# Week 2: Bubbles and Crashes

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October 2021

These notes consider Abreu and Brunnermeier's (2003) paper on the failure of rational arbitrage in asset markets. Recall that the "no-trade" theorem states that speculative bubbles cannot exist in a world with only rational traders even if there is asymmetric information, so long as these traders share a common prior. Believers in the efficient market hypothesis argue that even if there are also behavioral or boundedly rational traders in the market, the presence of rational arbitrageurs will still push asset prices to fundamental values. Various models have been put forward as to why this might not be the case (e.g. Delong et al. 1990; Shleifer and Vishny, 1997). The idea of AB's paper is that even when rational arbitrageurs are aware of mispricing, a lack of common knowledge may prevent them from coordinating their attacks on a buble. As a result, persistent mispricing may occur even in the presence of rational traders.

## 1 Model

The market is composed of behavioral traders, whose behavior we will not directly model, and rational traders. The idea is that the behavioral traders cause a bubble in asset prices, which can be punctured only if there is sufficient selling pressure from the rational traders.

The price process for stocks works as follows. Starting from time $t = 0$, stock prices rise exponentially with $p_t = e^{gt}$. We assume that $g > r$, the risk-free interest rate. At the outset, these prices coincide with fundamental values. However at some random time $t_0$, stock prices and fundamental values diverge. At any time $t > t_0$, fundamental value is given by $(1 - \beta(t - t_0))p_t$, where $\beta : [0, \overline{\tau}] \to [0, \overline{\beta}]$ is increasing. Thus, after $t_0$, only a fraction

*The notes are based on the ones that I got from Jonathan Levin. I am indebted to him for his help. I am also grateful to Andy Skrzypacz and Debraj Ray for letting me use theirs.

$1-\beta(\cdot)$ of the price is due to fundamentals and a fraction $\beta(\cdot)$ is due to the bubble. The time $t_0$ at which prices depart from fundamentals is distributed exponentially on $[0,\infty)$ so its cdf is $\Phi(t_0) = 1 - e^{-\lambda t_0}$.

Stock prices are kept on the price path $p_t$ by behavioral traders. To alter the price path, a significant number of rational traders must sell. Specifically, if a fraction $\kappa$ of rational traders sell, prices drop to fundamentals. If any fraction less than $\kappa$ sells, prices continue to grow at rate $g$. However, we assume that the bubble cannot grow to size greater than $\overline{\beta}$ — this occurs after $\beta(\overline{\tau})$ periods, so if at time $t_0+\overline{\tau}$ the bubble has not burst, it will burst exogenously with prices returning to fundamentals.

With perfect information among rational traders as to when prices depart from fundamentals (i.e. time $t_0$ is common knowledge), it is clear that the bubble cannot grow at all. Since each rational trader would want to sell just before the bubble collapsed, this "pre-emption" motive will unravel their sales from the final date $t_0+\overline{\tau}$ all the way back to $t_0$.

The key ingredient in the model is that rational traders do not all share the same information. Instead, once stock prices depart from fundamentals, rational arbitrageurs figure this out sequentially. Starting at $t_0$, a cohort of mass $1/\eta$ becomes aware at each moment. By time $t_0+\eta$, all arbitrageurs are aware that prices are above fundamental value. However, since $t_0$ is random, arbitrageurs do not know how many other arbitrageurs became aware of the bubble before them. In particular, if a trader wakes up at $t$, he learns only that $t_0 \in [t-\eta, t]$, and hence that all traders will become informed at some point between $t$ and $t+\eta$.

After time $t_0$, the price $p_t$ exceeds fundamentals, but only a few traders realize this. After time $t_0+\eta\kappa$, however, enough traders are aware of the mispricing to burst the bubble. Thus, AB say that there is a true "bubble" if mispricing exists beyond $t_0+\eta\kappa$.

### 1.1 Strategies and Equilibrium

We can identify each rational trader with the time $t_i \in [t_0, t_0+\eta]$ at which he becomes aware of the bubble. A strategy for trader $t_i$ is a function $\sigma(\cdot, t_i) : [0,\infty) \to \{0,1\}$, where $\sigma(t,1) = 0$ means "Hold at $t$" and 1 means "Sell".

AB show, and we will assume, that each trader uses a simple "cut-off" strategy, so that:

$$\sigma(t,t_i) = \begin{cases} 0 & \text{for all } t < T(t_i) \\ 1 & \text{for all } t \geq T(t_t) \end{cases} .$$

Thus, the strategy for trader $t_i$ is summarized by the time $T(t_i)$ at which he sells. We also follow AB in restricting attention to equilibria that satisfy a monotonicity property whereby traders who become aware of mispricing earlier also sell out earlier in equilibrium.

**Definition 1** *A* ***trading equilibrium*** $\{T(t_i)\}$ *is a perfect Bayesian equilibrium with the property that if* $t_i < t_j$*, then* $T(t_i) \leq T(t_j)$*.*

In this dynamic game, in which players get the information about the state of nature also as the game unfolds, the perfect Bayesian Equilibrium with the common belief that the information structure is as we defined it above, requires that: (i) players Bayesian update their beliefs, given the actual strategies of the opponents and signals about state of nature that they get, (ii) given their beliefs, players play optimally at any point of time.

## 1.2 Optimal Trading Strategies

Given selling times $\{T(t_i)\}$ satisfying the monotonicity property, a bubble that starts at time $t_0$ will burst at time:

$$T^*(t_0) = \min\left\{T(t_0 + \eta\kappa), t_0 + \overline{\tau}\right\}.$$

AB establish that in any trading equilibrium, the function $T^*(\cdot)$ is continuous and strictly increasing, and so is its inverse $T^{*-1}(\cdot)$.

Now, define $\Pi(t|t_i)$ to be trader $t_i$'s belief that the bubble will burst by time $t$, given the strategies of the other traders. Then:

$$\Pi(t|t_i) \equiv \Phi\left(T^{*-1}(t)|t_i\right),$$

and let $\pi(t|t_i)$ denote the corresponding density, where $\Phi(\cdot|t_i)$ is the distribution of beliefs of type $t_i$ about the start of the bubble (see below).

Thus the expected payoff to $t_i$ if he sells out at $t$ is:

$$\int_{t_i}^{t} e^{-rs}(1 - \beta(s - T^{*-1}(s)))p(s)\pi(s|t_i)ds + e^{-rt}p(t)(1 - \Pi(t|t_i)).$$

Differentiating with respect to $t$ yields the optimal sell-out time for $t_i$.

**Lemma 1** *Given random bursting time* $T^*(t_0)$*, trader* $t_i$ *optimally sells time* $t$ *such that:*

$$h(t|t_i) = \frac{\pi(t|t_i)}{1 - \Pi(t|t_i)} = \frac{g - r}{\beta(t - T^{*-1}(t))}.$$

To see the intuition, consider the benefits of attacking the buble at time $t$ rather than time $t + \Delta$. These benefits are equal to the probability the bubble will burst at $t$ times the profits from selling before the burst:

$$h(t|t_i) \cdot p(t) \cdot \beta(t - T^{*-1}(t))$$

Note that the bubble size at $t$ is $\beta(t - t_0)$ and if the bubble does burst at $t$, then $t_0 = T^{*-1}(t)$. On the other hand, the benefit of waiting a bit longer is $(g - r) \cdot p(t)$. Combining these terms yields the result.

### 1.3 Exogenous and Endogenous Crashes

We now have the optimal selling time for each trader in response to the random bursting time $T^*(t_0)$. Since the random bursting time is determined by these trading strategies, we are in position to characterize trading equilibria.

Suppose each trader believes the bubble will burst $\xi$ units of time after the bubble begins, i.e. at time $t_0+\xi$. Each trader has a different belief about the burst time, because each has a different belief about the start time.

Let's consider $i$'s belief about the start time. Let $S$ be the random start date for the bubble, and $S_i$ the random wake up time for agent $i$. Then $S$ has a cdf $\Phi(t) = 1 - e^{-\lambda t}$, and if $S = t_0$, then $S_i$ is uniformly distributed on $[t_0, t_0 + \eta]$. So

$$\begin{aligned}
\Pr(S \leq t_0 | S_i = t_i) &= \frac{\Pr(S_i = t_i \text{ and } S \leq t_0)}{\Pr(S_i = t_i)} \\
&= \frac{\int_0^{t_0} \Pr(S_i = t_i | S = s) \Pr(S = s) ds}{\int_0^{\infty} \Pr(S_i = t_i | S = s) \Pr(S = s) ds}
\end{aligned}$$

Now, note that

$$\Pr(S_i = t_i | S = s) = \begin{cases} \frac{1}{\eta} & \text{if } s \in [t_i - \eta, t_i]. \\ 0 & \text{otherwise} \end{cases}$$

and the density of $S$ is $\lambda e^{-\lambda t}$ so loosely, $\Pr(S = s) = \lambda e^{-\lambda s}$. So we have

$$\begin{aligned}
\Pr(S \leq t_0 | S_i = t_i) &= \frac{\int_{t_i-\eta}^{t_0} \frac{1}{\eta} \lambda e^{-\lambda s} ds}{\int_{t_i-\eta}^{t_i} \frac{1}{\eta} \lambda e^{-\lambda s} ds} \\
&= \frac{e^{-\lambda(t_i-\eta)} - e^{-\lambda t_0}}{e^{-\lambda(t_i-\eta)} - e^{-\lambda t_i}} \\
&= \frac{e^{\lambda\eta} - e^{-\lambda(t_i-t_0)}}{e^{\lambda\eta} - 1}.
\end{aligned}$$

Therefore trader $i$ believes the start time is randomly randomly distributed on $[t_i - \eta, t_i]$ with distribution

$$\Phi(t_0|t_i) = \frac{e^{\lambda\eta} - e^{\lambda(t_i - t_0)}}{e^{\lambda\eta} - 1}.$$

Thus, she believes that the bursting date $t_i + \tau$ has distribution:

$$\Pi(t_i + \tau|t_i) = \frac{e^{\lambda\eta} - e^{\lambda(\xi - \tau)}}{e^{\lambda\eta} - 1},$$

and hazard rate:

$$h(t_i + \tau|t_i) = \frac{\lambda}{1 - e^{-\lambda(\xi - \tau)}}.$$

The optimal selling Lemma says that $t_i$ should sell at some time $t_i + \tau$ such that:

$$h(t_i + \tau|t_i) = \frac{g - r}{\beta(t - T^{*-1}(t))} = \frac{g - r}{\beta(\xi)}.$$

Note that if $\xi = \overline{\tau}$, so the trader believes the bubble will burst exogenously, then $\beta(\xi) = \overline{\beta}$.

After re-arranging these two equalities involving the hazard rate, we find that if all traders expect the bubble to burst $\xi$ periods after $t_0$, they will sell $\tau$ periods after becoming aware of the bubble, where:

$$\tau = \xi - \frac{1}{\lambda} \ln \left( \frac{g - r}{g - r - \lambda\beta(\xi)} \right). \tag{1}$$

Moreover, if all traders sell $\tau$ periods after becoming aware of the bubble, the bubble will burst at:

$$\xi = \min \{\overline{\tau}, \eta\kappa + \tau\}. \tag{2}$$

A trading equilibrium is thus a pair $(\tau, \xi)$ satisfying these two equations. We thus two possibilities covering the cases where the bubble bursts exogenously and endogenously.

**Proposition 1** *There is a unique trading equilibrium.*

1. *If* $\frac{\lambda}{1-e^{-\lambda\eta\kappa}} \leq \frac{(g-r)}{\overline{\beta}}$*, then each trader sells out* $\tau^* = \overline{\tau} - \frac{1}{\lambda} \ln \left( \frac{g-r}{g-r-\lambda\overline{\beta}} \right)$ *periods after becoming aware of the bubble, and for all* $t_0$*, the bubble bursts exogenously at time* $t_0 + \overline{\tau}$*.*

2. *If* $\frac{\lambda}{1-e^{-\lambda\eta\kappa}} > \frac{(g-r)}{\overline{\beta}}$*, then each trader* $t_i$ *with* $t_i \geq \eta\kappa$ *sells out* $\tau^* = \beta^{-1}(\frac{g-r}{\lambda/(1-e^{-\lambda\eta\kappa})}) - \eta\kappa$ *periods after becoming aware of the bubble, and all traders* $t_i$ *with* $t_i < \eta\kappa$ *sells out* $\tau^* + \eta\kappa$ *periods after becoming aware of the bubble. Thus the bubble bursts endogenously at time:*

$$t_0 + \xi^* = t_0 + \beta^{-1}\left(\frac{1-e^{-\lambda\eta\kappa}}{\lambda} \cdot (g-r)\right) < t_0 + \overline{\tau}.$$

(See the paper to check the proof of uniqueness - we *assumed* that agents believe that bubble will always burst $\xi$ periods after it starts.) Thus, if traders' prior belief is that fundamentals will justify prices for a relatively long period of time, and the bubble will not grow too quickly, the bubble will last for the maximum length of time. On the other hand, if the bubble is expected to start quickly or grow rapidly, it will burst endogenously. However, even if it bursts endogeously, arbitrage trades are *delayed* by $\tau^*$ periods in equilibrium, so the bubble still grows significantly above fundamentals.

## 2 Comments

1. The uniqueness aspect of the equilibrium is analogous to global games analysis, but there is a difference in that this game combines aspects of coordination and competition. In particular, traders want to "attack" just before other traders, and not "invest" exactly when the others do. Thus, with perfect information, a backward induction argument yields immediate attack, rather than multiple coordination equilibria.

2. Backward induction fails to bite because although the existence of the bubble eventually becomes mutual knowledge, it does not become common knowledge. To see this, note that at time $t_0 + \eta$, all traders are aware, so the bubble is mutual knowledge. However, not until time $t_0 + 2\eta$ is the mutual knowlege of the bubble itself mutual knowledge. The bubble is $m$th order mutual knowledge at time $t_0 + m\eta$, but clearly never common knowledge.

3. An interesting point here is that due to the behavioral traders, rational traders actually benefit from their failure to coordinate and burst the bubble.

4. In the last section of their paper, AB show that public "news" can have a disproportionate effect on prices. The idea is that even though public

news announcements may not reveal any new information about fundamentals, they can create common knowledge that allows traders to coordinate. In particular, traders may learn from the news not about fundamentals, but about what other traders know about fundamentals and thus about how they are likely to trade.

5. Several of the criticisms that I raise in the discussion in class are addressed in the "improved" version by Doblas-Madrid, in EMA (see below).

## References


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