

Econ202A Notes

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1 Lecture 1—October 5, 2020

Emphasis will be on neoclassical models of macroeconomic growth.

Office hours after class. Cohort effects are huge; spend lots of time collaborating with classmates. Start with certainty; converge to a single point steady state. With stochastic elements, we converge to a distribution. In both cases, we're studying an optimization problem. The model we're studying is the solution to an optimization problem. We will then discuss how to interpret these solutions as competitive equilibria. In these models, what is chosen by the planner and the competitive outcome are the same. In more realistic models, this won't be the case—there will be distortions. The goal is not to be theoretically rigorous; the focus is on applied macroeconomics—applying macroeconomic tools to study **the real world**. How can we use our theoretical models to interpret the real world or do interesting policy experiments?

1.1 Basic Neoclassical Growth Model (Cass-Koopmans)

Economy consists of many identical infinitely lived households, each with the same preferences and endowments. There are several possible interpretations

- Representative Agent
- Benevolent Social Planner
- Infinitely Lived family

There is one production sector, and output is produced from capital and labor. Output can be either consumed or invested. There is only one type of output.

Investment becomes productive capital the following period:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1$$

subject to

$$c_t + i_t \leq y_t = F(k_t, n_t) \quad (\text{resource constraint})$$

$$k_{t+1} \leq (1 - \delta)k_t + i_t, \quad 0 < \delta \leq 1$$

$$0 \leq n_t \leq 1$$

$$k = \text{given}$$

Note that these are resource constraints, not budget constraints. They're different, and commonly confused. There are *no prices* here! δ is depreciation and β is a discount factor. The values for these terms are related to the length of time periods, t .

We make some assumptions about the objects in our model.

The production function $F : \mathbb{R}_+^2 \rightarrow \mathbb{R}^+$ is

1. continuously differentiable

2. Homogeneous of degree 1 \equiv constant returns to scale
3. Strictly quasi-concave
- 4.

$$\begin{aligned}
 F(0, n) &= 0 && \text{(capital is essential)} \\
 F_k &= \text{marginal product of capital} \\
 &> 0 \\
 F_n &> 0 && \text{(marginal product of labor)}
 \end{aligned}$$

5. INADA conditions

$$\lim_{k \rightarrow 0} F_k(k, 1) = \infty \quad \lim_{k \rightarrow \infty} F_k(k, 1) = 0$$

The utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is

- Bounded—important for dynamic programming
- Continuously differentiable
- Strictly concave
- $\lim_{c \rightarrow 0} u'(c) = \infty$

Note: We will use functional forms for F and u for almost everything in this class.

We make a few assumptions to simplify the planners problem:

1. $F_n > 0 \implies n_t = 1$ for all t and $u'(c) > 0$. We will introduce leisure in the utility function later.
2. $u'(c) > 0 \implies$ the resource constraint holds with equality,

$$c_t + i_t = F(k_t, n_t)$$

3. $\beta < 1 \implies$ a positive rate of return from giving up consumption today for consumption tomorrow; (actually $MP_k + 1 - \delta$)

$$\implies k_{t+1} = (1 - \delta)k_t + i_t$$

4. Let

$$f(k) \equiv F(k, 1) + (1 - \delta)k$$

We can now rewrite this problem as choosing a sequence of capital of stocks:

$$\begin{aligned}
 &\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}) \\
 &0 \leq k_{t+1} \leq f(k_t)
 \end{aligned}$$

where k_0 is given.

- This is called a "sequence problem" by Stokey and Lucas
- There are infinitely many choice variables, so this problem is very hard to solve.
- Instead we can use dynamic programming. We will solve the problem recursively, and therefore there's only one choice variable (next period's capital).

1.2 Dynamic Programming

Let

$$V(k_0) \equiv \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1})$$

given k_0 . We are maximizing the discounted utility given k_0 . Rewrite as

$$\begin{aligned} V(k_0) &= \max_{\{k_{t+1}\}_{t=0}^{\infty}} \left\{ u(f(k_0) - k_1) + \beta \sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t) - k_{t+1}) \right\} \\ &= \max_{k_1} \{ u(f(k_0) - k_1) + \beta V(k_1) \} \end{aligned}$$

where

$$V(k_1) = \max_{\{k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t) - k_{t+1})$$

1.2.1 Bellman's Equation

$$V(k) = \max_{k'} \{ u(f(k) - k') + \beta V(k') \}$$

This is a functional equation where the unknown is $V(k)$. $V(k)$ is called the *value function*. $u(f(k) - k')$ is called the "return function". Our First Order Condition is

$$u'(f(k) - k') = \beta V'(k').$$

We want to solve for

$$k' = g(k),$$

which is the *policy function*, often called the *optimal policy function* or *decision rule*. Later we'll prove that we can take FOCs because the function is differentiable.

1.2.2 Solving for $V(k)$

1. Guess a function $V_0(k)$.
2. Create a new function by applying a mapping to $V_0(k)$. We apply the operator T , so the new function is

$$T(v_0(k)) = \max_{k'} \{u(f(k) - k') +_\beta V_0(k')\}$$

3. Let

$$V_1(k) = T(V_0(k))$$

4. Repeat, forming a sequence of functions where

$$V_n(k) = T(V_{n-1}(k)),$$

creating (with a computer) the sequence $\{V_n\}_{n=0}^\infty$, where each V_n is a function.

5. Continue creating functions until $V_{n-1}(k)$ and $V_n(k)$ are the same, or close enough for the desired error term.

It turns out that this sequence is a Cauchy sequence, so we're guaranteed to get solutions that are arbitrarily close together. Look up contraction mappings, etc.

2 Lecture 2—October 7, 2020

The T we discussed yesterday is known as the *Bellman Operator*.

Very few cases exist where we can find a fixed point analytically. We can find a fixed point analytically when the return function is quadratic and the constraints are linear. If a closed form exists, we can find it using the *method of undetermined coefficients*. This is Problem 1 in our homework, and involves two steps.

1. Find the functional form for V . For example, if $V_n(k)$ is quadratic and $T(V_n(k))$ is quadratic, $V(k)$ must be quadratic.
2. Find the parameter of the fixed point.

If the operator takes in a quadratic function and outputs a quadratic function, we know the limit must be a quadratic function, so we can guess a quadratic solution and find the parameters. For the homework, think about log-linear functional forms.

Example. Suppose

$$V_n(k) = ak^2 + bk + c$$

and

$$T(V_n(k)) = \tilde{a}k^2 + \tilde{b}k + c.$$

T transforms a quadratic function into another quadratic function, so the fixed point is a quadratic function. How do we find the fixed point? It must be the case that

$$\begin{aligned}\tilde{a} &= f_1(a, b, c) \\ \tilde{b} &= f_2(a, b, c) \\ \tilde{c} &= f_3(a, b, c)\end{aligned}$$

Now we're solving a system of three equations in three unknowns. How do we know if v (the fixed point) exists and is unique? We will use a few concepts from Stokey and Lucas, Chapter 3.

Theorem (Contraction Mapping Theorem). *If (S, ρ) is a complete metric space and $T : S \rightarrow S$ is a contraction mapping with modulus β , then*

1. T has exactly one fixed point $V \in S$, and
2. for any $v_0 \in S$, $\rho(T^n v_0, V) \leq \beta^n \rho(v_0, V)$ for $n = 0, 1, 2, \dots$

Since we've defined β such that $0 \leq \beta < 1$, we can get arbitrarily close to the fixed point by sending $n \rightarrow \infty$.

- S is a sequence of functions
- ρ is a measure of distance between two points in S (a metric)
- A Cauchy sequence is a sequence $\{V_n\}_{n=1}^{\infty}$ of elements of S where, for all $\varepsilon > 0$, there exists N_ε such that $\rho(V_n, V_m) < \varepsilon$ for all $n, m > N_\varepsilon$.
- Complete—every Cauchy sequence in S converges to some element of S .

Definition (Contraction Mapping). A *contraction mapping* $T : S \rightarrow S$ is a contraction if, for some $\beta \in (0, 1)$,

$$\rho(Tx, Ty) \leq \beta \rho(x, y)$$

for any $x, y \in S$.

A *fixed point* is some $v \in S$ such that $Tv = v$.

Problem: While the theorem guarantees that iterating on Bellman's equation will provide a sequence of functions that converges to a unique fixed point, verifying the conditions of the theorem is hard.

Theorem (Blackwell's Sufficient Condition for a Contraction). *Let $X \subseteq \mathbb{R}^n$ and $B(X)$ be a space of bounded functions $f : X \rightarrow \mathbb{R}$ with the sup norm, $\|f\| = \sup_x |f(x)|$. Let $T : B(X) \rightarrow B(X)$ satisfy*

1.

$$f, g \in B(X) \wedge f(x) \leq g(x) \forall x \in X \implies (Tf)(x) \leq (Tg)(x) \forall x \in X \quad (\text{monotonicity})$$

In words, if we have two functions that satisfy the initial assumption on the left, the Bellman operator preserves this ordering.

2. There exists some $\beta \in (0, 1)$ such that

$$[T(f + a)](x) \leq (Tf)(x) + \beta a \quad (\text{discounting})$$

for all $f \in B(X)$ and $a \geq 0$.

Intuitively, T is the sum of two functions: the current return and the future return. We need these functions to get closer and closer to a fixed point. It is sufficient to replace a general bounded function with this scalar and then show that this is a contraction.

Exercise. Verify that the Bellman mapping

$$T : B(X) \rightarrow B(X)$$

from the neoclassical growth model satisfies monotonicity and discounting.

2.1 Envelope Condition and Euler Equation

The FOC for the right hand side of the Bellman equation is

$$u'(f(k) - g(k)) = \beta V'(g(k)),$$

where $k' = g(k)$. The goal here is to go from our sequence problem to the Euler equation. We can get the Euler equation FOC directly by taking derivatives of the sequence problem. Note that the Bellman operator maps concave functions to concave functions.

Theorem (Envelope Theorem). Suppose v is concave and w is concave and differentiable with

$$w(x_0) = v(x_0) \text{ and } w(x) \leq v(x)$$

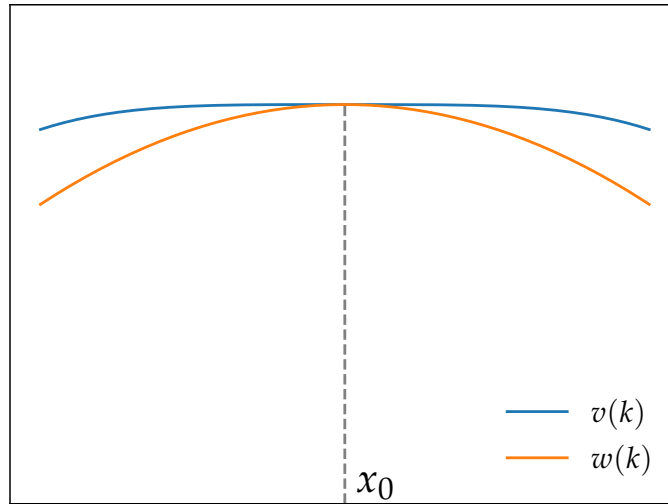
for all x in a neighborhood of x_0 . Then v is differentiable at x_0 and

$$v_i(x_0) = w_i(x_0).$$

v is a function we don't necessarily know, but we know it is concave. w is a function we are going to make up. We want it to be concave and differentiable, and we want the two functions to be exactly the same at x_0 . If we form w this way, we know we can take its derivative, since we designed it to be differentiable. If we take the i^{th} partial of w and evaluate it at x_0 , then its i^{th} partial derivative is exactly the same as the i^{th} partial of v . In the case of the Bellman mapping, let

$$\begin{aligned} w(k) &\equiv u(f(k) - g(k_0)) + \beta V(g(k_0)) \\ \implies w(k_0) &= (k_0) \\ w(k) &\leq v(k) \text{ for } k \text{ near } k_0. \end{aligned}$$

Intuitively, we're holding the future fixed and the only thing that's moving is the current capital stock. We only require this condition to hold locally; the picture below is a zoomed-in version of the Envelope Theorem. We aren't assuming anything about the behavior of either of these functions anywhere else.



Definition (Envelope Condition).

$$V'(k_0) = w'(k_0) = u'(f(k_0) - g(k_0))f'(k_0) \quad (\text{chain rule})$$

in general,

$$v'(k) = u'(f(k) - g(k))f'(k).$$

We can characterize the derivative of the value function without knowing the value function. Combine these two equations, and we get

$$u'(f(k) - g(k)) = \beta u'(f(g(k)) - g(g(k)))f'(g(k))$$

or

$$u'(f(k_t) - k_{t+1}) = \beta u'(f(k_{t+1}) - k_{t+2})f'(k_{t+1}).$$

This is the Euler Equation. The RHS is discounting the marginal utility of capital from next period. β is discounting, and u' is the marginal utility of capital.

It is important that k doesn't appear by itself (will show up later in class).

REVIEW TWO WAYS TO OBTAIN EULER EQUATION

Intuition for $f'(k_{t+1})$: $f(k_{t+1})$ is the output we can produce next period. f' is the derivative of that production function, and we can think of it as the slope of the production function, or the (limit of) additional output we get divided by the additional capital deployed. u' is the additional utility we get from an additional amount of consumption.

There is only one k_1 that will give a sequence, from the Euler equation, that satisfies the transversality condition. We have one Euler equation for each $t \geq 0$, so... infinitely many. This isn't good, since we have too many equations! But the Bellman equation doesn't have this problem. How can we get a similar solution for the Euler equations?

Definition (Transversality Condition).

$$\lim_{t \rightarrow \infty} \beta^t u'(f(k_t) - k_{t+1}) f'(k_t) \cdot k_t = 0$$

Working from right to left, this limit is the capital stock, multiplied by the marginal product of capital. Now we want to turn this into utility, so we multiply it by the utility of the output produced from this stock of capital. Finally, we have to discount this back to today, since this is future output/utility. Intuitively, this whole term is the present value of capital that we have in period t . As we send the limit to infinity, our future capital is worth nothing. Using this condition allows us to pin down the infinite sequence from the Euler equations. If we choose an arbitrary k_1 instead, we can end up with either negative consumption or a negative capital stock, neither of which makes sense. **VERIFY THIS NUMERICALLY**. Mathematically, there is only one k_1 that lies on the saddle path.

Theorem (Sufficiency of the Euler and transversality conditions; Stokey and Lucas 4.15). Let $X \subset \mathbb{R}_+^l$, and let F satisfy the following conditions:

1. X is a convex subset of \mathbb{R}^l , and the correspondence $\Gamma : X \rightarrow X$ is non-empty, compact-valued, and continuous.
2. The function $F : A \rightarrow \mathbb{R}$ is bounded and continuous, and $0 < \beta < 1$.
3. For each y , $F(\cdot, y)$ is strictly increasing in each of its first l arguments.
4. F is strictly concave.
5. F is continuously differentiable on the interior of A .

Then the sequence

$$\{x_{t+1}^*\}_{t=0}^\infty, \text{ with } x_{t+1}^* \in \text{int } \Gamma(x_t^*), t = 0, 1, \dots$$

is optimal for the problem

$$\begin{aligned} \sup_{\{x_{t+1}\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t F(x_t, x_{t+1}) \\ \text{s.t. } x_{t+1} \in \Gamma(x_t), t = 0, 1, 2, \dots, \\ \text{given } x_0 \in X \end{aligned} \tag{SP}$$

if the sequence satisfies

$$0 = F_y(x_t^*, x_{t+1}^*) + \beta F_x(x_{t+1}^*, x_{t+2}^*), \quad t = 0, 1, 2, \dots \tag{2}$$

$$0 = \lim_{t \rightarrow \infty} \beta^t F_x(x_t^*, x_{t+1}^*) \cdot x_t^* \tag{3}$$

Proof. Take x_0 as given, assume $\{x_t^*\} \in \Pi(x_0)$ satisfies 2 and 3, and let $\{x_t\} \in \Pi(x_0)$ be any feasible sequence. It is sufficient to show that the difference (call it D) between the objective function in SP evaluated at $\{x_t^*\}$ and $\{x_t\}$ is nonnegative.

Since F is continuous, concave, and differentiable,

$$\begin{aligned} D &= \lim_{T \rightarrow \infty} \sum_{t=0}^T \beta^t [F(x_t^*, x_{t+1}^*) - F(x_t, x_{t+1})] \\ &\geq \lim_{T \rightarrow \infty} \sum_{t=0}^T \beta^t [F_x(x_t^*, x_{t+1}^*) \cdot (x_t^* - x_t) + F_y(x_t^*, x_{t+1}^*) \cdot (x_{t+1}^* - x_{t+1})]. \end{aligned}$$

Since $x_0^* - x_0 = 0$, rearranging terms gives

$$D \geq \lim_{T \rightarrow \infty} \left\{ \sum_{t=0}^{T-1} \beta^t [F_y(x_t^*, x_{t+1}^*) + \beta F_x(x_{t+1}^*, x_{t+2}^*)] \cdot (x_{t+1}^* - x_{t+1}) + \beta^T F_y(x_T^*, x_{T+1}^*) \cdot (x_{T+1}^* - x_{T+1}) \right\}.$$

Since $\{x_t^*\}$ satisfies 2, all the terms in the summation are zero. Substituting from 2 into the last term and using 3 gives

$$\begin{aligned} D &\geq - \lim_{T \rightarrow \infty} \beta^T F_x(x_T^*, x_{T+1}^*) \cdot (x_T^* - x_T) \\ &\geq - \lim_{T \rightarrow \infty} \beta^T F_x(x_T^*, x_{T+1}^*) \cdot x_T^*. \end{aligned}$$

The last line uses the fact that $F_x \geq 0$ (one of our assumptions) and $x_t \geq 0$ for all t . It follows from 2 that $D \geq 0$, establishing the desired result. \square

2.2 Things to know for exam/course

1. Contraction mapping theorem
2. Won't be tested on complete metric space, Cauchy sequence; we'll be dealing with specific functional forms that meet our assumptions
3. We are expected to be able to verify the Blackwell Sufficient Conditions

3 TA Section—October 9, 2020 (Introduction to Dynamic Programming)

3.1 The Sequential Problem and its Solution

Consider the simple deterministic neoclassical growth model. We don't have prices, just the resource constraint and the capital accumulation equation. We have an infinitely-lived household, so the utility function depends on a sequence of consumption and is time-separable,

$$\sum_{t=0}^{\infty} \underbrace{\beta^t}_{\text{discount factor}} \underbrace{u(c_t)}_{\text{instantaneous utility}}.$$

The agent chooses a sequence of consumption, capital and investment,

$$\{c_t, k_{t+1}, i_t\}_{t=0}^{\infty}$$

subject to

$$\begin{aligned}
c_t + i_t &\leq \underbrace{F(k_t, n_t)}_{\text{predetermined output}} && \text{(resource constraint)} \\
k_{t+1} &= i_t + \underbrace{(1 - \delta)k_t}_{\substack{\text{undepreciated} \\ \text{capital}}} && \text{(capital accumulation)} \\
n_t &\in [0, 1] && \text{(labor)} \\
c_t, k_t &\geq 0 \quad \forall t && \text{(non-negativity constraints)} \\
k_0 &\text{ given}
\end{aligned}$$

We can substitute for investment in our initial problem to get

$$c_t + \{k_{t+1} - (1 - \delta)k_t\} = F_{t+1}.$$

We can further simplify this so that the only thing we're choosing is a sequence of capital levels. Assuming $\delta = 1$, we get

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t (f(k_t) - k_{t+1})$$

with $0 \leq k_{t+1} \leq f(k_t)$ and k_0 given. We can now derive the Euler equation. There are only two terms that depend on k_{t+1} ,

$$\dots \beta^t u(f(k_t) - k_{t+1}) + \beta^{t+1} u(f(k_{t+1}) - k_{t+2}) \dots$$

Now we can take the derivative with respect to k_{t+1} ,

$$\beta^t u'(f(k_t) - k_{t+1}) \cdot -1 + \beta^{t+1} u'(f(k_{t+1}) - k_{t+2}) \cdot f'(k_{t+1}).$$

Simplifying,

$$\underbrace{u'(f(k_t) - k_{t+1})}_{c_t} = \beta \underbrace{f'(k_{t+1})}_{MPk} \cdot \underbrace{u'(f(k_{t+1}) - k_{t+2})}_{c_{t+1}}.$$

In words, the marginal cost of forgone consumption today must be the same as the discounted value of tomorrow's marginal product of capital, plus the marginal utility from consuming tomorrow's output.

3.2 Analytic Solution (full depreciation)

$$\begin{aligned}
u(c) &= \log(c) \\
f(k) &= k^\alpha, \quad \alpha \in (0, 1) \\
u'(c) &= \frac{1}{c} \\
f'(k) &= \alpha k^{\alpha-1} \\
(k_t^\alpha - k_{t+1})^{-1} &= \beta \alpha k_{t+1}^{\alpha-1} (k_{t+1}^\alpha - k_{t+2})^{-1} && \text{(Euler Equation)} \\
\implies k_{t+1}^\alpha - k_{t+2} &= \beta \alpha k_{t+1}^{\alpha-1} (k_t^\alpha - k_{t+1})
\end{aligned}$$

We want

$$k_{t+1} = g(\underbrace{k_t}_{\text{given}}).$$

The limit of the solution to the finite horizon problem is equivalent to the unique solution to the infinite horizon problem.

$$k_{t+1}^\alpha - k_{t+2} = \beta\alpha k_{t+1}^{\alpha-1}(k_t^\alpha - k_{t+1}) \text{ for } t = 1, 2, \dots, T-1.$$

We have $\underbrace{k_0}_{\text{given}}, k_1, \dots, K_T$. This leaves us T variables and $T-1$ equations, so we need to consider the terminal condition,

$$\begin{aligned} \underbrace{\beta^t u'(c_T)}_{>0} \cdot k_{T+1} &= 0 \\ \implies k_{T+1} &= 0. \end{aligned}$$

Consider the Euler Equation for $t = T-1$,

$$\begin{aligned} k_T^\alpha - k_{T+1} &= \beta\alpha k_T^{\alpha-1}(k_{T-1}^\alpha - k_T) \\ k_T^\alpha &= \beta\alpha k_T^{\alpha-1}(k_{T-1}^\alpha - k_T) \\ k_T &= \beta\alpha(k_{T-1}^\alpha - k_T) \\ \implies k_T &= \frac{\beta\alpha}{1 + \beta\alpha} k_{T-1}^\alpha \end{aligned}$$

Iterate on $t = T-2$,

$$k_{T-1} = \frac{\alpha\beta(1 - \alpha\beta)}{(1 + \alpha\beta + (\alpha\beta)^2)} k_{T-2}$$

Taking the limit as $t \rightarrow \infty$,

$$\begin{aligned} k_{t+1} &= \frac{\alpha\beta(1 - (\alpha\beta)^{T_t})}{1 - (\alpha\beta)^{T-t+1}} k_t^\alpha \\ k_{t+1} &= \alpha\beta k_t^\alpha. \end{aligned}$$

The intuition is that

$$\begin{aligned} \text{savings} &= \text{investment} \\ &= \alpha\beta k_t^\alpha \\ s/y &= \alpha\beta \\ \text{consumption} &= (1 - \alpha\beta) k_t^\alpha. \end{aligned}$$

3.3 The Recursive Problem and its Solution

Define a real-valued function $V(k)$,

$$\begin{aligned} V(k_0) &= \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}) \\ \text{s.t. } &0 \leq k_{t+1} \leq f(k_t) \\ \text{given } &k_0 \end{aligned}$$

We want

$$V(k_0) = \text{Return} + \beta \cdot V(k_1).$$

$$V(k_0) = \max_{k_1} \max_{\{k_{t+1}\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1})$$

$$V(k_0) = \max_{k_1} \max_{\{k_{t+1}\}_{t=1}^{\infty}} u(f(k_0) - k_1) + \beta u(f(k_1) - k_2) + \dots$$

$$V(k_0) = \max_{k_1} \max_{\{k_{t+1}\}_{t=1}^{\infty}} \beta u(f(k_0) - k_1) + \beta \sum_{t=1}^{\infty} \beta^t \cdot u(f(k_1) - k_{t+1})$$

$$V(k_0) = \max_{k_1} \beta u(f(k_0) - k_1) + \underbrace{\max_{\{k_{t+1}\}_{t=1}^{\infty}} \beta u(f(k_0) - k_1) + \beta \sum_{t=1}^{\infty} \beta^t \cdot u(f(k_1) - k_{t+1})}_{\beta \max_{\{k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} u(f(k_t) - k_{t+1})}$$

$$V(k_0) = \max_{k_1} u(f(k_0) - k_1) + \beta V(k_1)$$

$$V(k) = \max_{k_1} u(f(k) - k') + \beta V(k')$$

Where k' denotes tomorrow's capital stock. The optimal k' is

$$\begin{aligned} -1 \cdot u'(f(k) - k') + \beta V'(k') &= 0 \\ u'(f(k) - k') &= \beta V'(k') \end{aligned}$$

Suppose

$$\begin{aligned} V(k) &= A + \frac{\alpha}{1 - \alpha\beta} \ln(k) \\ V'(k) &= \frac{\alpha}{1 - \alpha\beta} \cdot k^{-1} \\ V'(k') &= \frac{\alpha}{1 - \alpha\beta} (k')^{-1} \end{aligned}$$

3.4 Habit Persistence

$$u(c_t, c_{t-1})$$

c_{t-1} is given in t , so it is a state variable. For notational convenience, define $d \equiv c_{t-1}$.

$$V(k, d) = \max_{\{c, k'\}} u(c, d) + \beta V(k', d')$$

subject to the constraints

$$\begin{aligned} c + k' &= f(k) \\ d' &= c \end{aligned}$$

3.5 Value Function Iteration (HW in Matlab)