Week 2: Bayesian Games - Framework and Application to Global Games

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In the first part of these notes we introduce the Bayesian games framework that we will use for the rest of the semester. We discuss the relation of this framework to the explicit rendering of "higher-order beliefs", that is, players' beliefs, beliefs about opponents' beliefs, beliefs about beliefs about beliefs...

In the second part we investigate the importance of "higher-order beliefs" in the setting of coordination with uncertainty. We start with a particular class of "global coordination games" of incomplete information where players' beliefs are highly, but not perfectly, correlated. These games are interesting for several reasons. First, they capture in simple form the idea that in strategic settings where actions are conditioned on beliefs, in particular settings where coordination is important, players need to be concerned with what their opponents believe, what their opponents believe about their beliefs, and so on. Second, global games can allow us to refine equilibria in coordination games in a very strong way. In some models, we can use global games analysis to show that even if common knowledge of payoffs gives rise to multiple equilibria, there will be a unique equilibrium if the players' information is perturbed in even a "small" way. A recent applied literature has arisen using these techniques particularly in finance and macroeconomics (Morris and Shin, 2002, is a nice survey).

We then use related ideas to look at the difference between public and private information in situations where higher order beliefs matter greatly.

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1 Bayesian Games and Bayesian Nash Equilibrium

Bayesian Equilibrium extends the Nash Equilibrium to situations when there is incomplete information about the situation, game. In contrast to the latter, it requires an explicit model of uncertainty - how agents think about the situation that they are facing. This significantly narrows the scope of its application to the groups of rational agents.

Fix a set of players I and a set of payoff relevant states of nature Θ ("fundamentals"). The basic model of uncertainty in game theory is a variant of the following:

Definition 1 *A finite type space consists of:*

1. *Finite sets of types T_i, $i \leq I$,*

2. *Probability distrubution assignments $\mu_i : T_i \to \Delta(\Theta \times_{j \leq I} T_j)$, such that $\mu_i(t_i)(t_i) = 1$.*

Θ is a set of parameters that might affect the payoffs in a game. Each type t_i encodes (at least) the beliefs about Θ, and, by unpacking deeper, the beliefs about beliefs of the opponents about Θ etc. We assume that each type is convinced and right about herself. Note that we have not mentioned any game or play yet - type space can be used to model uncertainty in other, nonstrategic frameworks.

There are several variants of this definition that you will be encountering in the literature.

1. Uncertainty can be equivalently represented by a kind of information structure considered in previous classes. We define a state space $\Omega = \Theta \times_{j \leq I} T_j$ and private priors p_i over Ω. Beliefs of each type t_i over $\Theta \times_{j \leq I} T_j$ are described by the prior p_i conditioned on t_i. For finite type spaces we can assume that each type has positive probability under own prior (set e.g. $p_i(\theta, t_i, t_{-i}) = 1/|T_i| * \mu_i(t_i)(\theta, t_{-i})$).

2. Often, and in the original definitions by Harsanyi (1968), it is assumed that the probability distributions of the agents are obtained from a common prior p over $\Theta \times_{j \leq I} T_j$, conditioned on own type. We know from the previous class that this assumption is not without loss of generality. Moreover, the interpretation of prior in this setting (with no account of e.g. ex-ante signal sending by mediator) is problematic. (Compare with the various definitions of Correlated Equilibrium.)

3. It is often assumed that types not only model beliefs of the players, but are themselves all the payoff relevant parameters, $\Theta = \times_{i \leq I} T_i$. In this case we can simplify the beliefs $\mu_i : T_i \to \Delta(\Theta \times_{j \leq I} T_j)$ to $\mu_i : T_i \to \Delta(\times_{j \leq I} T_j)$. This assumtion is without loss of generality for the analysis of Bayesian games to be defined (it takes a fair amount of manipulating conditional expectations, see Harsanyi 1968), but makes the interpretation less straightforward.

You can go through all the formalities of the definitions above in the following examples.

Example 1 *Opponent with unknown strenght. There are two agents, row and column. Row's type is fixed, but the column player can be either "strong", with probability α, or "weak", with probability $1 - \alpha$. All this is commonly known (or believed) by the players.*

Example 2 *Market for lemons. A firm A might be taking over another firm B. The true value of B is known to B but unknown to A. A believes that the value is uniformly distributed on $[0, 1]$. It is also known that B's value will flourish under A's ownership: it will rise from x to $1.5x$. All this is commonly known (or believed).*

In order to arrive at the Bayesian game, we must complement the description of uncertainty with the description of possible actions and payoffs.

Definition 2 *A finite Bayesian Game consists of:*

1. *A finite type space.*
2. *For each player a finite set of actions S_i.*
3. *For each player a payoff function $u_i : \Theta \times_{j \leq I} S_j \to \mathbb{R}$.*

In general all the spaces - Θ, T_i or S_i - or the functions - u_i or f_i - need to be only measurable (with some extra change if $\{(\theta, t_1, .., t_i) | t_i = t_i'\}$ is not measurable).

A (pure) strategy for player i in a Bayesian game specifies her play for any of her types, i.e. it is a (measurable) function $f_i : T_i \to S_i$.

Definition 3 *A Bayesian Nash equilibrium for a Bayesian game is a strategy profile $f = (f_1, ..., f_I)$ such that for every i and every t_i*

$$\int_{\Theta \times T_{-i}} u_i(\theta, f(t_1, .., t_I) d\mu_i (t_i) (\theta, t_{-i}) \geq \int_{\Theta \times T_{-i}} u_i(\theta, s_i, f_{-i}(t_{-i})) d\mu_i (t_i) (\theta, t_{-i}),$$

for every action $s_i \in S_i$.

If the type space is finite, we can assume wlog that the type space has beliefs derived from priors p_i over $\Theta \times_{j \le I} T_j$ such that each type has positive probability under own prior (see 1. above). In this case Bayesian Nash equilibrium for a Bayesian game is a Nash equilibrium in (Bayesian game) strategies, where payoffs are the ex-ante expected payoffs.

Example 3 (1, cont.) *Imagine that each agent chooses whether to Fight or Yield, and the payoffs are as in the matrices:*

$$\begin{array}{ccc} \textit{Strong:} & F & Y \\ F & -1,1 & 1,0 \\ Y & 0,1 & 0,0 \end{array} \qquad \begin{array}{ccc} \textit{Weak:} & F & Y \\ F & 1,-1 & 1,0 \\ Y & 0,1 & 0,0 \end{array} .$$

To solve for BNE note that if column is strong, then it is dominant for him to Fight. Consequently, row gets -1 if column is strong and 1 if column is weak, so he gets $-\alpha + (1 - \alpha)$ for F and 0 for Y. Therefore we have single BNE in case $\alpha \neq 1/2$ and continuum of equilibria for $\alpha = 1/2$.

Example 4 (2, cont.) *Now, imagine that actions for A are the bids $y \in \mathbb{R}$, and for B each action is a yes-no decision as a function of the bid (note that here we model a situation with sequential decisions via a strategic game). The strategies in Bayesian game are the functions from types to those actions.*

As for the payoffs, if the type $x \in [0, 1]$ of B accepts bid y, A gets $1.5x - y$, while B gets y. If x rejects then A gets 0, while B gets x. It follows that x will accept if $x < y$ and reject if $x > y$. Therefore the expected payoff to A for bid $y \in [0, 1]$ is $\int_{[0,y)} 1.5x dx - y = -1/4y$. It follows that in any BNE there is no trade, even though for every realization of the B's value there are positive gains from trade!

The last example is essentially the Nobel price winning model of trade by Akerlof (1971), which opened up peoples eyes on the information economics, and proved irrelevance of many General Equilibrium-type results.

1.1 Modelling Beliefs and Higher Order Beliefs

In this section, we take a short detour to discuss the modeling of beliefs and "higher-order" beliefs. A running theme in the coordination models we will look at is the importance not only of the players' beliefs about underlying fundamentals (i.e. about the payoff relevant states), but also about the beliefs of others. The information about the beliefs of others (so as to ascertain their actions) can easily be more important than beliefs about fundamentals.[1]

However, in most of the models of incomplete information that you will see the type spaces will typically be very small: It will usually consist of types corresponding to some basic parameters of payoff uncertainty, and the beliefs stemming from some (common or private) prior over the parameters conditioned on own type - "signal" (see points 1.-3. above). In this sense the type will model explicitly only "first-order" beliefs about fundamentals, and the "higher-order" beliefs will usually be very simple.

This approach is mostly dictated by convenience and tractability of applied models, yet it brings out a question whether the type space formalism limits the diversity of higher order beliefs. In other words, if we start with an explicit model specifying 1st order beliefs about fundamentals, 2nd order beliefs and so on, can we find a type space and a profile of types that correspond to those beliefs? Harsanyi already hinted, while Mertens and Zamir (1985) and Brandenburger and Dekel (1993) proved that this is in fact true. In doing so, they show that it is possible to define a "universal" type space that would allow for all possible hierarchies of belief. These papers are quite technical, particularly Mertens and Zamir's, but it is worth sketching the ideas. I'll follow Brandenburger and Dekel for the case of two agents, glossing over virtually all of the technical issues.

The starting point is a space of possible fundamentals Θ (e.g. payoffs). Each player will have a first-order belief about fundamentals, that is some $t_1 \in T_1 = \Delta(\Theta)$. Each player will then have a second-order belief over fundamentals and the other player's first order beliefs, that is some $t_2 \in$

[1] The higher-order beliefs are - as we have seen - also relevant in *complete information* games, in the sense that player i has to form a belief about player j's action, j has to form a belief about i's belief about j's action, and so on. Nash equilibrium "cuts through" this infinite regress of beliefs by assuming the first order beliefs are correct and player are rational with respect to these correct beliefs. Rationalizability, on the other hand, makes use of higher order beliefs to eliminate strategies that only iteratively dominated. With incomplete information, there is a sense in which beliefs about beliefs must be tackled head on — because to the extent that j has private information about i's payoff, i cares *directly* about j's belief, as well as about j's action.

$T_2 = \Delta(\Theta \times \Delta^{I-1}(\Theta))$. Formally, define the spaces:

$$\begin{aligned} X_0 &= \Theta \\ X_1 &= X_0 \times \Delta^{I-1}(X_0) \\ &\vdots \\ X_n &= X_{n-1} \times \Delta^{I-1}(X_{n-1}) \end{aligned}$$

The space of each player's nth order beliefs is then $T_n = \Delta(X_{n-1})$. A *type* for player i is a hierarchy of beliefs $t^i = (t_1^i, t_2^i, ...) \in \times_{n=1}^{\infty} T_n$. Let $T = \times_{n=1}^{\infty} T_n$ denote the space of all possible types for a given player.

Under this formulation, i knows his own type but not the type of his opponent j. So perhaps we will need to specify a belief for i about j's type, a belief for j about i's belief and so forth. Brandenburger and Dekel show that one additional assumption, however, pins down i's belief about j's type.

Definition 4 *A type $t = (t_1, t_2, ...) \in T$ is* ***coherent*** *if for every $n \geq 2$, $marg_{X_{n-2}} t_n = t_{n-1}$, where $marg_{X_{n-2}}$ denotes the marginal probability distribution on the space X_{n-2}.*

This simply says that i's beliefs do not contradict one another. This leads to the following result, for which you will need to know that a *homeomorphism* is a 1-1 map that is continuous and has a continuous inverse. For this result, let T' denote the set of all coherent types. (If you are into such results, the proof by Brandenburger and Dekel is based on the celebrated Kolmogorof's extension of measure theorem).

Proposition 1 *There is a homeomorphism $f : T' \to \Delta^{I-1}(\Theta \times T)$.*

This result says that i's hierarchy of beliefs (his "type") also determines i's belief about j's type. In fact, beliefs of each type are such as "described by" this type. "Iterating" this argument Brandenburger and Dekel go on to show that if there is common certainty of coherency (where i "is certain about" something if he assigns probability 1 to it), then i's type will determine not only his belief about j's type, but his belief about j's belief about his type, and so on. To state this formally, let T'' denote the set of types for which there is common certainty of coherency.

Proposition 2 *There is a homeomorphism $g : T'' \to \Delta^{I-1}(\Theta \times T'')$.*

In other words, there is a single *universal type space* such that for every belief hierarchy with cc of coherency there is a type in the type space with beliefs as described by the hierarchy.

1.2 Comments

1. As noted above, few applied economic models explicitly model higher order uncertainty. Rather, types are drawn from a small subset of the universal type sapce. There is, however, a recent literature that asks whether standard models that have only first-order uncertainty lead to predictions that are *robust* to perturbations of higher order beliefs (see, for instance, Rubinstein, 1989, or its generalization in Weinstein and Yildiz 2007). Indeed, this is one way to view the global games analysis — as questioning the robustness of coordination equilibria that are not risk-dominant — even though the global games model retains the standard modeling approach.

2. Bergemann and Morris (2005) pursue the robustness line of inquiry in the context of mechanism design. They ask whether when a mechanism that implements some outcome in a standard model with first-order uncertainty will implement the same outcome if types are drawn from the universal type space constructed above. Their answer is that for a mechanism to be robust in this sense, it must implement the desired outcome as an *ex post* equilibrium (in which you do not want to deviate from your action even when you learn the types of the opponents), which in the case of private values is equivalent to *dominant strategy implementation.*

3. Recall our discussion of the models to represent knowledge and the role of A1-A5 axioms. We mentioned that for every set of knowledge formulas that are consistent with the axioms one can find a model and a state in which all those formulas hold. The universal type space result can be seen as its brother: given a set of formulas (that can be nested - hence hierarchies), which directly specify agents' probabilistic beliefs and satisfy assumption of cc of coherency, we can find a profile of types in a type space where all the formulas hold. Another question is whether one can characterize BNE over type spaces via Bayesian rationality conditions on belief hierarchies (Compare Proposition 7 in previous notes) - for further details see e.g. Forges 1994 , Liu 2006, Brandenburger, Dekel and Morris 2007, Sadzik 2007.

2 Global Games

We work with an example drawn from Morris and Shin (2002) and based on Carlsson and van Damme (1993). There are two players $i = 1, 2$ who choose one of two actions, "Invest" or "Not Invest". The payoffs are as follows:

	Invest	Not Invest
Invest	θ, θ	$\theta - 1, 0$
Not Invest	$0, \theta - 1$	$0, 0$

Not Invest is a safe action that yields zero, while Invest yields $\theta - 1$ if the opponent doesn't invest and θ if she does. If θ is known to the players, there are three possibilities:

- $\theta > 1$, Invest is a dominant strategy
- $\theta \in [0, 1]$, (Invest, Invest) and (NI,NI) are both NE.
- $\theta < 0$, Not invest is a dominant strategy

Suppose we introduce incomplete information by assuming that each player does not observe θ, but rather a private signal $x_i = \theta + \varepsilon_i$, where $\varepsilon_i \sim N(0, \sigma^2)$, each ε_i is independent. Suppose also that θ is drawn from a uniform distribution over the entire real line (i.e. players have an uninformative or improper prior on θ).

Proposition 3 *In the incomplete information game, there is a unique equilibrium in which both players invest if and only if* $x_i > x^* = \frac{1}{2}$.

Proof. Let's first verify that the stated equilibrium is in fact an equilibrium. First, observe that if player j uses the stated strategy, then i's payoff from investing conditional on having signal x_i is

$$\mathbb{E}\left[\theta \mid x_i\right] - \Pr\left[x_j \leq \frac{1}{2} \mid x_i\right] = x_i - \Phi\left(\frac{\frac{1}{2} - x_i}{\sqrt{2}\sigma}\right),$$

where we use the fact that $x_j | x_i \sim N(x_i, 2\sigma^2)$. This payoff increasing in x_i, and equal to zero when $x_i = 1/2$. So i's best response is to invest if and only if $x_i \geq 1/2$, verifying the equilibrium.

To prove uniqueness, we'll show that the stated profile is the only one that survives iterated deletion of dominated strategies. As a preliminary, consider i's payoff from investing conditional on having signal x_i and conditional on j playing the strategy "invest if and only if $x_j \geq k$" for some "switching point" k. The payoff is

$$\mathbb{E}\left[\theta \mid x_i\right] - \Pr\left[x_j \leq k \mid x_i\right] = x_i - \Phi\left(\frac{k - x_i}{\sqrt{2}\sigma}\right),$$

which is again increasing in x_i, and also decreasing in k. Let $b(k)$ be the unique value of x_i at which the payoff to investing is zero. Observe that $b(0) > 0$, $b(1) < 1$, that $b(\cdot)$ is strictly increasing in k, and that there is a unique value k^* that solves $b(k) = k$, namely $k^* = 1/2$.

We now show that if a strategy s_i survives n rounds of iterated deletion of strictly dominated strategies, then:

$$s_i(x) = \begin{cases} \text{Invest} & \text{if } x_i > b^{n-1}(1) \\ \text{Not Invest} & \text{if } x_i \leq b^{n-1}(0) \end{cases}.$$

Note that the value of $s_i(x)$ for values of x between $b^{n-1}(0)$ and $b^{n-1}(1)$ is not pinned down here.

This claim follows from an induction argument. Suppose $n = 1$. The worst case for i investing is that j never invests (i.e. uses a switching strategy with $k = \infty$). If j never invests, then i should invest if and only if $\mathbb{E}[\theta|x_i] > 1$, or in other words if $x_i \geq 1$ (recall that $\mathbb{E}[\theta|x_i] = x_i$). This means that Not Invest is dominated by Invest if and only if $x_i > 1$. Conversely, Invest is dominated by Not Invest if and only if $x_i < 0$. This verifies the claim for $n = 1$.

Now suppose the claim is true for n. Now the worst case for i investing is that j invests only if $x_j > b^{n-1}(1)$, i.e. plays a switching strategy with cut-off $k = b^{n-1}(1)$. In this case, i should invest if and only if $x_i > b(b^{n-1}(1)) = b^n(1)$, meaning that Not Invest is in fact dominated if $x_i > b^n(1)$. Conversely, Invest is dominated by Not Invest at this round if and only if $x_i < b(b^{n-1}(0)) = b^n(0)$. So we have proved the claim.

To complete the result, we observe that:

$$\lim_{n\to\infty} b^n(0) = \lim_{n\to\infty} b^n(1) = \frac{1}{2}.$$

Therefore iterated deletion of dominated strategies yields a unique profile in which both players invest if and only if their respective signal exceeds one-half. *Q.E.D.*

2.1 Generalizing the Model

The logic of this example can be generalized without much trouble to other two-action two-player supermodular games. In particular, suppose we have two players, each of who chooses an action $a \in \{0, 1\}$. For simplicity, they have symmetric payoffs $u : \{0, 1\} \times \{0, 1\} \times \mathbb{R} \to \mathbb{R}$, where $u(a_i, a_j, x_i)$ is i's payoff from choosing a_i, given that j chooses a_j, and that i's private signal is x_i. Signals are generated in the following way. First, nature selects a state $\theta \in \mathbb{R}$ drawn from the (improper) uniform distribution on $\mathbb{R}$. Player i then observes a signal $x_i = \theta + \sigma\varepsilon_i$, where ε_i has a continuous density $f(\cdot)$ with support $\mathbb{R}$. Call this game $G(\sigma)$.

Define the incremental returns to choosing $a = 1$ as:

$$\Delta(a_j, x) = u(1, a_j, x) - u(0, a_j, x).$$

We impose the following assumptions.

1. **Action Monotonicity**. Δ is increasing in a_j.

2. **State Monotonicity**. Δ is strictly increasing in x.

3. **Continuity.** Δ is continuous in x.

4. **Limit Dominance**. There exist $\underline{\theta}, \overline{\theta} \in \mathbb{R}$ such that $\Delta(a_j, x) < 0$ whenever $x < \underline{\theta}$, and $\Delta(a_j, x) > 0$ whenever $x \geq \overline{\theta}$.

Proposition 4 *Under (A1)–(A4), the essentially unique strategy profile that survives iterated deletion of strictly dominated strategies in $G(\sigma)$ satisfies $s(x) = 0$ for all $x < \theta^*$ and $s(x) = 1$ for all $x > \theta^*$, where θ^* is the unique solution to:*

$$(\Delta(1, \theta^*) + \Delta(0, \theta^*))/2 = 0.$$

Proof. Nearly identical to the one above. *Q.E.D.*

In the unique equilibrium, each player uses a cut-off strategy (we say essentially unique because the strategy is indeterminate at $x = \theta^*$. Moreover, the cut-off θ^* is such that if i receives the signal θ^*, and believes that j is equally likely to play 0 or 1, then i will be just indifferent between playing 0 and 1.

Note one difference between this game and the first one is that this has "private values" while the other game had "common values". This distinction is not so important, however. Suppose payoffs in the more general case were $u(a, \theta)$, rather than $u(a, x)$. We could simply define $\Delta(a_j, x)$ as the *expected* return to playing 1 rather than 0 after observing a signal x, given that one's opponent was playing a_j. When j is using a cut-off strategy, Δ satisfies all the same properties, so everything still goes through.

2.2 Discussion

We now discuss several features and extensions of the model.

1. (Inefficiency of the Unique Equilibrium). If we consider a sequence of incomplete information games with $\sigma \to 0$, then, since in equilibrium each player invests if and only if $x = \theta + \varepsilon \geq \frac{1}{2}$, in the limit players coordinate on (Invest,Invest) whenever $\theta > \frac{1}{2}$, and on (Not Invest,Not Invest) whenever $\theta \leq \frac{1}{2}$. Coordination on $(Invest, Invest)$, however, is *efficient* whenever $\theta > 0$. So the fact that the players act in a decentralized fashion means that they generally won't coordinate efficiently.

2. (Risk Dominance) In 2×2 symmetric games, an action is *risk-dominant* if it is a best-response given that one's opponent is mixing uniformly. In the underlying symmetric information game, investing is risk-dominant if $\theta \geq \frac{1}{2}$ and not investing is risk-dominant if $\theta \leq \frac{1}{2}$. Hence as $\sigma \to 0$, and we converge to the complete information game, the players play the risk dominant action.

3. (Generalizations) It is quite easy to duplicate the above analysis to more general 2×2 games with strategic complementarities, provided some technical conditions are satisfied (see Morris and Shin, 2002). Frankel, Morris and Pauzner (2002) extend the above result to asymmetric n-player many action games with strategic complementarities. They provide conditions under which, if there is only a small amount of noise, equilibrium will be unique. The selected equilibrium, however, may depend on the fine structure of the noise. Interestingly, however, if a game has a unique robust equilibrium, this equilibrium will be selected regardless of the noise structure. Frankel and Pauzner (2000) and Levin (2000) use global game arguments to identify unique equilibria in dynamic games with strategic complementarities. One such problem is on your homework.

4. (Equilibrium Refinements) More generally, note that if $0 < \theta < \frac{1}{2}$, (Invest, Invest) is an equilibrium of the complete information game, but not of the closely related incomplete information game (with σ small but positive). This can be related to the general problem of selecting more or less plausible equilibria in a given game — or *refining* the set of equilibria. A common approach in this regard is to consider a family of "nearby" games and ask if these games have equilibria that

are “close” to a given equilibrium of the original game. Kajii and Morris (1997) say that an ex-ante equilibrium distribution of actions of a given game is *robust* if it is an ex-ante equilibrium distributions of all "nearby" games of incomplete information - "nearby" roughly means that each agent is almost sure that the payoffs are as in complete information game. Some games have no robust equilibria, but Kajii and Morris show that some interesting classes of games do have robust equilibria. In contrast to that, Weinstein and Yildiz (2007) show that for any profile of types and any action profile $(a_1, ...a_I)$ one can find a sequence of "nearby" types for which $(a_1, ...a_I)$ is the unique rationalizable outcome - "nearby" meaning that the types agree on their k-level beliefs, for high k.

5. (Generalizations No 2) I do not know of any generalization of the "global games information structures" for the general class of games. It is worth spending some time to appreciate the problem: In the case when payoffs of *all* players are described by a single parameter (θ above), global games do a good job in descring the situation where "I believe payoffs are within small range around θ, I believe that others believe that payoffs are within small range around θ', with θ' close to θ, etc". How to generalize this intuition sensibly to all games? Incidentally, the two refinements mentioned in point 4 are based on the notion of "closeness" of beliefs that requires putting high probability on right beliefs (or probability one upto kth level), but still allow putting small probability on payoffs that are very off (your beliefs above kth level can be very off). Is it reasonable?

6. (Crises) One of the two main theories of economic crises (e.g. Obstfeld 1996) associates them with the multiplicity of equilibria, where large shifts in behavior correspond to a shift between two equilibria due to "market sentiments" or "animal spirits". The other (e.g. Morris and Shin) explains the shifts by the hyper-sensitivity of (often unique) equilibrium to parameter changes. For example, look at our model with very small σ, i.e. very precise private information. The observable equilibrium behavior will change drastically with small changes of θ around 1/2.

7. (Learning) The model predictions become less clear - often losing the uniqueness of equilibrium property - when we allow agents to play the game repeatedly and learn over time (Angeletos, Hellwig, Pavan 2007). First, they have an option to postpone their "investment" until

later, and second, they can learn about θ from the behavior of the opponents. For example, consider a model in which there is not 2, but continuum of players, to profitably coordinate at least θ_1 agents must "invest", and you observe only your payoffs. If in equilibrium in the first round people "invest" when their private signal, $\theta + \varepsilon_i$, is above $\overline{x}_1$, then if the first "investment" is not profitable, in the second round it becomes *common knowledge* between the players that θ is greater than $\overline{x}_1$ (recall the problems in the static case when there is common knowledge of θ).

8. (Applications) There has been a lot of interest in the application of global games to macroeconomics, in particular currency crises — the view being that currency pegs tend to fall when there is an attack by many investors, a situation that naturally gives rise to a coordination game. A twist in such settings is that the ability to observe prices may restore common knowledge (one analysis of this is by Angeletos and Werning, 2005).

3 Public and Private Information

In the coordination game above, players care about fundamentals (the value of θ) and also about the actions of their fellow player (the value of a_{-i}). Each player's signal is informative about both variables of interest. Of course, in the context of the above example, information is private. An interesting question that arises in a coordination setting concerns the role of *public* information. Intuitively, public information about fundamentals should be valuable in a coordination setting because (barring unfortunate coordination failure) it will allow for coordination on the more appropriate action. Morris and Shin (2003), however, show that this intuition fails in settings where players have private information as well as public information. The basic idea, as we will see shortly, is that players are "too sensitive" to the public information because even if it is not that informative about fundamentals, it tends to be quite informative about other players' beliefs and hence about their actions.

3.1 The "Beauty Contest" Model

Morris and Shin's model is based on a famous "beauty contest" parable in Keynes' *General Theory.*[2] There are a continuum of agents, indexed by $i \in [0, 1]$. Agent i chooses a_i, and receives a payoff:

$$u_i(a_i, a_{-i}, \theta) = -(1-r)(a_i - \theta)^2 - r(L_i - \overline{L}),$$

where $0 < r < 1$ and:

$$L_i = \int_0^1 (a_j - a_i)^2 dj \qquad \text{and} \qquad \overline{L} = \int_0^1 L_j dj \ .$$

Agent i wants to minimize the distance between his action and the true state θ and also minimize the distance between his action and the actions of the other agents. The parameter r weights these two parts of the objective function.

The beauty contest part of the game has a zero-sum flavor. If we define social welfare as the average of individual utilities:

$$W(a, \theta) = \int_0^1 u_i(a, \theta) di = -(1-r) \int_0^1 (a_i - \theta)^2 di.$$

From a social point of view, what matters is how close the individual actions are to θ, not to each other.

Each agent will maximize his utility by choosing a_i to minimize his expected loss:

$$a_i = (1-r)\mathbb{E}_i[\theta] + r\mathbb{E}_i[\overline{a}],$$

where $\overline{a} = \int_0^1 a_j dj$ is the population average action, and $\mathbb{E}_i$ is the expectation operator conditioning on i's information.

3.2 Public Information Benchmark

As a benchmark, consider the case where the agent's only have public information. Suppose everyone shares an improper uniform prior on θ and then observes a public signal:

$$y = \theta + \eta,$$

[2]Keynes drew an analogy between the stock market and a certain beauty contest in a London newspaper. The paper printed pictures of young women. Keynes said that making money in the stock market was like trying to pick the girl who the most people would vote for as most beatiful — what mattered was not the true beauty of the girls, but whether or not people would vote for them.

where $\eta \sim N(0, \sigma_\eta^2)$. Then $\mathbb{E}_i[\theta|y] = y$ by Bayesian updating, and the unique equilibrium has each agent choose:

$$a_i(y) = y.$$

In this equilibrium, the expected welfare is:

$$\mathbb{E}(W|\theta) = -(1-r)\mathbb{E}[(y-\theta)^2|\theta] = -(1-r)\sigma_\eta^2.$$

So clearly better public information unambiguously improves social welfare.

3.3 Public and Private Information

In contrast, now suppose that in addition to y, each agent observes a *private* signal:

$$x_i = \theta + \varepsilon_i,$$

where $\varepsilon_i \sim N(0, \sigma_\varepsilon^2)$, and moreover, ε_i and ε_j are independent for all $i \neq j$. In this set-up, agent i's information is the pair (x_i, y). She needs to use this information to forecast both the true state θ and the average action in the population.

By the wonderful properties of Bayes updating with normal random variables, we have:

$$\mathbb{E}_i[\theta|x_i, y] = \frac{h_\eta y + h_\varepsilon x_i}{h_\eta + h_\varepsilon},$$

where $h_\varepsilon = 1/\sigma_\varepsilon^2$ and $h_\eta = 1/\sigma_\eta^2$ are the precisions of ε and η.

We look for a linear equilibrium of the form:

$$a_i(x_i, y) = \kappa x_i + (1-\kappa)y.$$

If the equilibrium has this linear form, then:

$$\begin{aligned} \mathbb{E}_i[\overline{a}|x_i, y] &= \kappa \left(\frac{h_\eta y + h_\varepsilon x_i}{h_\eta + h_\varepsilon} \right) + (1-\kappa)y \\ &= \left(\frac{\kappa h_\varepsilon}{h_\eta + h_\varepsilon} \right) x_i + \left(1 - \frac{\kappa h_\varepsilon}{h_\eta + h_\varepsilon} \right) y. \end{aligned}$$

Agent i's optimal action is:

$$\begin{aligned} a_i(x_i, y) &= (1-r)\mathbb{E}_i[\theta] + r\mathbb{E}_i[\overline{a}] \\ &= \left(\frac{h_\varepsilon(r\kappa + 1 - r)}{h_\eta + h_\varepsilon} \right) x_i + \left(1 - \frac{h_\varepsilon(r\kappa + 1 - r)}{h_\eta + h_\varepsilon} \right) y, \end{aligned}$$

meaning we have a linear equilibrium with:

$$\kappa = \frac{h_\varepsilon(1-r)}{h_\eta + h_\varepsilon(1-r)}.$$

In this equilibrium, agent i's action is:

$$a_i(x_i, y) = \frac{h_\eta y + h_\varepsilon(1-r)x_i}{h_\eta + h_\varepsilon(1-r)}.$$

Morris and Shin (2003) thought that they verify that this is the unique equilibrium in their model. You can consult their paper for what they describe as a "brute force" proof. However, the proof turned out to have some gaps, and an alternative proof was given by Angeletos and Pavan (2007b).

3.4 Discussion and Welfare Properties

The key point to notice about the equilibrium behavior in the beauty contest model is that agents actions *over-weight* public information relative to its informativeness about economic fundamentals. In particular, both $\mathbb{E}_i[\theta|x_i, y]$ and $a_i(x_i, y)$ are linear combinations of x_i and y. In forming her expectation of θ, agent i puts a weight $h_\eta/(h_\eta + h_\varepsilon)$ on y. But in choosing her action, agent i puts a weight $h_\eta/(h_\eta + (1-r)h_\varepsilon)$ on y. Why the larger weight? Because even if x_i and y were to be equally informative about θ ($h_\eta = h_\varepsilon$), the public signal y would be *more informative* about other player's beliefs, and hence about their actions. An early version of Morris and Shin's paper referred to this as the "publicity multiplier".

Because agent's are forecasting other agents' actions — and hence other agents' beliefs and beliefs about beliefs and so on — public information is given disproportionate weight relative to its true informativeness about fundamentals. This can give rise to surprising welfare effects.

In particular, suppose we re-write the equilibrium strategy of each agent i as:

$$a_i = \theta + \frac{h_\eta \eta + h_\varepsilon(1-r)\varepsilon_i}{h_\eta + h_\varepsilon(1-r)}.$$

Then expected welfare is given by:$\mathbb{E}[W|\theta] = -(1-r)\frac{h_\eta + h_\varepsilon(1-r)^2}{(h_\eta + h_\varepsilon(1-r))^2}$.

An increase in h_ε, the informativeness of the private signals, has an unambiguously positive effect on social welfare. On the other hand, an increase in h_η, the informativeness of the public signal, has an effect:

$$\frac{\partial \mathbb{E}[W|\theta]}{\partial h_\eta} \overset{sign}{=} h_\eta - (2r-1)(1-r)h_\varepsilon.$$

Better public information is always beneficial is $r < 1/2$, so that the "beauty contest" incentive is relatively small. If the beauty contest component of payoffs is large, however, so that $r > 1/2$ and there is a significant element of coordination involved in the equilibrium, better public information improves welfare only if the public information is reasonably good relative to the quality of private information — if h_η is small relative to h_ε, an increase in h_η will cause the agents' too substitute toward y in choosing their actions and increase the variance in the population action around θ.

4 Information acquisition

Hellwig and Veldkamp (2007) ask the following question in the context of global games: suppose that prior to playing a "global game" agents invest in the precision of their information. Given that the subsequent "global game" exhibits complementarities/substitubility in actions, will the initial information acquisition game exhibit complementarity/substitubility? (For example, complementarity would mean that the benefits of my acquiring information grow with opponent acquiring more information.) We will look at the answer they provide in a version of the beauty contest model by Morris and Shin.

Just as above, there are a continuum of agents, indexed by $i \in [0, 1]$. The payoffs are slightly different: agent i chooses a_i, and receives a payoff:

$$u_i(a_i, a_{-i}, \theta) = -\frac{1}{(1-r)^2}(a_i - a^*)^2, \text{ where } a^* = (1-r)\,\theta + r\int_0^1 a_j dj,$$

The best response to the actions of the opponents is:

$$a_i = (1-r)\mathbb{E}_i[\theta] + r\mathbb{E}_i[\overline{a}] \tag{1}$$

. For example, if θ is common knowledge, one can verify that the unique equilibrium is for everybody to choose $\alpha_i = \theta$. The coefficient $r < 1$ measures complementarity/substitubility of actions: If $r > 0$, decisions are complementary, own action is increasing in the action of others. If $r < 0$, actions are substitutes.

We are, of course, interested in the case when θ is not commonly known. We will look at the following simple special case of Hellwig and Veldkamp's model. Suppose that each agent i can potentially observe a signal:

$$x_i = \theta + \eta + \varepsilon_i,$$

where θ is distributed $N(0,1)$, η is a public noise distributed $N\left(0,\sigma_\eta^2\right)$, and ε_i is a private noise distributed $N\left(0,\sigma_\varepsilon^2\right)$. Private noises are pairwise independent.

The game unfolds as follows. In the first stage each agent decides whether to observe x_i or not. More specifically, he can "buy" the probability $p_i \in [0,1]$ of observing x_i at a cost $c(p_i)$ for some convex cost function. At the second stage, given the information I_i that the agent has (either no information whatsoever, or the value of x_i) agent chooses his action $a_i(I_i)$.

4.1 Second Stage - actions.

Lets start by analyzing the stage when the agents have made already their choice regarding the information acquisition. The relevant state variable from agent's point of view is $\omega = [\theta,\eta]$, since he is trying to predict the true state and average action. If agent ends up not observing any x_i (denote it $I_i = \emptyset$), or he observes x_i then the conditional expectation of ω are, respectively:

$$\begin{aligned} E\left([\theta,\eta]\,|\emptyset\right) &= [0,0]\,, \\ E\left([\theta,\eta]\,|x_i\right) &= \left[\frac{x_i}{\sigma_\varepsilon^2}, \frac{\sigma_\eta x_i}{\sigma_\varepsilon^2}\right]. \end{aligned}$$

Also, we can write out the $2x2$ posterior covariance matrices $\Sigma\left([\theta,\eta]\,|I_i\right)$. Their exact values need not concern us here (check the paper). What is relevant is that $\Sigma\left([\theta,\eta]\,|x_i\right) = \Sigma\left([\theta,\eta]\,|x_i'\right)$ for any x_i, x_i', $\Sigma\left([\theta,\eta]\,|\emptyset\right)$ is a 2×2 identity matrix $\mathbf{1}$, ∘and the difference between $\Sigma\left([\theta,\eta]\,|\emptyset\right)$ and $\Sigma\left([\theta,\eta]\,|x_i\right)$ is a positive definite matrix. Consequently, if the fraction μ of the agents ends up buing the signal, then the average posterior covariance matrix is:

$$\Sigma_\mu\left([\theta,\eta]\right) = \mu\Sigma\left([\theta,\eta]\,|x_i\right) + (1-\mu)\,\Sigma\left([\theta,\eta]\,|\emptyset\right),$$

and so if $\mu' > \mu$ then $\Sigma_{\mu'}\left([\theta,\eta]'\right) - \Sigma_\mu\left([\theta,\eta]'\right)$ is positive definite.

Now, the equilibrium must feature best responses as in (**??**), but adapted to the new info setting, i.e.:

$$\begin{aligned} a_i\left(I_i\right) &= (1-r)E_i[\theta|I_i] + rE_i[\overline{a}|I_i] = \\ &= \int_\omega \left\{(1-r)\,\theta + r\int_{I'} a_j\left(I'\right)dP\left(I'|\omega\right)\right\} dP\left(\omega|I_i\right). \end{aligned}$$

If we let $\overline{E}\,[\cdot]$ be the average of the expectations across all the agents (and $\overline{E}^0\,[[\theta,\eta]] = [\theta,\eta]$, $\overline{E}^{n+1}\,[\cdot] = \overline{E}\left[\overline{E}^n\,[\cdot]\right]$), then the solution to the above equation can be alternatively expressed as:

$$a_i(I_i) = (1-r)[1 \ \ 0] \sum_{k=0}^{\infty} r^k E\left[\overline{E}^k\left[[\theta,\eta]'\right]|I_i\right].$$

Now, a bit of the algebra to the terms above gives us ($\mathbf{1}$ is a 2×2 identity matrix):

$$\begin{aligned}\overline{E}\left[[\theta,\eta]\right] &= \left[\mathbf{1}-\Sigma_\mu\left([\theta,\eta]\right)\right][\theta,\eta]',\\ \overline{E}^k\left[[\theta,\eta]'\right] &= \left[\mathbf{1}-\Sigma_\mu\left([\theta,\eta]\right)\right]^k[\theta,\eta]',\end{aligned}$$

and so, taking the sum of a geometric sequence:

$$a_i(I_i) = (1-r)[1 \ \ 0]\left[(1-r)\mathbf{1}+r\Sigma_\mu\left([\theta,\eta]'\right)\right]^{-1}E\left[[\theta,\eta]'|I_i\right].$$

As the last step, we can use the above formula to compute the expected utility from buying the probability $p\in[0,1]$ of the signal, given that the fraction μ of the agents ends up buing the signal:

$$EU(p,\mu) = -[1 \ \ 0]\left[(1-r)\mathbf{1}+r\Sigma_\mu\left([\theta,\eta]'\right)\right]^{-1}\Sigma_p\left([\theta,\eta]'\right)\left[(1-r)\mathbf{1}+r\Sigma_\mu\left([\theta,\eta]'\right)\right]^{-1}[1 \ \ 0]'.$$

4.2 First stage - information

This is really the last touch to the analysis. In the last equation above we have the explicit expression for expected utility. We also know that if $\mu'>\mu$ then $\Sigma_{\mu'}\left([\theta,\eta]'\right)-\Sigma_\mu\left([\theta,\eta]'\right)$ is positive definite. Putting those facts together, plus some bit of algebra, we get the main proposition:

Proposition 5 *Consider $\mu,\mu',p,p'\in[0,1]$ with $\mu'>\mu$, $p'>p$.*

(i) If there is no strategic interaction ($r=0$) then value of additional information is independent of other's informational choices:

$$EU(p',\mu')-EU(p,\mu') = EU(p',\mu)-EU(p,\mu).$$

(ii) If decisions are complementary ($r>0$) then there is complementarity in information acquisition:

$$EU(p',\mu')-EU(p,\mu') > EU(p',\mu)-EU(p,\mu).$$

(iii) If decisions are substitutes ($r<0$) then there is substitutability in information acquisition:

$$EU(p',\mu')-EU(p,\mu') < EU(p',\mu)-EU(p,\mu).$$

After the tedious algebra, few words of intuition. If the actions are complementary, high investment in information by the others makes decisions covary more with the fundamentals, and so the own investment in information is more profitable if and only if the decisions are complementary.

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