Single Variable Calculus I

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Function of Single Variable

In this lecture and the next, we review basic facts about functions of single variable and study some applications.

- This serves as an introduction and a preview for functions of several variables.
- Some functions of interest are indeed functions of single variable. We are often interested in the effect of one variable over another keeping the other variables fixed (ex. demand function $D(p)$).

A function of single variable maps a number to another number. We use the notation $f : X \to \Re$ to represent function f.

A few definitions (we always use bold letters for definitions):

- X is the **domain** of f, where f is defined. X is usually an interval and often $\Re$ (real line) or $\Re_+$ (nonnegative number) or $\Re_{++}$ (strictly positive number).
- The set of values f can take on its domain is called the **range** of f.
- The set of (x, y) (in $\Re^2$) that f passes through is called the **graph** of f.

For example, for $f(x) = x^2$, we can take $\Re$ as its domain. The range of f is $\Re_+$. The graph of f is $\left\{(x, y) \in \Re^2 |\ y = x^2\right\}$.

What type of functions do we use?

The simplest kind of function is **linear (affine) function** such as $2x$, $3x+5$.
More generally, it can be expressed as $f(x) = ax + b$, where a is the **slope** and b is the y-**intercept** (the value at $x = 0$).

Any two inputs pins down one linear function.

- If any two points $(x', y') \neq (x'', y'')$ in the graph of $f(x) = ax + b$ are given, then you can derive a and b by $a = \frac{y''-y'}{x''-x'}$ and $b = y' - ax'$.
- If slope a and one point (x', y') of the graph are given, you can derive f by solving $a = \frac{f(x)-y'}{x-x'}$, hence $f(x) = y' + a(x - x')$.

Other typical functions:

- **Polynomial**: $2x^2$, $3x^3 - 2x^2 + 4x - 1$,
  $f(x) = a_k x^k + a_{k-1}x^{k-1} + ... + a_1 x + a_0$ is a polynomial function with **degree** k (with $a_k \neq 0$), where $a_k, ..., a_0$ are **coefficients**. A linear function is a special polynomial with degree 1 or 0.
- **Exponential Function**: 2^x, e^x, where $e = 2.718...$ is the exponential constant.
- **Logarithmic Function**: $\ln x$. It is the **inverse** of e^x, i.e. $f(x) = \ln x$ is defined as the unique number that satisfies $x = e^{f(x)}$.

Continuity

It is usually reasonable to assume that a small change of one variable leads to a small change of another, i.e. $f(x')$ should be close to $f(x)$ when x' is close to x. Such f is called a **continuous** function. All the examples we saw, such as polynomials and exponential functions, are continuous functions.

We introduce a more rigorous language to define continuity (and then derivative afterward).

Sequence, Convergence, and Limit

- A **sequence** of numbers $x_1, x_2, ... \in \Re$ is denoted by $\{x_n\}_n$.
- A sequence $\{x_n\}_n$ is **bounded** if there exists a number $K \in \Re$ such that $|x_n| \leq K$ for every n ($|x_n|$ is the absolute value of x_n).
- A sequence $\{x_n\}_n$ **converges to** $x^* \in \Re$ if, for any $\epsilon > 0$, there exists an integer N such that $|x_n - x^*| < \epsilon$ for every $n \geq N$. We write this as $\lim_{n\to\infty} x_n = x^*$ or $x_n \to x^*$. x^* is a **limit** of $\{x_n\}_n$.

A sequence may or may not converge. $1, 2, 1, 2, 1, 2, , ,$ is not a convergent sequence. $x_n = 1/n$ is converging to $x^* = 0$.

Properties of Convergent Sequences

Some useful facts about convergent sequences:

- A convergent sequence has only one limit.
- A convergent sequence is bounded.
- If $x_n \leq K$ for every n, then $x^* \leq K$.

A **subsequence** of a sequence is a sequence that is a “subset” of the sequence.

For example, $1, 1, 1, 1, ...$ is a subsequence of $1, 2, 1, 2, ...$

- Every subsequence of a convergent sequence has the same limit.
- A bounded sequence has a convergent subsequence (Bolzano-Weierstrass theorem).

All those properties except the BW-theorem easily follow from the definition.

Continuous Function

Now we define continuity using sequence and limit.

Continuous Function

Function $f : X \to \Re$ is **continuous at** $x \in X$ if $x_n \to x$ for $x_n \in X$, then $f(x_n) \to f(x)$. f is **continuous** if it is continuous at every x in its domain X.

Note: An equivalent definition with $\epsilon - \delta$: $f : X \to \Re$ is continuous at $x \in X$ if, for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x') - f(x)| < \epsilon$ for any $x' \in X$ such that $|x' - x| < \delta$.

An example of **discontinuous** function: $f(x) = \begin{cases} 2x & \text{for } x < 0 \\ 2x+1 & \text{for } x \geq 0 \end{cases}$.

f is not continuous at $x = 0$. f is continuous at all other points.

Monotonic Function

Monotonic function is a function for which the effect of x over $y = f(x)$ is always the same in sign. It often naturally arises in Economics, is particularly useful when functions are not continuous.

Monotonic Function

$f : X \to \Re$ is **increasing (strictly increasing)** if $f(x') \geq (>) f(x)$ for any $x' \geq (>) x$, **decreasing (strictly decreasing)** if $f(x') \leq (<) f(x)$ for any $x' \geq (>) x$.

For example, $f(x) = e^x$ is a strictly increasing function.

Slope of Nonlinear Function

Consider a change from x to $x + \Delta x$ and the associated change of value from $f(x)$ to $f(x + \Delta x)$. For a linear function, the ratio of these changes $\frac{f(x+\Delta x)-f(x)}{\Delta x}$ is its slope. For general nonlinear functions, this ration depends on Δx and x.

The **derivative/slope** of function f at x is this ratio evaluated at x as Δx goes to 0.

Derivative

The formal definition of differentiability and derivative:

Differentiable Function and Derivative

A function f on (a, b) is **differentiable** at $x \in (a, b)$ if $\frac{f(x+\Delta x)-f(x)}{\Delta x}$ converges to the same number for any sequence $\Delta x(\neq 0)$ such that $|\Delta x| \to 0$. This number is the **derivative** of f at x and denoted by $\frac{df(x)}{dx}$ or $f'(x)$. f is differentiable on (a, b) if it is differentiable at every $x \in (a, b)$.

Note: If a function is differentiable at x, it must be continuous at x by definition.

Example: You probably know the derivatives of the following standard functions.

- $(x^k)' = kx^{k-1}$, where $k \in \Re$.
- $(a^x)' = a^x \ln a$ (in particular, $(e^x)' = e^x$).
- $(\ln x)' = \frac{1}{x}$.

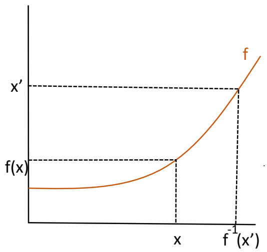
Some Useful Rules

For differentiable f and g, $f+g$ is differentiable and its derivative is given by $(f+g)' = f' + g'$. Here are some other useful rules about derivatives.

- **Product Rule:** $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
- **Quotient Rule:** $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$
- **Chain Rule:** Consider a **composite function** $h(x) = g(f(x))$. If f is differentiable at x and g is differentiable at $f(x)$, then h is differentiable at x and its derivative is given by $h'(x) = g'(f(x)) \cdot f'(x)$.

Inverse and its Derivative

- The **inverse** function f^{-1} of f is defined for each x (in the range of f) as the unique value that satisfies $f(f^{-1}(x)) = x$.
- Ex. **inverse demand function** $D(p)$.



If f is differentiable and has an inverse f^{-1}, then f^{-1} is differentiable. The derivative of f^{-1} is the reciprocal of f' (to check this, apply the chain rule to $f\left(f^{-1}(x)\right) = x$ to obtain $f^{-1\prime}(x) = \frac{1}{f'(f^{-1}(x))}$ (assuming $f'(f^{-1}(x)) \neq 0$)).

Elasticity

The slope of a function depends on the unit of variables. For example, if $D(p) = -ap + b$ is a demand function in dollar, the demand function in terms of cents would be $-\frac{a}{100}p + b$.

We may want to define a measure of relative changes that is independent of the choice of units. **Elasticity** measures the percentage change of a variable with respect to a percentage change of another variable.

Suppose that two variables x, y satisfy $y = f(x)$. The x-**elasticity of** y at (x_0, y_0) is given by

$$\lim_{\Delta x(\neq 0)\to 0} \frac{\frac{\Delta y}{y_0}}{\frac{\Delta x}{x_0}} = \lim_{\Delta x(\neq 0)\to 0} \frac{\frac{f(x_0+\Delta x)-f(x_0)}{f(x_0)}}{\frac{\Delta x}{x_0}} = f'(x_0)\frac{x_0}{f(x_0)}$$

We often express variables in natural logs to find the elasticity more easily. It turns out that $\frac{d\ln y}{d\ln x}$ is exactly the x-elasticity of y.

For example, the **price elasticity of demand** for a demand function $\ln q = \alpha \ln p + ...$ is α.

This can be shown as follows. Let $y = f(x)$, $X = \ln x$, and $Y = \ln y$. Then $Y = \ln f(e^X)$, so we can apply the chain rule to get

$$\frac{dY}{dX} = (\ln y)' \times f'(x) \times \frac{de^X}{dX} = \frac{1}{f(x)} f'(x) x$$

Exercises

1. What is the linear function on $\Re$ which has slope 5 and passes through $(2, 5)$?

2. Let $\{x_n\}_n$ and $\{y_n\}_n$ be two convergent sequences. Show the following.

    1. $\lim (x_n y_n) = \lim x_n \lim y_n$
    2. $\lim \left(\frac{x_n}{y_n}\right) = \frac{\lim x_n}{\lim y_n}$, assuming $\lim y_n \neq 0$.

3. Show that a bounded increasing sequence $x_1 \leq x_2 \leq \cdots \leq K < \infty$ must be a convergent sequence (use BWT).

Exercises

1. Let $f : \Re \to \Re$ and $g : \Re \to \Re$ be continuous functions. Show that $h(x) := g(f(x))$ is a continuous function.
2. What is the derivative of the following functions?
   - $f(x) = e^{2x}$
   - $f(x) = \frac{3x^2-2}{2x+1}$
   - $f(x) = \ln\left(\frac{1}{x}\right)$
3. Let $f(x) = x^3 + 4x^2 + 4x$ on $\Re_+$.
   - What is $f^{-1}(32)$?
   - What is the derivative of f^{-1} at 32?

Exercises

1. Derive the price elasticity of demand function $q = -\frac{p}{3} + 8$ as a function of $p \in (0, 24)$.
2. Derive the x-elasticity of $f(x) = 3x^2$ and show that it does not depend on x. More generally, discuss why any function $f : \Re_+ \to \Re_+$ with constant elasticity can be expressed as Ax^B with some $A > 0$ and $B \in \Re$.
3. Show that the x-elasticity of $f(x)g(x)$ is the sum of the x-elasticity of $f(x)$ and the x-elasticity of $g(x)$.