ECON 202A : Week 4

Paula Beltran*

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In the previous notes, we discuss and characterized the *Recursive Competitive Equilibrium*. The models we have considered in class are stationary since there is no source of growth. In these notes we will discuss and show you how to stationarize a model with *long run growth*. We then turn to review some practice problems.

1 Balanced Growth

In this section we discuss how to handle the neoclassical growth models with exogenous growth. In principle, the introduction of exogenous technological progress and population growth complicates the use of dynamic programming techniques because the problem becomes non-stationary. That is, the value of variables depend crucially on time and the agent confronts a problem which is no more recursive. Our goal is to recast the non-stationary problem, using a change of variables, so that our dynamic programming tools are once again applicable.

In past lectures, we have seen that, starting from any nonzero initial capital stock, the neoclassical growth model without exogenous technological progress exhibits transitional growth as the capital stock monotonically converges to its unique steady state level. Importantly, this transitional growth is endogenous, and is characterized by non-constant growth rates, as the economy's growth rate declines as it approaches the steady state. Moreover, it is a short-run phenomenon, because eventually the capital stock will converge to its unique steady state value and will remain there forever. That is, the long-run growth rate will be zero.

Meanwhile, in these notes, we concentrate on the model with exogenous growth in the long-run. In this model, the technology progresses exogenously, which makes the economy grow not only as a transition to the steady state, but also "at the steady state". To be empirically relevant, we would like the model to match some facts found in the data, which will be introduced in the next section. After that, we will cover how to fully characterize the balanced growth path of the economy. To do this, we must solve for the growth rates of all

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growing variables in the economy; solve for the steady state values of these variables; and finally, reconstruct the original model variables from their steady state values and growth rates.

1.1 Empirical Facts and Restrictions on Functions

Nicholas Kaldor (1957) provides a set of empirical regularities[1] about economic growth as follows:

(1) output per worker shows continuing growth,

(2) capital per worker shows continuing growth,

(3) the rate of return on capital is steady,

(4) the capital-output ratio is steady,

(5) labor and capital receive constant shares of total income,[2]

(6) there are wide differences in the rate of growth of productivity across countries.

To be consistent with those facts, we will often consider introducing exogenous growth through labor-augmenting technological progress. This is because, given our assumption of constant returns to scale, in addition to the facts that capital and output growth at roughly the same rates and per capita hours worked are constant, it must be that technological progress occurs by augmenting labor.

Also, it has been shown in class that, in order for the model to exhibit balanced growth, the preference of the households must be characterized by one of the following utility functions[3]:

$$\begin{aligned} U(c,l) &= c^{1-\sigma}v(l) \qquad \text{for } \sigma \neq 1, \\ U(c,l) &= \log(c) + v(l) \qquad \text{for } \sigma = 1. \end{aligned}$$

That is, the utility must exhibit constant relative risk aversion on consumption. Also, it must be additively separable with leisure when CRRA parameter is 1. Otherwise, consumption and leisure must be non-separable.

1.2 Characterizing the Balanced Growth Path

In this section, we cover how to fully characterize the balanced growth path of an economy. Although I will consider an economy in which all variables end up growing at the same rate,

[1] These facts are usually called "Kaldor's stylized facts"

[2] For some developing countries, labor income share shows some trends.

[3] See King, Plosser and Rebelo, 1988, JME

this will not always the case! So we should not begin with the assumption that all variables grow at the same rate.

$$\max \sum_{t=0}^{\infty} \beta^t N_t U(c_t, 1-h_t)$$

subject to

$$N_t c_t + N_t i_t = N_t k_t^{\alpha} \left(\gamma^t h_t\right)^{1-\alpha} \ , \ \gamma > 1 \tag{1}$$

$$N_{t+1} k_{t+1} = N_t i_t + (1-\delta) N_t k_t \tag{2}$$

$$N_{t+1} = \eta N_t \ , \ \eta > 1$$

$$N_0 = 1$$

$$0 \leq h_t \leq 1$$

$$k_0 \text{ given}$$

This economy will exhibit unbounded growth, so the planner's decision depends on time t. Therefore, we cannot construct a recursive formulation of this model. So we are going to divide every variable by its growth rate and then construct a stationary version of this problem so that we can use the same technique of Bellman equation that we have done so far.

Before I move on, let me eliminate N_t of this model using the assumption that $N_0 = 1$ and $N_{t+1} = \eta N_t$. Combining those two, we have $N_t = \eta^t$. Therefore, the preference becomes:

$$\max \sum_{t=0}^{\infty} (\beta\eta)^t U(c_t, 1-h_t)$$

and (1) becomes

$$c_t + i_t = k_t^{\alpha} \left(\gamma^t h_t\right)^{1-\alpha} \tag{3}$$

and finally, (2) becomes

$$\eta k_{t+1} = i_t + (1-\delta) k_t. \tag{4}$$

Combining (3) and (4) yields

$$c_t + \eta k_{t+1} = k_t^{\alpha} \left(\gamma^t h_t\right)^{1-\alpha} + (1-\delta) k_t. \tag{5}$$

1.3 Construction of the Stationary Problem

I will follow a simple process to stationarize this problem which will allow us to solve the associated Bellman equation:

1. Let every variable grow at a different rate, define "hatted" variables which will be constant along the steady state.

2. Substitute these definitions into the resource constraints to solve for the growth rates necessary to get rid of time dependences.

3. Plug these growth rates into the objective function, and rewrite the objective (make sure it is well-defined and non-explosive).

Following the above steps, let's detrend the problem:

Step 1: Let g_c and g_k be the growth factors of c_t and k_t in the steady state[4]. Define

$$\hat{c}_t = c_t / g_c^t \quad , \quad \hat{k}_t = k_t / g_k^t$$

and leave h_t as is[5]. By construction, $\hat{c}_t$ and $\hat{k}_t$ are stationary (i.e. they do not grow over time.). Here, we conjecture that consumption and capital grow at constant but potentially different rates and that hours worked does not grow over time. That is, we do not impose that both consumption and capital grow at the same rate (though we know they will in this simple model).

Step 2: Using the change of variables in the per capita resource constraint (5),

$$g_c^t \hat{c}_t + \eta g_k^{t+1} \hat{k}_{t+1} = \left(g_k^t \hat{k}_t\right)^{\alpha} \left(\gamma^t h_t\right)^{1-\alpha} + (1-\delta) g_k^t \hat{k}^t$$

Dividing both sides by g_c^t gives us:

$$\hat{c}_t + \eta g_k \left(\frac{g_k}{g_c}\right)^t \hat{k}_{t+1} = \left(\frac{g_k^{\alpha} \gamma^{1-\alpha}}{g_c}\right)^t \hat{k}_t^{\alpha} h_t^{1-\alpha} + (1-\delta) \left(\frac{g_k}{g_c}\right)^t \hat{k}_t$$

In order for this constraint to be stationary, it must be that all terms that are raised to the t power are equal to 1. This requires that $g_k = g_c$ and $(g_k^{\alpha} \gamma^{1-\alpha})/g_c = 1$, or that $g_c = g_k = \gamma$. Imposing this requirement, we can write the stationary resource constraint as

$$\hat{c}_t + \eta \gamma \hat{k}_{t+1} = \hat{k}^{\alpha} h_t^{1-\alpha} + (1-\delta) \hat{k}_t. \tag{6}$$

Step 3: Suppose the utility takes the following form

$$U(c_t, 1-h_t) = \frac{\left(c_t^{\psi} (1-h_t)^{1-\psi}\right)^{1-\sigma}}{1-\sigma}.$$

Substitute the hatted variables. Then the objective function becomes

$$\sum_{t=0}^{\infty} (\beta \eta)^t \frac{\left[\left(\gamma^t \hat{c}_t\right)^{\psi} (1-h_t)^{1-\psi}\right]^{1-\sigma}}{1-\sigma},$$

which is equal to

$$\sum_{t=0}^{\infty} \left(\beta \eta \gamma^{\psi(1-\sigma)}\right)^t \frac{\left(\hat{c}_t^{\psi} (1-h_t)^{1-\psi}\right)^{1-\sigma}}{1-\sigma}.$$

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[4]Here, "growth factor" means one plus growth rate.

[5]If h_t grows at a constant rate at the steady state, then its growth rate must be zero, or $g_h^t = 1$. This is because the domain of h_t is $[0, 1]$, a bounded set. If $g_h^t \neq 1$, then h_t will collapse to zero or explode beyond one.

So, what we did is like combining the growing term with the discount factor to obtain a modified discount factor of $\left(\beta\eta\gamma^{\psi(1-\sigma)}\right)^t$. In order for the household to discount future utility, we require $\beta\eta\gamma^{\psi(1-\sigma)} < 1$.

We have thus recast the non-stationary sequence problem as a problem with both stationary preferences and a stationary constraint. The Bellman equation gives us the dynamic program

$$V(\hat{k}) = \max_{\hat{k}',h} \left[\frac{\left(\hat{c}^{\psi}(1-h)^{1-\psi}\right)^{1-\sigma}}{1-\sigma} + \beta\eta\gamma^{\psi(1-\sigma)}V(\hat{k}') \right]$$

subject to

$$\hat{c} + \eta\gamma\hat{k}' = \hat{k}^{\alpha}h^{1-\alpha} + (1-\delta)\hat{k}.$$

This recursive problem is similar to the previous problems we have seen in this class. All hatted variables are stationary by construction, and thus, we can apply all the tools that we have learned to analyze dynamic programs.

2 Steady State

In order to characterize the steady state, it is important to ensure that you have enough equilibrium equations to uniquely pin down the steady state variables. These equations will consist of the first-order conditions for optimality, as well as any additional unused constraints of the economy. Consider the Lagrangian form of the above dynamic program, where λ represents the Lagrange multiplier,

$$\mathcal{L} = \frac{\left(\hat{c}^{\psi}(1-h)^{1-\psi}\right)^{1-\sigma}}{1-\sigma} + \beta\eta\gamma^{\psi(1-\sigma)}V(\hat{k}') + \lambda\left[\hat{k}^{\alpha}h^{1-\alpha} + (1-\delta)\hat{k} - \hat{c} - \eta\gamma\hat{k}'\right].$$

Then, the first-order conditions are:

$$[\hat{c}]: \qquad \psi\hat{c}^{\psi(1-\sigma)-1}(1-h)^{(1-\psi)(1-\sigma)} = \lambda \tag{7}$$

$$[h]: \qquad (1-\psi)\hat{c}^{\psi(1-\sigma)}(1-h)^{(1-\psi)(1-\sigma)-1} = \lambda(1-\alpha)\hat{k}^{\alpha}h^{-\alpha} \tag{8}$$

$$[\hat{k}']: \qquad \lambda\eta\gamma = \beta\eta\gamma^{\psi(1-\sigma)}V'(\hat{k}') \tag{9}$$

The envelope condition is given by

$$V'(\hat{k}) = \frac{\partial\mathcal{L}}{\partial\hat{k}} = \lambda\left(\alpha\hat{k}^{\alpha-1}h^{1-\alpha} + 1 - \delta\right). \tag{10}$$

Combining (9) and (10) gives the intertemporal Euler equation

$$\lambda\gamma = \beta\gamma^{\psi(1-\sigma)}\lambda'\left[\alpha\left(\hat{k}'\right)^{\alpha-1}\left(h'\right)^{1-\alpha} + 1 - \delta\right]. \tag{11}$$

Substitute (7) into (8) and (11), and evaluating at steady state, i.e. $h = h' = \overline{h}$, $\hat{k} = \hat{k}' = \overline{k}$, $\hat{c} = \hat{c}' = \overline{c}$, gives the static labor-leisure condition and intertemporal Euler equation evaluated at steady state

$$(1-\psi)\bar{c} = \psi(1-\alpha)(1-\bar{h})\bar{k}^{\alpha}\bar{h}^{-\alpha}, \tag{12}$$
$$1 = \beta\gamma^{\psi(1-\sigma)-1}\left(\alpha\bar{k}^{\alpha-1}\bar{h}^{1-\alpha} + 1 - \delta\right). \tag{13}$$

In addition to (12) and (13), we have the resource constraint evaluated at steady state

$$\bar{c} + \eta\gamma\bar{k} = \bar{k}^{\alpha}\bar{h}^{1-\alpha} + (1-\delta)\bar{k}. \tag{14}$$

Equations (12)∼(14) represent three equations in the three unknowns $\{\bar{c}, \bar{k}, \bar{h}\}$.

Finally, to fully characterize the balanced growth path of this economy, it must be that

$$\begin{aligned} c_t &= \bar{c}\gamma^t, \\ k_t &= \bar{k}\gamma^t, \\ h_t &= \bar{h}. \end{aligned}$$

These fully characterizes the asymptotic behavior of this economy with exogenous, labor-augmenting technological progress. All growing variables will grow at the constant rate γ and hours worked will be constant along the balanced growth path. Also, using the production function and capital law of motion, you can back out y_t and i_t. (You should check these grow at rate γ.)

3 Additional Exercises

3.1 Midterm 2019-Q2

Consider a one-sector stochastic growth model where preferences are described by

$$E\sum_{t=0}^{\infty}\beta^t U(c_t)$$

where $U(c_t)$ is a concave function and $0 < \beta < 1$. Output is produced using a constant returns to scale production function, $z_t F(k_t, h_t)$, where z_t is shock to technology observed at the beginning of period t. The stochastic process for z is a two state Markov chain with transition matrix P and where the unconditional mean of z is one. The representative household is endowed with one unit of time (labor). Output produced in period t can be consumed that period or saved. One unit of output saved provides one unit of productive capital the following period. The capital stock is assumed to depreciate at the rate $0 < \delta < 1$.

A.Write the planner's problem for this economy as a dynamic program.

The DPP, since there is no utility for leisure and so $h = 1$, is given by:

$$V(k, z) = \max_{k'} \left[U\left(zF(k, h) + (1-\delta)k - k'\right) + \beta E\left[V\left(k', z'\right) \mid z\right]\right]$$

subject to $z' \in \{z_L, z_H\}$, and z' evolves according to P where wlog I have rearranged the space of z in an increasing fashion. Since the shocks are discrete, and assuming

$$P = \begin{bmatrix} P_{LL} & P_{LH} \\ P_{HL} & P_{HH} \end{bmatrix}$$

we can also define one Bellman equation for each shock:

$$V(k, z) = \begin{cases} V^L(k) \equiv V\left(k, z_L\right) = \max_{k'}\left[U\left(z_L F(k, h) + (1-\delta)k - k'\right) + \beta\left[p_{LL} V^L\left(k'\right) + p_{LH} V^H\left(k'\right)\right]\right. \\ V^H(k) \equiv V\left(k, z_H\right) = \max_{k'}\left[U\left(z_L F(k, h) + (1-\delta)k - k'\right) + \beta\left[p_{HL} V^L\left(k'\right) + p_{HH} V^H\left(k'\right)\right]\right. \end{cases} \tag{15}$$

B. Characterize the nonstochastic steady state for this model

The first order condition for hours worked implies that $h = 1$ given there is no utility for leisure. After deriving the first order condition:

$$U_c\left(zF(k, 1) + (1-\delta)k - k'\right) = \beta E\left[V_k\left(k', z'\right) \mid z\right]$$

and the envelope condition

$$V_k\left(k', z'\right) = \left(z' F_k\left(k', 1\right) + (1-\delta)\right) U_c\left(z' F\left(k', 1\right) + (1-\delta)k' - k''\right)$$

Combining, bringing back the t subscript and assuming that $z = 1$ and non-stochastic, we have:

$$U_c\left(F\left(k_t, 1\right) + (1-\delta)k_t - k_{t+1}\right) = \beta U_c\left(F\left(k_{t+1}, 1\right) + (1-\delta)k_{t+1} - k_{t+2}\right)\left(F_k\left(k_{t+1}, 1\right) + (1-\delta)\right)$$

In the steady state, $c_t = \bar{c}$ and $k_t = \bar{k}$ for all t. This implies that the steady state level of capital is given by:

$$\beta\left(F_k(\bar{k}, 1) + (1-\delta)\right) = 1$$

and the steady state level of consumption is given by:

$$\bar{c} = F(\bar{k}, 1) + \delta\bar{k}$$

C. Assume that capital can only be held in discrete units such as individual houses of equal size (say $\phi > 0$ units of output). Describe an algorithm for solving

the dynamic programming problem in part A that could be implemented on a computer. You do not have to actually write a program; just outline the algorithm in sufficient detail that your "research assistant" would be able to write the program without knowledge of dynamic programming.

We observe that the Blackwell sufficient conditions are satisfied and thus we know that there is a solution to the dynamic program, that can be obtained using value function iteration.

In this case, we can only do it using numerical methods. So the research assistant needs to know the steps that if he follows them, the numerical value function iteration will stop and give the solution that yields then correct numerical value function and the numerical policy function.

As a reminder the general problem, we try to solve the following:

We do not discretize the space at which capital belongs, since in the problem is already assumed that capital needs to belong in a uniform grid, with step size ϕ. In addition, the stochastic process is discretized so we do not need to do this either.

The theorem above tells us that we will converge from any initial guess, however given that we calculated the steady state, we will use the steady state as our initial guess(it usually saves at least one iteration.)

Steps

- Setting the grid:

  So lets know describe the grid, since we are looking, for a policy that depends on z, k and assuming that the number of real numbers between zero and the maximum sustainable capital is equal to M, we generate a sequence of $2M$ values for our capital grid, this is just the capital grid replicated twice once for z_L and once for z_H

- Initializing the Value function:

  We create the initial guess for the value function, either a grid of zeros or as described above after calculating steady state capital $\bar{k}$, we can calculate $V\left(z_L, \bar{k}\right)$ and $V\left(z_H, \bar{k}\right)$ and use as an initial guess a sequence that consists of M values of $V\left(z_L, \bar{k}\right)$ and M values of $V\left(z_H, \bar{k}\right)$

- Set the tolerance level and maximum number of iterations:

  We need to specify a tolerance for when to stop searching over value functions. In particular, my objective is going to be the norm of the of the absolute value of the difference between two V^n and V^{n+1}. One norm that can be used is the square root of the sum of squared deviations. We need to set a tolerance for this for when the loop should stop. I set this tolerance at 0.0001. We also can specify the maximum number of iterations after which the loop should stop, in case we might create an infinite loop.

- VF iteration:

  We still need to implement the loop that will be our numerical value function iteration. In each step, for each value in z, k, we find the k that maximizes the Bellman equation given my guess of the value function.There are several ways to do this, one of which I describe below. Outside of the loop I have a "while" statement, which tells MATLAB to keep repeating the text as long as the difference between value functions is greater than the tolerance and the number of iteration, a variable that is increasing by 1 in each iteration, is below the maximum number of iterations.

  The way to create the next guess V^{n+1} based on the current guess V^n, which can be called v0 is: for each spot i in the state space, we find the argmax of the Bellman equation. We collect the optimized values into the new value function (called v1), and we also find the policy function associated with this choice. After the loop over the possible values of the state I calculate the difference and write out the iteration number. This process can be equivalently done using a loop or matrices.

D. Can the exact invariant distribution for the model of part C be computed (rather than approximated)? How?

Given that the state variables ,(z, k), and thus control variables can only take a finite number of points, we are trying to solve a discrete Markov Chain decision process. Up to rounding errors, we can solve discrete MPDs exactly. In other words the policy function we got from the previous problem is exact(up to rounding errors).

We have the policy function $g(z, k)$ for each combination of z, k. Define the expanded state space S for the combinations of z, k as follows:

$S = \quad [S_1 = (z_1, k_1) \quad \ldots S_m = (z_1, k_m) \quad S_{m+1} = (z_2, k_1) \quad \ldots S_{2m} = (z_2, k_m)]$

We define the transition matrix probabilities

$$\hat{P}_{ij} = P\left(k' = S_i, z' = S_j \mid k, z\right) = P\left(z' = S_j(1) \mid z = S_i(1)\right) \times P\left(k' = S_j(2) \mid (z, k) = S_i\right)$$

where $S_k(i)$ the ith element of the vector that is the kth element of our grid. Having the new transition matrix, we can use the software to compute the eigenvalues, and the invariant distribution. If the unit eigenvalue is unique, we have a unique invariant distribution.

E. Suppose now that z is governed by a three state Markov chain.

a. Provide an example of a transition matrix P such that z has is a unique invariant distribution and one transient state.

Let's assume that the state space for z is z_1, z_2, z_3, then an example that z_1 is a transient state is one where we know that there is always zero probability to return to the state and positive probability to leave from state z_1. Then strictly positive probabilities for every other entry in the transition matrix will ensure a unique invariant distribution and one ergodic set z_2, z_3 For example,

$$P' = \begin{bmatrix} .8 & .1 & .1 \\ 0 & .8 & .2 \\ 0 & .2 & .8 \end{bmatrix}$$

Then the invariant distribution is $\pi = \begin{bmatrix} 0 & .5 & .5 \end{bmatrix}$.

b. Provide an example of a transition matrix P *such that there are three ergodic sets. Characterize the invariant distribution(s) in this case. Given an initial distribution, π_0, what will be the limit distribution? Explain. (12 points)*

Since all of z_1, z_2 and z_3 have to be ergodic sets. There is a unique matrix delivering this result:

$$P' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In this case, there is an infinite amount of invariant distribution that is a convex combination of $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$. Which implies all $0 \geq q, r \leq 1$ such that $\pi = \begin{bmatrix} q & r & 1-q-r \end{bmatrix}$ deliver an invariant distribution. Any initial distribution π_0 will have π_0 as the limiting distribution.