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IDENTIFICATION OF STANDARD AUCTION MODELS

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This paper presents new identification results for models of first-price, second-price, ascending (English), and descending (Dutch) auctions. We consider a general specification of the latent demand and information structure, nesting both private values and common values models, and allowing correlated types as well as ex ante asymmetry. We address identification of a series of nested models and derive testable restrictions enabling discrimination between models on the basis of observed data. The simplest model—symmetric independent private values—is nonparametrically identified even if only the transaction price from each auction is observed. For richer models, identification and testable restrictions may be obtained when additional information of one or more of the following types is available: (i) the identity of the winning bidder or other bidders; (ii) one or more bids in addition to the transaction price; (iii) exogenous variation in the number of bidders; (iv) bidder-specific covariates. While many private values (PV) models are nonparametrically identified and testable with commonly available data, identification of common values (CV) models requires stringent assumptions. Nonetheless, the PV model can be tested against the CV alternative, even when neither model is identified.

KEYWORDS: Auctions, nonparametric identification and testing, private values, common values, asymmetric bidders, unobserved bids, order statistics.

1. INTRODUCTION

THIS PAPER DERIVES new results regarding nonparametric identification and testing of models of first-price sealed-bid, second-price sealed-bid, ascending (English), and descending (Dutch) auctions. The theory literature has focused on several different specifications of the economic primitives in these auctions. In private values models, each bidder knows the value he places on winning the object, but not the values of his opponents. In common values models, information about the value of the object is spread among bidders. Within these classes of models, bidders may be symmetric or asymmetric, and bidders' information may be independent or correlated. We consider a general specification of bidders' preferences and information, nesting private values and common values models, and allowing both correlated private information and ex ante bidder asymmetry.

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We address identification of a series of these models and derive testable implications that enable discrimination between models.

Identification depends critically on what types of data are available—something that varies by application. Prior research on nonparametric identification of auction models has focused on symmetric first-price auctions under the assumption that all bids are observed (see, e.g., the recent survey by Perrigne and Vuong (1999)). However, second-price and ascending auctions are the most common in practice, particularly with the rising popularity of internet auctions (Lucking-Reiley (2000)).[2] Further, while all bids may be observable to the econometrician in some applications, ascending auctions by design end when the next-to-last bidder drops out, leaving the planned exit price of the winner unobserved. Likewise, a Dutch (descending) auction ends as soon as the first bid is made. Even in sealed-bid auctions, researchers may have access only to a subset of the submitted bids—e.g., the winning bid or the top two bids.

For the simplest independent private values (IPV) model, a model studied extensively in the prior literature using parametric methods, we show that the transaction price alone is sufficient for nonparametric identification. For richer private and common values models, however, omitting even one order statistic from the sample of bids can create challenges. We show that without additional structure neither an affiliated private values nor an affiliated common values model is identified from bids at an ascending auction or any other standard auction in which one or more bids is unobserved. However, other types of data commonly available in practice (such as bidder identities or bidder-specific covariates) can enable identification of richer models even when one or more bids is unobserved.

Work in the empirical literature on auctions has typically proceeded by first assuming a model of bidder demand and information (such as IPV) based on qualitative features of the application. However, because different assumptions can lead to very different results and policy implications, a formal basis for evaluating alternative models would be preferred and could lead to greater confidence in empirical results. We show that the assumptions of standard models often can be tested, using data available in many applications considered previously in the literature. Furthermore, a failure of identification need not preclude testing. We show, for example, that when there is exogenous variation in the number of bidders, the private values model can be tested against the common values alternative, even when neither model is identified.

Several other factors provide additional motivation for our focus on nonparametric identification and testing. First, given the effects that ad hoc parametric

[2] The prevalence of ascending auctions is well known. Second-price auctions have been used to allocate public resources such as radio spectrum (Crandall (1998)). In ascending auctions, the use of agents (human or software) who bid according to pre-specified cutoff prices results in an auction game equivalent to a second-price sealed bid auction. Bajari and Hortaçsu (2000) and Roth and Ockenfels (2002) have argued that bids at certain internet auctions can be viewed as coming from standard second-price sealed bid auctions.

assumptions can have on empirical results, one is naturally interested in knowing what can be estimated without such assumptions. Determining the conditions under which the distribution of observables uniquely determines the primitive distributions of interest is a critical step toward answering this question and developing nonparametric estimators. Indeed, in many cases our identification arguments suggest estimation approaches. Of course, the question of identifiability of a model is fundamentally distinct from the choice of approximation used for estimation. When a model is nonparametrically identified, one can view a parametric specification as a parsimonious approximation rather than a maintained hypothesis about the true structure (see, e.g., Roehrig (1988)). Conversely, nonidentification results can both demonstrate why the data fail to enable inferences of certain types and suggest the range of models that could generate the same observables—something that may be valuable for policy-makers interpreting results obtained with strong identifying assumptions. Finally, a variety of positive and normative issues, including market design, the optimal use of reserve prices, the effects of mergers between bidders, and the effect of increased bidder participation on revenues, depend critically on which model describes the environment and on the specific distributions characterizing the demand and information structure. Hence, our results address central challenges facing researchers hoping to evaluate the structure of demand at auctions in order to guide policy.

Our work contributes to a growing applied and theoretical literature on structural econometrics of auctions.[3] Several parametric estimation approaches have been proposed within the IPV framework.[4] Nonparametric methods have been developed by Guerre, Perrigne, and Vuong (2000) for first-price auctions and by Haile and Tamer (2002) for ascending auctions. While a few papers have considered structural estimation outside the IPV paradigm,[5] each either relies on parametric distributional assumptions or addresses only first-price auctions in which all bids are observed. The same is true of the handful of papers proposing tests to distinguish common and private values models.[6] No prior work has considered nonparametric identification and testing of the standard alternative models of ascending and second-price sealed-bid auctions. To our knowledge, the only prior work addressing nonparametric identification when some bids are unobserved applies to symmetric IPV first-price auctions (Guerre, Perrigne, and Vuong (1995)).

The remainder of the paper is organized as follows. We first describe our general framework and review equilibrium predictions. In Section 3 we consider

[3] Recent surveys include Hendricks and Paarsch (1995), Laffont (1997), Perrigne and Vuong (1999), and Hendricks and Porter (2000).

[4] See, e.g., Paarsch (1992b), Laffont and Vuong (1993), Laffont, Ossard, and Vuong (1995), Donald and Paarsch (1996), Baldwin, Marshall, and Richard (1997), Deltas and Chakraborty (1997), Donald, Paarsch, and Robert (1999), Bajari and Ye (2000), and Haile (2001).

[5] See, e.g., Paarsch (1992a), Li, Perrigne, and Vuong (2000, 2002), Hong and Shum (2002), and Bajari and Hortaçsu (2000).

[6] See Paarsch (1992a), Hendricks, Pinkse, and Porter (2002), and Haile, Hong, and Shum (2000).

identification and testing of private values models, beginning with second-price sealed-bid and ascending auctions. At the end of this section we extend many of these results to first-price auctions in which some bids are unobserved. Section 4 then takes up the case of common values, where a scarcity of positive identification results motivates development of tests for distinguishing common and private values models in Section 5. Section 6 discusses the robustness of our results to bidder uncertainty regarding the number of opponents they face and to the seller's use of a reserve price. Section 7 concludes.

2. THE MODEL

2.1. *Primitives and Equilibrium Strategies*

Consider an auction of a single indivisible good with $n \geq 2$ risk-neutral bidders. In our base model we assume that the number of bidders is common knowledge and there is no reserve price (we relax these assumptions in Section 6). Each bidder $i = 1, \ldots, n$ would receive utility $U_i - p$ from winning the object at price p. Following the literature, we use the terms "utility," "valuation," and "value" interchangeably. Let $F_{U_i}(\cdot)$ and $F_{\mathbf{U}}(\cdot)$ denote the distributions of U_i and $\mathbf{U} = (U_1, \ldots, U_n)$, respectively.

Each bidder i's private information consists of a signal X_i that is affiliated with U_i. Let $F_{\mathbf{X}}(\cdot)$ denote the joint distribution of $\mathbf{X} = (X_1, \ldots, X_n)$. When we refer to models with *symmetric* bidders we assume the joint distribution of $(\mathbf{U}, \mathbf{X})$ is exchangeable with respect to the bidder indices, so that $F_{\mathbf{U}}(\cdot)$ and $F_{\mathbf{X}}(\cdot)$ are exchangeable; in this case, the marginal distributions $F_{U_i}(\cdot)$ and $F_{X_i}(\cdot)$ can be written $F_U(\cdot)$ and $F_X(\cdot)$. When we treat models allowing *asymmetric* bidders we drop the exchangeability assumption. For a sample of generic random variables $\mathbf{S} = (S_1, \ldots, S_n)$ drawn from a distribution $F_{\mathbf{S}}(\cdot)$, we denote by $S^{(j:n)}$ the jth order statistic, with, e.g., $S^{(n:n)}$ denoting the maximum. Similarly, $F_S^{(j:n)}(\cdot)$ denotes the marginal distribution of $S^{(j:n)}$.

Our framework nests a wide range of specifications of the underlying demand and information structure, falling into two classes of models:

PRIVATE VALUES (PV): $U_i = X_i \; \forall i$.

COMMON VALUES (CV): For all i and j, U_i and X_j are strictly affiliated conditional on any $\chi \subset \{X_k\}_{k \neq j}$, but are not perfectly correlated.

In a private values model, no bidder has private information relevant to another's expected utility. In contrast, in a common values model bidder i would update her beliefs about her utility, U_i, if she observed X_j in addition to her own signal X_i. Thus, there is a "winner's curse" in a common values model. Roughly, winning the auction reveals (in equilibrium) to the winner that her signal was more optimistic than those of her opponents; rational bidders anticipate this information when forming expectations of the utility they would receive by winning.[7] Note that CV models allow utilities to differ across bidders; however, in the special case of *pure common values* (discussed below), $U_i = V$ for all i.[8]

The value and signal distributions are assumed to be common knowledge among the bidders. In a first-price (second-price) sealed-bid auction bidders submit bids simultaneously, with the object going to the high bidder at a price equal to his bid (to the second-highest bid). For ascending auctions[9] we assume the standard "button auction" model of Milgrom and Weber (1982), where bidders exit observably and irreversibly as the price rises exogenously until only one bidder remains.[10]

Throughout the paper we restrict attention to (perfect) Bayesian Nash equilibria in weakly undominated strategies, denoted by $\beta_i(\cdot)$ for each i, and further to symmetric equilibria when bidders are ex ante symmetric. In the second-price auction, each bidder i's equilibrium bid when $X_i = x_i$ solves

$$b_i = \beta_i(x_i) = E\Big[U_i \mid X_i = x_i, \max_{j \neq i} \beta_j(X_j) = b_i\Big]. \tag{1}$$

In a PV auction, this becomes $b_i = x_i = u_i$. In a CV auction, strict affiliation implies that $\beta_i(\cdot)$ is strictly increasing; hence, when bidders are symmetric, $\beta_i(x_i) = E[U_i \mid X_i = \max_{j \neq i} X_j = x_i]$.

Equilibrium strategies are similar for an ascending auction, where a "bid" is a planned price at which to exit. However, two complications arise. First, as the auction proceeds bidders condition on the signals of opponents who have already dropped out, as inferred from their exit prices. Second, in the symmetric model there are multiple symmetric equilibria in weakly undominated strategies (Bikhchandani, Haile, and Riley (2002)). In any such equilibrium, however, if i is one of the last two bidders to exit, his exit price b_i solves

$$b_i = \beta_i(x_i) = E[U_i \mid X_i = x_i, \beta_j(X_j) = b_i \ \forall j \notin \{i \cup L_i\}, X_k = x_k \ \forall k \in L_i], \tag{2}$$

[7] The strict affiliation assumption in our definition of common values is a restriction ensuring that the winner's curse arises. Up to this simplifying assumption our PV and CV definitions define a partition of Milgrom and Weber's (1982) general affiliated values model, although they impose the additional restrictions of symmetry and affiliation of $(\mathbf{U}, \mathbf{X})$.

[8] Sometimes the class of models we refer to as CV are referred to as models with "interdependent values," with the term "common values" reserved for the pure common values model. While both taxonomies are used in the literature, we follow that which emphasizes the distinction between *statistical* properties (independence, affiliation, etc.) of bidders' private information and the *economic* nature (private vs. common value) of this information.

[9] We use the terms "English auction" and "ascending auction" interchangeably. For reviews of standard auction models see, e.g., Milgrom and Weber (1982), McAfee and McMillan (1987a), or Klemperer (1999).

[10] This is a stylized model of an ascending auction that, for example, rules out jump bidding. This model will match actual practice better in some applications than others. Some ascending auctions are designed with "activity rules" specifically to replicate features of the button auction (e.g., the FCC spectrum auctions discussed in McAfee and McMillan (1996)). For an empirical model of English auctions avoiding the button auction structure, see Haile and Tamer (2001, 2002).

where L_i denotes the set of bidders who exit before i. With private values, all bidders use this strategy, which reduces to $b_i = x_i = u_i$. Note that the auction ends at the price $b^{(n-1:n)}$.

In the first-price auction or the (strategically equivalent) Dutch auction, after observing $X_i = x_i$, bidder i solves (letting $B_j = \beta_j(X_j)$)

$$\max_{b_i}\left(E[U_i \mid X_i = x_i, \max_{j\neq i} B_j \leq b_i] - b_i\right) \Pr\left(\max_{j\neq i} B_j \leq b_i | X_i = x_i\right). \tag{3}$$

For first-price auctions (only) we make the additional assumptions that $F_{\mathbf{X}}(\cdot)$ (i) is affiliated (at least weakly) and (ii) has an associated positive joint density $f_{\mathbf{X}}(\cdot)$. For PV models of the first-price auction we also assume (iii) X_i has common support for all i. For the first-price auction models we consider below,[11] (i)–(iii) ensure existence of an equilibrium in which (a) strategies are strictly increasing and (b) the supports of B_i and $\max_{j\neq i} B_j$ are identical;[12] however, (i)–(iii) are otherwise unrelated to our identification arguments.

When equilibrium bidding strategies are strictly increasing, the equilibrium bids $(B_1, \ldots, B_n)$ have the same information content as the signals $(X_1, \ldots, X_n)$. Define

$$\zeta_i(x; n) = E\left[U_i \mid X_i = x, \max_{j\neq i} B_j = \beta_i(x)\right].$$

If bidders are symmetric, $\zeta_i(x; n) = \zeta(x; n) = E[U_i \mid X_i = \max_{j\neq i} X_j = x]$. For almost every signal x_i of bidder i, a necessary condition for b_i to be an optimal bid in a first-price auction is[13]

$$b_i + \frac{\Pr(\max_{j\neq i} B_j \leq b_i | B_i = b_i)}{\frac{\partial}{\partial z}\Pr(\max_{j\neq i} B_j \leq z | B_i = b_i)\,|_{z=b_i}} = \zeta_i(x_i; n). \tag{4}$$

[11] Below, our results focus on symmetric bidders when considering the CV model of the first-price auction.

[12] For the symmetric PV and CV models, there exists a symmetric equilibrium in strictly increasing, differentiable strategies (Milgrom and Weber (1982)). More generally, Athey (2001) shows that as long as each bidder's best response to nondecreasing opponent strategies is itself nondecreasing, a pure strategy Nash equilibrium exists, where for almost all x_i, the distribution over opponent bids is continuous at $\beta_i(x_i)$. A sufficient condition for such monotonicity in a PV auction is affiliation. For PV auctions, it can be shown that the common support restriction is sufficient (but not necessary) to ensure that the equilibrium strategies satisfy (a) and (b) above. In CV models, the conditions required for monotone best responses are more stringent when there are more than two asymmetric bidders, but Athey (2001) shows that they are satisfied in the "mineral rights" model we discuss in Section 4.

[13] Our assumptions guarantee that the derivative in (4) exists and is positive at $b_i = \beta_i(x_i)$ for almost all x_i. Further, (a) and (b) above ensure that (4) must hold for almost every x_i. A Bayesian Nash equilibrium in a game with atomless type distributions only specifies behavior up to a set of signals of measure zero. Likewise, since behavior on a set of measure zero does not affect any of the analysis, we can ignore measure zero differences in bid functions.

2.2. *Observables*

We consider situations in which the researcher has access to a sample of observations from independent auctions. The joint distribution of $(\mathbf{U}, \mathbf{X})$ (conditional on auction-specific covariates, if any) is fixed across auctions, with each auction representing an independent draw from this distribution.[14] We let B_i denote the bid made by player i, $H_{B_i}(\cdot)$ its distribution, and $H_{\mathbf{B}}(\cdot)$ the joint distribution of bids. The econometrician always observes the number of bidders and the transaction price, which equals either $B^{(n-1:n)}$ or $B^{(n:n)}$, depending on the type of auction. If bidders are asymmetric, we assume that the set of bidders who participate in each auction is observed (this holds trivially if the same bidders participate in all auctions). In addition, the following may or may not be observed: (i) other bids; (ii) the bidder identities associated with one or more bids, with $I^{(m:n)}$ giving the identity of the bidder bidding $B^{(m:n)}$; (iii) auction-specific covariates, such as the ex ante appraised value of the object or the ex post realization of the value of the object; or (iv) bidder-specific covariates.

Bids other than the transaction price are observed in some but not all applications.[15] Bidder identities and bidder-specific covariates such as firm size, location, or inventories are often observed, particularly for government auctions, as are auction-specific covariates such as the appraised value or other characteristics of the good for sale. Measures of the realized value of the good are observed in government auctions of mineral leases and timber contracts.[16] In other cases resale prices can provide measures of realized values (e.g., McAfee, Takacs, and Vincent (1999)).

Finally, we allow the possibility that the number of bidders varies exogenously. To formalize this, let $\mathbf{S} = (\mathbf{U}, \mathbf{X})$. For $\mathcal{P} \subset \{1, 2, \dots\}$ let $F_{\mathbf{S}}^{\mathcal{P}}(\cdot)$ denote the joint distribution of $\{S_i\}_{i \in \mathcal{P}}$ when $\mathcal{P}$ is the set of bidders participating in the auction. For $\mathcal{P}' \subset \mathcal{P}$, let $F_{\mathbf{S}}^{\mathcal{P}'|\mathcal{P}}(\cdot)$ denote the marginal distribution of $\{S_i\}_{i \in \mathcal{P}'}$ when $\mathcal{P}$ is the set of participants. Typically the number of bidders, $n = |\mathcal{P}|$, varies across auctions. We say that participation is *exogenous* if for all $\mathcal{P}$ and $\mathcal{P}' \subset \mathcal{P}$, $F_{\mathbf{S}}^{\mathcal{P}'|\mathcal{P}}(\cdot) = F_{\mathbf{S}}^{\mathcal{P}'}(\cdot)$. This exogenous variation could arise if (outside the formal model described above) there were a pool of potential bidders who received random shocks to the cost of participating that were independent of U_i and X_i. Bidders with favorable shocks would then participate and learn X_i. With no reserve price, all bidders who learn X_i would place a bid in the auction, yielding

[14] These assumptions are standard in the literature. Such data might arise as a result of noncooperative bidding in auctions for procurement contracts or natural resources, where the underlying competitive environment is stationary over the sample period and contracts are small from the perspective of the bidders. In some applications, where the same bidders participate in multiple auctions over time, we might expect dependence of bidders' information and/or willingness to pay on outcomes of prior auctions and expectations of future opportunities (e.g., Jofre-Bonet and Pesendorfer (2000)). Examining the empirical implications of such models is a valuable direction for future research.

[15] In oral "open outcry" auctions we may lack confidence in the interpretation of losing bids below the transaction price even when they are observed.

[16] See Hendricks and Porter (1988), Hendricks, Pinkse, and Porter (2002), and Athey and Levin (2001).

exogenous variation in n. Such variation can also arise from participation restrictions by the seller (e.g., in government auctions), by design in field experiments (e.g., Engelbrecht-Wiggans, List, and Lucking-Reiley (1999)), or from variation in the length of internet auctions, where more potential bidders may become aware of longer auctions.

2.3. *Identification*

A model is identified if, given the implications of equilibrium behavior in a particular auction game, the joint distribution of bidders' utilities and signals is uniquely determined by the joint distribution of observables. More formally, define a *model* as a pair $(\mathbb{F}, \Gamma)$, where $\mathbb{F}$ is a set of joint distributions over the vector of latent random variables, Γ is a collection of mappings $\gamma : \mathbb{F} \to \mathbb{H}$, and $\mathbb{H}$ is the set of all joint distributions over the vector of observable random variables. Implicit in the specification of a model is the assumption that it contains the true $(\mathcal{F}, \gamma)$ generating the observables.[17]

DEFINITION: A model $(\mathbb{F}, \Gamma)$ is *identified* iff for every $(F, \widehat{F}) \in \mathbb{F}^2$ and $(\gamma, \hat{\gamma}) \in \Gamma^2$, $\gamma(F) = \hat{\gamma}(\widehat{F})$ implies $(F, \gamma) = (\widehat{F}, \hat{\gamma})$.

A model is testable if equilibrium behavior in that model implies refutable restrictions on the distribution of observables.[18]

DEFINITION: A model $(\mathbb{F}, \Gamma)$ is *testable* iff $\bigcup_{\gamma \in \Gamma, \mathcal{F} \in \mathbb{F}} \gamma(\mathcal{F}) \neq \mathbb{H}$.

We emphasize that throughout our analysis we maintain the assumption of equilibrium bidding. Hence, failures of the predictions of a particular model are interpreted as violations of assumptions regarding model primitives, not as a failure of bidders to follow equilibrium strategies. Of course, failures of the latter type are also of interest and could lead to rejections as well.

3. PRIVATE VALUES MODELS

We first consider identification of a series of private values models, showing how richer models require richer data sets for identification and testing. We focus initially on ascending and second-price auctions, and these auction forms are assumed unless otherwise stated. In these auctions, the equilibrium bid function is just the identity function, so the identification question reduces to that of whether the joint distribution of valuations can be determined when only certain order

[17] For our purposes, we typically consider a single equilibrium so that Γ is a singleton. In such cases we often refer explicitly to identification of $\mathcal{F}$, showing that $\mathcal{F}$ is uniquely determined in $\mathbb{F}$ by Γ and the observed $\mathcal{H} \in \mathbb{H}$. A few results address *partial* identifiability, referring explicitly to the identified components of $\mathcal{F}$.

[18] Just as addressing identification precedes the development and evaluation of estimators, showing that a model is testable leaves open important details regarding appropriate testing procedures.

statistics are observed. At the end of this section we consider first-price sealed-bid and Dutch auctions where, although bidding strategies are more complicated, many of the same ideas can be applied.

3.1. *Independent Private Values*

One of the most widely studied auction models is the IPV model.

INDEPENDENT PRIVATE VALUES (IPV): $X_i = U_i \ \ \forall i$, with $(X_1, \ldots, X_n)$ mutually independent.

Much empirical work has focused on the symmetric IPV model. We begin by showing that in this model, the underlying distribution of valuations is nonparametrically identified even when only one bid per auction is observed. Furthermore, the model can be tested if more than one bid is observed or there is exogenous variation in the number of bidders.

THEOREM 1: *In the symmetric IPV model,* (i) $F_{\mathbf{U}}(\cdot)$ *is identified from the transaction price.* (ii) *The model is testable if either* (a) *more than one bid per auction is observed or* (b) *transaction prices are observed from auctions with exogenously varying numbers of bidders.*

PROOF: (i) The ith order statistic from an i.i.d. sample of size n from an arbitrary distribution $F(\cdot)$ has distribution (see, for example, Arnold, Balakrishnan, and Nagaraja (1992))

$$F^{(i:n)}(z) = \frac{n!}{(n-i)!\,(i-1)!} \int_0^{F(z)} t^{i-1}(1-t)^{n-i}\,dt. \tag{5}$$

Because the right-hand side of (5) is strictly increasing in $F(z)$, $F(z)$ is uniquely determined by $F^{(i:n)}(z)$ for any $(i:n)$. Hence, $F_U(\cdot)$ is identified whenever any distribution $F_U^{(i:n)}(\cdot)$ is. Since the observed transaction price is equal to the order statistic $U^{(n-1:n)}$ and the marginal distribution $F_U(\cdot)$ completely determines $F_{\mathbf{U}}(\cdot)$, the result follows.

(ii) Under the IPV assumption, values of $F_U(u)$ implied by the distributions of different order statistics ($U^{(i:n)}$ and $U^{(j:m)}$ with $\mathbf{1}\{i=j, n=m\}=0$) must be identical for all u, a testable restriction. *Q.E.D.*

Although the identification result is an immediate implication of known properties of order statistics, nonparametric identification of even this simplest model of second-price and English auctions has not been previously established. Indeed, all prior structural econometric work on second-price and English auctions has relied on parametric distributional assumptions even in the symmetric IPV model (e.g., Donald and Paarsch (1996), Paarsch (1997)). Theorem 1 shows that parametric approaches will not always be necessary and that the assumptions of this model of bidder demand can be tested even if only the transaction price is observed. Further, the result provides an approach to testing the symmetric IPV assumption, based on detecting violations of (5). Since (5) holds only for i.i.d. random variables, the parent distributions implied by different $F^{(j:n)}(\cdot)$ need not be the same if bidders' signals are not independent or bidders are asymmetric.[19,20]

Theorem 1 relies on both the independence and exchangeability properties of the symmetric IPV model. However, identification from a single bid (in particular, the transaction price) still holds when exchangeability is dropped as long as the identity of the winning bidder is also observed. We establish this by applying results from the literatures on competing risks (Berman (1963)) and reliability theory (Meilijson (1981)).[21]

THEOREM 2: *In the asymmetric IPV model, assume that each* $F_{U_i}(\cdot)$ *is continuous and that* $\text{supp}[F_{U_i}(\cdot)]$ *is the same for all* i. *Each* $F_{U_i}(\cdot)$ *is identified if either*:

(a) *the transaction price* $(B^{(n-1:n)})$ *and identity of the winner are observed*; *or*

(b) *in a second-price auction, the highest bid* $(B^{(n:n)})$ *and identity of the winner are observed*.

Given (a) *or* (b), *the model is testable if either*:

(i) *participation is exogenous and each bidder* i *participates in auctions with at least two different sets of opponents*; *or*

(ii) *in a second-price auction, both* (a) *and* (b) *hold*.

PROOF: Identification given (a) follows from Meilijson (1981, Theorem 1 and Section 4). Identification given (b) follows from Theorem 7.3.1 and Remark 7.3.1 in Prakasa-Rao (1992). Each of these arguments holds when the set of participants is fixed. Hence, with exogenous variation in participation, each $F_{U_i}(\cdot)$ is uniquely determined by data from each $\mathcal{P} \ni i$, and equality of the distributions $F_{U_i}(\cdot)$ obtained from each such $\mathcal{P}$ is a testable restriction. Similarly, if both (a) and (b) hold, the two versions of the distribution $F_{U_i}(\cdot)$ implied by the distributions of $(B^{(n-1:n)}, I^{(n:n)})$ and $(B^{(n:n)}, I^{(n:n)})$ must be identical, providing a testable restriction. *Q.E.D.*

[19] Bidding in many ascending auctions is free-form. In such cases the IPV button-auction model could be rejected even when the symmetric IPV assumptions hold because recorded bids understate the true willingness-to-pay of the bidders. In that case, one could compare estimates obtained from low-ranked bids to those obtained from high-ranked bids to assess the magnitude of the bias that arises from bidders' failing to bid at prices as high as their true valuations.

[20] It may be surprising that independence is testable using only one bid per auction. However, (5) specifies a particular way in which the distribution of $B^{(n-1:n)}$ must vary with n. This restriction fails in natural examples with asymmetric or strictly affiliated values.

[21] In competing risks models, a system fails as soon as one of its components fails, enabling observation of the first failure time and the identity of the component that fails. Up to a trivial translation between minimum and maximum, this is formally equivalent to observing the bid and identity of the highest bidder in an auction. In a more general reliability model, a "coherent system" fails when certain combinations of components fail. In such cases one might observe only "autopsy statistics," consisting of the time of system failure and the set of components that have failed by the time of system failure. This is formally equivalent to observing the price at which the auction ends and the set of bidders who made bids at or below this price.

3.2. *Auction-Specific Covariates*

Several generalizations of the IPV model have been considered in the literature. For example, Li, Perrigne, and Vuong (2000) consider a model in which $U_i = V + A_i \forall i$, with $(V, \mathbf{A})$ mutually independent.[22,23] In this model, each A_i is analogous to an i.i.d. measurement error on the variable V. If all bids are observed in a second-price auction, identification is then immediate from existing results (Kotlarski (1966), Prakasa-Rao (1992), Li and Vuong (1998)).[24] The identification argument holds even if bidders are asymmetric as long as bidder identities are observed, but fails when not all bids are observed. Observed bids correspond to realizations of order statistics $U^{(j:n)} = A^{(j:n)} + V$. Because order statistics are dependent even when the underlying random variables are independent, the "measurement errors" $\{A^{(j:n)}\}$ underlying the observed bids are dependent, precluding application of methods from the measurement error literature that rely on independence.[25]

Identification holds, however, if the conditioning variable V is determined by observable covariates; e.g., $V = g_0(\mathbf{W}_0)$ for some (unknown) function g_0. This is a natural structure in many applications: while the idiosyncratic components of bidders' valuations are independent, correlation of valuations at each auction arises through variation in the observable attributes of the objects sold.

THEOREM 3: *Let private values be given by $U_i = g_i(A_i, \mathbf{W}_0)$ where each $g_i(\cdot,\cdot)$ is an unknown function, $F_{A_i}(\cdot)$ is continuous and strictly increasing, and supp$[F_{A_i}]$ is the same for all i. Assume $\mathbf{W}_0$ and the transaction price are observed. If bidders are asymmetric, assume that the identity of the winner is also observed.*

(i) *If $A_1, \ldots, A_n$ are independent conditional on $\mathbf{W}_0$, then each $F_{\mathbf{U}}(\cdot|\mathbf{w}_0)$ is identified.*

(ii) *Let $g_i(A_i, \mathbf{W}_0) = A_i + g_0(\mathbf{W}_0)$ $\forall i$, with $g_0(\cdot)$ an unknown function. If $A_1, \ldots, A_n$ are mutually independent and independent of $\mathbf{W}_0$, then $F_{\mathbf{A}}(\cdot)$ and $g_0(\cdot)$ are identified up to location.*

(iii) *The symmetric models in* (i) *and* (ii) *are testable if more than one bid is observed in each auction or the transaction price is observed in auctions with exogenously varying numbers of bidders.*

[22] A closely related model is one in which V is observed by bidders but not by the econometrician. The two models are equivalent in the case of a second-price or ascending auction. In a first-price auction, however, the distinction is important. Since the realization of V would affect a bidder's beliefs about opponents' signals, bids generally will not satisfy (4) if V is observed by bidders.

[23] In Athey and Haile (2000) we showed that this structure can match the first two moments of any affiliated exchangeable distribution of values, but imposes restrictions on third moments.

[24] Li, Perrigne, and Vuong (2000) applied this approach to the case of symmetric first-price auctions.

[25] When $(V, \mathbf{A})$ are mutually independent, it seems plausible that, since each $A^{(j:n)}$ is an order statistic from an i.i.d. sample from a common parent distribution, there may be sufficient structure to identify the model from only two order statistics $U^{(j:n)}, U^{(k:n)}$. However, we have not obtained such a result. It is interesting to note that the difference $U^{(j:n)} - U^{(k:n)} = A^{(j:n)} - A^{(k:n)}$ does not identify $F_A(\cdot)$ up to location (Arnold, Balakrishnan, and Nagaraja (1992, p. 143)).

(iv) *The asymmetric models in* (i) *and* (ii) *are testable in a second-price sealed-bid auction if winner's identity and bid are observed*.

PROOF: (i) For fixed $\mathbf{w}_0, U_1, \ldots, U_n$ are independent, so we can apply Theorem 1 (in the symmetric case) or Theorem 2 (in the asymmetric case). (ii) For each $\mathbf{w}_0$, in equilibrium

$$H_B^{(n-1:n)}(b \mid \mathbf{w}_0) = \Pr\left(A^{(n-1:n)} \leq b - g_0(\mathbf{w}_0)\right) = F_A^{(n-1:n)}(b - g_0(\mathbf{w}_0)).$$

Using standard arguments, variation in b and $\mathbf{w}_0$ identifies $g_0(\cdot)$ up to a location normalization. Then, identification of $F_{\mathbf{A}}(\cdot)$ follows from Theorem 1 under symmetry and from Theorem 2 when bidders are asymmetric. (iii) Suppose $B^{(i:n)}$ and $B^{(j:m)}$ are observed and $\mathbf{1}\{i = j, n = m\} = 0$. Let $F_U(\cdot|\mathbf{w}_0; i, n)$ and $F_U(\cdot|\mathbf{w}_0; j, m)$ $[F_A(\cdot; i, n)$ and $F_A(\cdot; j, m)]$ denote the marginal distributions implied by the bid distributions $H_B^{(i:n)}(\cdot|\mathbf{w}_0)$ and $H_B^{(j:m)}(\cdot|\mathbf{w}_0)$, respectively, using part (i) [part (ii)]. A testable restriction of the model is $F_U(\cdot|\mathbf{w}_0; i, n) = F_U(\cdot|\mathbf{w}_0; j, m)$ $[F_A(\cdot; i, n) = F_A(\cdot; j, m)]$. (iv) Follows from a similar argument. *Q.E.D.*

3.3. *Unrestricted Private Values*

For datasets with heterogeneous objects, the preceding model of conditionally independent private values above is clearly more realistic than the IPV model. The testing approaches proposed above can help ascertain whether the conditional independence assumption is valid. When it is not (e.g., if there is unobserved auction-specific heterogeneity), one must consider a richer class of models. Here we consider PV models without restriction on the correlation structure of bidders' valuations.

For fixed n, any set of observed bids can be rationalized in the private values framework (Laffont and Vuong (1996)): simply let the distribution of values equal the distribution of bids. Thus, the unrestricted PV model is identified from observation of all bids in a sealed-bid auction, but untestable without further information. Below we derive both positive and negative identification results for the unrestricted private values model for cases in which some bids are unobserved.

3.3.1. *The PV Model Is Not Identified From Incomplete Sets of Bids*

With data consisting only of bids, the unrestricted PV model is not identified in an ascending auction, nor in a second-price auction unless all bids are observed.[26] This is true even if bidders are symmetric.[27]

[26] This generalizes classic results from the literature on competing risks; in particular, Cox (1959) and Tsiatis (1975) show that a joint distribution of competing risks is not identified from that of the first order statistic alone.

[27] This result does not impose affiliation of $\mathbf{U}$. If affiliation holds weakly, it is potentially disturbed by small perturbations of the distribution. The result can be generalized to the case where we restrict $\mathbf{U}$ to be strictly affiliated, as long as we take a "small enough" perturbation when constructing the counterexample.

THEOREM 4: *In the symmetric PV model*: (i) $F_{\mathbf{U}}(\cdot)$ *is not identified from the vector of bids in an ascending auction*. (ii) $F_{\mathbf{U}}(\cdot)$ *is not identified in a second-price auction unless all bids are observed*.

PROOF: Suppose that $[0, 5]^n$ is the interior of the support of $\mathbf{U}$ and that $F_{\mathbf{U}}(\cdot)$ has an associated density $f_{\mathbf{U}}(\cdot)$ that is positive throughout this region. Suppose that for some $k \in \{1, \dots, n\}$ a subset of $\{U^{(j:n)} : j \neq k\}$ is observed but $U^{(k:n)}$ is unobserved. Define a set of partitions of bidder indices

$$\begin{aligned}\mathscr{S}^k = \big\{(S_1, S_{k-1}, S_{n-k}) : {} & S_1 \cup S_{k-1} \cup S_{n-k} \\ & = \{1, \dots, n\}, |S_1| = 1, |S_{k-1}| = k-1, |S_{n-k}| = n-k\big\}.\end{aligned}$$

Then, for $S \in \mathscr{S}^k$ and $0 < \varepsilon < 1/2$, define

$$\begin{aligned} & c(\mathbf{u}; S, \varepsilon) \\ & \quad \equiv \mathbf{1}\{u_i \in [3-\varepsilon, 3+\varepsilon], i \in S_1\} \cdot \mathbf{1}\{u_i \in [1-\varepsilon, 1+\varepsilon], \forall i \in S_{k-1}\} \\ & \qquad \cdot \mathbf{1}\{u_i \in [4-\varepsilon, 4+\varepsilon], \forall i \in S_{n-k}\} - \mathbf{1}\{u_i \in [2-\varepsilon, 2+\varepsilon], i \in S_1\} \\ & \qquad \cdot \mathbf{1}\{u_i \in [1-\varepsilon, 1+\varepsilon], \forall i \in S_{k-1}\} \cdot \mathbf{1}\{u_i \in [4-\varepsilon, 4+\varepsilon], \forall i \in S_{n-k}\}. \end{aligned}$$

For sufficiently small $\gamma > 0$, $\breve{f}_{\mathbf{U}}(\cdot) \equiv f_{\mathbf{U}}(\cdot) + \gamma \sum_{S \in \mathscr{S}^k} c(\cdot; S, \varepsilon)$ is a PDF, with the function c shifting probability weight from some regions to others. If $k = n-1$, probability weight shifts from a neighborhood of $(1, \dots, 1, 2, 4)$ to a neighborhood of $(1, \dots, 1, 3, 4)$, and similarly for all permutations of these vector pairs. With $k = n-1$ this change in the underlying joint distribution preserves exchangeability and does not change the joint distribution of the observable order statistics. Similar perturbations can be constructed for any k. *Q.E.D.*

3.3.2. *Identification and Testing Using Bidder Covariates*

Availability of bidder-specific covariates[28] can yield identification even in the unrestricted private values model. To show this, we begin with approaches from the literatures on competing risks (e.g., Peterson (1976), Heckman and Honoré (1989), Han and Hausman (1990)) and the Roy model (Heckman and Honoré (1990), Heckman and Smith (1998)). In these models either a lowest or highest order statistic is observed. Observing an extreme order statistic (minimum or maximum) yields a relatively simple identification problem since its distribution provides direct information about the underlying joint distribution. For example, if $U_i = g_i(\mathbf{W}_i) + A_i$, then

$$\Pr(B^{(n:n)} \leq b | \mathbf{w}) = F_{\mathbf{A}}(b - g_1(\mathbf{w}_1), \dots, b - g_n(\mathbf{w}_n)),$$

[28] Examples of such covariates include the distance from the firm to a construction site or tract of timber, a contractor's backlog of jobs won in previous auctions, or measures of demand in the home markets of bidders at wholesale used car auctions.

enabling one to "trace out" $F_{\mathbf{A}}(\cdot)$ through variation in b and $\mathbf{w}$. Thus, when we observe either the highest bid in a second-price auction, or the lowest bid in an ascending auction, existing results (Heckman and Honoré (1990)) can be applied if, as in this prior literature, we also observe the identity of the auction winner/loser.[29] However, such results are of dubious value in the case of an ascending auction (unless $n=2$), as they require observation of the maximum or minimum bid. Inference from nonextremal order statistics is more difficult, and has not been considered in the prior literature. However, the following result shows that even if only the transaction price is observed, we can uncover the underlying joint distribution of values when bidder-specific covariates with sufficient variation are available. Further, the model is testable if more than one bid is observed in each auction.

THEOREM 5: *In the asymmetric PV model, assume* (a) $U_i = g_i(\mathbf{W}_i) + A_i\,\forall\, i$; (b) $F_{\mathbf{A}}(\cdot)$ *has support equal to* $\mathbb{R}^n$ *and a differentiable density*; (c) $(A_i, \mathbf{W}_j)$ *are independent for all* i, j; (d) $supp(g_1(\mathbf{W}_1), \ldots, g_n(\mathbf{W}_n)) = \mathbb{R}^n$; (e) $\forall\, i$, $g_i(\cdot)$ *is differentiable, and* $\lim_{\mathbf{w}_i \to -\infty} g_i(\mathbf{w}_i) = -\infty$. *Then*:

(i) $F_{\mathbf{A}}(\cdot)$ *and each* $g_i(\cdot), i = 1, \ldots, n$, *are identified up to a location normalization from observation of the transaction price and* $\mathbf{W}$.

(ii) *The model is testable if more than one bid per auction is observed.*

PROOF: (i) For simplicity, let each $\mathbf{W}_i$ be a scalar W_i. For $T \subset \{1, 2, \ldots, n\}$ define

$$\overline{F}^T_{\mathbf{A}}(a_1, \ldots, a_n) \equiv \Pr(A_i > a_i\,\forall\, i \in T, A_j \le a_j\,\forall\, j \notin T),$$

$$\overline{F}^T_{\mathbf{A}, A_i}(a_1, \ldots, a_n) = \frac{\partial}{\partial a_i} \overline{F}^T_{\mathbf{A}}(a_1, \ldots, a_n), \quad \text{and}$$

$$\mathbf{z} = (b - g_1(w_1), \ldots, b - g_n(w_n)).$$

Then for $0 \le m \le n-1$

$$\Pr(B^{(n-m:n)} \le b \mid \mathbf{w}) = \sum_{T \subseteq \{1, \ldots, n\} \text{ s.t. } |T| = m} \sum_{i \notin T} \int_{-\infty}^{b} \overline{F}^T_{\mathbf{A}, A_i}(\tilde{b} - g_1(w_1), \ldots, \tilde{b} - g_n(w_n))\,d\tilde{b}$$

[29] To see the relation between these models, recall that in the Roy model (auction model) a worker (auctioneer) selects the sector (bidder) offering the highest wage (bid). We observe only the maximum wage (bid) and the identity of the corresponding sector (bidder). Observation of the minimum, as in the competing risks model, is isomorphic.

where we sum over the possible identities of the top $m+1$ bidders. Differentiating yields

$$\frac{\partial}{\partial b}\frac{\partial^n}{\partial w_1\cdots\partial w_n}\Pr(B^{(n-m:n)}\le b\mid \mathbf{w})$$
$$=\sum_{T\subseteq\{1,\ldots,n\}\text{ s.t. }|T|=m}\ \sum_{i\notin T}(-1)^m\prod_{j=1}^{n}(-g_j'(w_j))\frac{\partial}{\partial a_i}f_{\mathbf{A}}(\mathbf{a})\Big|_{\mathbf{a}=\mathbf{z}}$$
$$=\binom{n-1}{m}(-1)^m\prod_{j=1}^{n}(-g_j'(w_j))\sum_{i=1}^{n}\frac{\partial}{\partial a_i}f_{\mathbf{A}}(\mathbf{a})\Big|_{\mathbf{a}=\mathbf{z}}$$

since there are $\binom{n-1}{m}$ sets T of size m that exclude i. Using $\lim_{b\to-\infty}F_{\mathbf{A}}(\mathbf{z})=0$ and integrating,

$$\text{(6)}\qquad \frac{\partial^n}{\partial w_1\cdots\partial w_n}\big(\Pr(B^{(n-m:n)}\le b\mid \mathbf{w})\big)=(-1)^m\binom{n-1}{m}\prod_{j=1}^{n}(-g_j'(w_j))f_{\mathbf{A}}(\mathbf{z}).$$

Observe further that

$$\text{(7)}\qquad \frac{\partial^n}{\partial w_1\cdots\partial w_n}F_{\mathbf{A}}(b-g_1(w_1),\ldots,b-g_n(w_n))=\prod_{j=1}^{n}(-g_j'(w_j))f_{\mathbf{A}}(\mathbf{z}).$$

Noting that $\lim_{\mathbf{w}\to(-\infty,\ldots,-\infty)}F_{\mathbf{A}}(b-g_1(w_1),\ldots,b-g_n(w_n))=1$, the fundamental theorem of calculus and equations (6) and (7) then imply that $F_{\mathbf{A}}(b-g_1(w_1),\ldots,b-g_n(w_n))$ is identified from observation of $B^{(n-m:n)}$, and is equal to

$$1+\frac{1}{(-1)^m\binom{n-1}{m}}\int_{-\infty}^{w_n}\cdots\int_{-\infty}^{w_1}\frac{\partial^n}{\partial w_1\cdots\partial w_n}\Pr\big(B^{(n-m:n)}\le b\mid \mathbf{w}\big)dw_1\cdots dw_n.$$

Note that $\lim_{\mathbf{w}_{-i}\to(-\infty,\ldots,-\infty)}F_{\mathbf{A}}(b-g_1(w_1),\ldots,b-g_n(w_n))=F_{A_i}(b-g_i(w_i))$. For each i, then, variation in b and w_i at this limit identifies $g_i(\cdot)$.[30] With knowledge of the $g_i(\cdot)$'s, we can then determine $F_{\mathbf{A}}(\cdot)$ at any point $(a_1,\ldots,a_n)$ through appropriate choices of b and $\mathbf{w}$. (ii) Since the argument in (i) applies for any order statistic $B^{(n-m:n)}$, observation of two order statistics leads to two expressions for $F_{\mathbf{A}}$; their equality is a testable restriction. *Q.E.D.*

3.4. *First-Price Auctions with Some Bids Unobserved*

We now turn to first-price auctions. Nonparametric identification of first-price auction models has been studied extensively for the case in which bidders

[30] For an alternative argument that relies on observing the identities of the top two bidders, but does not require holding $\mathbf{w}_{-i}$ at $(-\infty,\ldots,-\infty)$ while varying w_i, see Athey and Haile (2000). The approach requires varying w_i and w_j in a way that holds the probability that $I^{(n:n)}=i$ and $I^{(n-1:n)}=j$ fixed, and examining the effect of such variation on $\Pr(B^{(n-1:n)}\le b\mid \mathbf{w},I^{(n:n)}=i,I^{(n-1:n)}=j)=\Pr(A_j+g_j(w_j)\le b\mid \mathbf{w},I^{(n:n)}=i,I^{(n-1:n)}=j)$.

are symmetric and all bids are observed. Hence we focus on the complementary cases. Equation (4), which expresses the latent expectation $\zeta_i(x_i; n)$ in terms of observable bids, has been widely exploited in the literature.[31] Since $\Pr(\max_{j\neq i} B_j \leq z | B_i = b)$ is observable when all bids are observed, (4) can be used to identify $F_{\mathbf{U}}(\cdot)$ (Laffont and Vuong (1993, 1996)). For the symmetric IPV model of first-price auctions, Guerre, Perrigne, and Vuong (1995) show that, using (4), $F_U(\cdot)$ is identified from the transaction price $B^{(n:n)}$ alone, since $\Pr(\max_{j\neq i} B_j \leq b | B_i = b) = H_B(b)^{n-1}$ and $H_B(b) = (H_B^{(n:n)}(b))^{1/n}$. However, when bidders are asymmetric, another approach is required. Our results below provide a solution, establishing that many of the preceding results extend to first-price auctions.

THEOREM 6: *In the PV model of the first-price auction, suppose that the transaction price is observed. If bidders are asymmetric, assume that the identity of the winner is also observed. Then*:

(i) *In the IPV model*, $F_{\mathbf{U}}(\cdot)$ *is identified*.

(ii) *If* $U_i = g_i(A_i, \mathbf{W}_0)$, *where each* $g_i(\cdot, \cdot)$ *is an unknown function*, $\mathbf{W}_0$ *is observed* (*by bidders as well as the econometrician*), *and* $A_1, \ldots, A_n$ *are independent conditional on* $\mathbf{W}_0$, *then each* $F_{\mathbf{U}}(\cdot \mid \mathbf{w}_0)$ *is identified*.

(iii) *If* $U_i = g_0(\mathbf{W}_0) + A_i, \mathbf{W}_0$ *is observed, and* $A_1, \ldots, A_n$ *are mutually independent and independent of* $\mathbf{W}_0$, *then each* $F_{\mathbf{A_i}}(\cdot)$ *and* $g_0(\cdot)$ *are identified up to location*.

(iv) *If bidders are symmetric*, *then the models in* (i)–(iii) *are testable if more than one bid from each auction is observed*, *or if there is exogenous variation in the number of bidders*.

(v) *If bidders are asymmetric*, *then the models in* (i)–(iii) *are testable if* $B^{(n-1:n)}$ *is observed*, *or if there is exogenous variation in the number of bidders*.

PROOF: (i) For the symmetric case, identification follows from Guerre, Perrigne, and Vuong (1995, Corollary 2). Under asymmetry, we observe the joint distribution of $(B^{(n:n)}, I^{(n:n)})$. Since bids are independent, the proof of Theorem 2 (part b) implies that each $H_{B_i}(\cdot)$ is identified.[32] These marginal distributions uniquely determine, for each i and b, $\Pr(\max_{j\neq i} B_j \leq b)$ and, therefore, the inverse bid function $\beta_i^{-1}(\cdot)$ defined in (4). Since $\zeta_i(X_i, n) = X_i = U_i$, this identifies each $F_{U_i}(\cdot)$. (ii) For fixed $\mathbf{w}_0$ the bids (and also the valuations) are independent, so we can apply part (i). (iii) Following the argument in part (i), for any $\mathbf{w}_0$ both $H_{B_i}(b \mid \mathbf{w}_0)$ and $\beta_i^{-1}(b; \mathbf{w}_0)$ are uniquely determined for all b. Since

$$\begin{aligned} H_{B_i}(b \mid \mathbf{w}_0) &= \Pr(\beta_i(A_i + g_0(\mathbf{w}_0); \mathbf{w}_0) \leq b \mid \mathbf{w}_0) \\ &= F_{A_i}(\beta_i^{-1}(b - g_0(\mathbf{w}_0); \mathbf{w}_0)), \end{aligned}$$

[31] For private values, it has been used for the symmetric IPV model (Elyakime et al. (1994)), the affiliated PV model (Li, Perrigne, and Vuong (2002)), and the asymmetric PV model (Campo, Perrigne, and Vuong (2002)); for the pure CV model, it has been applied by Hendricks, Pinkse, and Porter (2002) and Li, Perrigne, and Vuong (2000).

[32] Our maintained assumptions about first-price auctions imply that the assumptions of Theorem 2 are satisfied.

variation in b and $\mathbf{w}_0$ then determines $F_{A_i}(\cdot)$ and $g_0(\cdot)$ up to location by standard arguments. (iv) Because valuations are i.i.d. conditional on $\mathbf{w}_0$ and each bidder uses the same strictly increasing bid function, bids are also i.i.d. conditional on $\mathbf{w}_0$. Hence (5) describes the relation between $H_B(b \mid \mathbf{w}_0)$ and any $H_B^{(j:n)}(b \mid \mathbf{w}_0)$. Since $H_B^{(j:n)}(b \mid \mathbf{w}_0)$ is observed for at least two values of j (or n), this relation is testable. (v) Following the logic above and Theorem 2, each $F_{U_i}(\cdot \mid \mathbf{w}_0)$ (or $F_{A_i}(\cdot \mid \mathbf{w}_0)$) is identified when $(B^{(n-1:n)}, I^{(n:n)})$ are observed, in which case the overidentifying restriction is testable. Exogenous variation in n is analogous. *Q.E.D.*

Despite the prior attention in the literature to symmetric IPV first-price auctions with all bids observed, specification testing has been limited to verifying monotonicity of the right-hand side of (4). Unfortunately, this restriction holds under many natural alternatives to the IPV model. Theorem 6 provides testable restrictions that typically fail under affiliated private or common values. Note also that the assumptions required in the cases of asymmetric models contrast with those for existing identification results for asymmetric PV models (Laffont and Vuong (1996)), which rely on observation of all bids. As mentioned above, Dutch auctions are strategically equivalent to first-price auctions but have the feature that the transaction price is the observable bid; thus, these results apply directly to Dutch auctions as well.[33]

Now consider the more general affiliated private values model. Although many first-price auction data sets either contain only the transaction price or else all bids, there are reasons for an auctioneer (or auction participants) to maintain records of the top two bids. In procurement auctions, the top bidder may default or be disqualified, in which case the second-highest bidder will often receive the contract. Further, auction participants often refer to the difference between the top two bids as "money left on the table," a measure that has intuitive appeal as information relevant to bidding strategies. The following result shows that observation of the top two bids is sufficient for determination of the equilibrium bid functions, using (4). This enables partial identification of the affiliated private values model from limited data.

LEMMA 1: *Assume the affiliated PV model of the first-price auction, and assume that the two highest bids are observed. If bidders are asymmetric, assume the identity of the winner* ($I^{(n:n)}$) *is also observed. Then the equilibrium bid functions* $\beta_i(\cdot)$, $i = 1, \ldots, n$, *are identified.*

PROOF: Consider the more general asymmetric case. Take $i = 1$ without loss of generality and $b_1 \in \operatorname{supp}[H_{B_1}(\cdot)]$. For almost all such b_1 (using Bayes' rule, and

[33] Prior studies of (symmetric) Dutch auctions include Laffont and Vuong (1993) and Elyakime et al. (1994).

canceling common terms)

$$\begin{aligned}
&\frac{\Pr(\max_{j\neq 1} B_j \le b_1 \mid B_1 = b_1)}{\frac{\partial}{\partial x}\Pr(\max_{j\neq 1} B_j \le x \mid B_1 = b_1)\,|_{x=b_1}} \\
&\quad = \frac{\frac{\partial}{\partial y}\Pr(\max_{j\neq 1} B_j \le b_1, B_1 \le y)\big|_{y=b_1}}{\frac{\partial^2}{\partial x \partial y}\Pr(\max_{j\neq 1} B_j \le x, B_1 \le y)\big|_{x=y=b_1}} \\
&\quad = \frac{\frac{\partial}{\partial y} H_{\mathbf{B}}(y, b_1, \ldots, b_1)\big|_{y=b_1}}{\sum_{j\neq 1}\frac{\partial^2}{\partial y \partial z_j} H_{\mathbf{B}}(y, z_2, \ldots, z_n)\big|_{y=z_2=\cdots=z_n=b_1}} \\
&\quad = \frac{\frac{\partial}{\partial x}\Pr(B^{(n:n)} \le x, I^{(n:n)} = 1)|_{x=b_1}}{\frac{\partial^2}{\partial x \partial y}\Pr(B^{(n-1:n)} \le x, B^{(n:n)} \le y, I^{(n:n)} = 1)\big|_{x=y=b_1}}.
\end{aligned}$$

Since the last expression is the ratio of two observable functions, the right-hand side of (4) is identified almost everywhere, which gives bidder 1's (inverse) equilibrium bid function up to a set of measure zero. *Q.E.D.*

This immediately implies the following result.

THEOREM 7: *Assume the symmetric affiliated PV model of the first-price auction and that the two highest bids are observed. Then the joint distribution of* $(U^{(n:n)}, U^{(n-1:n)})$ *is identified.*

Theorem 7 establishes identification of the joint distribution of the top two bidder valuations in symmetric first-price auctions.[34] Although $F_{\mathbf{U}}(\cdot)$ is not uniquely determined by the distribution of $(U^{(n:n)}, U^{(n-1:n)})$, the latter distribution is sufficient for some important policy simulations, including evaluation of reserve prices or simulation of outcomes under many alternative selling mechanisms. Below, we show that this also enables testing of private versus common values.

However, other policy questions (for example, the design of the optimal auction) require knowledge of the full joint distribution $F_{\mathbf{U}}(\cdot)$. Thus, we next consider whether this distribution is identified from incomplete sets of bids. One might conjecture that knowledge (through Lemma 1) of the bid functions $\beta_i(\cdot)$, which incorporate strategic responses to the distribution of opponents' bids, could enable us to "fill in" missing bids to obtain identification. However, as argued in the proof of Lemma 1, bid functions depend only on the top two order statistics of the bids and, therefore, provide no information about the distribution of lower bids. Thus, the negative result of Theorem 4 extends to first-price auctions.[35]

[34] The situation for asymmetric bidders is more complex because, with asymmetric bidding strategies, the top two bids are not necessarily made by the bidders with the top two valuations.

[35] To see how this follows from Theorem 4, fix $F_{\mathbf{U}}(\cdot)$. Suppose that the jth bid is unobserved, and that β is the equilibrium bidding function. These together imply a distribution of bids $H_{\mathbf{B}}(\cdot)$. Now (following Theorem 4), construct another value distribution, $\widetilde{F}_{\mathbf{U}}(\cdot)$, such that $\widetilde{F}_{\mathbf{U}}(\cdot)$ and $F_{\mathbf{U}}(\cdot)$ induce

COROLLARY 1: *In the symmetric affiliated PV model of the first-price auction, assume the two highest bids are observed.* $F_{\mathbf{U}}(\cdot)$ *is not identified if any other bid is unobserved.*

This negative result makes the identification of bid functions discussed above even more valuable. Although parametric assumptions may be needed to identify the affiliated private values model when lower-ranked bids are missing, it may still be possible to estimate the bid functions nonparametrically (based on Lemma 1) and compare these, as a specification test, to estimates obtained through parametric restrictions.

4. COMMON VALUES MODELS

Identification of a common values model requires unique determination of the joint distribution $F_{\mathbf{X},\mathbf{U}}(\cdot)$ from the observables. The full joint distribution of signals and values is required for policy questions such as determination of an optimal reserve price. It is also of interest because it contains information about the extent of bidders' residual uncertainty about their values after observing their signals, as well as the extent to which bidder i's opponents have information about his valuation U_i. Thus, for example, only with knowledge of $F_{\mathbf{X},\mathbf{U}}(\cdot)$ can the extent of the winner's curse be quantified.

Because bidder behavior depends only on the information content of the signals, which is preserved by monotone transformations, the scaling of $\mathbf{X}$ is arbitrary. One natural normalization of signals satisfies

$$E\Big[U_i | X_i = \max_{j\neq i} X_j = x, n\Big] = x. \tag{8}$$

With this normalization, the equilibrium bidding strategy in a symmetric second-price auction is $b(x_i) = x_i$. Hence, in the symmetric CV model, the distribution $F_{\mathbf{X}}(\cdot)$ is just identified (up to a normalization of signals) in a second-price sealed-bid auction when all bids are observed.[36] However, unobserved bids are problematic: the nonidentification result of Theorem 4 carries over immediately to identification of $F_{\mathbf{X}}(\cdot)$ in the CV model. Furthermore, even if all bids are observed in a second price auction, the bid distribution identifies only $F_{\mathbf{X}}(\cdot)$, providing no further information about $F_{\mathbf{X},\mathbf{U}}(\cdot)$ (even in a pure CV model).

the same distributions of order statistics except for the jth. But then, since $\Pr(\max_{j\neq i} U_j \leq x | U_i \leq y)|_{x=y}$ is the same under $\widetilde{F}_{\mathbf{U}}(\cdot)$ and $F_{\mathbf{U}}(\cdot)$, β will still be a best response for bidder i to strategies of β by all opponents, when values are drawn from $\widetilde{F}_{\mathbf{U}}(\cdot)$. Finally, β and the value distribution $\widetilde{F}_{\mathbf{U}}(\cdot)$ together generate a distribution of bids, $\widetilde{H}_{\mathbf{B}}(\cdot)$, where $\widetilde{H}_{\mathbf{B}}(\cdot)$ and $H_{\mathbf{B}}(\cdot)$ induce the same distributions of order statistics except for the jth. Thus, $\widetilde{H}_{\mathbf{B}}(\cdot)$ and $H_{\mathbf{B}}(\cdot)$ are observationally equivalent.

[36] With pure common values and ex post observability of the value V of the good, observing all bids enables both identification of the model and testing (Hendricks, Pinkse, and Porter (2002)). One estimate of $E[V \mid B_i = \max_{j\neq i} B_j = b_i, n]$ can be constructed using equilibrium bidding strategies together with observation of the bids. A second estimate of this quantity can be obtained directly from the joint distribution of the ex post value V and the bids. The two estimates can then be compared.

Hence, the data are insufficient to provide answers to most policy questions. We summarize these results in the following corollary.

COROLLARY 2: *In the CV model*, (i) $F_{\mathbf{X}}(\cdot)$ *is not identified from bids in a second-price sealed-bid auction unless all bids are observed*; (ii) $F_{\mathbf{X},\mathbf{U}}(\cdot)$ *is not identified from observed bids in a second-price sealed-bid auction*.

This result serves to qualify some prior studies in which some bids are unobserved, as it implies that parametric assumptions play an essential role in determining the results.[37]

In order to identify $F_{\mathbf{X},\mathbf{U}}(\cdot)$, additional structure and/or data are required. One natural CV structure is Milgrom and Weber's (1982) "mineral rights model"—a symmetric pure CV model in which signals are independent conditional on the common value. Li, Perrigne, and Vuong (2000) have studied such a model, assuming, in addition, that signals have an additively separable structure and providing a set of conditions under which this structure survives the rescaling (8), although in general it does not. In particular, they assume that for each n there exist two known constants $(C, D) \in \mathbb{R} \times \mathbb{R}_+$ and random variables $(A_1, \ldots, A_n)$ with joint distribution $F_{\mathbf{A}}(\cdot)$ such that, with the normalization $E[V | X_i = \max_{j \neq i} X_j = x, n] = x$, $X_i = C + D(V + A_i)\,\forall i$. Further, $(V, \mathbf{A})$ are mutually independent. Li, Perrigne, and Vuong (2000) construct several examples satisfying these requirements and establish that this model is identified in first-price auctions when all bids are observed. It follows immediately that the model is also identified in second-price auctions when all bids are observed.

Even with these strong assumptions, identification is problematic when some bids are unobserved. Letting $C = 0$ and $D = 1$ for simplicity, bids reveal order statistics of the form $X^{(i:n)} = V + A^{(i:n)}$. Since order statistics are correlated even when the underlying random variables are independent, the identification approach based on the measurement error literature followed by Li, Perrigne, and Vuong (2000) fails (recall the related discussion in Section 3.2). In the case of pure common values, a solution exists if we observe the ex post value V. Then, if $U_i = V + A_i$ and $\mathbf{A}$ are independent conditional on V, the model is identified from the transaction price in first- or second-price sealed-bid auctions, or in ascending auctions with two bidders (for details see Athey and Haile (2000)). Of course, the range of applications in which an accurate ex post measure is available may be limited.

Ascending auctions are even more difficult. While a normalization like (8) can be applied to signals in the *initial* phase of an ascending auction (the period before any bidders have dropped out), no single normalization can induce the simple strategy $b(x) = x$ throughout the auction, since bidders modify their strategies each time an opponent exits. The exact forms of these modifications depend

[37] Hong and Shum (2002) and Bajari and Hortaçsu (2000) use the normal distribution to estimate common values models of, respectively, ascending auctions (for which we obtain an even stronger nonidentification result below) and second-price auctions in which the top bid is unobserved.

on the joint distribution of signals and values. While we might hope that this dependence would enable observed bids to provide information about this joint distribution, it also creates serious challenges. Further complications arise from the fact that, when $n > 2$, there is a multiplicity of symmetric equilibria in weakly undominated strategies, implying that there is no unique interpretation of bids below the transaction price.

The following result establishes that the CV model is generally not identified in ascending auctions. Here we ignore the multiplicity of equilibria and assume a special case of a pure CV model in which signals are i.i.d. Even this very special CV model is not identified.

THEOREM 8: *In an ascending auction, assume the pure CV model, i.i.d. signals X_i, and select the equilibrium characterized by Milgrom and Weber* (1982). *The model is not identified* (*even up to a normalization of signals*) *from the observable bids.*

PROOF: Take $n = 3$ and consider two models. In both, signals are uniform on $[0, 1]$. In the first,

$$V = v(x_1, x_2, x_3) = \frac{\sum_i x_i}{3},$$

while in the second model

$$V = \hat{v}(x_1, x_2, x_3) = \frac{x^{(1:3)}}{3} + \frac{x^{(2:3)}}{6} + \frac{x^{(3:3)}}{2}.$$

Because in both models $E[V \mid X_1 = X_2 = X_3 = x] = x$, equilibrium bidding in the initial phase of the auction is identical in the two models; i.e., $H_B^{(1:3)}(b) = F_X^{(1:3)}(b) = 1 - (1 - b)^3$ in both cases. Similarly, since $b^{(2:3)} = E[V \mid X^{(1:3)} = b^{(1:3)}, X^{(3:3)} = X^{(2:3)} = x^{(2:3)}]$, the fact that $\hat{v}(x, y, y) = v(x, y, y)$ for all x and y implies that $H_B^{(2:3)}(\cdot \mid B^{(1:3)})$ is identical under the two models. Since $H_B^{(1:3)}(\cdot)$ and $H_B^{(2:3)}(\cdot \mid B^{(1:3)})$ completely determine the joint distribution of the observable bids, the two models are observationally equivalent.[38] *Q.E.D.*

This is a strong negative result for CV ascending auctions. Even ignoring the equilibrium selection problem and possible doubts about the interpretation of losing bids in an ascending auction, this most restrictive of CV models is not identified.

5. TESTS OF PRIVATE VERSUS COMMON VALUES

Our negative results for identification of CV models provide additional motivation for determining whether auction data enable testing between the PV and CV paradigms. The distinction between private and common values is fundamental

[38] Policy implications, such as the optimal reserve price, differ across the two models.

in the theoretical literature on auctions and many other types of markets. This distinction is also important for policy. The existing literature on structural analysis of auctions is fairly discouraging about the possibility of empirically discriminating between these models. The problem was first studied by Paarsch (1992a), who proposed an approach for testing between the symmetric IPV model and the mineral rights model; however, the approach relied heavily on parametric distributional assumptions. Laffont and Vuong (1996) have shown that when the number of bidders is fixed in a sealed-bid auction, the PV and CV models cannot be distinguished even when all bids are observed. One might expect the problem to be even more difficult in ascending auctions, where we never observe all bids and the CV model admits a continuum of equilibria.

A testing approach discussed in the prior literature is based on the fact that the winner's curse arises only in CV auctions. Since the severity of the winner's curse increases with the number of competitors a bidder faces, several empirical studies have tested for common values by examining whether bids decrease with the number of bidders (e.g., Paarsch (1992a; 1992b)). However, there are problems with this approach. In a first-price auction, when strategic responses to changes in the level of competition are accounted for, bids can increase or decrease in the number of bidders under both the PV and CV paradigms (Pinkse and Tan (2000)). This complication is avoided in second-price and ascending auctions, although another problem arises in these and any other auctions in which not all bids are observed: the distribution of an order statistic such as $U^{(n-1:n)}$ varies with n even when there is no winner's curse, again confounding the effects of interest.

In spite of these difficulties, we show that it is possible to use exogenous variation in the number of bidders to test for the winner's curse. We first consider the case of symmetric bidders, where the distribution $F_{\mathrm{U}}(\cdot)$ is exchangeable. Exchangeability implies the marginal distributions of the order statistics $U^{(j:n)}$ must satisfy (see, e.g., David (1981 p. 105))

$$\frac{n-r}{n}F_U^{(r:n)}(u)+\frac{r}{n}F_U^{(r+1:n)}(u)=F_U^{(r:n-1)}(u) \quad \forall\, u, r\le n-1. \tag{9}$$

To see the intuition for (9), observe that if one bidder is dropped at random from a set of n bidders, there is probability r/n that the eliminated bidder has one of the r lowest valuations, and probability $(n-r)/n$ that the bidder has one of the $n-r$ highest. Using these probabilities as weights, the distribution of $U^{(r:n-1)}$ is a weighted average of the distributions of $U^{(r+1:n)}$ and $U^{(r:n)}$.

Using (9), we are able to isolate the effect of an exogenous change in n. In a PV model, the number of bidders n has no effect on valuations; hence, the distributions of the order statistics of these valuations (obtained directly or indirectly from the distributions of bids) must obey the recurrence relation (9). In a CV second-price auction the distribution of transaction prices from auctions with $n-1$ bidders stochastically dominates the appropriate convex combination of bid distributions from n-bidder auctions, due to the effect of the winner's curse discussed above. The case of ascending auctions is more complicated, both because bidders update their strategies in response to the realizations of opponents' types inferred as the auction proceeds and because there are multiple equilibria. Nonetheless, the PV model is testable against the CV alternative in both types of auctions.

THEOREM 9: *In a second price sealed-bid or ascending auction, the symmetric PV model is testable against the symmetric CV alternative if we observe the transaction price* $B^{(m-1:m)}$ *from auction with* $m \geq 2$ *bidders and bids* $B^{(m-1:n)}, \ldots, B^{(n-1:n)}$ *from auctions with* $n > m$ *bidders. In the case of a second-price auction, it is also sufficient to observe* $B^{(m:m)}$ *at auctions with* m *bidders and bids* $B^{(m:n)}, \ldots, B^{(n:n)}$ *from the n-bidder auctions.*

PROOF: First consider a second-price sealed-bid auction and consider the case $m = n - 1$ (the argument is similar for other cases). Recall from (1) that each player i bids

$$b_i = E\Big[U_i \mid X_i = \max_{j \neq i} X_j = x_i\Big] \equiv b(x_i; n).$$

In a PV model, $b(\cdot; n)$ does not depend on n. Bids are then fixed monotonic transformations of exchangeable signals, so bids are also exchangeable. By (9), we must then have

$$\frac{2}{n}\Pr(B^{(n-2:n)} \leq b) + \frac{n-2}{n}\Pr(B^{(n-1:n)} \leq b) = \Pr(B^{(n-2:n-1)} \leq b). \tag{10}$$

Under the CV alternative (taking $i = 1$ without loss of generality and exploiting exchangeability)

$$\begin{aligned} b(x_1; n) &= E[U_1 \mid X_1 = X_2 = x_1, X_j \leq x_1, j = 3, \ldots, n] \\ &< E[U_1 \mid X_1 = X_2 = x_1, X_j \leq x_1, j = 3, \ldots, n-1] \\ &= b(x_1; n-1) \end{aligned} \tag{11}$$

with the strict inequality following from the fact that $E[U_1 | X_1, \ldots, X_n]$ strictly increases in each X_i, due to strict affiliation of (U_1, X_i) conditional on any subset of $\{X_j\}_{j \neq i}$. Hence, $b(x_1; n)$ strictly decreases in n, implying

$$\frac{2}{n}\Pr(B^{(n-2:n)} \leq b) + \frac{n-2}{n}\Pr(B^{(n-1:n)} \leq b) > \Pr(B^{(n-2:n-1)} \leq b). \tag{12}$$

A test of the equality of distributions in (10) against the first-order stochastic dominance relation in (12) then provides a test of the PV model against the CV alternative.

Now consider an ascending auction, again taking the case $m = n - 1$. In a PV model, equilibrium bidding is exactly as in the second-price auction, implying that (10) holds. This directly implies a recurrence relation between means:

$$E[B^{(n-2:n-1)}] = \frac{2}{n}E[B^{(n-2:n)}] + \frac{n-2}{n}E[B^{(n-1:n)}]. \tag{13}$$

Now assume a CV model. In an auction with $n-1$ bidders the transaction price is

$$B^{(n-2:n-1)} = \widetilde{B}^{n-1}_{I(n-2:n-1)} \tag{14}$$

where $\widetilde{B}^{n}_{I(j:k)}$ denotes the random variable equal to the equilibrium bid made in an n-bidder auction by the bidder whose signal is jth lowest in a sample of k bidders.[39] Letting $I = I^{(n-2:n-1)}$, we have

$$\begin{aligned}
\widetilde{B}^{n-1}_{I} &= E_{U_I}[U_I|X_I, \{X^{(n-1:n-1)} = X_I\}, X^{(j:n-1)} j=1,\dots,n-3] \\
&= E_{X_n}\big[E_{U_I}[U_I|X_I, X_n, \{X^{(n-1:n-1)} = X_I\}, X^{(j:n-1)} j=1,\dots,n-3]\big] \\
&> \Pr(X_n > X_I) E_{X_n > X_I}\big[E_{U_I}[U_I|X_I, \{X_n = X^{(n-1:n-1)} = X_I\}, \\
&\qquad\qquad X^{(j:n-1)} j=1,\dots,n-3]\big] \\
&\quad + \Pr(X_n \le X_I) E_{X_n \le X_I}\big[E_{U_I}[U_I|X_I, X_n, \{X^{(n-1:n-1)} = X_I\}, \\
&\qquad\qquad X^{(j:n-1)} j=1,\dots,n-3]\big] \\
&\ge E_{X_n}[\widetilde{B}^{n}_{I}].
\end{aligned} \tag{15}$$

The strict inequality above follows from affiliation. To understand the final weak inequality, assume for the moment that bidders follow the equilibrium strategies specified by Milgrom and Weber (1982), where a bidder i with type x_i who has seen k of his opponents exit bids

$$b(x_i) = E[U_i|X^{(r:n)} = X_i = x_i, \forall r > k; X^{(j:n)} = x^{(j:n)}, \forall j \le k].$$

Then if $x_n < x_I$, in an n-bidder auction bidder n will drop out before bidder I, revealing x_n. If $x_n > x_I$, however, bidder I will drop out before bidder n, and I's exit price will be based on an expectation that conditions on all remaining bidders', *including n*, having signal x_I. Hence the final weak inequality above holds with equality in the Milgrom-Weber equilibrium. Bikhchandani, Haile, and Riley (2002) show that (a) $B^{(n-2:n-1)}$ is the same in all symmetric separating equilibria and (b) bids in the Milgrom-Weber equilibrium are maximal among those in all such equilibria; thus, the final inequality holds in all such equilibria. Taking expectations over $X_1, \dots, X_{n-1}$ in (15) then gives

$$E\big[\widetilde{B}^{n-1}_{I(n-2:n-1)}\big] > E\big[\widetilde{B}^{n}_{I(n-2:n-1)}\big] \tag{16}$$

i.e., the expected bid made by the top losing bidder in an $(n-1)$-bidder auction is larger than the expectation of the bid the same bidder would make in an n-bidder auction.

[39] Note that the bid made by a given bidder depends on his own signal, on the rank of this signal among those of all n bidders (since this determines which signals he will infer from opponents' exits before he exits himself), and on the realizations of lower-ranked signals.

Now note that exchangeability and the argument used to derive (9) imply

$$\Pr\big(\widetilde{B}^n_{I(n-2:n-1)} \le b\big) = \frac{2}{n}\Pr\big(\widetilde{B}^n_{I(n-2:n)} \le b\big) + \frac{n-2}{n}\Pr\big(\widetilde{B}^n_{I(n-1:n)} \le b\big)$$

since when a bidder is dropped at random from a sample of n bidders, $I^{(n-2:n-1)} = I^{(n-2:n)}$ with probability $2/n$ and $I^{(n-2:n-1)} = I^{(n-1:n)}$ with probability $(n-2)/n$. This implies

$$E\big[\widetilde{B}^n_{I(n-2:n-1)}\big] = \frac{2}{n}E\big[\widetilde{B}^n_{I(n-2:n)}\big] + \frac{n-2}{n}E\big[\widetilde{B}^n_{I(n-1:n)}\big].$$

With (16), this gives

$$E\big[\widetilde{B}^{n-1}_{I(n-2:n-1)}\big] > \frac{2}{n}E\big[\widetilde{B}^n_{I(n-2:n)}\big] + \frac{n-2}{n}E\big[\widetilde{B}^n_{I(n-1:n)}\big],$$

i.e.,

$$E\big[B^{(n-2:n-1)}\big] > \frac{2}{n}E\big[B^{(n-2:n)}\big] + \frac{n-2}{n}E\big[B^{(n-1:n)}\big]. \tag{17}$$

Hence a test of the null hypothesis of (13) against (17) provides a test of PV against the CV alternative. *Q.E.D.*

This result implies, for example, that the PV model is testable against the CV alternative whenever we observe the top two (or the second and third highest) bids from auctions with n and $n-1$ bidders, holding all else fixed. Strikingly, this result holds without restriction on which equilibrium (or equilibria) describe(s) actual behavior in ascending auctions.[40]

While Theorem 9 requires symmetric bidders, we can extend the result to the case in which the distributions are completely unrestricted. The following result exploits the fact that by taking random draws from samples of arbitrary random variables, one obtains a sample of exchangeable random variables, enabling use of (9) (Balasubramanian and Balakrishnan (1994)).

THEOREM 10: *In a second-price auction, take any $\mathcal{P}_n \subset \mathbb{N}$ such that $|\mathcal{P}_n| = n \ge 3$ and the probability that $\mathcal{P}_n$ is the set of participating bidders is positive. If for some $m < n, m \ge 2$, there is positive probability of participation by every $\mathcal{P}_m \subset \mathcal{P}_n$ such that $|\mathcal{P}_m| = m$, then if we observe bids $B^{(m-1:n)}, \ldots, B^{(n-1:n)}$ in auctions with n bidders and the transaction price in auctions with m bidders, the (unrestricted, asymmetric) private values model is testable against the CV alternative.*[41] *Under the same assumptions, the PV model is testable in an ascending auction.*

[40] However, we repeat the caveat that for many ascending auctions, a plausible alternative hypothesis is that bids $B^{(n-2:n)}$ and below do not always reflect the full willingness to pay of losing bidders, although $B^{(n-1:n)}$ does (since only two bidders are active when that bid is placed). In that case this testing approach could suggest the CV model even in a PV setting. This problem does not arise in second-price sealed-bid auctions or in first-price auctions, which we discuss below.

[41] As with Theorem 9, it is also sufficient to observe $B^{(m:m)}$ at auctions with m bidders and bids $B^{(m:n)}, \ldots, B^{(n:n)}$ from the n-bidder auctions.

PROOF: Let $U_1, \ldots, U_n$ be the random variables corresponding to the valuations of the bidders in $\mathcal{P}_n$, and let $Y_1, \ldots, Y_m$ be a sample of size $m < n$ drawn without replacement from $\{U_1, \ldots, U_n\}$ using a discrete uniform distribution.[42] Then $Y_1, \ldots, Y_m$ are exchangeable, implying that the distributions of the order statistics from this sample must satisfy (9). Define

$$\overline{F}_U^{(r:m)}(u) = \frac{1}{\binom{n}{m}} \sum_{\mathcal{P}_m \subset \mathcal{P}_n : |\mathcal{P}_m| = m} F_U^{(r:\mathcal{P}_m)}(u)$$

where $F_U^{(r:\mathcal{P}_m)}(\cdot)$ is the distribution of the rth order statistic from $\{U_i, i \in \mathcal{P}_m\}$. $\overline{F}_U^{(r:m)}(\cdot)$ is the distribution of the rth order statistic of $\{Y_1, \ldots, Y_m\}$. So for $r < n$ we must have

$$\text{(18)} \qquad \frac{n-r}{n} \overline{F}_U^{(r:n)}(y) + \frac{r}{n} \overline{F}_U^{(r+1:n)}(y) = \overline{F}_U^{(r:n-1)}(y).$$

Since $\overline{F}_U^{(l:n)}(u) = F_U^{(l:n)}(u)$ for $l \leq n$, this simplifies to

$$\text{(19)} \qquad \frac{n-r}{n} F_U^{(r:n)}(u) + \frac{r}{n} F_U^{(r+1:n)}(u) = \overline{F}_U^{(r:n-1)}(u).$$

Following the argument in Theorem 9 one can confirm that the equalities (18) and (19) are replaced by strict inequalities in a CV model of a second-price sealed-bid auction. Hence, if $m = n-1$, (19) can be tested against the CV alternative. Similarly, for $m < n-1$, repeated application of (19) enables testing of (18). In an ascending auction, the same argument provides a testable implication of the PV model.[43] *Q.E.D.*

Theorems 9 and 10 imply that the observational equivalence between the PV and CV models noted in Laffont and Vuong (1996) and Li, Perrigne, and Vuong (2000, 2002) is eliminated when one observes exogenous variation in the number of bidders.[44] Similarly, the top two bids in a first-price auction give us enough information to test the symmetric PV model against the symmetric CV alternative.[45]

[42] Recall that when bidders are asymmetric we assume that the identities of the participating bidders are observable.

[43] In a CV ascending auction with asymmetric bidders, a full characterization of the set of equilibria has not been given in the literature. If bidding follows the (equilibrium) strategies in (2), the argument in Theorem 9 can be extended to enable testing specifically against the CV alternative. Whether this argument carries over regardless of the equilibrium selection, as in the symmetric case, is an open question.

[44] Other approaches for empirically distinguishing these models based on observation of all bids at first-price auctions are given in Hendricks, Pinkse, and Porter (1999) and Haile, Hong, and Shum (2000). The latter also exploits variation in the number of bidders to detect the winner's curse and develops formal statistical tests.

[45] As with the preceding results, the testing approach can be extended to the case in which different sets of bids are observed and/or data are available only from auctions with nonconsecutive numbers of bidders.

THEOREM 11: *In the first-price auction, if the top two bids are observed in auctions with n and $n-1$ bidders, where $n \geq 3$, then the symmetric affiliated private values model is testable against the symmetric CV alternative.*

PROOF: The proof of Lemma 1 and the fact that $\zeta(x;n)$ strictly increases in x imply that the observables uniquely determine the distributions $F_{\zeta,n-1}^{(n-1:n-1)}(\cdot)$, $F_{\zeta,n}^{(n-1:n)}(\cdot)$, and $F_{\zeta,n}^{(n:n)}(\cdot)$ of the random variables $\zeta(X^{(n-1:n-1)};n-1)$, $\zeta(X^{(n-1:n)};n)$, and $\zeta(X^{(n:n)};n)$. Under the PV hypothesis, $\zeta(x_i;n)=x_i=u_i$, so these distributions are $F_U^{(n-1:n-1)}(\cdot)$, $F_U^{(n-1:n)}(\cdot)$, and $F_U^{(n:n)}(\cdot)$, which must satisfy (9). Now consider the symmetric CV alternative. Let $\zeta(X^{(n-1:n-1)};n)$ denote the value of the random variable $\zeta(X_i,n)$ when i has the highest signal among the bidders who remain after one bidder in an n-bidder auction is removed exogenously. Let $F_{\zeta,n}^{(n-1:n-1)}(\cdot)$ denote the distribution of $\zeta(X^{(n-1:n-1)};n)$. Since $\zeta(x;n)$ is strictly increasing in x, the random variables $\zeta(X_i;n)$ are exchangeable, so (9) and (11) imply that for all

$$\frac{n-1}{n}F_{\zeta,n}^{(n:n)}(t)+\frac{1}{n}F_{\zeta,n}^{(n-1:n)}(t) > F_{\zeta,n}^{(n-1:n-1)}(t). \qquad Q.E.D.$$

The proof of Theorem 11 has two steps. First, using Lemma 1, we identify the equilibrium bid functions, which vary with n for strategic reasons even under the PV null. Second, we use (9) to account for the way in which the distributions of order statistics change with the number of bidders. This allows us to isolate the effects of the winner's curse, so that variation in the number of bidders has unambiguous (and mutually exclusive) consequences under the null and alternative hypotheses.

6. EXTENSIONS

6.1. *Bidder Uncertainty Over the Number of Opponents*

6.1.1. *Second-Price Auctions*

Our analysis of CV models required an assumption that bidders know the number of competitors they face. In an ascending auction, this assumption may be natural. In a sealed-bid auction, bidders might not know the number of competitors when submitting their bids.[46] However, as long as bidders (but not necessarily the econometrician) observe an informative signal of n, the testing approach in Theorem 9 can still be applied.

THEOREM 12: *In the second-price sealed-bid auction, suppose bidders do not know n but observe a public signal η (unobserved to the econometrician) that is strictly affiliated with n. Then the symmetric PV model is testable against the symmetric CV alternative if we observe the transaction price $B^{(m-1:m)}$ in auctions with m bidders and bids $B^{(m-1:n)},\dots,B^{(n-1:n)}$ in auctions with $n>m$ bidders.*

[46] Matthews (1987) and McAfee and McMillan (1987b) provide theoretical analyses of auctions with a stochastic number of bidders.

PROOF: Since (10) is unaffected by imperfect observability of n in a PV model, it is sufficient to show that (12) still holds in all CV models. Let $\pi(\eta|n)$ denote the conditional distribution of the signal η. Assume $m = n-1$ (the argument is similar for other cases). Given η, each player i views n as a random variable and bids

$$b_i = E_n\Big[E\Big[U_i|X_i = \max_{j\neq i} X_j = x_i\Big]\Big|\eta\Big] \equiv \hat{b}(x_i;\eta).$$

Taking $i=1$, in a CV model the inequality (11) and strict affiliation imply that for any $\eta_2 > \eta_1$, $\hat{b}(x_1;\eta_2) < \hat{b}(x_1;\eta_1)$. Therefore, strict affiliation of n and η implies:

$$\text{(20)} \qquad \begin{aligned} \Pr(B^{(n-2:n)} \le b) &= \int_{-\infty}^{\infty} \Pr(\hat{b}(X^{(n-2:n)};\eta) \le b)\, d\pi(\eta|n) \\ &> \int_{-\infty}^{\infty} \Pr(\hat{b}(X^{(n-2:n)};\eta) \le b)\, d\pi(\eta|n-1) \end{aligned}$$

and, similarly,

$$\text{(21)} \qquad \Pr(B^{(n-1:n)} \le b) > \int_{-\infty}^{\infty} \Pr(\hat{b}(X^{(n-1:n)};\eta) \le b)\, d\pi(\eta|n-1).$$

Using (9) and strict monotonicity of $\hat{b}(\cdot;\eta)$, we obtain the testable stochastic dominance relation

$$\begin{aligned} \Pr(B^{(n-2:n-1)} \le b) &= \int_{-\infty}^{\infty} \Pr(\hat{b}(X^{(n-2:n-1)};\eta) \le b)\, d\pi(\eta|n-1) \\ &= \int_{-\infty}^{\infty} \bigg[\frac{2}{n}\Pr(\hat{b}(X^{(n-2:n)};\eta) \le b) \\ &\qquad + \frac{n-2}{n}\Pr(\hat{b}(X^{(n-1:n)};\eta) \le b)\bigg]\, d\pi(\eta|n-1) \\ &< \frac{2}{n}\Pr(B^{(n-2:n)} \le b) + \frac{n-2}{n}\Pr(B^{(n-1:n)} \le b) \end{aligned}$$

where the inequality follows from (20) and (21). *Q.E.D.*

6.1.2. *First-Price Auctions*

Bidder uncertainty over the number of opponents is a more difficult problem in a first-price auction. In the case of private values, each bidder i solves

$$\max_b (u_i - b)\Pr\Big(\max_{j\neq i} B_j \le b | U_i = u_i, \eta\Big)$$

giving first-order condition

$$\text{(22)} \qquad b_i + \frac{\Pr(\max_{j\neq i} B_j \le b_i | B_i = b_i, \eta)}{\frac{\partial}{\partial x}\Pr(\max_{j\neq i} B_j \le x | B_i = b_i, \eta)|_{x=b_i}} = u_i.$$

If the econometrician observes a set of auctions in which η is fixed, this relation between bids and valuations can be used in essentially the same way that (4) was used above. For example, in the symmetric IPV case, observation of the winning bid in auctions with fixed η is still sufficient to identify $F_{\mathrm{U}}(\cdot)$. Let $\tilde{\pi}(n|\eta)$ denote the probability that there are n bidders when signal η is observed. Fixing η and letting B^{win} denote the winning bid, we observe

$$\Pr(B^{win} \leq b|\eta) = \sum_{n=2}^{\infty} \tilde{\pi}(n|\eta)\Pr(B^{(n:n)} \leq b|\eta) = \sum_{n=2}^{\infty} \tilde{\pi}(n|\eta)\Pr(B_i \leq b|\eta)^n. \tag{23}$$

Since the right-hand side of (23) strictly increases in $\Pr(B_i \leq b|\eta)$ and $\tilde{\pi}(n|\eta)$ is observed directly, $\Pr(B_i \leq b|\eta)$ is identified. This determines $\Pr(\max_{j\neq i} B_j \leq b_i|\eta)$, identifying (through (22)) the distribution $U^{(n:n)}$ for each n such that $\tilde{\pi}(n|\eta) > 0$. Equation (5) then determines $F_U(\cdot)$ and, therefore, $F_{\mathrm{U}}(\cdot)$.

Our identification results for other private values models of first-price auctions can be extended in similar fashion. Testing of the PV hypothesis can be achieved by comparing distributions of $U^{(j:n)}$ for different values of η: under the PV hypothesis, these distributions will be identical; with common values we recover the distribution of $E[U_i|X^{(j:n)} = \max_{k\neq i} X_k = x_i, \eta]$ rather than that of $U^{(j:n)}$. This distribution in auctions in which signal $\eta = \eta_1$ is observed will first-order stochastically dominate that in auctions in which signal $\eta = \eta_2 > \eta_1$ is observed.

6.2. *Reserve Prices*

In many auctions the seller announces a reserve price for the auction. When the reserve price r_0 is in the interior of the support of bidders' valuations, with positive probability some potential bidders will be unwilling to bid, creating a discrepancy between the number of potential bidders, p, and the number of participating bidders, n.[47] For simplicity, consider a symmetric IPV auction. Clearly, no auction can reveal information about $F_U(u)$ for $u < r_0$ without parametric assumptions; however, our results extend to identify the truncated distribution

$$F_U(\cdot|r_0) = \frac{F_U(\cdot) - F_U(r_0)}{1 - F_U(r_0)}.$$

COROLLARY 3: *In the symmetric IPV model, consider first-price, second-price, or ascending auction with a reserve price,* r_0, *such that* $F_U(r_0) \in (0,1)$.

(i) *If only one bid from each auction is observed,* $F_U(\cdot|r_0)$ *is identified on* $[r_0, \infty]$.

[47] Bajari and Hortaçsu (2000) and Donald, Paarsch, and Robert (1999) use parametric models to simultaneously estimate a model of entry and bidding. Guerre, Perrigne, and Vuong (2000) include a discussion of binding reserve prices in the IPV model. See also the recent paper by Li (2000). Hendricks, Pinkse, and Porter (2002) estimate a model of stochastic participation and reserve prices in pure common value auctions.

(ii) *If either* (a) *two bids from each auction are observed or* (b) *a single bid is observed in auctions with different numbers of participating bidders, the model is testable*.

(iii) *If* p *is fixed and the number of participating bidders is observed*, p *and* $F_U(r_0)$ *are identified*.

PROOF: Because each potential bidder i participates when $x_i > r_0$, participating bidders' valuations are i.i.d. draws from $F_U(\cdot|r_0)$. Parts (i) and (ii) then follow from Theorems 1 and 6. Because the participation rule for each potential bidder is binomial with parameter $\lambda = F_U(r_0)$, both p and $F_U(r_0)$ are uniquely determined by the distribution of n (Guerre, Perrigne, and Vuong (2000)). *Q.E.D.*

Similar extensions can be made for identification of the other models considered above. The following result shows how our tests of the PV model extend to auctions with reserve prices.[48]

COROLLARY 4: *In first-price, second-price, or ascending auctions with reserve price* r_0, *the symmetric PV model is testable if we observe* $B^{(j:n)}$ *and* $B^{(j+1:n)}$ $(2 \leq j < n)$ *in all n-bidder auctions and* $B^{(j:n-1)}$ *in all* $(n-1)$*-bidder auctions*.

PROOF: Participants draw their valuations $U_1, \ldots, U_n$ from the distribution $F_{\mathrm{U}}(\cdot)$ truncated at $(r_0, \ldots, r_0)$. Because exchangeability is preserved by this truncation, the recurrence relation (9) still holds under the PV hypothesis. *Q.E.D.*

7. CONCLUSIONS

While much empirical work in the broad area of demand estimation relies on parametric assumptions, recent work by Laffont and Vuong (1996), Guerre, Perrigne, and Vuong (2000) (and others) has shown that nonparametric methods can be used in some auction settings. Our results complement this work by considering standard auction forms beyond the first-price auction, environments in which not all bids are observable, and data in addition to bids that are often available in practice. In addition, while relatively little attention has been given to nonparametric testing of the assumptions underlying standard models of bidder demand and information, we have shown that data available in many applications enable testing of these assumptions against interesting alternatives. Such testing can be used to evaluate the suitability of models selected for particular applications and should raise the confidence one has in the results obtained through structural analysis of auction data.

[48] Here we do not specify outcomes under the CV alternative. With a binding reserve price, the set of types willing to participate changes with n in a CV model. One can also test the PV hypothesis $\zeta(x_0; m) = r_0$ against the alternative $\zeta(x_0; m) > r_0$ implied by the CV model (Milgrom and Weber (1982)). This testing approach has been proposed for first-price auctions by Hendricks, Pinkse, and Porter (2002).

While some qualitative policy questions depend primarily on which model best describes an economic environment (for example, the choice of auction format depends crucially on the distinction between private and common values), others require detailed knowledge of the distribution of bidder information. We establish that one of the most commonly used models, independent (perhaps conditional on covariates) private values, is identified from the transaction price alone in standard auctions. Our results for more general private values models are mixed. While the unrestricted private values model is not identified from bids alone in ascending auctions (or any other auctions in which some bids are unobserved), additional data beyond bids can enable identification of PV models allowing correlated values and ex ante asymmetry.

Our identification results for common values models are generally negative. We have shown that identification from observable bids fails for a large class of demand structures in an ascending auction, and holds in a second-price sealed-bid auction only under stringent conditions on the latent demand structure and the types of data available. However, when there is exogenous variation in the number of bidders, the private values model can be *tested* against the common values alternative, even when neither model is identified, as long as two or more bids are observed from each auction.

We have focused exclusively on identification and testable restrictions. In general, identification is necessary but not sufficient for existence of a consistent estimator. While many of our identification proofs suggest straightforward estimation and testing strategies, we have left development and evaluation of estimators and test statistics for future work, along with their application to bidding data. Finally, identification is an open question for other auction models of practical relevance, including models of sequential and simultaneous auctions of multiple goods.[49]

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