# Answer for Selected Exercises

## 1 Single Variable Calculus I

- Let $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = y^*$. We would like to show $\lim_{n\to\infty} x_n y_n = x^* y^*$. Note that $x^*y^* - x_n y_n = x^*(y^* - y_n) + (x^* - x_n) y_n$. So $|x^*y^* - x_n y_n| \le |x^*(y^* - y_n)| + |(x^* - x_n) y_n|$ by the triangle inequality. Since $\{y_n\}_n$ is a convergent sequence, hence bounded, there exists $K$ s.t. $|(x^* - x_n) y_n| \le |x^* - x_n| |y_n| \le |(x^* - x_n)| K$. Then $|x^*(y^* - y_n)| \le |x^*| |(y^* - y_n)| \to 0$ and $|(x^* - x_n) y_n| \le |(x^* - x_n)| K \to 0$. Therefore $|x^*y^* - x_n y_n| \to 0$.

- Such a sequence is between $x_1$ and $K$, hence bounded. So there is a convergent subsequence $x_{n(k)}$ such that $\lim_{k\to\infty} x_{n(k)} = x^* (\le K)$ by the Bolzano-Weierstrass theorem. This means that, for any $\epsilon > 0$, there is $\hat{k}$ such that $\left|x_{n(k)} - x^*\right| < \epsilon$ for any $k \ge \hat{k}$. But then $|x_n - x^*| < \epsilon$ holds for any $n \ge n(k)$ because $x_n$ must be between $x_{n(k)}$ and $x^*$. (**Note:** You do not need BWT to prove this. Let $x^* = \sup\{x_n, n = 1, 2, ...\} (\le K)$ be the least upper bound of these points (i.e. the smallest point that is at least as large as any of $x_n$). Then for any $\epsilon > 0$, there exists $N$ such that $x_N > x^* - \epsilon$ (otherwise $x^*$ is not the "least"), hence $x_n > x^* - \epsilon$ for any $n \ge N$. As $x_n \le x^*$ for all $n$, this sequence converges to $x^*$).

- As $x_n \to x^*$, $f(x_n) \to f(x^*)$ by the continuity of $f$. Then $h(x_n) = g(f(x_n)) \to g(f(x^*)) = h(x^*)$ by the continuity of $g$. Hence $h$ is continuous.

- For $f(x) = x^3 + 4x^2 + 4x$,
  - $f(2) = 32$. So $f^{-1}(32) = 2$.
  - The derivative of the inverse at 32 is the reciprocal of the derivative of $f$ at 2, which is $\frac{1}{f'(2)} = \frac{1}{32}$.

- $\frac{dq}{dp}\frac{p}{q} = -\frac{1}{3}\frac{p}{-\frac{1}{3}p + 8} = -\frac{p}{24-p}$

- If the elasticity of some variable $y$ with respect to $x$ is constant and independent of $x$, then it must be represented as $\ln y = a \ln x + b$ for some number $a, b$, where $a$ is the parameter for constant elasticity. Solving this for $y$, we can obtain $y = Ax^B$, where $A = e^b$ and $B = a$.

## 2 Single Variable Calculus II

- Since $f$ is $\mathcal{C}^1$ (the derivative is continuous) and $f'(x) > 0$ at $x$, there exists $\varepsilon > 0$ such that $f'(x') > 0$ for all $x'$ such that $|x - x'| < \varepsilon$. For any $\widehat{x} < \widetilde{x}$ in $(x - \varepsilon, x + \varepsilon)$, $f(\widetilde{x}) - f(\widehat{x}) = f'(t)(\widetilde{x} - \widehat{x}) > 0$ for some $t \in (\widehat{x}, \widetilde{x})$ by the mean value theorem, so $f(\widetilde{x}) > f(\widehat{x})$. Hence $f$ is strictly increasing on $(x - \varepsilon, x + \varepsilon)$. Consider the following converse: suppose that $f$ is strictly increasing in $(x - \varepsilon, x + \varepsilon)$, does this imply $f'(x) > 0$? No. $f(x) = x^3$ is strictly increasing in $(-\epsilon, \epsilon)$ for any $\epsilon > 0$, but $f'(0) = 0$.

- Since $f$ is $\mathcal{C}^2$, $f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x + \frac{f''(x_0 + t\Delta x)}{2}(\Delta x)^2$ for some $t \in (0,1)$ (which depends on $\Delta x$). Let $A + B\Delta x + C(\Delta x)^2$ be an arbitrary second order degree polynomial in $\Delta x$ and consider
$$\frac{\left|f(x_0 + \Delta x) - \left(A + B\Delta x + C(\Delta x)^2\right)\right|}{(\Delta x)^2}.$$
If $A \neq f(x_0)$, then the numerator converges to $|f(x_0) - A| \neq 0$ as $\Delta x \to 0$. Hence this ratio does not converge to 0. So $A = f(x_0)$ must hold. Similarly, suppose that $B \neq f'(x_0)$. Then the numerator divided by $\Delta x$ converges to $|f'(x_0) - B| \neq 0$. Hence the numerator divided by $(\Delta x)^2$ does not converge to 0. So $B = f'(x_0)$ must hold. Finally suppose that $C \neq f''(x_0)$, then this ratio converges to $\left|\frac{f''(x_0)}{2} - C\right|$ as the second order derivative $f''$ is continuous by assumption. So $C = \frac{f''(x_0)}{2}$ must hold to make this limit 0.

## 3 Optimization with Single Variable

- Two optimization problems.
  1. The FOC $f'(x) = \frac{1}{x} - 3 = 0$ must be satisfied at the maximum point. The only point satisfying FOC is $x^* = \frac{1}{3}$. Note that $f'x) > 0$ for $x < x^*$ and $f'x) < 0$ for $x > x^*$. So $x^*$ indeed achieves the maximum value (or Theorem 2 can be applied for this question to conclude that $x^*$ is the maximum point).
  2. First, there must be a maximum $x^*$ because this function converges to $-\infty$ as $x \to \infty$.There are two possibilities: (1) $x^* = 0$ & $f'(0) \leq 0$ and (2) $x^* > 0$ & $f'(x^*) = 0$. Since $f'(0) = -5$, (1) is indeed satisfied. For (2), note that $f'(x) = (-3x + 5)(x - 1)$. So (2) is satisfied at 1 and 5/3. Since $f''(1) = 2$, $x = 1$ corresponds to a strict local minimum. So it boils down to the competition between 0 and 5/3. If we compare the value of $f$ directly, we get $f(0) > f(5/3)$. So $x^* = 0$ maximizes $f$.

- There are three possible cases: (1) $x^* = 0$ & $f'(0) \leq 0$, (2) $x^* \in (0, 100/p)$ & $f'(x^*) = 0$, and (3) $x^* = 100/p$ and $f'(100/p) \geq 0$. The derivative of utility $f'(x)$ is $\frac{2}{(x+1)^2} - p$. So (1) is satisfied at 0 if and only if $p \geq 2$. (2) is satisfied if and only if $\sqrt{\frac{2}{p}} - 1 \in (0, 100/p)$. This is satisfied when when $p \in (0,2)$ and $0 \leq p^2 + 198p + 100^2$. But the second inequality is strictly satisfied for any $p \in (0,2)$. (3) is satisfied when $\frac{2}{(100/p+1)^2} \geq 0$. But this equality becomes $0 \geq p^2 + 198p + 100^2$, which is never satisfied. Note that only (1) holds for $p \geq 2$ and (2) holds for $p \in (0,2)$. Since the solution for this problem exists, the maximum point is 0 when $p \geq 2$ and $\sqrt{\frac{2}{p}} - 1$ when $p \in (0,2)$.

- Three properties of concavity/convexity:

  1. Take any $x, y$ and $\lambda \in [0,1]$. Since $f$ is concave, $f((1-\lambda)x + \lambda y) \geq (1-\lambda) f(x) + \lambda f(y)$. Then $-f((1-\lambda)x + \lambda y) \leq (1-\lambda)(-f(x)) + \lambda(-f(y))$. Hence $-f$ is convex.
  2. Take any $x, y$ and $\lambda \in [0,1]$. $(f+g)((1-\lambda)x + \lambda y) = f((1-\lambda)x + \lambda y) + g((1-\lambda)x + \lambda y)$ by definition. Since $f$ and $g$ is concave, this is at least as large as

$$\begin{aligned} & (1-\lambda) f(x) + \lambda f(y) + (1-\lambda) g(x) + \lambda g(y) \\ = \; & (1-\lambda)(f+g)(x) + \lambda (f+g)(y). \end{aligned}$$

     So $f + g$ is concave.
  3. We show that the slope of the function is the same everywhere. Since $f$ is concave and convex, $f((1-\lambda)x + \lambda y) = (1-\lambda) f(x) + \lambda f(y)$ for any $x, y$ and $\lambda \in [0,1]$. Rewrite this as $\frac{f(x+\lambda(y-x)) - f(x)}{\lambda(y-x)} = \frac{f(y)-f(x)}{y-x}$. This means that the slope between $(x, f(x))$ and $(y, f(y))$ and the slope between $(x, f(x))$ and $((1-\lambda)x + \lambda y, f((1-\lambda)x + \lambda y))$ are the same for any $y \neq x$ and for any $\lambda \in (0,1]$ for a given $x$. Pick any $x_0$ and let $a_0$ be this slope. Now take any two points $x' < x''$, we show that the slope between these two points must be $a_0$ as well. This is true if $x'$ or $x''$ is $x_0$ by definition. If not, this follows from the fact that both the slope between $x_0$ and $x'$ and the slope between $x_0$ and $x''$ is $a_0$.