Math Camp Notes

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Abstract

These are my notes from UCLA summer math camp with Ichiro Obara. I've copied down the definitions that I thought would be useful throughout my PhD studies, usually because I learned something new, I think the definition will be used in coursework, or because it's something I was supposed to learn a long time ago but keep forgetting. If you have any questions, or if you notice any errors, please submit a pull request or email `christopherackerman104@gmail.com`.

Contents

1 Single Variable Calculus I

Definition (Graph). *The set of* (x, y) *that* f *passes through is called the graph of* f*.*

Definition (Convergent Sequence). *A sequence* x_{nn} *converges to* $x^* \in \mathbb{R}$ *if, for any* $\varepsilon > 0$*, there exists an integer* N *such that* $|x_n| \leq K$ *for every* n*.*

Definition (Continuous Function). *Function* $f : X \to \mathbb{R}$ *is continuous at* $x \in X$ *if* $x_n \to x$ *for* $x_n \in X \implies f(x_n) \to f(x)$*.* f *is continuous if it is continuous at every* x *in its domain* X*.*

Definition (Differentiable Function and Derivative). *A function* f *on* (a, b) *is differentiable at* $x \in (a, b)$ *if* $\frac{f(x+\Delta x)-f(x)}{\Delta x}$ *converges to the same number for any sequence* $\Delta x (\neq 0)$ *such that* $|\Delta x| \to 0$*. This number is the derivative of* f *at* x*.* f *is differentiable on* (a, b) *if it is differentiable at every* $x \in (a, b)$*.*

1.1 Rules for Taking Derivatives of Simple Analytic Functions

Definition (Product Rule).

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

Definition (Quotient Rule).

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Definition (Chain Rule). *Consider a composite function*

$$h(x) = g(f(x)).$$

If f *is differentiable at* x *and* g *is differentiable at* $f(x)$*, then* h *is differentiable at* x *and its derivative is given by*

$$h'(x) = g'(f(x)) \cdot f'(x).$$

Definition (Inverse Function). *The inverse function* f^{-1} *of* f *is defined for each* x *in the range of* f *as the unique value that satisfies*

$$f(f^{-1}(x)) = x.$$

If f *is differentiable and has an inverse* f^{-1}*, then* f^{-1} *is differentiable and the derivative of* f^{-1} *is the reciprocal of* f'*.*

Definition (elasticity). *Elasticity measures the percentage change of a variable with respect to a percentage change of another variable. Suppose that two variables,* x *and* y*, satisfy* $y = f(x)$*. The* x*-elasticity of* y *at* (x_0, y_0) *is given by*

$$\lim_{\Delta x \to 0} \frac{\frac{\Delta y}{y_0}}{\frac{\Delta x}{x_0}} = \lim_{\Delta x \to 0} \frac{\frac{f(x_0+\Delta x)-f(x_0)}{f(x_0)}}{\frac{\Delta x}{x_0}} = f'(x_0)\frac{x_0}{f(x_0)}.$$

We often express variables in natural logs to find the elasticity more easily. In logs, the x*-elasticity of* y *is*

$$\frac{d \ln y}{d \ln x}.$$

1.2 Exercises and Solutions

Question 1. *Let* $\{x_n\}_n$ *and* $\{y_n\}_n$ *be two convergent sequences. Show that*

1. $\lim(x_n y_n) = \lim x_n \lim y_n$,
2. $\lim\left(\frac{x_n}{y_n}\right) = \frac{\lim x_n}{\lim y_n}$, *assuming* $\lim y_n \neq 0$.

Answer.

1. Note that

$$xy - x_n y_n = x(y - y_n) + (x - x_n)y_n.$$

So,

$$|xy - x_n y_n| \leq |x(y - y_n)| + |(x - x_n)y_n|. \qquad \text{(Triangle Inequality)}$$

Since $\{y_n\}_n$ is a convergent (thus bounded) sequence, there exists a K such that

$$\begin{aligned} |(x - x_n)y_n| &\leq |x - x_n||y_n| \\ &\leq |(x - x_n)|K. \end{aligned}$$

Do we need to repeat the argument for K here for $\{x_n\}$?

Then,

$$|x(y - y_n)| \leq |x||(y - y_n)| \to 0,$$

and

$$|(x - x_n)y_n| \leq |(x - x_n)|K \to 0.$$

Therefore,

$$|xy - x_n y_n| \to 0.$$

2. Choose m so that $|y_n - y| < \frac{1}{2}y$ for $n \geq m$. Then,

$$|y_n| > \frac{1}{2}|y|. \qquad (n \geq m)$$

Given $\varepsilon > 0$, there is an integer $N > m$ such that $n \geq N$ implies

$$|y_n - y| < \frac{1}{2}|s|^2\varepsilon.$$

Hence, for $n \geq N$,

$$\begin{aligned}\left|\frac{1}{y_n} - \frac{1}{y}\right| &= \left|\frac{y_n - y}{y_n y}\right| \\ &< \frac{2}{|y|^2}|y_n - y| \\ &< \varepsilon.\end{aligned}$$

Substitute this result into the previous proof to get the desired result.

□

Question 2. *Show that a bounded increasing sequence $x_1 \leq x_2 \leq \ldots \leq K < \infty$ must be a convergent sequence (use Bolzano-Weierstrasse).*

Answer.

Since $\{x_n\}_n$ is bounded between x_1 and K, it has a convergent subsequence by BWT. Let $\{x_n\}_k$ denote this subsequence, and let x^* be its limit, and note that $x^* \leq K$. By the definition of a limit, for any $\varepsilon > 0$, there exists a $\hat{k}$ such that

$$|x_{n(k)} - x^*| < \varepsilon \text{ for } k \geq \hat{k}.$$

Because the original sequence is increasing, this also holds for any $n \geq n(k)$ in the original sequence, completing the proof.

□

Question 3. *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Show that $h(x) \equiv g(f(x))$ is a continuous function.*

Answer. Start with $f(x)$ and take the limits one at a time. Use continuity at each step to show that the composite function h is also continuous. □

Question 4. *Derive the price elasticity of $q = -\frac{p}{3} + 8$ as a function of p. Note that the answer will only be defined for $p \in (0, 24)$.*

Answer.

Start with the definition,

$$\begin{aligned}
\varepsilon &= f'(p)\frac{p}{f(p)} \\
f(p) &= \frac{-p}{3} + 8 \\
f'(p) &= -\frac{1}{3} \\
\varepsilon &= \frac{-1}{3} \cdot \frac{p}{\frac{-p}{3} + 8} \\
&= -\frac{1}{3} \cdot \frac{3p}{-p + 24} \\
&= \frac{-p}{24 - p}
\end{aligned}$$

□

Question 5. *Derive the x-elasticity of $f(x) = 3x^2$ and show that it does not depend on x. More generally, discuss why any function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with constant elasticity can be expressed as Ax^B with some $A > 0$ and $B \in \mathbb{R}$.*

Answer.

Let's follow the same steps as the last question.

$$\begin{aligned}
\varepsilon &= f'(x)\frac{x}{f(x)} \\
f(x) &= 3x^2 \\
f'(x) &= 6x \\
\varepsilon &= \frac{6x \cdot x}{3x^2} \\
&= 2.
\end{aligned}$$

For the second part of the question, I'm going to prove the generalized form of the previous result.

$$\begin{aligned}
f(x) &= Ax^B \\
f'(x) &= ABx^{B-1} \\
\varepsilon &= \frac{ABx^{B-1} \cdot x}{Ax^B} \\
&= \frac{ABx^B}{Ax^B} \\
&= B.
\end{aligned}$$

This approach is different from Ichiro's solution and goes in the opposite direction. It doesn't have the same intuition from the log transform, but I think it's still sufficient.

□

Question 6. *Show that the x-elasticity of $f(x)g(x)$ is the sum of the x-elasticity of $f(x)$ and the x-elasticity of $g(x)$.*

Answer.

Take the log transform of the function. More generally, this is a nice way to get separable forms of multiplicative equations.

$$\begin{aligned}\frac{d\ln fg}{d\ln x} &= \frac{d\ln f + d\ln g}{d\ln x} \\ &= \frac{d\ln f}{d\ln x} + \frac{d\ln g}{d\ln x}.\end{aligned}$$

□

2 Single Variable Calculus II

3 Introduction to Optimization