Econ 139 – Midterm Suggested Solutions
Spring 2019

As a general rule for exams, you may use any theorems stated in lecture without proof, unless you are explicitly directed otherwise.

Problem 1. True or false. For each statement, please carefully justify your answer (no credit without justification). (22 points)

1. Consider an economy with two agents trading three goods, where the agents have increasing and concave utility functions U^1 and U^2 over all three goods. A Pareto optimal allocation in this economy is an allocation where the marginal rate of substitution (MRS) between any pair of goods is the same for both agents. (8 points)

   Solution: True. A Pareto optimal allocation for a particular value of λ is the solution to the "social planning" problem

$$\max_{x^1,x^2} \; U^1(x^1) + \lambda U^2(x^2)$$
$$\text{s.t.} \quad x_1^1 + x_1^2 = \omega_1, \;\; x_2^1 + x_2^2 = \omega_2, \;\; x_3^1 + x_3^2 = \omega_3,$$

   where $x^1 = (x_1^1, x_3^1, x_3^1)$, $x^2 = (x_1^2, x_3^2, x_3^2)$, and ω_1, ω_2, ω_3 are the economy-wide endowments of goods 1, 2, and 3, respectively. Using the constraints, we can rewrite this as

$$\max_{x_1^1,x_2^1,x_3^1} U^1(x_1^1, x_2^1, x_3^1) + \lambda U^2(\omega_1 - x_1^1, \omega_2 - x_2^1, \omega_2 - x_3^1)$$

   The first order conditons yield

$$\frac{U_1^1}{U_1^2} = \frac{U_2^1}{U_2^2} = \frac{U_3^1}{U_3^2} = \lambda$$

   where $U_a^b = \partial U^b / \partial x_a^1$. Rearranging (and multiplying through by minus one) gives

$$-\frac{U_1^1}{U_2^1} = -\frac{U_1^2}{U_2^2} \text{ and } -\frac{U_1^1}{U_3^1} = -\frac{U_1^2}{U_3^2} \text{ and } \quad -\frac{U_2^1}{U_3^1} = -\frac{U_2^2}{U_3^2}.$$

2. A VNM expected utility investor, with a strictly increasing and strictly convex utility function u over payoffs, and initial wealth $W \geq 0$, considers an investment in an asset with uncertain payoff $\tilde{X}$. For this investor, the certainty equivalent of the investment $CE(\tilde{X})$ is less than or equal to the expected payoff $\mathbb{E}[\tilde{X}]$. (7 points)

Solution: False. Since the investor's utility function u over payoffs is strictly convex, by Jensen's inequality it follows that

$$u(W + \mathbb{E}[\tilde{X}]) < \mathbb{E}[u(W + \tilde{X})]$$

The certainty equivalent $CE(\tilde{X})$ satisfies the equality

$$u(W + CE(\tilde{X})) = \mathbb{E}[u(W + \tilde{X})]$$

Hence, we must have $CE(\tilde{X}) > \mathbb{E}[\tilde{X}]$.

3. If $\succeq$ is rational, then $\succ$ is transitive, that is, $x \succ y$ and $y \succ z$ imply $x \succ z$. (7 points)

Solution: True. Suppose $x \succ y$ and $y \succ z$. This means we have the following

$$x \succeq y \text{ and } y \not\succeq x$$

and

$$y \succeq z \text{ and } z \not\succeq y$$

Hence, to show $x \succ z$ we need to show

$$x \succeq z \text{ and } z \not\succeq x.$$

Since $\succeq$ is rational, it is complete and transitive. Then $x \succeq y$, $y \succeq z$, and transtivity of $\succeq$ imply $x \succeq z$. Now suppose $z \succeq x$, then $x \succeq y$ and transtivity of $\succeq$ imply $z \succeq y$. This contradicts our supposition that $y \succ z$. Hence we must have $z \not\succeq x$.

Problem 2. Evaluating risky investments. (21 points, 7 points each)

Consider an economic environment in which there are two possible future states: a good state that occurs with probability $\pi = 1/2$ and a bad state that occurs with probability $1 - \pi = 1/2$. Three assets are traded, with payoffs in the two states tabulated below:

State	Asset 1	Asset 2	Asset 3
Good	25	20	15
Bad	5	10	15

1. Rank these assets, or determine they can not be ranked, in terms of

   (a) State-by-state dominance

   (b) First order stochastic dominance

   (c) Second order stochastic dominance

   Solution: No asset dominates another on a state-by-state basis.

   In addition, no asset first order stochastically dominates another. To see this, consider the CDFs of the assets

Payoff	F_1	F_2	F_3
5	0.5	0	0.0
10	0.5	0.5	0.0
15	0.5	0.5	1.0
20	0.5	1.0	1.0
25	1.0	1.0	1.0

   For one asset to first order stochastically dominate another, we must have $F_A(x) \leq F_B(x), \forall x$.

   Asset 1 is a mean-preserving spread of asset 2 and of asset 3. Asset 2 is a mean-preserving spread of asset 3. Hence, by second order stochastic dominace asset 3 is preferred to asset 1 and asset 2, and asset 2 is preferred to asset 1.

2. Rank these assets, or determine that they can not be ranked, by the mean-variance criterion.

   Solution: Since all assets have the same mean, the ranking using the mean-variance criterion is based on the variances of the assets. Asset 1 has a higher variance than asset 2 and asset 3. Asset 2 has a higher variance than asset 3. Hence, by the mean-variance criterion asset 3 is preferred to asset 1 and asset 2, and asset 2 is preferred to asset 1.

3. How will a VNM expected utility investor with Bernoulli utility function over wealth $u(W) = \ln(W)$ allocate their investable wealth among these three assets (assuming they allocate all of their investable wealth among these assets and they are not allowed to borrow)?

   Solution: If asset A second order stochastically dominates asset B, we know that *for all* increasing and concave Bernoulli utility functions the expected utility of asset A is higher than the expected utility of asset B. Since $\ln(W)$ is an increasing and concave function, the VNM expected utility of asset 3 is higher than the VNM expected utility of both assets 1 and 2. Therefore, a VNM expected utility investor with Bernoulli utility function over wealth $u(W) = \ln(W)$ will place all of their investable wealth in asset 3.

Problem 3. Expected utility and portfolio allocation. (21 points, 7 points each)

Consider the portfolio allocation problem faced by an investor who has initial wealth W_0. The investor allocates the amount a to stocks, which provide return r_G in a good state that occurs with probability π, and return r_B in a bad state that occurs with probability $1 - \pi$. The investor allocates the remaining amount $W_0 - a$ to a risk-free bond, which provides return r_f in both states. Assume $r_G > r_f > r_B$ throughout.

Suppose that the investor's preferences can be described by a VNM expected utility function, with Bernoulli utility function

$$u(W) = \ln(W).$$

1. Write down the expected utility maximization problem (in terms of π, W_0, a, r_G, r_B, and r_f) and the first order condition for the investor, where the decision variable is the amount the investor allocates to stocks a.

   Solution: The expected utility maximization problem is

$$\max_a \ \pi \ln(W_0(1 + r_f) + a(r_G - r_f)) + (1 - \pi) \ln(W_0(1 + r_f) + a(r_B - r_f))$$

   The first order condition with respect to a is

$$\frac{\pi(r_G - r_f)}{W_0(1 + r_f) + a^*(r_G - r_f)} + \frac{(1 - \pi)(r_B - r_f)}{W_0(1 + r_f) + a^*(r_B - r_f)} = 0$$

   or

$$\frac{\pi(r_G - r_f)}{W_0(1 + r_f) + a^*(r_G - r_f)} = \frac{(1 - \pi)(r_f - r_B)}{W_0(1 + r_f) + a^*(r_B - r_f)}$$

2. Suppose the good state occurs with probability $\pi = 2/3$ and the bad state occurs with probability $1 - \pi = 1/3$. Further suppose that initial wealth is $W_0 = 100$, the risk-free rate is $r_f = 0.05$ (5 percent), and stocks return $r_G = 0.55$ (a 55 percent gain) in the good state and $r_B = -0.45$ (a 45 percent loss) in the bad state. Use your first order condition from point 1 to find the numerical value of a^*.

   Solution: Plugging the given values into the first order condition gives

$$\frac{(2/3)(0.5)}{105 + 0.5a^*} = \frac{(1/3)(0.5)}{105 - 0.5a^*}$$

   Solving for a^* yields

$$2(105 - 0.5a^*) = 105 + 0.5a^*$$

   or

$$a^* = 70.$$

3. Suppose now that, as in point 2, $\pi = 2/3$, $1 - \pi = 1/3$, $r_f = 0.05$, $r_G = 0.55$, and $r_B = -0.45$, but the investor's initial wealth is $W_0 = 200$. What is the numerical value of a^* with this larger value of W_0?

   Solution: As we discussed in class, the logarithmic Bernoulli utility function implies that the investor's coefficient of relative risk aversion is constant and equal to one, and with constant relative risk aversion, the optimal choice of a scales up and down proportionally with initial wealth W_0. Hence, since W_0 doubled, we can conclude that $a^* = 140$. Alternatively, this can be verified using $W_0 = 200$ in the fist order condition.

Problem 4. Risk aversion and savings behaviour. (36 points, 6 points each)

Recall our intertemporal choice problem:

$$\max_{c_0,c_1} \quad u(c_0) + \gamma u(c_1)$$
$$\text{s.t.} \quad c_0 + \frac{c_1}{1+r} = M_0 + \frac{M_1}{1+r}$$

where (c_0, c_1) and (M_0, M_1) are consumption and income at $t = 0$ and $t = 1$, respectively, $\gamma = 1/(1+\delta)$, where δ is the individual's "impatience" parameter, and r is the economy-wide saving rate. Assume $u' > 0$ and $u'' < 0$ throughout.

Now suppose that $t = 1$ represents retirement, so that $M_1 = 0$, and individuals can only save (i.e., they are unable to borrow). Further suppose that all savings must be invested in an asset with risky net return $\tilde{r}$ and risky gross return $\tilde{R} = 1 + \tilde{r}$.

1. Write down the individual's intertemporal expected utility maximization problem, where the decision varibale is the amount of savings – use s to represent savings.

   Solution: In terms of savings, we have

   $$\begin{aligned} c_0 &= M_0 - s \\ \tilde{c}_1 &= s(1+\tilde{r}) = s\tilde{R} \end{aligned}$$

   We can write the expected utility maximization problem as

   $$\max_{s} \quad \mathbb{E}[u(M_0 - s) + \gamma u(s\tilde{R})]$$
   $$\text{s.t.} \quad 0 \le s \le M_0$$

   equivalently,

   $$\max_{s} \quad u(M_0 - s) + \mathbb{E}[\gamma u(s\tilde{R})]$$
   $$\text{s.t.} \quad 0 \le s \le M_0$$

2. Write down the first order condition with repect to s. How will s^* change in response to a change in $\mathbb{E}[\gamma u'(s^*\tilde{R})\tilde{R}]$ if the asset becomes riskier?

   Solution: The first order condition with respect to s is

   $$-u'(M_0 - s^*) + \mathbb{E}[\gamma u'(s\tilde{R})\tilde{R}] = 0$$

which can be rewritten as

$$\mathbb{E}[\gamma u'(s^*\tilde{R})\tilde{R}] = u'(M_0 - s^*).$$

Due to the concavity of u, s^* will increase if $\mathbb{E}[\gamma u'(s^*\tilde{R})\tilde{R}]$ increases, decrease if $\mathbb{E}[\gamma u'(s^*\tilde{R})\tilde{R}]$ decreases, and stay the same if $\mathbb{E}[\gamma u'(s^*\tilde{R})\tilde{R}]$ does not change.

For the remainder of the problem, consider two possible distributions of return $\tilde{r}_A$ and $\tilde{r}_B$, where

$$\tilde{r}_B = \tilde{r}_A + \tilde{\varepsilon},$$

with $\mathbb{E}[\tilde{\varepsilon}] = 0$ and $\mathbf{V}ar(\tilde{\varepsilon}) > 0$. Let s_A^* and s_B^* be the solutions to our expected utility maximization problem under the return distributions for $\tilde{r}_A$ and $\tilde{r}_B$, respectively.

3. Suppose the individual has the Bernoulli utility function over consumption

$$u(c) = c - bc^2,$$

where $b > 0$ and $c < 1/(2b)$. Without reference to any theorems discussed in lecture, determine whether s_A^* is greater than, equal to, or less than s_B^* for this individual. Show your work.

Solution: In this case we have

$$u'(c) = 1 - 2bc.$$

Using this in the first order condition yields

$$\mathbb{E}[\gamma(1 - 2b(s^*\tilde{R}))\tilde{R}] = 1 - 2b(M_0 - s^*).$$

Solving for s^* gives

$$s^* = \frac{\gamma\mathbb{E}[\tilde{R}] + 2bM_0 - 1}{2b(1 + \gamma\mathbb{E}[\tilde{R}^2])}$$

We know that $\mathbf{V}ar(\tilde{R}_B) > \mathbf{V}ar(\tilde{R}_A)$ and $\mathbb{E}(\tilde{R}_B) = \mathbb{E}(\tilde{R}_A)$. Since we can write

$$\mathbf{V}ar(\tilde{R}) = \mathbb{E}[\tilde{R}^2] - \mathbb{E}[\tilde{R}]^2,$$

we conclude that $\mathbb{E}[\tilde{R}_B^2] > \mathbb{E}[\tilde{R}_A^2]$ and therefore, $s_B^* < s_A^*$.

4. Suppose the individual has the Bernoulli utility function over consumption

$$u(c) = \ln(c).$$

Without reference to any theorems discussed in lecture, determine whether s_A^* is greater than, equal to, or less than s_B^* for this individual. Show your work.

Solution: In this case we have

$$u'(c) = \frac{1}{c}$$

Using this in the first order condition yields

$$\mathbb{E}\left[\gamma \frac{1}{s^* \tilde{R}} \tilde{R}\right] = \frac{1}{M_0 - s^*}.$$

Solving for s^* gives

$$s^* = \frac{\gamma M_0}{1+\gamma}$$

In this case s^* is not sensitive to the riskiness of the asset. Therefore, $s_B^* = s_A^*$.

5. Finally, suppose the individual has the Bernoulli utility function over consumption

$$u(c) = \frac{c^{1-\beta}}{1-\beta},$$

where $\beta > 1$. Without reference to any theorems discussed in lecture, determine whether s_A^* is greater than, equal to, or less than s_B^* for this individual. Show your work.

Solution: In this case we have

$$u'(c) = c^{-\beta}$$

Using this in the first order condition yields

$$\mathbb{E}\left[\gamma (s^* \tilde{R})^{-\beta} \tilde{R}\right] = (M_0 - s^*)^{-\beta}.$$

Letting $\gamma(s^*\tilde{R})^{-\beta}\tilde{R} \equiv g(\tilde{R})$, we need to determine if g is a concave, linear, or convex function of $\tilde{R}$. The first derivative of g with respect to $\tilde{R}$ is given by

$$\begin{aligned} g'(\tilde{R}) &= \gamma(s^*\tilde{R})^{-\beta} - \gamma\beta(s^*\tilde{R})^{-\beta-1}(s^*\tilde{R}) \\ &= \gamma(s^*\tilde{R})^{-\beta}(1-\beta) \end{aligned}$$

Then the second derivative of g with respect to $\tilde{R}$ is given by

$$g''(\tilde{R}) = -\gamma\beta(1-\beta)(s^*\tilde{R})^{-\beta-1} > 0,$$

since $\beta > 1$. Hence, g is a convex function of $\tilde{R}$. This means that an increase in the riskiness of the asset (via a mean preserving spread) will increase $\mathbb{E}\left[\gamma(s^*\tilde{R})^{-\beta}\tilde{R}\right]$ which leads to an increase in s^*. Therefore, $s_B^* > s_A^*$.

6. Verify your answer to point 5 using Kimball's relative prudence coefficient and corresponding theorem. Show your work.

Solution: Kimball's relative prudence coefficient is defined as

$$P_R(c) = -\frac{u'''(c)}{u''c}c.$$

For power utility we have

$$\begin{aligned} u(c) &= \frac{c^{1-\beta}}{1-\beta} \\ u'(c) &= c^{-\beta} \\ u''(c) &= -\beta c^{-\beta-1} \\ u'''(c) &= \beta(\beta+1)c^{-\beta-2} \end{aligned}$$

The relative prudence coefficient is then

$$\begin{aligned} P_R(c) &= -\frac{\beta(\beta+1)c^{-\beta-2}}{-\beta c^{-\beta-1}}c \\ &= \frac{(\beta+1)c^{-\beta-1}}{c^{-\beta-1}} \\ &= \beta+1. \end{aligned}$$

Since $\beta > 1$ (by supposition), we have $P_R(c) > 2$. By the theorem given in lecture, this implies $s_B^* > s_A^*$, which confirms our answer to point 5.