Econometrica, Vol. 71, No. 1 (January, 2003), 173–204

BUBBLES AND CRASHES

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We present a model in which an asset bubble can persist despite the presence of rational arbitrageurs. The resilience of the bubble stems from the inability of arbitrageurs to temporarily coordinate their selling strategies. This *synchronization problem* together with the individual incentive to *time the market* results in the persistence of bubbles over a substantial period. Since the derived trading equilibrium is unique, our model rationalizes the existence of bubbles in a strong sense. The model also provides a natural setting in which news events, by enabling synchronization, can have a disproportionate impact relative to their intrinsic informational content.

KEYWORDS: Bubbles, crashes, temporal coordination, synchronization, market timing, 'overreaction', limits to arbitrage, behavioral finance.

1. INTRODUCTION

NOTORIOUS EXAMPLES OF BUBBLES followed by crashes include the Dutch tulip mania of the 1630's, the South Sea bubble of 1719–1720 and more recently the Internet bubble which peaked in early 2000. Standard neoclassical theory precludes the existence of bubbles, by backwards induction arguments in finite horizon models and a transversality condition in infinite horizon models (cf. Santos and Woodford (1997)). Similar conclusions emerge from 'no trade theorems' in settings with asymmetric information (cf. Milgrom and Stokey (1982), Tirole (1982)).[2] All agents in these models are assumed to be rational. The *efficient markets hypothesis* also implies the absence of bubbles. Many proponents of this view (see, for instance, Fama (1965)) are quite willing to admit that behavioral/boundedly rational traders are active in the market place. However, they argue that the existence of sufficiently many well-informed arbitrageurs guarantees that any potential mispricing induced by behavioral traders will be corrected.

We develop a model that challenges the efficient markets perspective. In particular, we argue that bubbles can survive despite the presence of rational arbitrageurs who are collectively both well-informed and well-financed. The backdrop of our analysis is a world in which there are *some* 'behavioral' agents variously subject to animal spirits, fads and fashions, overconfidence and related psychological biases that might lead to momentum trading, trend chasing, and the like.

[1] We thank seminar participants at various universities and conferences, anonymous referees, and especially Haluk Ergin, Ming Huang, Sebastian Ludmer, Stephen Morris, José Scheinkman, Andrei Shleifer, and an editor for helpful comments. The authors acknowledge financial support from the National Science Foundation and Princeton's CEPS program.

[2] A more extensive review of the literature on bubbles can be found in Brunnermeier (2001).

There is by now a large literature that documents and models such behavior.[3] We do not investigate why behavioral biases needing rational corrections arise in the first place; for this we rely on the body of work very partially documented in footnote 3. We study the impact of rational arbitrage in this setting.

We suppose that rational arbitrageurs understand that the market will eventually collapse but meanwhile would like to ride the bubble as it continues to grow and generate high returns. Ideally, they would like to exit the market just prior to the crash, to "beat the gun" in Keynes' colorful phrase. However, market timing is a difficult task. Our arbitrageurs realize that they will, for a variety of reasons, come up with different solutions to this optimal timing problem. This dispersion of exit strategies and the consequent lack of synchronization are precisely what permit the bubble to grow, despite the fact that the bubble bursts as soon as a sufficient mass of traders sells out. This scenario is the starting point of our work.

We present a model that formalizes the synchronization problem described above. Our approach emphasizes two elements: *dispersion of opinion* among rational arbitrageurs and the need for *coordination*.

We assume that the price surpasses the fundamental value at a random point in time t_0. Thereafter, rational arbitrageurs become sequentially aware that the price has departed from fundamentals. Arbitrageurs do not know whether they have learnt this information early or late relative to other rational arbitrageurs. In addition to its literal interpretation, the assumption of sequential awareness can be viewed as a metaphor for a variety of factors such as asymmetric information, differences in viewpoints, etc. that, in our model, find expression in the feature we seek to explore and emphasize, that is, temporal miscoordination. Arbitrageurs have common priors about the underlying structure of the model.

The coordination element in our model is that selling pressure only bursts the bubble when a sufficient mass of arbitrageurs have sold out. Arbitrageurs face financial constraints that limit their stock holding as well as their maximum short-position. This limits the price impact of each arbitrageur. Large price movements can only occur if the accumulated selling pressure exceeds some threshold κ. In other words, a permanent shift in price levels requires a coordinated attack. In this respect, our model shares some features with the static second generation models of currency attacks in the international finance literature (Obstfeld (1996)).

Thus, our model has both elements of cooperation and of competition. On the one hand, at least a fraction κ of arbitrageurs need to sell out in order for the bubble to burst; this is the coordination/cooperative aspect. On the other hand, arbitrageurs are competitive since at most a fraction $\kappa < 1$ of them can leave the market prior to the crash.

In the equilibrium of our model, arbitrageurs stay in the market until the subjective probability that the bubble will burst in the next trading round is sufficiently high. Arbitrageurs who get out of the market just prior to the crash

[3] See, for instance, Daniel, Hirshleifer, and Subrahmanyam (1998), Hirshleifer (2001), Odean (1998), Thaler (1991), Shiller (2000), and Shleifer (2000).

make the highest profit. Arbitrageurs who leave the market very early make some profit, but forgo much of the higher rate of appreciation of the bubble. Arbitrageurs who stay in the market too long lose all capital gains that result from the bubble's appreciation.

Our results regarding market timing seem to fit well with popular accounts of the behavior of hedge fund managers during the recent internet bubble.[4,5] For example, when Stanley Druckenmiller, who managed George Soros' $8.2 billion Quantum Fund, was asked why he didn't get out of internet stocks earlier even though he knew that technology stocks were overvalued, he replied that he thought the party wasn't going to end so quickly. In his words "We thought it was the eighth inning, and it was the ninth." Faced with mounting losses, Druckenmiller resigned as Quantum's fund manager in April 2000.[6]

Rational arbitrageurs ride the bubble even though they know that the bubble will burst for exogenous reasons by some time $t_0 + \bar{\tau}$ if it has not succumbed to endogenous selling pressure prior to that time. Here t_0 is the unknown time at which the price path surpasses the fundamental value and arbitrageurs start getting aware of the mispricing. In equilibrium each arbitrageur sells out after waiting for some interval after she becomes aware of the mispricing. If arbitrageurs' opinions are sufficiently dispersed, there exists an equilibrium in which the bubble never bursts prior to $t_0 + \bar{\tau}$. Even long after the bubble begins and after all agents are aware of the bubble, it is nevertheless the case that endogenous selling pressure is never high enough to burst the bubble. For more moderate levels of dispersion of opinion (or equivalently for smaller κ) endogenous selling pressure advances the date at which the bubble eventually collapses. Nevertheless, the bubble grows for a substantial period of time. Moreover, these equilibria are *unique* (modulo a natural refinement). Thus, there is a striking failure of the backwards induction argument that would yield immediate collapse in a standard model. The synchronization 'problem' arbitrageurs face is from their point of view a blessing rather than a curse. After becoming aware of the bubble, they always have the option to sell out, but rather optimally choose to ride the bubble over some interval.

Our model provides a natural setting in which news events can have a disproportionate impact relative to their intrinsic informational content. This is because

[4] Brunnermeier and Nagel (2002) provide a detailed documentation of hedge funds' stock holdings.

[5] Kindleberger (1978) notes that even Isaac Newton tried to ride the South Sea Bubble in 1720. He got out of the market at £7,000 after making a £3,500 profit, but he decided to re-enter it thereby losing £20,000 at the end. Frustrated with his experience, he concluded: "I can calculate the motions of the heavenly bodies, but not the madness of people."

[6] However, fund managers can also not afford to simply stay away from a rapidly growing bubble. Julian Roberts, manager of the legendary Tiger Hedge Fund, refused to invest in technology stocks since he thought they were overvalued. The Tiger Fund was dissolved in 1999 because its returns could not keep up with the returns generated by dotcom stocks. A Wall Street analyst who has dealt with both hedge fund managers vividly summarized the situation: "Julian said, 'This is irrational and I won't play,' and they carried him out feet first. Druckenmiller said, 'This is irrational and I will play,' and they carried him out feet first." *New York Times*, April 29, 2000, "Another Technology Victim; Top Soros Fund Manager Says He 'Overplayed' Hand."

news events make it possible for agents to synchronize their exit strategies. Of course, large price drops are themselves significant synchronizing events, and we investigate how an initial price drop may lead to a full-fledged collapse. Thus the model yields a rudimentary theory of 'overreaction' and 'price cascades' and suggests a rationale for psychological benchmarks such as 'resistance lines.' In addition, our model provides a framework for understanding fads in information such as the (over-)emphasis on trade figures in the eighties and on interest rates in the nineties.

Overall, the idea that the bursting of a bubble requires synchronized action by rational arbitrageurs, who might lack both the incentive and the ability to act in a coordinated way, has important implications. It provides a theoretical argument for the existence and persistence of bubbles. It undermines the central presumption of the efficient markets perspective that not all agents need to be rational for prices to be efficient, and hence provides further support for behavioral finance models that do not explicitly model rational arbitrageurs. More generally, our model suggests a new mechanism that limits the effectiveness of arbitrage. The latter might be relevant in other contexts; see Abreu and Brunnermeier (2002) for an application of the framework developed in this paper.

The remainder of the paper is organized as follows. Section 2 illustrates how the analysis relates to the literature. In Section 3 we introduce the primitives of the model and define the equilibrium. Section 4 shows that all arbitrageurs employ trigger strategies in any trading equilibrium. Section 5 demonstrates that if the dispersion of opinion among arbitrageurs is sufficiently large, they never burst the finite horizon bubble and it only crashes for exogenous reasons at its maximum size. For smaller dispersion of opinion the bubble also persists but arbitrageurs burst it before the end of the horizon. Section 6 investigates the role of synchronizing events including the special case of price events. Section 7 concludes.

2. RELATED LITERATURE

The "no-trade theorems" provide sufficient conditions for the absence of speculative bubbles. Allen, Morris, and Postlewaite (1993) develop "contra-positives" of the no-trade theorems highlighting necessary conditions for the existence of bubbles. They call a mispricing a bubble if it is mutual knowledge that the price is too high. Of course, mutual knowledge (that is, all traders know) does not imply common knowledge (that is, all traders know, that all traders know, and so on ad infinitum). They provide illustrative examples that satisfy their conditions and support bubbles. In Allen and Gorton (1993) fund managers "churn bubbles" at the expense of their less informed client investors (who do not know the skill of the fund manager). They take on an overvalued asset even though they know that they might be 'last in line' and hence unable to unload the asset. The order of trades is random and exogenous in their model. In equilibrium, our sequential awareness assumption also leads to sequential trading and uncertainty about the timing of one's trades relative to other arbitrageurs' trades.

The papers described above assume that all agents are fully rational. In contrast, our work falls within the class of models in which rational arbitrageurs interact with boundedly rational behavioral traders. In DeLong, Shleifer, Summers, and Waldmann (1990b) rational arbitrageurs push up the price after some initial good news in order to induce behavioral feedback traders to aggressively buy stocks in the next period. This delayed reaction by feedback traders allows the arbitrageurs to unload their position at a profit. In DeLong, Shleifer, Summers, and Waldmann (1990a) arbitrageurs' risk aversion and short horizons limit their ability to correct the mispricing. In contrast, in our model arbitrageurs initially do not even attempt to lean against the mispricing even though they are risk-neutral and infinitely lived. In Shleifer and Vishny (1997), professional fund managers forgo profitable long-run arbitrage opportunities because the price might depart even further from the fundamental value in the intermediate term. In that case, the fund manager would have to report intermediate losses causing client investors to withdraw part of their money which forces her to liquidate at a loss. Knowing this might happen in advance, the fund manager only partially exploits the arbitrage opportunity.

In these papers rational arbitrageurs do not have the collective ability to correct mispricing either because of their risk aversion or because of exogenously assumed capital constraints. In contrast in our paper, the aggregate resources of all arbitrageurs are sufficient to bring the price back to its fundamental value. Thus, the weakness of arbitrage in our model is particularly striking because arbitrageurs can jointly correct the mispricing, but it nevertheless persists.

Our assumption that a critical mass of speculators is needed to burst a bubble has its roots in the currency attack literature. Obstfeld (1996) highlights the necessity of coordination among speculators to break a currency peg and points out the resulting multiplicity of equilibria in a setting with symmetric information. Morris and Shin (1998) introduce asymmetric information and derive a unique equilibrium by applying the *global games* approach of Carlsson and van Damme (1993). Both currency attack models are static in the sense that speculators only decide whether to attack now or never. In contrast, our model is dynamic. The dispersion of opinion among arbitrageurs, which we model as sequential awareness, can be related to the idea of "asynchronous clocks." The latter were first introduced in the computer science literature (see Halpern and Moses (1990)). Morris (1995) applies the concept to a dynamic coordination problem in labor economics. There are several differences between our papers. His model satisfies strategic complementarity and the global games approach applies. Our model has elements of both coordination and competition. The latter leads to a *preemption motive* that plays a central role in our analysis. It is important for Morris' result that players can only condition on their individual clocks and not on calendar time or on their payoffs, whereas in our model they can. In particular, traders learn from the existence of the bubble in our setting. In the richer strategy set of our model strategic complementarity is not satisfied and the global games approach does not apply. See footnote 15 for a discussion of this point. In the typical global game application the symmetric information game has multiple equilibria, and any degree of asymmetric information leads to a unique equilibrium. Thus, there is a discontinuity at the point where asymmetry vanishes. In contrast, in our dynamic model the preemption motive leads to a unique equilibrium even under symmetric information. Furthermore, as the extent of asymmetric information goes to zero, the equilibrium outcome converges to the symmetric information (no bubble) outcome.

3. THE MODEL

Historically, bubbles have often emerged in periods of productivity enhancing structural change. Examples include the railway boom, the electricity boom, and the recent internet and telecommunication boom. In the latter case, and in many of the historical examples, sophisticated market participants gradually understood that the immediate economic impact of these structural changes was limited and that their full implementation would take a long time. They also realized that only a few of the companies engaged in the new technology would survive in the long run. On the other hand, less sophisticated traders over-optimistically believed that a 'paradigm shift' or a 'new economy' would lead to permanently higher growth rates.

We assume the price process depicted in Figure 1. This price process reflects the scenario outlined above, and may be interpreted as follows. Prior to $t = 0$

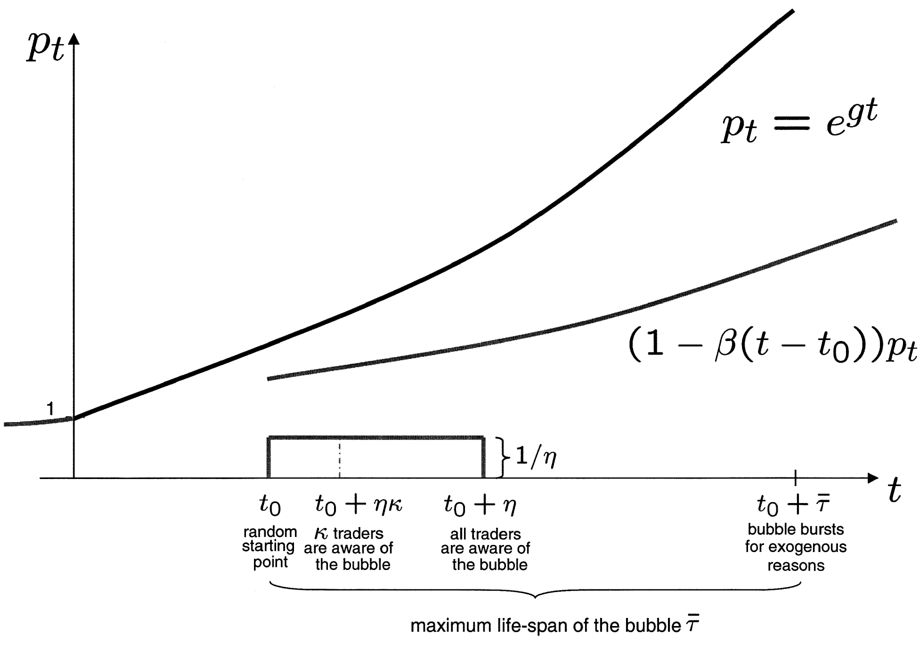

FIGURE 1.—Illustration of price paths.

the stock price index coincides with its fundamental value, which grows at the risk-free interest rate r, and rational arbitrageurs are fully invested in the stock market. Without loss of generality, we normalize the starting point to $t = 0$ and the stock market price at $t = 0$ to $p_0 = 1$. From $t = 0$ onwards the stock price p_t grows at a rate of $g > r$, that is $p_t = e^{gt}$.[7] This higher growth rate may be viewed as emerging from a series of unusual positive shocks that gradually make investors more and more optimistic about future prospects. Until some random time t_0, the higher price increase is justified by the fundamental development.[8] We assume that t_0 is exponentially distributed on $[0, \infty)$ with the cumulative distribution function $\Phi(t_0) = 1 - e^{-\lambda t_0}$. Nevertheless, the price continues to increase at the faster rate g even after t_0. Hence, from t_0 onwards, only some fraction $(1 - \beta(\cdot))$ of the price is justified by fundamentals, while the fraction $\beta(\cdot)$ reflects the "bubble component." We assume that $\beta(\cdot) : [0, \bar{\tau}] \mapsto [0, \bar{\beta}]$ is a strictly increasing and continuous function of $t - t_0$, the time elapsed since the price departed from fundamentals. For the special case where the fundamental value $e^{gt_0 + r(t-t_0)}$ coincides with the price path at t_0 and always grows at a rate r thereafter, $\beta(t - t_0) = 1 - e^{-(g-r)(t-t_0)}$.

The price $p_t = e^{gt}$ is kept above its fundamental value by "irrationally exuberant" behavioral traders. They believe in a "new economy paradigm" and think that the price will grow at a rate of g in perpetuity.[9] As soon as the cumulative selling pressure by rational arbitrageurs exceeds κ, the absorption capacity of behavioral traders, the price drops by a fraction $\beta(\cdot)$ to its post-crash price. From this point onwards, the price grows at a rate of r.[10] Even if the selling pressure never exceeds κ we assume that the bubble bursts for exogenous reasons as soon as it reaches its maximum size $\bar{\beta}$. This translates to a final date, since $\beta(\cdot)$ is strictly increasing. Let us denote this date by $t_0 + \bar{\tau}$. Note that this assumption of a final date is arguably the least conducive to the persistence of bubbles. In a classical model it would lead to an immediate collapse for the usual backwards programming reasons.

Another important element of our analysis is that rational arbitrageurs become sequentially informed that the fundamental value has not kept up with the growth of the stock price index. More specifically, a new cohort of rational arbitrageurs of mass $1/\eta$ becomes 'aware' of the mispricing in each instant t from t_0 until $t_0 + \eta$. That is, $[t_0, t_0 + \eta]$ forms the 'awareness window.' Since t_0 is random, an

[7] The analysis can be directly extended to a setting that incorporates a stochastic price process with a finite number of possible price paths and where the price grows in expectations at a rate of g.

[8] Kindleberger (1978) refers to the phase of fundamentally good news as 'displacement.'

[9] The price process we assume is a modeling simplification that facilitates a clean and simple analysis. One, admittedly simplistic, setting that rationalizes this process is a world with behavioral traders who are risk-neutral, but wealth constrained, and who require a rate of return of g in order to invest in the stock market. If they are overconfident (in the sense that they are mistakenly convinced that they will receive early warning of a change in fundamentals), they will purchase any quantity of shares up to their aggregate absorption capacity κ whenever the price falls below e^{gt}. See also the remark at the end of this section.

[10] We refer to fundamental value as the price that will emerge after the bubble bursts. Notice that for $\beta(0) = 0$, there is no drop in fundamentals at t_0.

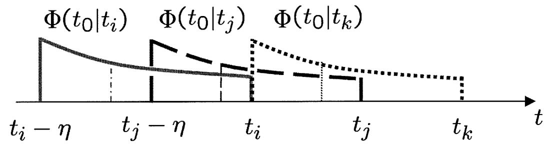

FIGURE 2.—Sequential awareness.

individual arbitrageur does not know how many other arbitrageurs have received the signal before or after her. An agent who becomes aware of the bubble at t_i has a posterior distribution for t_0 with support $[t_i - \eta, t_i]$. Each agent views the market from the relative perspective of her own t_i. We will refer to the arbitrageur who learns of the mispricing at time t_i as arbitrageur t_i. From the perspective of arbitrageur t_i the truncated distribution of t_0 is

$$\Phi(t_0|t_i) = \frac{e^{\lambda\eta} - e^{\lambda(t_i - t_0)}}{e^{\lambda\eta} - 1}.$$

Figure 2 depicts the distributions of t_0 for arbitrageurs t_i, t_j, and t_k, where $t_k = t_i + \eta$, so that arbitrageur t_k is the last possible type to become aware of the bubble given arbitrageur t_i's information. Viewed more abstractly, the set of possible types of arbitrageurs is $[0, \infty)$. Nature's choice of t_0 determines the 'active' types $t_i \in [t_0, t_0 + \eta]$ in the economy. As noted earlier, we view this specification as a modeling device that captures temporal miscoordination arising from differences of opinion and information. We assume that

$$\frac{\lambda}{1 - e^{-\lambda\eta\kappa}} < \frac{g - r}{\beta(\eta\kappa)}.$$

This guarantees that arbitrageurs do not wish to sell out before they become aware of the mispricing. When, for instance, λ is very high or $(g - r)$ very small, the model is uninteresting in that arbitrageurs will sell out right away.

The price $p_t = e^{gt}$ exceeds its post-crash (fundamental) value from t_0 onwards. However, only a few arbitrageurs are aware of the mispricing at this point. From $t_0 + \eta\kappa$ onwards, the mispricing is known to a large enough mass of arbitrageurs who are collectively able to correct it. We label any persistent mispricing *beyond* $t_0 + \eta\kappa$ a *bubble*.

Given such an environment, we want to identify the best strategy of a rational arbitrageur t_i. Each arbitrageur can sell all or part of her stock holding or even go short until she reaches a certain limit where her financial constraint is binding. Each trader can also buy back shares. A trader may exit from and return to the market multiple times. However, each arbitrageur is limited in the number of shares she can go short or long. Without loss of generality, we can normalize the action space to be the continuum between $[0, 1]$, where 0 indicates the maximum long position and 1 the maximum short position each arbitrageur can take on. This sign convention is convenient given the focus of our analysis on selling pressure.

Let $\sigma(t, t_i)$ denote the selling pressure of arbitrageur t_i at t and the function $\sigma : [0, \infty) \times [0, \infty) \mapsto [0, 1]$ a strategy profile. The strategy of a trader who became aware of the bubble at time t_i is given by the mapping $\sigma(\cdot, t_i) : [0, t_i + \bar{\tau}] \mapsto [0, 1]$. Note that $[1 - \sigma(t, t_i)]$ is trader t_i's stock holding at time t. For arbitrary σ, the function $\sigma(t, \cdot)$ need not be measurable in t_i. We confine attention to strategy profiles that guarantee a measurable function $\sigma(t, \cdot)$. Notice that no individual deviation from such a profile has any impact on the measurability property. The aggregate selling pressure of all arbitrageurs at time $t \geq t_0$ is given by $s(t, t_0) = \int_{t_0}^{\min\{t, t_0+\eta\}} \sigma(t, t_i)\, dt_i$. Let

$$T^*(t_0) = \inf\{t | s(t, t_0) \geq \kappa \text{ or } t = t_0 + \bar{\tau}\}$$

denote the bursting time of the bubble for a given realization of t_0. Recall $\Phi(\cdot|t_i)$ denotes arbitrageur t_i's beliefs about t_0 given that $t_0 \in [t_i - \eta, t_i]$. Hence, her beliefs about the bursting date are given by

$$\Pi(t|t_i) = \int_{T^*(t_0)<t} d\Phi(t_0|t_i).$$

Given the structure of the game, the actions of the other traders affect trader t_i's payoff only if these actions cause the bubble to burst. Each arbitrageur's payoff depends on the prices at which she sells and buys her shares minus the transactions costs for executing the order. The execution prices of arbitrageurs' orders are either the pre-crash price $p(t)$ or the post-crash price $(1 - \beta(t - t_0))p(t)$. In the special case where an arbitrageur submits her sell order exactly at the instant when the bubble bursts, her order is executed at the pre-crash price $p(t) = e^{gt}$ as long as the accumulated selling pressure is smaller than or equal to κ. If the accumulated selling pressure exceeds κ, then only the first randomly chosen orders will be executed at the pre-crash price while the remaining orders are only executed at the post-crash price. In other words, the *expected* execution price

$$(1 - \alpha)p(t) + \alpha(1 - \beta(t - t_0))p(t)$$

is a convex combination of both prices, where $\alpha > 0$ if the selling pressure is strictly larger than κ, and $\alpha = 0$ if the selling pressure is less than or equal to κ at the time of the bursting of the bubble.

We assume that arbitrageurs incur transactions costs whenever they alter their positions. This implies that each arbitrageur will only change her stock holdings at most a finite number of times; strategies entailing an infinite number of changes will be strictly dominated. Consequently, we confine attention to strategies that involve at most an (arbitrary) finite number of changes of position. We rule out prohibitively high transactions costs that preclude arbitrageurs from ever selling out despite the fear of a bursting bubble. It is algebraically convenient to assume that the present value of transactions cost is a constant c. That is, transactions cost at t equals ce^{rt}. This formulation together with the former assumption guarantees that equilibrium behavior is independent of c, the size of the transactions cost.

A general specification of the payoff function is notationally cumbersome and involves a recursive formulation; it is given in the Appendix. For the special case that arbitrageur t_i remains fully invested in the market until she completely sells out at t and remains out of the market thereafter (i.e. does not re-enter the market), arbitrageur t_i's expected payoff for selling out at t is

$$\int_{t_i}^{t} e^{-rs}(1-\beta(s-T^{*-1}(s)))p(s)d\Pi(s|t_i)+e^{-rt}p(t)(1-\Pi(t|t_i))-c$$

provided that the selling pressure at t does not strictly exceed κ and that $T^*(\cdot)$ is strictly increasing. It will turn out that in the unique equilibrium trader t_i adopts precisely such a "trigger-strategy," that the selling pressure never strictly exceeds κ, that $T^*(\cdot)$ is strictly increasing, and consequently $T^{*-1}(\cdot)$ is well defined. The latter implies that if the bubble bursts at s, the bubble component of the asset price is precisely $\beta(s-T^{*-1}(s))p(s)$.[11]

REMARK: Our assumption that the bubble only bursts when the selling pressure exceeds κ implies that rational agents do not become aware of selling pressure by other rational agents until it crosses this threshold. We relax this simplifying assumption somewhat in Section 6 by allowing for intermediate price drops prior to the final crash. Nevertheless, we do not fully endogenize the price process. Note that in our model, the central uncertainty is about the one-dimensional random variable t_0. A nonstochastic price process that is strictly monotonic in selling pressure would immediately reveal t_0. A more complete model would entail multi-dimensional uncertainty and noisy prices that preclude all relevant asymmetric information in the model from being inferred with certainty from the current price level. We believe that our principal results would be qualitatively preserved in such a setting, though their precise expression would be substantially more complicated.

4. PRELIMINARY ANALYSIS

This section shows that without loss of generality we can restrict the analysis to trigger strategies, that is, to trading strategies in which an agent who sells out at t continues to attack the bubble at all times thereafter. We also derive the *sell out condition* according to which each arbitrageur sells her shares exactly at the moment when the temptation to 'ride the bubble' balances the fear of its imminent collapse (Lemma 7).

We use the following notion of equilibrium.

DEFINITION 1: A *trading equilibrium* is defined as a Perfect Bayesian Nash Equilibrium in which whenever a trader's stock holding is less than her maximum,

[11] Much of the analysis in Section 4 is used to establish these properties and in particular that, without loss of generality, attention may be restricted to trigger-strategies. Readers who wish to omit these details may, after absorbing the *sell-out condition* of Lemma 7, move directly to Section 5.

then the trader (correctly) believes that the stock holding of all traders who became aware of the bubble prior to her are also at less than their respective maximum long positions.

This definition entails a restriction on beliefs that is a natural one in our setting since the earlier an arbitrageur becomes aware of the mispricing, the lower is her estimate of the fundamental value, and the more inclined she is, ceteris paribus, to sell out. Indeed, an immediate conjecture is that this property of beliefs is an *implication* of equilibrium, but we have not been able to prove this. Note that we are not a priori restricting attention to trigger-strategies: rather this restriction emerges as a result of our analysis.

Lemma 1 states that, in equilibrium, an arbitrageur is either fully invested in the market, $\sigma(t, t_i) = 0$, or at her maximum short-position, $\sigma(t, t_i) = 1$.

LEMMA 1 (No Partial Purchases or Sell Outs): $\sigma(t, t_i) \in \{0, 1\}\ \forall t, t_i$.

This Lemma, 'in effect,' reduces the per period action space to 0 or 1. It facilitates the analysis since aggregate selling pressure is now simply given by the mass of traders who are 'out' of the market. It is a consequence of risk-neutrality and the fixed component of transactions costs. However, it is not essential for the main results of our paper to hold. Risk-averse arbitrageurs would gradually leave the market. This would make it necessary to keep track of each arbitrageur's position to calculate the aggregate selling pressure. All this would add complexity without a corresponding increase in insight.

Lemma 1, together with the definition of a trading equilibrium, immediately implies Corollary 1. It states that when arbitrageur t_i sells out her shares, all arbitrageurs t_j where $t_j \leq t_i$ also have already sold, or will at that moment sell, all their shares. We refer to this feature as the 'cut-off' property.

COROLLARY 1 (Cut-off Property): $\sigma(t, t_i) = 1 \Rightarrow \sigma(t, t_j) = 1\ \ \forall t_j \leq t_i$ *and* $\sigma(t, t_i) = 0 \Rightarrow \sigma(t, t_j) = 0\ \forall t_j \geq t_i$.

DEFINITION 2: The function $T(t_i) = \inf\{t | \sigma(t, t_i) > 0\}$ denotes the *first instant* at which arbitrageur t_i sells any of her shares.

Arbitrageur $t_0 + \eta\kappa$ reduces her holdings for the first time at $T(t_0 + \eta\kappa)$. Since, by Corollary 1, all arbitrageurs who became aware of the mispricing prior to $t_0 + \eta\kappa$ are also completely out of the market at $T(t_0 + \eta\kappa)$, the bubble bursts when trader $t_0 + \eta\kappa$ sells out her shares, provided that it did not already burst earlier for exogenous reasons.

COROLLARY 2: *The bubble bursts at* $T^*(t_0) = \min\{T(t_0 + \eta\kappa), t_0 + \bar{\tau}\}$.

We will show that in any equilibrium, each arbitrageur t_i can rule out certain realizations of t_0 at the time when she first sells out her shares. For this purpose, we define $\underline{t_0^{\text{supp}}}(t_i)$.

DEFINITION 3: The function $\underline{t_0^{\text{supp}}}(t_i)$ denotes the *lower bound of the support* of trader t_i's posterior beliefs about t_0, at $T(t_i)$.

Lemma 2 derives a lower bound for $\underline{t_0^{\text{supp}}}(t_i)$.

LEMMA 2 (Preemption): *In equilibrium, arbitrageur t_i believes at time $T(t_i)$ that at most a mass κ of arbitrageurs became aware of the bubble prior to her. That is,* $\underline{t_0^{\text{supp}}}(t_i) \geq t_i - \eta\kappa$.

To see this, suppose to the contrary that arbitrageur t_i believes that t_0 can also be strictly smaller than $t_i - \eta\kappa$. For these t_0's, the selling pressure at $T(t_i)$ strictly exceeds κ. Consequently, if the bubble bursts at $T(t_i)$, arbitrageur t_i does not receive the pre-crash price for her holding of the asset. Furthermore, arbitrageur t_i has to believe that the bubble bursts with strictly positive probability at $T(t_i)$. Hence, she has a strict incentive to preempt and sell out slightly prior to $T(t_i)$. In contrast to the proof in the Appendix, this intuitive argument implicitly assumes $T(t_i) > t_i$.

The Preemption Lemma allows us to derive further properties of the bursting time $T^*(\cdot)$. Lemmas 3 and 4 in the Appendix show that $T^*(\cdot)$ is strictly increasing and continuous. It follows that $T^{*-1}(\cdot) : [T^*(0), \infty) \mapsto [0, \infty)$, the inverse of $T^*(\cdot)$, is well defined. Lemma 5 in the Appendix establishes that $T(\cdot)$ is also continuous. Let $B^c(t)$ denote the event that the bubble has not burst until time t.

LEMMA 6 (Zero Probability): *For all $t_i > 0$, arbitrageur t_i believes that the bubble bursts with probability zero at the instant $T(t_i)$. That is,* $\Pr[T^{*-1}(T(t_i))|t_i, B^c(T(t_i))] = 0$ *for all $t_i > 0$.*

The following proposition establishes that in any equilibrium arbitrageurs necessarily use trigger-strategies. That is, each arbitrageur t_i sells out at $T(t_i)$ and never re-enters the market.

PROPOSITION 1 (Trigger-strategy): *In equilibrium, arbitrageur t_i maintains the maximum short position for all $t \geq T(t_i)$, until the bubble bursts.*

PROOF: Suppose not, and that there exists an equilibrium in which arbitrageur t_i sells out at $T(t_i)$ and re-enters the market later. Given transaction costs $c > 0$ and the preceding lemma, in equilibrium arbitrageur t_i must stay out of the market for a strictly positive time interval. She will stay out of the market at least until type $t_i + \varepsilon$ exits the market, for some $\varepsilon > 0$ (independent of t_i).[12] By the cut-off property arbitrageur t_i cannot re-enter the market until after arbitrageur

[12] Exiting the market and paying the transaction cost c can only be justified if the bubble bursts with a certain probability. Since the bubble bursts with zero probability exactly when trader t_i enters, she has to stay out of the market until at least a certain mass of (younger) traders ε also exit the market. Note ε is independent of t_i.

$t_i+\varepsilon$ first re-enters. The same reasoning applies to arbitrageur $t_i+\varepsilon$ with respect to arbitrageur $t_i+2\varepsilon$. Proceeding this way we conclude that arbitrageur t_i stays out of the market until the bubble bursts at $t_0+\bar{\tau}$ or arbitrageur $t_i+\eta\kappa$ exits and then *re*-enters the market. Of course, the latest possible date at which the bubble bursts from arbitrageur t_i's viewpoint is when $t_i+\eta\kappa$ exits the market. *Q.E.D.*

We have proved that $T^{*-1}(\cdot)$ exists and is strictly increasing and continuous. We confine attention to equilibria for which the latter function is *absolutely* continuous such that $\Pi(t)=\Phi(T^{*-1}(t))$ is also absolutely continuous. Let $\pi(t)$ denote its associated density. Recall that $\Phi(t_0)=1-e^{-\lambda t_0}$ and $\Pi(t|t_i)$ is arbitrageur t_i's conditional cumulative distribution function of the bursting date at time t_i. Similarly, $\pi(t|t_i)$ denotes the associated conditional density.

The trigger-strategy characterization greatly simplifies the analysis of equilibrium and the specification of payoffs. As noted earlier the payoff to selling out at time t is

$$\int_{t_i}^{t} e^{-rs}(1-\beta(s-T^{*-1}(s)))p(s)\pi(s|t_i)\,ds+e^{-rt}p(t)(1-\Pi(t|t_i))-c.$$

Differentiating the payoff function with respect to t yields the sell-out condition stated in Lemma 7. Note that

$$h(t|t_i)=\frac{\pi(t|t_i)}{1-\Pi(t|t_i)}$$

is the hazard rate that the bubble will burst at t.

LEMMA 7 (Sell-out Condition): *If arbitrageur t_i's subjective hazard rate is smaller than the 'cost-benefit ratio,' i.e.*

$$h(t|t_i)<\frac{g-r}{\beta(t-T^{*-1}(t))},$$

trader t_i will choose to hold the maximum long position at t. Conversely, if

$$h(t|t_i)>\frac{g-r}{\beta(t-T^{*-1}(t))},$$

she will trade to the maximum short position.[13]

[13] One could easily extend the analysis to capture the *reputational penalty*, due to relative performance evaluation, that institutional investors face for staying out of the market while the bubble grows at the expected rate g. In a setting with a 'reputational penalty' equal to a fraction k of the price level, the term $\int_t^{t_i+\bar{\tau}}[\int_t^s e^{-ru}kp(u)\,du]\pi(s|t_i)\,ds$ needs to be added to the payoff and the sell-out condition generalizes to

$$h(t|t_i)<(>)\frac{g-r-k}{\beta(t-T^{*-1}(t))}.$$

The reputational term does not modify the structure of the results, but would be relevant in obtaining realistic calibrations of the model.

Recall that if the bubble bursts at t its size is precisely $\beta(t-T^{*-1}(t))p(t)$. To understand the sell-out condition intuitively, consider the first-order benefits and costs of attack at t versus $t+\Delta$, respectively. The benefits are given by $\Delta h(t|t_i)[p(t)\beta(t-T^{*-1}(t))]$, the size of the bubble times the probability that the bubble will burst over the small interval Δ. In the case that the bubble does not burst, the costs of being out of the market for a short interval Δ are

$$(1-\Delta h(t|t_i))\left(\frac{p(t+\Delta)-p(t)}{\Delta}-rp(t)\right)\Delta.$$

Note that

$$\frac{p(t+\Delta)-p(t)}{\Delta}-rp(t)>0$$

since the bubble appreciates faster than the riskfree rate. Dividing by $\Delta p(t)$ and letting $\Delta\to 0$ yields the attack condition. Lemma 7 also conforms with our earlier result that trader t_i either wholeheartedly attacks or holds the maximum long position.

5. PERSISTENCE OF BUBBLES

The impossibility of bubbles emerging in standard asset pricing models is most transparent in a finite horizon setting in which they are ruled out by a straightforward backward induction argument. Interestingly, a similar argument involving the "iterative removal of non-best-response symmetric trigger-strategies" provides some intuition for the main results of this section.

Even if no arbitrageur sells her shares, the bubble ultimately bursts at $t_0+\bar{\tau}$ by assumption. Since each arbitrageur becomes aware of the mispricing only after t_0, she knows for sure that the bubble will never last beyond $t_i+\bar{\tau}$. But it might burst even before this time since from arbitrageur t_i's point of view, the ultimate bursting date $t_0+\bar{\tau}$ is distributed between $t_i+\bar{\tau}-\eta$ and $t_i+\bar{\tau}$. If the bubble bursts at $t_0+\bar{\tau}$, arbitrageur t_i's best response is to exit the market before $t_i+\bar{\tau}$. More specifically, let $t_i+\tau^1$ be arbitrageur t_i's best response exit time if she conjectures that other arbitrageurs never attack, and consequently that the bubble bursts only at $t_0+\bar{\tau}$. The conjecture that all arbitrageurs ride the bubble for τ^1 periods leads to a new bursting date and a new best response exit time $t_i+\tau^2$. Proceeding in this way, we inductively obtain τ^3, τ^4 and so on. In the standard model this yields $\lim_{n\to\infty}\tau^n=0$, which precludes the emergence of bubbles. We show in Section 5.1 that this backward induction procedure has no bite if

$$\frac{\lambda}{1-e^{-\lambda\eta\kappa}}\le\frac{g-r}{\bar{\beta}}.$$

In particular, $\lim_{n\to\infty}\tau^n=\tau^1$ and furthermore, the bubble only bursts at $t_0+\bar{\tau}$. Conversely, for

$$\frac{\lambda}{1-e^{-\lambda\eta\kappa}}>\frac{g-r}{\bar{\beta}},$$

we show in Section 5.2 that $\lim_{n\to\infty} \tau^n < \tau^1$ but $\lim_{n\to\infty} \tau^n > 0$. Hence, this backward procedure does bite, but not as much as in the classical case. Note that this induction argument is restricted to symmetric strategies. The formal analysis below employs a different reasoning that also addresses nonsymmetric strategies.

Sequential awareness blunts the competition amongst arbitrageurs upon which the efficient markets view is premised and leads to sufficient ambiguity amongst arbitrageurs for a strategy of market timing to be profitably pursued. Hence, from the arbitrageurs' point of view, the lack of synchronization is not a "problem" but rather a blessing.

5.1. *Exogenous Crashes*

Recall that arbitrageurs become aware of the bubble sequentially in a random order and furthermore have a nondegenerate posterior distribution over t_0. All arbitrageurs become 'aware' of the bubble during the interval $[t_0, t_0+\eta]$, where we have interpreted η to be a measure of differences in opinion and other heterogeneities across players. From $t_0+\eta\kappa$ onwards, more than κ arbitrageurs are aware of the bubble and have collectively the ability to burst it.

We show that if

$$\frac{\lambda}{1-e^{-\lambda\eta\kappa}} \leq \frac{g-r}{\bar{\beta}},$$

then in the unique trading equilibrium the bubble only bursts for exogenous reasons when it reaches its maximum size $\bar{\beta}$ relative to the price. In this case the endogenous selling pressure of the rational arbitrageurs has absolutely no influence on the time at which the bubble bursts.

DEFINITION 4: The function $\tau(t_i) := T(t_i) - t_i$ denotes the *length of time* arbitrageur t_i chooses to ride the bubble subsequent to becoming aware of the mispricing.

It is worth noting that our result holds despite the fact that it is possible within our model for traders to coordinate selling out on particular dates, say Friday, 13th of April, 2001 by adopting (asymmetric) strategies that entail nonconstant $\tau(t_i)$.

PROPOSITION 2: *Suppose*

$$\frac{\lambda}{1-e^{-\lambda\eta\kappa}} \leq \frac{g-r}{\bar{\beta}}.$$

Then there exists a unique trading equilibrium. In this equilibrium all traders sell out

$$\tau^1 = \bar{\tau} - \frac{1}{\lambda}\ln\left(\frac{g-r}{g-r-\lambda\bar{\beta}}\right) < \bar{\tau}$$

periods after they become aware of the bubble and stay out of the market thereafter. Nevertheless, for all t_0, the bubble bursts for exogenous reasons precisely when it reaches its maximum possible size $\bar{\beta}$.

PROOF: *Step 1*: τ^1 *defines a symmetric equilibrium*. Suppose arbitrageur t_i believes that the bubble bursts at $t_0+\zeta$. Conditioning on $t_0 \in [t_i-\eta, t_i]$ her posterior distribution over t_0 is

$$\Phi(t_0|t_i) = \frac{e^{\lambda\eta} - e^{\lambda(t_i-t_0)}}{e^{\lambda\eta}-1}$$

and her posterior distribution over bursting dates $t = t_i + \tau$ is

$$\Pi(t_i+\tau|t_i) = \frac{e^{\lambda\eta} - e^{\lambda(\zeta-\tau)}}{e^{\lambda\eta}-1}$$

at t_i. The corresponding hazard rate is

$$h(t_i+\tau|t_i) := \frac{\pi(t_i+\tau|t_i)}{1-\Pi(t_i+\tau|t_i)} = \frac{\lambda}{1-e^{-\lambda(\zeta-\tau)}}.$$

Suppose that the bubble bursts for exogenous reasons. Then $\zeta = \bar{\tau}$ in the expression for the hazard rate. The latter is depicted in Figure 3 as an increasing function of τ. Since the exogenous bursting date $t_0+\bar{\tau}$ is independent of τ, so is the cost-benefit ratio of the sell-out condition (Lemma 7). This ratio equals $(g-r)/\bar{\beta}$ and is represented by the horizontal line in Figure 3. The point of intersection τ^1 solves $h(t_i+\tau^1|t_i) = (g-r)/\bar{\beta}$. Clearly,

$$\tau^1 = \bar{\tau} - \frac{1}{\lambda}\ln\left(\frac{g-r}{g-r-\lambda\bar{\beta}}\right).$$

By the sell-out lemma arbitrageur t_i is optimally out of the market for $\tau > \tau^1$ and conversely for $\tau < \tau^1$ she holds on to her shares. Under our assumed condition

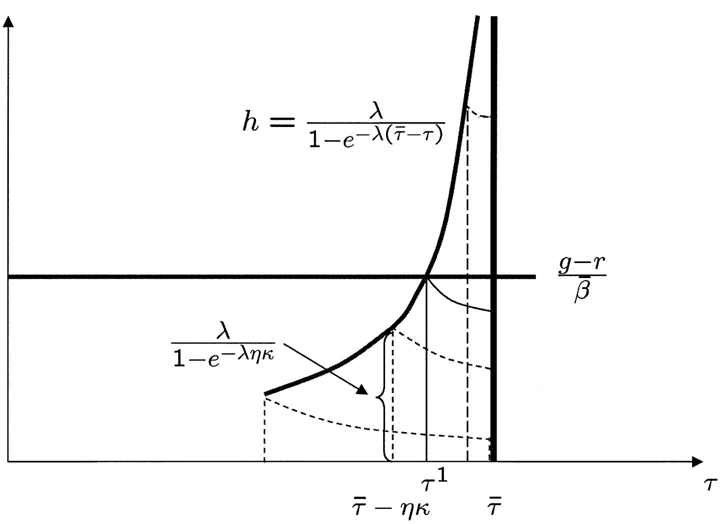

FIGURE 3.—Exogenous crash.

$t_0+\tau^1+\eta\kappa > t_0+\bar{\tau}$. Hence, the symmetric trigger τ^1 does define an equilibrium, which results in an exogenous crash at $t_0+\bar{\tau}$.

Step 2: *Uniqueness.* Let us suppose that there is another equilibrium. Consider $\underline{t}_j = \arg\min_{t_i}\{\tau(t_i)\}$.[14] Since $\tau(t_i) \le \tau^1$ for all t_i, it must be the case that $\tau(\underline{t}_j) < \tau^1$. Consider $\underline{t_0^{\text{supp}}}(\underline{t}_j)$.

(i) By Lemma 2 (Preemption) $\underline{t_0^{\text{supp}}}(\underline{t}_j) \ge t_j - \eta\kappa$.

(ii) If $\underline{t_0^{\text{supp}}}(\underline{t}_j) > t_j - \eta\kappa$, then arbitrageur $\underline{t}_j$ does not delay selling out at $t_j + \tau(\underline{t}_j)$ only out of fear of an *exogenous* bursting of the bubble. By the argument of the first part, it follows that $\tau(\underline{t}_j) = \tau^1$. This contradicts the initial assumption that $\tau(\underline{t}_j) < \tau^1$.

(iii) Finally, suppose $\underline{t_0^{\text{supp}}}(\underline{t}_j) = t_j - \eta\kappa$. In this case the hazard rate that the bubble bursts, at the time when $\underline{t}_j$ sells out, is at most $\lambda/(1-e^{-\lambda\eta\kappa})$ since $\underline{t}_j = \arg\min\{\tau(t_i)\}$. Since this is in turn less than $(g-r)/\bar{\beta}$, the sell-out condition is violated. *Q.E.D.*

In equilibrium, each arbitrageur optimally rides the bubble sufficiently long that by the end of the horizon less than κ will have sold out, so that the bubble indeed bursts purely for exogenous reasons.

We note here that the global games approach does not apply in our setting, since our game does not satisfy strategic complementarity. This is both because the assumption that $\kappa < 1$ introduces a competitive element and the fact that traders infer information from the fact that the bubble still exists.[15]

5.2. *Endogenous Crashes*

The previous section demonstrates that arbitrageurs never burst the bubble if the dispersion of opinion among them, η, and the absorption capacity of behavioral traders, κ, are sufficiently large. In this section we examine the opposite case when this condition is not satisfied. We show that our proposed backward iteration procedure does bite in this case. When no other arbitrageur ever sells out, the bubble bursts at $t_0+\bar{\tau}$, which induces arbitrageurs to sell out at $t_i+\tau^1$. In this scenario, the bubble will burst at $t_0+\tau^1+\eta\kappa$, which is now strictly earlier than $t_0+\bar{\tau}$ given the assumed smaller parameter values for η and κ. Given that the bubble bursts by $t_0+\tau^1+\eta\kappa$ at the latest, arbitrageurs seek to sell out even

[14] Whenever arg min and arg max are not defined, the corresponding arguments can be rewritten in terms of infimums and supremums respectively.

[15] This is probably best illustrated by means of an example, wherein we restrict the strategy space to trigger strategies. Consider a trader t_i who starts attacking the bubble at $t' = t_i + \tau_i$, provided that all other traders attack immediately when they become aware of the bubble. Given this strategy profile, trader t_i can infer a lower bound for t_0 from the fact that the bubble still exists. Compare this with a situation where other traders do not start attacking immediately when they become aware of the bubble but only at, say, t'. In this case trader t_i cannot derive a lower bound for t_0 from the existence of the bubble. Consequently, she has a greater incentive to attack the bubble at t'. This is exactly the opposite of what strategic complementarity would prescribe.

earlier, $\tau^2 < \tau^1$ periods after they became aware of the bubble. Proceeding in this way leads to a decreasing sequence $\tau^1, \tau^2, \tau^3, \ldots$ which converges to some τ^* that in fact defines the unique symmetric trigger-strategy Perfect Bayesian Nash equilibrium. The iteration of this argument does not eliminate bubbles. The reason is that the iterative procedure loses bite gradually. As the bursting date advances, the size of the bubble also diminishes, which in turn increases the incentive to ride the bubble by reducing the cost of not exiting prior to the crash. Though the bubble bursts for endogenous reasons it may survive for a substantial period; at the time of the crash the bubble component is

$$\beta^* = \frac{1-e^{-\lambda\eta\kappa}}{\lambda}(g-r).$$

By an appropriate choice of parameter values, the latter term can be made arbitrarily close to $\bar{\beta}$, which in turn can be chosen to be close to 1. Note that for the special case where the fundamental value is e^{gt_0} at t_0 and grows at a rate of r thereafter, $\beta(t-t_0) = 1-e^{-(g-r)(t-t_0)}$ and

$$\tau^* = \frac{1}{g-r}\ln\left(\frac{\lambda}{\lambda-(g-r)(1-e^{-\lambda\eta\kappa})}\right) - \eta\kappa.$$

Proposition 3 derives a symmetric equilibrium, where each arbitrageur sells her shares τ^* periods after she becomes aware of the bubble. It also demonstrates that this trading equilibrium is unique.

PROPOSITION 3: *Suppose*

$$\frac{\lambda}{1-e^{-\lambda\eta\kappa}} > \frac{g-r}{\bar{\beta}}.$$

Then there exists a unique trading equilibrium, in which arbitrageurs t_i with $t_i \geq \eta\kappa$ leave the market

$$\tau^* = \beta^{-1}\left(\frac{g-r}{\frac{\lambda}{1-e^{-\lambda\eta\kappa}}}\right) - \eta\kappa$$

periods after they become aware of the bubble. All arbitrageurs t_i such that $t_i < \eta\kappa$ sell out at $\eta\kappa + \tau^*$. Hence, the bubble bursts when it is a fraction

$$\beta^* = \frac{1-e^{-\lambda\eta\kappa}}{\lambda}(g-r)$$

of the pre-crash price.

PROOF: *Step 1*: *τ^* defines a symmetric equilibrium.* Suppose that all arbitrageurs with $t_i \geq \eta\kappa$ sell out their shares at $t_i + \tau$ and arbitrageurs with $t_i < \eta\kappa$ at $\eta\kappa + \tau$ for some $\tau \in (0, \bar{\tau} - \eta\kappa)$. Then the bubble bursts at $t_0 + \zeta$ for endogenous reasons, where $\zeta = \eta\kappa + \tau$. Substituting this into the hazard rate $\lambda/(1-e^{-\lambda(\zeta-\tau)})$ as derived in Step 1 of Proposition 2 yields the equilibrium hazard rate $h^* = \lambda/(1-e^{-\lambda\eta\kappa})$. It is independent of τ and represented by the horizontal line in Figure 4.

The cost-benefit ratio $(g-r)/\beta(\eta\kappa+\tau)$ of the sell-out condition (Lemma 7) is declining in τ, since $\beta(\eta\kappa+\tau)$ is (by assumption) an increasing function. See Figure 4. In equilibrium

$$\frac{\lambda}{1-e^{-\lambda\eta\kappa}} = \frac{g-r}{\beta(\eta\kappa+\tau^*)}.$$

Hence,

$$\tau^* = \beta^{-1}\left(\frac{g-r}{\frac{\lambda}{1-e^{-\lambda\eta\kappa}}}\right) - \eta\kappa.$$

Note that the assumption that

$$\frac{\lambda}{1-e^{-\lambda\eta\kappa}} < \frac{g-r}{\beta(\eta\kappa)}$$

(see Section 3) implies that $\tau^* > 0$, while the condition

$$\frac{\lambda}{1-e^{-\lambda\eta\kappa}} > \frac{g-r}{\bar{\beta}}$$

implies that $\tau^* < \bar{\tau} - \eta\kappa$. Hence, τ^* indeed defines a symmetric equilibrium. The next steps establish that this equilibrium is unique.

Step 2: *The bubble always bursts for endogenous reasons when*

$$\frac{\lambda}{1-e^{-\lambda\eta\kappa}} > \frac{g-r}{\bar{\beta}}.$$

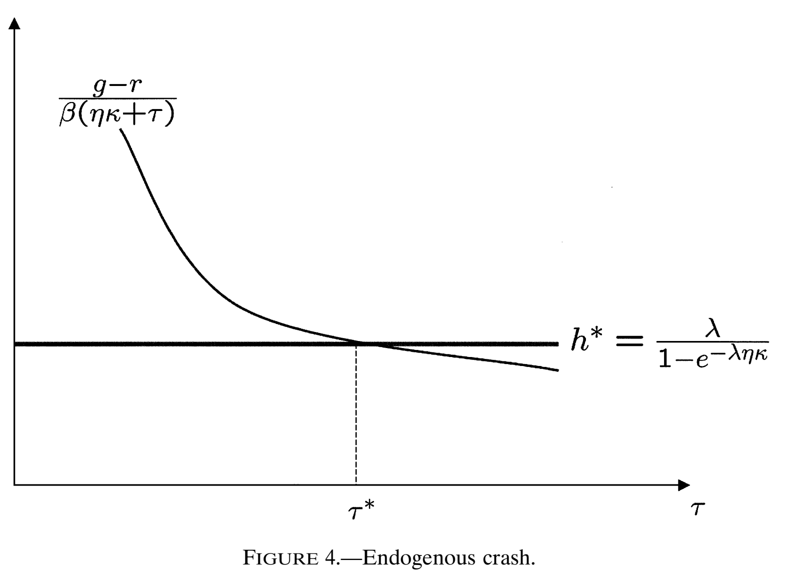

FIGURE 4.—Endogenous crash.

Let τ^1 solve

$$h(t_i+\tau^1|t_i)=\frac{\lambda}{1-e^{-\lambda(\bar{\tau}-\tau^1)}}=\frac{g-r}{\bar{\beta}}.$$

Each trader would exit the market at $t_i+\tau^1$ if she believed that the bubble would burst for exogenous reasons when it reached its maximum possible size $\bar{\beta}$. Under our assumptions $t_0+\tau^1+\eta\kappa<t_0+\bar{\tau}$. Hence, the bubble does not always burst for endogenous reasons only if $\tau(t_i)>\tau^1$ for at least some t_i. Consider arbitrageur t_j, where $t_j=\arg\max_{t_i}\{\tau(t_i)\}$. We establish Step 2 by showing that $\tau(t_j)>\tau^1$ leads to a contradiction. Observe that:

(i) By the Preemption Lemma $\underline{t_0^{\text{supp}}}(t_j)\geq t_j-\eta\kappa$.

(ii) If $\underline{t_0^{\text{supp}}}(t_j)>t_j-\eta\kappa$, then arbitrageur t_j's hazard rate at $T(t_j)>t_j+\tau^1$ that the bubble will burst for exogenous reasons strictly exceeds $(g-r)/\bar{\beta}$. Since this is also the case at $T(t_j)-\varepsilon$ for sufficiently small $\varepsilon>0$, she has an incentive to sell out strictly prior to $T(t_j)$.

(iii) Finally, suppose $\underline{t_0^{\text{supp}}}(t_j)=t_j-\eta\kappa$. Since $\tau(t_j)\geq\tau(t_i)$ for all t_i, the hazard rate that the bubble will burst at $T(t_j)$ is at least $\lambda/(1-e^{-\lambda\eta\kappa})$. Since the bubble bursts at $t_0+\eta\kappa+\tau(t_j)>t_0+\eta\kappa+\tau^1$, the size of the bubble $\beta(\eta\kappa+\tau(t_j))$ exceeds $\beta(\eta\kappa+\tau^1)$. It follows that the sell-out condition for arbitrageur t_j is violated at $T(t_j)$.

Step 3: *Minimum and maximum of $\tau(t_i)$ coincide for $t_i\geq\eta\kappa$.* By Step 2, $T(t_0+\eta\kappa)<t_0+\bar{\tau}\ \forall t_0\geq 0$. That is, the bubble only bursts for *endogenous* reasons. By the Preemption Lemma, $\underline{t_0^{\text{supp}}}(t_i)\geq t_i-\eta\kappa$. Furthermore, $\underline{t_0^{\text{supp}}}(t_i)>t_i-\eta\kappa$ can be ruled out since arbitrageur t_i would be strictly better off selling out at some $\varepsilon>0$ time after her ostensible equilibrium sell out date $T(t_i)$. (Recall that by Lemma 5, $T(\cdot)$ is continuous.) Hence, given $\underline{t_0^{\text{supp}}}(t_i)=t_i-\eta\kappa$, arbitrageur t_i's conditional density of t_0 at $T(t_i)$ is

$$\phi(t_i-\eta\kappa|t_i,B^c(T(t_i)))=\frac{\lambda e^{\lambda\eta\kappa}}{e^{\lambda\eta\kappa}-1},$$

which is independent of t_i. Let $\underline{t}_i\in\arg\min\tau(t_i)$ and $\bar{t}_i\in\arg\max\tau(t_i)$ and suppose that $\max\tau(t_i)>\min\tau(t_i)$. By the continuity of $T(\cdot)$ shown in Lemma 5 and the definitions of $\underline{t}_i$ and $\bar{t}_i$, it follows that

$$\Pi(T(\underline{t}_i)+\Delta|\underline{t}_i,B^c(T(\underline{t}_i)))<\Pi(T(\bar{t}_i)+\Delta|\bar{t}_i,B^c(T(\bar{t}_i)))\quad\text{for all }\ \Delta>0,$$

and conversely for $\Delta<0$. Consequently,

$$h(T(\underline{t}_i)|\underline{t}_i,B^c(T(\underline{t}_i)))<h(T(\bar{t}_i)|\bar{t}_i,B^c(T(\bar{t}_i))).$$

However,

$$\beta(\eta\kappa+\tau(\underline{t}_i))\leq\beta(\eta\kappa+\tau(\bar{t}_i)).$$

Thus, the sell-out condition cannot be satisfied for both arbitrageurs $\underline{t}_i$ and $\bar{t}_i$, a contradiction.

Step 4: For $t_i < \eta\kappa, T(t_i) = T(\eta\kappa)$. Clearly, no t_i should sell out prior to $T(\eta\kappa)$ and by the cut-off property will sell out at $T(\eta\kappa)$. *Q.E.D.*

Notice that the maximum 'relative' bubble size β^* increases as the dispersion of opinions among arbitrageurs η increases. The comparative statics for the absorption capacity of the behavioral traders κ are the same as for η. A larger κ requires more coordination among arbitrageurs and thus extends the bubble size. Finally, β^* is also increasing in the 'excess growth rate' of the bubble $(g-r)$. The faster the bubble appreciates, the more tempting it is to ride it.

REMARKS:

- The usual backward induction argument ruling out bubbles begins at a terminal date that is common knowledge. In our model, prior to the actual bursting of the bubble it is never common knowledge even among a mass $\kappa < 1$ of the arbitrageurs that a bubble exists. To see this, note that at $t_0 + \eta\kappa$, (a mass of) κ traders know of the bubble. That is, at $t_0 + \eta\kappa$ the existence of the bubble is mutual knowledge among κ traders. But they do not know that κ traders know of the bubble. This is the case only at $t_0 + 2\eta\kappa$. More generally, nth level mutual knowledge among κ arbitrageurs is achieved precisely at $t_0 + n\eta\kappa$. See Rubinstein (1989) for an early exploration of the crucial distinction between common knowledge and high levels of mutual knowledge.
- Note that in contrast to Tirole (1982) and Milgrom and Stokey (1982), our trading game is not a zero-sum game. The profits of rational arbitrageurs come at the expense of the behavioral traders.

6. SYNCHRONIZING EVENTS

A central theme of our analysis is that dispersion of opinion makes it difficult for arbitrageurs to synchronize selling out of the bubble asset. This "difficulty" provides crucial cover for profitable 'riding of the bubble.' A natural conjecture is that the presence of synchronizing events punctures this cover. Perhaps the most obvious such events are dates like Friday, April 13th, 2001. In fact, the analysis of the preceding section allows arbitrageurs to condition on absolute time. Nevertheless, equilibrium was found to be unique and symmetric: all arbitrageurs wait the same number of periods τ^* after becoming aware of the mispricing before attacking. Hence, perfectly anticipated events have no impact on the analysis. In Section 6.1 we consider unanticipated synchronizing events. The important implications of this analysis are that:

(i) news may have an impact disproportionate to any intrinsic (fundamental) informational content;

(ii) fads and fashions may arise in the use of different kinds of information;

(iii) while synchronizing events facilitate "coordinated attacks," when such attacks fail, they lead to a temporary strengthening of the bubble.

Finally in Section 6.2 we consider price events. These may also serve as synchronization devices and lead to full correction (a price cascade) or market rebounds.

6.1. *Uninformative Events*

Since the primary emphasis of this section is to understand the role of 'news' in causing price changes *beyond* its informational content, we restrict our formal analysis to nonfundamental events that serve as pure coordination devices. In particular, we extend the earlier model to allow for the arrival of synchronizing signals at a Poisson arrival rate θ.

It is natural to assume that only arbitrageurs who are aware of the bubble observe synchronizing events. Indeed, we assume that such events are only observed by traders who became aware of the bubble more than $\tau_e \geq 0$ periods ago. The stronger assumption that $\tau_e > 0$ captures in a simple way the idea that arbitrageurs become more wary the longer they have been aware of the bubble, and, hence, look out more vigilantly for signals that might precipitate a bursting of the bubble. An alternative justification for some arbitrageurs not recognizing synchronizing events is suggested by the second remark at the end of this subsection.

Synchronizing events both allow arbitrageurs to synchronize sell outs and also convey valuable information in the event that attacks are unsuccessful. The bubble will survive a synchronized sell out if an insufficient number of arbitrageurs have observed the synchronizing event, that is, if t_0 is sufficiently high. Let t_e be the date of such a synchronizing event. By the cut-off property, there exists $\tilde{\tau}_e \geq \tau_e$ such that all arbitrageurs t_i who observed the sunspot at least $\tilde{\tau}_e$ periods after becoming aware of the mispricing, sell out after observing the synchronizing event at t_e. If the bubble does *not* burst then all arbitrageurs learn that $t_0 > t_e - \tilde{\tau}_e - \eta\kappa$. It follows that all arbitrageurs t_i with $t_i \leq t_e - \tilde{\tau}_e$ will re-enter the market until type $t_e - \tilde{\tau}_e$ first exits in equilibrium, or until the next synchronizing event occurs. Even traders who left the market prior to the arrival of the synchronizing event buy back their shares.

In contrast to the previous sections, there is no hope of finding a unique equilibrium in this generalized setting. For example, the equilibrium behavior of the previous section would be exactly replicated if all arbitrageurs were to simply ignore all synchronizing events, and there are potentially numerous intermediate levels of responsiveness to synchronizing signals. We focus on *responsive equilibria*.

DEFINITION 6: A *'responsive equilibrium'* is a trading equilibrium in which each arbitrageur believes that all other traders will synchronize (sell out) at each synchronizing event.

This raises the bar for our analysis since bubbles are harder to sustain in a responsive equilibrium. This is the polar opposite of the equilibrium in which all arbitrageurs ignore all synchronizing events.

We establish that there exists a unique responsive equilibrium. It is symmetric in the sense that as long as an arbitrageur does not observe a synchronizing event, each arbitrageur stays in the market for a fixed number τ^{**} periods after becoming aware of the bubble. We assume that $\theta < (g-r)/\bar{\beta}$ to ensure that the possible occurrence of a synchronizing event does not by itself justify selling out. As in Subsection 5.2 (endogenous crashes), we focus on the parameter values that satisfy

$$\frac{g-r}{\bar{\beta}} < \frac{\lambda}{1-e^{-\lambda\eta\kappa}} < \frac{g-r}{\beta(\eta\kappa)}.$$

The latter inequality guarantees that no arbitrageur finds it optimal to sell out even before she becomes aware of the mispricing.

PROPOSITION 4: *There exists a unique responsive equilibrium. In this equilibrium, each arbitrageur t_i always sells out at the instances of synchronizing events $t_e \geq t_i + \tau_e$. Furthermore, she stays out of the market for all $t \geq t_i + \tau^{**}$ except in the event that the last synchronized attack failed in which case she re-enters the market for the interval $t \in (t_e, t_e + \tau^{**} - \tau_e)$, unless a new synchronizing event occurs in the interim.*

Absent the arrival of a synchronizing event the bubble will burst at $t_0 + \eta\kappa + \tau^{**}$. The arrival of synchronizing events might accelerate the bursting date. After a failed sell out at the latest synchronizing event t_e, even traders who started selling out prior to t_e, that is, traders with $t_i < t_e - \tau^{**}$, buy back shares. Sell outs only resume at $t_e + \tau^{**} - \tau_e$. ‘Market sentiment’ bounces back after a failed synchronized attack. However, even in the event of failed attacks and re-entry by arbitrageurs into the market, the bubble bursts no later than $t_0 + \eta\kappa + \tau^{**}$.

The proof of Proposition 4 is summarized in Appendix A.2. Notice that given the structure of our setup, the equilibrium is fully characterized by τ^{**}: In any trading equilibrium, all arbitrageurs who sold out after the last synchronizing event will re-enter the market after a failed attack and sell out again exactly when the ‘first’ trader, who did not participate in the synchronized sell out, exits the market.

REMARKS:

- The news events above are special in that they convey no information about the value of the asset. Nevertheless, they facilitate synchronization. An immediate consequence is that, in general, news events may have an impact quite out of proportion to their fundamental content.
- The issue of what events are considered by market participants to be synchronizing events is a subtle one. There is obviously an unlimited supply of potential synchronizing events (such as the weather, the levels of various economic indicators, etc.). But which do participants view as relevant? Their very abundance emphasizes the difficulty and suggests a higher level synchronization problem, and the possibility of multiple equilibria.
- The above issue is closely related to *fads and fashion* in the acquisition and use of information. For example, trade figures drove the market during the 1980’s. In contrast, in the late 1990’s Alan Greenspan’s statements moved

stock prices, while trade figures were ignored. The model in this section may be extended to allow for multiple types of signals, A, B, C, etc. In some equilibria, arbitrageurs may condition on A and C, in others only on B, etc. More interestingly, some equilibria may initially condition on A and then after some histories switch to B and subsequently back to A, etc. Such a model would yield a rudimentary theory of fads and fashion. These themes are reminiscent of Keynes' (1936) comparison of the stock market with a beauty contest:

> professional investment may be likened to those newspaper competitions in which the competitors have to pick out the six prettiest faces from a hundred photographs, the prize being awarded to the competitor whose choice most nearly corresponds to the average preferences of the competitors as a whole; so that each competitor has to pick, not those faces which he himself finds prettiest, but those which he thinks likeliest to catch the fancy of the other competitors, all of whom are looking at the problem from the same point of view.

6.2. *Price Events—Price Cascades and Market Rebounds*

Arguably, the most visible synchronizing events on Wall Street are large past price movements or breaks through psychological resistance lines. In this section, we allow random temporary price drops to occur, which are possibly due to mood changes by behavioral traders. This enables us to illustrate how a large price decline might either lead to a full blown crash or to a rebound. In the latter event the bubble is strengthened in the sense that *all* arbitrageurs are 'in the market' for some interval, including those who had previously exited prior to the price shock.

In our simple and highly stylized model, exogenous price drops occur with a Poisson density θ_p at the end of a random trading round t. We suppose that behavioral traders' mood changes temporarily and that they are only willing to hold on to shares if the price is less than or equal to $(1-\gamma_p)p_t$. If the bubble does not burst in the 'subsequent' trading round, behavioral traders regain their confidence and are willing to sell and buy at a price of p_t until their absorption capacity κ is reached or another price drop occurs. Arbitrageurs who exit the market immediately after a price drop receive $(1-\gamma_p)p_t$ per share. Should the synchronized attack after a price drop fail, arbitrageurs can only buy back their shares at a price of p_t. In other words, leaving the market even only for an instant is very costly despite the absence of transactions costs, which we set equal to zero. Hence, only traders who are sufficiently certain that the bubble will burst after the price drop will leave the stock market. Let $H_p^n := (t_p^{(1)}, \ldots, t_p^{(n)})$ denote a history of past temporary price drops. Proposition 5 states that only traders who became aware of the mispricing more than $\tau_p(H_p^n)$ periods earlier will choose to leave the market and attack the bubble after a price drop. Notice that, although $\tau_p(H_p^n)$ is derived endogenously for the subgame after a price drop, it serves the same role as the exogenous τ_e in subsection 6.1. The structure of the equilibrium is similar to the unique responsive equilibrium of the preceding subsection, with the main difference being that all arbitrageurs know H_p^n, and H_p^n affects $\tau_p(H_p^n)$. Analogous to the previous subsection, τ^{***} denotes how long each arbitrageur rides the bubble after t_i if there has been no price drop so far.

PROPOSITION 5: *There exists an equilibrium* $(\tau_p(\cdot), \tau^{***})$ *in which arbitrageur* t_i *exits the market after a price drop at* $t_p^{(n+1)}$ *if* $t_p^{(n+1)} \geq t_i + \tau_p(H_p^n)$. *Furthermore, she is out of the market at all* $t \geq t_i + \tau^{***}$ *except in the event that the last attack failed, in which case she re-enters the market for the interval* $t \in (t_p^{(n+1)}, t_p^{(n+1)} + \tau^{***} - \tau_p(H_p^n))$.

Proposition 5 shows that a price drop that is not followed by a crash leads to a rebound and temporarily strengthens the bubble. In this case all arbitrageurs can rule out that the bubble will burst within $(t_p^{(n+1)}, t_p^{(n+1)} + \tau^{***} - \tau_p(H_p^n))$ for endogenous reasons. Within this time interval, the price grows at a rate of g with certainty, except if another exogenous price drop occurs. Consequently, all arbitrageurs re-enter the market after a failed attack and buy back shares at a price of p_t, even if they have sold them an instant earlier for $(1-\gamma^p)p_t$. Furthermore, the failed attack at $t_p^{(n)}$ increases the endogenous threshold $\tau_p(H_p^n)$. More specifically, a failed attack at $t_p^{(n)}$ makes arbitrageurs more cautious about the prospects of mounting a successful attack after a price drop at $t_p^{(n+1)} > t_p^{(n)}$. In particular, if $t_p^{(n+1)}$ is close to a failed synchronized sell out at $t_p^{(n)}$, arbitrageurs will not find it optimal to exit again at $t_p^{(n+1)}$.[16]

7. CONCLUSION

This paper argues that bubbles can persist even though all rational arbitrageurs know that the price is too high and they jointly have the ability to correct the mispricing. Though the bubble will ultimately burst, in the intermediate term, there can be a large and long-lasting departure from fundamental values. A central, and we believe, realistic, assumption of our model is that there is dispersion of opinion among rational arbitrageurs concerning the timing of the bubble. This assumption serves both as a general metaphor for differences of opinion, information and beliefs among traders, and, more literally, as a reduced-form modeling of the temporal expression of heterogeneities amongst traders. While it is well understood that appropriate departures from common knowledge will permit bubbles to persist, we believe that our particular formulation is both natural and parsimonious. The model provides a setting in which 'overreaction' and self-feeding price drops leading to full-fledged crashes will naturally arise. It also provides a framework that allows one to rationalize phenomena such as 'resistance lines' and fads in information gathering.

[16] The equilibrium described in Proposition 5 is also maximally responsive in that price drops are responded to 'maximally' in the chronological order in which they appear. However, the bubble is more likely to burst at $t_p^{(n+1)}$ in an equilibrium in which the earlier event at $t_p^{(n)}$ was not responded to than in the equilibrium where arbitrageurs sold out at $t_p^{(n)}$ and the attack failed. We do not have a uniformly most responsive equilibrium in this section, but rather one that is responsive in chronological order to the maximum extent possible. In short, the above equilibrium is not necessarily the one in which the bubble bursts earliest for any possible sequence of price drops. Note that the same issue also arises in a setting with unanticipated public events and strictly positive transactions costs $c > 0$. We abstracted from these effects in Section 6.1 by assuming $c = 0$.

Many of the assumptions of our simple model may be viewed as being conducive to arbitrage. In particular, we assume that all professionals are in agreement that assets are overvalued, while arguably there are substantial differences in opinion even amongst professionals regarding the possibility that current valuations indeed reflect a new era of higher productivity growth, lower wages, and inflation, et cetera. Presumably incorporating these realistic complications would reinforce our conclusions.

The dynamic game developed in this paper might have useful applications in other settings. These include models of currency attacks, bank-runs, and R&D investment games.

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Manuscript received December, 2001; final revision received May, 2002.

APPENDIX A: General Payoff Specification

Let $B(t) = \{t_0 | T^*(t_0) < t\}$ be the set of t_0 that lead to a bursting of the bubble strictly prior to t. Let $B^c(t)$ denote the complement of $B(t)$ and $b(t) = \{t_0 | T^*(t_0) = t\}$ be the set of t_0 that lead to a bursting exactly at t.

The value at time t^1 of a position consisting of $x^1(t^1)$ stocks that is maintained unchanged until the bubble bursts is (from the viewpoint of arbitrageur t_i):

$$V^1((x^1, t^1)|t_i) = x^1 \int_{B(t_i+\bar{\tau})} e^{-r(T^*(t_0)-t^1)}[1-\beta(T^*(t_0)-t_0)]p(T^*(t_0))\, d\Phi(t_0|t_i, B^c(t^1)),$$

where

$$\Phi(\cdot|t_i, B^c(t^1)) = \frac{\Phi(\cdot|t_i)}{1-\Phi(t^1|t_i)}$$

is arbitrageur t_i's distribution over t_0, conditional upon the time t_i when arbitrageur t_i became aware of the mispricing, and conditional upon the existence of the bubble at t^1.

Proceeding iteratively from the final change of position, the value at $t^2 < t^1$ of x^2 stocks of which $(x^2 - x^1)$ are sold at t^1 is given by

$$\begin{aligned} &V^2((x^2, t^2), (x^1, t^1)|t_i) \\ &\quad = x^2 \int_{B(t^1)\setminus B(t^2)} e^{-r(T^*(t_0)-t^2)}[1-\beta(T^*(t_0)-t_0)]p(T^*(t_0))d\Phi(t_0|t_i, B^c(t^2)) \\ &\qquad + [1-\Phi(B(t^1)|t_i, B^c(t^2))]\{-e^{rt^2}c + e^{-r(t^1-t^2)}[V^1((x^1, t^1)|t_i) \\ &\qquad\qquad + (x^2-x^1)p(t^1) - A(t^1|t_i, B^c(t^2))]\}. \end{aligned}$$

The adjustment term $A(t^1|t_i, B^c(t^2))$ takes care of the special case where the bubble bursts exactly at t^1 and the order of (x^2-x^1) shares is not executed at the price $p(t^1)$ but at $[1-\alpha_t(t_0)\beta(t-t_0)]p(t)$. The latter term depends on the uncertain value of t_0. More precisely,

$$A(t^1|t_i, B^c(t^2)) = (x^2-x^1)p(t^1)\int_{b(t^1)} \alpha_t(t_0)\beta(t-t_0)d\Phi(t_0|t_i, B^c(t^2)).$$

Replacing superscript 2 by $k+1$ and superscript 1 with superscript k gives us a general recursive definition of the payoff function. Arbitrageur t_i's expected payoff from a strategy involving $K-1$ changes in her portfolio is then given by the function

$$V^K((x^K, t^K), \ldots, (x^1, t^1)),$$

where $(x^K = 1, t^K = t_i)$ denotes her initial position.

APPENDIX B: DETAILS OF SECTION 4

LEMMA 1 (No partial purchases or sell-outs): $\sigma(t, t_i) \in \{0, 1\}\ \forall t, t_i$.

PROOF: Consider an equilibrium strategy $\sigma(\cdot, t_i)$ that involves a change in position at t^k from the preceding position adopted at $t^{k+1} < t^k$. (The initial position is represented by K.) Suppose that $x^k(t^k) \notin \{0, 1\}$ and $\sigma(\cdot, t_i)$ is optimal. The expected payoff from t^{k+1} onwards is given by $V^{k+1}(\cdot)$ plus the value of bond holdings. Notice that $V^{k+1}(\cdot)$, excluding the transaction costs $ce^{rt^{k+1}}$, is linear in x^k and furthermore must be strictly positive. Hence, for $x^k - x^{k+1} > (<)0$ the payoff is strictly dominated by $x^k = 1$ $(x^k = 0)$. This contradicts the initial presumption that $\sigma(\cdot, t_i)$ is optimal. *Q.E.D.*

LEMMA 2 (Preemption): *In equilibrium, arbitrageur t_i believes at time $T(t_i)$ that at most a mass κ of arbitrageurs became aware of the bubble prior to her. That is,* $\underline{t_0^{\text{supp}}}(t_i) \geq t_i - \eta\kappa$.

PROOF: Let $\underline{t_0^{\text{supp}}}(t_i) = t_i - \eta\kappa - a$. For arbitrageur t_i to believe that $a > 0$, it is necessarily the case that for all $t_j \in [t_i - a, t_i)$, arbitrageurs t_j do not sell out prior to $T(t_i)$, since otherwise by Lemma 1 and Corollary 1 the bubble would have burst already. Furthermore, the aggregate selling pressure at $t = T(t_i)$ is $s_{t_0, t} > \kappa$ with strictly positive probability, since (by Lemma 1 and Corollary 1) all traders $t_k \leq t_i$ will also be fully out of the market at $T(t_i)$. Hence, with strictly positive probability all traders t_j who first sell out at $T(t_i)$ will only receive the post-crash price for their sales. Consequently, all traders $t_j \in [t_i - a, t_i)$ have an incentive to preempt (attack slightly earlier) the possible crash at $T(t_i)$, a contradiction. *Q.E.D.*

LEMMA 3 (Bursting Time): *The function $T^*(\cdot)$ is strictly increasing.*

PROOF: Recall $T^*(t_0) = \min\{T(t_0 + \eta\kappa), t_0 + \bar{\tau}\}$. Since $T(\cdot)$ is weakly increasing by the cut-off property, so is $T^*(\cdot)$. Now suppose that there exists $\bar{t}_0 > \underline{t}_0$ such that $T^*(\underline{t}_0) = T^*(\bar{t}_0)$. Clearly, $\underline{t}_0 + \bar{\tau} \geq T^*(\underline{t}_0)$. It follows that for all $t_0 \in (\underline{t}_0, \bar{t}_0]$, $T(t_0 + \eta\kappa) = T^*(t_0) = T^*(\underline{t}_0)$. Let $t_i = t_0 + \eta\kappa$. By Lemma 2 $\underline{t_0^{\text{supp}}}(t_i) \geq t_0$. However, $\underline{t_0^{\text{supp}}}(t_i) \leq \underline{t}_0 < t_0 \equiv t_i - \eta\kappa$, a contradiction. *Q.E.D.*

LEMMA 4 (Continuity of T^*): *The function $T^* : [0, \infty) \to [0, \infty)$ is continuous.*

PROOF: (i) Since T^* is increasing, we only need to rule out upward jumps. Suppose that there is an upward jump at t_0. Then for small enough $\varepsilon > 0$ and $s \in (t_0 - \varepsilon, t_0)$, $T^*(s) = T(s + \eta\kappa)$. Let $\overline{T}^* = \lim_{s \searrow t_0} T^*(s)$ and $\underline{T}^* = \lim_{s \nearrow t_0} T^*(s)$, and suppose $\varepsilon < \overline{T}^* - \underline{T}^*$. Recall that transactions cost ce^{rt} is incurred for any change of position at t. This precludes an arbitrageur from changing position at t and t' if the probability of a crash between t and t' is small relative to c. Hence, for small enough $\varepsilon > 0$, type $t_i = t_0 + \eta\kappa - \varepsilon$ is strictly better off selling out at $\overline{T}^* - \varepsilon > \underline{T}^*$ than at $T(t_i) < \underline{T}^*$, a contradiction. *Q.E.D.*

LEMMA 5 (Continuity of T): *The function $T : [0, \infty) \to [0, \infty)$ is continuous.*

PROOF: This argument is almost identical in structure to the proof of Lemma 4 and is omitted.

LEMMA 6 (Zero Probability): *For all $t_i > 0$, arbitrageur t_i believes that the bubble bursts with probability zero at the instant $T(t_i)$. That is,* $\Pr[T^{*-1}(T(t_i)) | t_i, B^c(T(t_i))] = 0$ *for all* $t_i > 0$.

PROOF: Observe that the conditional c.d.f.

$$\Phi[t_0|t_i, B^c(T(t_i))] = \frac{\Phi(t_0) - \Phi(\underline{t_0^{\text{supp}}}(t_i))}{\Phi(t_i) - \Phi(\underline{t_0^{\text{supp}}}(t_i))}.$$

Clearly, the latter is continuous. *Q.E.D.*

APPENDIX C: DETAILS OF SECTION 6

C.1. *Restriction to Interim-Trigger-Strategies*

Let an *interim-trigger-strategy* be one for which an agent follows a trigger-strategy *between* two successive synchronizing events. We argue that, in equilibrium, agents use interim-trigger-strategies.

Let $H(t|t_i) := (t_e^{(1)}, \ldots, t_e^{(n)})$ denote a history of past events at times $t_e^{(1)} < t_e^{(2)} < \cdots < t_e^{(n)}$ strictly prior to t and observed by arbitrageur t_i. Let $t_e^{(n)} \in H(t|t_i)$ be the time of the most recent synchronizing event observed by arbitrageur t_i. Let t_a equal t_i if arbitrageur t_i has not observed any synchronizing event, and equal $t_e^{(n)}$ otherwise. Denote arbitrageur t_i's posterior distribution over t_0 at $t = t_a$ by $\Phi(\cdot|t_i, \widetilde{B}^c(t_a), H(t_a|t_i))$. In other words, trader t_i conditions upon her awareness date, the fact that the bubble did not burst prior to t_a and the history of observed past synchronizing events if $t_a = t_e^{(n)}$. The event that the bubble did not burst prior to t_a, $\widetilde{B}^c(t_a)$, depends on the realization of t_0 and also on the sequence of synchronizing events. To simplify notation we denote $\{t_i, \widetilde{B}^c(t_a), H(t_a|t_i)\}$ by $\mathcal{I}_{t_i}(t_a)$.

DEFINITION 5: (i) The function $T_e(t_i) = \inf\{t|\sigma(t, t_i) > 0, H(t|t_i) = \varnothing\}$ denotes the *first instant at which arbitrageur t_i sells out* any of her shares conditional upon not having observed a synchronizing event thus far.

(ii) The function $T_e^*(t_0) = \min\{T_e(t_0 + \eta\kappa), t_0 + \bar{\tau}\}$ specifies the *date at which the bubble bursts*, absent the observation of a synchronizing event by trader $t_0 + \eta\kappa$.

The analysis for $T(\cdot)$ and $T^*(\cdot)$ of Section 4 applies to $T_e(\cdot)$ and $T_e^*(\cdot)$ in this section. In particular, $T_e^*(\cdot)$ is strictly increasing and continuous. The arguments are analogous and are not repeated here. Since each arbitrageur incurs positive transactions cost c when selling her shares, and the bubble bursts with zero probability at each instant, she has to leave the market at least for an interval. Arbitrageur t_i stays out of the market at least until arbitrageur $t_i + \varepsilon$ exits the market. Recall that the occurrence of synchronization events alone does not justify the loss in appreciation from exiting over a certain interval, since $\theta\bar{\beta} < g - r$. Hence, a generalized version of Proposition 1 applies *between* synchronization events, leading to what we have termed interim-trigger-strategies. Furthermore, the distribution function that the bubble will burst in the absence of further synchronizing events is given by

$$\Pi(t|\mathcal{I}_i(t_a)) = \Phi(T_e^{*-1}(t)|\mathcal{I}_i(t_a)).$$

Since in Section 6 arbitrageurs re-enter the market in equilibrium, transactions costs are potentially incurred repeatedly. Keeping track of all transactions costs complicates the analysis in uninteresting ways. In what follows we exploit the strategic restrictions obtained above (i.e. interim-trigger-strategies), while setting transactions costs to zero.

C.2. *Generalized Sell-out Condition*

LEMMA 8 (Generalized Sell-out Condition): *Suppose arbitrageur t_i has not observed a failed sell out. Then she sells out at t satisfying*

$$h(t|\mathcal{I}_{t_i})\beta(t - T_e^{*-1}(t)) + \theta \int_{t_0 \le t - \tau_e - \eta\kappa} (\alpha_{t_0,t}\beta(t - t_0))\, d\Phi(t_0|\mathcal{I}_{t_i}) = g - r.$$

The proof is similar to that of Lemma 7. The first term reflects the possibility that the bubble might burst in the next instant due to endogenous selling pressure. From trader t_i's viewpoint this occurs at the hazard rate

$$h(t|t_i) = \frac{\pi(t|\mathcal{I}_{t_i})}{1 - \Pi(t|\mathcal{I}_{t_i})}.$$

In this case, the relative size of the bubble is $\beta(t - T_e^{*-1}(t))$. Arbitrageur t_i also has to take into account the possibility that an unexpected synchronizing announcement might occur at t with a density of θ. The synchronizing event is followed by a crash if $t_0 \le t - \eta\kappa - \tau_e$. In this case the relative size of the bubble $\beta(t - t_0)$ depends on the realization of t_0. Note that in the case of a synchronizing event, trader t_i still has a chance to leave the market at the pre-crash price with probability $(1 - \alpha_{t_0,t})$. Hence, one has to multiply the advantage of attacking prior to the arrival of a synchronizing event by $\alpha_{t_0,t}$. Recall that $(1 - \alpha_{t_0,t})$ is the fraction of orders executed at the pre-crash price over all orders submitted immediately after a synchronizing event. The exact payoff specification and a more formal proof of this lemma can be found in an earlier version of our paper (Abreu and Brunnermeier (2001)).

C.3. *Proof of Proposition 4*

Step 1: *Derive symmetric equilibrium* τ^{**}. 1.1. Recall that $\tau^{**} = T_e(t_i) - t_i \ \forall\, t_i$ denotes the number of periods an arbitrageur waits to sell out if she does not observe any synchronizing event. Note that τ^{**} fully characterizes the symmetric responsive equilibrium outlined in Proposition 4. In any trading equilibrium all arbitrageurs who sold out after the last synchronizing event will re-enter the market after a failed attack and sell out again exactly when the ‘first’ trader, who did not participate in this common sell out, exits the market. This follows directly from the cut-off property. Hence, we can restrict our attention to the trading activity of traders who have not observed a synchronized event as yet. By describing their trading strategy we fully characterize any trading equilibrium.

1.2. There exists a unique symmetric trading equilibrium. Note that each arbitrageur t_i's posterior about t_0 at $T_e(t_i)$ exactly coincides with the posterior she had at $T(t_i)$ in a setting without synchronizing events. That is,

$$\Phi(t_0|t_i, B^c(T_e(t_i)), H(T_e(t_i)|t_i) = \varnothing) = \Phi(t_0|t_i, B^c(T(t_i))).$$

To see this, observe that in any symmetric equilibrium, the support of t_0 at $T_e(t_i)$ is also $[t_i - \eta\kappa, t_i]$ (as in Section 5). Furthermore, any unobserved synchronizing event at $t_e < t_i + \tau_e$ would not have led to a bursting of the bubble, since $t_e < t_0 + \eta\kappa + \tau_e$ for all possible $t_0 \in [t_i - \eta\kappa, t_i]$. Hence, synchronizing events do not serve to distinguish between t_0 in the latter interval. Since the posteriors are the same, so is the hazard rate $h(t|\mathcal{I}_{t_i})$ at the time that trader t_i sells out in either setting.

1.3. Clearly, if $\tau_e \ge \tau^{**}$, then synchronizing events have no impact on the bursting of the bubble, since trader $t_0 + \eta\kappa$ does not observe a signal prior to $t_0 + \eta\kappa + \tau^{**}$. Hence in this case $\tau^{**} = \tau^*$. Suppose now that $\tau_e < \tau^*$.

Define

$$\varphi(\tau) = h(t_i + \tau|\mathcal{I}_{t_i})\beta(\tau + \eta\kappa) + \theta \int_{t_0 \le t_i + \tau - \tau_e - \eta\kappa} (\alpha_{t_0,s}\beta(t_i + \tau - t_0))\, d\Phi(t_0|\mathcal{I}_{t_i}) - (g - r).$$

For equilibrium τ^{**} it is necessary that $\varphi(\tau^{**}) = 0$. We argue that τ^{**} such that $\varphi(\tau^{**}) = 0$ is (i) unique, and (ii) exists. By Step 1.2 $h(t_i + \tau^{**}|\mathcal{I}_{t_i})$ is constant across equilibrium τ^{**}. However, $\beta(\cdot)$ is strictly increasing, $\Phi(t_0|\mathcal{I}_{t_i})$ is the same, and the upper bound of the integral is increasing in τ. Thus, $\varphi(\tau^{**})$ is strictly increasing in equilibrium τ^{**}. Uniqueness follows directly.

For $\tau < \tau_e$ the sell out condition reduces to $h(t_i + \tau|\mathcal{I}_{t_i})\beta(\tau + \eta\kappa) - g - r = 0$. There does not exist a $\tau < \tau_e$ that solves this equation. This follows directly from the fact that we focus on $\tau_e < \tau^*$

and that τ^* solves the preceding equation uniquely (as established in Section 5). Hence, $\varphi(\tau) < 0$ for $\tau \le \tau_e$. Moreover, $\lim_{\tau \nearrow \bar{\tau}} \varphi(\tau) \to \infty$, since $h(t_i + \tau | \mathcal{I}_{t_i}) \to \infty$. Continuity of $\varphi(\tau)$ and the intermediate value theorem imply existence of τ^{**}. Steps 1.4 and 1.5 show that the unique τ^{**} indeed defines an equilibrium.

1.4. Immediately after a synchronizing event each arbitrageur who observes the synchronizing event is assumed to sell out in a 'responsive equilibrium.' Hence, from each arbitrageur's point of view, a bubble bursts with strictly positive probability at each t_e in a responsive equilibrium. Given these beliefs, and the fact that an instantaneous attack is costless for $c = 0$, it is indeed (strictly) optimal for an arbitrageur who observes the synchronizing event to sell out at this time.

1.5. To fully specify all relevant strategies, it only remains to consider continuation strategies after a failed sell out attempt. After a failed attack, arbitrageurs learn that fewer than κ traders have observed the synchronizing event. That is, $t_0 > t_e - \tau_e - \eta\kappa$. Since all other arbitrageurs who did not observe the synchronizing event only sell out at $t_i + \tau^{**}$, all arbitrageurs who observed the synchronizing event at t_e can rule out the possibility that the bubble bursts prior to $t_e + \tau_e^{**} - \tau_e$ provided that no new synchronizing event occurs. Note that the bubble will not burst for exogenous reasons prior to $t_e + \tau^{**} - \tau_e$, since the endogenous bursting time $t_0 + \eta\kappa + \tau^{**}$ occurs strictly before $t_0 + \eta\kappa + \tau^* < t_0 + \bar{\tau}$. The bubble might only burst prior to $t_e + \tau^{**} - \tau_e$ if a new synchronizing event occurs. After $t = t_e + \tau_e^{**} - \tau_e$, the analysis coincides with a setting without a synchronizing event at t_e. At this point all arbitrageurs who had participated in the failed sell out and subsequently re-entered the market, exit again.

The *next two steps* of the proof establish that equilibria are necessarily symmetric. They are analogous to Steps 2 and 3 of the proof of Proposition 3 and are not repeated here. See Abreu and Brunnermeier (2001) for more details.

C.4. Proof of Proposition 5

LEMMA 9: *There exists a function $\tau_p(\cdot)$ such that for all $n = 1, 2, \ldots$ and all histories of price drops H_p^n, after a price drop at $t_p^{(n+1)}$, all traders t_i who became aware of the mispricing prior to $t_p^{(n+1)} - \tau_p(H_p^n)$ leave the market and all other arbitrageurs remain in the market.*

PROOF: Recall that τ^{***} denotes the (post-awareness) time interval after which each arbitrageur leaves the market in the absence of a price shock. We proceed inductively, defining $\tau_p(H_p^k)$ given $\tau_p(H_p^l)$ where $l = 1, 2, \ldots, k-1$. We are looking for the smallest τ_p such that in a trading equilibrium arbitrageur $t_i = t_p^{(k)} - \tau_p$ is indifferent between exiting and staying in the market. At $t_p^{(k)}$, arbitrageurs with $t_i < t_p - \tau^{***}$ are already out of the market provided that $t_p^{(k)} > \max\{t_p^{(k-j)} + \tau^{***} - \tau_p(H_p^{(k-j-1)})\}_{j=1,2,\ldots}$. (When the latter condition is violated, traders have returned to the market because of a prior failed synchronized sell out.) Arbitrageur $t_i = t_p^{(k)} - \tau_p$ is indifferent between staying in the market or not if

$$\int_{t_0 \le t_p^{(k)} - \tau_p\left(H_p^{(k-1)}\right)} [-\gamma_p + \beta(t_p - t_0)](1 - \tilde{\alpha}_{t,t_0})\, d\Phi(t_0 | \mathcal{I}_{t_i,t_p}) + \int_{t_0 > t_p^{(k)} - \tau_p\left(H_p^{(k-1)}\right)} (-\gamma_p)\, d\Phi(t_0 | \mathcal{I}_{t_i,t_p}) = 0,$$

where $\mathcal{I}_{t_i,t_p}$ denotes the information set $\{t_i = t_p^{(k)} - \tau_p, t_p, H_p^{(k-1)}\}$. That is,

$$\int_{t_0 \le t_p^{(k-1)} - \tau_p\left(H_p^{(k-1)}\right)} [\beta(t - t_0)][1 - \tilde{\alpha}_{t,t_0}]\, d\Phi(t_0 | \mathcal{I}_{t_i,t_p}) - \gamma_p = 0.$$

Denote the left-hand side by $f_p^{(k)}(\tau_p | \mathcal{I}_{t_i,t_p})$. Note that if the bubble does not burst, then all arbitrageurs sell their shares at the price of $(1 - \gamma_p)p_t$. If the bubble does burst, only the first $[\kappa - (1/\eta)(t_p^{(k)} - t_0 - \tau^{***})]$ orders (κ orders) are executed at $(1 - \gamma_p)p_t$ if $t_p^{(k)} \ge (<) \max\{t_p^{(k-j)} + \tau^{***} - \tau_p(H_p^{(k-j-1)})\}_{j=1,2,\ldots}$. The term $(1 - \tilde{\alpha}_{t,t_0})$ reflects this fact. If $f_p^{(k)}(0 | \mathcal{I}_{t_i,t_p}) \ge 0$, then $\tau_p(H_p^{(k-1)}) = 0$. If $f_p^{(k)}(0 | \mathcal{I}_{t_i,t_p}) < 0$ look for $\min \tau_p$ such that $f_p^{(k)}(\tau_p | \mathcal{I}_{t_i,t_p}) \ge 0$. If such a τ_p does not exist, set $\tau_p = t_p$.

It remains to check that all arbitrageurs with $t_i < t_p^{(k)} - \tau_p(H_p^{(k-1)})$ strictly prefer to leave the market and all traders with $t_i > t_p^{(k)} - \tau_p(H_p^{(k-1)})$ prefer to remain in the market. By looking at trader t_i's distribution $\Phi(t_0|\cdot)$, it is easy to check that $f_p^{(k)}(\tau_p|\mathcal{I}_{t_i,t_p})$ is decreasing in t_i. Notice that each trader can rule out any $t_0 < t_i - \eta$. From the fact that the bubble did not burst before $t_p^{(k)}$ for endogenous reasons all arbitrageurs can rule out $t_0 < t_p^{(k)} - \tau^{***} - \eta\kappa$. Finally, since the bubble survived all sell out attempts after previous price drops, $t_0 \geq \max\{t_p^{(k-j)} - \tau_p(H_p^{(k-j-1)})\}_{j=1,2,\ldots}$. For all traders with $t_i - \eta \leq \max\{t_p^{(k)} - \tau^{***} - \eta\kappa, \max\{t_p^{(k-j)} - \tau_p(H_p^{(k-j-1)})\}_{j=1,2,\ldots}\}$, the lower bound is the same, while the upper bound is given by t_i. Hence, for arbitrageurs with lower t_i the density on t_0 for which $t_0 \leq t_p - \tau_p(H_p^{(k-1)})$ is higher. In addition, the (conditional) exponential distribution has the nice property that the relative likelihood of possible states in the support is the same across arbitrageurs. The distribution $\Phi(\cdot)$ for arbitrageurs whose support is $[t_i - \eta, t_i]$ is symmetric in t_i. Hence, $f_p^{(k)}(\tau_p|\mathcal{I}_{t_i,t_p})$ is also decreasing in t_i in this case. *Q.E.D.*

After establishing the critical value $\tau_p(H_p^{(k-1)})$ in Lemma 9, the rest of the proof of Proposition 5 is analogous to the proof of Proposition 4. There are three differences: all arbitrageurs observe each price drop, the first orders after the price drop are only executed at a price of $(1-\gamma^p)p_t$ instead of the pre-crash price p_t, and all $\tau_p(H_p^{(k-1)})$ are history dependent. Note that $\tau_p(H_p^{(k-1)})$ is known to all traders in equilibrium since all arbitrageurs can observe the past price process.

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