Week 1: Solution Concepts

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These notes discuss some of the central solution concepts for normal-form games: Nash and Correlated Equilibrium, Iterated Deletion of Strictly Dominated Strategies, Rationalizability and Self Confirming Equilibrium.

1 Nash Equilibrium

Nash equilibrium captures the idea that players ought to do best they can given the strategies chosen by the other players.

Example 1 Prisoners' Dilemma

	C	D
C	$1, 1$	$-1, 2$
D	$2, -1$	$0, 0$

The unique Nash Equilibrium is (D, D).

Example 2 Battle of the Sexes

	B	F
B	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

There are two pure Nash equilibria (B, B) and (F, F) and a mixed strategy equilibrium where Row plays $\frac{2}{3}B + \frac{1}{3}F$ and Column plays $\frac{1}{3}B + \frac{2}{3}F$.

Definition 1 *A normal form game G consists of*

*The notes are based on the ones that I got from Jonathan Levin. I am indebted to him for his help. I am also grateful to Andy Skrzypacz and Debraj Ray for letting me use theirs.

1. *A set of players* $i = 1, 2, ..., I$.

2. *Strategy sets* $S_1, ..., S_I$*; let* $S = S_1 \times ... \times S_I$.

3. *Payoff functions: for each* $i = 1, ..., I$, $u_i : S \to \mathbb{R}$

A mixed strategy for player i, $\sigma_i \in \Delta(S_i)$, is a probability distribution on S_i. A pure strategy places all probability weight on a single action.

Definition 2 *A strategy profile* $(\sigma_1, ..., \sigma_I)$ *is a* ***Nash equilibrium*** *of* G *if for every* i*, and every* $s_i \in S_i$,

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(s_i, \sigma_{-i}).$$

A natural question, given the wide use of Nash equilibrium, is whether or why one should expect Nash behavior. One should think of Nash equilibrium as capturing, broadly speaking, a steady state behavior in a game. For example, if the behavior is shaped by the process of evolution or learning, and the process gets "locked-in" at some behavior that never changes afterwards, it must be Nash behavior. Similarly, if players reach an agreement on a strategy profile before independently choosing their actions, then no player will have reason to deviate only if the agreed profile is a Nash equilibrium.

Note that we say nothing here how/if this steady state behavior would be reached in the first place. We do not examine explicitly the process by which the steady state is reached, such as the process of evolution, learning, communication towards the agreement, or rational deliberation. In what follows we will take a closer look at the problem of communication, rational deliberation and learning, and show that rather than to NE, they lead to the solution concepts of Correlated Equilibrium, Rationalizaility and Self-Confirming Equilibrium, respectively.

1.1 Existence of Nash Equilibrium

The simplest result regarding the existence is the following, due to Nash.

Theorem 1 *Every game with finite strategy sets has a Nash equilibrium, possibly in mixed strategies.*

Proof. Let $\Delta S = \times_{i=1}^{I} \Delta(S_i)$ be the product of sets of all mixed strategy profiles. For each $\sigma \in \Delta S$ and $i \leq I$ define

$$B_i(\sigma) := \{\sigma_i' \in \Delta(S_i) | \sigma_i' \text{ is a best reply to } \sigma_{-i}\}.$$

B_i function is nonempty and convex. Define $B : \Delta S \twoheadrightarrow \Delta S$ by $B(\sigma) := \times_{i=1}^{I} B_i(\sigma)$. Due to Kakutani's fixed point theorem the correspondence B has a fixed point. ■

The conditions on a game to have a nonempty set of Nash equilibria have been analyzed extensively. First, the proof above easily generalizes to a case when S_i are not finite but compact metric and u_i are continuous (use not Kakutani, but Glicksburg fixed point theorem). Moreover, if S_i are compact and convex, u_i continuous and quasiconcave in s_i, then a Nash equilibrium exists in pure strategies. Dasgupta and Maskin (1986) and Reny (1999) investigate the case of discontinuous payoffs.

2 Correlated Equilibrium

2.1 Equilibria as a Self-Enforcing Agreements

Let's start with the account of Nash equilibrium as a self-enforcing agreement. Consider Battle of the Sexes (BOS). Here, it's easy to imagine the players jointly deciding to attend the Ballet or Football and then playing either (B, B) or (F, F), since neither wants to attend either event alone. However, a little imagination suggests that Nash equilibrium does not allow the players sufficient freedom to communicate.

Example 2, cont. Suppose in BOS, the players flip a coin and go to the Ballet if the coin is Heads, the Football game if Tails. That is, they just randomize between two different Nash equilibria. This coin flip allows a payoff $(\frac{3}{2}, \frac{3}{2})$ that is *not* a Nash equilibrium payoff.

So at the very least, under the account of play as a self-enforcing agreement reached in communication, one might want to allow for randomizations between Nash equilibria. Moreover, the coin flip is only a primitive way to communicate prior to play. A more general form of communication is to find a mediator who can perform clever randomizations, as in the next example.

Example 3 This game has three Nash equilibria (U, L), (D, R) and $(\frac{1}{2}U + \frac{1}{2}D, \frac{1}{2}L + \frac{1}{2}R)$ with payoffs $(5, 1)$, $(1, 5)$ and $(\frac{5}{2}, \frac{5}{2})$.

	L	R
U	$5, 1$	$0, 0$
D	$4, 4$	$1, 5$

Suppose the players find a mediator who chooses $x \in \{1, 2, 3\}$ with equal probability $\frac{1}{3}$. She then sends the following messages:

- If $x = 1 \Rightarrow$ tells Row to play U, Column to play L.
- If $x = 2 \Rightarrow$ tells Row to play D, Column to play L.
- If $x = 3 \Rightarrow$ tells Row to play D, Column to play R.

Claim. No player can do better then to follow the mediator's advice.

Proof. We need to check the incentives of each player.

- If Row hears U, believes Column will play $L \Rightarrow$ play U.
- If Row hears D, believes Column will play L, R with $\frac{1}{2}, \frac{1}{2}$ probability $\Rightarrow$ play D.
- If Column hears L, believes Row will play U, D with $\frac{1}{2}, \frac{1}{2}$ probability $\Rightarrow$ play L.
- If Column hears R, believes Row will play $D \Rightarrow$ play R.

■

Thus the players will follow the mediator's suggestion. With the mediator in place, expected payoffs are $(\frac{10}{3}, \frac{10}{3})$, strictly higher than the players could get by randomizing between Nash equilibria. To summarize, if we try to explicitly model the communication before playing the game, the plausible outcomes do not agree with the Nash behavior of the game.

2.2 Correlated Equilibrium

The notion of correlated equilibrium builds on the mediator story.

Definition 3 *A **correlating mechanism** $(\Omega, \{\mathcal{H}_i\}, p)$ consists of:*

- *A finite set of states* Ω
- *A probability distribution* p *on* Ω.
- *For each player* i*, a partition of* Ω*, denoted* $\mathcal{H}_i$*. Let* h_i *be a function that assigns to each state* $\omega \in \Omega$ *the element of* i*'s partition to which it belongs.*

Example 2, cont. In the BOS example with the coin flip, the states are $\Omega = \{\text{Heads, Tails}\}$, the probability measure is uniform on Ω, and Row and Column have the same partition, $\{\{\text{Heads}\}, \{\text{Tails}\}\}$.

Example 3, cont. In this example, the set of states is $\Omega = \{1, 2, 3\}$, the probability measure is again uniform on Ω, Row's partition is $\{\{1\}, \{2, 3\}\}$, and Column's partition is $\{\{1, 2\}, \{3\}\}$.

Definition 4 *A* ***correlated strategy*** *for* i *is a function* $f_i : \Omega \to S_i$ *that is measurable with respect to* i*'s information partition. That is, if* $h_i(\omega) = h_i(\omega')$ *then* $f_i(\omega) = f_i(\omega')$.

Definition 5 *A strategy profile* $(f_1, ..., f_I)$ *is a* ***correlated equilibrium*** *relative to the mechanism* $(\Omega, \{\mathcal{H}_i\}, p)$ *if for every* i *and every correlated strategy* $\tilde{f}_i$*:*

$$\sum_{\omega\in\Omega} u_i\left(f_i(\omega), f_{-i}(\omega)\right) p\left(\omega\right) \geq \sum_{\omega\in\Omega} u_i\left(\tilde{f}(\omega), f_{-i}(\omega)\right) p\left(\omega\right) \tag{1}$$

This definition requires that f_i maximize i's *ex ante* payoff. That is, it treats the strategy as a contingent plan to be implemented after learning the partition element. Note that this is equivalent to f_i maximizing i's *interim* payoff for each element of $\mathcal{H}_i$ that occurs with positive probability — that is, for all i,ω, and every $s_i' \in S_i$,

$$\sum_{\omega'\in h_i(\omega)} u_i(f_i(\omega), f_{-i}(\omega'))p(\omega'|h_i(\omega)) \geq \sum_{\omega'\in h_i(\omega)} u_i(s_i', f_{-i}(\omega'))p(\omega'|h_i(\omega))$$

Here, $p(\omega'|h_i(\omega))$ is the conditional probability on ω' given that the true state is in $h_i(\omega)$. By Bayes' Rule,

$$p(\omega'|h_i(\omega)) = \frac{p\left(h_i(\omega)|\omega'\right)p(\omega')}{\sum_{\omega''\in h_i(\omega)} p\left(h_i(\omega)|\omega''\right)p(\omega'')} = \frac{p(\omega')}{p(h_i(\omega))}$$

The definition of CE corresponds to the mediator story, but it's not very convenient. To search for all the correlated equilibria, one needs to consider millions of mechanisms.

Fortunately, it turns out that we can focus on a special kind of correlating mechanism, called a *direct mechanism*. We will show that for any correlated equilibrium arising from some correlating mechanism, there is a correlated equilibrium arising from the direct mechanism that is precisely equivalent in terms of behavioral outcomes. Thus by focusing on one special class of mechanism, we can capture all possible correlated equilibria.

Definition 6 *A* ***direct mechanism*** *has* $\Omega = S$*,* $h_i(s) = \{s' \in S | s'_i = s_i\}$*, and some probability distribution* q *over pure strategy profiles.*

Proposition 1 *Suppose* f *is a correlated equilibrium relative to* $(\Omega, \{\mathcal{H}_i\}, p)$*. Define* $q(s) \equiv p(\{f(\omega) = s\})$*. Then the strategy profile* $\tilde{f}$ *with* $\tilde{f}_i(s) = s_i$ *for all* $i, s \in S$ *is a correlated equilibrium relative to the direct mechanism* $(S, \{\widetilde{\mathcal{H}}_i\}, q)$*.*

Proof. Suppose that under the direct mechanism $(S, \{\widetilde{\mathcal{H}}_i\}, q)$ s_i is recommended to i with positive probability, so $q(s_i, s_{-i}) > 0$ for some s_{-i}. We check that player i cannot benefit from choosing another strategy s'_i when s_i is suggested. If s_i is recommended, then i's expected payoff from playing s'_i is:

$$\sum_{s_{-i} \in S_{-i}} u_i(s'_i, s_{-i}) q(s_{-i}|s_i).$$

The result is trivial if there is only one $h_i(\omega)$ in the original mechanism for which $f_i(h_i(\omega)) = s_i$. In this case, conditioning on s_i is the same as conditioning on $h_i(\omega)$ in the original mechanism. More generally, we substitute for q to obtain:

$$\frac{1}{p(f_i(\omega) = s_i)} \cdot \sum_{\omega | f_i(\omega) = s_i} u_i(s'_i, f_{-i}(\omega)) p(\omega).$$

Re-arranging to separate elements of $\mathcal{H}_i$ at which s_i is optimal:

$$\frac{1}{p(f_i(\omega) = s_i)} \cdot \sum_{H_i \in \mathcal{H}_i | f_i(H_i) = s_i} p(H_i) \left[\sum_{\omega \in H_i} u_i(s'_i, f_{-i}(\omega)) p(\omega | H_i) \right]$$

Since f is a correlated equilibrium relative to $(\Omega, \{\mathcal{H}_i\}, p)$, each bracketed term for which $p(H_i) > 0$ is maximized at $f_i(H_i) = s_i$. So s_i is optimal given recommendation s_i. ■

Thus what really matters in correlated equilibrium is the probability distribution over strategy profiles. We refer to any probability distribution q over strategy profiles that arises as the result of a correlated equilibrium as a *correlated equilibrium distribution (c.e.d.).*

Example 2, cont. In the BOS example, $\frac{1}{2}(B,B), \frac{1}{2}(F,F)$ is a c.e.d.

Example 3, cont. In this example $\frac{1}{3}(U,L), \frac{1}{3}(D,L), \frac{1}{3}(D,R)$ is a c.e.d.

The next proposition summarizes the characterization of correlated equilibrium distributions.

Proposition 2 *The distribution $q \in \Delta(S)$ is a correlated equilibrium distribution if and only if for all i, every s_i with $q(s_i) > 0$ and every $s_i' \in S_i$,*

$$\sum_{s_{-i} \in S_{-i}} u_i\left(s_i, s_{-i}\right) q\left(s_{-i}|s_i\right) \geq \sum_{s_{-i} \in S_{-i}} u_i(s_i', s_{-i}) q(s_{-i}|s_i). \tag{2}$$

Proof. ($\Leftarrow$) Suppose q satisfies (2). Then the "obedient" profile f with $f_i(s) = s_i$ is a correlated equilibrium given the direct mechanism $(S, \{\mathcal{H}_i\}, q)$ since (2) says precisely that with this mechanism s_i is optimal for i given recommendation s_i. ($\Rightarrow$) Conversely, if q arises from a correlated equilibrium, the previous result says that the obedient profile must be a correlated equilibrium relative to the direct mechanism $(S, \{\mathcal{H}_i\}, q)$. Thus for all i and all recommendations s_i occuring with positive probability, s_i must be optimal — i.e. (2) must hold. ■

Consider a few properties of correlated equilibrium.

Property 1 Any Nash equilibrium is a correlated equilibrium.

Proof. Need to ask if (2) holds for the probability distribution q over outcomes induced by the NE. For a pure equilibrium s^*, we have $q(s_{-i}^*|s_i^*) = 1$ and $q(s_{-i}|s_i^*) = 0$ for any $s_{-i} \neq s_{-i}^*$. Therefore (2) requires for all i, s_i :

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).$$

This is precisely the definition of NE. For a mixed equilibrium, σ^*, we have that for any s_i^* in the support of σ_i^*, $q(s_{-i}|s_i^*) = \sigma_{-i}(s_{-i})$. This follows from the fact that in a mixed NE, the players mix independently. Therefore (2) requires that for all i, s_i^* in the support of σ_i^*, and s_i,

$$\sum_{s_{-i} \in S_{-i}} u_i\left(s_i^*, s_{-i}\right) \sigma_{-i}\left(s_{-i}\right) \geq \sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \sigma_{-i}(s_{-i}),$$

again, the definition of a mixed NE. ■

Property 2 Correlated equilibria exist in finite games.

Proof. Any NE is a CE, and use Theorem 1. Hart and Schmeidler (1989) show the existence of CE directly, exploiting the fact that a CE is just a probability distribution q satisfying a system of linear inequalities (2). Their proof does not appeal to fixed point results! This also has implications regarding the complexity of computing CE and NE. ■

Property 3 The sets of correlated equilibrium distributions and payoffs are convex.

Proof. Left as an exercise.

2.3 Comments

1. The difference between mixed strategy Nash equilibria and correlated equilibria is that mixing is *independent* in NE. With more than two players, it may be important in CE that one player believes others are correlating their strategies. Consider the following example from Aumann (1987) with three players: Row, Column and Matrix.

$0,0,3$	$0,0,0$
$1,0,0$	$0,0,0$

$2,2,2$	$0,0,0$
$0,0,0$	$2,2,2$

$0,0,0$	$0,0,0$
$0,1,0$	$0,0,3$

   No NE gives any player more than 1, but there is a CE that gives everyone 2. Matrix picks middle, and Row and Column pick (Up,Left) and (Down,Right) each with probability $\frac{1}{2}$. The key here is that Matrix must expect Row to pick Up precisely when Column picks Left.

2. Note that in CE, however, each agent uses a pure strategy — he just is uncertain about others' strategies. So this seems a bit different than mixed NE if one views a mixed strategy as an explicit randomization in behavior by each agent i. However, another view of mixed NE is that it's not i's actual choice that matters, but j's beliefs about i's choice. On this account, we view σ_i as what others expect of i, and i as simply doing some (pure strategy) best response to σ_{-i}. This view, which is consistent with CE, was developed by Harsanyi (1973), who introduced small privately observed payoff perturbations so that in pure strategy BNE, players would be uncertain about others behavior. His "purification theorem" showed that these pure strategy BNE are observably equivalent to mixed NE of the unperturbed game if the perturbations are small and independent.

3. Returning to our pre-play communication account, one might ask if a mediator is actually needed, or if the players could just communicate by flipping coins and talking. With two players, it should be clear from the example above that the mediator is crucial in allowing for messages that are not common knowledge. However, Barany (1992) shows that if $I \geq 4$, then any correlated equilibrium payoff (with rational numbers) can be achieved as the Nash equilibrium of an extended game where prior to play the players communicate through cheap talk. Gerardi (2001) shows the same can be done as a sequential equilibrium provided $I \geq 5$. For the case of two players, Amitai (1996) and Aumann and Hart (2003) characterize the set of attainable payoffs if players can communicate freely, but without a mediator, prior to playing the game.

4. There is tons of applications of the Correlated Equilibrium, which range from proper appplications and "application" within theory. Of the latter kind, there is a recent strand of papers that characterize robust mechanisms in incomplete information games (check out Bergmann, Brooks and Morris; Du; Brooks and Du; Roessler Szentes etc). Turns out that, in an incomplete information setting, it is easier to characterize all, or worst CE for a game, than NE of the game. In terms of real applications, a recent CalTech postdoc (blanking out on his Russian name, argh - email me please), recently had a beautiful JMP on the application in voting. Roughly: if peaple play NE, then the probability that you are pivotal (you matter) is minimal, so why do people bother? Turns out that if people play a correlated equilibrium, with news media signal serving as a correlating device, the behavior is much more sensible.

3 Rationalizability and Iterated Dominance

Bernheim (1984) and Pearce (1984) investigated the question of whether one should expect rational players to introspect their way to Nash equilibrium play. They argued that even if rationality was common knowledge, this should not generally be expected. Their account takes a view of strategic behavior that is deeply rooted in single-agent decision theory.

To discuss these ideas, it's useful to explicitly define rationality.

Definition 7 *A player is rational if he chooses a strategy that maximizes his expected payoff given his belief about opponents' strategies.*

Note that assessing rationality requires defining beliefs, something that the formal definition of Nash equilibrium does not require. Therefore, as a matter of interpretation, if we're talking about economic agents playing a game, we might say that Nash equilibrium arises when each player is rational and know his opponents' action profile. But we could also talk about Nash equilibrium in an evolutionary model of fish populations without ever mentioning rationality.

3.1 (Correlated) Rationalizability

Rationalizability is motivated by two requirements on strategic behavior.

1. Players maximize with respect to their beliefs about what opponents will do (i.e. are rational).
2. The above is common knowledge among the players.

Example 4 In this game (from Bernheim, 1984), there is a unique Nash equilibrium (a_2, b_2). Nevertheless a_1, a_3, b_1, b_3 can all be rationalized.

	b_1	b_2	b_3	b_4
a_1	$0,7$	$2,5$	$7,0$	$0,1$
a_2	$5,2$	$3,3$	$5,2$	$0,1$
a_3	$7,0$	$2,5$	$0,7$	$0,1$
a_4	$0,0$	$0,-2$	$0,0$	$10,-1$

- Row will play a_1 if Column plays b_3
- Column will play b_3 if Row plays a_3
- Row will play a_3 if Column plays b_1
- Column will play b_1 if Row plays a_1

This "chain of justification" rationalizes a_1, a_3, b_1, b_3. Of course a_2 and b_2 rationalize each other. However, b_4 cannot be rationalized, and since no rational player would play b_4, a_4 can't be rationalized.

Definition 8 *A subset $B_1 \times ... \times B_I \subset S$ is a **best reply set** if for all i and all $s_i \in B_i$, there exists $\sigma_{-i} \in \Delta(B_{-i})$ to which s_i is a best reply.*

- In the definition, note that σ_{-i} can reflect correlation — it need not be a mixed strategy profile for the opponents. This allows for more "rationalizing" than if opponents mix independently. More on this later.

Definition 9 *The set of* ***correlated rationalizable strategies*** *is the component by component union of all best reply sets:*

$$R = R_1 \times ... \times R_I = \bigcup_{\alpha} B_1^{\alpha} \times ... \times B_I^{\alpha}$$

where each $B^{\alpha} = B_1^{\alpha} \times ... \times B_I^{\alpha}$ is a best reply set.

Proposition 3 *R is the maximal best reply set.*

Proof. Suppose $s_i \in R_i$. Then $s_i \in B_i^{\alpha}$ for some α. So s_i is a best reply to some $\sigma_{-i} \in \Delta(B_{-i}^{\alpha}) \subset \Delta(R_{-i})$. So R_i is a best reply set. Since it contains all others, it is maximal. ■

3.2 Iterated Strict Dominance

In contrast to asking what players might do, iterated strict dominance asks what players *won't* do, and what they won't do conditional on other players not doing certain things, and so on. Recall that a strategy s_i is *strictly dominated* if there is some mixed strategy σ_i such that $u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$ for all $s_i \in S_{-i}$, and that iterated dominance applies this definition repeatedly.

- Let $S_i^0 = S_i$
- Let $S_i^k = \left\{ \begin{array}{c} s_i \in S_i^{k-1} | \text{There is no } \sigma_i \in \Delta(S_i^{k-1}) \text{ s.t.} \\ u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}^{k-1} \end{array} \right\}$
- Let $S_i^{\infty} = \bigcap_{k=1}^{\infty} S_i^k$.

Iterated strict dominance never eliminates Nash equilibrium strategies, or any strategy played with positive probability in a correlated equilibrium (proof left as an exercise!). Indeed it is often quite weak. Most games, including many games with a unique Nash equilibrium, are not dominance solvable.

Example 4, cont. In this example, b_4 is strictly dominated. Eliminating b_4 means that a_4 is also strictly dominated. But no other strategy can be eliminated.

Proposition 4 *In finite games, iterated strict dominance and correlated rationalizability give the same solution set, i.e.* $S_i^\infty = R_i$.

This result is suggested by the following Lemma (proved in the last section of these notes).

Lemma 1 *A pure strategy in a finite game is a best response to some beliefs about opponent play if and only if it is not strictly dominated.*

Proof. (Proposition) $R \subset S^\infty$. If $s_i \in R_i$, then s_i is a best response to some belief over R_{-i}. Since $R_{-i} \subset S_{-i}$, Lemma 1 implies that s_i is not strictly dominated. Thus $R_i \subset S_i^2$ for all i. Iterating this argument implies that $R_i \subset S_i^k$ for all i, k, so $R_i \subset S_i^\infty$.

$S^\infty \subset R$. It suffices to show that S^∞ is a best-reply set. By definition, no strategy in S^∞ is strictly dominated in the game in which the set of actions is S^∞ (otherwise, the process of elimination would continue). Thus, any $s_i \in S_i^\infty$ must be a best response to some beliefs over S_{-i}^∞. ■

In the next class we will develop a formal model of knowledge and common knowledge and will show that correlated rationalizability (and so ISD) is characterized by the postulates (1) and (2) from he begining of the section. We will also show that it is identical with an appropriare weaker notion of CE.

3.3 Comments

1. Bernheim (1984) and Pearce (1984) originally defined rationalizability assuming that players would expect opponents to mix independently. So B is a best reply set if $\forall s_i \in B_i$, there is some $\sigma_{-i} \in \times_{j \neq i} \Delta(S_j)$ to which s_i is a best reply. For $I = 2$, this makes no difference, but when $I \geq 3$, their concept refines ISD (it rules out more strategies).

2. There is an alternative definition of correlated rationalizable strategies, in which they are defined inductively, similarly as in the case of ISD. That is, $R_i^0 = S_i$, $R_i^k = \{s_i \in R_i^{k-1} | \text{There is some } \sigma_{-i} \in \Delta(R_{-i}^{k-1}) \text{ s.t. } u_i(s_i, \sigma_{-i}) \geq u_i(\sigma_i, \sigma_{-i}) \text{ for all } \sigma_i \in \Delta(R_i^{k-1})\}$, $R_i^\infty = \bigcap_{k=1}^\infty R_i^k$. The two definitions are equivalent, $R_i = R_i^\infty$, by the argument as in the proof of proposition 4.

   This argument is the first instance of a more general important result, which says that a set of elements surviving an iterated procedure (formally: iterated application of a monotone set function) agrees with the largest fixed point (for this function). We will see a second instance of it in the next class, when we show the equivalence of two definitions of common knowledge, and the third in the class on repeated games with imperfect public monitoring, when we follow closely the technique of Abreu, Pearce and Stacchetti (1990). You will prove this result in your problem set.

3. You might wonder: who cares about solution concepts in XXI century? Wrong. One of the best JMP in the last 2-3 years was one by Shengwu Li (AER, "Obvious-Strategy Proofness"), landing Harvard job. Basically, in an IPV single unit auction theory, 2nd price auction and English auction are strategically equivalent, and truthful bid is weakly dominant in both. Yet in practice, people behave properly in English and less so in 2nd Price. Shengwu introduces a refinement of strategy proofness (i.e., dominance) that is satisfied by English, yet not 2nd price (read in his intro). Here is a project that will land you a job at Harvard: turns out that when you look at multiunit auctions, the dynamic equivalent of English auction is not even strategy-proof, and at best unique rationalizable one (see HW), while truthful bidding in sealed bid auction is. So the "strategic difficulty" ranking is reversed. What is going on, can the dynamic mechanism be improved? Are our concepts wrong? How do people act in the two auctions, in multi-unit settings? When you introduce proper incomplete information and interdependence, the whole hell breaks loose...

3.4 Appendix: Omitted Proof

For completeness, this section provides a proof of the Lemma equating dominated strategies with those that are never a best response. The proof requires a separation argument, so let's first recall the Duality Theorem for linear programming. To do this, start with the following problem:

$$\begin{aligned} &\min_{x\in\mathbb{R}^n} \sum_{j=1}^{n} c_j x_j \\ &\text{s.t. } \sum_{j=1}^{n} a_{ij} x_j \geq b_i \qquad \forall i = 1, ..., m \end{aligned} \tag{3}$$

This problem has the same solution as

$$\max_{y\in\mathbb{R}_+^m}\left(\min_{x\in\mathbb{R}^n}\left(\sum_{j=1}^n c_jx_j+\sum_{i=1}^m y_i\left(b_i-\sum_{j=1}^n a_{ij}x_j\right)\right)\right). \tag{4}$$

The solution to (4) is clearly lower than the one to (3) as one can always use the minimizer of (3) in (4); In the other direction one uses y that is the vector of associated shadow values. Rearranging terms, we obtain

$$\max_{y\in\mathbb{R}_+^m}\left(\min_{x\in\mathbb{R}^n}\left(\sum_{j=1}^n\left(c_j-\sum_{i=1}^m y_ia_{ij}\right)x_j+\sum_{i=1}^m y_ib_i\right)\right). \tag{5}$$

Swapping the order of optimization (minmax theorem) give us

$$\min_{x\in\mathbb{R}^n}\left(\max_{y\in\mathbb{R}_+^m}\left(\sum_{j=1}^n\left(c_j-\sum_{i=1}^m y_ia_{ij}\right)x_j+\sum_{i=1}^m y_ib_i\right)\right), \tag{6}$$

which, again, can be related to the following "dual" problem:

$$\begin{aligned}&\max_{y\in\mathbb{R}_+^m}\sum_{i=1}^m y_ib_i \\ &\text{s.t.}\quad\left(c_j-\sum_{i=1}^m y_ia_{ij}\right)=0\qquad\forall j=1,...,n.\end{aligned} \tag{7}$$

Theorem 2 *Suppose problems (3) and (7) are feasible (i.e. have non-empty constraint sets). Then their solutions are the same.*

We use the duality theorem to prove the desired Lemma.

Lemma 2 *A pure strategy in a finite game is a best response to some beliefs about opponent play if and only if it is not strictly dominated.*

Proof. (Myerson, 1991). Let $s_i\in S_i$ be given. Our proof will be based on a comparison of two linear programming problems.*Problem I:*

$$
\begin{aligned}
&\min_{\sigma_{-i},\delta} \delta \\
\text{s.t.} \quad & \sigma_{-i} \in \mathbb{R}^{S_{-i}}, \ \ \delta \in \mathbb{R} \\
& \sigma_{-i}(s_{-i}) \geq 0 \qquad \forall s_{-i} \in S_{-i} \\
& \sum_{s_{-i}} \sigma_{-i}(s_{-i}) \geq 1 \qquad \text{and} \qquad -\sum_{s_{-i}} \sigma_{-i}(s_{-i}) \geq -1 \\
& \delta + \sum_{s_{-i}} \sigma_{-i}(s_{-i}) \left[u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) \right] \geq 0 \qquad \forall s_i' \in S_i
\end{aligned}
$$

Observe that s_i is a best response to some beliefs over opponent play if any only if the solution to this problem is less than or equal to zero.*Problem II:*

$$
\begin{aligned}
&\max_{\eta,\varepsilon_1,\varepsilon_2,\sigma_i} \varepsilon_1 - \varepsilon_2 \\
\text{s.t.} \quad & \varepsilon_1, \varepsilon_2 \in \mathbb{R}_+, \, \sigma_i \in \mathbb{R}_+^{S_i}, \, \eta \in \mathbb{R}_+^{S_{-i}} \\
& \sum_{s_i'} \sigma_i(s_i') = 1 \\
& \eta(s_{-i}) + \varepsilon_1 - \varepsilon_2 + \sum_{s_i'} \sigma_i(s_i') \left[u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) \right] = 0 \qquad \forall s_{-i} \in S_{-i}
\end{aligned}
$$

Observe that s_i is strictly dominated if and only if the solution to the problem II is strictly greater than zero — i.e. s_i is *not* strictly dominated if and only if the solution to this problem is less than or equal to zero.

Finally, the Duality Theorem for linear programming says that so long as these two problems are feasible (have non-empty constraint sets), their solutions must be the samee. This means that s_i is a best response against σ_{-i}, which is the argument solving Problem I, establishing the result. ■

4 Self-Confirming Equilibria

The third possible foundation for equilibrium is learning. Lets just ask what might happen as the end result of a learning processes, given that a learning process settles down into steady-state play.[1] Fudenberg and Levine (1993) suggest that a natural end-result of learning is what they call *self-confirming equilibria*. In a self-confirming equilibrium:

[1]Be careful here: read the previous sentence again and think for a while. See the last comment at the end of the section.

1. Players maximize with respect to their beliefs about what opponents will do (i.e. are rational).

2. Beliefs cannot conflict with the empirical evidence (i.e. must match the empirical distribution of play).

The difference with Nash equilibrium and rationalizability lies in the restriction on beliefs. In a Nash equilibrium, players hold correct beliefs about opponents' strategies and hence about their behavior. By contrast, with rationalizability, beliefs need not be correct, they just can't conflict with rationality. With SCE, beliefs need to be consistent with available data.

4.1 Examples of Self-Confirming Equilibria

In a simultaneous game, assuming actions are observed after every period, every Nash equilibrium is self-confirming. Moroeover, any self-confirming equilibrium is Nash.

Example 5 Consider matching pennies. It is an SCE for both players to mix 50/50 and to both believe the other is mixing 50/50. On the other hand, if player i believes anything else, he must play a pure strategy. But then player j must believe i will play this strategy or else he would eventually be proved wrong. So the only SCE is the same as the NE.

In extensive form games, the situation is different, as the next example shows.

Example 6 (Fudenberg-Kreps, 1993) Consider the three player game below.

In this game, there is a self-confirming equilibrium where (A_1, A_2) is played. In this equilibrium, player 1 expects player 3 to play R, while player 2 expects 3 to play L. Given these beliefs, the optimal strategy for player 1 is to play A_1, while the optimal strategy or player 2 is to play A_2. Player 3's beliefs and strategy can be arbitrary so long as the strategy is optimal given beliefs.

The key point here is that there is *no* Nash equilibrium where (A_1, A_2) is played. The reason is that in any Nash equilibrium, players 1 and 2 *must* have the *same* (correct) beliefs about player 3's strategy. But if they have the same beliefs, then at least one of them must want to play D.

The distinction between Nash and self-confirming equilibria in extensive form games arises because players do not get to observe all the relevant information about their opponents' behavior. The same issue can arise in simultaneous-move games as well if the information feedback that players' receive is limited. For instance, in sealed-bid auctions it is relatively common to announce only the winner and possibly not even the winning price, so the information available to form beliefs is rather limited.

4.2 Formal Definition of SCE

To define self-confirming equilibrium in extensive form games, let s_i denote a strategy for player i, and σ_i a mixture over such strategies. Let H_i denote the set of information sets at which i moves, and $H(s_i, \sigma_{-i})$ denote the set of information sets that can be reached if player i plays s_i and opponents play σ_{-i}. Let $\pi_i(h_i|\sigma_i)$ denote the mixture over actions that results at information set h_i, if player i is using the strategy σ_i (i.e. π_i is the behavior strategy induced by the mixed strategy σ_i). Let μ_i denote a belief over $\Pi_{-i} = \times_{j \neq i} \Pi_j$ the product set of other players' behavior strategies.

Definition 10 *A profile σ is a Nash equilibrium if for each $s_i \in$support(σ_i), there exists a belief μ_i such that (i) s_i maximizes i's expected payoff given beliefs μ_i, and (ii) player i's beliefs are correct, for all $h_j \in H_{-i}$*

$$\mu_i \left[\{ \pi_{-i} \mid \pi_j(h_j) = \pi_j(h_j|\sigma_j) \} \right] = 1.$$

The way to read this is that for each information set at which some player $j \neq i$ moves, player i's belief puts a point mass on the probability distribution that exactly coincides with the distribution induced by j's strategy. Thus i has correct beliefs at all opponent information sets.

Definition 11 *A profile σ is a Self-Confirming equilibrium if for each $s_i \in$support(σ_i), there exists a belief μ_i such that (i) s_i maximizes i's expected payoffs given beliefs μ_i, and (ii) player i's beliefs are empirically correct, for all histories $h_j \in H(s_i, \sigma_{-i})$ and all $j \neq i$*

$$\mu_i \left[\{ \pi_{-i} \mid \pi_j(h_j) = \pi_j(h_j|\sigma_j) \} \right] = 1.$$

The difference is that in self-confirming equilibrium, beliefs must be correct only for reachable histories. This definition assumes that the players observe their opponents' actions perfectly, in contrast to the brief discussion above; it's not hard to generalize the definition. Note that this definition formally encompasses games of incomplete information (where Nature moves first); Dekel et. al (2004) study SCE in these games.

4.3 Comments

1. You might play with the requirements on the beliefs of the players even more. For example, Esponda (2009) defines the equilibrium concept related to SCE, but assuming, roughly, that the players do not perceive the correlations between the play of the opponents, or the opponents play and the moves of nature. This has very strong implications on the very many strategic models of imperfect information (such as any auctions or trade with common value component), in which the correlation offen plays a crucial role.

2. While the stories behind the Rationalizability and (mediator based) Correlated Equilibrium sound as plausible, possible stories explaining the process of reaching the solution concept, it is not true with SCE. Just like a NE, SCE has a flavor of only a necessary condition: *if* the steady state is reached, it should be a SCE. But will learning in a strategic, game-theoretic environment lead to a steady state to begin with? For negative answers to this question see Nachbar (1997) or Sadzik (2008)

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