Optimization with Single Variable

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Optimization with Single Variable

Many problems in Economics can be expressed as some type of optimization problem.

Here we study the simplest optimization problem with one variable over an interval such as:

$$\max_{x \in [a,b]} f(x)$$

or equivalently

$$\max_{x \in \Re} f(x) \text{ s.t. } x \geq a \ \& \ x \leq b$$

Throughout we assume that every function is a $\mathcal{C}^1$ function unless mentioned otherwise.

Example: Profit Maximization by Monopoly

Consider a monopoly firm which has the price setting power. If it sets the price of its product to $p \geq 0$, then it can sell $D(p)$ units of products. Each unit of product costs c to produce and there is a fixed cost F for production independent of the scale of production.

This monopoly firm's profit maximization problem is (with the possibility of shutdown)

$$\max\left\{\max_{p\geq 0}(p-c)D(p)-F, 0\right\},$$

Maximum and Minimum

When x^* maximizes/minimizes f over X, x^* is a **maximum/minimum point** and achieves the maximum value of f over X.

Definition: Maximum and Minimum

- $x^* \in X$ is a **maximum point** of f on X if $f(x^*) \geq f(x)$ for all $x \in X$.
- $x^* \in X$ is a **(strict) local maximum point** of f on X if there is $\epsilon > 0$ and $f(x^*) \geq (>) f(x)$ for all $x \in X$ such that $|x - x^*| < \epsilon$.
- $x^* \in X$ is a **minimum point** of f on X if $f(x^*) \leq f(x)$ for all $x \in X$.
- $x^* \in X$ is a **(strict) local minimum point** of f on X if there is $\epsilon > 0$ and $f(x^*) \leq (<) f(x)$ for all $x \in X$ such that $|x - x^*| < \epsilon$.

Unconstrained Optimization: First Order Condition

How to solve optimization problems? We first consider optimization problems over open intervals (**unconstrained optimization problems**).

Remember this?

- $f'(x) > 0 \Rightarrow f$ is strictly increasing around x.
- $f'(x) < 0 \Rightarrow f$ is strictly decreasing around x.

So we have the following simple but useful result.

Theorem: First Order Condition

If f achieves the maximum (or the minimum) value at x^* on (a, b), then $f'(x^*) = 0$.

$f'(x^*) = 0$ is called the **first order condition (FOC)**.

Comment:

Assume that there is an optimal solution. Since FOC is necessary for optimality,

- FOC has only one solution $\Rightarrow$ the solution must be the optimal one.
- FOC has multiple solutions $\Rightarrow$ the optimal solution must be one of them.

$\rightarrow$ the question of finding the maximum is reduced to the question of solving the FOC equation.

Example: Monopoly Price

Consider a profit maximization problem $\max_{p \in \Re_{++}} (p - c)\, D(p)$

The profit maximizing price $p^* > 0$ satisfies the following first order condition:

$$D(p) + (p - c)\, D'(p) = 0$$

Hence $-\mathcal{E}_D(p^*) = \frac{p^*}{p^* - c}$: $(-)$ the price elasticity of demand $=$ the reciprocal of the profit margin at the optimal price p^*.

Constrained Optimization

Next we consider optimization problems over closed and bounded intervals:

$X = [a, b]$ (**constrained optimization problems**).

We first note that the existence of a maximum point (and a minimum point) is guaranteed in this case.

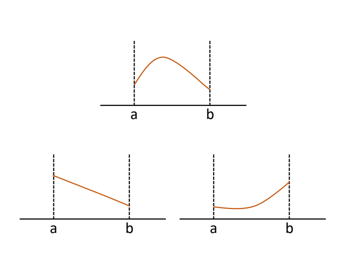
Extreme Value Theorem

If $f : \Re \to \Re$ is continuous, then it has a maximum point and a minimum point over any closed and bounded interval $[a, b]$.

(The proof based on the BW theorem is postponed until we study a more general case.)

Necessary Condition for Maximum

Suppose that x^* maximizes f on $[a, b]$. What conditions would x^* need to satisfy? Three possible cases.



- If $x^* \in (a, b)$, then $f'(x^*) = 0$ must be satisfied.
- If $x^* = a$, then the derivative $f'(a)$ must be nonpositive.
- If $x^* = b$, then the derivative $f'(b)$ must be nonnegative.

Necessary Condition for Maximum

Putting these conditions together, we have a necessary condition for optimality in this case.

Theorem: Necessary Condition for Constrained Problem

Suppose that x^* is a maximum point for f on $[a, b]$. Then one of the following conditions must hold:

- $x^* \in (a, b)$ and $f'(x^*) = 0$
- $x^* = a$ and $f'(x^*) \leq 0$
- $x^* = b$ and $f'(x^*) \geq 0$

Comment:

Consider the case with $x^* = b$. The optimality condition can be expressed by an equation $f'(x) - \lambda = 0$, where $\lambda = f'(b) \geq 0$. The following interpretations would be useful later...

- λ can be regarded as an artificial "penalty" or marginal cost that is associated with going beyond the constraint $x \leq b$. The original constrained problem was transformed into another unconstrained problem with the penalty.
- λ represents the value of relaxing the constraint $x \leq b$. If b increases by $\Delta b > 0$, then the maximized value of f would increase roughly by $\lambda \Delta b$.

Second Order Condition

FOC is a <u>necessary condition</u> for a maximum (or a minimum), but not a <u>sufficient condition</u>. A point satisfying FOC can be a (local) maximum, a (local) minimum, or neither.

Second Order Condition

If f is twice differentiable, then we can say a bit more about f's behavior around the point satisfying FOC.

Theorem: Second Order Condition

Suppose that f is a $\mathcal{C}^2$ function and $f'(x^*) = 0$ at $x^* \in (a, b)$.

- If $f''(x^*) < 0$, then x^* is a strict local maximum point for f.
- If $f''(x^*) > 0$, then x^* is a strict local minimum point for f.

Proof Detail

Concavity and Sufficiency

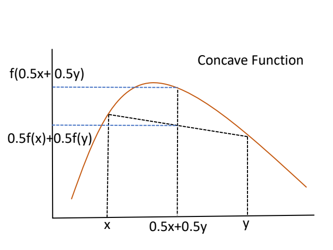
When can we say x^* satisfying FOC is a maximum point, not just a local maximum point?

One sufficiency condition is that f'' is always negative. More generally, FOC is sufficient if f is a **concave** function.

Let $f : X \to \Re$ be a function on some interval X.

Concave and Convex Function

- f is **concave** if $f((1-\lambda)x + \lambda y) \geq (1-\lambda)f(x) + \lambda f(y)$ for every $x, y \in X$ and $\lambda \in [0, 1]$.
- f is **strictly concave** if $f((1-\lambda)x + \lambda y) > (1-\lambda)f(x) + \lambda f(y)$ for every $x \neq y \in X$ and $\lambda \in (0, 1)$.
- f is **convex** $f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$ for every $x, y \in X$ and $\lambda \in [0, 1]$.
- f is **strictly convex** if $f((1-\lambda)x + \lambda y) < (1-\lambda)f(x) + \lambda f(y)$ for every $x \neq y \in X$ and $\lambda \in (0, 1)$.



A few useful facts about concave function:

- If f is differentiable, f is concave $\Leftrightarrow f(x) \leq f(x_0) + f'(x_0)(x - x_0)$ for any $x \in X$ given any x_0.
- If f is twice differentiable, f is concave $\Leftrightarrow f''(x) \leq 0$ for all $x \in X$.

Note: Similar results hold for convex functions, which are useful for minimization. Note that f is concave if and only if $-f$ is convex.

FOC is Sufficient with Concavity

Theorem 1: Sufficiency

Suppose that f is a concave function on (a, b) and $f'(x^*) = 0$ at $x^* \in (a, b)$.

Then f achieves the maximum value at x^* on (a, b).

Proof

Since f is concave, $f(x) \leq f(x^*) + f'(x^*)(x - x^*) = f(x^*)$ for every $x \in (a, b)$.

Hence x^* is a maximum point of f on (a, b).

Another Sufficiency Condition

Here is another condition that guarantees the sufficiency of FOC.

Theorem 2: Sufficiency

Suppose that x^* is the only point in (a, b) that satisfies the FOC $f'(x^*) = 0$. If $f''(x^*) < 0$, then x^* must be a maximum point for f on (a, b).

Intuition: Since x^* is a strict local maximum, if there is any point $x \neq x^* \in (a, b)$ that achieves a higher value than $f(x^*)$, f must be flat somewhere between x and x^*. ▸ Proof Detail

Exercises

1. Consider the following type of maximization problem: $\max_{x\in(a,b]} f(x)$. What would be the necessary condition that the optimal solution x^* must satisfy?

2. Solve the following optimization problems.
   - $\max_{x\in\Re_{++}} f(x) = \ln x - 3x$
   - $\max_{x\in\Re_{+}} f(x) = -x^3 + 4x^2 - 5x - 2$

3. A consumer spends \$$x$ out of \$100 to buy some product at price $p > 0$. Suppose that her utility when purchasing $x \in \left[0, \frac{100}{p}\right]$ units of the product is given by $\frac{2x}{x+1} + 100 - px$. Find her utility-maximizing consumption $x(p)$ as a function of price.

Exercises

1. Prove the following statements:
   - If f is a concave function, then $-f$ is a convex function.
   - If f and g are concave, then $f+g$ is a concave.
   - If f is concave and convex, then f must be a linear function.
2. You would like to find a linear model $y = a + bx$ that explains the data $(x_t, y_t), t = 1, ..., T$ well. It makes sense to pick a and b to satisfy $\overline{y} = a + b\overline{x}$ for the average $\overline{x} = \frac{\sum x_t}{T}$ and $\overline{y} = \frac{\sum y_t}{T}$. Then your model is effectively one parameter model $\tilde{y} = b\tilde{x}$, where $\tilde{z} = z - \overline{z}$ for $z = x, y$. Find the parameter b^* that minimizes the (sum of squared) error $\sum_{t=1}^{T}(\tilde{y}_t - b\tilde{x}_t)^2$.
3. Solve $\max_{x\in\Re} 2xe^{-x}$.

Appendix: Proofs

Proof of SOC

- The 1st statement: If $f''(x^*) < 0$, then f' is strictly decreasing in $(x^* - \epsilon, x^* + \epsilon)$ for some $\epsilon > 0$. So $f'(x) < 0$ below x^* and $f' > 0$ above x^* for any x within ϵ from x^*. Now suppose that, say, there exists $x' \in (x^*, x^* + \epsilon)$ such that $f(x^*) \leq f(x')$. Then there must be $\hat{x} \in (x^*, x')$ such that $f(x') = f(x^*) + f'(\hat{x})(x' - x^*)$ by the mean value theorem. But this contradicts $f'(\hat{x}) < 0$.
- The proof of the 2nd statement is exactly the same.

Appendix: Proofs

Proof of Theorem 2

- Suppose that x^* is not a maximum point. Then there exists a point such as, say, $x' \in (x^*, b)$ with $f(x') > f(x^*)$.
- Since $f''(x^*) < 0$, $f(x) < f(x^*)$ for nearby x above x^*. By the extreme value theorem, there must be a minimum point $\hat{x} \in (x^*, x')$ of f over $[x^*, x']$.
- Then $f'(\hat{x}) = 0$ must hold, which is a contradiction.