Single Variable Calculus II

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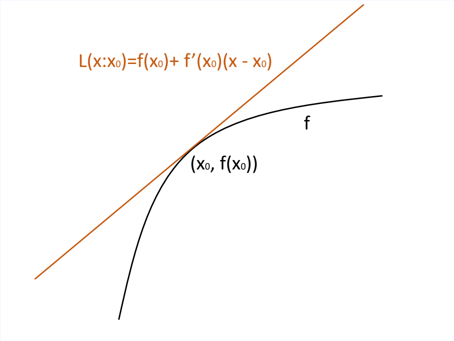
Summer Math Camp, 2020

Linear Approximation



The derivative of f at x_0 can be used to construct a linear approximation of f around x_0: $L(x : x_0) = f(x_0) + f'(x_0)(x - x_0)$. It is a straight line that is tangent to f at x_0.

How good is this approximation?.

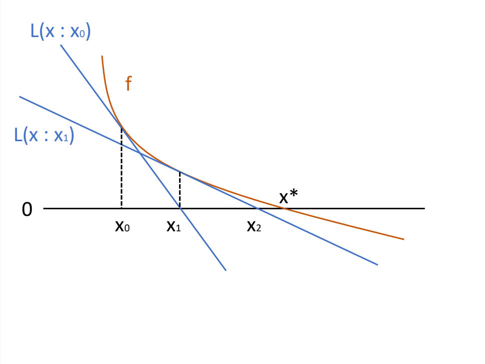
- Move x by Δx and evaluate the "error" $R(\Delta x; x_0) = |f(x_0 + \Delta x) - L(x_0 + \Delta x : x_0)|$.
- $R(\Delta x; x_0) = |f(x_0 + \Delta x) - f(x_0) - f'(x_0)\Delta x|$, so $\frac{R(\Delta x; x_0)}{\Delta x}$ converges to 0 as $\Delta x \to 0$. That is, the error goes to 0 faster than Δx goes to 0.
  - We often write this as $f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x + o(\Delta x)$, where $o(\cdot)$ is called "little o" and represents a term that satisfies $\frac{o(h)}{h} \to 0$ for $h \to 0$.

$L(x : x_0)$ is the best linear approximation of f around x_0.

Newton's Method

Newton's method is an example of algorithm that uses linear approximations.

- You want to find x^* to solve $f(x) = 0$.
- Guess any point x_0. Find a solution for the linear approximation ($L(x : x_0) = 0$) instead of $f(x) = 0$, which gives us $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$.
- Repeat this to obtain $x_1, x_2, \ldots$ by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. If x_n happens to converge to some number x^*, then x^* must satisfy $f(x^*) = 0$.



Mean Value Theorem

We like to refine our intuition about linear approximation and obtain an even better approximation. The mean value theorem is a first step towards that goal.

Suppose that x increases to $x + \Delta x > x$ and $f(x)$ moves to $f(x + \Delta x)$. The rate of changes is $\frac{f(x+\Delta x)-f(x)}{\Delta x}$. For differentiable f, this rate should be somewhere between the maximum derivative (= steepest slope) and the minimum derivative of f on $(x, x + \Delta x)$ intuitively.

Mean Value Theorem

The mean value theorem says that this rate is the derivative of f at some point $t \in (x, x + \Delta x)$ when the derivative of f is continuous.

Mean Value Theorem

Suppose that $f : \Re \to \Re$ is differentiable and f' is continuous. Pick any point $x \in \Re$ and $\Delta x > 0$. Then there exists $t \in (x, x + \Delta x)$ such that

$$f(x + \Delta x) - f(x) = f'(t)\Delta x$$

Let's reexamine the relative speed of $R(\Delta x; x_0) \to 0$ vs. $\Delta x \to 0$.

- By MVT, $R(\Delta x; x_0) = |(f'(t) - f'(x_0))\,\Delta x|$ for some $t \in (x, x + \Delta x)$. So $\frac{R(\Delta x; x_0)}{\Delta x}$ converges to 0 at the speed where $f'(t)$ converges to $f'(x_0)$.
- If f' is differentiable, then $|f'(t) - f'(x_0)| \sim |f''(x_0)(t - x_0)| \leq |f''(x_0)\Delta x|$. So, intuitively, we can tell that $R(\Delta x; x_0)$ converges to 0 at the speed of $(\Delta x)^2$ in this case.

Higher Order Derivatives

As we noticed, a derivative is itself a function, so it is possible to consider the derivative of a derivative, which we call the second derivative. Then we can take the third derivative and so on...

A function is **continuously differentiable** if its derivative is continuous. A function is k-**times continuously differentiable** if it's k-times differentiable and its kth derivative is continuous. We denote the set of k-times continuously differentiable functions by $\mathcal{C}^k$

Polynomials, exponential functions, and log functions are all C^∞ functions. C^0 function means a continuous function.

Approximation by Higher Order Polynomials

For any $f \in \mathcal{C}^n$, $L^k(x : x_0) = f(x_0) + \sum_{i=1}^{k} \frac{f^{(i)}}{i!}(x - x_0)^i$ is the k**th order Taylor polynomial** of f around x_0, where $f^{(i)}$ is the ith derivative of f.

Higher order polynomials are more flexible than linear functions, so they provide a better local approximation.

Taylor's Theorem

We can generalize the MVT to higher order polynomials in the following way. ▸ Proof Detail

Taylor's Theorem

Suppose that $f : \Re \to \Re$ is a $\mathcal{C}^k$ function and $(k+1)$-times differentiable. Pick any point $x \in \Re$ and $\Delta x > 0$. Then there exists $t \in (x, x + \Delta x)$ such that

$$f(x + \Delta x) = L^k(x + \Delta x : x) + \frac{f^{(k+1)}(t)}{(k+1)!}(\Delta x)^{k+1}$$

Comment

- If f is C^2 function, then $R(\Delta x; x_0) = \left|\frac{f''(t)}{2}(\Delta x)^2\right|$. So $R(\Delta x; x_0)$ indeed converges to 0 at the speed of $(\Delta x)^2$. More generally, if f is C^{k+1} function, then the difference between $f(x_0 + \Delta x)$ and the kth order Taylor polynomial $R_k(\Delta x; x_0) = \left|f(x_0 + \Delta x) - L^k(\Delta x; x_0)\right|$ converges to 0 at the speed of $(\Delta x)^{k+1}$.
- One important implication of this formula: whether f is strictly increasing or strictly decreasing locally around x_0 is determined by the sign of $f'(x_0)$ (as shown in the next exercise). If $f'(x_0) = 0$, then it is determined by the sign of $f''(x_0)$, and so on. This is because the effect of each derivative dominates the effect of all the higher derivatives locally.

Exercises

1. Let f be a $\mathcal{C}^1$ function and suppose that $f'(x) > 0$ at x. Show that there exists $\epsilon > 0$ such that f is strictly increasing in $(x-\epsilon, x+\epsilon)$. Is the converse true?

2. **L'Hospital's rule:** Suppose that f and g are differentiable at x', $f(x') = g(x') = 0$, and $g'(x') \neq 0$. Show that $\lim_{x \to x'} \frac{f(x)}{g(x)} = \frac{f'(x')}{g'(x')}$.

3. For $\mathcal{C}^2$ function f, verify that $L^2(x_0 + \Delta x : x_0) = f(x_0) + f'(x_0)\Delta x + \frac{f''(x_0)}{2}(\Delta x)^2$ is the only 2nd order polynomial that satisfies $\frac{\left|f(x_0+\Delta x) - L^2(x_0+\Delta x : x_0)\right|}{(\Delta x)^2} \to 0$ as $\Delta x \to 0$.

Appendix: Proof of Taylor's Theorem

- Fix any $\Delta x > 0$. For $k = 1$, let M be such that $f(x_0 + \Delta x) - L^1(x_0 + \Delta x : x_0) = M(\Delta x)^2$. The RHS evaluates the residual of the linear approximation in terms of $(\Delta x)^2$. Define a function $g(h) = f(x_0 + h) - L^1(x_0 + h : x_0) - Mh^2$. By definition $g(0) = g(\Delta x) = 0$. Hence $g'(s) = 0$ for some $s \in (0, \Delta x)$ by MVT. Note that $g'(0) = f'(x_0) - f'(x_0) = 0$, hence $g''(t) = 0$ for some $t \in (0, s)$ by applying MVT again. As $g''(t) = f''(x_0 + t) - 2M$, we have $M = \frac{f''(x_0+t)}{2}$.
- Similarly, for $k = 2$, define $M = \frac{f(x_0+\Delta x) - L^2(x_0+\Delta x : x_0)}{(\Delta x)^3}$ so that $M(\Delta x)^3$ is the residual with respect to the second order Taylor polynomial. Then we can obtain $M = \frac{f^{(3)}(x_0+t)}{3!}$ for some $t \in (0, \Delta x)$ by applying the MVT three times. This argument works for any $k > 2$.