201C HW1

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1 Question 1

a) $w_0 = 0$ because we never want to pay more than the outside option. For w_1, we want the agent to be just indifferent to $q \in \{L, H\}$. Let's find the wage offer that makes the expected utility exactly the same.

$$\mathbb{E}[u \mid a = L] = 0$$
$$\mathbb{E}[u \mid a = H] = pu(w_1) + (1-p)o - c(H)$$
$$0 = pu(w_1) - c(H)$$
$$\frac{c(H)}{p} = u(w_1)$$
$$\implies w_1 = u^{-1}\left(\frac{c(H)}{p}\right)$$

This makes sense. The wage for high output needs to compensate the agent for his effort, but since high effort doesn't always produce high output and we don't directly observe effort, we need to overweight the compensation for high output to account for times when the agent exerts high effort but doesn't produce high output.

b) We can break this problem into two steps. First, we can figure out how much higher w_H must be relative to w_L to make the agent indifferent between the two effort levels. Then, we can set absolute levels of w_H and w_L so that the IR constraint is just satisfied and the agent is indifferent between accepting the contract and taking the outside option. Assuming we fix a NE where the other player is playing H, then

$$\mathbb{P}(\text{win} \mid H, H)u(w_H) + \mathbb{P}(\text{lose} \mid H, H)u(w_L) - c \geq \mathbb{P}(\text{win} \mid H, L)u(w_H) + (1 - \mathbb{P}(\text{win} \mid H, L))u(w_L)$$
$$\text{(IC)}$$

$$\implies \frac{u(w_H)}{2} + \frac{u(w_L)}{2} - c = \frac{1-p}{2}u(w_H) + \frac{1-p}{2}u(w_L)$$
$$= 0$$
$$\mathbb{P}(\text{win} \mid H, H)u(w_H) + \mathbb{P}(\text{lose} \mid H, H)u(w_L) \geq 0 \qquad \text{(IR)}$$
$$\implies \frac{u(w_H)}{2} + \frac{u(w_L)}{2} = c$$
$$\frac{u(w_H)}{2} + \frac{u(w_L)}{2} - c = 0 \qquad \text{(IR 1)}$$
$$\implies u(w_L) = 2c - u(w_H)$$
$$(1-p)u(w_H) = (-1-p)u(w_L) \qquad \text{(IR 2)}$$
$$\implies u(w_H) = \frac{c(1+p)}{p}$$
$$w_H = u^{-1}\left(\frac{c(1+p)}{p}\right)$$
$$u(w_L) = 2c - \frac{c(1+p)}{p}$$
$$\implies w_L = u^{-1}\left(\frac{c(p-1)}{p}\right)$$

Both agents are playing H, so the output is the same in part (a) and part (b). To show that profit is lower in part (b), we just need to show that the wages paid to the agents are higher in part (b). Since the agents have the same outside option in both part (a) and part (b), the question is how much, in dollar terms, the principal must pay to set the expected utility of accepting a contract equal to zero. This means that both of the contracts must have the same expected value for the agent, in utility terms.

In part (a) the only relevant term is w_H because the agent always receives w_H, so the distribution is degenerate. In part (b), the agent receives w_L half the time and w_H half the time, so the contract in part (b) is a mean-preserving spread of the contract in part (a), in utility terms. But the agent is risk averse, so the principal must increase the high wage more, in dollar terms, than he decreases the low wage, so the expected dollar value of the contract in part (b) is higher than the dollar value of the contract in part (a), and the principal makes lower profits.

c)

$$\mathbb{P}(w_H \mid H)u(w_H) + \mathbb{P}(w_L \mid L)u(w_L) - c \geq \mathbb{P}(w_H \mid L)u(w_H) + \mathbb{P}(w_L \mid L)u(w_L)$$
$$\implies (1-p)^2 u(w_H) + (1-(1-p)^2)u(w_L) - c \geq pu(w_H) + (1-p)u(w_L)$$

At the optimal w_L, w_H this holds with equality,

$$\implies (1-p)^2 u(w_H) + (1-(1-p)^2)u(w_L) - c = pu(w_H) + (1-p)u(w_L)$$
$$(1-p)^2 u(w_H) + (1-(1-p)^2)u(w_L) - c = 0 \tag{IR 1}$$
$$\implies u(w_H) = \frac{c - (1-(1-p)^2)u(w_L)}{(1-p)^2}$$
$$pu(w_H) + (1-p)u(w_L) = 0 \tag{IR 2}$$
$$\implies u(w_H) = u(w_L) - \frac{1}{p}u(w_L)$$
$$u(w_L) - \frac{1}{p}u(w_L) = \frac{c - (1-(1-p)^2)u(w_L)}{(1-p)^2}$$
$$(1-p)^2 u(w_L) - \frac{(1-p)^2}{p}u(w_L) = c - (1-(1-p)^2)u(w_L)$$
$$(1 - 2p + p^2)u(w_L) - \frac{1 - 2p + p^2}{p}u(w_L) = c - 2pu(w_L) + p^2 u(w_L)$$
$$u(w_L) - \frac{1}{p}u(w_L) + 2u(w_L) - pu(w_L) = c$$
$$u(w_L) = \frac{c}{3 - \frac{1}{p} - p}$$
$$w_L = u^{-1}\left(\frac{c}{3 - \frac{1}{p} - p}\right)$$
$$u(w_H) = u(w_L) - \frac{1}{p}u(w_L)$$
$$= \frac{c}{3 - \frac{1}{p} - p} - \frac{1}{p}\left(\frac{c}{3 - \frac{1}{p} - p}\right)$$
$$w_H = u^{-1}\left((1 - \frac{1}{p})\left(\frac{c}{3 - \frac{1}{p} - p}\right)\right)$$

d) Regardless of whether the NE is $\{H, H\}$ or $\{L, L\}$, each agent will win half of the time.

$$\mathbb{E}[u_i \mid H, H] = u\left(\frac{1}{2}w_H\right) - u\left(\frac{1}{2}w_L\right) - c(H)$$
$$\mathbb{E}[u_i \mid L, L] = u\left(\frac{1}{2}w_H\right) - u\left(\frac{1}{2}w_L\right) - 0$$

The payoffs are the same in both NE, but the costs are higher for $\{H, H\}$, so both players prefer L, L to H, H. To show that $\{L, L\}$ is a SPNE, take $\delta = 1$ and assume that both players play Grim trigger

strategies; each player plays L as long as the other play plays L, but switches to H if the other player switches. All subgames are identical since the game is infinite.

$$\mathbb{E}[u \mid a_i = \{L\}_{t=1}^{\infty}] = u\left(\sum_{t=1}^{\infty} \frac{1}{2}w_H + \frac{1}{2}w_L\right)$$

Let's check a deviation to $a_i^t = H$ for some $t' < \infty$. Since both players are playing Grim trigger, this means that they'll play H, H for the rest of the game

$$\mathbb{E}[u \mid a_i = \{H\}_{t=t'}^{\infty}] = \underbrace{u\left(pw_h + \frac{1}{2}(1-p)w_L + \frac{1}{2}(1-p)w_H\right)}_{\text{this period}} + u\left(\sum_{t=t'}^{\infty} \frac{1}{2}w_h + \frac{1}{2}w_L\right) - \sum_{t=t'}^{\infty} c(H)$$

Let's check if this is a profitable deviation with $\delta = 1$. Agent i does not want to deviate to H, H if and only if

$$u\left(\sum_{t=t'}^{\infty} \frac{1}{2}w_H + \frac{1}{2}w_L\right) > u\left(\sum_{t=t'}^{\infty} \frac{1}{2}w_H + \frac{1}{2}w_L\right) + u\left(pw_H + \frac{1}{2}(1-p)w_L + \frac{1}{2}(1-p)w_H\right) - \sum_{t=t'}^{\infty} c(H)$$

$$\sum_{t=t'}^{\infty} c(H) > u\left(pw_H + \frac{1}{2}(1-p)w_L + \frac{1}{2}(1-p)w_H\right)$$

The undiscounted cost of high effort, the term on the left, is infinite, while the one-period benefit of deviating, the term on the right, is finite. Therefore L, L is a SPNE for each subgame.

e) TODO

4

2 Question 2

a) The competitive market means that all firms are making the same offers to the worker, so she never wants to quit and take an identical offer at another firm. On the other hand, since there's a competitive market no firm can make positive profits, since if one firm is making positive profits then another firm can offer slightly more compensation and take the worker away.

b) Because there are competitive firms, we want to offer the agent their expected output each period so that expected profits are equal to 0. In the first period we don't know anything about the agent, so their expected output is their expected type, which is $\frac{1}{2}0 + \frac{1}{2}2 = 1$, so $w_1 = 1$. In period 2, we've seen output in period 1 so we perfectly know the agent's type. Since period 1 output is perfectly revealing about the agent's type, we can now write $w_2(y^1)$ as a function of the agent's type, and the zero-profit condition gives $w_2 = y_2 = \theta$. Therefore, the risk-neutral equilibrium wage offer is

$$\{1, \theta\}.$$

But now we want to insure the agent against (some) of the probability that they will be a low type. We still need to pay them $w = \theta = 2$ when they are high type, but we can spread the compensation between the first period and the low-type second period. To maintain the zero profit condition, we can take ε away from first-period pay and add 2ε to second-period pay when the agent has low time (which only happens half the time). The optimal insurance yields

$$\left\{ \begin{array}{ll} \frac{2}{3}, \ 2 & \text{if } \theta = 2 \\ \frac{2}{3}, \ \frac{2}{3} & \text{if } \theta = 0 \end{array} \right\}$$

c) Here we again want to write a contract based on the agent's type, but the information structure is different. Seeing $y = 4$ is perfectly informative about the agent's type, but seeing $y = 0$ is not. The more we see $y = 0$, the more likely it is that the agent is type $\theta = 0$. Using Bayes rule, we can write $\mathbb{P}(\theta = t \mid y^t)$ as

$$y^t = \{0\} \implies \mathbb{P}(\theta = 2) = \frac{1}{3}$$
$$y^t = \{0, 0\} \implies \mathbb{P}(\theta = 2) = \frac{1}{9}$$
$$y^t = \{0^t\} \implies \mathbb{P}(\theta = 2) = \frac{1}{3^t}$$

We can also write expected profits as

$$\mathbb{E}[\Pi] = 2\mathbb{P}(\theta = 2)$$

To write the optimal contract, we can break it into three parts.

(a) Before seeing any output, we offer the unconditional expected output,

$$\mathbb{E}[y_1] = \frac{1}{2}(y \mid \theta = 0) + \frac{1}{2}(y \mid \theta = 2)$$
$$= 0 + \frac{1}{2}\left(\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 4\right)$$
$$= 1$$
$$\implies w_1 = 1$$

(b) If we see output 0 in the first period, or more generally if we see a vector of zeroes of length t, then we start to think that the agent is *probably* type $\theta = 0$, but we can't be sure. So we want to reduce their compensation to account for the increased probability they're low-type. Formally, for any history s^t where $y_t = 0 \forall t' < t$, we offer the wage

$$w(y(s^t)) = 2 \cdot \frac{1}{3^t},$$

5

where we're again offering expected output, but now discounting it by the probability that the agent is actually high type.

(c) Finally, the observation $y_t = 4$ is perfectly information and unambiguously tells us that $\theta = 2$. Therefore, if we see $y = 4$ for some t', then

$$w(s^t) = 2 \quad \forall t > t'$$

The last step is to provide insurance by spreading the 1 in the first period across all low periods, while maintaining the zero profit condition.

3 Question 3

a) z satisfies the equation

$$u(w) + \delta U_+ = \frac{u(z)}{1 - \delta}$$

b)

$$\Pi(u) = \max_{\delta, U_+, w} \mathbb{E}[q - w + \delta \Pi(U_+)] \text{ s.t.}$$

$$\Pi_+ \geq 0$$

$$u(w) + \delta U_+ \geq \frac{u(z)}{1 - \delta}$$

$$u(w) \geq 0$$

$$[u(w) + \delta U_+]F(z) \geq U$$

$$\max_{z, U_+, w} [q - w + \delta \Pi(U_+)]F(z)$$

c)

$$\mathcal{L} = [q - w - \delta \Pi(U_+)]F(z) + \lambda \left[u(w) + \delta \Pi(U_+) - \frac{u(z)}{1 - \delta} \right] + \mu \left[[u(w) + \delta(U_+)]F(z) - U \right]$$

FOCs:

$$0 = -F(z) + u'(w)[\lambda + \mu F(z)] \tag{w}$$

$$\mu \left(\frac{u'(z)}{1 - \delta} \right) = [q - w + \delta \Pi(U_+)]f(z) \tag{z}$$

$$\mu = \frac{f(z)(1 - \delta)}{u'(z)}[q - w + \delta \Pi(U_+)]$$

$$0 = \Pi'(U_+)F(z) + [\lambda + \mu F(z)]$$

$$-\Pi'(U_+) = \frac{1}{u'(w)}$$

$$= \frac{\lambda + \mu F(z)}{F(z)}$$

$$= \frac{\left[\frac{(1-\delta)f(z)}{u'(z)}(q - w + \delta \Pi(U_+)) + \mu F(z) \right]}{F(z)}$$

$$= \frac{(1 - \delta)f(z)}{u'(z)F(z)}(q - w + \delta \Pi(U_+)) + \mu$$

Envelope condition:

$$\Pi'(u) = -\mu$$

$$\frac{\Pi(u)}{F(z)} = q - w - \delta\Pi(U_+)$$

$$= \frac{(1-\delta)f(z)}{'(z)F(z)}\left(\frac{\Pi(u)}{F(z)}\right) - \Pi'(u)$$

$$= \frac{1-\delta}{u'(z)}\Pi(u)\frac{f(z)}{F(z)^2} - \Pi'(u)$$

4   Question 4

a)

$$\mathbb{E}[U^I \mid \theta] = \mathbb{E}[y - t \mid \theta]$$
$$\geq \mathbb{E}[U^{NI} \mid \theta]$$
$$= \theta$$
$$\mathbb{E}[y - t(y,\tilde{\theta}) \mid \hat{\theta}] \geq \theta$$

for some $\hat{\theta}$

$$\implies \mathbb{E}[y \mid \hat{\theta}] - t(y,\tilde{\theta}) \geq$$
$$\iff \mathbb{E}[v] \geq \mathbb{E}[t(y,\tilde{\theta}) \mid \hat{\theta}].$$

This equation holds with equality at $\hat{\theta}$, since $\hat{\theta}$ is the largest θ where it is true. We now want to show

$$\mathbb{E}[v] \geq \mathbb{E}[t(y,\tilde{\theta}) \mid \theta' < \hat{\theta}]$$
$$\iff \mathbb{E}[t(y,\tilde{\theta}) \mid \hat{\theta}] \geq \mathbb{E}[t(y,\tilde{\theta}) \mid \theta']$$
$$\equiv \int f_{y|\hat{\theta}}t(y,\tilde{\theta})dy \geq \int f_{y|\theta'}t(y,\tilde{\theta})$$

MLRP implies that this inequality is always true, so any $\theta' < \hat{\theta}$, all the way down to $\underline{\theta}$, wants to invest.