Integration of a distribution which is exponential in vertical direction and linearly varying with radius

$$F(r,z) = (const_1 \cdot r + const_2)\, e^{-\frac{z}{z_s}}$$

$$I = 4\pi \int_0^{r_{in}} \int_0^{z_0} (const_1 \cdot r + const_2)\, e^{-\frac{z}{z_s}}\, r\, dr\, dz =$$

$$= \left(4\pi\, const_1 \cdot \frac{r^3}{3}\Big|_0^{r_{in}} + 4\pi\, const_2 \cdot \frac{r^2}{2}\Big|_0^{r_{in}}\right)\left(1 - e^{-\frac{z_0}{z_s}}\right) \cdot z_s =$$

$$= \left(\frac{4\pi}{3}\, const_1 \cdot r_{in}^3 + 2\pi\, const_2 \cdot r_{in}^2\right) z_s \left(1 - e^{-\frac{z_0}{z_s}}\right)$$

$$\begin{cases} F(r_{in}, z) = A_0 e^{-\frac{r_{in}}{h_s}} \cancel{e^{-\frac{z}{z_s}}} = (const_1 \cdot r_{in} + const_2)\, \cancel{e^{-\frac{z}{z_s}}} \\ F(0, z) = \dfrac{F(r_{in}, z)}{2} = \dfrac{A_0}{2} e^{-\frac{r_{in}}{h_s}} \cancel{e^{-\frac{z}{z_s}}} = const_2\, \cancel{e^{-\frac{z}{z_s}}} \end{cases}$$

for the case the value of the function is equal to that of an exponential at $r = r_{in}$ and decreases half that value at $r = 0$

$$const_2 = \frac{A_0}{2} e^{-\frac{r_{in}}{h_s}}$$

$$A_0 e^{-\frac{r_{in}}{h_s}} = const_1 \cdot r_{in} + \frac{A_0}{2} e^{-\frac{r_{in}}{h_s}}$$

$$const_1 = \frac{A_0}{2} \frac{1}{r_{in}} e^{-\frac{r_{in}}{\lambda_s}}$$

$$F(r,z) = \frac{1}{2} A_0 \, e^{-\frac{r_{in}}{\lambda_s}} \left(\frac{r}{r_{in}} + 1 \right) e^{-\frac{z}{z_s}}$$

$$\underline{I} = \left(\frac{4\pi}{3} \frac{A_0}{2} \frac{1}{r_{in}} e^{-\frac{r_{in}}{\lambda_s}} \cdot r_{in}^3 + 2\pi \cdot \frac{A_0}{2} e^{-\frac{r_{in}}{\lambda_s}} r_{in}^2 \right) z_s \left(1 - e^{-\frac{z_0}{z_s}}\right)$$

$$\underline{I} = \frac{5\pi}{3} A_0 \, e^{-\frac{r_{in}}{\lambda_s}} r_{in}^2 \, z_s \left(1 - e^{-\frac{z_0}{z_s}}\right)$$