ECON 202A : The Recursive Competitive Equilibrium

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In the previous notes, we determined how to solve a sequential and *dynamic programming problem*. We then had a look to the neoclassical problem, and show that we could obtain the Euler equation and the steady state by using the *Envelope Theorem*, without the need to solve for the Bellman equation. We show how to use of the *Lagrangian* to solve for the Value function.

In this lecture we define the recursive competitive equilibrium for the neoclassical model and show this equivalence and we discuss the equivalence between the social planner's problem and the decentralized problem. Then we solve some additional exercises.

Zero Profit and Constant Returns to Scale

In this section I show that any CRS production function guarantees zero profit in a competitive equilibrium of the neoclassical growth model. Let $y = F(k, h)$ be a production function which is homogeneous of degree one. By definition, for any $\mu > 0$,

$$\mu F(k, h) = F(\mu k, \mu h).$$

Differentiating both sides with respect to μ,

$$F(k, h) = F_1(\mu k, \mu h)k + F_2(\mu k, \mu h)h. \tag{1}$$

Note that (1) holds for any $\mu > 0$. Therefore, it also holds when $\mu = 1$ and

$$F(k, h) = F_1(k, h)k + F_2(k, h)h. \tag{2}$$

Now we can revisit the firm's profit-maximization problem.

$$\max_{k_t^f, h_t^f} \left[F(k_t^f, h_t^f) - w_t h_t^f - r_t k_t^f \right]$$

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The first-order conditions are derived as

$$\begin{aligned} w_t &= F_2(k_{f,t}, h_{f,t}), && (3) \\ r_t &= F_1(k_{f,t}, h_{f,t}), && (4) \end{aligned}$$

and market clearing conditions imply $k_t^f = k_t$ and $h_t^f = h_t$. Plugging these, coupled with (3) and (4), into (2):

$$\begin{aligned} \pi_t &= F(k_{f,t}, h_{f,t}) - w_t h_{f,t} - r_t k_{f,t} \\ &= F(k_t, h_t) - F_2(k_t, h_t)h_t - F_1(k_t, h_t)k_t \\ &= 0 \qquad \text{by (2)} \end{aligned}$$

Therefore, the equilibrium profit is zero, and the whole output produced by the firm is paid to the household as either labor income or capital income. As a consequence, the ownership of the firm is not an important matter.

1 The Recursive Competitive Equilibrium

As an alternative to the date-zero Arrow-Debreu or sequence of markets equilibrium definitions, in this class we will focus on defining a recursive competitive equilibrium for decentralized economies. As previously, to construct a sensible definition of equilibrium, it is crucial to identify the *economic agents* populating the economy, the *problems* that these agents solve, the *choice and states variables*, and the number of *markets* (and, as a result, prices) present in the economy.

A critical assumption in our equilibrium definition is that agents act **competitively**. That is, even though there is a single household and single firm, they do not act as if they can affect aggregate variables or prices. Consequently, agents take aggregate variables and prices as given when solving their respective problems. A result of this assumption is that the **state vector** will expand in the decentralization. No longer is it sufficient for the household to only know its capital stock. Rather, since the household must be able to forecast movements in relative prices over time, it must know something about the aggregate capital stock, and how this stock evolves.

Given the need to forecast prices to solve their problem, households must form **perceptions** about the evolution of the aggregate state. To see this, note that prices, such as the rental rate of capital, depend on the aggregate level of the capital stock in any given period, which was chosen in the previous period and thus fixed. The household's Euler equation shows that it must know something about next period's rental rate in order to choose an optimal intertemporal allocation of resources.

As a result, the perceptions about the evolution of the aggregate state are particularly important for a household to act optimally. These perceptions may prove to be incorrect out of equilibrium. But, our definition of equilibrium requires that these perceptions are accurate in equilibrium. This assumption is sometimes referred to as aggregate consistency, and it is where we impose **rational expectations** in the equilibrium definition.

Using this reasoning, the household's **state vector** will expand to include both the household's capital stock and the aggregate capital stock (k, K). We now state the problems faced by each agent in the model, beginning with the representative firm.

2 Neoclassical model

2.1 Firm's Problem

The representative firm produces and sells the final good at the normalized price of one. It rents labor and capital services from the household each period, implying that it faces a static problem. Given the assumption of perfect competition, the firm takes the prices of labor and capital services - the wage and rental rate of capital, respectively - as given, and maximizes its profit given these factor prices. The firm's problem is

$$\max_{\{k^f, h^f\}} \left[F(k^f, h^f) - wh^f - rk^f \right]$$

The first-order conditions for this problem provide the familiar result that labor and capital services are paid their marginal products, i.e.

$$\begin{aligned} w &= F_2(k^f, h^f), \\ r &= F_1(k^f, h^f). \end{aligned}$$

From class we have seen that these prices are a function of the aggregate state. Let's examine why this must be the case. In equilibrium, we will require that markets clear. This implies that the aggregate supply of the capital stock must equal the aggregate demand for capital, or denoting the aggregate supply of capital as K, $k^f = K$. Similarly, market clearing in the labor market requires $h^f = h$, where h is labor supply. But, h is an endogenous variable derived from the household's problem. Its value must be characterized by some function of the state variables, $h(k, K)$. In equilibrium, however, $k = K$. Thus, market clearing implies that in equilibrium the firm's labor demand is a function only of the current aggregate capital stock: $h^f = h(K, K)$. Using these insights in the above first-order conditions, we can see that the wage and rental rate will only be functions of the current aggregate capital stock,

$$\begin{aligned} w(K) &= F_2(K, h(K, K)), \\ r(K) &= F_1(K, h(K, K)). \end{aligned}$$

This reasoning implies that prices must only be a function of aggregate state variables. We will use this fact below when formulating the household's problem.

2.2 Household's Problem

The representative household derives resources from supplying its labor and renting its capital stock to the representative firm. As argued above, it must know both its beginning of period capital stock and the aggregate capital stock at the start of the period. Furthermore, it must have some perceptions about the law of motion of the aggregate capital stock. Denote this perception as: $K' = G(K)$. The household's problem is

$$V(k,K) = \max_{\{h,k'\}} \Big[U(c, 1-h) + \beta V(k', K') \Big]$$

subject to

$$\begin{aligned}
& c + k' = w(K)h + r(K)k + (1-\delta)k, \\
& K' = G(K), \\
& k_0\ ,\ K_0 \quad \text{given.}
\end{aligned}$$

To see that this is a well-defined dynamic programming problem, notice that, given k and K, by choosing h and k', c is determined directly from the budget constraint. The lone remaining concern is the determination of K'. However, given the perception of the aggregate law of motion and the state variables K, K' is determined from the perspective of the household.

2.3 Recursive Competitive Equilibrium

Our definition of recursive competitive equilibrium will require that, given decision rules by economic agents and the objects these agents take as given, everyone must solve his/her problem optimally and markets must clear. That is, for each market, supply must equal demand. The recursive nature of the problem places an additional restriction on how agents can perceive the evolution of variables that are outside of their direct control. For this, we impose rational expectations on the formation of these perceptions in equilibrium. In particular, these perceptions must be correct in equilibrium.

A **recursive competitive equilibrium** for the standard neoclassical growth model is

(1) Household's decision rules $k'(k,K)$, $h(k,K)$

(2) Firm's decision rules $k^f(K)$, $h^f(K)$

(3) Pricing functions $w(K)$, $r(K)$

(4) Aggregate perception $K' = G(K)$

such that:

(a) Given (3) and (4), (1) solves the household's dynamic programming problem

(b) Given (3), (2) solves the firm's problem

(c) Markets clear

- Labor market: $h^f(K) = h(K, K)$
- Capital rental market: $k^f(K) = K$

(d) Perceptions are correct: $k'(K, K) = G(K)$

A few remarks are necessary about the definition of a recursive competitive equilibrium. First, you should explicitly state the problem that each agent solves. Second, each object and its arguments must be precisely described. As such, the state variables on which decision rules and the value function depend should be made precise and explicit. In other words, you should not leave the argument ambiguous, such as $V(\cdot)$, but should instead write out all of the state variables on which functions depend. Third, the distinction between capital and lowercase letters is very important. Notice in (c) that when markets clear, the household's choice of the capital stock is equivalent to the aggregate capital stock. That is, $h^f(K) = h(K, K)$, not $h^f = h(k, K)$. Fourth, be precise about what the agents take as given. Finally, I would recommend not straying very far from this form of the definition.

Equivalence between the Social Planner's Problem and the Decentralized Problem

It is of prime importance to keep in mind that the solution to the social planner's problem is equivalent to the solution to the decentralized version of the problem with a representative household and representative firm. This is true in general because our standard model does not have any frictions, wedges or distortions. To actually prove this statement a hardworking graduate student would derive a set of equations that characterize the solution to each problem and then show that these conditions are in fact equivalent. As a result, the allocations that solve the two problems must also be equivalent. In essence, you would prove that the fundamental theorems of welfare economics apply to the present problem. [1]

[1] We could show the equivalence in a sequential equilibrium.

(i) **The First Welfare Theorem:** A competitive equilibrium is Pareto efficient.

(ii) **The Second Welfare Theorem:** Any Pareto efficient can be supported as a competitive equilibrium with an appropriate choice of transfers and prices.

You will see more on these theorems in 201A, but bare in mind that the conditions for these theorems hold in our case[2]. The first welfare theorem guarantees that, in our case, the competitive equilibrium is Pareto-efficient, the second one, in some sense, makes the equivalence argument more explicit, since the allocations chosen by the social planner are efficient in the Pareto sense by definition, and moreover the allocation is unique. In different environments where distortions are present or we have incomplete information (markets), the welfare theorems may not hold.

3 Exercises

<Midterm 2010, Question 2>

Consider a neoclassical growth economy with three sectors. One sector produces the market consumption good, c_M, using a constant returns to scale technology, $F^1(k_1, h_1)$. The second sector produces the investment good, i, using the constant returns to scale technology, $F^2(k_2, h_2)$. If you need to do so, you can assume that the function F^2 is invertible where $f(i, k_2)$ is the hours required to produce i units of the investment good given that k_2 units of capital have been allocated to the sector. The third sector is a non-market sector in which households produce a consumption good, c_H, in there homes. Denote the technology for this sector by $F^3(k_3, h_3)$.

Suppose that the economy is populated by a large number of identical households with preferences given by, $\sum_{t=0}^{\infty} \beta^t U(c_{Mt}, c_{Ht}, l_t)$, where $0 < \beta < 1$. Households are endowed with one unit of time that can be allocated to working in the three sectors and to leisure l_t.

Assume that the stock of capital in each sector depreciated at a common rate δ and that it takes one period to produce productive capital. Hence, the total stock of capital, k_t, evolves according to $k_{t+1} = (1-\delta)k_t + i_t$ and can be freely moved across sectors.

A. Formulate the dynamic programming problem that would be solved by a social planner in this economy.

Solution: The full characterization of the *sequential problem* is

$$\max_{\{k_{1t}, k_{2t}, h_t, h_{1t}, h_{2t}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_{Mt}, c_{Ht}, l_t)$$

[2]In case you wonder, these conditions correspond to: utility increasing and convex, production function increasing and convex and the economy is free of distortionary taxation, externalities, market power, all agents have perfect information. In the presence of risk, we also must include complete markets, meaning, agents have the ability to insure against every potential risk.

subject to

$$
\begin{aligned}
c_{Mt} &= F^1(k_{1t}, h_{1t}),\\
i_t &= F^2(k_{2t}, h_{2t}),\\
c_{Ht} &= F^3(k_{3t}, h_{3t}),\\
k_{t+1} &= (1-\delta)k_t + i_t,\\
k_t &= k_{1t} + k_{2t} + k_{3t},\\
h_t &= h_{1t} + h_{2t} + h_{3t},\\
1 &= h_t + l_t,\\
k_0 & \quad \text{given.}
\end{aligned}
$$

There can be several ways of representing this with DPP. For example, the following is one way:

$$V(k) = \max_{k_1,k_2,h,h_1,h_2} \left[U\left(F^1(k_1,h_1), F^3(k-k_1-k_2, h-h_1-h_2), 1-h\right) + \beta V((1-\delta)k + F^2(k_2,h_2)) \right]$$

However, it is more convenient to take k' as one of the choice variables for a neat solution of part C, as follows[3]:

$$V(k) = \max_{k_1,k_2,h,h_1,k'} \left[U\left(F^1(k_1,h_1), F^3\left(k-k_1-k_2, h-h_1-f(k'-(1-\delta)k, k_2)\right), 1-h\right) + \beta V(k') \right]$$

where f is the inverse function for h_2 implied by

$$k' = (1-\delta)k + F^2(k_2, h_2).$$

B. Repeat part (A) under the assumption that capital cannot be moved across sectors.

Solution: The sequential problem is:

$$\max_{\{h_{1t},h_{2t},h_{3t},k_{1,t+1},k_{2,t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_{Mt}, c_{Ht}, 1-h_t)$$

subject to

$$
\begin{aligned}
c_{Mt} &= F^1(k_{1t}, h_{1t})\\
i_t &= F^2(k_{2t}, h_{2t})\\
c_{Ht} &= F^3(k_{3t}, h_{3t})\\
i_t &= i_{1t} + i_{2t} + i_{3t}\\
k_{i,t+1} &= (1-\delta)k_{it} + i_{it} \qquad \text{for } i = 1,2,3\\
h_t &= h_{1t} + h_{2t} + h_{3t}\\
k_{i0} & \quad \text{given, for } i = 1,2,3
\end{aligned}
$$

[3] If k' is not a choice variable and is substituted by other variables, then it is very likely that the expression for k' contains k. Then, the envelope condition, the expression for $V'(\cdot)$, will contain $V'(\cdot)$ again, then it is a bit complex to eliminate $V'(\cdot)$ from the set of equilibrium-characterizing equations.

The key point is that, due to immobility of capital across sectors, now the capital for each sector evolves separately. Therefore, the level of capital stock in each sector is now a state of the value function. Then, a representation of the social planner's DPP is

$$V(k_1,k_2,k_3) = \max_{h_1,h_2,h_3,k_1',k_2'} \Big[U(F^1(k_1,h_1),F^3(k_3,h_3),1-h_1-h_2-h_3)+\beta V(k_1',k_2',k_3')\Big]$$

subject to

$$\begin{aligned} &k_1'+k_2'+k_3' = (1-\delta)(k_1+k_2+k_3)+F^2(k_2,h_2) \\ &k_i'-(1-\delta)k_i \geq 0 \quad \text{for } i=1,2,3 \end{aligned}$$

Note that, in this economy, capital is sector-specific. Therefore, if $k_i' < (1-\delta)k_i$ for some i, then we choose k_i' lower than the undepreciated capital of the sector i. This is disinvesting in sector i and converting it to another sector. Since this is not allowed in this economy, we need the nonnegativity condition for the investment in each sector.

C. Derive the static first-order conditions and Euler equations that characterize the solution to the planner's problem (for this part and the rest of the problem, assume capital can be moved across sectors).

Solutions: Let me rewrite the DPP in part (A):

$$V(k) = \max_{k_1,k_2,h,h_1,k'} \Big[U\left(F^1(k_1,h_1),F^3\left(k-k_1-k_2,h-h_1-f(k'-(1-\delta)k,k_2)\right),1-h\right)+\beta V(k')$$

Then the first-order conditions are:

$$\begin{aligned}
&[k_1]: && U_1(c_M,c_H,1-h)F_1^1(k_1,h_1) = U_2(c_M,c_H,1-h)F_1^3(k_3,h_3) && (5)\\
&[k_2]: && U_2(c_M,c_H,1-h)\Big[-F_1^3(k_3,h_3)-F_2^3(k_3,h_3)f_2(i,k_2)\Big] = 0 && (6)\\
&[h]: && U_2(c_M,c_H,1-h)F_2^3(k_3,h_3) = U_3(c_M,c_H,1-h) && (7)\\
&[h_1]: && U_1(c_M,c_H,1-h)F_2^1(k_1,h_1) = U_2(c_M,c_H,1-h)F_2^3(k_3,h_3) && (8)\\
&[k']: && U_2(c_M,c_H,1-h)F_2^3(k_3,h_3)f_1(i,k_2) = \beta V'(k') && (9)
\end{aligned}$$

The envelope condition is:

$$V'(k) = U_2(c_M,c_H,1-h)\Big[F_1^3(k_3,h_3)+F_2^3(k_3,h_3)f_1(i,k_2)(1-\delta)\Big] \tag{10}$$

Combining (9) and (10) yields the following Euler equation:

$$U_2(c_M,c_H,1-h)F_2^3(k_3,h_3)f_1(i,k_2) = \beta U_2(c_M',c_H',1-h')\Big[F_1^3(k_3',h_3')+F_2^3(k_3',h_3')f_1(i',k_2')(1-\delta)\Big] \tag{11}$$

Since we introduced variables $\{c_M, c_H, k_3, h_3, i\}$, we need the following resource constraints to pin down those variables:

$$c_M = F^1(k_1, h_1) \tag{12}$$
$$c_H = F^3(k_3, h_3) \tag{13}$$
$$k_3 = k - k_1 - k_2 \tag{14}$$
$$h_3 = h - h_1 - f(i, k_2) \tag{15}$$
$$i = k' - (1-\delta)k \tag{16}$$

The optimal solution is characterized by (5)∼(8) and (11)∼(16), which are 10 equations. The unknowns are (if you are asked for steady state) $\{c_M, c_H, k_1, k_2, k_3, h, h_1, h_3, k, i\}$, confirming that we are done after we apply the steady state conditions in the 10 equations. Similarly, if you are asked for sequences that solve the growth model you just need to substitute the t subscripts again since you have the same number of equations and unknowns.

D. Consider a decentralized version of this economy where households own capital and rent it to firms, household supply labor to the firms, and *there is a market for privately issued one period bonds*. Define a recursive competitive equilibrium for this economy. Be sure to carefully state the dynamic program solved by households and the problem solved by firms. Also, assume in your decentralization that non-market goods, like the name says, are not traded in a market.

Solutions: Assume that there is one single representative household and one single firm in each sector.

<u>Household's Problem</u>

$$V(K, k, b) = \max_{k', k_3, h, h_3, b'} \left[U(c_M, c_H, 1-h) + \beta V(K', k', b') \right]$$

subject to

$$\begin{aligned}
&c_M + p(K)i + q(K)b' = r(K)(k - k_3) + w(K)(h - h_3) + b \\
&c_H = F^3(k_3, h_3) \\
&k' = (1-\delta)k + i \\
&K' = G(K) \qquad \textit{(Perception for aggregate state)}
\end{aligned}$$

where $p(K)$ and $q(K)$ are the prices of the investment goods and bonds, respectively.

<u>Consumption-good Firm's Problem</u>

$$\max_{k^c, h^c} \left[F^1(k^c, h^c) - r(K)k^c - w(K)h^c \right]$$

Investment-good Firm's Problem

$$\max_{k^i,h^i}\left[p(K)F^2(k^i,h^i)-r(K)k^i-w(K)h^i\right]$$

Then a RCE can be defined as:

(1) HH decision rules: $k'(K,k,b)$, $k_3(K,k,b)$, $h(K,k,b)$, $h_3(K,k,b)$, $b'(K,k,b)$

(2) Consumption-good firm's choice: $k^c(K)$, $h^c(K)$

(3) Investment-good firm's choice: $k^i(K)$, $h^i(K)$

(4) Pricing functions: $p(K)$, $q(K)$, $r(K)$, $w(K)$

(5) Perception of HH: $G(K)$

satisfying the follows.

- Given (4) and (5), (1) solves the HH DPP.
- Given (4), (2) solves the consumption-good firm's problem.
- Given (4), (3) solves the investment-good firm's problem.
- Markets clear:
  - investment good market: $F^2(k^i(K),h^i(K))=k'(K,K,0)-(1-\delta)K$
  - labor market: $h(K,K,0)-h_3(K,K,0)=h^c(K)+h^i(K)$
  - capital rental market: $K-k_3(K,K,0)=k^c(K)+k^i(K)$
  - bond market: $0=b'(K,K,0)$
- Aggregate consistency: $G(K)=k'(K,K,0)$

4 Appendix

4.1 Social planner's problem

Consider an economy that lasts for infinite number of periods. Each period, a benevolent social planner chooses consumption allocations, hours worked, and capital to carry into the following period, in order to maximize the discounted value of lifetime utility of a representative for a representative household subject to the usual resource constraint and the law of motion for the capital stock ($k_{t+1}=i_t+(1-\delta)k_t$). We assume that the household has utility over consumption and leisure and that the household is endowed with a fixed amount of time normalized to 1. Denoting leisure as l_t and hours worked as h_t, this assumption implies $l_t+h_t\leq 1$. We also assume that the production function takes the following Cobb-Douglas form: $F(k,h)=k^\alpha h^{1-\alpha}$ with $\alpha\in(0,1)$. By substituting the capital law of motion into the resource constraint for current period's investment, the planner's problem can be written recursively as follows:

$$V(k)=\max_{c,h,k'}\ U(c,1-h)+\beta V(k') \tag{17}$$

subject to

$$c + k' = k^\alpha h^{1-\alpha} + (1-\delta)k \tag{18}$$

$$\tag{19}$$

We can construct a Lagrangian with the Lagrange multiplier at time t given by λ_t:

$$\mathcal{L} = U(c, 1-h) + \beta V(k') + \lambda\Big[k^\alpha h^{1-\alpha} + (1-\delta)k - c - k'\Big] \tag{20}$$

The first-order conditions for this problem with respect to c, h, and k' and the envelope with respect to k are given by, respectively,

$$U_1(c, 1-h) = \lambda \tag{21}$$

$$U_2(c, 1-h) = \lambda(1-\alpha)k^\alpha h^{-\alpha} \tag{22}$$

$$\beta V'(k') = \lambda \tag{23}$$

$$V'(k') = \lambda'[\alpha k'^{\alpha-1} h'^{\alpha} + 1 - \delta] \tag{24}$$

Eliminating λ by substituting (21) into (22) to (24), combining equations (21)$\sim$(24) reduce to two familiar equations after substituting for the t subscript: the intratemporal labor-leisure condition and the Euler equation:

$$\frac{U_2(c_t, 1-h_t)}{U_1(c_t, 1-h_t)} = (1-\alpha)k_t^\alpha h_t^{-\alpha} \tag{25}$$

$$U_1(c_t, 1-h_t) = \beta U_1(c_{t+1}, 1-h_{t+1})\Big[\alpha k_{t+1}^{\alpha-1} h_{t+1}^{1-\alpha} + 1 - \delta\Big] \tag{26}$$

The labor-leisure condition is a static condition that equates the marginal rate of substitution between leisure and consumption (the LHS of (25)) with the marginal product of labor (the RHS of (25)). These two equations, coupled with the resource constraint (18), fully characterize a solution to this problem.

In the next section, we present the decentralized version of this problem. Our goal is to establish the set of equations characterizing a solution to the decentralized version and show that it is identical to the set of equations we just found to characterize the solution to the social planner's problem.

4.1.1 The Decentralized Economy

We consider the decentralization from class, except that I assume only one representative household for convenience. The economy is populated by a representative household and a representative firm, and the household is assumed to own the whole shares of the firm, so firm's profit (or loss) belongs to the household. Both entities take the prices in the economy as given when solving their maximization problems. The household maximizes its discounted value of utility over consumption and leisure subject to its budget constraint.[4] Its income is derived from labor income ($w_t h_t$) where it receives a wage w_t for each unit of labor supplied, income from renting the capital it owns to the firm for production ($r_t k_t$), where it receives a rental rate r_t for each unit of capital it rents, and profit (π_t) of the firm that it owns.

The household's problem is given by:

$$\max \sum_{t=0}^{T} \beta^t U(c_t, 1-h_t) \tag{27}$$

subject to

$$c_t + k_{t+1} = w_t h_t + r_t k_t + \pi_t + (1-\delta)k_t \tag{28}$$

Again, we can rewrite this problem in a recursive form in the following manner:

$$V(k,K) = \max_{c,h,k'} \; U(c, 1-h) + \beta V(k', K') \tag{29}$$

$$c + k' = w(K)h + r(K)k + (1-\delta)k \tag{30}$$

$$K' = G(K)(\text{Aggregate Perception}) \tag{31}$$

The first-order conditions for this problem with respect to consumption, hours worked, and next period's capital and the envelope condition are, respectively:

$$U_1(c, 1-h) = \lambda \tag{32}$$

$$U_2(c, 1-h) = \lambda w(K) \tag{33}$$

$$\beta V_1(k', K') = \lambda \tag{34}$$

$$V_1(k', K') = \lambda'[r(K') + 1 - \delta] \tag{35}$$

We can again substitute the Lagrange multiplier out of equations (33) $\sim$ (35) to reduce the equations (32)$\sim$(35) to the labor-leisure condition and Euler equation in the aggregate level:

$$\frac{U_2(c(K,K), 1-h(K,K))}{U_1(c(K,K), 1-h(K,K))} = w(K) \tag{36}$$

$$U_1(c(K,K), 1-h(K,K)) = \beta U_1(c(k'(K,K),K'), 1-h(k'(K,K),K'))(r(K')+1-\delta) \tag{37}$$

Note that (36) and (37) look very similar to (25) and (26) from the social planner's problem. The only difference is that w and r are replaced by the marginal product of labor and capital in the social planner's problem.

[4] As repeatedly emphasized in lecture, the distinction between budget constraint and resource constraint is important in the decentralization. The resource constraint states that the uses of physical resources (consumption and investment in the standard model, for example) must be equal to the sources of physical resources (aggregate production). The household's budget constraint, on the other hand, states that total household expenditure equals total household income, where appropriate care is taken to identify relative prices in this constraint.

Firm's Problem

The perfectly competitive representative firm solves a static profit maximization problem each period. It operates a constant returns to scale production function that takes labor and capital as inputs. The firm's problem is the following:

The representative firm produces and sells the final good at the normalized price of one. It rents labor and capital services from the household each period, implying that it faces a static problem. Given the assumption of perfect competition, the firm takes the prices of labor and capital services - the wage and rental rate of capital, respectively - as given, and maximizes its profit given these factor prices. The firm's problem is:

$$\max_{\{k^f,h^f\}}\left[F(k^f,h^f)-w(K)h^f-r(K)k^f\right]$$

The first-order conditions for this problem provide the familiar result that labor and capital services are paid their marginal products:

$$\begin{aligned} w(K) &= F_2(k^f,h^f) \\ r(K) &= F_1(k^f,h^f) \end{aligned}$$

From class we have seen that in a recursive competitive equilibrium: Markets clear:

- Labor market: $h^f(K)=h(K,K)$
- Capital rental market: $k^f(K)=K$

Perceptions are correct: $k'(K,K)=G(K)$

Substituting in the FOCs of the firm, we get:

$$\begin{aligned} w(K) &= F_2(K,h(K,K)) \\ r(K) &= F_1(K,h(K,K)) \end{aligned}$$

Moving the equation for interest rate one period forward and then substituting the aggregate consistency equation, we get the equilibrium equation for interest rate:

$$\begin{aligned} r(K') &= F_1(K',h(K',K')) \\ r(K') &= F_1(k'(K,K),h'(k'(K,K),k'(K,K))) \end{aligned}$$

This reasoning implies that prices must only be a function of aggregate state variables. We will use this fact below when substituting in the household's optimality conditions.

Now, substituting these first-order conditions from the firm's problem in the household's first-order conditions ((36) and (37)) and replacing the t subscript, we have:

$$\frac{U_2(c_t,1-h_t)}{U_1(c_t,1-h_t)}=(1-\alpha)k_t^{\alpha}h_t^{-\alpha} \tag{38}$$

$$U_1(c_t,1-h_t)=\beta U_1(c_{t+1},1-h_{t+1})\left[\alpha k_{t+1}^{\alpha-1}h_{t+1}^{1-\alpha}+1-\delta\right] \tag{39}$$

which are now identical to (25) and (26) from the social planner's problem.

We have now used the market clearing conditions and the first-order conditions from both the household and firm's problems in the household's first-order conditions. We will next use the market clearing conditions and the firm's first-order conditions in the household's budget constraint. Note that, in equilibrium:

$$\begin{aligned} w(K)h(K,K) + r(K)K &= \Big[(1-\alpha)K^\alpha h(K,K)^{-\alpha}\Big]h(K,K) + \Big[\alpha K^{\alpha-1}h(K,K)^{1-\alpha}\Big]K \\ &= (1-\alpha)K^\alpha h(K,K)^{1-\alpha} + \alpha K^\alpha h(K,K)^{1-\alpha} \\ &= K^\alpha h(K,K)^{1-\alpha} \end{aligned} \tag{40}$$

Therefore, the equilibrium profit is zero, and thus the household's total income for each period t reduces to $w_t h_t + r_t k_t + \pi_t = k_t^\alpha h_t^{1-\alpha}$. Using this result, we can rewrite the household's budget constraint (28) as follows:

$$c_t + k_{t+1} = k_t^\alpha h_t^{1-\alpha} + (1-\delta)k_t \tag{41}$$

which is identical to the resource constraint from the social planner's problem (equation (18)). Hence, the optimal allocations derived from each must be identical[5], and we have therefore proven the result that the solution to the social planner's problem is identical to the solution of the decentralized economy.

[5] In addition, one should prove that the transversality conditions of the two problems are identical, which is a good exercise so try do it yourself.