ECON 202A : Introduction to Dynamic Programming

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These notes are intended to introduce the basic concept of *dynamic programming*, the mathematical tool for solving dynamic economic problems in a recursive way.

1 The Sequential Problem and its Solution

Consider the simple deterministic neoclassical growth model. In this model, there is an infinitely-lived, representative household who chooses an infinite sequence for consumption, investment and next period's capital stock $\{c_t, i_t, k_{t+1}\}_{t=0}^{\infty}$ in order to maximize its lifetime utility, subject to a resource constraint and law of motion for the capital stock, given the initial capital stock k_0. That is, the representative solves the following maximization problem:

$$\max_{\{c_t,i_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to

$$\begin{aligned}
c_t + i_t &= F(k_t, n_t) \qquad \forall t,\\
k_{t+1} &= i_t + (1-\delta)k_t \qquad \forall t,\\
c_t, k_t &\geq 0 \qquad \forall t,\\
n_t &\in [0,1] \qquad \forall t,\\
&k_0 \text{ given.}
\end{aligned}$$

Here $\beta \in (0,1)$ denotes the household's subjective discount factor for the future utility, δ is the depreciation rate for capital, i_t is investment, and n_t is the hours worked by the household. $U(c_t)$ is the instantaneous utility function and $F(k_t, n_t)$ is the production function. Assume that those functions satisfy the properties discussed in lecture.[1] Since the marginal product of labor and the marginal utility of consumption are always positive according to

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[1]So I write the resource constraint and capital law of motion with equalities.

the assumptions, we know that $n_t = 1$ for all t in any solution. Otherwise, the household will lose the opportunity to obtain more output from production, while there is no benefit from lower hours worked. Therefore, now $F(k_t, n_t) = F(k_t, 1)$ in any solution, so let me define

$$f(k_t) \equiv F(k_t, 1).$$

For simplicity, I assume that capital fully depreciates each period (i.e. $\delta = 1$) so that the law of motion of the capital stock is simplified to $i_t = k_{t+1}$. With these assumptions, we can substitute c_t and i_t with functions of the capital stock of today and tomorrow. Therefore, the problem is simplified as:

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1})$$

subject to

$$\begin{aligned} &0 \leq k_{t+1} \leq f(k_t) \qquad \forall t, \\ &k_0 \quad \text{given.} \end{aligned}$$

Therefore, now all we have to do is to solve the infinite sequence of capital stock only. The first-order condition of this problem with respect to k_{t+1} is called *Euler Equation* , which has the following form:

$$U'(f(k_t) - k_{t+1}) = \beta U'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}). \tag{1}$$

This equation shows the intertemporal condition for optimality: the marginal cost of forgone consumption by increasing k_{t+1} (LHS) must equal the present value of marginal benefit of the household tomorrow by the higher capital stock tomorrow (RHS).

In order to obtain an analytic solution, let me specify the functional forms of the utility function and the production function as follows:

$$U(c) = \log(c), \qquad f(k) = k^{\alpha}, \quad \alpha \in (0, 1).$$

That is, log utility and Cobb-Douglas technology are assumed.[2] Then, (1) becomes:

$$\frac{1}{k_t^{\alpha} - k_{t+1}} = \beta \frac{\alpha k_{t+1}^{\alpha-1}}{k_{t+1}^{\alpha} - k_{t+2}}. \tag{2}$$

So we obtain a nonlinear second-order difference equation. Our goal is to derive a relationship between k_t and k_{t+1} for all t, so that we can solve for the entire sequence of capital stock with k_0 given. To do this, I use the fact that *the limit of the solution to the finite horizon problem* is equivalent to the unique solution to *the infinite horizon problem.* Proving that

[2]This kind of specification, together with the assumption of full depreciation of capital stock, makes it possible to solve the optimal policy function in closed form by hand. Closed form solutions do not exist in most of the cases though.

this conjecture is correct involves establishing the legitimacy of interchanging the operators "max" and "$\lim_{T\to\infty}$", which is more challenging than one might guess.[3] Assume T is the terminal period of the finite horizon problem. Since (2) is a second-order equation, we need not only an initial condition (k_0 given) but also a terminal condition to solve the problem. Let $k_{T+1} = 0$ be the terminal condition.[4] Then, considering (2) at $t = T - 1$ with using this terminal condition, we have:

$$\frac{1}{k_{T-1}^{\alpha} - k_T} = \beta \frac{\alpha k_T^{\alpha-1}}{k_T^{\alpha}}. \tag{3}$$

or, we have $k_T = \frac{\alpha\beta}{1+\alpha\beta} k_{T-1}^{\alpha}$. Using this value for k_T, we can return to the Euler Equation evaluated at $t = T - 2$. Doing so and solving for k_{T-1}, we get $k_{T-1} = \frac{\alpha\beta(1+\alpha\beta)}{1+\alpha\beta+(\alpha\beta)^2} k_{T-2}^{\alpha}$. Continuing this procedure, we can find the expression for the relationship between k_t and k_{t+1} for an arbitrary $t < T$ as follows:

$$k_{t+1} = \frac{\alpha\beta(1-(\alpha\beta)^{T-t})}{1-(\alpha\beta)^{T-t+1}} k_t^{\alpha}. \tag{4}$$

Hence, we can solve for the optimal sequence of capital stock in an infinite-horizon model by taking the limit as $T \to \infty$. Since $\alpha, \beta \in (0, 1)$, we find the solution to the Euler Equation:

$$k_{t+1} = \alpha\beta k_t^{\alpha}. \tag{5}$$

Given k_0, therefore, we can solve for the entire sequence of $\{k_{t+1}\}_{t=0}^{\infty}$. After doing so, we can solve for the sequence of consumption using the resource constraint as well.

2 The Recursive Problem and its Solution

2.1 The Recursive Formulation

This section introduces how to construct and solve the optimization problem in the previous section recursively. Define a real-valued function $V(k)$ as follows:

$$V(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1}) \tag{6}$$

subject to

$$\begin{aligned} &0 \le k_{t+1} \le f(k_t) \qquad \forall t. \\ &k_0 \quad \text{given} \end{aligned}$$

Here, $V(\cdot)$ is called the *value function*. It represents the maximized value of the lifetime utility of the household as a function of the initial capital stock k_0. Separating the infinite

[3] See Stokey and Lucas, page 12.

[4] It actually is $\beta^T U'(c_T) k_{T+1} = 0$ which corresponds to the transversality condition in an infinite-horizon framework. By strict monotonicity of $U(c) : \beta^T U'(c_T) > 0 \Rightarrow k_{T+1} = 0$

sum and distributing the max operator, (6) can be rewritten as:

$$
\begin{aligned}
V(k_0) &= \max_{k_1} \max_{\{k_{t+1}\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1}) \\
&= \max_{k_1} [U(f(k_0) - k_1) + \beta \max_{\{k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} U(f(k_t) - k_{t+1})].
\end{aligned}
$$

Note that the second maximum term has the same form as (6) with one period forwarded. Therefore, (6) can be written as the following recursive form:[5]

$$
V(k_0) = \max_{k_1} \Big[U(f(k_0) - k_1) + \beta V(k_1) \Big]. \tag{7}
$$

Hence, now the problem is reduced and k_1 is the only choice variable on the right-hand side. In order to underline the recursive structure of this expression, let k denote the capital stock today and k' denote the capital stock tomorrow. Then, we can rewrite (7) as:

$$
V(k) = \max_{k'} \Big[U(f(k) - k') + \beta V(k') \Big]. \tag{8}
$$

The functional form of $V(\cdot)$ is what we want to solve for, so (8) is a functional equation. This functional equation is called the *Bellman Equation*. The instantaneous utility $U(\cdot)$ is called the *return function*, and k is called the *state variable*. The state variable completely summarizes all information that is necessary for the household to make an optimal decision in the current period.

In order to confirm that a Bellman equation is well constructed, check to see:

1. All the right-hand side variables must be either state variables or choice variables. Otherwise, that is not a well-defined Bellman equation.

2. A sequential problem can have multiple number of Bellman equation representations that are equivalent.

You may wonder why do we care about this function V if we actually care about the solution for k' as a function of k. The reason is the following, for the moment, assume that you know V and moreover, you know V is differentiable, so the first order condition for (8) defines k' implicitly:

$$
U'\left(f(k) - k'\right) = \beta V'(k') \tag{9}
$$

In particular, let's assume that $U(x) = \log(x)$, $f(k) = k^\alpha$ and that you know that $V(k) = A + \frac{\alpha}{1-\alpha\beta} \log(k)$, with A some constant, then (9) implies:

$$
\frac{1}{k^\alpha - k'} = \frac{\alpha\beta}{1-\alpha\beta} \frac{1}{k'} \Rightarrow k' = \alpha\beta k^\alpha \tag{10}
$$

Same as (5). So, as you can see, solving for V can be really helpful to solve for the policy function.

[5]This separation and recursive formulation is what is called "Principle of Optimality"

These are examples of recursive formulations:

Example 1 The Bellman equation (8) can be rewritten as follows:

$$V(k) = \max_{k',c} \left[U(c) + \beta V(k') \right] \tag{11}$$

subject to

$$c + k' = f(k).$$

Here, k is the state variable and c and k' are choice variables. (11) is a well-defined Bellman equation that represents the same problem as (8).

Example 2: Habit Persistence Consider a modified neoclassical growth model in which the household's utility is a function of not only its current consumption but also its consumption last period: that is, the utility function is $U(c_t, c_{t-1})$. This is what we mean by habit persistence. This specification implies that last period's consumption level directly influences the household's current utility. As a result, the household must know last period's consumption to optimally solve its problem today. In other words, c_{t-1} must be a state variable in this case. The corresponding Bellman equation can be written as:

$$V(k, c_{-1}) = \max_{c,k'} \left[U(c, c_{-1}) + \beta V(k', c) \right]$$

subject to

$$\begin{aligned} c + i &= f(k), \\ k' &= i + (1 - \delta)k, \end{aligned}$$

where c_{-1} denotes last period's consumption, with time subscript omitted.

Let's go back to the problem of solving (8) and showing its solution is equivalent to that of the sequential problem. Unlike the sequential problem where the solution consists of the infinite sequences, the recursive formulation asks us to solve for the value function and the optimal policy function between two consecutive periods, which governs the evolution of the state. Let $g(k)$ be the optimal policy function, i.e. the optimal choice of the capital stock next period given the current capital stock, or $k' = g(k)$.

To motivate the solution method that will come, let me define the *Bellman operator* $T(\cdot)$. T maps from functions to functions, and is defined as follows:

$$V_{m+1}(k) = T(V_m(k)) = \max_{k'} \left[U(f(k) - k') + \beta V_m(k') \right] \tag{12}$$

That is, given a function $V_m(\cdot)$, T returns another function $V_{m+1}(\cdot)$. Therefore, solving (8) is equivalent to finding a *fixed point* of the mapping T.

2.2 Contraction Mapping Theorem and Blackwell's Sufficient Conditions

To solve the Bellman equation, we need to rely on some mathematical results, which are introduced in Stokey and Lucas chapter 3. Here are definitions that we need.

Definition A *metric space* is a set X, together with a metric (distance function) $\rho : X \times X \to \mathbb{R}$, such that for all $x, y, z \in X$:

1. $\rho(x, y) \geq 0$ with equality iff $x = y$ *(non-negativity)*
2. $\rho(x, y) = \rho(y, x)$ *(symmetry)*
3. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ *(triangular inequality)*

Definition A sequence $\{x_n\}_{n=0}^{\infty}$ in X is a *Cauchy sequence* if for each $\epsilon > 0$, there exists N_ϵ such that $\rho(x_n, x_m) < \epsilon$ for all $n, m \geq N_\epsilon$.

Definition A metric space (X, ρ) is *complete* if every Cauchy sequence in X converges to some element in X.

Now we are ready to define a contraction mapping.

Definition Let (X, ρ) be a metric space and $T : X \to X$ be a function mapping X onto itself. T is a *contraction mapping* (with *modulus* β) if for some β in $(0, 1)$, $\rho(T(x), T(y)) \leq \beta\rho(x, y)$, for all $x, y \in X$.

That is, a mapping T is a contraction mapping if, when you apply it to two elements in the metric space X, the distance between the transformed elements is reduced. Now I introduce the contraction mapping theorem.

Theorem *(Contraction Mapping Theorem)* If (X, ρ) is a *complete metric space* and $T : X \to X$ is a *contraction mapping with modulus* β, then:

1. T has exactly one fixed point v in X, and,
2. for any $v_0 \in X$, $\rho(T^n(v_0), v) \leq \beta^n \rho(v_0, v)$.

Therefore, the Contraction Mapping Theorem provides the conditions under which the unique fixed point of the mapping T in (12) exists. Instead of directly showing that T is a contraction mapping, we usually use Blackwell's sufficient conditions for a contraction mapping to examine whether the Bellman equation has a solution.

Theorem (Blackwell's Sufficient Conditions for a Contraction) Let $X \subset \mathbb{R}^l$, and let $B(X)$ be a space of bounded functions $f : X \to \mathbb{R}$. Let $T : B(X) \to B(X)$ be an operator satisfying:

1. *(monotonicity)* $f, g \in B(X)$ and $f(x) \leq g(x)$, for all $x \in X$, implies $T(f(x)) \leq T(g(x))$, for all $x \in X$, and

2. *(discounting)* there exists some $\beta \in (0,1)$ such that $T(f(x)+a) \leq T(f(x)) + \beta a$ for all $f \in B(X)$, $a \geq 0$, and $x \in X$.

Then, T is a contraction mapping with modulus β.

Let me check if the operator T in (12) satisfies Blackwell's sufficient conditions to ensure that our operator is indeed a contraction mapping.

1. **Monotonicity**: Assume that $v(k) \geq w(k)$ for all $k \in \mathbb{R}_+$, then

$$\begin{aligned} T(v) &= \max_{k'} \Big[U(f(k)-k') + \beta v(k')\Big] \\ &\geq \max_{k'} \Big[U(f(k)-k') + \beta w(k')\Big] \\ &= T(w) \end{aligned}$$

2. **Discounting**: Let a be an arbitrary constant function. Then,

$$\begin{aligned} T(v+a) &= \max_{k'} \Big[U(f(k)-k') + \beta(v(k')+a)\Big] \\ &= \max_{k'} \Big[U(f(k)-k') + \beta v(k') + \beta a\Big] \\ &= \max_{k'} \Big[U(f(k)-k') + \beta v(k')\Big] + \beta a \\ &= T(v) + \beta a \end{aligned}$$

Therefore, our operator T satisfies Blackwell's sufficient conditions, and thus it is a contraction mapping. This guarantees that the Bellman equation has a unique fixed point which can be reached by starting with any initial guess that is a bounded continuous function. Moreover, the theorem indicates that we can arrive at the fixed point by iteration.

2.3 Solving for the Value Function

2.3.1 Using Value Function Iteration

A guideline to conduct value function iteration is below:

1. Define the operator

$$T(V(k)) = \max_{0 \leq k' \leq f(k)} \Big[U(f(k)-k') + \beta V(k')\Big]$$

2. Check if the operator T satisfies Blackwell's sufficient conditions. If so, T is a contraction mapping and it has a unique fixed point.

3. Make a initial guess of the value function, say V_0. Any bounded continuous function works, e.g. $V_0(k) = 0$ for all k.[6]

4. Iterate on

$$V_{m+1}(k) = T(V_m(k)) = \max_{0 \leq k' \leq f(k)} \left[U(f(k) - k') + \beta V_m(k') \right] \tag{13}$$

until the distance between V_{m+1} and V_m is negligible. The sup norm is often used to measure the distance between the two functions. Then, the fixed point is $V(k) = \lim_{m \to \infty} V_m(k)$

As mentioned in lecture, the value function $V(k)$ does not have a closed-form representation in most cases. However, there are some cases where the value function has a clear closed form. In this case, we can solve for the functional form using the next procedure.

2.3.2 Using Method of Undetermined Coefficients

1∼3. Same as above.

4. Iterate on (13). Confirm that a certain functional form appears repeatedly.

5. Once the functional form has converged, state a general functional form (with unknown coefficients) and solve the problem as a function of these coefficients and the structural parameters of the model. You will get a 'new' value function. Finally, match your unknown coefficients with the ones of the new value function and solve for them as a function of the known parameters.

Let me apply this method to the problem defined by (11). So, first I guess $V_0(k) = 0$ for all k. Then,

$$\begin{aligned} V_1(k) = T(V_0(k)) &= \max_{0 \leq k' \leq f(k)} \left[\log(k^\alpha - k') + \beta V_0(k') \right] \\ &= \max_{0 \leq k' \leq f(k)} \log(k^\alpha - k') \end{aligned}$$

It is straightforward that the optimal choice is $k' = 0$. Plugging this into the objective function, V_1 has of the form:

$$V_1(k) = \log(k^\alpha) = \alpha \log(k)$$

Using this new guess, we have:

$$\begin{aligned} V_2(k) = T(V_1(k)) &= \max_{0 \leq k' \leq f(k)} \left[\log(k^\alpha - k') + \beta V_1(k') \right] \\ &= \max_{0 \leq k' \leq f(k)} \left[\log(k^\alpha - k') + \beta \alpha \log(k') \right] \end{aligned}$$

[6] In a computer, you need to define a grid for k

Then, the first-order condition is:

$$\frac{1}{k^\alpha - k'} = \frac{\alpha\beta}{k'} \qquad \Rightarrow \qquad k' = \frac{\alpha\beta}{1+\alpha\beta}k^\alpha$$

Plugging this into the objective function and conducting some algebra, we get:

$$\begin{aligned} V_2(k) &= \log\left(k^\alpha - \frac{\alpha\beta}{1+\alpha\beta}k^\alpha\right) + \alpha\beta\log\left(\frac{\alpha\beta}{1+\alpha\beta}k^\alpha\right) \\ &= \underbrace{\alpha\beta\log(\alpha\beta) - (1+\alpha\beta)\log(1+\alpha\beta)}_{C} + \underbrace{\alpha(1+\alpha\beta)}_{F}\log(k) \\ &\equiv C + F\log(k) \end{aligned}$$

Based on this, we can make the general guess that $V(k) = A + B\log(k)$, for some A and B, then we have:

$$V(k) = A + B\log(k) \quad = \quad \max_{0\le k' \le f(k)} \Big[\log\left(k^\alpha - k'\right) + \beta\left(A + B\log(k')\right) \Big]$$

FOC implies $k' = \frac{\beta B}{1+\beta B}k^\alpha$, plug this back into (11) and arranging terms we have

$$V(k) = A + B\log(k) = \left[\beta A + \log\left(\frac{1}{1+\beta B}\right) + \beta B\log\left(\frac{\beta B}{1+\beta B}\right)\right] + \Big[\alpha + \alpha\beta B\Big]\log(k) \quad (14)$$

So, it must be the case that $A = \beta A + \log\left(\frac{1}{1+\beta B}\right) + \beta B\log\left(\frac{\beta B}{1+\beta B}\right)$ and most importantly, $B = \alpha + \alpha\beta B$. Then, $B = \frac{\alpha}{1-\alpha\beta}$, with that we can solve for A. Thus

$$V(k) = A + \frac{\alpha}{1-\alpha\beta}\log(k) \qquad (15)$$

Finally, using (9) or the fact that $k' = \frac{\beta B}{1+\beta B}k^\alpha$, the policy function is:

$$k' = \alpha\beta k^\alpha$$

This expression is the same as (5), which we derived from the sequential problem. We have therefore shown how to solve a dynamic programming problem these tools and that the solution to this reformulated problem is identical to the solution to the original sequential problem.[7]

3 Clever Guesses for Method of Undetermined Coefficients

[7]It is worth to mention that Value Function Iteration (almost) always works, but it is quite slow because of the max operator. There are other methods that give faster solutions e.g. policy function iteration, Howard's improvement, endogenous grid method, etc.

Deterministic

States	Return	Good Guess	Notes
(k)	$\log(c)$	$E + F\log(k)$	
(k,c_{-1})	$\log(c) + A\log(c_{-1})$	$E + F\log(k) + G\log(c_{-1})$	
(k)	$\log(c) - Bh$	$E + F\log(k)$	don't include intratemporal variables[8]
(k,c_{-1})	$\log(c) + A\log(c_{-1}) - Bh$	$E + F\log(k) + G\log(c_{-1})$	

Stochastic

States	Return	Good Guess	Notes
(k,z) - z AR(1)	$\log(c)$	$E + F\log(k) + Gz$	The objective will look like $\log(e^z k^\theta - k')$, so the guess is linear in z
(k,z) - z AR(1)	$\log(c) - Bh$	$E + F\log(k) + Gz$	
(k,z) - z 2-State Markov	$\log(c)$	$v_0^L(k) = E_L + F\log(k)$ and $v_0^H(k) = E_H + B\log(k)$	Split guess over high/low states
(k,z) - z 2-State Markov	$\log(c) - Bh$	$v_0^L(k) = E_L + F\log(k)$ and $v_0^H(k) = E_H + B\log(k)$	