Networks Pset #1

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Question 3

If we make the two leftmost red nodes green, the resulting green nodes are $\frac{2}{3}$-cohesive, and the remaining red nodes are $\frac{3}{4}$-cohesive. The original green nodes only have green neighbors, and the two new ones each have 2 green neighbors and a single red neighbor, so the green subnetwork is $\frac{2}{3}$-cohesive. Two red nodes are connected to a green node, but they are each connected to three red nodes, so the new red subnetwork is $\frac{3}{4}$-cohesive.

Question 4

a) The highest $q^*(g)$ we can achieve is $\frac{1}{2}$, and the line and circle networks will both achieve $q^*(g) = \frac{1}{2}$.

b) The lowest $q^*(g)$ we can achieve is $\frac{1}{9}$, and the complete graph achieves this $q^*(g)$.

Question 5

Without a network, the monopolist solves the problem

$$\max_p \; p \cdot q \qquad \text{(Assume } mc = 0\text{)}$$
$$\theta \sim U[0,1] \implies q = 1 - p$$
$$\max_p \; p \cdot (1-p)$$
$$\frac{\partial}{\partial p} p - p^2 = 0 \qquad \text{(FOC)}$$
$$1 - 2p = 0$$
$$p = \frac{1}{2}$$

Now let's put this on a network. We just need to cook up a counterexample, so here's a funny network that will work. Put all $\theta < \frac{1}{2} + \delta, \delta > 0$ into singletons. Put all $\theta \geq \frac{1}{2} + \delta$ in a completely connected network. I guess the best economic "story" for this is that all of the rich guys live in a gated community and hang out together, and all the poor guys are on their own somewhere else. So not *too* crazy. Now, as we send $\varepsilon \to 0$ we only hit the giant component (all of the $\theta \in \left[\frac{1}{2} + \delta, 1\right]$). Let's see how the monopolist's price without a network does, and if we can do better.

$$\begin{aligned} \pi &= p \cdot q \\ &= p \cdot \left(1 - \left[\frac{1}{2} + \delta\right]\right) \\ &= \left(\frac{1}{2}\right)^2 - \frac{\delta}{2} \end{aligned}$$

*I worked on this problem set with Aristotle Magganas, Antonio Martner and Koki Okumura.

But the monopolist can do better! $\delta > 0 \implies \theta > p$ for all the guys that are actually buying this product. Everyone is getting some consumer surplus. We can squeeze *all* of this surplus out of the guy with $\underline{\theta} = \frac{1}{2} + \delta$, and we can squeeze *some* surplus (δ) out of everyone that's buying. Basically, increasing the price to $p' \in \left(\frac{1}{2}, \frac{1}{2} + \delta\right]$ doesn't cause anybody to stop buying the product, but does give a higher per-unit price to the monopolist.

$$\begin{aligned}
\pi' &= p' \cdot q \\
&= \underbrace{\left(\frac{1}{2} + \delta\right)}_{\text{higher price}} \underbrace{\left(1 - \left[\frac{1}{2} + \delta\right]\right)}_{\text{same quantity}} \\
&= \left(\frac{1}{2}\right)^2 - \frac{\delta}{2} + \underbrace{\delta\left(\frac{1}{2} - \delta\right)}_{\text{additional profit}} \\
&> \pi
\end{aligned}$$

This holds for all $\delta \in \left(0, \frac{1}{2}\right)$.

Question 7

Proof by contradiction and induction. Choose agent i_1 in a finite network, and suppose he is *not* in a strongly connected or closed group, and does *not* have a directed path to an agent in a strongly connected group. Then, there exist two nodes i_2 and i_3 such that

1. there is a path from $i_1 \to i_2$
2. i_2 is not in a strongly connected and closed group
3. there is a path from $i_2 \to i_3$
4. i_3 is not in a strongly connected and closed group

By induction we can keep going, but then the network isn't finite. Therefore, agent i_1 is either in a strongly connected and closed group or has a directed path to an agent in a strongly connected and closed group. And if the agent is either in a strongly connected and closed group or has a directed path into one, there must be at least one strongly connected and closed group.