Module 2: Bond Analytics

Roadmap…

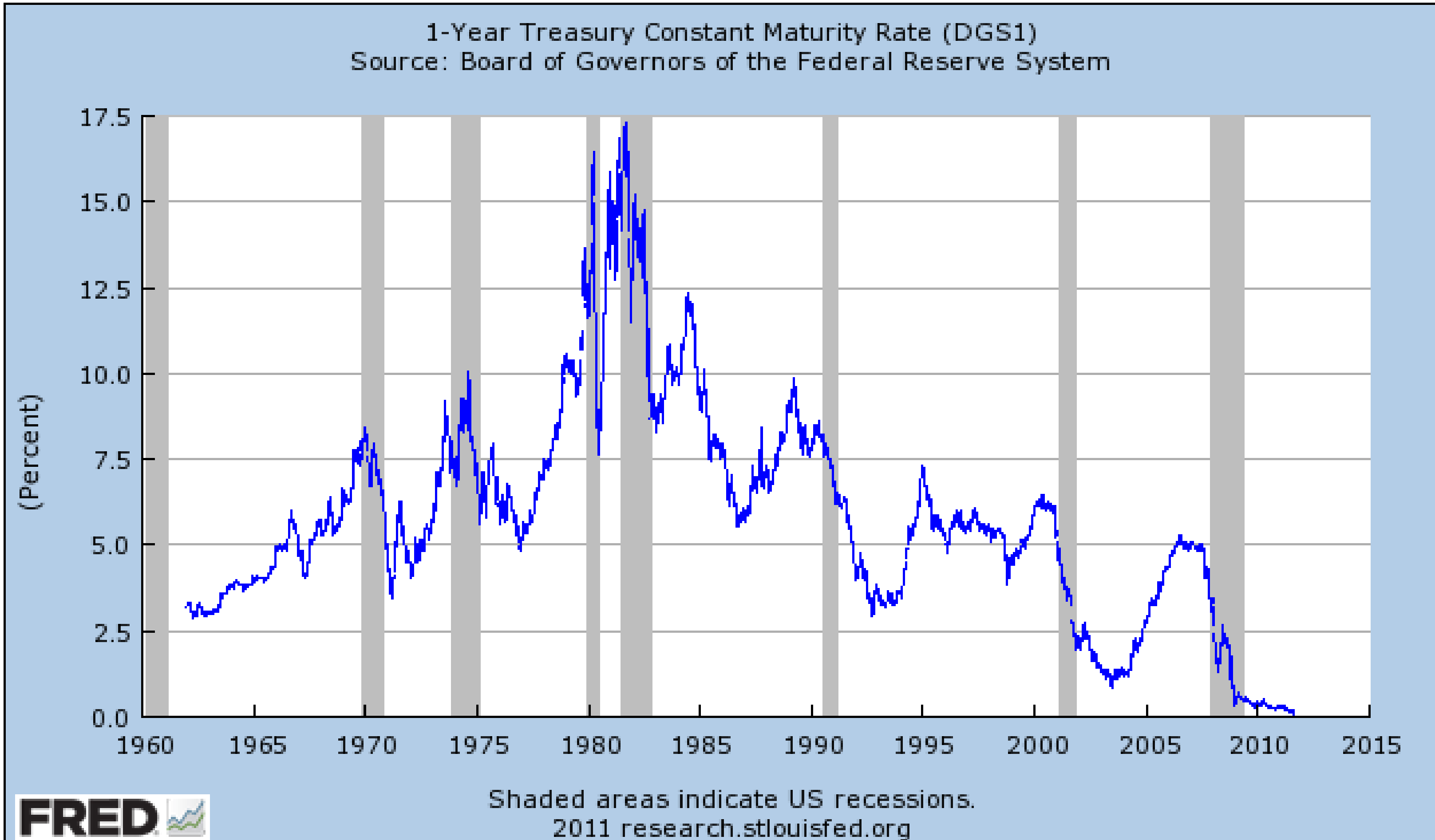
- We want to examine the relationship between risk and reward in the bond market.
  - The reward will be the yield when we purchase.
  - The risk is that price will change with yields change.
- Before we look at this relationship, we need to learn the terminology.
- We will also look at how yields have changed in the past.

Terminology

- Yield: percent per year (units)
- Basis Point (b.p.): equal to 0.01%
- Price: percent of Par (units)
  - Par: Price = 100
  - Premium: Price > 100
  - Discount: Price < 100
- Duration: <u>years</u> (units)
  - Just because the units are years, don't think of this as a measure of time – think of it as a <u>measure of risk</u>.

Basis Points

- Q: Why do we talk about basis points, rather than percentage points?
- A: A change of 1% in interest rates is a huge change.
- For One-Year Rates (Data from 10/25/2011)
  - Average absolute daily change: 4.87 b.p. per day
  - Greatest 1-day increase (since 2001): 52 b.p. (Sep. 2001)
  - Greatest 1-day decrease (since 2001): -50 b.p (Sep. 2001)
  - Standard Deviation (since 2001): 4.95 b.p
  - # days >= 50 b.p. shift: 60 out of 12,186 (0.48%)
  - # days >= 25 b.p. shift: 285 out of 12,186 (2.30%)


1-Year Treasury Constant Maturity Rate (DGS1)
Source: Board of Governors of the Federal Reserve System
17.5
15.0
12.5
10.0
7.5
5.0
2.5
0.0
(Percent)
1960
1965
1970
1975
1980
1985
1990
1995
2000
2005
2010
2015
Shaded areas indicate US recessions.
2011 research.stlouisfed.org
FRED

Pricing

- We use present value formulas to find the value of a bond.
- We can then find the value as a percent of face value.
- This value is then split into Price and Accrued Interest.

Pricing

- Yield to Price
  - y – Annual Yield
  - C_k – k^th Cash Flow
  - t_k – Time in years to Cash Flow
  - PV – Price (including accrued interest)

$$PV = \sum_{k=1}^{n} \frac{C_k}{\left(1 + \frac{y}{2}\right)^{2t_k}}$$

Pricing

- Most bonds pay the same coupon twice per year until maturity and then repay the principal at maturity.
- This means that cash flows are (Coupon/2) every six months with 100% of the principal added at maturity.
- We could price coupons using the Annuity formula and then add in the PV of the principal payment.

Example: Ten Year Auction

- Issued on 8/15/2011
- Matures on 8/15/2021
- Coupon Rate = 2.125%
- Yield at Issue = 2.140%
- Price at Issue = 99.865607
- Yield on 8/29/2011 = 2.280%

Finding the Price of Ten Year

Start Date	8/15/2011
Settlement	8/30/2011
Days to Settle	15
Next Coupon	2/15/2012
Days in Period	184
Coupon	2.125%
Yield	2.280%
Present Value	98.71216%
Accrued Interest	0.08662%
Price	98.62555%

AI is calculated by multiplying half the coupon by the number of days to settlement, then dividing by the number of days in the period

Date	Periods	Cash Flow	PV
2/15/2012	0.9185	1.063%	1.05150%
8/15/2012	1.9185	1.063%	1.03964%
2/15/2013	2.9185	1.063%	1.02792%
8/15/2013	3.9185	1.063%	1.01634%
2/15/2014	4.9185	1.063%	1.00488%
8/15/2014	5.9185	1.063%	0.99356%
2/15/2015	6.9185	1.063%	0.98236%
8/15/2015	7.9185	1.063%	0.97128%
2/15/2016	8.9185	1.063%	0.96034%
8/15/2016	9.9185	1.063%	0.94951%
2/15/2017	10.9185	1.063%	0.93881%
8/15/2017	11.9185	1.063%	0.92823%
2/15/2018	12.9185	1.063%	0.91777%
8/15/2018	13.9185	1.063%	0.90742%
2/15/2019	14.9185	1.063%	0.89719%
8/15/2019	15.9185	1.063%	0.88708%
2/15/2020	16.9185	1.063%	0.87708%
8/15/2020	17.9185	1.063%	0.86720%
2/15/2021	18.9185	1.063%	0.85742%
8/15/2021	19.9185	101.063%	80.63663%

Finding the Price of Ten Year

- We calculate the price on the settlement date of 08/30/2011. The delivery date is 08/29/2011 -- one day before the settlement of 08/30/2011. For example, we consider the coupon date of 02/15/2012
  - The number of days from the settlement to the next coupon date is 184-15=169. Thus the number of period = 169/184 = 0.9185.
  - The cash flow at coupon date (except the maturity) = 2.125%/2 = 1.0625% (at maturity it will be 1+1.0625% = 1.01625%).
  - The present value of cash flow = $1.0625\%/(1+2.280\%/2)^{0.9185}$ = 1.051495% (here we use the yield of 2.28% on the delivery date).
  - Accrued interest (AI) = 15/184*1.0625 = 0.086617%.
  - Present value of bond = sum (all PV of cash flows) = 98.71216%.
  - Price = present value of bond – accrued interest = 98.71216% - 0.086617% = 98.6155%

Price Yield Relationship

- When yield increases, bond value goes down.
- When yield decreases, bond value goes up.
- Why?
  - Let’s look at one economic reason and two mathematical reasons.

Price Yield Relationship

- **Economic Rational**:
  - Suppose that you bought a bond at going rate (coupon rate = yield).
  - If Yields go down, you will not be willing to sell this for the price you paid, since the coupon rate is now above the going rate. You would demand a premium since they would get higher interest payments.
  - If Yields go up, you will not be able to sell this for the price you paid, since the coupon rate is now below the going rate. Buyers would demand a discount.
  - The same reasoning works for premiums and discounts.

Price Yield Relationship

- Mathematical Rational #1:
  - Each term of the pricing function looks like:

$$\frac{C_k}{\left(1+\frac{y}{2}\right)^{2t_k}}$$

  - Increasing the yield increases the denominator value, so every term decreases in value.

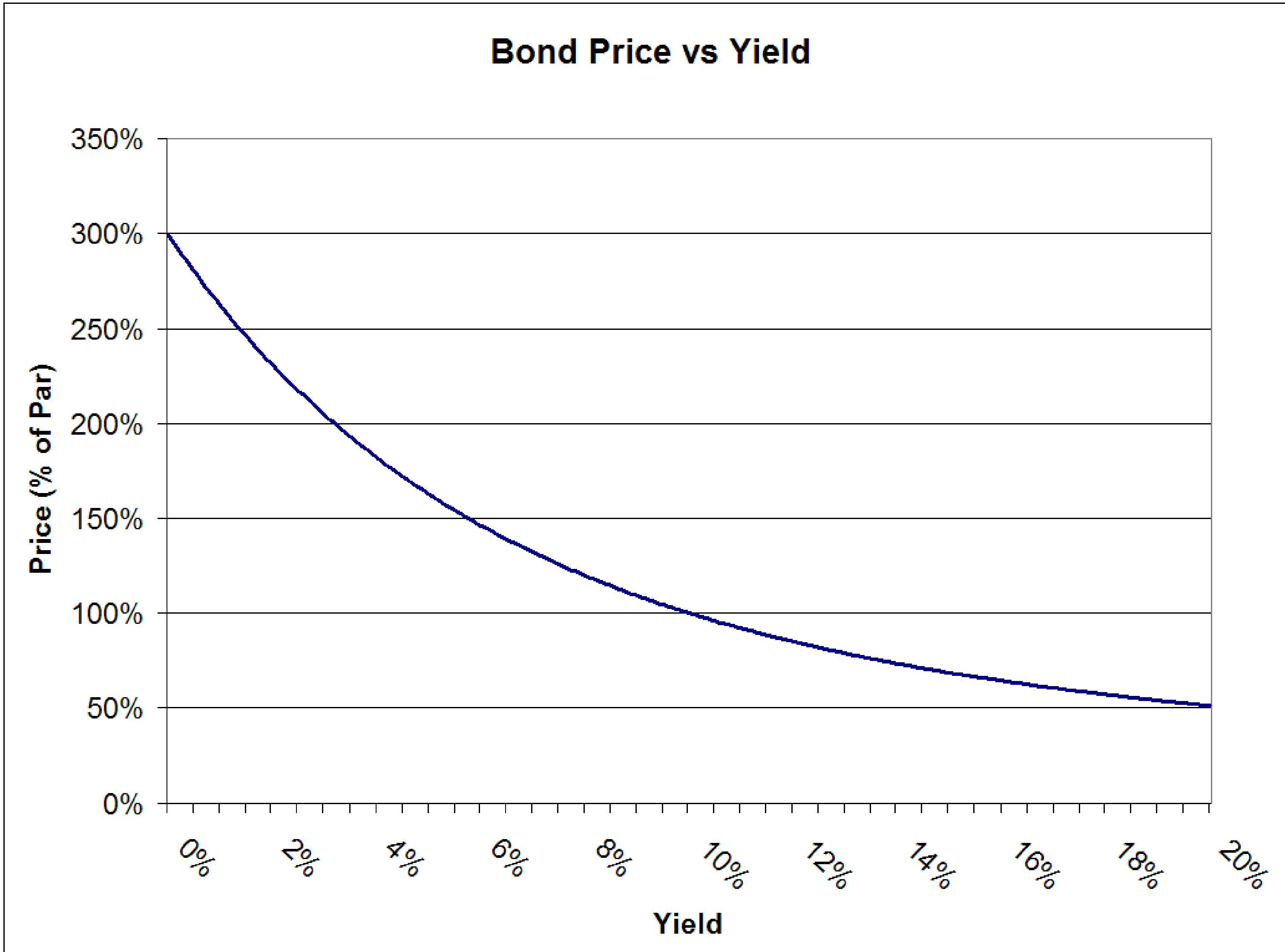
Price Yield Relationship

- Mathematical Rational #2:
  - Each term of the pricing function looks like:

$$\frac{C_k}{\left(1+\frac{y}{2}\right)^{2t_k}}$$

  - Note that the first derivative of each of these terms is negative – so the Pricing Function is decreasing

$$\frac{d}{dy}\left(\frac{C_k}{\left(1+\frac{y}{2}\right)^{2t_k}}\right)=-\frac{t_k\cdot C_k}{\left(1+\frac{y}{2}\right)^{2t_k+1}}$$



Bond Price vs Yield
350%
300%
250%
200%
150%
100%
50%
0%
Price (% of Par)
0%
2%
4%
6%
8%
10%
12%
14%
16%
18%
20%
Yield

Bond Dynamics

- What happens to price when yields change?
  - We saw that prices move in the opposite direction.
- Is there a good measure of how much price changes when yield changes?
- How can we compare the price risk of one bond against another?

Macaulay Duration

- Macaulay Duration of a zero coupon bond is the time to maturity.
- Macaulay Duration of a bond is the time-to-cash-flow weighted by the PV of the flow

$$D_{mac} = \frac{1}{P} \cdot \sum_{k=1}^{n} t_k \cdot PV(C_k)$$

Modified Duration

- Modified Duration
  - Percent change in price for a small change in yield

$$D_{\mathrm{mod}} = \frac{1}{\left(1+\frac{y}{2}\right)} D_{mac}$$

$$D_{\mathrm{mod}} = \frac{1}{P \cdot \left(1+\frac{y}{2}\right)} \cdot \sum_{k=1}^{n} t_k \cdot PV\left(C_k\right)$$

  - Note: Technically, yield should be divided by the number of compounding periods per year. In US, the standard convention is two periods per year.

Modified Duration

- Modified Duration is a measure of risk:
  - What economists call "Price Elasticity".
  - It is also equal to the derivative of the price-yield function divided by the price

  If $P(y)$ is the price for a given yield, then

$$D_{\text{mod}} = -\frac{\left(\frac{\partial P(y)}{\partial y}\right)}{P(y)}$$

  - Hence it is the proportional percent change in price for a small change in yield.

Macaulay vs. Modified

- Why are both definitions around?
  - History – Macaulay came first.
  - Modified is more useful.
- Is it useful to think of Duration as a weighted time until cash is received?
  - No!
- Is it useful to think of Duration as a measurement of risk?
  - Yes!

Duration in Years

- Q: Why is the units measure in years?
- A: Think of duration as the negative of change in price over change in yield. Price is in percent, and yield is in percent per year – so we get:

$$\frac{\%}{\left(\dfrac{\%}{Years}\right)} = \frac{\%}{1} \times \frac{Years}{\%} = Years$$

Estimating Price Sensitivity

- Since Modified Duration estimates percent change in price, we have

$$\frac{\Delta P}{P} \approx -D_{\text{mod}} \cdot \Delta y$$

- or

$$\Delta P \approx -D_{\text{mod}} \cdot P \cdot \Delta y$$

Example

- Suppose we have a bond with a value of 95, a yield of 5% and a modified duration of 6 years. If the yield increases by 7 b.p. (basis points), how much does value change?

$$\Delta P \approx -6 * 95 * 0.07\% = -0.399$$

Historical View

- Macaulay Duration (1938)
  - PV of cash flow weighted by time.
- Modified Duration
  - Sensitivity to yield changes for bonds with fixed cash flows.
- Effective Duration
  - Sensitivity to cash flows.
  - Sometimes called 'Option-Adjusted Duration' (for bonds with embedded option).
  - Equal to Modified when cash flows are fixed.

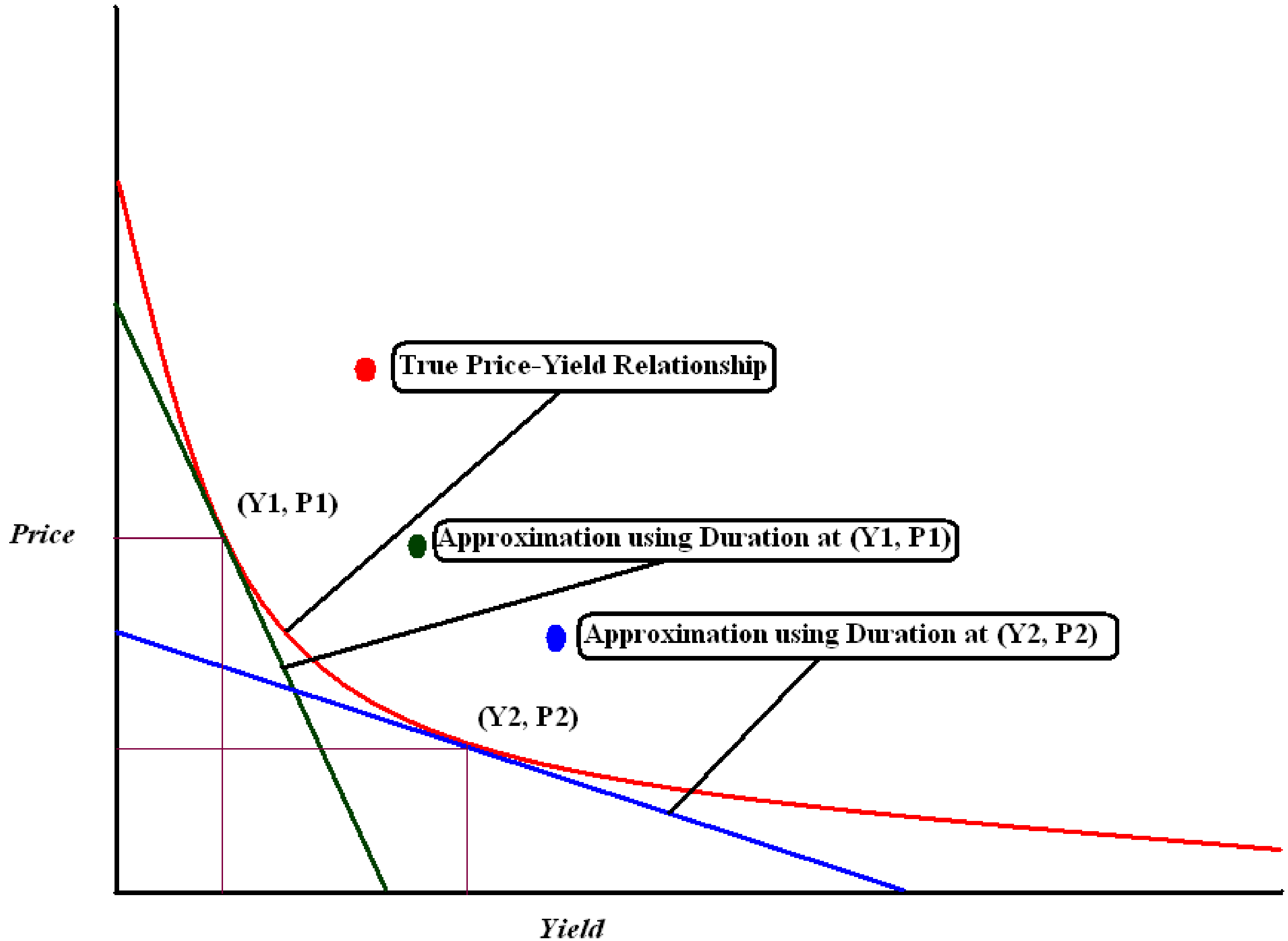
Duration

- What happens to Modified Duration if we change:
  - Maturity
    - Duration increases if maturity increases (longer duration), WHY?
  - Coupon
    - Duration decreases if coupon increases (shorter duration), WHY?
  - Yield
    - Duration decreases as Yield Increases, WHY?
- Note – effects similar for modified duration in “normal” range.

Using Duration

- How good is the approximation using Modified Duration?
  - Is the value we get only close for small yield changes or does it work for big yield changes, too?
  - It is dependent on Yield?

Using Duration

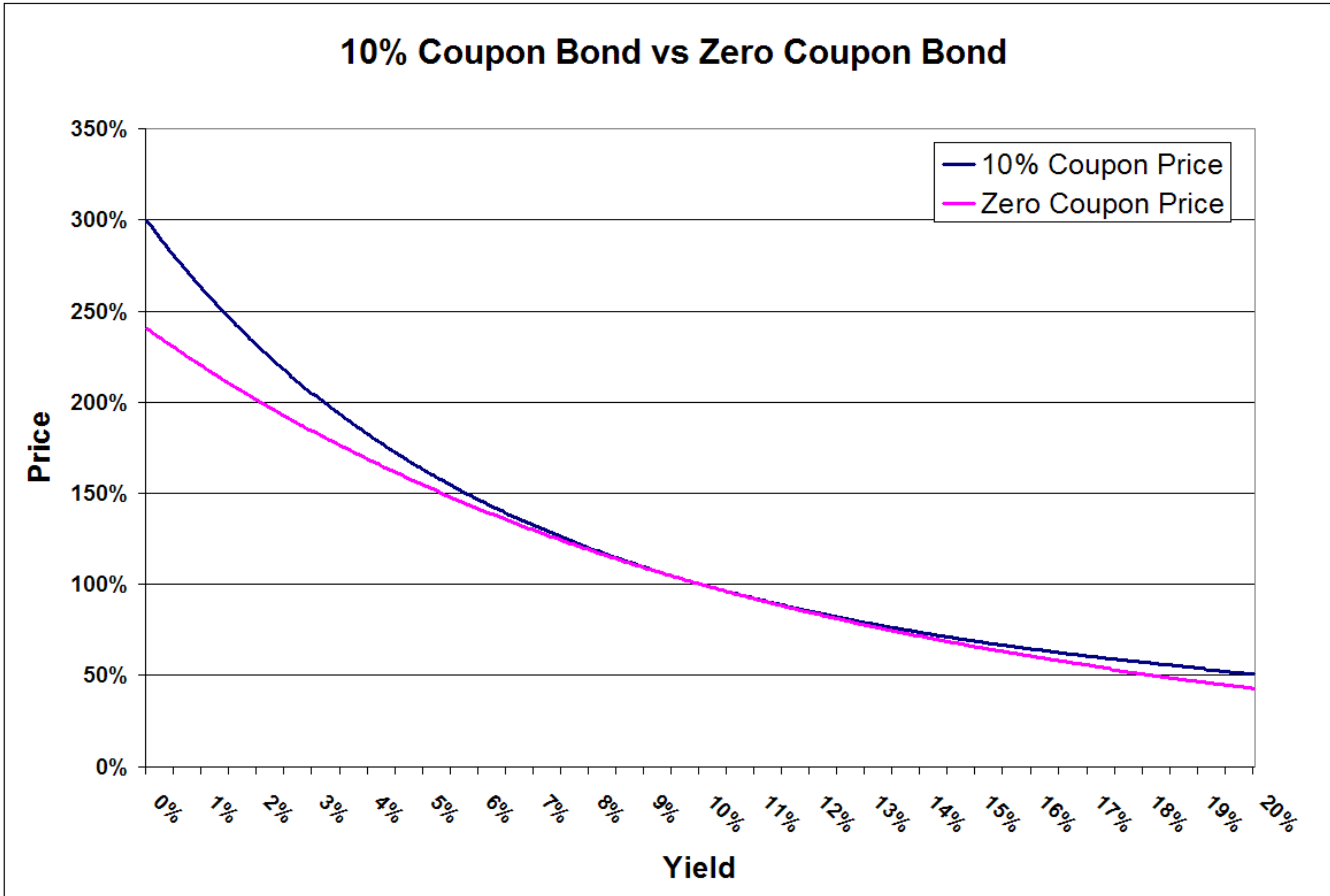


Using Duration

- Using duration to approximate new values gives us values along the tangent line to the price-yield curve.
- We see that for very small changes in yield, the approximation is good.
- For large values, the approximation moves away from the true value.
- For larger yields, the tangent line is more flat, and gives a better approximation.

Convexity

- When yield changes are large, the change predicted by duration is inaccurate.
- Duration ignores the fact that the Price-Yield curve is curved.
- Convexity can be thought of as a measure of <u>how much Duration changes when yields change</u>.
- The graph on the next slide shows the effect of convexity.
- We leave the calculation of convexity and the application to bond pricing for future discussion.



10% Coupon Bond vs Zero Coupon Bond
10% Coupon Price
Zero Coupon Price
Price
350%
300%
250%
200%
150%
100%
50%
0%
0%
1%
2%
3%
4%
5%
6%
7%
8%
9%
10%
11%
12%
13%
14%
15%
16%
17%
18%
19%
20%
Yield

Modified Duration

- Since modified duration is a measure of risk, and investors don't like risk – does this mean we ought to prefer investing in short duration securities? In other words – is there a "right" duration for our investment?
- It depends on our objectives and on investment horizon, e.g., if we need money in one month, we could:
  - Invest for one month in a 30-day T-Bill
  - Invest in a 10-year zero coupon bond and sell it after one month.
- What risks do we have with these investments?

Modified Duration

- Consider another example – if we need money in ten years, we could:
  - Invest for one month in a 30-day T-Bill, then take the proceeds and invest for another 30 days – and so on.
  - Invest in a ten year zero coupon bond.
  - What risks do we have with these investments?

Modified Duration

- In the first example
  - We face no risks for our one month investment.
  - With our ten year investment, we face the risk that the price could change.
- In the second investment:
  - We face no risk in our ten year investment.
  - With the one month investment, we face the risk that rates could go down and we have to reinvest at a lower rate.

Modified Duration

- With bond market investments we have to balance the price risk with the reinvestment risk.
- Matching the duration of the investment to the holding period minimizes these risks.

Examples

- For a 30-year 5% coupon bond:
  - Find the price with a yield of 4.5%.
  - Find the price with a yield of 4.4%.
  - Find the Duration.
  - Using the duration – what is the estimated price difference (between the first two questions)?
  - How close is this to the actual difference? How is it biased?
  - Find the yield if the price is 102.50.
- We can do this in Excel.

Duration

- Let’s Revisit a Question from earlier:
  - What happens to Modified Duration if we change:
    - Maturity?
    - Coupon?
    - Yield?
  - Let’s create an Excel File to get answers

What Did We Learn?

- Yield measures the reward of a bond.
- The duration measures risk, giving us a close approximation of how much price changes for a small yield change.
- Due to convexity, the approximation is better when yield changes are smaller
  - We can use convexity to get a better approximation for large changes.

Bond Valuation Problems

Problem 1

- Find the price and modified duration of a <u>Zero Coupon</u> Bond that matures in 12.75 years that has a yield of 5%
  - Note that there are two compounding periods per year.
  - Find the price to five decimal places
  - Find Modified Duration to three decimal places.

Problem 1

- Since there are two compounding periods per year, there are 25.5 periods (2*Time) until it matures.
- Use the PV formula for a single cash flow to find the price.
- There is one cash flow of 100% of the face value

$$P = \frac{100}{\left(1+\frac{y}{2}\right)^{2\cdot t}} = \frac{100}{\left(1+\frac{0.05}{2}\right)^{25.5}} = \frac{100}{(1.025)^{25.5}} = 53.27720$$

Problem 1

- Here is the general formula for Modified Duration.
- Note that for Zero Coupon Bonds:
  - There is only one term.
  - The PV of the single cash flow is equal to P – so the problem simplifies

$$D_{\text{mod}} = \frac{\sum_{k=1}^{n} t_k \cdot PV(C_k)}{P \cdot \left(1 + \frac{y}{2}\right)}$$

Problem 1

- The modified duration of a Zero Coupon Bond is simply the time to maturity divided by the one period return

$$D_{Mod} = \frac{t}{\left(1+\frac{y}{2}\right)} = \frac{12.75}{1.025} = 12.439$$

Problem 2

- Find the price of a 27 year bond with a coupon of 4% and a yield of 4.2%
  - Note that there are two compounding periods per year.
  - Find the price to five decimal places.
  - Hint: Use the annuity formula to find the value of the coupons.

Problem 2

- Step 1: find the value of the coupons using the annuity formula.

$$r = \frac{y}{2} = 0.021$$

$$C = \frac{Coupon}{2} = 2$$

$$n = 2 \cdot time = 54$$

$$PV = \frac{C}{r} - \frac{C}{r \cdot (1+r)^n}$$

$$PV = \frac{2}{0.021} - \frac{2}{0.021 \cdot (1.021)^{54}}$$

$$PV = 95.23910 - 31.00421 = 64.23389$$

Problem 2

- Step 2: find the present value of the principal

$$PV = \frac{100}{\left(1+\frac{y}{2}\right)^n} = \frac{100}{1.021^{54}} = 32.55442$$

- Step 3: add the value of the principal to the value of the coupons to get

P = 96.78831<100, WHY?

Problem 3

- A bond fund manager has a portfolio worth \$165,000,000.
- It has a modified duration of 4.3 years and a yield of 5.52%
  - Assume two compounding periods per year.
- If the portfolio yield increases by four basis points, estimate the change in the value of the portfolio.

Problem 3

- Use the following formula:

$$\Delta P \approx -D_{Mod} \cdot P \cdot \Delta y$$

$$D_{Mod} = 4.3$$

$$P = 165{,}000{,}000$$

$$\Delta y = 0.0004$$

$$\Delta P \approx -4.3 \cdot 165MM * 0.0004 = -283{,}800 < 0, WHY?$$

Problem 4

- Use the following Zero Coupon Bond information to find the price of a two-year 6% Treasury Bond.

Maturity	Yield	Price
0.50	2.00%	99.00990
1.00	2.20%	97.83577
1.50	2.40%	96.48470
2.00	2.50%	95.15243

Problem 4

- Step 1: Find the cash flows for the bond
- Step 2: Multiply the cash flows by the Zero Coupon Bond prices to get the PV for each flow
- Step 3: Add the PVs to get the value of the bond.

Maturity	Yield	Price	Cash Flow	PV
0.50	2.00%	99.00990	3.00	2.97030
1.00	2.20%	97.83577	3.00	2.93507
1.50	2.40%	96.48470	3.00	2.89454
2.00	2.50%	95.15243	103.00	98.00700
				106.80691

Problem 5

- Find the cash flows, price and modified duration of a One Year Treasury bond.
- Assume that it has a coupon rate of 3% per year and a yield of 3.2% .
- Assume that there are two compounding periods per year.

Problem 5

- The interest payment is 1.5% every six months.
- The final payment includes a 1.5% interest payment plus 100% of the principal (101.5%).
- The PV is found by calculating the PV of each of the two cash flows and adding them up.
- We also have an intermediate step of multiplying the time to each cash flow by its PV.

Problem 5

- Find the cash flows, price and modified duration of a one Year Treasury bond. Assume that it has a coupon rate of 3% per year, a yield of 3.2% and that there are two compounding periods per year.

$$P = \sum_{k=1}^{N} PV(C_k) = \sum_{k=1}^{N} \frac{C_k}{\left(1 + \frac{y}{2}\right)^{2 \cdot t_k}}$$

$$D_{\text{mod}} = \frac{1}{P \cdot \left(1 + \frac{y}{2}\right)} \cdot \sum_{k=1}^{N} PV(C_k) \cdot t_k$$

Problem 5

- Find cash flows, price and modified duration of a one Year Treasury bond. Assume that it has coupon rate = 3% per year, yield = 3.2% and that there are two compounding periods per year.

# Periods	Time	CF	PV	PV*Time
1	0.5	1.50	1.47638	0.73819
2	1.0	101.50	98.32832	98.32832
		Sum	99.80470	99.06651

D(Mod)	0.977
Price	99.80470

Price Yield Relationship

Price-Yield

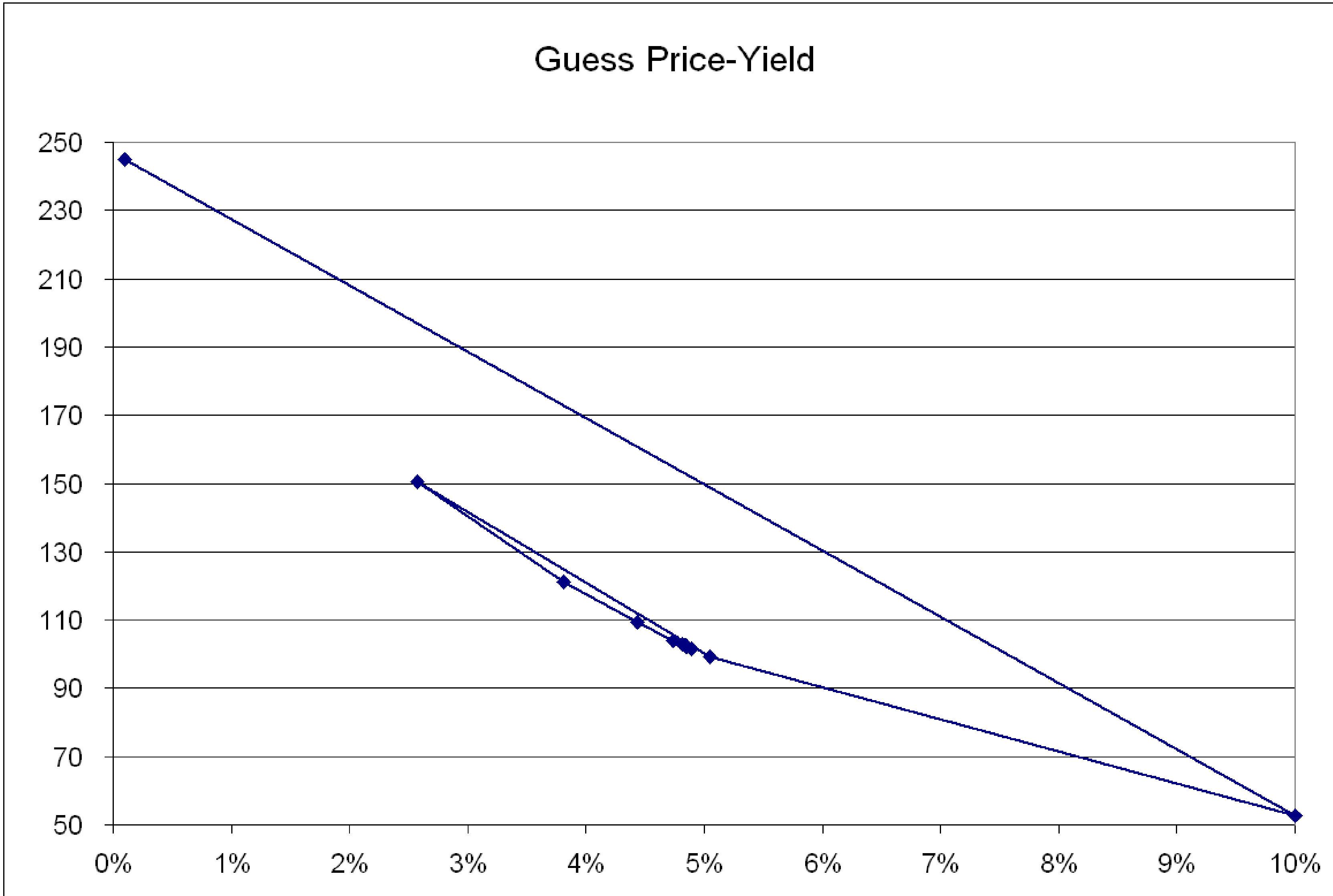
- We have discussed how it is easy to find a price if you know the yield.
- But it is harder to find the yield if you know the price.
- To find the yield of a bond, we just guess at it. We see how far off we are, then refine the guess.
- We stop when we are close enough.

Binary Search

- The naïve approach is to start off with two guesses – one that is too high and one that is too low
- As the next guess, choose the midpoint.
- Determine if the true price is between the new guess and the old high guess or the old low guess.
- Based on the answer, find a new midpoint

Maturity	30
Coupon	5.000%
Yield	4.500%
r	2.250%
Price	102.50000

Count	Yield 1	Price 1	Yield 2	Price 2	New Guess	Guess Price	Difference	High/Low
1	0.100%	244.78	10.000%	52.68	5.050%	99.23167	3.26833	1
2	0.100%	244.78	5.050%	99.23	2.575%	150.46450	47.96450	0
3	2.575%	150.46	5.050%	99.23	3.813%	121.11584	18.61584	0
4	3.813%	121.12	5.050%	99.23	4.431%	109.38864	6.88864	0
5	4.431%	109.39	5.050%	99.23	4.741%	104.12964	1.62964	0
6	4.741%	104.13	5.050%	99.23	4.895%	101.63735	0.86265	1
7	4.741%	104.13	4.895%	101.64	4.818%	102.87244	0.37244	0
8	4.818%	102.87	4.895%	101.64	4.857%	102.25216	0.24784	1
9	4.818%	102.87	4.857%	102.25	4.837%	102.56162	0.06162	0
10	4.837%	102.56	4.857%	102.25	4.847%	102.40672	0.09328	1
11	4.837%	102.56	4.847%	102.41	4.842%	102.48412	0.01588	1
12	4.837%	102.56	4.842%	102.48	4.840%	102.52286	0.02286	0
13	4.840%	102.52	4.842%	102.48	4.841%	102.50349	0.00349	0
14	4.841%	102.50	4.842%	102.48	4.842%	102.49381	0.00619	1
15	4.841%	102.50	4.842%	102.49	4.841%	102.49865	0.00135	1
16	4.841%	102.50	4.841%	102.50	4.841%	102.50107	0.00107	0
17	4.841%	102.50	4.841%	102.50	4.841%	102.49986	0.00014	1
18	4.841%	102.50	4.841%	102.50	4.841%	102.50046	0.00046	0
19	4.841%	102.50	4.841%	102.50	4.841%	102.50016	0.00016	0
20	4.841%	102.50	4.841%	102.50	4.841%	102.50001	0.00001	0
21	4.841%	102.50	4.841%	102.50	4.841%	102.49993	0.00007	1
22	4.841%	102.50	4.841%	102.50	4.841%	102.49997	0.00003	1
23	4.841%	102.50	4.841%	102.50	4.841%	102.49999	0.00001	1
24	**4.841%**	**102.50**	**4.841%**	**102.50**	**4.841%**	**102.50000**	**0.00000**	**1**
25	4.841%	102.50	4.841%	102.50	4.841%	102.50000	0.00000	0
26	4.841%	102.50	4.841%	102.50	4.841%	102.50000	0.00000	0


Guess Price-Yield
250
230
210
190
170
150
130
110
90
70
50
0%
1%
2%
3%
4%
5%
6%
7%
8%
9%
10%

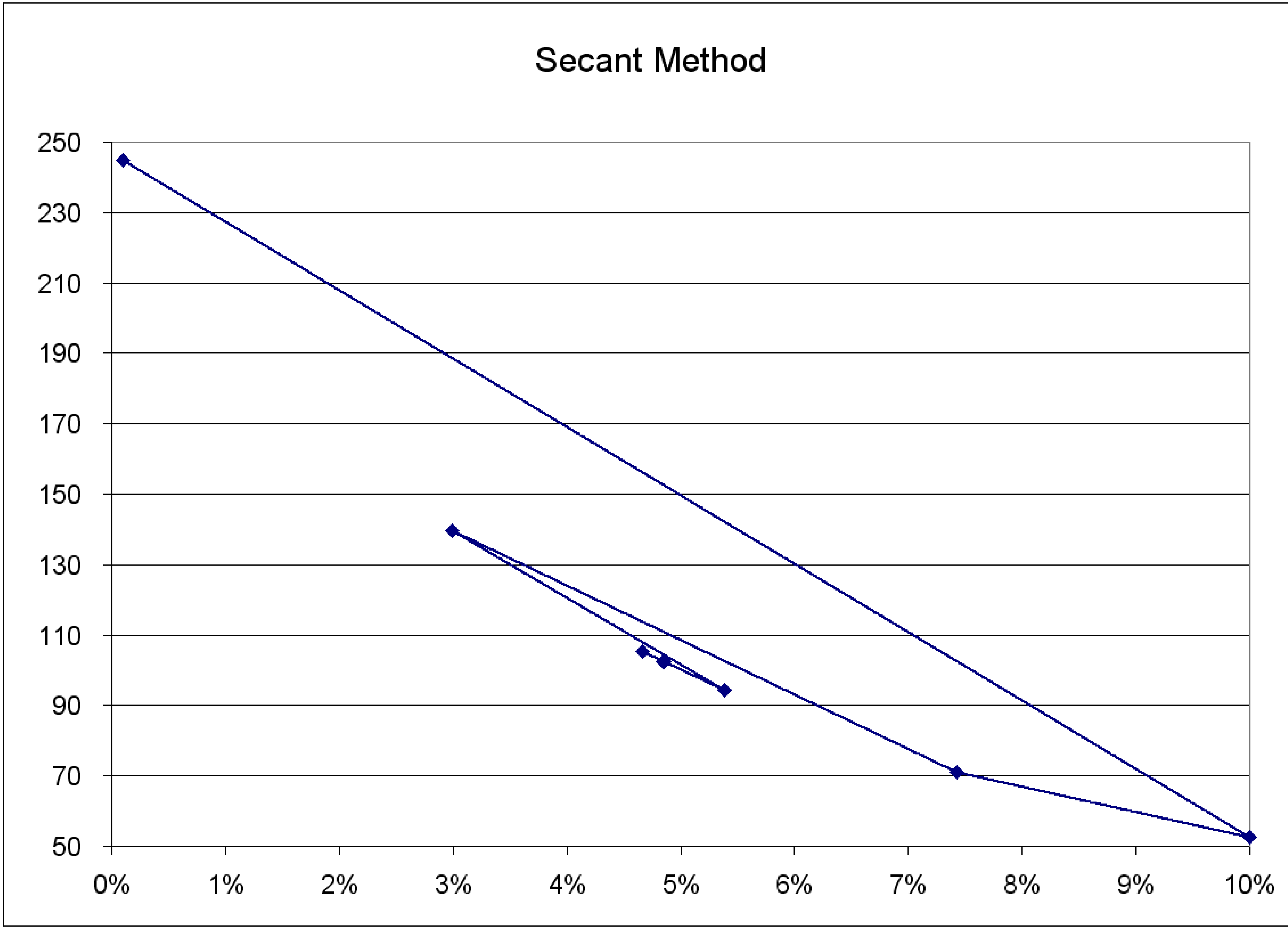
Secant Method

- 24 steps can be computer intensive when doing calculations for thousands of bonds.
- A better method is the Secant Method
  - This method uses interpolation to get the next guess
  - It is the method used by major Wall Street firms.

$$y_g = y_l + (y_h - y_l)\left(\frac{P - P_h}{P_l - P_h}\right)$$

Maturity	30
Coupon	5.000%
Yield	4.500%
r	2.250%
Price	102.50000

Count	Yield 1	Price 1	Yield 2	Price 2	New Guess	Guess Price	Difference
1	0.100%	244.78	10.000%	52.68	7.432%	70.93844	31.56
2	10.000%	52.68	7.432%	70.94	2.995%	139.50922	37.01
3	7.432%	70.94	2.995%	139.51	5.390%	94.23359	8.27
4	7.432%	70.94	5.390%	94.23	4.665%	105.37962	2.88
5	5.390%	94.23	4.665%	105.38	4.852%	102.32120	0.18
6	4.665%	105.38	4.852%	102.32	4.841%	102.49641	0.00
7	**4.852%**	**102.32**	**4.841%**	**102.50**	**4.841%**	**102.50000**	**0.00**
8	4.841%	102.50	4.841%	102.50	4.841%	102.50000	0.00
9	4.841%	102.50	4.841%	102.50	4.841%	102.50000	0.00
10	4.841%	102.50	4.841%	102.50	4.841%	102.50000	0.00


Secant Method
250
230
210
190
170
150
130
110
90
70
50
0%
1%
2%
3%
4%
5%
6%
7%
8%
9%
10%

Finding Yields in Excel

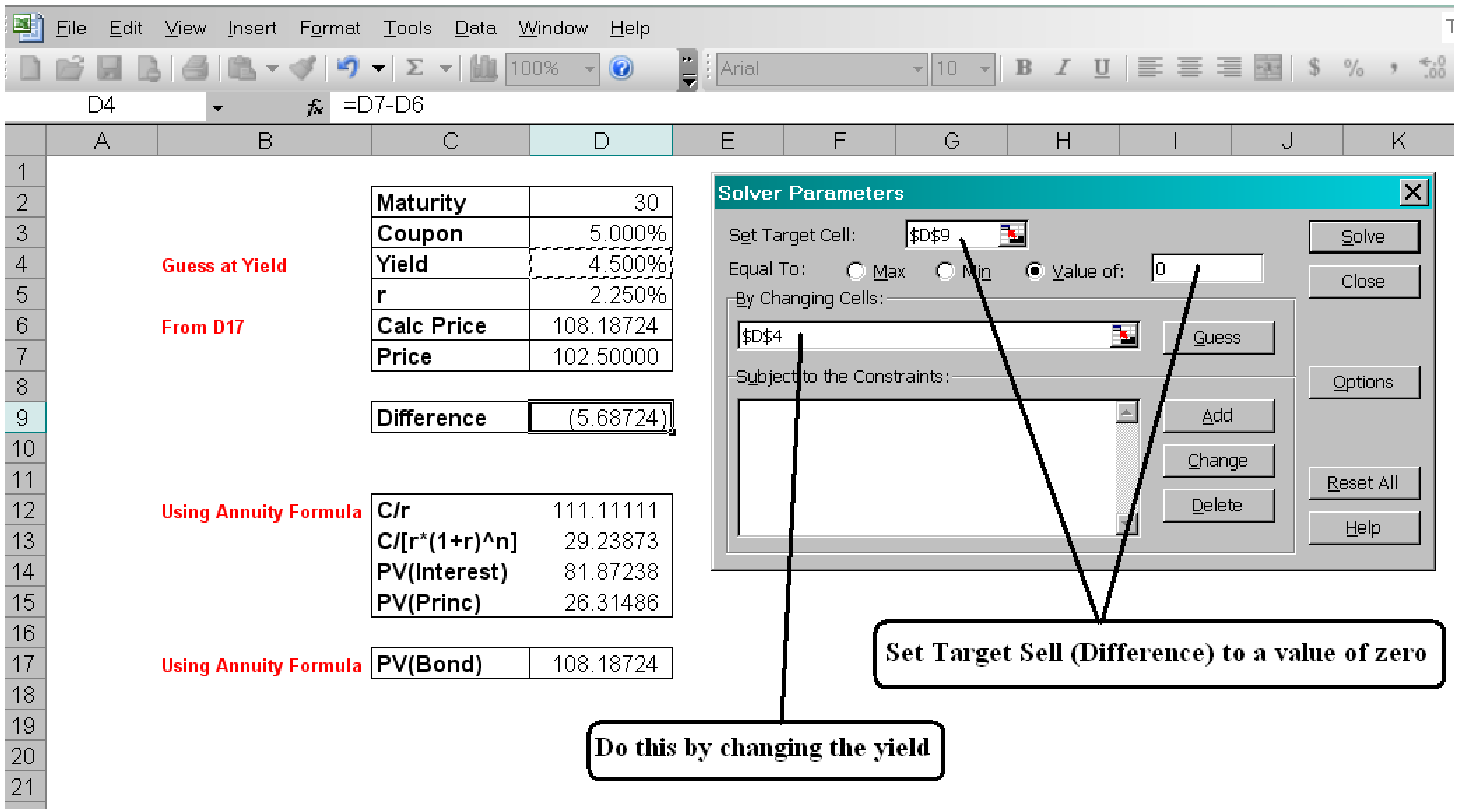
- In Excel, we can use Solver to find the yield
- Set up a spread sheet that finds the price of a bond given a guessed yield. Calculate the difference from the actual price.

	Maturity	30
	Coupon	5.000%
Guess at Yield	Yield	4.500%
	r	2.250%
From D17	Calc Price	108.18724
	Price	102.50000

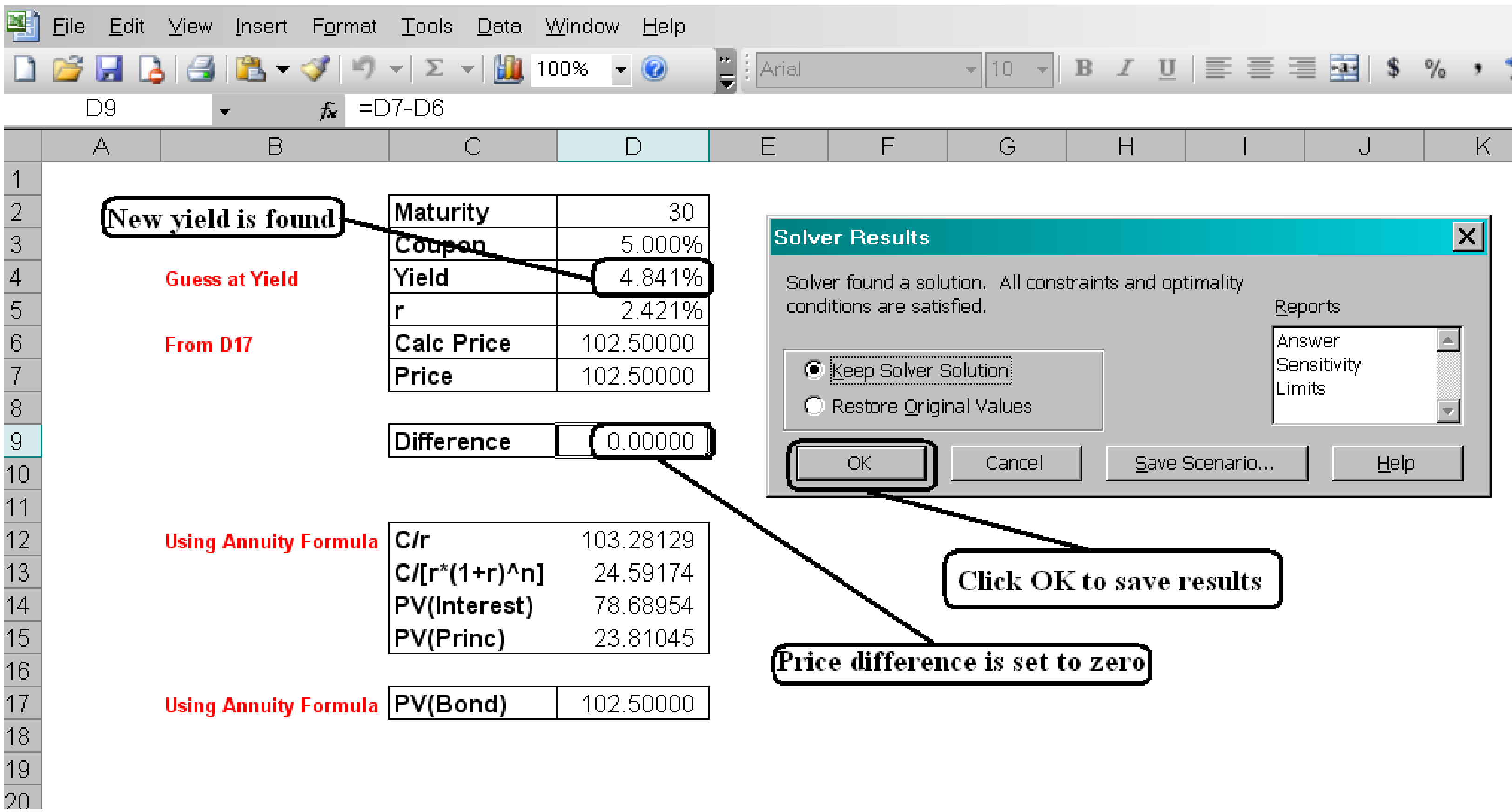
Difference	(5.68724)

Using Annuity Formula	C/r	111.11111
	C/[r*(1+r)^n]	29.23873
	PV(Interest)	81.87238
	PV(Princ)	26.31486

Using Annuity Formula	PV(Bond)	108.18724



File Edit View Insert Format Tools Data Window Help
D4
=D7-D6
Maturity 30
Coupon 5.000%
Guess at Yield
Yield 4.500%
r 2.250%
From D17
Calc Price 108.18724
Price 102.50000
Difference (5.68724)
Using Annuity Formula
C/r 111.11111
C/[r*(1+r)^n] 29.23873
PV(Interest) 81.87238
PV(Princ) 26.31486
Using Annuity Formula
PV(Bond) 108.18724
Solver Parameters
Set Target Cell: D9
Equal To: Max Min Value of: 0
By Changing Cells:
D4
Subject to the Constraints:
Solve
Close
Guess
Options
Add
Change
Delete
Reset All
Help
Set Target Sell (Difference) to a value of zero
Do this by changing the yield


File Edit View Insert Format Tools Data Window Help
100%
Arial
10
D9
=D7-D6
New yield is found
Guess at Yield
From D17
Maturity 30
Coupon 5.000%
Yield 4.841%
r 2.421%
Calc Price 102.50000
Price 102.50000
Difference 0.00000
Using Annuity Formula
C/r 103.28129
C/[r*(1+r)^n] 24.59174
PV(Interest) 78.68954
PV(Princ) 23.81045
Using Annuity Formula
PV(Bond) 102.50000
Solver Results
Solver found a solution. All constraints and optimality conditions are satisfied.
Reports
Answer
Sensitivity
Limits
Keep Solver Solution
Restore Original Values
OK
Cancel
Save Scenario...
Help
Click OK to save results
Price difference is set to zero