

# Econ202A Homework #2

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1. In this economy, assume that  $r = \delta$ . Prove Hall's Corollary 1 and 2, and 4. In addition, how would you go about estimating the implied regression in Corollary 4? Solving the constrained maximization problem we have the following

$$\mathcal{L} = E_t \sum_{\tau=0}^{T-\tau} \frac{u(c_{t+\tau})}{(1+\delta)^\tau} + \lambda \left( \sum_{\tau=0}^{T-\tau} \frac{c_{t+\tau}}{(1+r)^\tau} - \sum_{\tau=0}^{T-\tau} \frac{w_{t+\tau}}{(1+r)^\tau} - A_t \right)$$

The FOC with respect to  $c_{t+\tau}$  is then given by

$$[c_{t+\tau}] : E_t \frac{u'(c_{t+\tau})}{(1+\delta)^\tau} + \lambda \frac{1}{(1+r)^\tau} = 0$$

$$\tau = 0 \Rightarrow E_t[u'(c_t)] = u'(c_t) = -\lambda$$

$$\tau = 1 \Rightarrow E_t[u'(c_{t+1})] = -\lambda \frac{1+\delta}{1+r}$$

From this two conditions we get

$$E_t[u'(c_{t+1})] = \frac{1+\delta}{1+r} u'(c_t)$$

**Corollary.** *No information available in period  $t$  apart from the level of consumption,  $c_t$ , helps predict future consumption,  $c_{t+1}$ , in the sense of affecting the expected value of marginal utility. In particular, income or wealth in periods  $t$  or earlier are irrelevant, once  $c_t$  is known.*

*Proof.* Once we obtain the equation above, we can conclude that both income or wealth do not predict the future level of consumption in the sense of affecting the expected value of marginal utility. Moreover, considering the case where  $r = \delta$ , we get

$$E_t[u'(c_{t+1})] = u'(c_t)$$

which means that there are no other factors expect current level of consumption that affect the future consumption (assuming that we have time separable utility function, depending on consumption only).  $\square$

**Corollary.** *Marginal utility obeys the regression relation,  $u'(c_{t+1}) = \gamma u'(c_t) + \varepsilon_{t+1}$ , where  $\gamma = (1+\delta)/(1+r)$  and  $\varepsilon_{t+1}$  is a true regression disturbance; that is,  $E_t[\varepsilon_{t+1}] = 0$ .*

*Proof.* Taking the conditional expectation at period  $t$  we get

$$E_t[u'(c_{t+1})] = \gamma E_t[u'(c_t)] + E_t[\varepsilon_{t+1}]$$

Since the consumption at period  $t$  is known,  $E_t[u'(c_t)] = u'(c_t)$ . The relation from constrained maximization problem is that  $E_t[u'(c_{t+1})] = (1+\delta)/(1+r) u'(c_t)$ . Then, indeed, the marginal utility can be expressed as the above stated regression relation with  $\gamma = (1+\delta)/(1+r)$  and  $E_t[\varepsilon_{t+1}] = E_t[u'(c_{t+1})] - \gamma E_t[u'(c_t)] = E_t[u'(c_{t+1})] - (1+\delta)/(1+r) u'(c_t) = 0$ .  $\square$

**Corollary** (Corollary 4 from Hall's paper). *If the utility function has the constant elasticity of substitution form,  $u(c_t) = c_t^{(\sigma-1)/\sigma}$ , then the following statistical model describes the evolution of consumption:  $c_{t+1}^{-1/\sigma} = \gamma c_t^{-1/\sigma} + \varepsilon_{t+1}$ .*

*Proof.* From Corollary 2 we have that the marginal utility of consumption can be represented as the following regression relation

$$u'(c_{t+1}) = \gamma u'(c_t) + \varepsilon_{t+1}$$

Calculating the marginal utility for the constant elasticity of substitution utility form we get

$$\frac{\sigma-1}{\sigma} c_{t+1}^{-1/\sigma} = \gamma \frac{\sigma-1}{\sigma} c_t^{-1/\sigma} + \varepsilon_{t+1}$$

Since  $(\sigma-1)/\sigma$  is a constant, we can rewrite this regression as follows (without loss of generality)

$$c_{t+1}^{-1/\sigma} = \gamma c_t^{-1/\sigma} + \varepsilon_{t+1}^*$$

where  $\varepsilon_{t+1}^* = \sigma/(\sigma-1)\varepsilon_{t+1}$ , and therefore,  $E_t[\varepsilon_{t+1}^*] = E_t[\sigma/(\sigma-1)\varepsilon_{t+1}] = \sigma/(\sigma-1)E_t[\varepsilon_{t+1}] = 0$ .  $\square$

For estimating this regression we propose the log transformation

$$\ln c_{t+1} = \mu + \gamma \ln c_t + \epsilon_{t+1}$$

Then we can apply OLS method to estimate this regression, which is asymptotically equivalent to MLE estimator. In this case, the coefficient will be interpreted as a  $\gamma$ -percentage increase in future consumption in response to a 1-percentage increase in the current consumption.

2. Explain the economic intuition for why the stochastic process for income is irrelevant in terms of being able to forecast future consumption.

Considering stochastic process for income implies that the deviations from some stationary level are unexpected, thus they do not affect consumption (and are actually disturbance). All the expected deviations (change of trend) are incorporated in current consumption, and do not have any additional information that can help predict the future consumption.

3. Explain the economic intuition why if  $r < \delta$ , then consumption evolves as a random walk with positive drift, in which there is a constant term in the regression that is negative.

Having that marginal utility obeys a random walk apart from the trend, we can consider small deviations from the steady state consumption and expand the implicit equation for  $c_{t+1}$  and  $c_t$  in a Taylor series.

$$E_t[u'(\bar{c}) + u''(\bar{c})(c_{t+1} - \bar{c})] = \frac{1 + \delta}{1 + r}[u'(\bar{c}) + u''(\bar{c})(c_t - \bar{c})]$$

which can be rewritten as

$$c_{t+1} = \left( \frac{1 + \delta}{1 + r} - 1 \right) \left( \frac{u'(\bar{c})}{u''(\bar{c})} - \bar{c} \right) + \frac{1 + \delta}{1 + r} c_t + \varepsilon_{t+1}$$

Since  $r < \delta$ , we conclude that  $(1 + \delta)/(1 + r) - 1 > 0$ . Also, we know that by assumption made in Hall's paper utility function is strictly concave, thus  $u''(\bar{c}) < 0$  and  $u'(\bar{c})/u''(\bar{c}) - \bar{c} < 0$ . Hence, this regression can be rewritten as follows

$$c_{t+1} = \mu + \gamma c_t + \varepsilon_{t+1}$$

which is a random walk with a positive drift  $\gamma$  and negative constant  $\mu$ .

Intuitively, this result can be obtained from the fact that under the permanent income hypothesis individual chooses current consumption by estimating the future ability to consume. Thus, any information and changes in income at time  $t$  are unexpected, and make changes in future consumption unpredictable. Also, since the interest rate is lower than the rate of subjective time preference, individuals prefer to consume now, and future consumption responds more than 1 for 1 to a change in current consumption.

4. Obtain quarterly real consumption (in chained dollars) from the U.S. national income and product accounts from 1950 through 2019. Fit the following regression:

$$\ln(c_t) = \mu + \lambda \ln(c_{t-1}) + u_t$$

Attached here is the code used to download and estimate this model.

Listing 1: Testing the Life Cycle-Permanent Income Hypothesis

```

1 function testLifeCycleHypothesis()
2 % Tests the Life Cycle-Permanent Income Hypothesis
3 %
4 % Tests the theory laid out in Hall JPE 1978. Download the
   quarterly real
5 % consumption from FRED and fits a AR(1) model to see if it is
   reasonable.
6 %
7 % Part of HW 2 in 202 A with Chris Ackerman, Ekaterina Gurkova
   , and Ben Prie
8
9 % Ali Haider Ismail, 2020
10
11 %% Setup
12 close all;
13 status = license('test', 'datafeed_toolbox');
14 if ~status
15     error('Datafeed toolbox does not exist. Cannot download data
       dynamically from Fred');
16 end
17 startDate = '01/01/1950';
18 endDate = '12/31/2019';
19 series = 'PCECC96';
20 url = 'https://fred.stlouisfed.org/';
21 c = fred(url);
22 modelOrder = 1;
23
24 %% Download the data
25 % Dwnload Quarterly Real PCE from fred within appropriate
   dates
26 rawData = fetch(c, series, startDate, endDate);
27 % Clean up the data that was downloaded
28 dates = datetime(rawData.Data(:, 1), 'ConvertFrom', 'datenum')
   ;
29 data = timetable(rawData.Data(:, 2), log(rawData.Data(:, 2)),
   'RowTimes', dates);
30 data.Properties.VariableNames = {'RPCE', 'lnRPCE'};
31 Y = data{:, 2};

```

```

32 |
33 | %% Fit the model
34 | % Set model modelOrder AR lags, 0 multiplicative components,
    |     and 0 MA lags
35 | Mdl = arima(modelOrder, 0, 0);
36 | EstMdl = estimate(Mdl, Y);
37 |
38 | %% Test that the model fits well
39 | % Estimate the R squared
40 | [~, ~, E] = calculateR2(EstMdl, Y);
41 |
42 | % Obtain a plot of the autocorrelation in the data
43 | autocorr(Y);
44 | exportgraphics(gcf, 'data-autocorrelation-plot.pdf');
45 | close;
46 |
47 | % Check that there is autocorrelation in the residuals
48 | estimate(Mdl, E);
49 | autocorr(E);
50 | exportgraphics(gcf, 'residual-autocorrelation-plot.pdf');
51 | close;
52 |
53 | %% Appendix 1 – what would a true AR(1) look like
54 | testMdl = arima('Constant', EstMdl.Constant, 'AR', EstMdl.AR);
55 | testMdl.Variance = EstMdl.Variance;
56 | [testY, testE] = simulate(testMdl, 1000);
57 | autocorr(testY)
58 | exportgraphics(gcf, 'data-simulated-autocorrelation-plot.pdf')
    |     ;
59 | close;
60 |
61 | autocorr(testE)
62 | exportgraphics(gcf, 'residual-simulated-autocorrelation-plot.
    |     pdf');
63 | close;
64 |
65 | %% Appendix 2 – does using AR(2) fit better?
66 | % Based on Hall's paper, if the true model is AR(1), AR(2)
    |     should not do better
67 | modelOrder = 2;
68 | Mdl = arima(modelOrder, 0, 0);
69 | EstMdl = estimate(Mdl, Y);
70 | calculateR2(EstMdl, Y);
71 |
72 | %% Run the Box-Ljung test

```

```

73 % see Matlab documentation and Lee's notes for more info
74 stdE = E/sqrt(EstMdl.Variance); % Standardized residuals
75 lags = 10;
76 dof = lags - modelOrder; % One autoregressive parameter
77 [~, pValue] = lbqtest(stdE, 'Lags', lags, 'DOF', dof);
78 fprintf(['The pValue of whether to reject the null hypothesis
          that there\n' ...
          'is no autocorrelation for 10 lags in the residuals is %f\n'
          n'], pValue);
80
81 end
82
83 function [Rsqr, Rsqadj, E] = calculateR2(EstMdl, Y)
84 % Calculates the R^2 and adjusted R^2
85 %
86 % Inputs:
87 %   EstMdl - Arima model
88 %   Estimated Model output from estimate function
89 %   Y - numeric column data or numeric matrix
90 %   Response data
91 %
92 % Source:
93 %   https://stackoverflow.com/a/56497638/5101261
94
95 % Ali Haider Ismail, 2020
96
97 %% Set up
98 n = length(Y);
99
100 %% Get residuals
101 E = infer(EstMdl, Y);
102
103 %% Compute statistics
104 SSquares = dot(E,E);
105 Stotal = dot(Y - mean(Y), Y - mean(Y));
106 Rsqr = 1 - SSquares/Stotal;
107 Rsqadj = 1 - (1-Rsqr)*(n-1)/(n-2);
108
109 fprintf('R-Squared is %f, Adjusted R-Squared is %f\n', Rsqr,
        Rsqadj);
110
111 end

```

The relevant output from Matlab is copied below. The first output comes from fitting testing the model fit of the AR(1) on the data.



Listing 2: Matlab output from AR(1) model on data

1	Effective Sample Size: 280			
2	Number of Estimated Parameters: 3			
3	LogLikelihood: 954.097			
4	AIC: -1902.19			
5	BIC: -1891.29			
6				
7		Value	StandardError	TStatistic
8		PValue		
9				
10	Constant	0.020958	0.0056916	3.6822
11	AR{1}	0.99846	0.0007193	1388.1
12	Variance	6.4235e-05	2.7222e-06	23.597
		0.0002312		
		4.1773e-123		

Here is the output from testing to see if an AR(1) model fits the residual from the above model.

Listing 3: Matlab output from AR(1) model on residuals

1	ARIMA(1,0,0) Model (Gaussian Distribution):			
2				
3		Value	StandardError	TStatistic
4		PValue		
5				
6	Constant	6.1197e-05	0.00049822	0.12283
7	AR{1}	0.060598	0.034226	1.7705
8	Variance	6.4005e-05	2.8928e-06	22.125
		0.90224		
		0.076641		
		1.8103e-108		

5. Do you think that this is a reasonable statistical model of the log of consumption? (Your answer to this question may include a discussion regarding the value of the autoregressive coefficient, the R-square, and whether there is autocorrelation in the  $u_t$  residuals.)

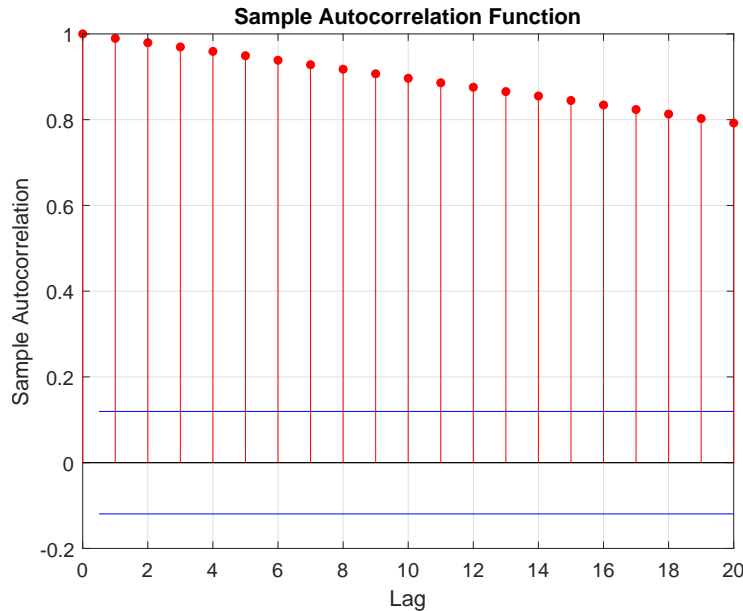


Figure 1: Question 4 - Autocorrelation plot of data

The model fits reasonably well. Firstly, the value of the value of the autoregressive coefficient in listing 2 (i.e the code output) is highly significant (with a p-value near 0 even), along with strong significance in the constant and variance. The R squared is 0.999857 (with the adjusted R being very similar since we are only estimating one lag) suggesting that the model explains the data very well. We also do a rudimentary test to see if the error is white noise in listing 3 which shows that we cannot reject the null hypothesis at the 5% significance that residuals have an AR(1) structure.

In figure 1 we see that the data clearly has a lagged structure. It is to be expected that with an AR(1) with a value for  $\rho$  close to 1, the lagged effects of the shock should be persistent. That is, we should *expect* non-zero auto correlation at all lags, which is in contrast to, say, an MA(q) process which only has non-zero autocorrelation for the first  $q$  lags.

The autocorrelations of the residual of the above model are plotted in figure 2. This figure shows that most of the lags are within the confidence intervals around 0 and hence looks reasonably like white noise.

To verify our intuitions, we simulate an AR model with the same sample moments as the data in figures 3 and 4. Both confirm our findings that the data fits an AR(1) reasonably well.

There are some minor discrepancies when considering the Box-Ljung test that are

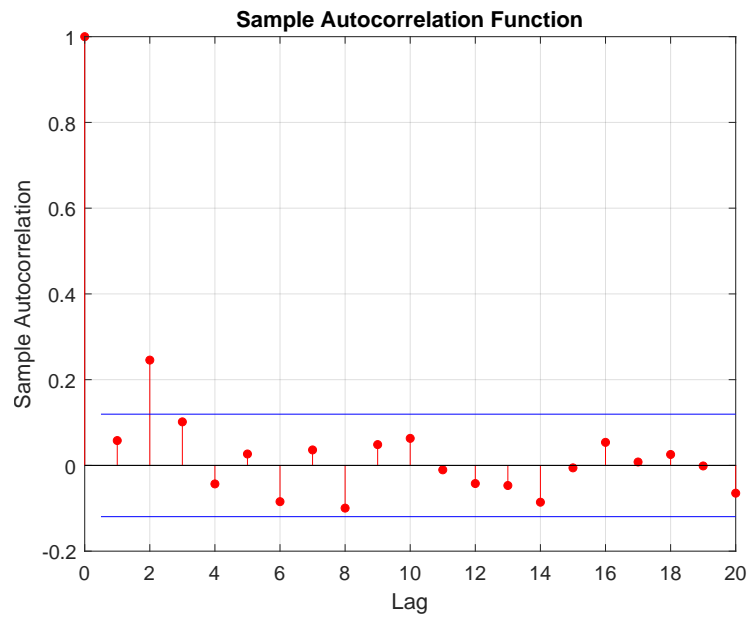


Figure 2: Question 4 - Autocorrelation plot of residuals

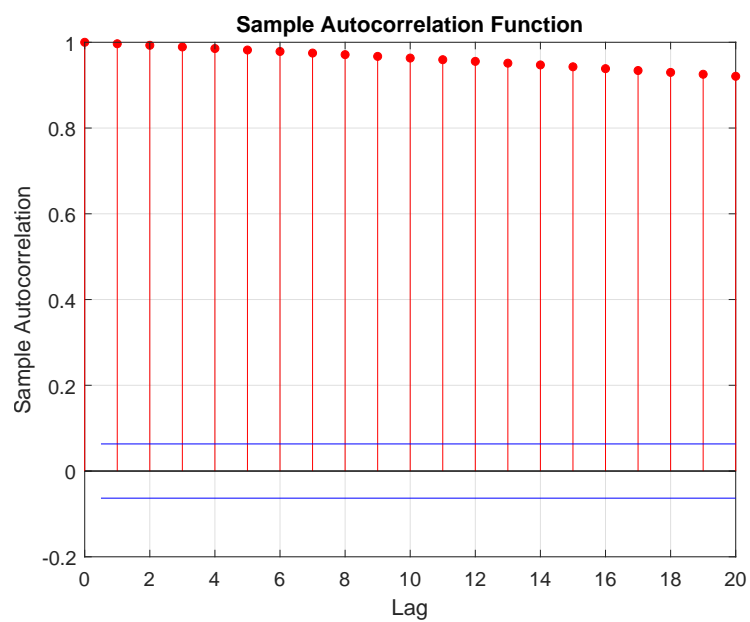


Figure 3: Question 4 - Autocorrelation plot of simulated data

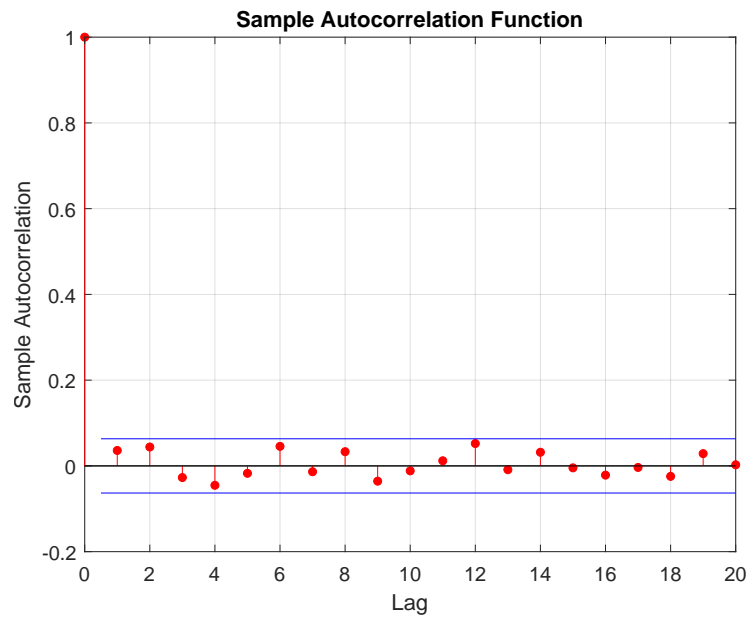


Figure 4: Question 4 - Autocorrelation plot of simulated residuals

persistent even with more lags. Despite that, we still believe that given the evidence that an  $AR(1)$  for consumption is a *reasonable* model.

Next, consider the following economy.

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to

$$\begin{aligned} z_t A_t^{1-\theta} k_t^\theta + (1-\delta)k_t &= c_t + k_{t+1} \\ A_t &= (1+\gamma)^t, \quad t = 0, 1, \dots \\ \ln(z_t) &= \rho \ln(z_{t-1}) + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2) \end{aligned}$$

Assume that the time period is annual. Construct a detrended version of this economy and show the first order conditions. Choose  $\beta$  so that the return to capital in the steady state of the detrended economy is five percent, choose  $\theta$  so that capital's share of income is 30 percent, and choose a depreciation rate such that the share of investment to GDP in the steady state is 20 percent. Choose  $\rho = 0.95$ ,  $\sigma_\varepsilon^2 = .002$  and  $\gamma = 0.02$ .

Rearranging terms, we have

$$\begin{aligned} k_{t+1} &= A_t^{1-\theta} k_t^\theta + (1-\delta)k_t - c_t \\ Y_t &= A_t^{1-\theta} k_t^\theta \\ c_t &= (1-\theta)A_t^{1-\theta} k_t^\theta \end{aligned}$$

To detrend, divide by  $A_t$ . Let's define a few new variables,

$$\begin{aligned} \hat{k}_t &= \frac{K_t}{A_t} \\ \hat{y}_t &= \frac{Y_t}{A_t} \\ \hat{c}_t &= \frac{C_t}{A_t}. \end{aligned}$$

Now, we can substitute these back into the original equations.

$$\begin{aligned} A\hat{k}_{t+1} &= \hat{y}_t + (1-\delta)\hat{k}_t - \hat{c}_t \\ 1 + \gamma\hat{k}_{t+1} &= \hat{y}_t + (1-\delta)\hat{k}_t - \hat{c}_t \\ \hat{y}_t &= \hat{k}_t^\theta \\ \hat{c}_t &= (1-\theta)\hat{y}_t. \end{aligned}$$

First order conditions give us

$$\frac{1}{\hat{c}_t} = \frac{\beta}{1+\gamma} E_t \left\{ \frac{1}{\hat{c}_{t+1}} \left[ \frac{\theta \hat{y}_{t+1}}{\hat{k}_{t+1}} + 1 - \delta \right] \right\}.$$

In the steady state, we have

$$\frac{\bar{c}}{\bar{y}} = \frac{1 + \gamma - \beta(1 - \delta) - \theta\beta(1 + \gamma - 1 + \delta)}{1 + \gamma - \beta(1 - \delta)}. \quad (*)$$

Now let's solve for parameters. We're given  $\gamma = 0.02$ , and we have to figure out  $\beta$ ,  $\theta$  and  $\delta$ . Since we have Cobb Douglas production,  $\theta = 0.3$ . To solve for  $\beta$ , note that the 5% return implies

$$\begin{aligned} \beta &= \frac{1}{1.05} \\ &= 0.95238. \end{aligned}$$

To solve for  $\delta$ , we're going to use equation \*. We're told that investment in the steady state is 20% of GDP, so that implies that consumption is 80% of GDP,

$$\begin{aligned} 0.8 &= \frac{1.02 - 0.95238(1 - \delta) - 0.3 \cdot 0.95238(1.02 - 1 + \delta)}{1.02 - 0.95238(1 - \delta)} \\ \implies \delta &= .082. \end{aligned}$$

6. Log-linearize this model around its deterministic steady state. (For simplicity, assume that  $z$  in the steady state is 1).

$$\text{Define } \tilde{x} \equiv \log \left( \frac{\hat{x}}{\bar{x}} \right).$$

From the Euler equation, we have

$$\frac{\hat{c}_{t+1}}{\hat{c}_t} = \frac{\beta}{A} E_t [\theta z_{t+1} \hat{K}_{t+1}^{\theta-1} + 1 - \delta].$$

Substituting our log linearization into the left-hand side, we have

$$\begin{aligned} \frac{\bar{c} \exp(\tilde{c}_{t+1})}{\bar{c} \exp(\tilde{c}_t)} &\approx (1 + \tilde{c}_{t+1})(1 - \tilde{c}_t) & (\text{LHS}) \\ &\approx 1 + \tilde{c}_{t+1} - \tilde{c}_t. & (\text{LHS}) \end{aligned}$$

Doing the same thing on the right-hand side, we have

$$\frac{\beta}{A} E_t [\theta \bar{z} (1 + \tilde{z}_{t+1}) \bar{K} (1 + (\theta - 1) \tilde{K}_{t+1}) + 1 - \delta] = \frac{\beta}{A} E_t [\theta \bar{z} \bar{K}^{\theta-1} (\theta - 1) \tilde{K}_{t+1} + \theta \bar{z} \bar{K}^{\theta-1} \tilde{z}_{t+1} + 1 - \delta]$$

In the steady state,

$$\begin{aligned} 1 &= \frac{\beta}{A} (\theta \bar{z} \bar{K}^{\theta-1} + 1 - \delta), \\ \bar{z} &= 1. \end{aligned}$$

We can use these to simplify the log-linearized Euler equation:

$$\tilde{c}_{t+1} - \tilde{c}_t = \frac{\beta}{A} E_t [\theta \bar{K}^{\theta-1} (\theta - 1) \tilde{K}_{t+1} + \theta \bar{K}^{\theta-1} \tilde{z}_{t+1}].$$

Now, let's do the same thing to the budget constraint.

$$\begin{aligned} \hat{c}_t + A \hat{K}_{t+1} &= z_t \hat{K}_t^\theta + (1 - \delta) \hat{K}_t \\ \bar{c} (1 + \tilde{c}_t) + A \bar{K} (1 + \tilde{K}_{t+1}) &= \bar{c} + A \bar{K} + \bar{c} \tilde{c}_t + A \bar{K} \tilde{K}_{t+1} & (\text{LHS}) \\ \bar{z} (1 + \tilde{z}_t) \bar{K}^\theta (1 + \theta \tilde{K}_t) + (1 - \delta) \bar{K} (1 + \tilde{K}_t) &= \bar{z} \bar{K}^\theta + \bar{z} \bar{K}^\theta \theta \tilde{K}_t + \bar{z} \bar{K}^\theta \tilde{z}_t + (1 - \delta) \bar{K} + (1 - \delta) \bar{K} \tilde{K}_t. & (\text{RHS}) \end{aligned}$$

In the steady state,

$$\begin{aligned} \bar{c} + A \bar{K} &= \bar{z} \bar{K}^\theta + (1 - \delta) \bar{K}, \\ \bar{z} &= 1, \end{aligned}$$

so we can simplify this expression to

$$\bar{c}\tilde{c}_t + A\bar{K}\tilde{K}_{t+1} = \bar{K}^\theta\theta\tilde{K}_t + \bar{K}^\theta\tilde{z}_t + (1-\delta)\bar{K}\tilde{K}_t,$$

or

$$\tilde{k}_{t+1} = \frac{\bar{K}^{\theta-1}}{A}\theta\tilde{K}_t + \frac{\bar{K}^{\theta-1}}{A}\tilde{z}_t + \frac{1-\delta}{A}\hat{K}_t - \frac{\bar{c}}{A\bar{K}}\tilde{c}_t.$$

Finally, for the stochastic process,

$$\begin{aligned}\ln(z_t) &= \rho \ln(z_{t-1}) + \varepsilon_t \\ \ln(\bar{z} \exp(\tilde{z}_t)) &= \rho \ln(\bar{z} \exp(\tilde{z}_{t-1})) + \varepsilon_t \\ \ln(\bar{z}) + \tilde{z}_t &= \rho \ln(\bar{z}) + \rho \tilde{z}_{t-1} + \varepsilon_t \\ \implies \tilde{z}_t &= \rho \tilde{z}_{t-1},\end{aligned}$$

or

$$\tilde{z}_{t+1} = \rho \tilde{z}_t.$$

Putting everything together, the log-linearized version of this economy is

$$\begin{aligned}\tilde{c}_{t+1} &= E_t \left\{ \frac{\beta\theta\bar{K}^{\theta-1}}{A} \left( (\theta-1) \left[ \frac{\theta\bar{K}^{\theta-1} + 1 - \delta}{A}\tilde{K}_t + \frac{\bar{K}^{\theta-1}}{A}\tilde{z}_t - \frac{\bar{c}}{A\bar{K}}\tilde{c}_t \right] + \rho\tilde{z}_t \right) + \tilde{c}_t \right\} \\ \hat{k}_{t+1} &= \frac{\theta\bar{K}^{\theta-1} + 1 - \delta}{A}\hat{K}_t + \frac{\bar{K}^{\theta-1}}{A}\tilde{z}_t - \frac{\bar{c}}{A\bar{K}}\tilde{c}_t \\ \tilde{z}_{t+1} &= \rho\tilde{z}_t.\end{aligned}$$



7. Use the formula of Blanchard and Kahn to show that there is a unique stationary solution to the linearized system.

Listing 4: Using BK to show there is a unique stationary solution

```

1 clear;
2 m = 2; % number of pre-determined variables: k, z
3 n = 1; % number of forward-looking variables: c
4
5 % values for calibration
6 barR = 1.05;
7 ItoYratio = 0.2;
8 RKtoYratio = 0.3;
9 ybar = 1;
10 CtoYratio = 1 - ItoYratio;
11
12 % calibration
13 bbeta = 1 / barR;
14 ttheta = RKtoYratio;
15 ggamma = 0.02;
16 ddelta = 1 - ((1 + ggamma) * (CtoYratio - 1 + bbeta * ttheta))
    / (bbeta * ttheta + CtoYratio * bbeta - bbeta);
17
18 % steady state
19 Kss = (bbeta * ttheta / ((1 + ggamma) - bbeta * (1 - ddelta)))
    ^ (1 - ttheta);
20 Yss = Kss^ttheta;
21 Css = Yss + (1 - ddelta - (1 + ggamma)) * Kss;
22 Rss = barR;
23
24 % define matrices
25
26 rrho = 0.95;
27 M31 = 0;
28 M32 = 0;
29 M33 = rrho;
30
31 M23 = Kss^(ttheta - 1) / (1 + ggamma);
32 M22 = (ttheta * Kss^(ttheta - 1) + 1 - ddelta) / (1 + ggamma);
33 M21 = -Css / ((1 + ggamma) * Kss);
34
35 tmp = (bbeta * ttheta * Kss^(ttheta - 1) / (1 + ggamma));
36 M13 = tmp * ((ttheta - 1) * M23 + M33);
37 M12 = tmp * (ttheta - 1) * (ttheta * Kss^(ttheta - 1) + 1 -
    ddelta) / (1 + ggamma);
38 M11 = 1 - tmp * (ttheta - 1) * Css / ((1 + ggamma) * Kss);

```

```

39
40 % diagonalize
41 M = [M11, M12, M13; M21, M22, M23; M31, M32, M33];
42 [Gamma, Lambda] = eig(M);
43 Lambda = diag(Lambda);
44 % sort eigenvalues in ascending order
45 [unused, order] = sort(abs(Lambda), 'ascend');
46 % reorder J and make diagonal again
47 Lambda = diag(Lambda(order));
48 % reorder eigenvectors
49 Gamma = Gamma(:, order);
50
51 % check number of eigenvalues outside unit circle equal to n
52 if(sum(abs(diag(Lambda)) > 1) ~= n)
53     return;
54 end
55
56 if(sum(abs(diag(Lambda)) > 1) == n)
57     disp('By Blanchard Khan proposition 1, we have a unique solution. ');
58 end
59
60 % partition matrices
61 Gammainv = inv(Gamma);
62 G11 = Gammainv(1 : m, 1 : n);
63 G12 = Gammainv(1 : m, n + 1 : m + n);
64 G21 = Gammainv(m + 1 : m + n, 1 : n);
65 G22 = Gammainv(m + 1 : m + n, n + 1 : m + n);
66 Lambda1 = Lambda(1 : m, 1 : m);
67 Lambda2 = Lambda(m + 1 : m + n, m + 1 : m + n);
68
69 % state variables solution
70 %  $E x_{t+t} = H x_t$ 
71 H = inv(-G11 * inv(G21) * G22 + G12) * Lambda1 * (-G11 * inv(
    G21) * G22 + G12);
72 % Jump Variables : Policy function
73 THETA = -inv(G21) * G22;
74
75 % Simulation
76 Mtilde11 = M11;
77 Mtilde12 = [M12, M13];
78 Mtilde21 = [M21 ; M31];
79 Mtilde22 = [M22, M23; M32, M33];
80 A12 = Mtilde11 * THETA + Mtilde12;
81 A22 = Mtilde21 * THETA + Mtilde22;
82 A = [zeros(n, size(n, n)), A12 ;

```

```

83     zeros(m, n), A22 ];
84 T =100; % number of periods
85 x2v(1, 1) = 0;
86 x2v(1, 2) = 0;
87 x1v(1, 1) = THETA * x2v (1 ,:)' ;
88 X = zeros(T, m + n);
89 error = normrnd(0, 0.01, [T, 1]);
90 X(1, :) = [x1v ,x2v];
91 for i = 2 : 1 : T
92 X(i ,:) = (A * X(i - 1, :) ' + [0, 0, error(i)]')';
93 end
94 GDP = Yss * exp((X(:, 2) * ttheta + X(:, 3)));
95 plot(linspace(0, T, T), GDP, 'LineWidth', 3, 'Color', 'k')
96 grid on
97 xlabel('Time')
98 ylabel('GDP t')
99 hold on
100 plot(linspace(0, T, T), GDP * 0 + Yss, 'LineWidth', 3, 'Color'
    , 'r')
101 legend('GDP (levels)', 'Yss')

```

8. Using a random number generator (Matlab has a built-in function for this), draw 1100 values of  $\varepsilon$  to construct the  $z$  process. Using these values of  $z$ , and assuming that  $k_0$  is equal to its steady state value, use the linearized system to construct 1100 values of output, consumption, and investment.

The answer to this problem is written in Python.

```

1  import matplotlib
2  import matplotlib.pyplot as plt
3  import numpy as np
4  import pandas as pd
5  matplotlib.rcParams['text.usetex'] = True
6
7
8  def simulate_log_linear_economy(
9      Y_ss=1.15188802402873,
10     rho=0.95,
11     beta=0.952380952380952,
12     delta=0.082,
13     mu=0,
14     sigma=0.002,
15     n_obs=1100,
16     initial_k_tilde=0,
17     initial_c_tilde=0,
18     initial_z_tilde=0,
19     my_seed=100
20 ):
21     """
22     Takes in parameter values.
23     Returns dataframe with a simulated log-linear economy.
24     All  $x_t$  are defined as log deviations from the steady state.
25     """
26     # initialize values for c, k, and z
27     # lists are vectors with one observation for each period.
28     c = [initial_c_tilde]
29     k = [initial_k_tilde]
30     z = [initial_z_tilde]
31     c_tilde = initial_c_tilde
32     k_tilde = initial_k_tilde
33     z_tilde = initial_z_tilde
34     t = 0
35     # set seed; draw random shocks
36     np.random.seed(my_seed)
37     epsilon = np.random.normal(mu, sigma, n_obs)
38     while t < n_obs:
39         # use the analytic formulas to calculate next period values

```

```

40     A = (1 + gamma) ** t
41     c_t = (beta * theta * K_ss ** (theta - 1)) * (1 / A) * \
42           ((theta - 1) * \
43            (((theta * K_ss ** (theta - 1) + 1 - delta)/A) * k_tilde + \
44             (K_ss ** (theta - 1))/A * z_tilde - \
45             C_ss/(A * K_ss) * c_tilde)) + \
46             rho * z_tilde + \
47             c_tilde
48
49     k_t = ((theta * K_ss ** (theta - 1) + 1 - delta)/A) * k_tilde + \
50           ((K_ss ** (theta - 1))/A) * z_tilde - \
51           (C_ss/(A * K_ss)) * c_tilde
52
53     z_t = rho * z_tilde + epsilon[t]
54
55     # store the new values in each variable's list
56     c.append(c_t)
57     k.append(k_t)
58     z.append(z_t)
59
60     # update the old values with the new values
61     k_tilde = k_t
62     c_tilde = c_t
63     z_tilde = z_t
64
65     # advance time period
66     t += 1
67
68     # send the variables to a dataframe
69     economy = pd.DataFrame(index=range(n_obs + 1))
70     economy['c'] = c
71     economy['k'] = k
72     economy['z'] = z
73     return economy
74
75
76 def graph_log_linear_economy(
77     log_linear_economy,
78     filename='log-linear-simulations.pdf'
79 ):
80     fig, ax = plt.subplots(1, 1)
81     ax.plot(log_linear_economy.index, log_linear_economy['c'],
82            label=r'$\tilde{c}$')
83     ax.plot(log_linear_economy.index, log_linear_economy['k'],
84            label=r'$\tilde{k}$')

```

```

85     ax.plot(log_linear_economy.index, log_linear_economy['z'],
86             label=r'$\tilde{z}$')
87     ax.legend(frameon=False)
88     if filename:
89         plt.savefig(filename)
90
91
92     def remove_log_linearization(log_linear_economy):
93         my_economy = pd.DataFrame(index=log_linear_economy.index)
94         my_economy['A'] = [(1 + gamma)**t for t in my_economy.index]
95         my_economy['C'] = my_economy['A'] * C_ss * np.exp(log_linear_economy['c'])
96         my_economy['K'] = my_economy['A'] * K_ss * np.exp(log_linear_economy['k'])
97         my_economy['Y'] = my_economy['A'] ** (1 - theta) * my_economy['K'] ** theta
98         my_economy['I'] = my_economy['Y'] - my_economy['C']
99         return my_economy
100
101
102     def graph_my_economy(my_economy, filename='my-economy-simulations.pdf'):
103         fig, ax = plt.subplots(1, 1)
104         ax.plot(my_economy.index, my_economy['C'],
105                 label=r'$C$')
106         ax.plot(my_economy.index, my_economy['K'],
107                 label=r'$K$')
108         ax.plot(my_economy.index, my_economy['Y'],
109                 label=r'$Y$')
110         ax.plot(my_economy.index, my_economy['I'],
111                 label=r'$I$')
112         ax.legend(frameon=False)
113         if filename:
114             plt.savefig(filename)
115
116
117     if __name__ == '__main__':
118         # These parameter values used in multiple functions
119         C_ss=0.988469588504255
120         K_ss=1.60214152474980
121         theta=0.3
122         gamma=0.02
123         log_linear_economy = simulate_log_linear_economy()
124         graph_log_linear_economy(log_linear_economy)
125         my_economy = remove_log_linearization(log_linear_economy)
126         graph_my_economy(my_economy)
127         my_economy.to_csv('simulated_economy.csv')
128         log_linear_economy.to_csv('simulated_log_linear_economy.csv')

```

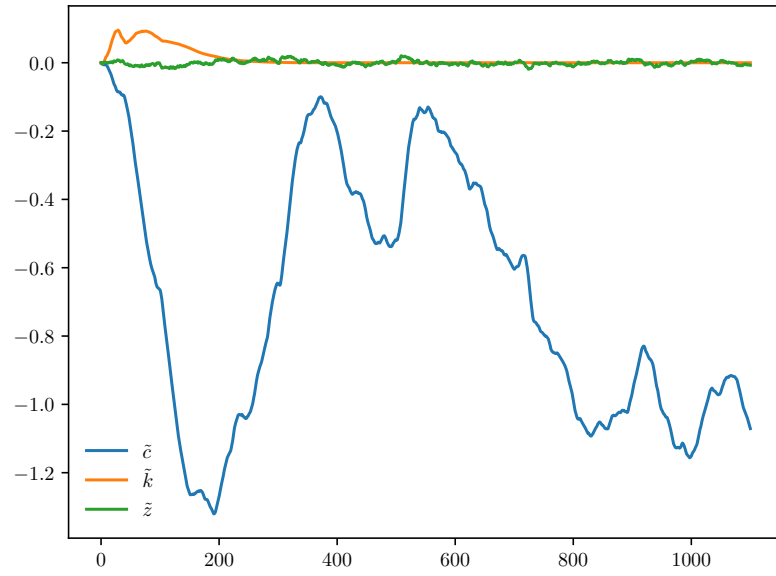


Figure 5: Simulated log deviations

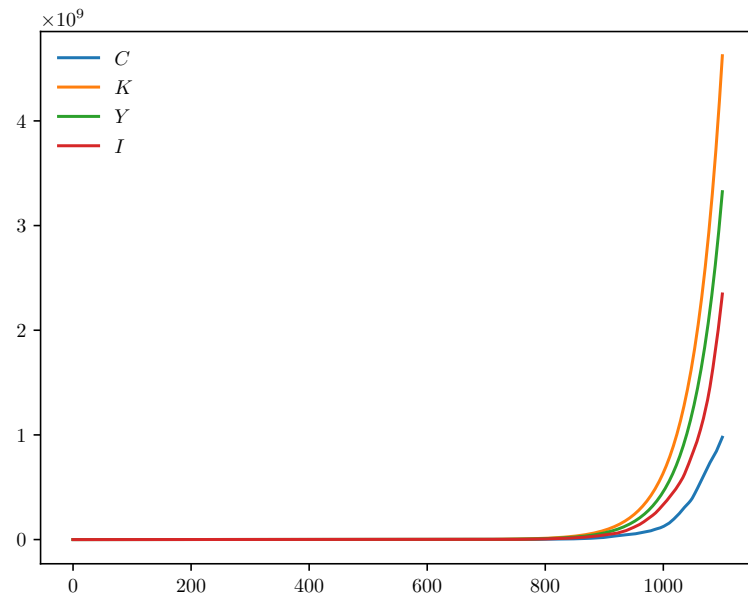


Figure 6: Simulated Consumption, Investment, Output, and Capital

9. Discard the first 100 observations, and then fit an AR(1) process to the log of consumption, measured as the log-deviation of consumption from the steady state value. Report the value of the AR(1) coefficient in the regression, and evaluate whether there is autocorrelation in the residuals.

Listing 5: Fitting AR(1) to Simulated Data

```

1 readtable('simulated_log_linear_economy.csv');
2
3 %%drop first 100 obs
4 ans(1:101,:) = [];
5 c = ans(:, 2);
6
7 %% Fit the model
8 % Set model 1 AR lags, 0 multiplicative components, and 0 MA
   lags
9 Mdl2 = arima(1, 0, 0);
10 EstMdl2 = estimate(Mdl2, c);
11
12 %infer the residuals
13 res=infer(EstMdl2, c);
14
15 %plot autocorrelation in residuals;
16 autocorr(res);

```

ARIMA(1,0,0) Model (Gaussian Distribution):

	Value	StandardError	TStatistic	PValue
<b>Constant</b>	-0.00039845	0.00054316	-0.73358	0.46321
<b>AR{1}</b>	1	0.00069589	1437	0
<b>Variance</b>	4.3745e-05	1.9158e-06	22.833	2.1341e-115



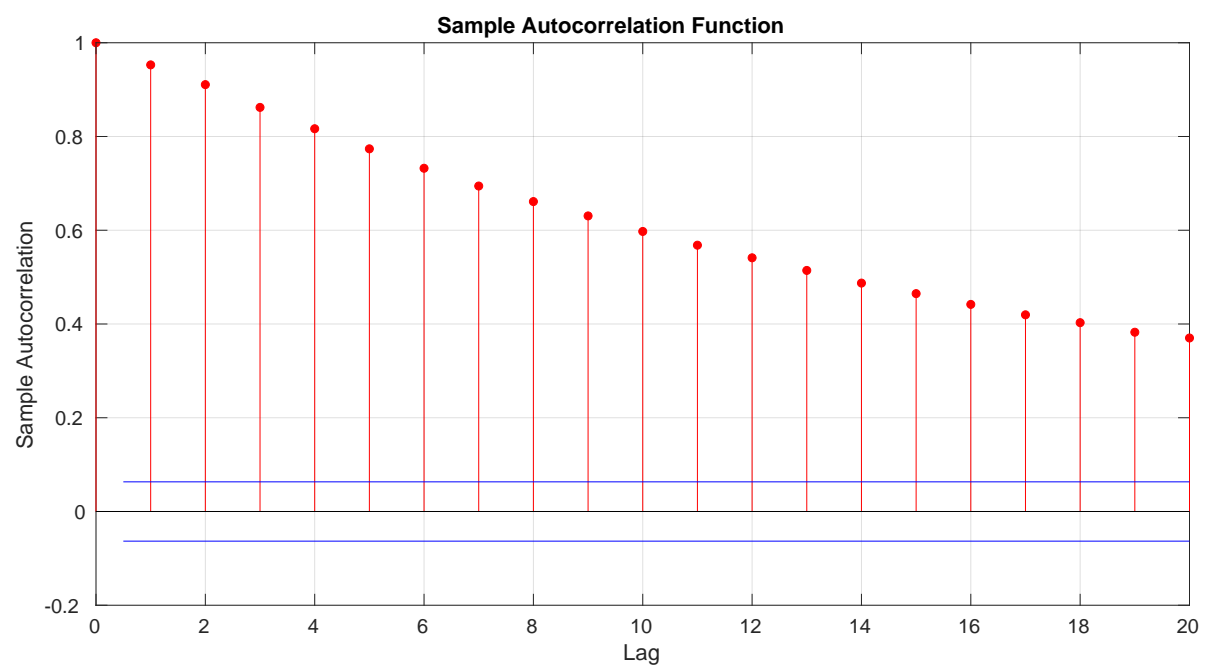


Figure 7: Question 9 - Residual Autocorrelation

10. Compare the regression coefficient in (9) and your assessment of the autocorrelation in the residuals, to your answers in (4) and (5). Does the RBC model provide a good approximation to consumption dynamics? What does it tell us about using consumption data to try to discriminate between the Hall