}).$$

Denote by

$$p(w_j|w_1, w_2, \ldots, w_{j-1})dw_j = P(W(t_j) \in dw_j|W(t_1) = w_1, W(t_2) = w_2, ..., W(t_{j-1}) = w_{j-1}).$$

Thus, we use this principle to iteratively deduce that

$$\begin{aligned}
& p(w_1, w_2, w_3, \ldots, w_n) \\
= \; & p(w_n|w_1, w_2, \ldots, w_{n-1})p(w_1, w_2, \ldots, w_{n-1}) \\
= \; & p(w_n|w_{n-1})p(w_{n-1}|w_1, w_2, \ldots, w_{n-2})p(w_1, w_2, \ldots, w_{n-2}) \\
= \; & \ldots \\
= \; & p(w_n|w_{n-1})p(w_{n-1}|w_{n-2}) \ldots p(w_2|w_1)p(w_1).
\end{aligned}$$

An alternative way to obtain the joint density $p(w_n, w_{n-1}, w_{n-2}, \ldots, w_1)$ is to calculate the mean vector and the covariance matrix. It is easy to know that the mean vector is zero, i.e.,

$$(EW(t_n), EW(t_{n-1}), \ldots, EW(t_1)) = (0, 0, \ldots, 0).$$

It is easy to find the covariance matrix as

$$\Sigma = (\operatorname{cov}(W(t_i), W(t_j)))_{n\times n} = (\min(t_i, t_j))_{n\times n}.$$

By using the formula of the joint density of a multivariate normal distribution, we have

$$p(\mathbf{w}) = p(w_1, w_2, w_3, \ldots, w_n) = (2\pi)^{-\frac{n}{2}} (\det \Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\mathbf{w}^\top \Sigma^{-1}\mathbf{w}\right) \tag{4}$$

Then, by factorization, we can obtain (1). We can see that this involves a lot of efforts.

The factorization procedure is as follows. First, we factorize the determinant $\det \Sigma$:

$$\begin{aligned}
\det \Sigma &= \begin{vmatrix} t_1 & t_1 & \cdots & t_1 & t_1 \\ t_1 & t_2 & \cdots & t_2 & t_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_1 & t_2 & \cdots & t_{n-1} & t_{n-1} \\ t_1 & t_2 & \cdots & t_{n-1} & t_n \end{vmatrix} \\
&= \begin{vmatrix} t_1 & t_1 & \cdots & t_1 & t_1 \\ 0 & t_2 - t_1 & \cdots & t_2 - t_1 & t_2 - t_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t_{n-1} - t_{n-2} & t_{n-1} - t_{n-2} \\ 0 & 0 & \cdots & 0 & t_n - t_{n-1} \end{vmatrix} \\
&= (t_n - t_{n-1})(t_{n-1} - t_{n-2}) \cdots\cdots (t_2 - t_1)t_1.
\end{aligned}$$

Second, we factorize $\exp\left(-\frac{1}{2}w^\top\Sigma^{-1}w\right)$, which is equivalent to decomposing $w^\top\Sigma^{-1}w$ to the sum of n terms. We have

$$\Sigma^{-1} = \begin{bmatrix} \frac{1}{t_1}+\frac{1}{t_2-t_1} & -\frac{1}{t_2-t_1} & 0 & \cdots & 0 & 0 \\ -\frac{1}{t_2-t_1} & \frac{1}{t_2-t_1}+\frac{1}{t_3-t_2} & -\frac{1}{t_3-t_2} & \cdots & 0 & 0 \\ 0 & -\frac{1}{t_3-t_2} & \frac{1}{t_3-t_2}+\frac{1}{t_4-t_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{t_{n-1}-t_{n-2}}+\frac{1}{t_n-t_{n-1}} & -\frac{1}{t_n-t_{n-1}} \\ 0 & 0 & 0 & \cdots & -\frac{1}{t_n-t_{n-1}} & \frac{1}{t_n-t_{n-1}} \end{bmatrix}.$$

Let

$$\Gamma = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

then

$$\Gamma^\top w = \begin{bmatrix} w_1 \\ w_2 - w_1 \\ \vdots \\ w_{n-1} - w_{n-2} \\ w_n - w_{n-1} \end{bmatrix}.$$

Let

$$\Upsilon = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

then

$$\Gamma\Upsilon = I,$$

and

$$\Upsilon\Sigma^{-1}\Upsilon^\top = \begin{bmatrix} \frac{1}{t_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{t_2-t_1} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{t_3-t_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{t_n-t_{n-1}} \end{bmatrix}.$$

Therefore,

$$\begin{aligned}
& \mathbf{w}^\top \Sigma^{-1} \mathbf{w} \\
= \; & \mathbf{w}^\top (\Gamma\Upsilon) \Sigma^{-1} (\Gamma\Upsilon)^\top \mathbf{w} \\
= \; & (\Gamma^\top \mathbf{w})^\top (\Upsilon \Sigma^{-1} \Upsilon^\top)(\Gamma^\top \mathbf{w}) \\
= \; & \begin{bmatrix} w_1 \\ w_2 - w_1 \\ \vdots \\ w_{n-1} - w_{n-2} \\ w_n - w_{n-1} \end{bmatrix}^\top \begin{bmatrix} \frac{1}{t_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{t_2 - t_1} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{t_3 - t_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{t_n - t_{n-1}} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 - w_1 \\ \vdots \\ w_{n-1} - w_{n-2} \\ w_n - w_{n-1} \end{bmatrix} \\
= \; & \frac{w_1^2}{t_1} + \frac{(w_2 - w_1)^2}{t_2 - t_1} + \cdots + \frac{(w_n - w_{n-1})^2}{t_n - t_{n-1}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& (2\pi)^{-\frac{n}{2}} (\det \Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} w^\top \Sigma^{-1} w\right) \\
= \; & (2\pi)^{-\frac{n}{2}} ((t_n - t_{n-1})(t_{n-1} - t_{n-2}) \cdots (t_2 - t_1) t_1)^{-\frac{1}{2}} \\
& \times \exp\left(-\frac{1}{2}(\frac{w_1^2}{t_1} + \frac{(w_2 - w_1)^2}{t_2 - t_1} + \cdots + \frac{(w_n - w_{n-1})^2}{t_n - t_{n-1}})\right) && (5) \\
= \; & (2\pi)^{-\frac{1}{2}} (t_n - t_{n-1})^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{(w_n - w_{n-1})^2}{t_n - t_{n-1}}\right) \cdots (2\pi)^{-\frac{1}{2}} (t_2 - t_1)^{-\frac{1}{2}} && (6) \\
& \times \exp\left(-\frac{1}{2} \frac{(w_2 - w_1)^2}{t_2 - t_1}\right) \cdot (2\pi)^{-\frac{1}{2}} t_1^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{w_1^2}{t_1}\right). && (7)
\end{aligned}$$

The transition density from $W(t_{i-1})$ to $W(t_i)$, $i = 1, 2, \ldots, n$, is

$$\begin{aligned}
p(w_i | w_{i-1}) &= p(w_i, w_{i-1}) / p(w_{i-1}) \\
&= (2\pi)^{-\frac{1}{2}} (t_i - t_{i-1})^{-\frac{1}{2}} \exp\left(-\frac{(w_i - w_{i-1})^2}{2(t_i - t_{i-1})}\right). && (8)
\end{aligned}$$

By combining (4), (6) and (8), we have

$$p(w_1, w_2, w_3, \ldots, w_n) = p(w_n | w_{n-1}) p(w_{n-1} | w_{n-2}) \ldots p(w_2 | w_1) p(w_1).$$