Inference with an Incomplete Model of English Auctions

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While English auctions are the most common in practice, their rules typically lack sufficient structure to yield a tractable theoretical model without significant abstractions. Rather than relying on one stylized model to provide an exact interpretation of the data, we explore an incomplete model based on two simple assumptions: bidders neither bid more than their valuations nor let an opponent win at a price they would be willing to beat. Focusing on the symmetric independent private values paradigm, we show that this limited structure enables construction of informative bounds on the distribution function characterizing bidder demand, on the optimal reserve price, and on the effects of observable covariates on bidder valuations. If the standard theoretical model happens to be the true model, our bounds collapse to the true features of interest. In contrast, when the true data-generating process deviates in seemingly small ways from that implied by equilibrium in the standard theoretical model, existing methods can yield misleading results that need not even lie within our bounds. We report results from Monte Carlo experiments illustrating the performance of our approach and comparing it to others. We apply our ap-

proach to U.S. Forest Service timber auctions to evaluate reserve price policy.

I.   Introduction

We propose a new approach to empirical analysis of the most common type of auction: the "English" or "oral ascending bid" auction, in which bidders offer progressively higher prices until only one bidder remains. The chief challenge we face is the fact that the free-form structure of most English auctions is not easily captured in a theoretical model that might provide a mapping between the observable bids and the latent demand structure of interest. All existing models of English auctions rely on strong abstractions for tractability. In the overwhelmingly dominant model (Milgrom and Weber 1982), each bidder continuously affirms her participation by holding down a button while the price rises continuously and exogenously. If bidders know their valuations, each drops out (in the dominant strategy equilibrium) by releasing her button when either the price reaches her valuation or all her opponents have exited. A bid in this "button auction" model is a price at which to exit. In practice, however, there may be no observables equivalent to the bids envisioned in the theory. At most real English auctions, bidders call out or affirm bids whenever they wish and need not indicate whether they are "in" or "out" as the auction proceeds. Prices typically rise in jumps of varying sizes, and active bidding by a player's opponents may eliminate any incentives for her to make a bid close to her valuation, or even to bid at all. Hence, while the standard model serves well in illuminating strategic forces that arise in an English auction, it may serve poorly in providing an exact interpretation of bidding data.

This mismatch threatens hopes of using theory to relate observed bids to the underlying distributions that characterize bidder demand and information—something essential for addressing a number of positive and normative questions of interest. For example, simulating outcomes under alternative selling mechanisms or applying results from the literature on market design requires information about these distributions, which the theory treats as known to the seller. Even simpler goals such as measuring the dispersion in bidders' private information or assessing the effects of product characteristics on bidders' willingness to pay require inference on these primitive distributions. The importance of these objectives has thus far led researchers to accept the button auction model as an approximation of the true data-generating process (see, e.g., Paarsch 1992b, 1997; Donald and Paarsch 1996; Baldwin, Marshall, and Richard 1997; Hong and Shum 1999; Haile 2001; Athey and Haile 2002).

We take a different approach. We focus on the symmetric independent private values model, which provides the simplest model of bidder demand and has dominated the prior literature. However, rather than committing to the interpretation of bids implied by one particular model of the auction itself, we rely on an incomplete model consisting of two simple assumptions.

ASSUMPTION 1. Bidders do not bid more than they are willing to pay.

ASSUMPTION 2. Bidders do not allow an opponent to win at a price they are willing to beat.

The motivation for assumption 1 is clear: every bid made in an English auction is potentially a winning bid, so no bidder should make a bid above his valuation. Assumption 2 is motivated by "the essential feature of the English auction" (McAfee and McMillan 1987): the ability of bidders to observe and respond to the current best bid with higher bids of their own.[1] Given such an ability, assumption 2 requires that bidders not pass up opportunities to make a profit.[2]

While these two assumptions have obvious intuitive appeal,[3] they are also weak restrictions in several precise senses. First, both assumptions hold in the dominant strategy equilibrium of the standard model—indeed, in all its symmetric separating equilibria (Bikhchandani et al. 2002). However, they also allow other types of bidding behavior. In contrast to equilibrium in the button auction model, for example, these assumptions allow for jump bidding, for the possibility that a bidder's highest bid lies below his valuation (or even that some bidders do not bid at all), and for the ranking of bidders according to their highest

[1] Notable exceptions are Internet auctions with fixed closing times, where it may be possible for a bidder to "slip in" a winning bid at the last moment, leaving no time for an opponent to respond (Roth and Ockenfels 2002). In most English auctions (including some Internet auctions), a practice similar to a "going once, going twice, sold!" announcement is followed to ensure that bidders have an opportunity to respond before the auction closes.

[2] In the button auction model and most other models of English auctions, assumption 2 is implied by elimination of weakly dominated strategies. The converse is not true. Bikhchandani, Haile, and Riley (2002), e.g., show that with private values, the button auction model itself has a continuum of symmetric perfect Bayesian equilibria in weakly dominated strategies, all satisfying assumptions 1 and 2. Violations of assumption 2 without weakly dominated strategies can occur in models in which submitting each bid is costly (Avery 1998; Daniel and Hirshleifer 1998). The bidding rules in these models can enable one bidder to infer from early bids of her opponent that she is very unlikely to win, making it rational for her to allow her opponent to win at a price below her own valuation. While these models provide useful insights, their empirical relevance is uncertain in most applications, including those we study below, since all bidders are physically present for the entire auction. Bidding costs may be more important, however, in Internet auctions, which typically last several days and require bidders to log in (or remain on-line) to make each bid (Easley and Tenorio 2001).

[3] These are also two of the assumptions made by Yamey (1972) and Rothkopf and Harstad (1994) in axiomatic analyses of English auctions with discrete bid increments. The latter paper also suggests the potential for discrete bid increments to lead to rational jump bidding.

bids to differ from that according to their valuations. Further, assumptions 1 and 2 imply neither (i) a unique distribution of bids given a distribution of valuations nor (ii) a unique distribution of valuations given a distribution of bids (see the examples in App. B). The latter property implies that the distribution of bidder valuations is not identified, motivating our exploration of bounds. The former property implies that assumptions 1 and 2 form an *incomplete* model of bidding. Our objective is to investigate the empirical content of this model—to see what can be learned about the underlying demand structure on the basis of only observed bids and these two restrictions on their interpretation.

   Our approach is motivated by a key tension in empirical economics: on one hand, structure from economic theory is often essential to obtaining useful estimates; on the other hand, interpreting data as though they were generated by one particular stylized model will be misleading if the true model turns out to be significantly different. Assumptions 1 and 2 can be interpreted as necessary conditions for equilibrium in a *class* of complete models. These necessary conditions define the *sets* of observable outcomes consistent with each realization of the primitives.[4] Our approach to inference exploits this relation between primitives and observables in essentially the same way that one exploits equilibrium mappings in standard structural estimation of economic models. This type of robust structural approach may be useful in other environments as well.[5]

   Here this approach enables construction of informative bounds on the distribution function characterizing bidder demand and information. Each bid provides a lower bound on the corresponding bidder's valuation. Each winning bid provides an upper bound (up to the minimum bid increment) on losing bidders' valuations. Exploiting relations between distributions of order statistics and the underlying parent distribution, we use this information to construct bounds on the cumulative distribution function (CDF) of interest. Estimation of these bounds is both nonparametric and simple. Simulations and an application show that the estimated bounds can be very tight in practice. In fact, if the button auction model captures the true data-generating process, the bounds collapse to the true distribution. Hence, there is a sense in which nothing is lost by taking this more robust approach. In contrast, when

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[4] For a very different class of (oligopoly) models, Sutton (1991, 1997, 1998) uses similar assumptions to derive bounds on the set of predicted outcomes given a specification of primitives.

[5] This approach is similar in spirit to work on incomplete structural models in other contexts (e.g., Jovanovic 1989; Tamer, in press) and other models in which one can identify bounds on the parameters or functions of interest, including Varian (1982, 1984), Manski (1990, 1995), Horowitz and Manski (1995), and Hotz, Mullin, and Sanders (1997).

the true data-generating process deviates from that of the button auction model in seemingly small ways, existing methods can yield misleading results. In fact, the distribution implied by observed bids and a misspecified complete model (even one satisfying our assumptions) need not lie within our bounds.

We also show how to construct bounds on the optimal reserve price. This is possible despite the limited restrictions we place on bidder behavior, despite the fact that sellers may use (suboptimal) reserve prices in the auctions we observe, and despite the fact that the optimal reserve price depends on derivatives of the distribution function that we bound only in levels. Besides enabling construction of bounds on this key policy parameter, this analysis illustrates the subtlety that can be involved in the critical task of translating imperfect knowledge of structural parameters/distributions into meaningful statements about policy. Finally, we show how to use our approach to construct bounds on the effects auction covariates have on bidder valuations—something useful for inference on hedonic models of valuations, testing for premia for certain products/sellers, or testing restrictions of the private values assumption.

Our work contributes to the literature on structural estimation of auction models begun by Paarsch (1992a). Hendricks and Paarsch (1995), Laffont (1997), Perrigne and Vuong (1999), and Hendricks and Porter (2000) offer surveys of the recent empirical literature on auctions. Much of this work has focused on parametric specifications of the distributions of bidder valuations. Our approach is nonparametric. Nonparametric methods for first-price auctions are developed by Li, Perrigne, and Vuong (2000, 2002) and Guerre, Perrigne, and Vuong (2000). Athey and Haile (2002) address nonparametric identification of standard auction models. We are unaware of prior work using nonparametric methods to study English auctions or of any method for inference on demand at English auctions that relaxes the assumptions of the button auction model.

Section II sets up the model and notation. In Section III we show how assumptions 1 and 2 identify bounds on the distribution of bidder valuations and develop estimators. Section IV addresses construction of bounds on the optimal reserve price. Section V then discusses incorporation of covariates accounting for heterogeneity in the distributions of valuations across auctions. Section VI presents the results of Monte Carlo experiments designed to evaluate our approach and compare it to others used previously. In Section VII we apply our methods to data from U.S. Forest Service timber auctions, focusing on reserve price policy.[6] We offer conclusions in Section VIII.

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[6] Prior work addressing reserve price policies in other applications includes McAfee and Vincent (1992), McAfee, Quan, and Vincent (1995), Li et al. (1997), Paarsch (1997), and Carter and Newman (1998).

II.   Model and Notation

Throughout we represent random variables in upper case and their realizations in lower case. We consider the standard symmetric independent private values paradigm in which for each auction there are $M \in \{2, \ldots, \bar{M}\}$ potential bidders. Each bidder $i \in \{1, \ldots, m\}$ draws her valuation V_i independently from a distribution $F_0(\cdot)$ with support $[\underline{v}, \bar{v}]$.[7] While we let this distribution vary with observables below, here we suppress this dependence for simplicity. Bidders know their own valuations but not those of their opponents. The seller, who places value v_0 on the object, first announces his intent to hold an auction and may set a reserve price r. Bidders then choose whether to participate. Assumptions 1 and 2 imply that all bidders with valuations above r participate. Let N denote the number of participating bidders. At the auction, the reserve price (or zero, if there is no reserve) is designated the initial bid, and monotonically increasing bids are then accepted from the participating bidders, subject to a minimum bid increment $\Delta \geq 0$. We assume $r + \Delta < \bar{v}$.

We leave the remaining details of the underlying model unspecified, including, for example, whether bidding is completely free-form or follows a more structured procedure, what motivates individuals to bid at one particular point in the auction rather than another, and how the seller ends the auction. We assume only that bidding satisfies assumptions 1 and 2. The highest price offered by each bidder is recorded as his "bid." If a bidder does not bid, the reserve price (or zero) is recorded as his bid. This is a common method of recording data at English auctions and exactly that used in the Forest Service auctions studied below.[8] The information available to the econometrician consists of these bids, the reserve price, and the minimum bid increment.

The structural feature of interest is the distribution $F_0(\cdot)$, which fully characterizes bidder demand and information. Note, however, that if the seller sets a binding reserve price $r > \underline{v}$, no auction can reveal anything about $F_0(v)$ on the truncated region of the support, that is, for $v < r$. In many auctions, the seller sets no reserve price; for such cases,

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[7] We explore other demand structures in Haile and Tamer (2001). Descriptions of standard auction models and demand structures are given in, e.g., McAfee and McMillan (1987), Wilson (1992), Klemperer (1999), and Athey and Haile (2002).

[8] Given our assumptions, these bids contain all the information revealed about $F_0(\cdot)$ during the auction. If only the bids of the top few bidders (at least two) were observed from each auction, our method could still be applied as long as we knew the number of participating bidders, i.e., those with valuations above the reserve price. As with all existing structural methods for auctions, this knowledge is needed to correctly interpret the observed order statistics of the bids.

let $r^0 = \underline{v}$. If the seller sets a reserve price above $\underline{v}$, let $r^0 = r$. Then define the truncated distribution

$$F(v) = \frac{F_0(v) - F_0(r^0)}{1 - F_0(r^0)}. \tag{1}$$

Conditional on participation, bidders have valuations that are independently and identically distributed (i.i.d.) draws from $F(\cdot)$. Because bids can reveal only information about $F(\cdot)$, we treat this distribution as the primitive of interest.[9] As we show below, $F(\cdot)$ is often sufficient for the positive and normative questions of interest.

For $j = 1, \ldots, n$, we let B_j denote bidder j's bid. The random variables $B_{1:n}, \ldots, B_{n:n}$ represent the order statistics of the bids, with $b_{i:n}$ denoting the realization of the ith lowest of the n bids. Let $G_{i:n}(\cdot)$ denote the distribution of $B_{i:n}$. Similarly, let $V_{1:n}, \ldots, V_{n:n}$ denote the ordered valuations, with each $V_{i:n} \sim F_{i:n}(\cdot)$. Note that $b_{i:n}$ need not be the bid made by the bidder with valuation $v_{i:n}$.

III.   Bounds on the Distribution of Valuations

A.   *Identification*

1.   Upper Bound

To obtain an upper bound on the distribution $F(\cdot)$, we use assumption 1, which can be written as $b_i \leq v_i$ for all i. If we let $G(b) = \Pr(B_i \leq b)$, this implies the first-order stochastic dominance relation

$$G(v) \geq F(v) \quad \forall v. \tag{2}$$

While (2) itself provides an upper bound on $F(\cdot)$, we can obtain a tighter bound by exploiting the fact that some order statistics $B_{j:n}$ may provide more precise information about valuations than others. To do this, first note that assumption 1 implies that the ith order statistic of the bids must lie below the ith order statistic of the valuations.[10]

LEMMA 1. $b_{i:n} \leq v_{i:n}$ for all i.

To use this result, for $H \in [0, 1]$, $n \in \{2, \ldots, \bar{M}\}$, and $i \in \{1, \ldots, n\}$,

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[9] If the reserve price is a random variable taking on values below $\underline{v}$ with positive probability, one obtains a random truncation model in which identification of bounds on the full distribution $F_0(\cdot)$ follows from the arguments below. A similar observation is made for the case of first-price auctions by Guerre et al. (2000). Estimation in such an environment is the subject of ongoing work.

[10] Suppose that $b_{i:n} > v_{i:n}$ for some $i \leq n$. Then there must be $n - i + 1$ bids exceeding the $(n - i + 1)$th-highest valuation, requiring a violation of assumption 1. This contradiction proves lemma 1.

define a strictly increasing differentiable function $\phi(H; i, n) : [0, 1] \rightarrow [0, 1]$ implicitly as the solution to

$$H = \frac{n!}{(n-i)!(i-1)!} \int_0^{\phi} s^{i-1}(1-s)^{n-i}ds. \tag{3}$$

The following lemma documents a well-known (e.g., Arnold, Balakrishnan, and Nagaraja 1992) and useful property of i.i.d. random variables: the distribution of any order statistic uniquely determines the parent distribution.

LEMMA 2. Given i.i.d. random variables $\{V_i\}_{i=1}^n$ drawn from $F(\cdot)$, the distribution $F_{i:n}(\cdot)$ of the ith order statistic $V_{i:n}$ is related to the parent distribution $F(\cdot)$ by[11]

$$F(v) = \phi(F_{i:n}(v); i, n). \tag{4}$$

We do not observe the realizations of any order statistic $V_{i:n}$ nor, therefore, any distribution $F_{i:n}(\cdot)$ that would enable us to identify $F(\cdot)$ through (4); however, we can infer bounds on each of these distributions. In particular, lemma 1 implies

$$F_{i:n}(v) \leq G_{i:n}(v) \quad \forall i, n, v. \tag{5}$$

Applying the monotone transformation $\phi(\cdot; i, n)$ to each side of (5) and recalling (4) then gives the following result.

THEOREM 1. For all $v \in [r^0, \bar{v},]$

$$F(v) \leq F_U(v) \equiv \min_{n \in \{2,\ldots,\bar{M}\}, i \in \{1,\ldots,n\}} \phi(G_{i:n}(v); i, n).$$

To understand the appearance of the min in the definition of $F_U(v)$, note that, for a given v, (4) and (5) imply $\sum_{n=2}^{\bar{M}} n$ different bounds on $F(v)$—one for each pair of indices (i, n) in (5). In general, some of these bounds will be more informative than others. In particular, the closer $B_{i:n}$ tends to be to $V_{i:n}$ when $V_{i:n} = v$, the closer $\phi(G_{i:n}(v); i, n)$ will be to $F(v)$. Our assumptions say nothing about which of these bounds will be tightest; however, as indicated in theorem 1, the tightest bound is obtained directly by taking the minimum at each value of v. Figure 1 illustrates for the case $N \equiv 3$ using hypothetical distributions $G_{i:3}(\cdot)$, where the bold lower envelope is the bound $F_U(\cdot)$.

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[11] This can also be written as

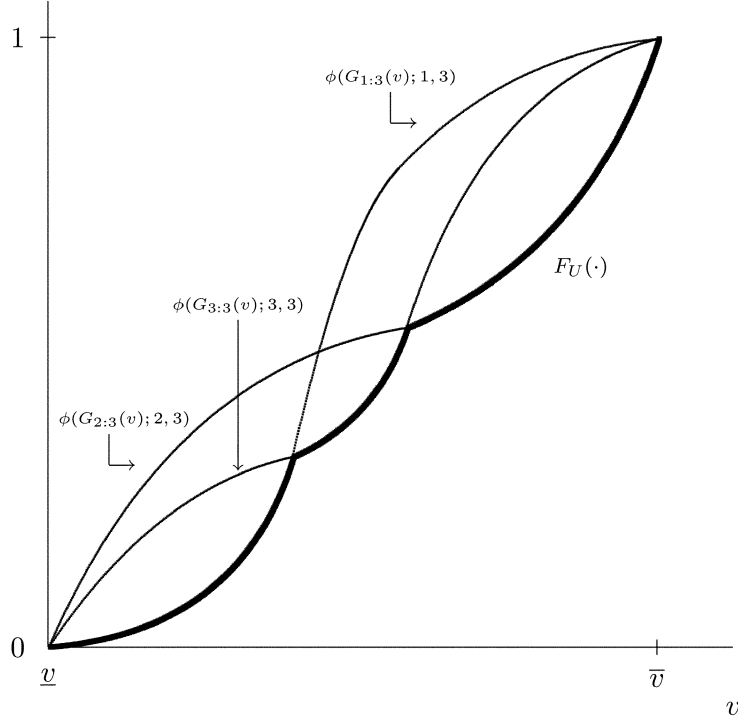$$F_{i:n}(v) = \sum_{j=i}^{n} \binom{n}{j} F(v)^j [1 - F(v)]^{n-j}.$$

FIG. 1

2. Lower Bound

Turning now to the lower bound on the distribution $F(\cdot)$, we follow an analogous approach. Rather than assumption 1, however, here we use assumption 2, which immediately implies

$$v_i \leq v_i^u \equiv \begin{cases} \bar{v} & b_i = b_{n:n} \\ b_{n:n} + \Delta & b_i < b_{n:n} \end{cases} \quad \forall i. \qquad (6)$$

Letting $\psi(\cdot)$ denote the CDF of the random variable V_i^u defined in (6), we then have $F(v) \geq \psi(v)$. Note, however, that $\{v_{n-1:n} < b_{n:n} + \Delta\}$ implies $\{v_{j:n} < b_{n:n} + \Delta \ \forall j < n-1\}$. Furthermore, the inequality $v_i < \bar{v}$ is completely uninformative. Hence (6) really consists of only one inequality.

LEMMA 3. $v_{n-1:n} \leq b_{n:n} + \Delta$.

Although this provides a nontrivial upper bound on the realization of only one order statistic of the valuations at each auction, the relation (4) enables us to use this limited information to construct a much more

informative lower bound than $\psi(\cdot)$. For $n = 2, \dots, \bar{M}$, let $G_{n:n}^{\Delta}(\cdot)$ denote the distribution of $B_{n:n} + \Delta$. Lemma 3 then implies

$$F_{n-1:n}(v) \geq G_{n:n}^{\Delta}(v) \quad \forall n, v. \tag{7}$$

Applying the monotone transformation $\phi(\cdot;\ n-1,\ n)$ to each side of (7) and recalling (4) gives the following result.

THEOREM 2. For all $v \in [r^0, \bar{v}]$,

$$F(v) \geq F_L(v) \equiv \max_{n} \phi(G_{n:n}^{\Delta}(v);\ n-1,\ n).$$

As before, a different lower bound, $\phi(G_{n:n}^{\Delta}(v);\ n-1,\ n)$, is obtained for each value of n, with the max function selecting the tightest bound at each value v.

B.   Remarks

We make two brief observations. First, the dominant strategy equilibrium of Milgrom and Weber's (1982) button auction model gives one example of bidding consistent with (but not implied by) assumptions 1 and 2. In this special case, the top losing bidder exits at his valuation, followed immediately by the winning bidder. Hence, $b_{n-1:n} = v_{n-1:n} = b_{n:n}$; that is, the upper and lower bounds on the order statistic $v_{n-1:n}$ are identical. Hence (since we must have $\Delta = 0$ for this to occur)

$$\phi(G_{n:n}^{\Delta}(v);\ n-1,\ n) = \phi(G_{n-1:n}(v);\ n-1,\ n),$$

implying that $F_L(\cdot)$ and $F_U(\cdot)$ are identical.

REMARK 1. In the dominant strategy equilibrium of the button auction, $F_L(v) = F_U(v)$ for all v.

Thus, whenever observed bids are consistent with equilibrium bidding in the button auction model, our bounds collapse to the true distribution. In fact, bids need not even be fully consistent with the button auction model for this to occur: whenever $v_{n-1:n} = b_{n:n}$ for some n, the lower bound implied by lemma 3 is the true distribution $F(\cdot)$; likewise, whenever $b_{i:n} = v_{i:n}$ for some (i, n), the upper bound obtained from lemma 1 is the true distribution. The model given as example 1 in Appendix B, for example, satisfies both conditions. Conversely, non-identical upper and lower bounds (up to the effects of sampling error) imply a rejection of the button auction model. Hence, there is a sense in which there is no cost to taking our approach rather than assuming the full structure of the standard model: only when the data are inconsistent with the standard model do we identify bounds on $F(\cdot)$ rather than identify $F(\cdot)$ itself.

A second observation is that our limited structure—the symmetric

independent private values model along with assumptions 1 and 2—implies a testable restriction.

REMARK 2. $F_L(v) \leq F_U(v)$ for all v.

This may be more useful than it initially appears. It should be clear that violations of assumptions 1 and 2 can lead to violation of this stochastic dominance relation by violating the inequalities (5) or (7) that underlie it. Furthermore, the critical relation (4) holds only for i.i.d. random variables. Hence, common value components, unobserved heterogeneity across auctions, or other sources of correlation in bidders' willingness to pay within auctions can cause this relation to fail—something we have confirmed both in simulations and with field data. While we do not pursue development of formal tests here, this restriction offers a principle by which such tests might be developed to evaluate the structure we assume in interpreting the data.

C.   Estimation

We suppose now that the researcher observes bids in auctions $t = 1, \ldots, T$, adding a subscript t as necessary to the variables defined above. In each auction, potential bidders draw their valuations independently from the distribution $F_0(\cdot)$. Using $\mathbf{1}[\cdot]$ to denote the indicator function, let $T_n = \sum_{t=1}^{T} \mathbf{1}[n_t = n]$ and define the empirical distribution functions

$$\hat{G}_{i:n}(v) \,=\, \frac{1}{T_n} \sum_{t=1}^{T} \mathbf{1}[n_t \,=\, n, \; b_{i:n_t} \leq v] \tag{8}$$

and

$$\hat{G}^{\Delta}_{n:n}(v) \,=\, \frac{1}{T_n} \sum_{t=1}^{T} \mathbf{1}[n_t \,=\, n, \; b_{n_t:n_t} + \Delta_t \leq v]. \tag{9}$$

For each v, consistent nonparametric estimators of $F_L(v)$ and $F_U(v)$ are easily obtained by substituting these empirical distributions for their population analogs in theorems 1 and 2, that is,

$$\hat{F}_U(v) \,\equiv\, \min_{n \in \{2,\ldots,\bar{M}\}, i \in \{1,\ldots,n\}} \phi(\hat{G}_{i:n}(v); \; i, \; n) \tag{10}$$

and

$$\hat{F}_L(v) \,\equiv\, \max_{n \in \{2,\ldots,\bar{M}\}} \phi(\hat{G}^{\Delta}_{n:n}(v); \; n-1, \; n). \tag{11}$$

Uniform consistency of these estimators is shown in the following theorem. In Haile and Tamer (2002), we also derive the asymptotic distribution and show that bootstrap confidence bands are consistent.

THEOREM 3. For $n = 2, \ldots, \bar{M}$, suppose that $T_n \to \infty$ and $T_n/T \to \lambda_n$ as $T \to \infty$, with $0 < \lambda_n < 1$. Then as $T \to \infty$, (a) $\hat{F}_U(v) \overset{\text{a.s.}}{\to} F_U(v)$ uniformly in v, and (b) $\hat{F}_L(v) \overset{\text{a.s.}}{\to} F_L(v)$ uniformly in v.

Proof. See Appendix A.

In practice, these estimators can be badly biased in small samples because of the concavity (convexity) of the min (max) function. Intuitively, by taking the minimum of the estimated upper bounds $\phi(\hat{G}_{i:n}(v); i, n)$ in (10), we tend to select an estimate with downward estimation error. The resulting bias can lead to estimated bounds that cross in finite samples—particularly if the population bounds $F_U(\cdot)$ and $F_L(\cdot)$ are close. To address this finite sample problem, we propose a simple modification of the estimators in theorem 3. For the upper bound, the estimator $\hat{F}_U(\cdot)$ above has the form $\min(\hat{y}_1, \ldots, \hat{y}_J)$, where each $\hat{y}_j \in [0, 1]$. We replace this minimum with a smooth weighted average that approximates the minimum:

$$\mu(\hat{y}_1, \ldots, \hat{y}_J; \rho_T) = \sum_{j=1}^{J} \hat{y}_j \left[\frac{\exp(\hat{y}_j \rho_T)}{\sum_{k=1}^{J} \exp(\hat{y}_k \rho_T)} \right]. \tag{12}$$

For $\rho \in \mathbb{R}$,

$$\mu(\hat{y}_1, \ldots, \hat{y}_J; \rho) > \min(\hat{y}_1, \ldots, \hat{y}_J),$$

although

$$\lim_{\rho \to -\infty} \mu(\hat{y}_1, \ldots, \hat{y}_J; \rho) = \min(\hat{y}_1, \ldots, \hat{y}_J).$$

Consistency of this modified estimator is obtained by letting the smoothing parameter ρ_T decrease to minus infinity at an appropriate rate as $T \to \infty$. Indeed, all the asymptotic properties of $\hat{F}_L(v)$ are then preserved (see App. C). However, in small samples, this substitution provides a simple and (in extensive Monte Carlo simulations) effective adjustment for the downward bias of $\hat{F}_U(v)$. We make an analogous adjustment to the max function in $\hat{F}_L(v)$, exploiting the fact that

$$\lim_{\rho \to +\infty} \mu(\hat{y}_1, \ldots, \hat{y}_J; \rho) = \max(\hat{y}_1, \ldots, \hat{y}_J).$$

Note that setting the smoothing parameter ρ_T to an arbitrarily large positive number for the upper bound and an arbitrarily large negative number for the lower bound ensures that the bounds will not cross—an important practical issue. This follows from the fact that because $\phi(\cdot; n-1, n)$ is monotonic,

$$\phi(\hat{G}_{n:n}^{\Delta}(v); n-1, n) \leq \phi(\hat{G}_{n-1:n}(v); n-1, n)$$

in any sample.

Finally, note that these nonparametric estimators are simple. Constructing the empirical distribution functions in (8) and (9) amounts

to calculating sample means of indicator functions. Numerical solution for $\phi(H;\ i,\ n)$ is also particularly simple because of the monotonicity of $\phi(\cdot;\ j,\ n)$.

IV.   Bounds on the Optimal Reserve Price

A.   Preliminaries

In many cases the key policy instrument for the seller is the reserve price (minimum acceptable bid). The trade-off the seller faces is straightforward: raising the reserve price will increase surplus extraction when only one bidder is willing to bid but will also raise the likelihood that the object fails to sell. The optimum depends on the distribution of valuations exactly as a monopolist's optimal price depends on the demand curve it faces (Bulow and Roberts 1989). However, deriving bounds on the optimal reserve price from bounds on $F(\cdot)$ is nontrivial because an increase (in the sense of first-order stochastic dominance) in bidder valuations need not raise the optimal reserve price, just as the monopoly price need not rise with an increase in demand. Hence, the optimal reserve need not lie between the prices that would be optimal if $F_L(\cdot)$ or $F_U(\cdot)$ were the true distribution.[12]

Our analysis here exploits results from the literature on optimal auctions (Myerson 1981; Riley and Samuelson 1981). Consistent with this literature we shall assume, for this section only, that $F(\cdot)$ is strictly increasing and continuously differentiable. We also make the following technical assumption, which is implied by the standard regularity condition of Myerson (1981).[13]

ASSUMPTION 3. $(p - v_0)[1 - F_0(p)]$ is strictly pseudo-concave[14] in p on $(\underline{v}, \bar{v})$.

Before proceeding, we must address two questions. The first is whether the optimal reserve price is well defined here. This might seem unlikely given the limited restrictions we place on bidder behavior. However, as long as we believe that bidders act rationally, assumptions 1 and 2 are sufficient to enable application of Myerson's revenue equivalence

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[12] The same issue would arise in constructing confidence bands around the optimal reserve price based on confidence bands around a nonparametric estimate of the distribution $F(\cdot)$, e.g., in the first-price auctions studied by Guerre et al. (2000). Our solution could be applied in that case as well.

[13] Myerson's condition is strict monotonicity of $p - \{[1 - F_0(p)]/f_0(p)\}$, where $f_0(\cdot)$ is the derivative of $F_0(\cdot)$. This condition is satisfied by many known distributions. We use the weaker assumption 3 primarily because imposing this restriction in our nonparametric analysis is more straightforward.

[14] A differentiable univariate function $\pi(\cdot)$ is strictly pseudo-concave on $\mathcal{A} \subset \mathbb{R}$ if, for all distinct x and y in $\mathcal{A}$, $\pi(y) < \pi(x)$ whenever $(y - x)\pi'(x) \leq 0$ (see, e.g., Avriel et al. 1988). This is equivalent to strict pseudo-monotonicity of $\pi'(\cdot)$, i.e., to the condition that $(y - x)\pi'(y) < 0$ whenever $(y - x)\pi'(x) \leq 0$ (Karamardian and Schaible 1990).

result. Following Myerson, let a *feasible auction mechanism* be any well-defined auction game with accompanying bidding strategies such that (i) bidding is voluntary and (ii) bidding strategies form a Nash equilibrium.

LEMMA 4. Suppose that the English auction can be represented by some feasible auction mechanism in which assumptions 1 and 2 hold. Then an English auction with reserve price r and no minimum bid increment is revenue equivalent to a second-price sealed-bid auction with reserve price r.

Proof. See Appendix A.

Hence, the optimal reserve price for a sealed-bid auction is also optimal for an English auction with $\Delta = 0$. Furthermore, under Myerson's regularity condition, lemma 4 and the optimality of the sealed-bid auction (with an optimal reserve) among all possible mechanisms (Myerson 1981) imply that a bid increment of zero is also optimal.[15] This is useful because, as is well known, under assumption 3, the optimal reserve price for a sealed-bid auction is identical to the optimal price for a monopolist with marginal cost v_0 and demand curve $q = 1 - F_0(p)$.[16]

COROLLARY 1. Under assumption 3, the optimal reserve price solves $\max_p (p - v_0)[1 - F_0(p)]$.

The second question is whether, when the seller sets a reserve price $r > \underline{v}$ in the auctions we observe, one can determine the optimal reserve price from the truncated distribution $F(\cdot) = [F_0(\cdot) - F_0(r)]/[1 - F_0(r)]$. This question is not special to the English auction or to our approach, but it has not been carefully addressed before. Clearly, if r exceeds the optimum, the truncated distribution $F(\cdot)$ cannot reveal the optimum except through arbitrary functional form assumptions. The most one could hope for is that when r is below the optimum, the truncated distribution $F(\cdot)$ still determines the optimum, whereas if r is above the optimum, properties of $F(\cdot)$ could at least reveal this fact. The following result shows that both of these hopes are fulfilled.

LEMMA 5. Given any univariate CDF $\Phi(\cdot)$, let

$$p^*(\Phi) \in \arg \max_{p \in \mathrm{supp}\Phi} (p - v_0)[1 - \Phi(p)].$$

[15] Under this regularity condition, the optimality of the sealed-bid auction provides an alternative motivation for the analysis that follows, even without lemma 4: we show how to construct bounds on the one unknown parameter of an optimal selling mechanism.

[16] See, e.g., Bulow and Roberts (1989). One can easily confirm that this problem has the same first-order condition as the reserve price problem

$$\max_{p \in [\underline{v}, \bar{v}]} n[1 - F_0(p)]F_0(p)^{n-1}(p - v_0) + \int_p^{\bar{v}} (v - v_0)n(n-1)f_0(v)[1 - F_0(v)]F_0(v)^{n-2}dv$$

and that the solution to each problem must be interior. Assumption 3 implies that there is only one interior solution to the first-order condition.

Under assumption 3, (i) if $r \leq p^*(F_0)$, $p^*(F_0) = p^*(F)$; (ii) if $r > p^*(F_0)$, then $p^*(F) = r$.

Proof. See Appendix A.

Hence, as long as the actual reserve price is below the optimum, we may ignore the distinction between $F(\cdot)$ and $F_0(\cdot)$.[17] This condition is almost certainly true in the timber auctions we study below (see Sec. VII*B*) and in many other auctions with no reserve prices or extremely low reserves. We proceed under the assumption that this condition holds. In practice, by part ii, finding that the lower bound on $p^*(F)$ exceeds r will provide evidence that this assumption is correct.

B.   Identification

Even with the results above, it is not obvious how bounds on $F(\cdot)$ could be used to construct bounds on the optimal reserve price, since this is characterized by the equation

$$p^* = v_0 + \frac{1 - F(p^*)}{f(p^*)} \tag{13}$$

yet bounds on $F(\cdot)$ place no restriction on $f(v)$ at any point. We are nonetheless able to obtain sharp bounds by exploiting corollary 1 and lemma 5, which imply that we can focus on the problem of identifying bounds on

$$p^* = \arg \max \pi(p),$$

where

$$\pi(p) = (p - v_0)[1 - F(p)].$$

---

[17] The assumption of a fixed reserve price r is made without loss of generality given our ability to condition on auction observables, including the reserve price r_t; however, the distribution bounded by $F_L(\cdot)$ and $F_U(\cdot)$ if we *do not* condition on the reserve price is the mixture

$$\tilde{F}(v) = \frac{1}{T} \sum_t \frac{F_0(v) - F_0(r_t)}{1 - F_0(r_t)}.$$

Under assumption 3, $p^*(\tilde{F})$ maximizes $(p - v_0)[1 - \tilde{F}(p)]$, which can be written as

$$\frac{1}{T} \sum_t (p - v_0) \left[1 - \frac{F_0(p) - F_0(r_t)}{1 - F_0(r_t)} \right].$$

As long as each $r_t \leq p^*(F_0)$, part i of lemma 5 implies that each term in this sum is maximized at $p^*(F_0)$, so that $p^*(\tilde{F}) = p^*(F_0)$; i.e., one need not condition on r_t to obtain valid bounds on the optimal reserve.
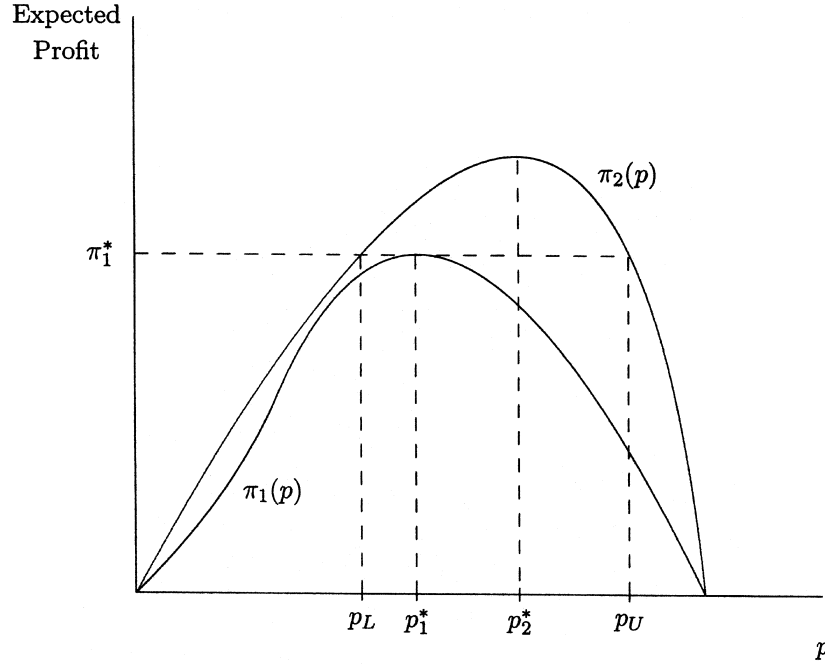
FIG. 2

Observe that bounds on the distribution $F(\cdot)$ imply bounds on the profit function $\pi(\cdot)$. In particular, let

$$\pi_1(p) = (p - v_0)[1 - F_U(p)],$$

$$\pi_2(p) = (p - v_0)[1 - F_L(p)], \tag{14}$$

so that, by theorems 1 and 2,

$$\pi_1(p) \leq \pi(p) \leq \pi_2(p) \quad \forall p. \tag{15}$$

Define $\pi_1^* = \sup_p \pi_1(p)$ and let $p_1^* \in \arg\sup_p \pi_1(p)$ and $p_2^* \in \arg\sup_p \pi_2(p)$. If $\pi_2(p_1^*) = \pi_1^*$ and either $\pi_1(\cdot)$ or $\pi_2(\cdot)$ has slope zero at p_1^*, assumption 3 implies that $p^* = p_1^*$. Likewise, if $\pi_2(p_1^*) = \pi_2(p_2^*) = \pi_1^*$, then $p^* = p_1^*$. For these trivial special cases we define degenerate upper and lower bounds $p_U = p_L = p_1^*$ for completeness. Otherwise define

$$p_L \equiv \sup\{p < p_1^* : \pi_2(p) \leq \pi_1^*\},$$

$$p_U \equiv \inf\{p > p_1^* : \pi_2(p) \leq \pi_1^*\}.$$

Note that $p_L \leq p_1^*$ and $p_U \geq p_1^*$ by construction. Figure 2 illustrates.
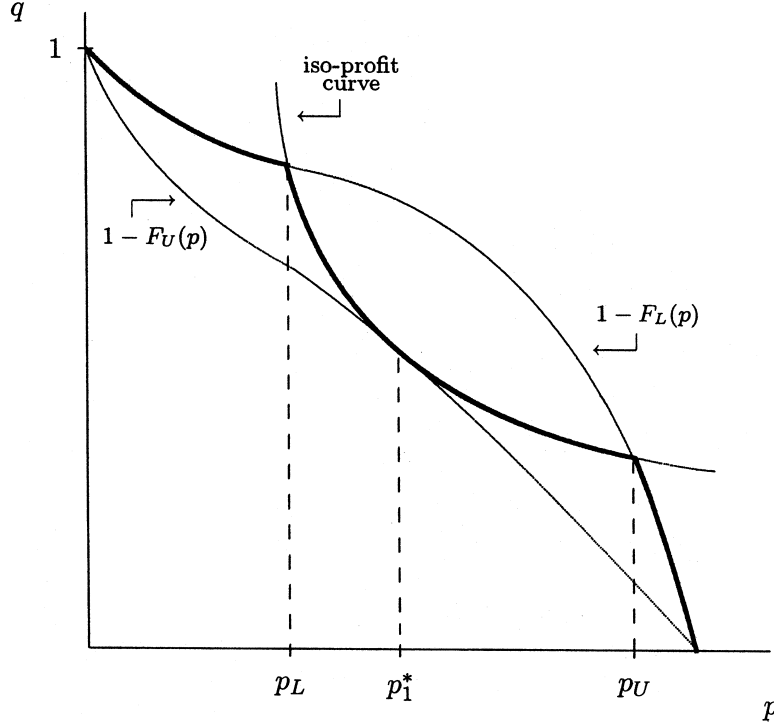
FIG. 3

The following result shows that p_L and p_U enclose the optimal reserve price and that these are the tightest bounds one can obtain from our bounds on $F(\cdot)$; that is, these are *sharp* bounds.

THEOREM 4. Let assumptions 1, 2, and 3 hold. Then $p^* \in [p_L, p_U]$. Given the bounds $F_L(\cdot)$ and $F_U(\cdot)$ on $F(\cdot)$, the bounds p_L and p_U are sharp.

Proof. See Appendix A.

Intuition for the first part of theorem 4 can be seen in figure 2. The true profit function $\pi(\cdot)$ must reach a peak of at least π_1^*, and such a peak cannot be reached at a price outside the interval $[p_L, p_U]$ since $\pi(\cdot)$ must lie below $\pi_2(\cdot)$. Intuition for the sharpness can be gained from figure 3, which shows the "demand curves" $q_j = 1 - F_j(p)$, $j \in \{U, L\}$, along with the iso-profit curve through the point $(p_1^*, 1 - F_U(p_1^*))$. Intersections of this iso-profit curve and the upper demand curve define p_L and p_U. Any downward-sloping demand curve lying between the two original demand curves is consistent with the upper and lower bounds. The bold curve illustrates one possibility. With this demand curve, any price in the interval $[p_L, p_U]$ maximizes profit. The proof shows that one can always construct a similar distribution function that also satisfies

assumption 3 and makes a price arbitrarily close to p_L (or to p_U) the *unique* optimum.

Note that, depending on the shape of $\pi_2(\cdot)$, the bounds (p_L, p_U) may be considerably wider than the interval (p_1^*, p_2^*), even when the bounds $\pi_1(\cdot)$ and $\pi_2(\cdot)$ on the profit function are close, that is, even when $F_L(\cdot)$ and $F_U(\cdot)$ are close. Of course, when $F_L(\cdot)$ and $F_U(\cdot)$ are close, bounds on the expected revenues from the auction must also be similar for reserve prices between p_L and p_U—something we shall see in simulations below.

C.   Estimation

To obtain estimates of the bounds p_L and p_U, we use the sample analogs $\hat{\pi}_1(p) = (p - v_0)[1 - \hat{F}_U(p)]$ and $\hat{\pi}_2(p) = (p - v_0)[1 - \hat{F}_L(p)]$ of the profit functions in (14). Define

$$\hat{\pi}_1^* = \sup_p \hat{\pi}_1(p),$$

$$\hat{p}_1^* \in \arg\sup_p \hat{\pi}_1(p),$$

$$\hat{p}_L = \sup\left\{ p < \hat{p}_1^* : \min_{\pi \in \hat{\pi}_2^c(p)} (\pi - \hat{\pi}_1^*)^2 \le \epsilon_T \right\},$$

$$\hat{p}_U = \inf\left\{ p > \hat{p}_1^* : \min_{\pi \in \hat{\pi}_2^c(p)} (\pi - \hat{\pi}_1^*)^2 \le \epsilon_T \right\},$$

where $\hat{\pi}_2^c(p)$ is the continuous correspondence defined by

$$\pi \in \hat{\pi}_2^c(p) \Leftrightarrow \pi \in \left[\lim_{p' \uparrow p} \hat{\pi}_2(p'), \ \lim_{p' \downarrow p} \hat{\pi}_2(p') \right], \tag{16}$$

and $\epsilon_T > 0$ goes to zero at an appropriate rate as $T \to \infty$ (more on this in App. A).[18] Theorem 5 shows that $\hat{p}_L$ and $\hat{p}_U$ are consistent estimators.

THEOREM 5. As $T \to \infty$, $\hat{p}_U$ and $\hat{p}_L$ converge in probability, respectively, to p_U and p_L.

Proof. See Appendix A.

[18] The estimators here are based on set estimation because $\pi_2(p)$ could be "flat" at π_1^*. Additional discussion of this type of estimator is given in Sec. V below and in Manski and Tamer (2002). If one assumes that $\pi_2(p)$ has nonzero slope at $p \in \{p_L, p_U\}$, then ϵ_T can be $o(1)$ (or set equal to zero), and standard consistency arguments apply. This results in a simpler estimator and may be preferred in practice.

V. Auction Heterogeneity

A. *Conditional Bounds*

In practice, objects sold by auction typically differ in observable dimensions—for example, the size of a tract of timber, the mileage of roads to be paved, or the location of real estate. Variation in these factors introduces correlation in bidders' valuations at each auction. However, conditional on these observables, heterogeneity in valuations reflects idiosyncratic factors (e.g., idiosyncratic tastes, cost shocks, or demand shocks) and may reasonably be assumed to be i.i.d. Our approach easily extends to cover such cases.

Let $\mathbf{X}_t$ be a vector of observable characteristics for auction t. It is straightforward to show that under the conditional independence assumption, our fully nonparametric results from the preceding sections carry through when the distributions $F_{i:n}(\cdot)$ and $G_{i:n}(\cdot)$ are replaced with conditional distributions $F_{i:n}(\cdot|\mathbf{X}_t)$ and $G_{i:n}(\cdot|\mathbf{X}_t)$, yielding bounds $F_L(\cdot|\mathbf{X}_t)$ and $F_U(\cdot|\mathbf{X}_t)$ on the conditional distributions $F(\cdot|\mathbf{X}_t)$ and so forth. These extensions follow standard methods from the literature on nonparametric econometrics and are therefore omitted here, although we provide additional detail in Haile and Tamer (2002).

B. *The Effects of Auction Covariates on Valuations*

Often one is directly interested in how valuations are affected by auction-specific observables such as characteristics of the object for sale, the terms of a government contract, or the identity or characteristics of the seller (e.g., his eBay rating) (see, e.g., Bryan et al. 2000; Houser and Wooders 2000; Bajari and Hortaçsu 2002). Our bounds on the conditional distributions $F(\cdot|\mathbf{X}_t)$ can be used to address such questions, following approaches developed in Manski and Tamer (2002). To do this, we add the following assumption about the conditional mean of bidders' valuations.

ASSUMPTION 4. $E[V_{it}|\mathbf{X}_t = \mathbf{x}_t] = l(\mathbf{x}_t, \boldsymbol{\beta}_0)$, where $\boldsymbol{\beta}_0 \in \mathscr{B}$, $\mathscr{B}$ is a compact subset of $\mathbb{R}^k$, and $l(\cdot, \cdot)$ is known.

Similar restrictions on other functionals of $F(\cdot|\mathbf{X})$ would also suffice. For example, we shall use the conditional median in our application below. The "link function" $l(\cdot, \cdot)$ may take any form; it may be the linear function used in most applied work or a polynomial, for example.

Assumption 4 places structure on the conditional mean of the valuations. With bounds $F_L(v|\mathbf{X}_t)$ and $F_U(v|\mathbf{X}_t)$ on the conditional distribution of valuations, $F(v|\mathbf{X}_t)$, we also know the range in which the conditional mean $l(\mathbf{x}, \boldsymbol{\beta}_0)$ can lie for each value of $\mathbf{x}$. To see this, let $\bar{S}_{\mathbf{x}}$ be a random variable with distribution $F_L(\cdot|\mathbf{x})$ and let $\underline{S}_{\mathbf{x}}$ be a random variable with distribution $F_U(\cdot|\mathbf{x})$. Then assumption 4, the fact that $F_L(v|\mathbf{x}) \leq F(v|\mathbf{x}) \leq$

$F_U(v|\mathbf{x})$ for all v, $\mathbf{x}$, and the definition of first-order stochastic dominance imply

$$E[\underline{S}_\mathbf{x}] \le l(\mathbf{x}, \boldsymbol{\beta}_0) \le E[\bar{S}_\mathbf{x}] \quad \forall \mathbf{x}; \tag{17}$$

that is, given $\mathbf{X}_t = \mathbf{x}$, the average draw from the true distribution must lie between the average draw from the corresponding bounds on the distribution. An estimation approach can then be built on the idea of excluding trial values for $\boldsymbol{\beta}_0$ that lead to violations of (17).

For a parameter value $\mathbf{b} \in \mathscr{B}$, define

$$V(\mathbf{b}) = \{\mathbf{x} : l(\mathbf{x}, \mathbf{b}) < E[\underline{S}_\mathbf{x}] \cup E[\bar{S}_\mathbf{x}] < l(\mathbf{x}, \mathbf{b})\}.$$

Given $\mathbf{b}$, this is the set of values of $\mathbf{x}$ for which $l(\mathbf{x}, \mathbf{b})$ fails to lie between the conditional means $E[\underline{S}_\mathbf{x}]$ and $E[\bar{S}_\mathbf{x}]$. If we observed such an $\mathbf{x}$, then we could rule out $\mathbf{b}$ as a candidate for the true parameter $\boldsymbol{\beta}_0$. Hence, the parameter $\boldsymbol{\beta}_0$ is *identified relative to* $\mathbf{b}$ if and only if $\Pr[V(\mathbf{b})] > 0$. The set of parameter values indistinguishable from the true $\boldsymbol{\beta}_0$ is

$$\Sigma = \{\mathbf{b} \in \mathscr{B} : \Pr[V(\mathbf{b})] = 0\}.$$

We shall refer to Σ as the *identified set*.

The parameter $\boldsymbol{\beta}_0$ is *point identified* (i.e., $\Sigma = \boldsymbol{\beta}_0$) iff $\Pr[V(\mathbf{b})] > 0$ for all $\mathbf{b} \in \mathscr{B} \backslash \boldsymbol{\beta}_0$. In general, the identified set Σ will be a nonempty subset of $\mathbb{R}^k$. However, sufficient conditions for point identification can be derived, even when the bounds $F_U(\cdot|\mathbf{X})$ and $F_L(\cdot|\mathbf{X})$ are not coincident. For example, if $l(\mathbf{x}, \mathbf{b}) = \mathbf{xb}$ and both $E[\underline{S}_\mathbf{x}]$ and $E[\bar{S}_\mathbf{x}]$ are linear in $\mathbf{X}$, all parameters on regressors with unbounded support are point identified. While such conditions obviously need not hold, the important point is that in practice even wide bounds on $F(\cdot|\mathbf{X})$ can yield tight bounds on the parameters of this flexible semiparametric model of bidder valuations.

In general, our problem consists of estimating the set of parameters consistent with the data. Estimation of this identified set is based on minimization of a criterion function that penalizes violations of (17). Let $\delta_1(\mathbf{x}, \mathbf{b}) = \mathbf{1}[E[\underline{S}_\mathbf{x}] > l(\mathbf{x}, \mathbf{b})]$ and $\delta_2(\mathbf{x}, \mathbf{b}) = \mathbf{1}[E[\bar{S}_\mathbf{x}] < l(\mathbf{x}, \mathbf{b})]$ and define

$$Q(\mathbf{b}) = \int \{[E[\underline{S}_\mathbf{x}] - l(\mathbf{x}, \mathbf{b})]^2 \delta_1(\mathbf{x}, \mathbf{b})$$
$$+ [l(\mathbf{x}, \mathbf{b}) - E[\bar{S}_\mathbf{x}]]^2 \delta_2(\mathbf{x}, \mathbf{b})\} dP_\mathbf{X}, \tag{18}$$

where $P_\mathbf{X}$ is the distribution of the conditioning variable(s) $\mathbf{X}$. Let $E_T[\underline{S}_\mathbf{x}]$ and $E_T[\bar{S}_\mathbf{x}]$ denote the means of a sample of draws from $\hat{F}_U(\cdot|\mathbf{x})$

and $\hat{F}_t(\cdot|\mathbf{x})$, respectively. Substituting these simulated values for their population counterparts in (18) gives the sample criterion function

$$Q_T(\mathbf{b}) = \frac{1}{T}\sum_{t=1}^{T}\{[E_T[\underline{S}_{\mathbf{x}_t}] - l(\mathbf{x}_t, \boldsymbol{\beta})]^2 \mathbf{1}[E_T[\underline{S}_{\mathbf{x}_t}] > l(\mathbf{x}_t, \boldsymbol{\beta})]$$

$$+ [l(\mathbf{x}_t, \boldsymbol{\beta}) - E_T[\bar{S}_{\mathbf{x}_t}]]^2 \mathbf{1}[l(\mathbf{x}_t, \boldsymbol{\beta}) > E_T[\bar{S}_{\mathbf{x}_t}]]\}.$$

A seemingly natural approach would be to minimize $Q_T(\mathbf{b})$. However, because we are estimating a set, this can lead to poor properties. In particular, the sample criterion function involves sampling error, and we want to avoid excluding parameter values that fail to minimize the criterion only because of sampling variation. To do this, we introduce a tolerance parameter $\epsilon_T > 0$ and define

$$\hat{\Sigma} \equiv \left\{\mathbf{b} \in \mathscr{B} : Q_T(\mathbf{b}) \leq \min_{\mathbf{c} \in \mathscr{B}} Q_T(\mathbf{c}) + \epsilon_T\right\}.$$

This is the set of parameter values that put the sample criterion function within ϵ_T of its minimum. By letting $\epsilon_T \to 0$, we ensure that as the sample size grows, every element in $\hat{\Sigma}$ is near some element of Σ. By limiting the rate at which $\epsilon_T \to 0$, we can also ensure that every element of Σ has a nearby value in $\hat{\Sigma}$.

To make this more precise, given two sets A and B, define

$$\rho(A, B) = \sup_{\mathbf{b} \in A} \inf_{\mathbf{b}^* \in B} |\mathbf{b} - \mathbf{b}^*|$$

so that the Hausdorff distance between the sets Σ and $\hat{\Sigma}$ is $\max\{\rho(\hat{\Sigma}, \Sigma), \rho(\Sigma, \hat{\Sigma})\}$. The following result provides sufficient conditions for (Hausdorff) convergence of $\hat{\Sigma}$ to Σ.

THEOREM 6. Let assumptions 1, 2, and 4 hold. Assume that (i) $E_T[\underline{S}_{\mathbf{x}}] \overset{\text{a.s.}}{\to} E[\underline{S}_{\mathbf{x}}]$, (ii) $E_T[\bar{S}_{\mathbf{x}}] \overset{\text{a.s.}}{\to} E[\bar{S}_{\mathbf{x}}]$, and (iii) there exists an integrable function $\Upsilon : \mathbb{R} \to \mathbb{R}$ that dominates

$$[E[\underline{S}_{\mathbf{x}}] - l(\mathbf{x}, \mathbf{b})]^2 \delta_1(\mathbf{X}, \mathbf{b}) + [E[\bar{S}_{\mathbf{x}}] - l(\mathbf{x}, \mathbf{b})]^2 \delta_2(\mathbf{X}, \mathbf{b}).$$

Then if $\epsilon_T \to 0$, $\rho(\hat{\Sigma}, \Sigma) \overset{\text{a.s.}}{\to} 0$. If $\sup_{\mathbf{b} \in \mathscr{B}}\{|Q_T(\mathbf{b}) - Q(\mathbf{b})|\}/\epsilon_T \to 0$, then $\rho(\Sigma, \hat{\Sigma}) \to 0$.

Proof. See Appendix A.

VI.   Monte Carlo Experiments

To examine the performance of our approach, we have conducted a number of Monte Carlo experiments. To generate artificial bidding data for each experiment, T_n samples of $n = 6$ valuations were first drawn from a known distribution. Bids were then generated as in example 2 of Appendix B, where the auctioneer iteratively selects a bidder at ran-

dom, who must then either raise the standing bid by one bid increment or drop out. We also consider a variation allowing jump bidding in a simplistic way: each time a bidder agrees to bid, with probability λ he jumps to a uniform draw between the standing bid (plus Δ) and his valuation, rather than raising the bid by Δ. This binomial draw on whether to jump bid is made independently at each bidding opportunity. These procedures enable us to construct bidding data that deviate (continuously) from equilibrium bids in the button auction model in ways that are common in field data. In particular, with nonzero values of Δ and λ, gaps between the top two bids may exist; bids may be poor approximations of valuations; and the ordering of bidders according to their bids may differ from that according to valuations. However, setting $\lambda = 0$ and letting $\Delta \to 0$ yields bids identical to equilibrium bids in the button auction model.

We performed 500 replications of each experiment. We summarize the results graphically, plotting the true distribution of valuations, along with pointwise mean estimates of the upper and lower bounds, fifth percentile of the lower bounds, and ninety-fifth percentile of the upper bounds. Figure 4 shows the results with a lognormal distribution of valuations, with $\lambda = 0$ and Δ equal to the maximum of one and 5 percent of the standing bid. We consider sample sizes ranging from $T_n = 25$ auctions to $T_n = 200$ auctions. Even in very small samples the estimators perform well, although the tightness of the bounds increases with T_n because of changes in the smoothing parameter ρ_T^U (we use a sequence diverging to $-\infty$ at rate $\sqrt{T_n}$). We also measured how frequently the estimated upper and lower bounds crossed on 5 percent or more of the interval $[0, 180]$. This occurred in between 0 and 3 percent of the replications for each value of T_n. This compares with frequencies of 16–33 percent obtained without the finite sample adjustment discussed in Section IIIC.

In figure 5, we examine the effects of varying the parameters Δ and λ, taking a sample size $T_n = 200$ in each case. Figure 5d allows the probability of making a jump bid to differ across bidders, depending on the realizations of their valuations. Roughly speaking, the bounds are less informative (wider) the farther actual bidding behavior is from that in a symmetric equilibrium of the button auction model. In particular, larger values of Δ or λ result in wider bounds. In figure 6 we illustrate the results for two other distributions, a χ^2 (fig. 6a, c) and a beta (fig. 6b, d). While the shapes of these distributions are quite different from the lognormal, the performance of the bounds is similar, as is the case for a range of other distributions we examined.

Table 1 summarizes the bounds on the optimal reserve price obtained using variations on the lognormal experiments above, all with $n = 6$ and $T_n = 200$. Although in each simulation $\lambda = 0$, implying that our
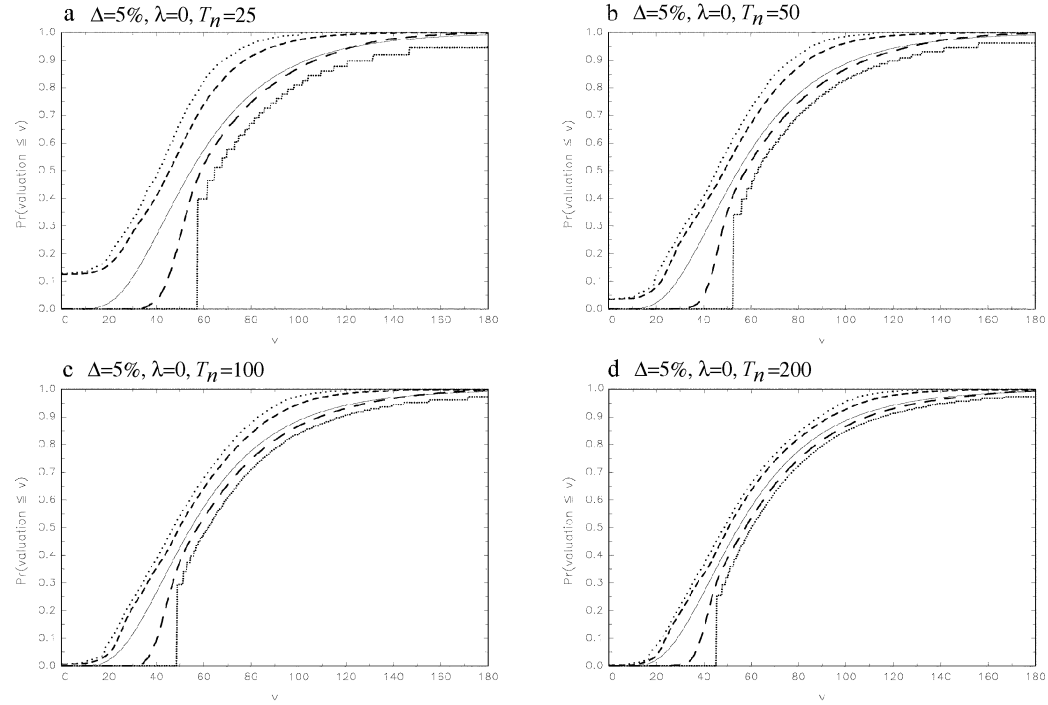
Fɪɢ. 4.—Monte Carlo results. Solid curves are true distribution functions, dashed curves are mean estimated upper and lower bounds, and dotted curves are the fifth and ninety-fifth percentiles.

a Δ=1%, λ=0, T_n=200

b Δ=10%, λ=0, T_n=200

c Δ=5%, λ=0.25, T_n=200

d Δ=5%, λ=$F(v)^4$, T_n=200

FIG. 5.—Monte Carlo results. Solid curves are true distribution functions, dashed curves are mean estimated upper and lower bounds, and dotted curves are the fifth and ninety-fifth percentiles.

a Δ=5%, λ=0, T_n=200

b Δ=5%, λ=0, T_n=200

c Δ=5%, λ=0.25, T_n=200

d Δ=5%, λ=0.25, T_n=200

FIG. 6.—Monte Carlo results. Solid curves are true distribution functions, dashed curves are mean estimated upper and lower bounds, and dotted curves are the fifth and ninety-fifth percentiles.
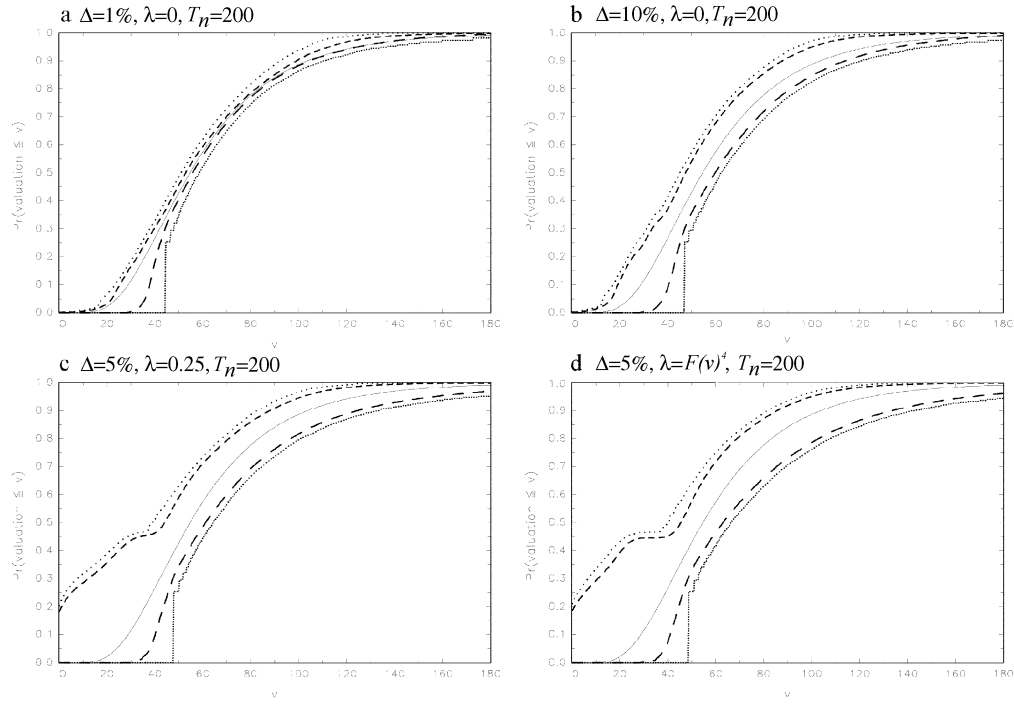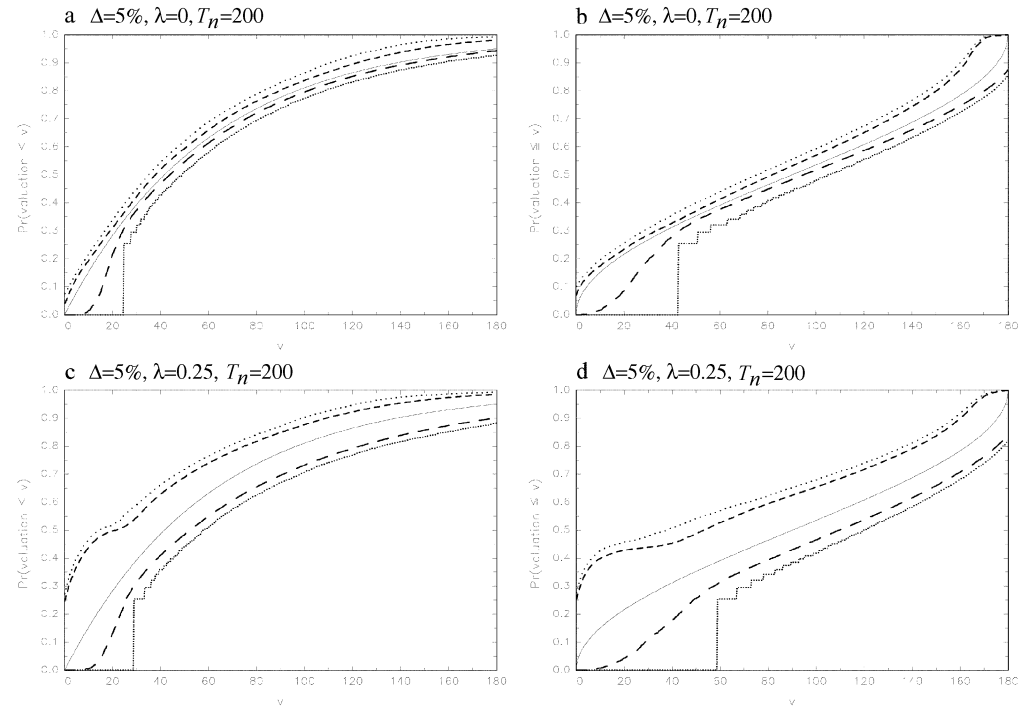
TABLE 1
MONTE CARLO SIMULATIONS: OPTIMAL RESERVE PRICE

Lognormal Parameters	$\mu=4, \sigma=.5$	$\mu=3, \sigma=1.0$	$\mu=5, \sigma=.25$
True p^*	42.1	27.2	112.6
$F(p^*)$	.30	.62	.13
Mean estimated bounds			
$[\hat{p}_L, \hat{p}_U]$	[28.4, 67.7]	[17.2, 50.3]	[82.9, 152.8]
90% confidence interval	[27.1, 70.3]	[15.2, 58.0]	[80.3, 157.3]

bounds on $F(\cdot)$ are quite tight (recall fig. 4*d*), the bounds on the optimal reserve price are fairly wide. Since our bounds on the optimal reserve price are sharp given the bounds on $F(\cdot)$, these results suggest the sensitivity of policy implications to seemingly small variation in the primitives. A virtue of the approach we propose here is that we can explicitly account for this in evaluating policy—something we shall do in the application below.

A.   Comparison to Existing Methods

We compare these results with those from experiments using the same data-generating processes but estimation approaches considered previously in the literature, both based on the button auction model. The first approach ignores winning bids and treats losing bids as though they were equilibrium bids in the button auction model. This approach was proposed by Donald and Paarsch (1996) and implemented by Paarsch (1997) and Hong and Shum (in press). Following this literature, we use a maximum likelihood implementation of this model, with the likelihood obtained directly from the joint density of the lowest $n_t - 1$ order statistics of the valuations at each auction. We refer to this as "model 1." In practice, one would not know the true parametric family; however, we shall ignore this source of potential misspecification and assume the true (lognormal) family. As figure 7 illustrates, even without misspecification of the parametric family, this method can perform badly. Here we report results using a sample size of $T_n = 200$, a bid increment of 5 percent, and $\lambda \in \{0, .25\}$. We compare the true lognormal distribution to that at the mean parameter estimates and include mean 90 percent confidence bands at each point calculated by the delta method. Even with no jump bidding ($\lambda = 0$), the performance of this model is poor, with the true CDF lying outside the 90 percent confidence bands on most of the support. When $\lambda = .25$, the results are even worse. The reason is that with minimum bid increments and jump bidding, bids often are well below the corresponding bidders' valuations, particularly in the case of lower-ranked bids. Interpreting
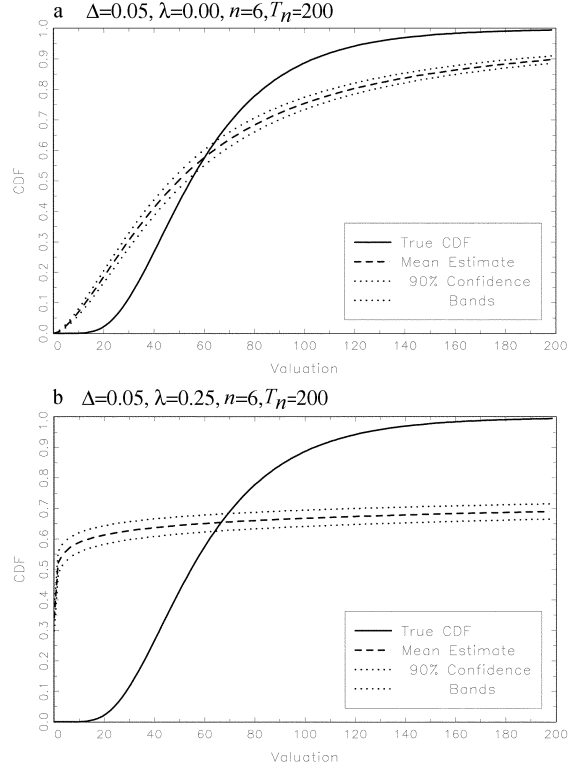
FIG. 7.—Monte Carlo results, model 1. Solid curves are true distributions, dashed curves are distributions implied by mean parameter estimates, and dotted curves are mean 90 percent confidence bands.

each bid as equal to the underlying valuation then results in a "stretching out" of the estimated distribution.

An alternative, "model 2," avoids misinterpretation of losing bids by ignoring them altogether and assuming only that the winning bid $b_{n_t:n_t}$ is equal (up to the minimum bid increment) to the second-highest valuation $v_{n_t-1:n_t}$ (see, e.g., Paarsch 1992b; Baldwin et al. 1997; Haile 2001). We use a nonparametric implementation suggested by the identification argument of Athey and Haile (2002, theorem 1). When $\Delta_t > 0$, there is ambiguity regarding the implementation of this model, since one could assume, for example,

$$b_{n_t:n_t} = v_{n_t-1:n_t} + \Delta_t \quad \text{or} \quad b_{n_t:n_t} = v_{n_t-1:n_t}$$

$$\text{or} \quad b_{n_t:n_t} = v_{n_t-1:n_t} - \Delta_t. \tag{19}$$

When N_t takes on only one value, the last of these options will result in
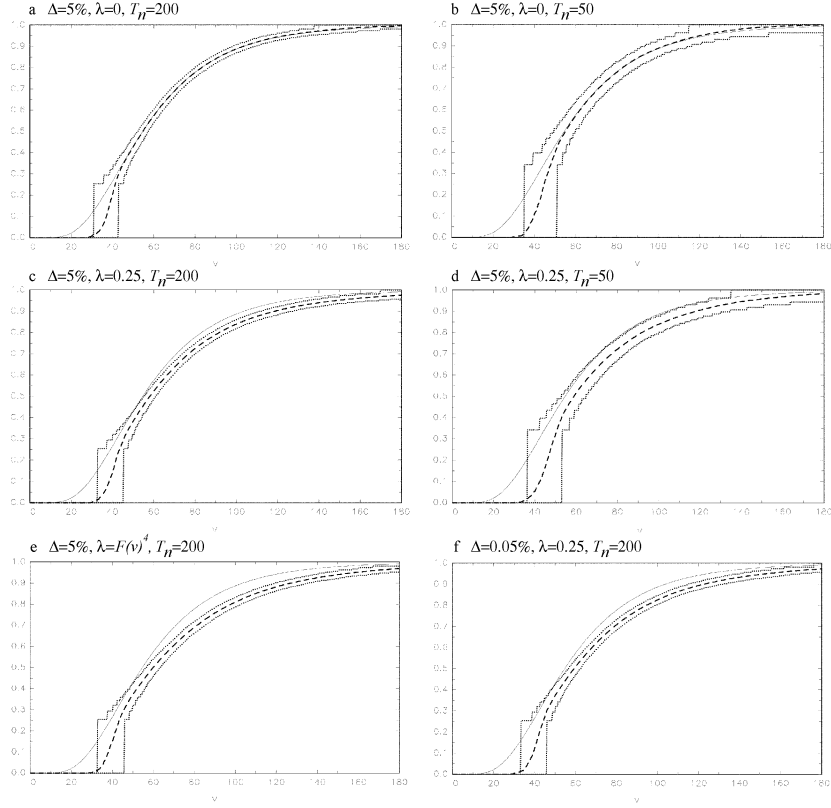
Fig. 8.—Monte Carlo results, model 2. Solid curves are true distributions, dashed curves are mean estimates, and dotted curves are fifth and ninety-fifth percentiles.

an estimator equal to $\hat{F}(v) = \phi(\hat{G}^{\Delta}_{n:n}(v);\ n-1,\ n) = \hat{F}_L(v)$; that is, it will give a point estimate *identical* to our estimate of the lower bound. When N_t takes on different values, one would construct an estimator equal to an average (over n) of each $\phi(\hat{G}^{\Delta}_{n:n}(v);\ n-1,\ n)$, leading to an estimate of a distribution lying *strictly below* our lower bound. We take the intermediate approach of assuming $b_{n_t:n_t} = v_{n_t-1:n_t}$, which is also consistent with the prior literature.

When the gap between each $b_{n_t:n_t}$ and $v_{n_t-1:n_t}$ is small, model 2 can give estimates that are close to the truth. Figure 8a illustrates. There the effect of ignoring the bid increment Δ_t almost perfectly offsets the difference between our lower bound and the true CDF. This results in very good estimates, at least above the thirtieth percentile. Because there are few realizations of $V_{n-1:n}$ in the left tail, however, the performance of the estimator is weaker below the thirtieth percentile. This can be

important in practice: here the true optimal reserve price lies at approximately the thirtieth percentile. As figure 8*b* illustrates, this problem is more serious with a sample size of 50—still a fairly large sample of homogeneous auctions. In figure 8*c, d,* and *e,* we see that this approach gives poor results when bidders jump bid with probability one-fourth or with probability $F(v)^4$. Here the true distribution often lies outside the 90 percent confidence bands. Figure 8*f* (by comparison with fig. 8*c*) illustrates the perverse fact that moving the true data-generating process closer to that of the button auction can lead to worse estimates. Here we shrink the minimum bid increment to 0.05 percent of the standing bid—roughly that of the timber auctions studied below. As suggested by the discussion above, this shifts the estimate of each $F(v)$ downward, that is, away from the true distribution. Finally, in figure 9 we repeat the experiments underlying figure 3, illustrating the performance of model 2 with alternative distributions.

An important conclusion from these simulations is that, despite the fact that equilibrium bidding in the button auction model obeys our two assumptions, estimates based on the button auction model can be quite misleading if the true data-generating process deviates from this structure, even in seemingly small ways. Furthermore, imposing the full structure of this model need not yield estimates lying within our bounds, even asymptotically. Model 2 appears to be much more robust than model 1, although even model 2 can deliver poor results.

VII.   Application to Forest Service Timber Auctions

A.   Background and Data

We apply our methods to data from auctions of timber-harvesting contracts held by the U.S. Forest Service. A contract gives the purchaser the obligation to harvest all included timber on the tract within a specified contract term, as well as rights to the harvested timber. We consider auctions held between 1982 and 1990 in Washington and Oregon ("region 6"). Bidders are primarily specialized sawmills, wood product conglomerates, and independent loggers. We shall assume that bidding in each auction is competitive and independent of all other auctions.[19]

As in Baldwin et al. (1997) and Haile (2001), we focus on a subset of sales for which the independent private values assumption is most compelling.[20] We consider only "scaled sales," where bids are species-

[19] Baldwin et al. (1997) consider possible collusion in this region in an earlier time period.

[20] Other studies assuming the symmetric independent private values model for timber auctions include Johnson (1979), Hansen (1985, 1986), Paarsch (1992*b*, 1997), Cummins (1994), Elyakime et al. (1994), Carter and Newman (1998), and Campo, Perrigne, and Vuong (2000).
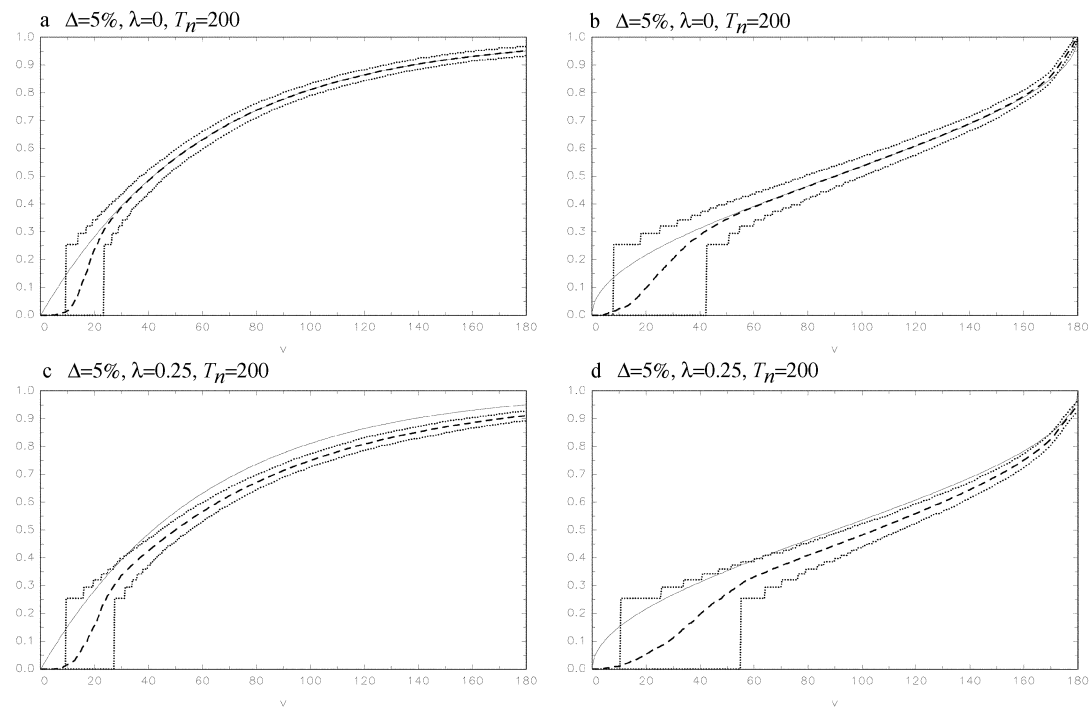
FIG. 9.—Monte Carlo results, model 2. Solid curves are true distributions, dashed curves are mean estimates, and dotted curves are fifth and ninety-fifth percentiles.

specific prices, and payments are made to the Forest Service on the basis of the quantities *actually* harvested.[21] We consider only contracts with terms of one year or less. Consequently, most bidder uncertainty regarding timber volumes and prices is eliminated or at least insured by the Forest Service. This reduces the gains to bidders from conducting their own cruises of the tracts. Indeed, for scaled sales, bidders usually do not undertake their own cruises (Natural Resources Management Corp. 1997). Furthermore, the restriction to sales with short contract terms, along with our restriction to sales after 1981, minimizes the likelihood that opportunities for resale/subcontracting introduce a common value element as in Haile (2001).[22] The features of these auctions leave little room for private information regarding any common factors determining bidder valuations. However, bidders are likely to have private information about private value elements such as their idiosyncratic demands, contracts for future sales, and their inventories of uncut timber from private sales (Baldwin et al. 1997; Haile 2001).

Before each sale in our sample, the Forest Service published a "cruise report," providing bidders with (among other things) estimates of timber volume, harvesting costs, costs of manufacturing end products, and revenues from end product sales. Records of these estimates enable us to condition on a large number of auction covariates capturing bidders' common information about the sale—something important to the validity of our assumption that valuations are independent conditional on observables. Forest Service officials also used these estimates to construct a reserve price, equal to the estimated selling value less estimated harvesting and manufacturing costs, and an allowance for profit and risk. Bidders were required to submit sealed bids of at least the reserve price to be eligible to bid in the auction. Hence, sales records indicate the registration of all bidders, including any who did not actually call out a bid at the auction.

When bidders gather for the auction, bidding opens at the highest sealed bid and then proceeds orally, with a minimum bid increment of 5 cents per thousand board feet (MBF), which is about $25 (1983

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[21] Athey and Levin (2001) point out that "skew bidding" can arise in scaled sales if bidders have private information about errors in the Forest Service estimates of the relative volumes of timber of different species. Such information would also introduce a common value element to the auction. As in Baldwin et al. (1997) and Haile (2001), we assume that there is little information of this sort in the auctions we consider and focus on the total bid made by each bidder, the same statistic used to determine the auction winner.

[22] Haile (2001) uses the opposite selection criterion to focus on resale. We also exclude from our sample salvage sales, sales set aside for small bidders, and contracts with road construction requirements.

TABLE 2
Gaps Between First- and Second-Highest Bids

Quantiles	High Bid	Gap	Minimum Increment	Gap ÷ Increment
10%	9,151	30	4.1	1.2
25%	22,041	92	10.1	6.9
50%	55,623	309	23.4	14.8
75%	127,475	858	52.1	20.0
90%	292,846	2,048	110.5	76.4

dollars) on the median tract.[23] Forest Service officials report that jump bidding is common. Table 2 provides some support, showing a gap between the highest and second-highest bid of several hundred dollars (roughly 10–20 times the minimum increment) in the majority of auctions. Since the cost of jump bidding—the risk that one wins with the jump bid and pays too much—is highest at the end of the auction, jump bidding is likely to be more significant early in the auctions. However, these gaps themselves are generally quite small relative to the total bid, suggesting that we may be able to obtain tight bounds.

B.   Reserve Price Policy

The Forest Service's mandated objective in setting a reserve price is to ensure that timber is sold at a "fair market value," defined as the value to an "average operator, rather than that of the most or least efficient" (U.S. Forest Service 1992). Many observers have argued that Forest Service reserve prices fall short of this criterion and are essentially non-binding floors (see, e.g., Mead, Schniepp, and Watson 1981, 1984; Haile 1996; Campo et al. 2000). Bidders, for example, claim that the reserve prices never prevent them from bidding on a tract (Baldwin et al. 1997). As discussed above, for our purposes it is sufficient to assume only that the actual reserve prices are below the profit-maximizing reserve prices.

There is an ongoing controversy over so-called below-cost sales—sales generating revenues insufficient to cover even the costs to the Forest Service of administering the contract (see, e.g., U.S. General Accounting Office 1984, 1990, 1991; U.S. Forest Service 1995). Obviously, this is possible only with reserve prices below profit-maximizing levels. However, reserve prices are not set with the goal of profit maximization nor

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[23] Forest Service rules actually require only that total bids rise as the auction proceeds, although local officials often specified discrete increments. In the time period we consider, the 5 cent increment was a common practice in this region. Sometimes increments of 1 cent per MBF were used, and many sales used no minimum increment. We use the 5 cent increment since this results in a more conservative bound, although variations of this magnitude have very little effect on the results: 5 cents represents about 0.05 percent of the average bid in our sample.

even based on costs at all. The Forest Service has a range of objectives, including "forest stewardship" and a mandate to provide a supply of timber to meet U.S. demand for wood and wood products. As a result, costs of the Forest Service timber sales program may exceed those of private timber producers, and some harvest decisions may be driven by goals of improving forest health or preventing catastrophic fires.

Determining whether these goals justify selling with suboptimal reserve prices, or even below cost, requires a careful evaluation of the costs and benefits of changing reserve prices. However, such an evaluation has been hindered in part by an inability to assess the effects of alternative reserve prices on outcomes. For example, a recent U.S. Forest Service (1995) report asserts that "studies indicate it is nearly impossible to use sale records to determine if marginal sales made in the past would have been purchased under a different [reserve price] structure." Our results below will address precisely this issue.

C. Results

We perform our analysis conditioning on a vector $\mathbf{X}_t$ of auction-specific covariates consisting of the year of the auction, an index of species concentration,[24] estimated manufacturing costs, estimated selling value, estimated harvesting costs, an indicator for the Forest Service geographical zone in which the tract lies, and a six-month inventory of timber sold in the same region. Given prior results suggesting correlation between the number of bidders and unobserved tract characteristics in other national forests (Haile 2001), we also condition on n_t.[25] Similar results are obtained if we also condition on the reserve price. This is not surprising since the reserve price was almost completely determined by a subset of the other covariates in this period. Table 3 presents summary statistics.

Figure 10 shows our estimates of the upper and lower bounds $F_U(\cdot|\mathbf{X}_t)$ and $F_L(\cdot|\mathbf{X}_t)$, along with bootstrap confidence bands (based on 500 replications), evaluated at the mean of the $\mathbf{X}_t$ vector.[26] The bounds

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[24] The index is equal to $\sum_j q_j^2$, where q_j is the estimated volume of species j timber on the tract. Because many bidders are specialized sawmills, a tract may be more attractive if it consists primarily of a single species.

[25] We include auctions with n_t between three and eight, which is equivalent to evaluating at the sample mean using a uniform kernel with bandwidth equal to 0.77 times the standard deviation. This conditioning turns out to have little effect in this sample.

[26] To construct the conditional empirical distribution functions, we use a product of Gaussian kernels with a bandwidth of 0.3 times the standard deviation for each component of $\mathbf{X}_t$, except in the case of the zone dummy, where we include sales only in zone 2. Small changes in the bandwidths have little effect on the results, although significantly wider bandwidths lead to sufficient heterogeneity in the included tracts that the estimated upper and lower bounds cross when using values of the smoothing parameters suggested by simulations (recall the discussion following remark 2).

TABLE 3
SUMMARY STATISTICS

	Mean	Standard Deviation	Minimum	Maximum
Number of bidders	5.7	3.0	2	12
Year	1985.2	2.6	1982	1990
Species concentration	.68	.23	.24	1.0
Manufacturing costs	190.3	43.0	56.7	286.5
Selling value	415.4	61.4	202.2	746.8
Harvesting cost	120.2	34.1	51.1	283.1
Six-month inventory*	1,364.4	376.5	286.4	2,084.3
Zone 2 dummy	.88		0	1

 * In millions of board feet.

are quite tight. The shape of the true distribution suggested by these bounds resembles a lognormal distribution, which has been used in several prior studies.

To construct estimates of bounds on the optimal reserve price, an estimate of v_0, the cost of allowing the harvest of the tract, is needed. We consider a range of possible values based on Forest Service estimates (U.S. Forest Service 1995; U.S. General Accounting Office 1999).[27] Table 4 shows the results of simulations used to evaluate the trade-offs between net revenues and the probability that a tract goes unsold with alternative reserve prices. Values of v_0 between \$20 and \$120 are considered and the implied bounds on the optimal reserve prices calculated. For each value of v_0, we consider three possible reserve prices: $\hat{p}_L$, $\hat{p}_U$, and the average of the two. The table reports simulated gains in profit per MBF relative to actual profits, using each value of v_0 as the measure of costs. This is done both assuming $F(\cdot|\mathbf{X}) = \hat{F}_L(\cdot|\mathbf{X})$ and assuming $F(\cdot|\mathbf{X}) = \hat{F}_U(\cdot|\mathbf{X})$, providing estimated bounds on the profit gains (losses) from using each reserve price considered. Note that lemma 4 enables us to use equilibrium bids in a second-price sealed-bid auction to obtain revenue predictions.

As foreshadowed by our simulations, despite the tightness of the bounds on $F(\cdot)$ in figure 8, the bounds on the optimal reserve price for each v_0 are fairly wide. Because the bounds on $F(\cdot)$ are tight, however, our estimates of the expected revenues obtained with reserve prices

[27] For sales in region 6 in 1993, the Forest Service estimated that costs of the timber sales program were between \$85 and \$113 per MBF (U.S. General Accounting Office 1999). On the basis of sales in 1990–92, nationwide cost-based reserve prices between \$18 and \$47 per MBF were suggested as appropriate (U.S. Forest Service 1995), depending on which timber sales program costs are to be covered by auction revenues. Both calculations include some costs that are sunk at the time of the auction and, therefore, should be excluded from v_0. However, other costs, such as forgone return on investment and adverse environmental impacts, are excluded. Obtaining more precise estimates of v_0, ideally as a function of tract characteristics, would be an important step toward a more definitive analysis of reserve price policies.
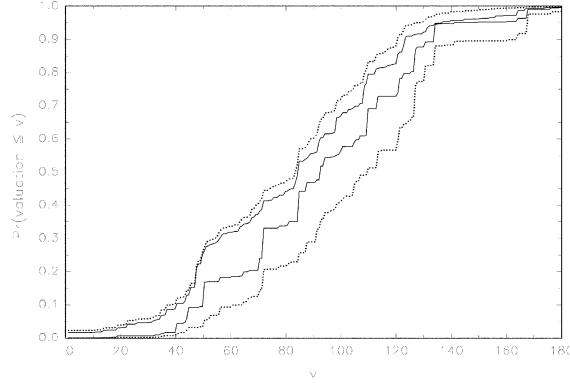
Fig. 10.—U.S. Forest Service timber auctions. Solid curves are estimated bounds, and dotted curves are bootstrap confidence bands.

between $\hat{p}_L$ and $\hat{p}_U$ differ little, with v_0 held fixed. The calculated bounds on the optimal reserve prices provide strong support for the assumption that the actual reserve price (around \$54) is well below the optimum. Even with $v_0 = 0$, the estimated lower bound on p^* is still slightly larger than the average actual reserve price. These results also suggest that, at least on average tracts in our sample, reserve prices could be raised considerably without causing many tracts to go unsold. Even if $F(\cdot) = \hat{F}_U(\cdot)$, a reserve price nearly twice the actual average would be required to drive the probability that a tract will go unsold past 15 percent—a key threshold given a Forest Service policy of ensuring that at least 85 percent of all offered timber volume is actually sold (U.S. Forest Service 1992).

The potential gains in profit from raising reserve prices obviously depend on v_0. With $v_0 = \$20$, for example, we estimate that gains would be less than 10 percent (and not necessarily positive) even when $F(\cdot) = F_L(\cdot)$.[28] With $v_0 = \$80$, however, the potential gains are much larger. In that case, the Forest Service might achieve net gains of \$10 per MBF or more, which would represent more than an 80 percent increase in profits. With opportunity costs above the average gross revenues of \$92.08 per MBF, sales typically lead to a net loss. Hence, for costs of \$100 or \$120, substantial gains (reductions in losses) from im-

[28] Note that, in general, revenues need not be higher with a given reserve price between p_L and p_U given one particular CDF between $F_L(\cdot)$ and $F_U(\cdot)$. However, if $\Delta = 0$ or if Myerson's regularity condition is assumed, then lemma 4 implies that we can rule out the optimality of reserve prices that yield a (statistically significant) reduction in expected revenues when $F(\cdot) = F_L(\cdot)$ is assumed. This follows from the fact that a rightward shift in $F(\cdot)$ raises expected revenues at any reserve price. In our simulations, reductions in expected revenues appear for a few reserve prices, but only when $F(\cdot) = F_U(\cdot)$ is assumed.

TABLE 4
SIMULATED OUTCOMES WITH ALTERNATIVE RESERVE PRICES

	RESERVE PRICE					
	p_L		$(p_L + p_U)/2$		p_U	
	Distribution of Valuations					
	F_L	F_U	F_L	F_U	F_L	F_U
Reserve price when $v_0 = \$20$	62.40		86.02		109.65	
Change in profit	6.96	−2.78	6.67	−2.74	1.74	−18.57
Pr(no bids)	.00	.02	.07	.12	.19	.41
Reserve price when $v_0 = \$40$	74.93		92.29		109.65	
Change in profit	7.64	−.61	7.61	−1.14	6.30	−10.04
Pr(no bids)	.03	.05	.11	.18	.19	.41
Reserve price when $v_0 = \$60$	85.67		103.39		121.11	
Change in profit	9.23	1.92	12.04	3.14	7.21	−6.05
Pr(no bids)	.07	.12	.15	.28	.35	.58
Reserve price when $v_0 = \$80$	98.20		112.34		126.48	
Change in profit	13.65	7.63	15.03	6.82	10.44	.96
Pr(no bids)	.13	.24	.28	.46	.46	.72
Reserve price when $v_0 = \$100$	111.09		122.54		134.00	
Change in profit	20.09	15.94	21.65	16.87	17.00	14.30
Pr(no bids)	.28	.45	.45	.60	.67	.80
Reserve price when $v_0 = \$120$	144.74		156.01		167.29	
Change in profit	32.06	31.31	33.72	31.64	31.56	28.87
Pr(no bids)	.84	.86	.84	.89	.88	.97

NOTE.—Profit and reserve price figures are given in 1983 dollars per MBF. See text for additional details.

posing higher reserve prices would be obtained by selling only tracts receiving unusually high bids. While revenue maximization is not the objective of the Forest Service timber sales program, these estimates suggest the magnitudes of revenues and costs that must be weighed against other objectives in determining optimal policy.

To evaluate the effects of auction observables on bidder valuations, we estimate the simple semiparametric model

$$v_{it} = \mathbf{X}_{it}\boldsymbol{\beta} + \epsilon_{it}$$

assuming $\mathrm{med}[\epsilon_{it} \mid X_{it}] = 0$. Table 5 presents estimated bounds on the parameter vector $\boldsymbol{\beta}$. Following Manski and Tamer (2002), we construct confidence intervals using the bootstrap. Since zero lies outside the 95 percent confidence interval for each coefficient, we can reject the hypothesis that any of these conditioning variables has no effect on valuations. The implied signs are as expected: larger inventories, higher harvesting costs, or higher manufacturing costs reduce valuations. Greater species concentration and higher selling value of end products

TABLE 5
FOREST SERVICE TIMBER AUCTIONS: SEMIPARAMETRIC MODEL OF BIDDER
VALUATIONS (Modified Minimum Distance Estimates)

	Interval Estimate	95% Bootstrapped Confidence Interval
Constant	[8.8, 15.12]	[2.33, 18.15]
Species concentration	[13.19, 13.64]	[11.14, 16.54]
Manufacturing cost	[−.85, −.81]	[−1.02, −.79]
Selling value	[.61, .71]	[.57, .96]
Harvesting cost	[−.54, −.51]	[−.59, −.48]
Six-month inventory	[−.026, −.025]	[−.030, −.021]
Number of bidders	[.81, 1.23]	[.66, 1.24]

lead to higher valuations. Moreover, the bounds are tight and the magnitudes are reasonable. For example, to a first approximation, the value of a contract is the selling value less harvesting and manufacturing costs. If this approximation were exact (up to bidders' idiosyncratic shocks), the corresponding coefficients would equal +1, −1, and −1, respectively, which are close to the estimated intervals. Finally, if the variation in the number of bidders were exogenous, a negative coefficient on this covariate would be implied by a common values model and a coefficient of zero by a private values model (see, e.g., Haile, Hong, and Shum 2000; Athey and Haile 2002). The positive but very small coefficient implied by our estimates is consistent with our assumption of private values and a small amount of unobserved heterogeneity correlated with the number of bidders.

VIII. Conclusion

Some theoretical models that serve well in capturing essential elements of behavior in a market may nonetheless fall short of providing a mapping between primitives and observables that can usefully be treated as exact by empirical researchers. This need not preclude the use of theory to provide a structure for interpreting data, nor preclude inference on the structural parameters and distributions essential for many policy questions. In some cases, useful inferences can be made by relying on weak assumptions—for example, axioms or necessary conditions for equilibrium in a class of models—that, while insufficient to fully characterize the mapping between primitives and observables, provide a robust structural framework for inference.

  We have considered one example of this approach, arguing that while standard theoretical models of English auctions can imply unpalatable identifying assumptions for many applications, useful inferences on the primitives characterizing the demand and information structure can be made on the basis of observed bids and weak restrictions on their in-

terpretation. Here this approach enables construction of bounds on the distributions of bidder valuations, on optimal reserve prices, on the range of revenues that might be attained with different reserve prices or other selling mechanisms, and on the effects of auction characteristics on valuations. The case for focusing on bounds is particularly compelling in this application: our bounds will be tight whenever the standard model is a good approximation of the true model, and they collapse to the true distribution if the button auction model *is* the true model. An open question is whether the bounds $F_L(\cdot)$ and $F_U(\cdot)$ exhaust all information in the data given our assumptions, that is, whether they are *sharp*. We provide further discussion of this issue in Appendix D.

We have focused on the symmetric independent private values model of bidder demand, which is both the simplest and most common in prior structural empirical work. However, this places restrictions on the demand structure that will not be appropriate in all applications. While the statistical techniques developed here exploit this structure, our fundamental approach—the interpretation of bids as bounds—is natural for other demand structures as well. Dropping the assumption of i.i.d. valuations requires development of different statistical techniques for mapping the information in observed bids to restrictions on the joint distribution of bidder valuations. However, in ongoing work, we are exploring identification and estimation of bounds on the joint distribution of bidder valuations in models allowing bidder asymmetry, affiliation of bidders' private values, and unobserved heterogeneity.

The case of common values is potentially even more challenging. Identification in common values auctions typically fails in complete models (Laffont and Vuong 1996; Athey and Haile 2002), and some of the same difficulties arise in identifying bounds. In the case of an English auction with common values, one key difficulty is the fact that each bidder's willingness to pay varies as the auction proceeds because of information she infers from others' behavior. Whether progress can be made by exploiting bounds—both on what the data reveal regarding bidders' willingness to pay and on how bidders' willingness to pay evolves over the course of the auction—is a question we leave for future work.

Appendix A

Proofs Omitted from the Text

Proof of Theorem 3

Consider the vector of moments

$$
E\begin{bmatrix} \mathbf{1}[B_{1:2} \le v] \\ \mathbf{1}[B_{2:2} \le v] \\ \vdots \\ \mathbf{1}[B_{\bar{M}:\bar{M}} \le v] \end{bmatrix} = \begin{bmatrix} G_{1:2}(v) \\ G_{2:2}(v) \\ \vdots \\ G_{\bar{M}:\bar{M}}(v) \end{bmatrix} \equiv \mathbf{G}(\mathbf{v}),
$$

and let $\hat{\mathbf{G}}_T(\mathbf{v})$ be the empirical analog of $\mathbf{G}(\mathbf{v})$,

$$
\hat{\mathbf{G}}_T(\mathbf{v}) = \begin{bmatrix} \dfrac{1}{T_2}\sum_{t=1}^{T}\mathbf{1}[n_t = 2, b_{1:n_t} \leq v] \\[2ex] \dfrac{1}{T_2}\sum_{t=1}^{T}\mathbf{1}[n_t = 2, b_{2:n_t} \leq v] \\[2ex] \vdots \\[2ex] \dfrac{1}{T_{\bar{M}}}\sum_{t=1}^{T}\mathbf{1}[n_t = \bar{M}, b_{1:n_t} \leq v] \\[2ex] \vdots \\[2ex] \dfrac{1}{T_{\bar{M}}}\sum_{t=1}^{T}\mathbf{1}[n_t = \bar{M}, b_{\bar{M}:n_t} \leq v] \end{bmatrix}.
$$

By the Glivenko-Cantelli theorem (e.g., Van der Vaart 1998, theorem 19.1), $\hat{\mathbf{G}}_T(\mathbf{v}) - \mathbf{G}(\mathbf{v}) = o_p(1)$ almost surely uniformly in $\mathbf{v}$. So by the continuous mapping theorem,

$$\phi(\hat{\mathbf{G}}_T(\mathbf{v})) - \phi(\mathbf{G}(\mathbf{v})) = o_p(1), \tag{A1}$$

where

$$\phi(\mathbf{G}(\mathbf{v})) = \begin{pmatrix} \phi(G_{1:2}(v);\ 1,\ 2) \\ \vdots \\ \phi(G_{\bar{M}:\bar{M}}(v);\ \bar{M},\ \bar{M}) \end{pmatrix},$$

and each component of $\phi(\cdot)$ is a continuously differentiable function from $[0, 1]$ to $[0, 1]$. The convergence in (A1) is also uniform in $\mathbf{v}$. Furthermore, since the min function is continuous, the continuous mapping theorem implies that, for each v,

$$\min_{n\in\{2,\dots,\bar{M}\}, i\in\{1,\dots,n\}} \phi(\hat{G}_{i:n}(v);\ i, n) - \min_{n\in\{2,\dots,\bar{M}\}, i\in\{1,\dots,n\}} \phi(G_{i:n}(v);\ i, n) = o_p(1).$$

Finally,

$$\sup_{v\in[\underline{v},\bar{v}]} |\hat{F}_U(v) - F_U(v)| \leq \sum_{i,n} \sup_{v\in[\underline{v},\bar{v}]} |\phi(\hat{G}_{i:n}(v);\ i, n) - \phi(G_{i:n}(v);\ i, n)| = o_p(1), \tag{A2}$$

where the inequality follows from the relation

$$|\min(\hat{a},\ \hat{b}) - \min(a,\ b)| \leq |\hat{a} - a| + |\hat{b} - b|.$$

Similar arguments apply for the lower bound. Q.E.D.

Proof of Lemma 4

The revenue equivalence theorem of Myerson (1981) shows that in the symmetric independent private values setting, any two feasible auction mechanisms are revenue equivalent if (*a*) they have the same allocation rule and (*b*) they

give the same surplus to a bidder with the lowest possible valuation. Fix a reserve price r. With a minimum bid increment of zero, assumptions 1 and 2 imply that (i) an English auction allocates the good efficiently except when no bidder has a valuation above r, in which case the seller retains the good; and (ii) a bidder with valuation $\underline{v}$ has expected payoff zero. Both conditions i and ii also hold in the dominant strategy equilibrium of a second-price sealed-bid auction with the same reserve price, implying that conditions a and b hold. Q.E.D.

Proof of Lemma 5

Part i: $p^*(F_0)$ solves

$$\max_{p \in [\underline{v}, \bar{v}]} [1 - F_0(p)](p - v_0), \tag{A3}$$

and $p^*(F)$ solves

$$\max_{p \in [r, \bar{v}]} [1 - F(p)](p - v_0). \tag{A4}$$

Plugging in the definition (1) reveals that these two objective functions are identical up to a positive multiplicative constant. With $r \leq p^*(F_0)$, this implies that (A3) and (A4) have the same (unique) solution. Part ii: The same argument implies that the slopes of these objective functions have the same sign for all $p \geq r$. Under assumption 3, this implies that $p^*(F) = r$ whenever $r > p^*(F_0)$. Q.E.D.

Proof of Theorem 4

Since $F(\cdot)$ is continuous and $\pi(p) \geq \pi_1(p)$ for all p, we must have

$$\pi(p^*) \geq \pi_1^*. \tag{A5}$$

Suppose $p^* < p_L$. There are two cases to consider. First, if $\pi_2(p^*) < \pi_1^*$, then

$$\pi(p^*) \leq \pi_2(p^*) < \pi_1^*,$$

contradicting (A5). In the second case, $\pi_2(p^*) \geq \pi_1^*$. Then by the definition of p_L, there must be some $\tilde{p} \in (p^*, p_1^*)$ such that $\pi_2(\tilde{p}) \leq \pi_1^*$. Since then $\pi(\tilde{p}) \leq \pi_2(\tilde{p}) \leq \pi_1^*$ whereas $\pi(p_1^*) \geq \pi_1^*$, this requires $\pi'(p) \geq 0$ for some $p > p^*$, contradicting assumption 3. An analogous argument rules out the optimality of any $p > p_U$.

To see that these bounds are sharp (for the nontrivial case in which $p_L < p_U$), we first show that, for arbitrarily small $\epsilon > 0$, there always exists a CDF $F^*(\cdot)$ within the bounds $F_L(\cdot)$ and $F_U(\cdot)$ that satisfies assumption 3 and such that $\pi^*(p) \equiv (p - v_0)[1 - F^*(p)]$ has a unique maximum at $p_L + \epsilon$. The definitions of p_L and p_U imply that, for small $\epsilon > 0$, $\pi_2(p_L + \epsilon) > \pi_1^*$. For such an ϵ, let

$$\tilde{\pi}_2(p) = \begin{cases} \pi_2(p) & \pi_2(p) \leq \pi_2(p_L + \epsilon) \\ \pi_2(p_L + \epsilon) & \text{otherwise.} \end{cases}$$

Now let $\bar{\pi}_2(\cdot)$ be the lower quasi-concave envelope of $\tilde{\pi}_2(\cdot)$ satisfying $\bar{\pi}_2(p_L + \epsilon) = \tilde{\pi}_2(p_L + \epsilon)$.[29] Define

$$\pi^*(p) = \begin{cases} \bar{\pi}_2(p) & p \le p_L + \epsilon \\ \alpha(p)\bar{\pi}_2(p_L + \epsilon) + [1 - \alpha(p)]\bar{\pi}_2(p_U) & p \in (p_L + \epsilon, \ p_U) \\ \bar{\pi}_2(p) & p \ge p_U, \end{cases} \quad (A6)$$

where $\alpha(\cdot)$ is a decreasing differentiable function with $\alpha(p_L + \epsilon) = 1$ and $\alpha(p_U) = 0$. For some such $\alpha(\cdot)$, $\pi^*(\cdot)$ lies between the profit functions $\pi_1(\cdot)$ and $\pi_2(\cdot)$, is quasi-concave, and has a unique maximum at $p_L + \epsilon$. Now let

$$F^*(p) = 1 - \frac{\pi^*(p)}{p - v_0}.$$

By construction, $F^*(p) \in [F_L(p), F_U(p)]$ for all p. For $p \notin (p_L + \epsilon, \ p_U)$, $F^*(p)$ is increasing by construction, with $F^*(\underline{v}) = 0$ and $F^*(\bar{v}) = 1$. To see that $F^*(p)$ is increasing on the interval $(p_L + \epsilon, \ p_U)$, note that here

$$\frac{dF^*(p)}{dp} = \frac{\alpha'(p)}{p - v_0}[\pi^*(p_U) - \pi^*(p_L + \epsilon)] + \frac{\alpha(p)}{(p - v_0)^2}\pi^*(p_L + \epsilon)$$

$$+ \frac{1 - \alpha(p)}{(p - v_0)^2}\pi^*(p_U). \quad (A7)$$

Since $\pi^*(p_U) \le \pi_1^* < \pi^*(p_L + \epsilon)$, this derivative is strictly positive. Hence, $F^*(p)$ is a valid CDF lying within the bounds. Finally, a slight perturbation of $F^*(\cdot)$, eliminating nondifferentiabilities and any regions with $d\pi^*(p)/dp = 0$ away from $p_L + \epsilon$, yields a differentiable strictly pseudo-concave profit function with a (unique) maximum at $p_L + \epsilon$. An analogous argument shows that one can construct another CDF within the bounds that yields $p_U - \epsilon$ as the unique optimum for arbitrarily small $\epsilon > 0$.[30] Q.E.D.

Proof of Theorem 5

Consider the following population objective function

$$Q(p) = \min_{\pi \in \pi_2^\epsilon(p)} (\pi - \pi_1^*)^2,$$

---

[29] This function is obtained from $\tilde{\pi}_2(\cdot)$ as follows. Suppose that $\tilde{\pi}_2(\cdot)$ has a local minimum at some price $\hat{p}$ below $p_L + \epsilon$. Let $a = \inf\{\tilde{p} : \tilde{\pi}_2(p) > \tilde{\pi}_2(\hat{p}) \ \forall p \in (\tilde{p}, \hat{p})\}$. For all $p \in (a, \hat{p})$, replace $\tilde{\pi}_2(p)$ with $\tilde{\pi}_2(\hat{p})$. For any local minima to the right of $p_L + \epsilon$, make a similar downward adjustment to $\tilde{\pi}_2(p)$. Repeating these procedures yields a quasi-concave function $\bar{\pi}_2(\cdot)$ with $\bar{\pi}_2(p) \le \tilde{\pi}_2(p)$ for all p and a global (but typically nonunique) maximum at $p_L + \epsilon$.

[30] A minor variation is required in that case to ensure that the constructed distribution is increasing. The reason is that $\pi_2(p_U - \epsilon)$ could be significantly larger than $\pi_1(p_1^*)$ (e.g., if $\pi_2(\cdot)$ jumps downward at p_U). Hence, instead of creating a revenue function defined by convex combinations of $\pi_2(p_L)$ and $\pi_2(p_U - \epsilon)$, as one would do in the analogue of (A6), select $\hat{p} \in [p_L, p_U - \epsilon)$ such that $\pi_2(p_U - \epsilon) - \pi_2(\hat{p})$ is a small positive number and substitute $\hat{p}$ for p_L in the construction.

where the correspondence $\pi_2^c(p)$ is defined by

$$\pi \in \pi_2^c(p) \Leftrightarrow \pi \in \left[\lim_{p'\uparrow p} \pi_2(p'), \lim_{p'\downarrow p} \pi_2(p')\right].$$

Note that at the continuity points of $\pi_2(p)$, $\pi_2^c(p) = \pi_2(p)$. By the definitions of p_L and p_U, we have $Q(p_L) = 0 = Q(p_U)$ and $Q(p) \geq 0$ otherwise. Using (16), define the sample analogue of $Q(p)$ as

$$Q_T(p) = \min_{\pi \in \hat{\pi}_2^c(p)} (\pi - \hat{\pi}_1^*)^2.$$

Given $\epsilon_T \geq 0$, consider the sets of prices $\{p : Q_T(p) \leq \epsilon_T\}$ that approximately minimize $Q_T(p)$. For a sequence $\{\epsilon_T\}$ converging to zero at an appropriate rate (slightly slower than $T^{-1/2}$), we show that these sets converge (in the Hausdorff metric) to the set that minimizes $Q(\cdot)$. Noting that we can write

$$p_L = \sup\{p < p_1^* : Q(p) = 0\},$$

$$p_U = \inf\{p > p_1^* : Q(p) = 0\}$$

will then give the result. To prove this set consistency, it suffices to show that $Q_T(\cdot)$ converges uniformly to $Q(\cdot)$ (see Manski and Tamer 2002). For uniform convergence of $Q_T(\cdot)$ to $Q(\cdot)$, it is sufficient to show that

$$\sup_p \min_{\pi \in \pi_2^c(p), \pi' \in \hat{\pi}_2^c(p)} |(\pi - \pi_1^*) - (\pi' - \hat{\pi}_1^*)| = 0.$$

The left-hand side is equal to

$$\sup_p |[\pi_2^c(p) - \hat{\pi}_2^c(p)] + (\hat{\pi}_1^* - \pi_1^*)| \leq \sup_p |\pi_2^c(p) - \hat{\pi}_2^c(p)| + |\hat{\pi}_1^* - \pi_1^*|$$

$$= o_p(1) + o_p(1),$$

where

$$|\pi_2^c(p) - \hat{\pi}_2^c(p)| = \min_{\pi \in \pi_2^c(p), \pi' \in \hat{\pi}_2^c(p)} |\pi - \pi'|,$$

and the equality above holds because $\sup_p |\pi_2(p) - \hat{\pi}_2(p)| = o_p(1)$ by theorem 3 and $\hat{\pi}_1(\hat{p}_1^*) - \pi_1(p_1^*)$ converges to zero in probability by uniform convergence and tightness of the process $\hat{\pi}_1(\cdot) - \pi_1(\cdot)$ (see Haile and Tamer 2002). Q.E.D.

Proof of Theorem 6

First note that $Q(\mathbf{b}) \geq 0$ for all $\mathbf{b} \in \mathscr{B}$, whereas $Q(\mathbf{b}) = 0$ if and only if $\mathbf{b} \in \Sigma$. To see this, note that if $\mathbf{b} \in \Sigma$, then $\delta_1(\mathbf{x}, \mathbf{b}) = \delta_2(\mathbf{x}, \mathbf{b}) = 0$. For $\mathbf{b} \notin \Sigma$,

$$[E[\underline{S}_{\mathbf{x}}] - l(\mathbf{x}, \mathbf{b})]^2 \delta_1(\mathbf{x}, \mathbf{b}) + [l(\mathbf{x}, \mathbf{b}) - E[\bar{S}_{\mathbf{x}}]]^2 \delta_2(\mathbf{x}, \mathbf{b}) > 0$$

for all $\mathbf{x} \in V(\mathbf{b})$, and $\Pr[V(\mathbf{b})] > 0$ for all $\mathbf{b} \notin \Sigma$. Now, given estimates $\hat{F}_L(\cdot|\mathbf{X})$ and $\hat{F}_U(\cdot|\mathbf{X})$, one can estimate the conditional means by simulation. Replace $E[\underline{S}_{\mathbf{x}}]$ and $E[\bar{S}_{\mathbf{x}}]$ by estimates $E_T[\underline{S}_{\mathbf{x}}]$ and $E_T[\bar{S}_{\mathbf{x}}]$ based on simulated draws from $\hat{F}_L(\cdot|\mathbf{X})$ and $\hat{F}_U(\cdot|\mathbf{X})$. These simulation estimators converge almost surely to their population counterparts since $\hat{F}_L(\cdot|\mathbf{X})$ and $\hat{F}_U(\cdot|\mathbf{X})$ converge almost surely to $F_L(\cdot|\mathbf{X})$ and $F_U(\cdot|\mathbf{X})$ (see, e.g., Stern [1997] and references therein). The argument

in the proof of theorem 5 in Manski and Tamer (2002) then completes this proof. Q.E.D.

Appendix B

Examples

In this Appendix we show that assumptions 1 and 2 are satisfied by several models of English auctions, but these assumptions imply neither a unique distribution of bids given a distribution of valuations nor a unique distribution of valuations given a distribution of bids. As noted in the text, one model yielding outcomes consistent with assumptions 1 and 2 is the standard model of Milgrom and Weber (1982). Below we provide two additional examples before turning to the question of whether assumptions 1 and 2 and the distribution of bids identify the distribution of valuations.

EXAMPLE 1. Consider Harstad and Rothkopf's (2000) "alternating recognition" auction, in which the seller begins with two randomly chosen bidders and holds a two-bidder button auction. When one bidder drops out, a replacement is randomly selected from the remaining bidders willing to participate. A bidder who refuses to participate or who drops out while participating in the button auction is never asked again to participate. This continues until there is no willing replacement, with the remaining bidder then declared the winner at the current price. Harstad and Rothkopf show that in the unique symmetric equilibrium of this game, assumptions 1 and 2 hold, the allocation is efficient, and the selling price is equal to the second-highest valuation. However, while each losing bidder exits the two-bidder button auction at his valuation if he participates, some bidders never participate (and therefore never bid) because the price rises above their valuations while others are bidding. Hence, when $n >$ 2, the distribution of bids will differ from that implied by the button auction model, given the same distribution of bidder valuations.

EXAMPLE 2. All bidders are initially identified as active, and the reserve price (minus at least one bid increment) is designated the initial standing bid. As long as at least two bidders are active, the seller picks one of the active bidders at random. This bidder may either raise the current standing bid by one bid increment or decline to bid. If the bidder declines, he becomes inactive and another bidder is selected. If the bidder accepts, the standing bid is raised by this increment and the process iterates. A bidder who accepts in one iteration is exempt from bidding until another player bids (so bidders are not asked to raise their own bids). A bidder is declared the winner when his is the standing bid and no other bidder is active. In the dominant strategy equilibrium of this game, each bidder agrees to bid when asked to do so if and only if his valuation exceeds the standing bid by at least Δ. This ensures that assumptions 1 and 2 are satisfied. However, a bidder need not be picked to bid when the standing bid is close to his valuation, implying that his highest bid need not be close to his valuation; indeed, he need not be picked to bid at all before the price rises above his valuation, in which case he would never make a bid.[31] Hence, the distribution of equilibrium bids need not match that for the button auction, given a distribution of valuations.

---

[31] Note that the allocation need not be efficient, since, e.g., when $v_{n:n} - v_{n-1:n} < \Delta$, the price could rise to $p \in (v_{n:n} - \Delta, v_{n-1:n})$ without any bid from the bidder with valuation $v_{n:n}$.

Showing that assumptions 1 and 2 are insufficient to identify the distribution of bidder valuations from the distribution of observed bids is straightforward. Consider a two-bidder auction in which valuations are independently distributed according to $F(\cdot)$ and bidding follows the simple rules $b_{1:2} = \underline{v}$ and $b_{2:2} = v_{1:2}$. These bidding rules are consistent with assumptions 1 and 2. The order statistics of the bids are then independent, with marginal distributions

$$G_{1:2}(b) = 1, \quad G_{2:2}(b) = F_{1:2}(b) \quad \forall b \in [\underline{v}, \bar{v}].$$

Now suppose that valuations are drawn independently from another distribution $\tilde{F}(\cdot)$ and bidding follows the rules $b_{1:2} = \underline{v}$ and $b_{2:2} = v_{2:2}$, again satisfying our assumptions. The order statistics of the bids are again independent, with marginal distributions

$$\tilde{G}_{1:2}(b) = 1, \quad \tilde{G}_{2:2}(b) = \tilde{F}_{2:2}(b) \quad \forall b \in [\underline{v}, \bar{v}].$$

If $\tilde{F}_{2:2}(b) = F_{1:2}(b)$, that is, if $\tilde{F}(b) = F_{1:2}(b)^{1/2}$, then these two models give the same distribution of the observable bids. Hence assumptions 1 and 2 and the distribution of bids do not uniquely determine the distribution of bidder valuations.

Appendix C

Asymptotic Equivalence of Alternative Estimators

Recalling the definition (12), let

$$\hat{F}_U^{\eta}(v) \equiv$$

$$\mu(\phi(\hat{G}_{1:2}(v); 1,2), \phi(\hat{G}_{2:2}(v); 2,2), \phi(\hat{G}_{1:3}(v); 1,3), \ldots, \phi(\hat{G}_{\bar{M}:\bar{M}}(v); \bar{M}, \bar{M}); \rho_T^U), \quad (C1)$$

and

$$\hat{F}_L^{\eta}(v) \equiv \mu(\phi(\hat{G}_{2:2}^{\Delta}(v); 1, 2), \phi(\hat{G}_{3:3}^{\Delta}(v); 2, 3), \ldots, \phi(\hat{G}_{\bar{M}:\bar{M}}^{\Delta}(v); \bar{M} - 1, \bar{M}); \rho_T^L). \quad (C2)$$

THEOREM 7. Assume that, for all $n = 2, \ldots, \bar{M}$, $T_n \equiv \sum_{t=1}^{T} \mathbf{1}[N_t = n] \to \infty$ and $T_n/T \to \lambda_n$ as $T \to \infty$, with $0 < \lambda_n < 1$. Let $\rho_T^U \to -\infty$ and $\rho_T^L \to \infty$, each at a rate exceeding $\log(\sqrt{T})$. Then as $T \to \infty$, (a) $\hat{F}_U^{\eta}(v) \overset{\text{a.s.}}{\to} F_U(v)$ uniformly in v, and (b) $\hat{F}_L^{\eta}(v) \overset{\text{a.s.}}{\to} F_L(v)$ uniformly in v. Furthermore, the asymptotic distributions of $\hat{F}_U^{\eta}(v)$ and $\hat{F}_L^{\eta}(v)$ are identical to those of $\hat{F}_U(v)$ and $\hat{F}_L(v)$.

Proof. Consider $\hat{F}_U^{\eta}(v)$ (the argument for $\hat{F}_L^{\eta}(v)$ is analogous). Let $\hat{y}_1, \ldots, \hat{y}_J$ denote the arguments of $\mu(\cdot; \rho_T^U)$ in (C1). Observe that

$$|\mu(\hat{y}_1, \ldots, \hat{y}_J; \rho_T^U) - \min(y_1, \ldots, y_J)| \leq |\mu(\hat{y}_1, \ldots, \hat{y}_J; \rho_T^U) - \min(\hat{y}_1, \ldots, \hat{y}_J)|$$

$$+ |\min(\hat{y}_1, \ldots, \hat{y}_J) - \min(y_1, \ldots, y_J)|. \quad (C3)$$

Rate $\sqrt{T}$ convergence to zero of the second term on the right-hand side of (C3) was shown in Haile and Tamer (2002) (extending the proof of theorem 3), where the asymptotic distribution of $\sqrt{T}[\min(\hat{y}_1, \ldots, \hat{y}_J) - \min(y_1, \ldots, y_J)]$ was also

derived. If the first term converges to zero at a rate faster that $\sqrt{T}$, then we can ignore this term asymptotically, giving the result. So observe that

$$
\begin{aligned}
\mu(\hat{y}_1, \ldots, \hat{y}_J; \rho_T^U) - \min(\hat{y}_1, \ldots, \hat{y}_J) &= \frac{\sum_{j=1}^J \hat{y}_j \exp(\rho_T^U \hat{y}_j)}{\sum_{j=1}^J \exp(\rho_T^U \hat{y}_j)} - \min(\hat{y}_1, \ldots, \hat{y}_J) \\
&= \frac{\hat{y}_1 \exp(\rho_T^U \hat{y}_1)}{\sum_{j=1}^J \exp(\rho_T^U \hat{y}_j)} + \cdots + \frac{\hat{y}_J \exp(\rho_T^U \hat{y}_J)}{\sum_{j=1}^J \exp(\rho_T^U \hat{y}_j)} \\
&\quad - \min(\hat{y}_1, \ldots, \hat{y}_J) \\
&= \frac{\hat{y}_1}{1 + \cdots + \exp[\rho_T^U(\hat{y}_J - \hat{y}_1)]} + \cdots \\
&\quad + \frac{\hat{y}_J}{\exp[\rho_T^U(\hat{y}_1 - \hat{y}_J)] + \cdots + 1} \\
&\quad - \min(\hat{y}_1, \ldots, \hat{y}_J). \quad\quad (\text{C4})
\end{aligned}
$$

Suppose without loss that $\min(y_1, \ldots, y_J) = y_1$. Then for all $j \neq 1$, $\hat{y}_j - \hat{y}_1$ becomes positive (but finite) at an exponentially fast rate.[32] So if $\rho_T^U \to -\infty$ faster than rate $\log(\sqrt{T})$, the first term in (C4) converges to $\hat{y}_1$ at a rate faster than $\sqrt{T}$. By the same argument, the remaining fractions go to zero at this rate. Q.E.D.

Appendix D

Sharpness

Bounds are sharp if they exhaust the information available from the data and assumptions. Sharp bounds reflect all the restrictions on the latent quantities of interest that one can obtain without additional assumptions. Hence, sharpness is a desirable theoretical property of bounds. Here, bounds will be sharp if they exploit all the information in the joint distribution of the order statistics of the bids, our assumption that valuations are i.i.d., and assumptions 1 and 2.

To simplify the exposition, we assume a fixed number of bidders and the absence of a binding reserve price here. Given n, define a *bidding rule*

$$
\Gamma(B_1, \ldots, B_n | V_1, \ldots, V_n) : \mathbb{R}^n \times [\underline{v}, \bar{v}]^n \to [0, 1]
$$

to be a conditional joint distribution function for the bids made, given a realization of bidders' valuations. Because valuations are i.i.d., the joint distribution of $\{V_{i:n}\}_{i=1}^n$ completely determines (and is completely determined by) the joint distribution of $\{V_i\}_{i=1}^n$. Hence, we may focus on bidding rules

$$
\Gamma^o(B_{1:n}, \ldots, B_{n:n} | V_{1:n}, \ldots, V_{n:n})
$$

that map the realizations of the *order statistics* of the valuations to a distribution

---

[32] The reason for this is the following. Haile and Tamer (2002) show that $\sqrt{T}[\phi(\hat{G}_{i:n}(v); i, n) - \phi(G_{i:n}(v); i, n)]$ has a normal asymptotic distribution for all v, i, n. Let the random variable a_T be such that $\sqrt{T}(a_T - a)$ is asymptotically normal with mean zero and variance one, with a strictly positive. Then $\Pr(a_T > 0) \to 1$ exponentially fast since

$$
\Pr(a_T > 0) = 1 - \Pr(\sqrt{T}(a_T - a) \leq -\sqrt{T}a) = 1 - \Phi(-\sqrt{T}a),
$$

where $\Phi(\cdot)$ is the standard normal CDF.

for the *order statistics* of the bids. Such a bidding rule satisfies assumptions 1 and 2 if

$$\mathrm{supp}\,\Gamma^o(\cdot|v_{1:n}, \ldots, v_{n:n}) \subseteq [v_{n-1:n} - \Delta, v_{n:n}] \times \prod_{j=1}^{n-1} [\underline{v}, \, v_{j:n}]. \tag{D1}$$

In general, there will be more than one Γ consistent with a given Γ^o; however, if Γ^o satisfies (D1), then at least one of these must satisfy assumptions 1 and 2—in particular, the one in which the bid $B_{j:n}$ is made by the bidder with valuation $V_{j:n}$.

Given a distribution of valuations $\tilde{F}$, let $\tilde{F}_{1,\ldots,n:n}(\cdot; \tilde{F})$ denote the implied joint distribution of the order statistics of the valuations. Let

$$\mathcal{G}_n(\cdot; \tilde{F}, \Gamma^o) = \int \cdots \int \Gamma^o(\cdot|v_{1:n}, \ldots, v_{n:n})\tilde{F}_{1,\ldots,n:n}(dv_{1:n}, \ldots, dv_{n:n}; \tilde{F})$$

denote the joint distribution of the order statistics of the bids implied by the bidding rule Γ^o and the distribution $\tilde{F}$. Finally, let $G_{1,\ldots,n:n}(\cdot)$ denote the observed distribution of the order statistics of the bids.

DEFINITION 1. Given assumptions 1 and 2, the symmetric independent private values assumption, and the joint distributions of the order statistics of the bids $G_{1,\ldots,n:n}$, the bounds F_L and F_U are sharp if, for every distribution $\tilde{F}$ satisfying $\tilde{F}(v) \in (F_L(v), F_U(v))$ for all v, there exists a bidding rule Γ^o satisfying assumptions 1 and 2 such that $\mathcal{G}_n(\cdot; \tilde{F}, \Gamma^o) = G_{1,\ldots,n:n}(\cdot)$.

For simplicity, consider the case $n = 3$ with $\Delta = 0$. Let $\tilde{F}$ be any CDF lying within the bounds F_L and F_U. Let

$$H(b_{1:3}, \, b_{2:3}, \, b_{3:3}, \, v_{1:3}, \, v_{2:3}, \, v_{3:3})$$

denote the joint distribution implied by Bayes' rule and the distributions

$$\Gamma(b_{1:3}, \, b_{2:3}, \, b_{3:3}|v_{1:3}, \, v_{2:3}, \, v_{3:3})$$

and

$$\tilde{F}_{1,2,3:3}(v_{1:3}, \, v_{2:3}, \, v_{3:3}; \tilde{F}).$$

Showing that the bounds are sharp means showing that one can always find a joint distribution $H(\cdot)$ that (A) has marginals $\tilde{F}_{1,2,3:3}(\cdot)$ and $G_{1,2,3:3}(\cdot)$ and (B) obeys the support restrictions implied by (D1).

If we ignore point B for a moment, this is a *copula* problem: we want to couple the two marginal distributions $\tilde{F}_{1,2,3:3}(\cdot)$ and $G_{1,2,3:3}(\cdot)$ to obtain a valid joint distribution that gives back each of these as the appropriate marginal distribution. Unfortunately, while much is known about the copula problem for the case of coupling two univariate marginals, the case of multivariate marginals is more subtle, with little of what is known in the univariate case generally extending to this multivariate case (see, e.g., Nelson 1999, sec. 3.4). Without the constraints implied by condition (2), one can always construct one valid joint CDF by letting

$$H(b_{1:3}, b_{2:3}, b_{3:3}, v_{1:3}, v_{2:3}, v_{3:3}) =$$

$$\tilde{F}_{1,2,3:3}(v_{1:3}, v_{2:3}, v_{3:3})G_{1,2,3:3}(b_{1:3}, b_{2:3}, b_{3:3}).$$

The constraints added by point B, however, make the problem more difficult. For the case of a one-bidder auction, assumption 2 has no bite and imposing

the restrictions of assumption 1 becomes fairly simple (see Smith [1983] for the solution of an equivalent problem). However, for the case of $n \geq 2$, we have been able to prove only a weaker result. In particular, instead of showing that any *joint* distribution $G_{1,2,3:3}$ can be rationalized by every distribution $\tilde{F}$ within the bounds (using a bidding rule satisfying assumptions 1 and 2), we show that every set of *marginal* distributions $G_{1:3}$, $G_{2:3}$, $G_{3:3}$ can be rationalized. We again consider $n = 3$ and $\Delta = 0$ for simplicity.

PROPOSITION 1. Given any $G_{1:3}$, $G_{2:3}$, and $G_{3:3}$, let F_L and F_U be the implied upper and lower bounds on F. Then for any distribution $\tilde{F}$ such that $\tilde{F}(v) \in [F_L(v), F_U(v)]$ for all v, there exists a bidding rule $\Gamma^o(b_{1:3}, b_{2:3}, b_{3:3}|v_{1:3}, v_{2:3}, v_{3:3})$ satisfying assumptions 1 and 2 such that, with valuations drawn from $\tilde{F}$ and bidding determined by Γ^o, the marginal distributions of the order statistics of the bids are $G_{1:3}$, $G_{2:3}$, and $G_{3:3}$.

Sketch of proof. Let $\tilde{F}_{j:3}$ denote the marginal distribution of the jth order statistic implied by $\tilde{F}(\cdot)$. Similarly, let $\tilde{F}_{i,j:3}(\cdot)$ denote the joint distribution of the ith and jth order statistics. We shall construct a bidding rule that consists of three bid functions, γ_1, γ_2, and γ_3. We define the first (that determining $B_{1:3}$) implicitly by the equation

$$G_{1:3}(\gamma_1(v)) = \tilde{F}_{1:3}(v) \quad \forall v.$$

Letting $G_{1:3}^{-1}(t) = \sup\{s : G_{1:3}(s) \leq t\}$ gives

$$\gamma_1(v) = G_{1:3}^{-1}(\tilde{F}_{1:3}(v)). \tag{D2}$$

By the definitions of the bounds and the fact that $\tilde{F}$ lies within the bounds, we know that

$$\tilde{F}_{1:3}(v) \leq G_{1:3}(v). \tag{D3}$$

Hence (D2) implies that $b_{1:3} = \gamma_1(v_{1:3}) \leq v_{1:3}$.

Next we construct a transformation $\gamma_2(v_{1:3}, v_{2:3})$ such that

$$G_{2:3}(v) = \int\limits_{v_{1:3}, v_{2:3} : \gamma_2(v_{1:3}, v_{2:3}) \leq v} \int \tilde{F}_{1,2:3}(dv_{1:3}, dv_{2:3}).$$

Since $\tilde{F}(\cdot)$ lies within the bounds, we know that

$$\tilde{F}_{2:3}(v) \leq G_{2:3}(v) \leq G_{1:3}(v).$$

So for an appropriately chosen $\alpha \in [0, 1]$, one such a transformation is $\gamma_2(v_{1:3}, v_{2:3}) = \alpha v_{2:3} + (1 - \alpha)\gamma_1(v_{1:3})$.

As for the last bid $B_{3:3}$, we know by the definitions of the bounds that

$$\tilde{F}_{3:3}(v) \leq G_{3:3}(v) \leq \tilde{F}_{2:3}(v).$$

We can therefore find a linear transformation $\gamma_3(v_{2:3}, v_{3:3}) = b_{3:3} = \beta v_{2:3} + (1 - \beta)v_{3:3}$ satisfying

$$\text{Pr}_{\tilde{F}}(\gamma_3(v_{2:3}, v_{3:3}) \leq b) = G_{3:3}(b)$$

with $\beta \in [0, 1]$. Then by construction,

$$v_{2:3} \leq \gamma_3(v_{2:3}, v_{3:3}) \leq v_{3:3},$$

ensuring that the vector function $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ satisfies assumptions 1 and 2. Q.E.D.

This result does not imply sharpness because we have ensured only that we can match the marginals $G_{1:3}(\cdot)$, $G_{2:3}(\cdot)$, and $G_{3:3}(\cdot)$, not the joint distribution $G_{1,2,3:3}(\cdot, \ \cdot, \ \cdot)$. However, any additional information that could be extracted from the data given our assumptions would require exploiting the correlation structure among bids implied by $G_{1,2,3:3}(\cdot, \ \cdot, \ \cdot)$. Matching an arbitrary correlation structure would likely require a stochastic bidding rule (rather than a set of deterministic bid functions like those above). This makes it particularly hard to "guess" a general solution, and a constructive approach based on copulas appears to require fundamental advances that are beyond what could be addressed in this paper. However, since assumptions 1 and 2 do not appear to have any implications for the correlation of bids, it seems possible that our bounds do in fact exhaust the information in these assumptions and the observable bids.

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