ECON 202A : Midterm Solution Fall, 2019

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# 1 Problem 1

Consider the following social planning problem:

$$\max \sum_{t=0}^{\infty} \beta^t U\left(c_t, 1-h_{1t}-h_{2t}\right) \tag{1}$$

where $0<\beta<1$ subject to

$$c_t = F^1\left(k_{1t}, h_{1t}\right)$$

$$k_{t+1} = F^2\left(k_{2t}, h_{2t}\right)$$

$$k_t = k_{1t}+k_{2t}$$

$$k_0 \text{ given.}$$

Here, the function $U$, $F^1$ and $F^2$ have all the usual properties. Assume that capital can be freely allocated across the two sectors.

a. Write this maximization problem as a dynamic program. (10 points)

The way one writes the dynamic program is not unique. Observe that in this problem the depreciation of capital is 100%, $\delta=1$.This problem can be simplified to:

$$\max_{\{k_{2t},h_{2t},h_{1t}\}_{t=0}} \left\{\sum_{t=0}^{\infty} \beta^t u\left[F^1\left(k_t-k_{2t}, h_t-h_{2t}\right), 1-h_t\right)\right] \tag{2}$$

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s.t.

$$k_{t+1} = F^2(k_{2t}, h_{2t}) \tag{3}$$

Then, one way to write the dynamic program is the following:

$$v(k) = \max_{k_2,h_2,h,k'\{u[F^1(k-k_2,h-h_2),1-h]+\beta v(k')\} \text{ s.t. } k'=F^2(k_2,h_2)} \tag{4}$$

or

$$v(k) = \max_{k_2,h_2,h} u\left[F^1(k-k_2, h-h_2), 1-h\right] + \beta v\left[F^2(k_2,h_2)\right] \tag{5}$$

b. Define a recursive competitive equilibrium for this economy. Be sure to state the problems solved by households and firms in your definition and define any additional variables you introduce. (15 points)

Remarks:

(1) Labor and capital are perfectly mobile across sectors, thus the wage rate and rental rate of capital are equalized across sectors. Therefore, the household doesn't care how the capital and labor supply is allocated across sectors. Only the total allocations matter for the household. The sectoral allocations are determined by the firms.

(2) Consumption and investment goods are no longer perfect substitutes because they are produced by different technologies. The price of consumption goods is still one (because we've chosen it as numeraire). However, the relative price of investment goods is no longer one, but some price q.

Now, we are ready to define the Recursive Competitive Equilibrium. The remarks, above, imply that the Household's SP can be written as:

$$\max_{\{c_t,k_t,h_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1-h_t) \tag{6}$$

subject to

$$c_t + q_t k_{t+1} = w_t h_t + r_t k_t \tag{7}$$

and $k_0$ given.

**Household's dynamic programming problem[(a)]:**

$$v(K,k) = \max_{h,k} \left\{u\left[w(K)h + r(K)k - q(K)k', 1-h\right] + \beta v\left(K', k'\right)\right\} \tag{8}$$

subject to $K' = G(K)$.

There are two firms in the economy:

1. Consumption good producing firm [(b)]:

$$\max_{k_1^f, h_1^f} F^1\left(k_1^f, h_1^f\right) - r(K)k_1^f - w(K)h_1^f \tag{9}$$

2. Investment good producing firm [(c)]:

$$\max_{k_2^f, h_2^f} q(K)F^2\left(k_2^f, h_2^f\right) - r(K)k_2^f - w(K)h_2^f \tag{10}$$

**Definition:** A recursive competitive equilibrium is:

1. A set of decision rules $k'(K,k), h'(K,k)$ for the household
2. A set of decision rules $k_1^f(K)$ and $h_1^f(K)$ for firm 1
3. A set of decision rules $k_2^f(K)$ and $h_2^f(K)$ for firm 2
4. A set of pricing functions $r(K), w(K)$ and $q(K)$
5. A law of motion for the aggregate state $K' = G(K)$.

such that

(i) Given (4) and (5), (1) solves the household's DPP (a)

(ii) Given (4), (2) solves the firm 1 's problem (b)

(iii) Given (4), (3) solves the firm 2's problem (c)

(iv) Markets clear:

$$\begin{array}{ll} \text{Labor market:} & h_1^f(K) + h_2^f(K) = h(K,K) \\ \text{Capital market:} & k_1^f(K) + k_2^f(K) = K \\ \text{Investment goods market:} & k'(K,K) = F^2\left[k_2^f(K), h_2^f(K)\right] \end{array} \tag{11}$$

By Walras's law the consumption goods market also clears

(v) Perceptions are correct:

$$k'(K,K) = G(K)$$

## 2 Problem 2

Consider a one-sector stochastic growth model where preferences are described by

$$E \sum_{t=0}^{\infty} \beta^t U(c_t) \tag{12}$$

where $U(c_t)$ is a concave function and $0 < \beta < 1$. Output is produced using a constant returns to scale production function, $z_t F(k_t, h_t)$, where $z_t$ is shock to technology observed at the beginning of period $t$. The stochastic process for $z$ is a two state Markov chain with transition matrix $P$ and where the unconditional mean of $z$ is one. The representative household is endowed with one unit of time (labor). Output produced in period $t$ can be consumed that period or saved. One unit of output saved provides one unit of productive capital the following period. The capital stock is assumed to depreciate at the rate $0 < \delta < 1$.

A. Write the planner's problem for this economy as a dynamic program.

The DPP, since there is no utility for leisure and so $h = 1$, is given by:

$$V(k, z) = \max_{k'} \left[ U\left( zF(k, h) + (1-\delta)k - k^{'} \right) + \beta E\left[ V\left(k', z'\right) | z \right] \right] \tag{13}$$

subject to

$$z' \in \{z_L, z_H\} \quad , \qquad \text{and } z' \text{ evolves according to } P \tag{14}$$

where wlog I have rearranged the space of $z$ in an increasing fashion. Since the shocks are discrete, and assuming $P = [p_{LL} \quad p_{LH} \quad ; p_{HL} \quad p_{HH}]$ we can also define one Bellman equation for each shock:

$$V(k, z) = \begin{cases} V^L(k) \equiv V(k, z_L) = \max_{k'} \left[ U\left( z_L F(k, h) + (1-\delta)k - k^{'} \right) + \beta \left[ p_{LL} V^L(k') + p_{LH} V^H(k') \right] \right] \\ V^H(k) \equiv V(k, z_H) = \max_{k'} \left[ U\left( z_L F(k, h) + (1-\delta)k - k^{'} \right) + \beta \left[ p_{HL} V^L(k') + p_{HH} V^H(k') \right] \right] \end{cases} \tag{15}$$

B. Characterize the nonstochastic steady state for this model (that is, provide a set of equations the solution to which is the nonstochastic steady state).

The first order condition for hours worked implies that $h = 1$ given there is no utility for leisure. After deriving the first order condition:

$$U_c\left( zF(k, 1) + (1-\delta)k - k^{'} \right) = \beta E\left[ V_k\left(k', z'\right) | z \right] \tag{16}$$

and the envelope condition

$$V_k\left(k', z'\right) = \left(z' F_k\left(k', 1\right) + (1-\delta)\right) U_c\left(z' F\left(k', 1\right) + (1-\delta)k' - k''\right) \tag{17}$$

Combining, bringing back the t subscript and assuming that $z = 1$ and non-stochastic, we have:

$$U_c\left(F\left(k_t, 1\right) + (1-\delta)k_t - k_{t+1}\right) = \beta U_c\left(F\left(k_{t+1}, 1\right) + (1-\delta)k_{t+1} - k_{t+2}\right)\left(F_k\left(k_{t+1}, 1\right) + (1-\delta)\right) \tag{18}$$

In the steady state, $c_t = \bar{c}$ and $k_t = \bar{k}$ for all $t$. This implies that the steady state level of capital is given by:

$$\beta\left(F_k\left(\bar{k}, 1\right) + (1-\delta)\right) = 1 \tag{19}$$

and the steady state level of consumption is given by:

$$\bar{c} = F\left(\bar{k}, 1\right) + \delta\bar{k} \tag{20}$$

C. Assume that capital can only be held in discrete units such as individual houses of equal size (say $\phi > 0$ units of output). Describe an algorithm for solving the dynamic programming problem in part A that could be implemented on a computer. You do not have to actually write a program; just outline the algorithm in sufficient detail that your "research assistant" would be able to write the program without knowledge of dynamic programming.

We observe that Blackwell sufficient conditions are satisfied and thus we know that there is a solution to the dynamic program, that can be obtained using value function iteration. In this case, we can only do it using numerical methods. So the research assistant needs to know the steps that if he follows them, the numerical value function iteration will stop and give the solution that yields then correct numerical value function and the numerical policy function. As a reminder the general problem, we try to solve is the following:

We do not discretize the space at which capital belongs, since in the problem is already assumed that capital needs to belong in a uniform grid, with step size $\phi$. In addition, the stochastic process is discretized so we do not need to do this either.

The theorem above tells us that we will converge from any initial guess, however given that we calculated the steady state, we will use the steady state as our initial guess(it usually saves at least one iteration.)

So lets know describe the grid, since we are looking, for a policy that depends on $z, k$ and assuming that the number of real numbers between zero and the maximum sustainable capital is equal to $M$, we generate a sequence of $2M$ values for our capital grid, this is just the capital grid replicated twice once for $z_L$ and once for $z_H$ and we create the initial guess for the value function, either a grid of zeros or as described above after calculating steady state capital $\bar{k}$, we can calculate $V(z_L, \bar{k})$ and $V(z_H, \bar{k})$ and use as an initial guess a sequence that consists of $M$ values of $V(z_L, \bar{k})$ and $M$ values of $V(z_H, \bar{k})$.

We need to specify a tolerance for when to stop searching over value functions. In particular, my objective is going to be the norm of the of the absolute value of the difference between two $V^n$ and $V^{n+1}$. One norm that can be used is the square root of the sum of squared deviations. We need to set a tolerance for this for when the loop should stop. I set this tolerance at 0.0001. We also can specify the maximum number of iterations after which the loop should stop, in case we might create an infinite loop.

Since, we decided about initialization and discretization, we still need to implement the loop that will be our numerical value function iteration. In each step, for each value in $z, k$, we find the k that maximizes the Bellman equation given my guess of the value function.There are several ways to do this, one of which I describe below. Outside of the loop I have a "while" statement, which tells MATLAB to keep repeating the text as long as the difference between value functions is greater than the tolerance and the number of iteration, a variable that is increasing by 1 in each iteration, is below the maximum number of iterations.

The way to create the next guess $V^{n+1}$ based on the current guess $V^n$, which can be called v0 is: for each spot i in the state space, we find the argmax of the Bellman equation. We collect the optimized values into the new value function (called v1), and we also find the policy function associated with this choice. After the loop over the possible values of the state I calculate the difference and write out the iteration number. This process can be equivalently done using a loop or matrices.

D. Can the exact invariant distribution for the model of part C be computed (rather than approximated)? How?.

Given that the state $z, k$ and thus control variables can only take a finite number of points this implies that we are trying to solve a discrete Markov Chain decision process, and, of course up to rounding errors, we can solve discrete MPDs exactly. In other words the policy function we got from the previous prooblem is exact( up to rounding errors).

We have the policy function $g(z, k)$ for each combination of $z, k$. Define the expanded state space $S$ for the combinations of $z, k$ as follows: $[S_1 \quad = (z_1, k_1) \quad \ldots S_m = (z_1, k_m) \quad S_{m+1} = (z_2, k_1) \quad \ldots S_{2m} = (z_2, k_m)]$ and the transition matrix probabilities $\hat{P}_{ij} = P(k' = S_i, z' =$

$S_j|k,z) = P(z' = S_j(1)|z = S_i(1)) \times P(k' = S_j(2)|(z,k) = S_i)$

where $S_k(i)$ the ith element of the vector that is the kth element of our grid. Having the new transition matrix, we can use the software to compute the eigenvalues, and the invariant distribution. If the unit eigenvalue is unique, we have a unique invariant distribution.

E. Suppose now that $z$ is governed by a three state Markov chain.

a. Provide an example of a transition matrix $P$ such that $z$ has is a unique invariant distribution and one transient state. (8 points)

Let's wlog assume that the state space for $z$ is $z_1, z_2, z_3$, then an example that $z_1$ is a transient state is one where we know that there is always zero probability to return to the state and positive probability to leave from state $z_1$. Then strictly positive probabilities for every other entry in the transition matrix will ensure a unique invariant distribution and one ergodic set $z_2, z_3$.

For example,

$$P' = \begin{bmatrix} .8 & .1 & .1 \\ 0 & .8 & .2 \\ 0 & .2 & .8 \end{bmatrix}$$

Then the invariant distribution is $\pi = [0 \quad .5 \quad .5]$.

b. Provide an example of a transition matrix P such that there are three ergodic sets. Characterize the invariant distribution(s) in this case. Given an initial distribution, $\pi_0$ , what will be the limit distribution? Explain. (12 points)

Since all of $z_1$, $z_2$ and $z_3$ have to be ergodic sets. There is a unique matrix delivering this result:

$$P' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In this case, there is an infinite amount of invariant distribution that is a convex combination of $[1 \quad 0 \quad 0]$, $[0 \quad 0 \quad 1]$ and $[0 \quad 1 \quad 0]$. Which implies all $0 \geq q, r \leq 1$ such that $\pi = [q \quad r \quad 1-q-r]$ deliver an invariant distribution. Any initial distribution $\pi_0$ will have $\pi_0$ as the limiting distribution.