Double Pendulum Simulation

Simulation of double pendulum to display chaotic behavior using Hamiltonian mechanics. “Chaotic” section contains two different sets of initial conditions (ics) corresponding to bars that are almost vertical, where the ics differ by a very small amount. Chaotic behavior is evident around the halfway point of the simulation. “Non-chaotic” section contains two different sets of ics corresponding to bars that are close to equilibrium position, where again the ics differ by a very small amount. This is non-chaotic, but one can increase the angle (using the parameter “a”) to find the height where the pendulum begins to show chaotic behavior.

To view each animation, be sure lines that define “ChaoticAni” or “NonChaoticAni” are unmuted (i.e. remove the semicolon).

```
(*define constants and Hamiltonian differential equations of motion*)
g = 1;
m = 1;
L = 1;
```

$$\text{eqns} = \Big\{\theta 1'[t] == \frac{6}{m * L^2} * \frac{(2 * p1[t] - 3 * \text{Cos}[\theta 1[t] - \theta 2[t]] * p2[t])}{16 - 9 * (\text{Cos}[\theta 1[t] - \theta 2[t]])^2},$$

$$\theta 2'[t] == \frac{6}{m * L^2} * \frac{(8 * p2[t] - 3 * \text{Cos}[\theta 1[t] - \theta 2[t]] * p1[t])}{16 - 9 * (\text{Cos}[\theta 1[t] - \theta 2[t]])^2},$$

$$p1'[t] == \frac{-1}{2} m * L^2 \left(\theta 1'[t] * \theta 2'[t] * \text{Sin}[\theta 1[t] - \theta 2[t]] + 3 * \frac{g}{L} * \text{Sin}[\theta 1[t]]\right),$$

$$p2'[t] == \frac{-1}{2} m * L^2 \left(-\theta 1'[t] * \theta 2'[t] * \text{Sin}[\theta 1[t] - \theta 2[t]] + \frac{g}{L} * \text{Sin}[\theta 2[t]]\right)\Big\};$$

```
(*define endpoints*)
x1[t_] = L * Sin[θ1[t]];
y1[t_] = -L * Cos[θ1[t]];
x2[t_] = L * Sin[θ2[t]] + x1[t];
y2[t_] = -L * Cos[θ2[t]] + y1[t];
```

Chaotic

Solutions 1,2 (both bars almost vertical)

```
(*define ics and solutions*)
ics1 = {θ1[0] == π, θ2[0] == 0.9 π, θ1'[0] == 0, θ2'[0] == 0};
soln1 = NDSolve[{eqns, ics1}, {θ1, θ2}, {t, 0, 1000}];
ics2 = {θ1[0] == π, θ2[0] == 0.9001 π, θ1'[0] == 0, θ2'[0] == 0};
soln2 = NDSolve[{eqns, ics2}, {θ1, θ2}, {t, 0, 1000}];

(*defining stuff for simulation*)
Animation1 = Animate[
   Graphics[{{Thickness[0.01], Line[{{0, 0}, {x1[t], y1[t]}, {x2[t], y2[t]}}]} /.
      soln1, {Blue, PointSize[0.0005],
      Line[Map[Function[Evaluate[{x2[#], y2[#]} /. soln1]], Range[0, t, 0.05]]]}},
    PlotRange → {{-2, 2}, {-2, 2}}, Axes → True, Ticks → False, ImageSize → 300],
   {t, 0, 50, 0.1}, SaveDefinitions → True, AnimationRate → 1];
Animation2 = Animate[Graphics[
    {{Thickness[0.01], Line[{{0, 0}, {x1[t], y1[t]}, {x2[t], y2[t]}}]} /. soln2,
     {Red, PointSize[0.0005],
      Line[Map[Function[Evaluate[{x2[#], y2[#]} /. soln2]], Range[0, t, 0.05]]]}},
    PlotRange → {{-2, 2}, {-2, 2}}, Axes → True, Ticks → False, ImageSize → 300],
   {t, 0, 50, 0.1}, SaveDefinitions → True, AnimationRate → 1];

(*defining stuff for Lyap plot*)
num1 = θ2[t] /. soln1;
num2 = θ2[t] /. soln2;
diff12 = num1 - num2;


(*Simulation of both solutions 1 and 2*)
ChaoticAni = GraphicsRow[{Animation1, Animation2}, ImageSize → 750];

(*Plot of Lyap exponent*)
ChaoticLyap =
  Plot[Log[Abs[diff12]], {t, 0, 50}, PlotLabel → "Lyap Exponent Sol 1 & 2"];
```

Non-chaotic

Solutions 3,4 (both bars almost horizontal when a=1)

```
(*define ics and solutions*)
a = 0.1;(*increase "a" to make angles bigger*)
ics3 = {θ1[0] == a * 0.500 π, θ2[0] == a * 0.500 π, θ1'[0] == 0, θ2'[0] == 0};
soln3 = NDSolve[{eqns, ics3}, {θ1, θ2}, {t, 0, 1000}];
ics4 = {θ1[0] == a * 0.500 π, θ2[0] == a * 0.501 π, θ1'[0] == 0, θ2'[0] == 0};
soln4 = NDSolve[{eqns, ics4}, {θ1, θ2}, {t, 0, 1000}];

(*defining stuff for simulation*)
Animation3 = Animate[
   Graphics[{{Thickness[0.01], Line[{{0, 0}, {x1[t], y1[t]}, {x2[t], y2[t]}}]} /.
      soln3, {Blue, PointSize[0.0005],
      Line[Map[Function[Evaluate[{x2[#], y2[#]} /. soln3]], Range[0, t, 0.025]]]}},
    PlotRange → {{-2, 2}, {-2, 2}}, Axes → True, Ticks → False, ImageSize → 300],
   {t, 0, 50, 0.025}, SaveDefinitions → True, AnimationRate → 1];
Animation4 = Animate[Graphics[
    {{Thickness[0.01], Line[{{0, 0}, {x1[t], y1[t]}, {x2[t], y2[t]}}]} /. soln4,
     {Red, PointSize[0.0005],
      Line[Map[Function[Evaluate[{x2[#], y2[#]} /. soln4]], Range[0, t, 0.025]]]}},
    PlotRange → {{-2, 2}, {-2, 2}}, Axes → True, Ticks → False, ImageSize → 300],
   {t, 0, 50, 0.025}, SaveDefinitions → True, AnimationRate → 1];

(*defining stuff for Lyap plot*)
num3 = θ2[t] /. soln3;
num4 = θ2[t] /. soln4;
diff34 = num3 - num4;



(*Simulation of both solutions 3 and 4*)
NonChaoticAni = GraphicsRow[{Animation3, Animation4}, ImageSize → 750];

(*Plot of Lyap exponent*)
NonChaoticLyap =
  Plot[Log[Abs[diff34]], {t, 0, 50}, PlotLabel → "Lyap Exponent Sol 3 & 4"];
```