

#### Mathematical Methods in Finance

# Lecture 7: Stochastic Differential Equations and Financial Applications

#### Fall 2013

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### Overview

- ► Stochastic Differential Equations (SDE)
- ► Examples in Financial Modeling

## Stochastic Differential Equations (SDEs)

▶ Definition: A one-dimensional Stochastic Differential Equation (SDE) is an equation of the form

$$dX(t) = \beta(t, X(t))dt + \gamma(t, X(t))dW(t). \tag{1}$$

- $ightharpoonup \beta(t,x)$ : drift;
- $\gamma(t,x)$ : diffusion;
- ▶ X(0) = x for  $t \ge 0$  and  $x \in \mathcal{R}$ : the initial condition.
- ► Similarly define SDEs with multiple driving Brownian motions
- ► Similarly define multidimensional SDEs



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## Two types of solutions

- ► A strong solution is a process that solves the dynamic (2) on a given probability space (the driving Brownian motion is given as an input);
- ► A weak solution consists of a probability space and a process on it that solves the dynamic (2).
- ▶ strong solution ⇒ weak solution



▶ Existence and Uniqueness of Strong Solution If there exist two constants C and D s.t. for any  $t \in [0, T]$  and  $x \in \mathcal{R}$ ,

- $|\beta(t,x)| + |\gamma(t,x)| \le C(1+|x|);$
- $|\beta(t,x) \beta(t,y)| + |\gamma(t,x) \gamma(t,y)| \le D|x y|.$

The SDE admits a unique strong solution!

- Generally speaking, a SDE is not easy to solve, but sometimes we can solve it explicitly.
- ► Sometimes, numerical computing (e.g. Monte Carlo simulation) are necessary!



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## Linear SDEs

SDE:

$$dX(t) = \beta(t, X(t))dt + \gamma(t, X(t))dW(t).$$
 (2)

with

- ▶ Drift  $\beta(t,x) = a(t) + b(t)x$ ;
- ▶ Diffusion  $\gamma(t,x) = \gamma(t) + \sigma(t)x$ ; condition.
- e.g. One-dimensional linear SDEs:

$$dX(t) = [a(t) + b(t)X(t)]dt + [\gamma(t) + \sigma(t)X(t)]dW(t).$$

## **Example 1: (Generalized) Geometric Brownian Motion for Modeling Asset Price**

ightharpoonup S(t) satisfies SDE:

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), S(0) = s_0$$

- ▶ Modeling issue:  $\alpha(t)$  is instantaneous mean rate of return, and  $\sigma(t)$  is volatility.
- ▶ Both  $\alpha(t)$  and  $\sigma(t)$  could be very general adapted stochastic processes.
- ▶ If  $\alpha(t)$  and  $\sigma(t)$  are both constants  $\Longrightarrow$  Black-Scholes-Merton model (1973)
- ► Explicit solution:

$$S(t) = s_0 e^{\int_0^t \sigma(u)dW(u) + \int_0^t \left(\alpha(u) - \frac{1}{2}\sigma^2(u)\right)du}.$$



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## Examples in Financial Modeling: the Vasicek Model

#### **Example 2: Vasicek Model for Interest Rate**

$$dR(t) = (\alpha - \beta R(t))dt + \sigma dW(t).$$

- ▶ When  $\alpha = 0$ , R(t) is called an Ornstein-Uhlenbeck
- Equivalently written as

$$dR(t) = \kappa(\theta - R(t))dt + \sigma dW(t).$$

process.

- $\blacktriangleright$   $\kappa$ : mean-reverting speed
- $\blacktriangleright$   $\theta$ : mean-reverting level

How to solve is?

If RHS does not involve R(t), the integral form of R(t) is ready. So our objective is to remove R(t) on the RHS.



## Recall from Ordinary Differential Equation

Recall ODE

$$\frac{df(x)}{dx} = -af(x) + g(x),$$

where g(x) is known. We have that

$$df(x) + af(x)dx = g(x)dx,$$

and

$$e^{ax}df(x) + ae^{ax}f(x)dx = e^{ax}g(x)dx,$$

i.e.,

$$d[e^{ax}f(x)] = e^{ax}g(x)dx.$$

**Therefore** 

$$f(x) = e^{-ax} \left[ f(0) + \int_0^x e^{as} g(s) ds \right].$$



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## Examples in Financial Modeling: the Vasicek Model

► Similarly, multiply

$$dR(t) = (\alpha - \beta R(t))dt + \sigma dW(t)$$

by  $e^{\beta t}$ . Then Itô lemma applies

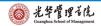
$$d\left[e^{\beta t}R(t)\right] = e^{\beta t}dR(t) + \beta e^{\beta t}R(t)dt = e^{\beta t}\alpha dt + e^{\beta t}\sigma dW(t)$$

► Integrating both sides yields

$$e^{\beta t}R(t) = R(0) + \frac{\alpha}{\beta} \left(e^{\beta t} - 1\right) + \int_0^t \sigma e^{\beta s} dW(s).$$

▶ Namely, a closed-form expression for R(t) is given by

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta} \left(1 - e^{-\beta t}\right) + \sigma e^{-\beta t} \int_0^t e^{\beta s} dW(s).$$



Normal Distribution:

$$R(t) \sim N\left(e^{-\beta t}R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right), \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta t}\right)\right).$$

- ► Disadvantage: possibility to be negative.
- Advantage: mean-reverting property.
  - $\beta$  (speed of mean reversion);
  - ▶  $\lim_{t\to+\infty} ER(t) = \frac{\alpha}{\beta}$  (long-term mean level);
  - ▶  $\lim_{t\to+\infty} Var(R(t)) = \frac{\sigma^2}{2\beta}$  (long-term variance).



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#### General Linear SDEs

Consider SDE

$$dX(t) = [a(t) + b(t)X(t)]dt + [\gamma(t) + \sigma(t)X(t)]dW(t), X(0) = X_0.$$

Apply Ito's rule to prove that

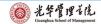
$$X(t) = Y(t) \left[ X_0 + \int_0^t (a(s) - \gamma(s)\sigma(s))Y(s)^{-1} ds + \int_0^t \gamma(s)Y(s)^{-1} dW(s) \right],$$

where

$$Y(t) = \exp\left\{ \int_0^t \left( b(s) - \frac{1}{2}\sigma(s)^2 \right) ds + \int_0^t \sigma(s)dW(s) \right\}.$$

**Question**: How to find the expectation and variance of X(t)?

Note: Previous examples are both special cases of linear SDEs.



#### **Example 3: Cox-Ingersoll-Ross (CIR) Model for Interest Rate**

$$dR(t) = (\alpha - \beta R(t))dt + \sigma \sqrt{R(t)}dW(t).$$

- ► So the advantage of CIR over Vasicek is its non-negativity.
- ► Widely used in modeling interest rate, stochastic volatility, stochastic intensity of credit default and other jumps.
- ▶ We cannot derive a closed form formula for R(t).
- ▶ However, we know R(t) assumes a noncentral Chi-square distribution.
- **Exercise:** Compute  $\mathbb{E}(R(t))$  and Var(R(t)) via Itô formula.



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## Examples in Financial Modeling: Multidimensional Geometric Brownian Motion

Example 4: Multidimensional Geometric Brownian Motion Model for Multiple Correlated Asset Prices, e.g., for two correlated assets

$$\frac{dS_1(t)}{S_1(t)} = \alpha_1 dt + \sigma_1 dW_1(t), 
\frac{dS_2(t)}{S_2(t)} = \alpha_2 dt + \sigma_2 [\rho dW_1(t) + \sqrt{1 - \rho^2} dW_2(t)],$$

where  $\{(W_1(t), W_2(t))\}$  is a standard two-dimensional Brownian motion.

Equivalent dynamics:

$$\frac{dS_1(t)}{S_1(t)} = \alpha_1 dt + \sigma_1 dW_1(t),$$

$$\frac{dS_2(t)}{S_2(t)} = \alpha_2 dt + \sigma_2 dW_3(t).$$



Here  $\{(W_1(t), W_3(t))\}$  is a two dimensional Brownian motion with  $Corr(W_1(t), W_3(t)) = \rho$ .

Apply Ito's formula to  $\log S_1(t)$  and  $\log S_2(t)$ , we find that

$$S_1(t) = S_1(0) \exp \left\{ \sigma_1 W_1(t) + \left( \alpha_1 - \frac{1}{2} \sigma_1^2 \right) t \right\}$$

$$S_2(t) = S_2(0) \exp \left\{ \sigma_2 \left[ \rho W_1(t) + \sqrt{1 - \rho^2} W_2(t) \right] + \left( \alpha_1 - \frac{1}{2} \sigma_1^2 \right) t \right\}$$

**Generalization:** multidimensional linear SDEs. Even in linear specifications, not all SDEs are explicitly solvable! This is not as simple as the one-dimensional linear SDEs.



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## More Examples

SDE provides us a powerful tool to describe the dynamics of financial market. For example, in order to incorporate the "volatility smile", a natural idea is to allow the change of volatility.

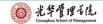
► Local volatility models (Dupire, Derman):

$$dS(t) = \mu S(t)dt + \sigma(t, S(t))S(t)dW(t)$$

► The stochastic volatility model (e.g. Heston (1993)):

$$dS(t) = \mu S(t)dt + \sqrt{V(t)}S(t)dW_1(t)$$
  
$$dV(t) = \kappa(\theta - V(t))dt + \sigma_v \sqrt{V(t)}dW_2(t).$$

▶ In practice, we may use more advanced models according to the special necessity, e.g. adding jumps, etc.



## Supplementary Material

Suggested Reading Material (We only need to focus on the material parallel to our course slides):

- ► Selected material from Shreve Vol. II: Examples 4.4.10, 4.4.11, Sections 6.1, 6.2
- ► Or equivalent material from Mikosch: 3.2, 3.3

Suggested Exercises (some of these exercises have been included in Homework Assignment #6; others are for your deeper understanding)

► Shreve Vol.II: 4.5, 4.8, 6.1, 6.6



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