Econ 203C HW1, Spring 2021

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*I worked on this homework with Paige Pearcy and Luna Shen.

Question 1

1.

$$
\begin{aligned}
F_{Y|X=x}(y,x) &= \mathbb{P}(Y \le y \mid X = x) \\
&= \mathbb{P}(\varepsilon \le y - g(x) \mid X = x) \\
&= \mathbb{P}\left(\frac{\varepsilon - \mu(x)}{\sqrt{\sigma^2(x)}} \le \frac{y - (g(x) + \mu(x))}{\sqrt{\sigma^2(x)}} \mid X = x\right) \\
&= \Phi\left(\frac{y - (g(x) + \mu(x))}{\sqrt{\sigma^2(x)}}\right) \\
Y \mid X &\sim \mathcal{N}(g(x) + \mu(x), \sigma^2(x))
\end{aligned}
$$

2. These results follow from properties of the normal distribution.

$$
\begin{aligned}
\mathbb{E}[Y \mid X = x] &= g(x) + \mu(x) \\
\operatorname{Var}(Y \mid X = x) &= \sigma(x)^2
\end{aligned}
$$

3. The distribution function for this model must be a normal distribution whose first and second moments match the values from the previous question. In this case,

$$
\begin{aligned}
m_1(x) &= \int y dF_{Y|X=x}(y,x) && \text{(first moment)} \\
&= g(x) + \mu(x) \\
m_2(x) &= \int y^2 dF_{Y|X=x}(y,x) && \text{(second moment)} \\
&= \sigma^2(x) + (g(x) + \mu(x))^2
\end{aligned}
$$

4. $\mu(x)$ is not identified. To see this, define

$$
\begin{aligned}
g^*(x) &\equiv g(x) - c \\
\mu^*(x) &\equiv \mu(x) + c \\
\implies g^*(x) + \mu^*(x) &= g(x) + \mu(x).
\end{aligned}
$$

$\sigma^2(x) = \operatorname{Var}(Y \mid X = x)$ is identified, because we observe Y and X.

Question 2

1.

$$
\begin{aligned}
F_{Y|X=x}(y,x) &= \mathbb{P}(Y \le y \mid X = x) \\
&= \mathbb{P}(\varepsilon \le y - \alpha^* \cdot g^*(x) \mid X = x) \\
&= \mathbb{P}\left(\frac{\varepsilon - \mu^*}{\sqrt{(\sigma^*)^2}} \le \frac{y - (\alpha^* \cdot g^*(x) + \mu^*)}{\sqrt{(\sigma^*)^2}} \mid X = x\right) \\
&= \Phi\left(\frac{y - (\alpha^* \cdot g^*(x) + \mu^*)}{\sqrt{(\sigma^*)^2}}\right)
\end{aligned}
$$

2. g^* is not identified within the class of continuous functions. If it were identified,

$$
\mathbb{E}[Y \mid X = x, \alpha^*, g^*, \mu^*, \sigma^*] = \mathbb{E}[Y \mid X = x, \tilde{\alpha}, \tilde{\mu}, \tilde{g}, \tilde{\sigma}] \implies \alpha^* g^*(x) + \mu^* = \tilde{\alpha}\tilde{g}(x) + \tilde{\mu}.
$$

To find a counterexample, take

$$
\begin{aligned}
\tilde{g}(x) &= g^*(x) + c \\
\tilde{\alpha} &= \alpha^* \\
\tilde{\mu} &= \mu^* - \alpha^* c \\
\implies \alpha^* g^*(x) + \mu^* &= \tilde{\alpha}\tilde{g}(x) + \tilde{\mu} \\
\tilde{g}(x) &\neq g^*(x)
\end{aligned}
$$

3. By the same reasoning, μ^* is not identified within the set of real numbers. To find a counterexample here, take

$$
\begin{aligned}
\tilde{\mu} &= \mu^* - \alpha^* t \\
\tilde{g}(x) &= g^*(x) + t \\
\tilde{\alpha} &= \alpha^*
\end{aligned}
$$

4.
$$\sigma^* = \operatorname{Var}(Y \mid X = x)$$

We observe Y and X, so $(\sigma^*)^2$ is identified.

5. With $\alpha^* = 1, \mu^* = 0$
$$\mathbb{E}[Y \mid X = x] = g^*(x),$$
so g^* is identified. $(\sigma^*)^2$ is still identified because we still observe X and Y.

6. In this case, μ is identified but α and g are not. To see that μ^* is identified, consider $\tilde{\alpha}, \tilde{g}(x)$ and $\tilde{\mu}$ such that
$$\alpha^* g^*(x) + \mu^* = \tilde{\alpha}\tilde{g}(x) + \tilde{\mu}.$$

$$
\begin{aligned}
\underbrace{\tilde{\alpha}\tilde{g}(\overline{x})}_{0} + \tilde{\mu} &= \underbrace{\alpha^* g^*(\overline{x})}_{0} + \mu^* \\
\implies \tilde{\mu} &= \mu^*,
\end{aligned}
$$

so μ^* is identified. To see that α and g are not identified, take

$$
\begin{aligned}
\tilde{\alpha} &\equiv \frac{1}{c}\alpha^* \\
\tilde{g}(x) &\equiv g^*(x)c
\end{aligned}
$$