Econ202A Homework #2

Chris Ackerman, Eketerina Gurkova, Ali Haider Ismail, and Ben Pirie

December 8, 2020

1. In this economy, assume that $r = \delta$. Prove Hall's Corollary 1 and 2, and 4. In addition, how would you go about estimating the implied regression in Corrolary 4?

2. Explain the economic intuition for why the stochastic process for income is irrelevant in terms of being able to forecast future consumption.

3. Explain the economic intuition why if $r < \delta$, then consumption evolves as a random walk with positive drift, in which there is a constant term in the regression that is negative.

4. Obtain quarterly real consumption (in chained dollars) from the U.S. national income and product accounts from 1950 through 2019. Fit the following regression:

$$\ln(c_t) = \mu + \lambda \ln(c_{t-1}) + u_t$$

5. Do you think that this is a reasonable statistical model of the log of consumption? (Your answer to this question may include a discussion regarding the value of the autoregressive coefficient, the R-square, and whether there is autocorrelation in the u_t residuals.)

Next, consider the following economy.

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to

$$\begin{aligned} z_t A_t^{1-\theta} k_t^\theta + (1-\delta)k_t &= c_t + k_{t+1} \\ A_t &= (1+\gamma)^t, \quad t = 0, 1, \ldots \\ \ln(z_t) &= \rho \ln(z_{t-1}) + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2) \end{aligned}$$

Assume that the time period is annual. Construct a detrended version of this economy and show the first order conditions. Choose β so that the return to capital in the steady state of the detrended economy is five percent, choose θ so that capital's share of income is 30 percent, and choose a depreciation rate such that the share of investment to GDP in the steady state is 20 percent. Choose $\rho = 0.95$, $\sigma_\varepsilon^2 = .002$ and $\gamma = 0.02$.

Rearranging terms, we have

$$k_{t+1} = A_t^{1-\theta} k_t^\theta + (1-\delta)k_t - c_t \tag{1}$$
$$Y_t = A_t^{1-\theta} k_t^\theta \tag{2}$$
$$c_t = (1-\theta) A_t^{1-\theta} k_t^\theta \tag{3}$$

To detrend, divide by A_t. Let's define a few new variables,

$$\hat{k}_t = \frac{K_t}{A_t} \tag{4}$$
$$\hat{y}_t = \frac{Y_t}{A_t} \tag{5}$$
$$\hat{c}_t = \frac{C_t}{A_t}. \tag{6}$$

Now, we can substitute these back into the original equations.

$$A\hat{k}_{t+1} = \hat{y}_t + (1-\delta)\hat{k}_t - \hat{c}_t \tag{7}$$
$$1 + \gamma\hat{k}_{t+1} = \hat{y}_t + (1-\delta)\hat{k}_t - \hat{c}_t \tag{8}$$
$$\hat{y}_t = k^\theta \tag{9}$$
$$\hat{c}_t = (1-\theta)\hat{y}_t. \tag{10}$$

First order conditions give us

$$\frac{1}{\hat{c}_t} = \frac{\beta}{1+\gamma} E_t \left\{ \frac{1}{\hat{c}_{t+1}} \left[\frac{\theta \hat{y}_{t+1}}{\hat{k}_{t+1}} + 1 - \delta \right] \right\}. \tag{11}$$

In the steady state, we have

$$\frac{\bar{c}}{\bar{y}} = \frac{1+\gamma-\beta(1-\delta)-\theta\beta(1+\gamma-1+\delta)}{1+\gamma-\beta(1-\delta)}. \tag{$*$}$$

Now let's solve for parameters. We're given $\gamma = 0.02$, and we have to figure out β, θ and δ. Since we have Cobb Douglas production, $\theta = 0.3$. To solve for β, note that the 5% return implies

$$\beta = \frac{1}{1.05} \tag{12}$$

$$= 0.95238. \tag{13}$$

To solve for δ, we're going to use equation $*$. We're told that investment in the steady state is 20% of GDP, so that implies that consumption is 80% of GDP,

$$0.8 = \frac{1.02-0.95238(1-\delta)-0.3\cdot 0.95238(1.02-1+\delta)}{1.02-0.95238(1-\delta)} \tag{14}$$

$$\implies \delta = .082. \tag{15}$$

6. Log-linearize this model around its deterministic steady state. (For simplicity, assume that z in the steady state is 1).

$$\text{Define } \tilde{x} \equiv \log\left(\frac{\hat{x}}{\overline{x}}\right).$$

From the Euler equation, we have

$$\frac{\hat{c}_{t+1}}{\hat{c}_t} = \frac{\beta}{A} E_t[\theta z_{t+1} \hat{K}_{t+1}^{\theta-1} + 1 - \delta].$$

Substituting our log linearization into the left-hand side, we have

$$\begin{aligned} \frac{\overline{c}\exp(\tilde{c}_{t+1})}{\overline{c}\exp(\tilde{c}_t)} &\approx (1+\tilde{c}_{t+1})(1-\tilde{c}_t) && \text{(LHS)} \\ &\approx 1 + \tilde{c}_{t+1} - \tilde{c}_t. && \text{(LHS)} \end{aligned}$$

Doing the same thing on the right-hand side, we have

$$\frac{\beta}{A} E_t[\theta \overline{z}(1+\tilde{z}_{t+1})\overline{K}(1+(\theta-1)\hat{K}_{t+1}) + 1 - \delta] = \frac{\beta}{A} E_t[\theta \overline{z}\overline{K}^{\theta-1}(\theta-1)\tilde{K}_{t+1} + \theta \overline{z}\overline{K}^{\theta-1}\tilde{z}_{t+1} + 1 - \delta]$$

In the steady state,

$$\begin{aligned} 1 &= \frac{\beta}{A}(\theta \overline{z}\overline{K}^{\theta-1} + 1 - \delta), \\ \overline{z} &= 1. \end{aligned}$$

We can use these to simplify the log-linearized Euler equation:

$$\tilde{c}_{t+1} - \tilde{c}_t = \frac{\beta}{A} E_t[\theta \overline{K}^{\theta-1}(\theta-1)\tilde{K}_{t+1} + \theta \overline{K}^{\theta-1}\tilde{z}_{t+1}].$$

Now, let's do the same thing to the budget constraint.

$$\begin{aligned} \hat{c}_t + A\hat{K}_{t+1} &= z_t \hat{K}_t^{\theta} + (1-\delta)\hat{K}_t \\ \overline{c}(1+\tilde{c}_t) + A\overline{K}(1+\hat{K}_{t+1}) &= \overline{c} + A\overline{K} + \overline{c}\tilde{c}_t + A\overline{K}\tilde{K}_{t+1} && \text{(LHS)} \\ \overline{z}(1+\tilde{z}_t)\overline{K}^{\theta}(1+\theta\tilde{K}_t) + (1-\delta)\overline{K}(1+\tilde{K}_t) &= \overline{z}\overline{K}^{\theta} + \overline{z}\overline{K}^{\theta}\theta\tilde{K}_t + \overline{z}\overline{K}^{\theta}\tilde{z}_t + (1-\delta)\overline{K} + (1-\delta)\overline{K}\tilde{K}_t. && \text{(RHS)} \end{aligned}$$

In the steady state,

$$\begin{aligned} \overline{c} + A\overline{K} &= \overline{z}\overline{K}^{\theta} + (1-\delta)\overline{K}, \\ \overline{z} &= 1, \end{aligned}$$

so we can simplify this expression to

$$\overline{c}\tilde{c}_t + A\overline{K}\tilde{K}_{t+1} = \overline{K}^{\theta}\theta\tilde{K}_t + \overline{K}^{\theta}\tilde{z}_t + (1-\delta)\overline{K}\tilde{K}_t,$$

or

$$\tilde{k}_{t+1} = \frac{\overline{K}^{\theta-1}}{A}\theta\tilde{K}_t + \frac{\overline{K}^{\theta-1}}{A}\tilde{z}_t + \frac{1-\delta}{A}\hat{K}_t - \frac{\overline{c}}{A\overline{K}}\tilde{c}_t.$$

Finally, for the stochastic process,

$$\begin{aligned}
\ln(z_t) &= \rho\ln(z_{t-1}) + \varepsilon_t \\
\ln(\overline{z}\exp(\tilde{z}_t)) &= \rho\ln(\overline{z}\exp(\tilde{z}_{t-1})) + \varepsilon_t \\
\ln(\overline{z}) + \tilde{z}_t &= \rho\ln(\overline{z}) + \rho\tilde{z}_{t-1} + \varepsilon_t \\
\implies \tilde{z}_t &= \rho\tilde{z}_{t-1},
\end{aligned}$$

or

$$\tilde{z}_{t+1} = \rho\tilde{z}_t.$$

Putting everything together, the log-linearized version of this economy is

$$\begin{aligned}
&\tilde{c}_{t+1} = E_t\left\{\frac{\beta\theta\overline{K}^{\theta-1}}{A}(\theta-1)\left[\frac{\theta\overline{K}^{\theta-1}+1-\delta}{A}\tilde{K}_t + \frac{\overline{K}^{\theta-1}}{A}\tilde{z}_t - \frac{\overline{c}}{A\overline{K}}\tilde{c}_t\right]\right\} \\
&\hat{k}_{t+1} = \frac{\theta\overline{K}^{\theta-1}+1-\delta}{A}\hat{K}_t + \frac{\overline{K}^{\theta-1}}{A}\tilde{z}_t - \frac{\overline{c}}{A\overline{K}}\tilde{c}_t \\
&\tilde{z}_{t+1} = \rho\tilde{z}_t.
\end{aligned}$$

7. Use the formula of Blanchard and Kahn to show that there is a unique stationary solution to the linearized system.

8. Using a random number generator (Matlab has a built-in function for this), draw 1100 values of ε to construct the z process. Using these values of z, and assuming that k_0 is equal to its steady state value, use the linearized system to construct 1100 values values of output, consumption, and investment.

I'm having some trouble with this question. I'm not sure if I'm doing something dumb, but I started with Ben's linearized equations and didn't know how to treat $\overline{k}$. The problem says that $k_0 = \overline{k}$, so $\tilde{k}$ should be zero, but $\overline{k}$ was in a bunch of other terms so here's what I've got so far. From earlier in the problem, we have

$$\overline{K}^{\theta-1} = \frac{\beta\theta\overline{z}}{A-\beta(1-\delta)}.$$

Substituting this into the expression for $\tilde{K}_{t+1}$ gives

$$\tilde{k}_{t+1} = \left[\theta\cdot\left(\frac{\beta\theta\overline{z}}{A-\beta(1-\delta)}\right)+1-\delta\right]\cdot A^{-1}\cdot\tilde{k}_t + \left(\frac{\beta\theta\overline{z}}{A-\beta(1-\delta)}\right)\cdot A^{-1}\cdot\tilde{z}_t - \underbrace{\frac{\overline{c}}{A\overline{K}}}_{\text{simplify}}\,\tilde{c}_t$$

To simplify this last term, start with

$$\begin{aligned}
\overline{c} &= \overline{y} + (1-\delta-A)\overline{kk} \\
\implies \frac{\overline{c}}{\overline{K}} &= \frac{\overline{y}+(1-\delta-A)\overline{K}}{\overline{K}} \\
&= \frac{\overline{y}}{\overline{K}} + 1-\delta-A \\
\frac{\overline{y}}{\overline{K}} &= \frac{\overline{K}}{\overline{Y}}\cdot 1^{-1} \\
&= \frac{A-\beta(1-\delta)}{\beta\theta\overline{z}} \\
\implies \frac{\overline{c}}{\overline{K}} &= \left(\frac{A-\beta(1-\delta)}{\beta\theta\overline{z}} + 1-\delta-A\right)
\end{aligned}$$

Putting this all together, the computationally tractable form of $\tilde{k}_{t+1}$ should be

$$\begin{aligned}
\tilde{k}_{t+1} = &\left[\theta\cdot\left(\frac{\beta\theta\overline{z}}{A-\beta(1-\delta)}\right)+1-\delta\right]\cdot A^{-1}\cdot\tilde{k}_t \\
&+\left(\frac{\beta\theta\overline{z}}{A-\beta(1-\delta)}\right)\cdot A^{-1}\cdot\tilde{z}_t \\
&-\left(\frac{A-\beta(1-\delta)}{\beta\theta\overline{z}}+1-\delta-A\right)\cdot A^{-1}\cdot\tilde{c}_t
\end{aligned}$$

$\overline{z}$ is 1, and everything else is a parameter except for the tilde variables, so this should work (I dropped t subscripts on A but that should be fine). I think this all looks okay but I'm not sure if there was an easier way to do this? If we do the same thing for consumption,

$$\begin{aligned}
\tilde{c}_{t+1} &= E_t \left\{ \frac{\beta\theta\overline{K}^{\theta-1}}{A} \cdot (\theta - 1)\tilde{k}_{t+1} \right\} \\
&= E_t \left\{ \beta\theta \cdot \left(\frac{\beta\theta\overline{z}}{A - \beta(1-\delta)} \right) \cdot \frac{\theta - 1}{A} \cdot \tilde{c}_{t+1} \right\} \\
&= \beta\theta \cdot \left(\frac{\beta\theta\overline{z}}{A - \beta(1-\delta)} \right) \cdot \frac{\theta - 1}{A} \cdot E_t \left\{ \tilde{k}_{t+1} \right\}
\end{aligned}$$

I'm a bit confused about this expectation, because it's taken at time t and all the terms in $\tilde{k}_{t+1}$ have t subscripts, so shouldn't they be known? But I thought that consumers chose their consumption based on the expected next-period shock.

9. Discard the first 100 observations, and then fit an AR(1) process to the log of consumption, measured as the log-deviation of consumption from the steady state value. Report the value of the AR(1) coefficient in the regression, and evaluate whether there is autocorrelation in the residuals.

10. Compare the regression coefficient in (9) and your assessment of the autocorrelation in the residuals, to your answers in (4) and (5). Does the RBC model provide a good approximation to consumption dynamics? What does it tell us about using consumption data to try to discriminate between the Hall