Networks Pset #1

Chris Ackerman*

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Question 1

$$p(n) \equiv \frac{\lambda}{n}, \quad \lambda \in (0, \infty)$$

The degree distribution is Poisson,

$$\begin{aligned} p_k &= \frac{\lambda^k}{k!} e^{-\lambda} \\ p &\equiv 1 - q \\ \therefore p &= \sum_{k=0}^{\infty} p_k p^k \\ \iff 1 - q &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} (1-q)^k \qquad (1) \\ \sum_{k=0}^{\infty} \frac{(\lambda(1-q))^k}{k!} &= e^{\lambda(1-q)} \end{aligned}$$

so the labeled equation becomes

$$\begin{aligned} 1 - q &= e^{-\lambda q} \\ \iff q &= 1 - e^{-\lambda q} \end{aligned}$$

If $\lambda < 1$, then the average degree is less than one and there is no giant component ($q = 0$). Then the extinction probability is 1. If instead the average degree is greater than 1, there is a giant component $q \in (0, 1)$ and the extinction probability is $p \in (0, 1) < 1$.

Question 3

Fix $p \in (0, 1)$ and some network q on k nodes. Now consider the sequence of Erdos-Renyi networks $G(n, p)$. Partition the n nodes into as many separable groups of k nodes as possible, and consider the subnetworks that form in each group. Let ℓ be the number of links in network g. The probability that none of these subnetworks matches the network g is

$$\left(1 - p^{\ell}(1-p)^{\frac{k(k-1)}{2} - \ell}\right)^n \underset{n \to \infty}{\to} 0$$

Thus the probability that there is a copy of the random network goes to 1 as $n \to \infty$.

*I worked on this problem set with Aristotle Magganas, Antonio Martner and Koki Okumura.

Question 4

Definition (Connected Network)**.** *For any two nodes in the network, there exists a path between them.*

Definition (Disconnected Network)**.** *A network with at least one vertex that does not have a path.*

Suppose g is a disconnected network. Take $a, b \in g$ and suppose they are in different components. Then, $\nexists (a, b) \in g$. By the definition of the complement of a network, (a, b) must be an edge in g'. Thus there is a path between a and b.

Suppose instead that $(a, b) \in g$. Then a, b are in the same component of g. Since g is disconnected, $\exists c$ s.t. $(a, c) \notin g \wedge (b, c) \notin g$. Therefore, (a, c, b) must bea path from node a to b in g'.

Question 8

For even n, the efficient network is a collection of dyads ij. But it is not pairwise stable.

1. First we will check efficiency. This is straightforward (the question isn't asking us to prove uniqueness, just to show that the dyad network *is* efficient, so step 1 is to maximize social welfare, and then step 2 is to show that the dyad network achieves this number).

   Definition (Efficiency, Jackson p.15)**.** *A network g is efficient if it maximizes $\sum_i u_i(g)$.*

   The maximal possible utilitarian welfare in the coauthor model is

$$\begin{aligned}\sum_{i \in N} u_i(g) &= \sum_{i:d_i(g)>0} \sum_{j:d_j(g)>0} \left[\frac{1}{d_i(g)} + \frac{1}{d_j(g)} + \frac{1}{d_i(g)d_j(g)}\right] \\ &\leq 2N + \sum_{i:d_i(g)>0} \sum_{j:d_j(g)>0} + \frac{1}{d_i(g)d_j(g)}\end{aligned}$$

   Since we're only summing over nodes with at least one connection,

$$\begin{aligned}\frac{1}{d_i(g)d_j(g)} &\leq \frac{1}{1 \cdot 1} \\ \implies \sum_{i:d_i(g)>0} \sum_{j:d_j(g)>0} + \frac{1}{d_i(g)d_j(g)} &\leq N \\ \implies \sum_{i \in N} u_i(g) &\leq 3N\end{aligned}$$

   Now, we just need to show that the dyad network achieves this upper bound. Note that $d_i = d_j = 1$ for every node in the dyad network, so

$$\begin{aligned}\sum_{i \in N} u_i(g) &= \sum_{i:d_i(g)>0} \sum_{j:d_j(g)>0} \left[\frac{1}{d_i(g)} + \frac{1}{d_j(g)} + \frac{1}{d_i(g)d_j(g)}\right] \\ &= \sum_i \sum_j \frac{1}{1} + \frac{1}{1} + \frac{1}{1} \\ &= 3N \quad \square\end{aligned}$$

2. Now pairwise stability. The procedure is similar; check the definition and then show that each individual in the dyad network wants to deviate (add at least 1 more connection).

   Definition (Pairwise Stability, Jackson p. 16)**.**

   (a) No agent can increase his or her utility by deleting a link that he or she is directly involved in.

(b) No two agents can both benefit (at least one strictly) by adding a link between themselves.

We're going to find a counterexample to (b) by looking at two agents who link together, so that each of them have two coauthors, and one of their coauthors has 1 coauthor, while the other (the guy deviating) has two coauthors. The payoffs should be strictly higher for both players, since everything is symmetric. Define g as the dyad network and g' is the network where two authors have chosen to write another paper together. We're going to look at u_i, the utility of one of the guys who deviates.

$$
\begin{aligned}
u_i(g) &= \sum_{j:ij\in g} \left(\frac{1}{d_i(g)} + \frac{1}{d_j(g)} + \frac{1}{d_i(g)d_j(g)} \right) \\
&= 1 + 1 + 1 \\
&= 3 \\
u_i(g') &= \sum_{j:ij\in g'} \left(\frac{1}{d_i(g)} + \frac{1}{d_j(g)} + \frac{1}{d_i(g)d_j(g)} \right) \\
&= \frac{1}{2} + \frac{1}{1} + \frac{1}{2} \\
&\quad + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} \\
&= \frac{13}{4} \\
&> u_i(g) \Rightarrow\Leftarrow
\end{aligned}
$$