122 Notes

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Definition. *A normal form game G consists of:*

1. *A finite set $\mathcal{I} = \{1, 2, \ldots, I\}$ of players,*
2. *Strategy sets $S_1, S_2, \ldots, S_I$,*
3. *Payoff functions $u_i : S_1 \times S_2 \times \ldots \times S_I \to \mathbb{R}$*

1.0.1 Example: Prisoner's Dilemma

$$\begin{aligned}
\mathcal{I} &= \{\text{Lev, Igor}\} \\
S_1 = S_2 &= \{\underline{\text{C}}\text{ooperate, } \underline{\text{D}}\text{efect}\} \\
u_{\text{Lev}}(C, C) &= 1 \\
u_{\text{Lev}}(D, C) &= 2 \\
u_{\text{Lev}}(D, D) &= -1 \\
&\vdots
\end{aligned}$$

	C	D
C	1, 1	-1, 2
D	2, -1	0, 0

The row player is Player 1; the column player is Player 2. Each cell represents the payoffs as a pair $u_1(s_1, s_2), u_2(s_1, s_2)$.

1.0.2 Meet in Cambridge

$$\begin{aligned}
\mathcal{I} &= \{1, 2\} \\
S_1 = S_2 &= \{\text{all places in Cambridge}\} \\
u_1(s_1, s_2) &= \begin{cases} 1 & \text{if } s_1 = s_2 \\ 0 & \text{if } s_1 \neq s_2 \end{cases}
\end{aligned}$$

Definition. *Player i plays as if rational with belief μ_i if $s_i \in \arg\max_{s_i \in S_i} \mathbb{E}_\mu(s_i', s_{-i})$. Here μ_i is a probability distribution on S_{-i}. This can equivalently be written as*

$$s_i \textit{ maximizes } \sum_{s_{-i} \in S_{-i}} u_i(s_i', s_{-i}) \mu_i(s_{-i}).$$

Definition. *Strategy s_i is strictly dominated if*

$$\exists s_i' \in S_i \textit{ with } u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}.$$

1.0.3 Left/Middle/Right

	L	M	R
U	2, 2	1, 1	4, 0
D	1, 2	4, 1	3, d

Looking at the initial game, we can rule out Middle (the plays in red) since that strategy is strictly dominated for Player 2. Once we have eliminated this column, we can look at the restricted game and eliminate "U" for Player 1 (the blue row). Finally, we can see that (U, L) is the only equilibrium.

Definition (Iterated Dominance)**.**

$$S_i^0 = S_i$$

$$S_i^1 = \underbrace{\{s_i \in S_i^0 \mid \nexists s_i' \in S_i^0 \ s.t.\ u(s_i', s_{-i}) > u_i(s_i, s_{-i} \ \forall s_{-i} \in S_{-i}^0\}}_{\textit{Everything that isn't strictly dominated}}$$

$$\vdots$$

$$S_i^{k+1} = \underbrace{\{s_i \in S_i^k \mid \nexists s_i' \in S_i^k \ s.t.\ u(s_i', s_{-i}) > u_i(s_i, s_{-i} \ \overbrace{\forall s_{-i} \in S_{-i}^k}^{\textit{assuming opponent is rational}}\}}_{\textit{Everything that isn't strictly dominated}}$$

$$S_i^\infty = \bigcap_{k=1}^{\infty} S_i^k 1$$

In The Left/Middle/Right example,

$$\begin{array}{rclrcl} S_1^0 & = & \{U, D\} & S_2^0 & = & \{L, M, R\} \\ S_1^1 & = & \{U, D\} & S_2^1 & = & \{L, R\} \\ S_1^2 & = & \{U\} & S_2^2 & = & \{L, R\} \\ S_1^3 & = & \{U\} & S_2^3 & = & \{L\} \\ & & & \vdots & & \\ S_1^k & = & \{U\} & S_2^k & = & \{L\} \\ \therefore S_1^\infty & = & \{U\} & S_2^\infty & = & \{L\} \end{array}$$

Definition. *A game G is solvable by pure strategy iterated strict dominance (PSISD) if S^∞ contains a single strategy profile.*

Definition. *A strategy profile is a vector $(s_1, s_2, \ldots, s_I) s_i \in S_{-i} \forall i$.*

1.0.4 Heads/Tails/$200

	200	H	T
200	200, 200	0, 200	0, 200
H	200, 0	300, 0	0, 300
T	200, 0	0, 300	300, 0

This game is not solvable by pure strategy iterated strict dominance.

Definition. *A strategy profile $s^* = \{s_1^*, s_2^*, \ldots, s_I^*\}$ is a pure-strategy Nash Equilibrium (PSNE) of G if and only if*

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i^*, s_{-i}^*) \quad \forall i.$$

Definition (Best Response Correspondence)**.**

$$BR_i(s_{-i}) = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$$

Proposition. *Strategy profile s^* is a pure-strategy Nash equilibrium if and only if*

$$s_i \in BR_i(s_{-i}^*) \quad \forall i.$$

Claim. *(U, L) is the unique PSNE in the Up-Down, Left-Middle-Right game from earlier.*

Proof. What to show:

1. (U, L) is a PSNE.

   Proof.

$$\begin{aligned} u_1(U, L) &= 2 \\ &\geq u_1(D, L) \\ u_2(U, L) &= 2 \\ &> u_2(U, M) \\ &> u_2(U, R) \end{aligned}$$

   □

2. Nothing else is a PSNE.

   Proof. To see there are no other Nash equilibria, find the best response correspondence:

$$\begin{array}{rclcrcl} BR_1(L) &=& U & & BR_2(U) &=& L \\ BR_1(M) &=& D & & BR_2(D) &=& R \\ BR_1(R) &=& U & & & & \end{array}$$

   (U, L) is the only pair where $s_i \in BR(s_{-i}) \forall i$. □

□

Proposition. *If s^* is a PSNE¡ then $s_i^* \in s_i^\infty \forall i$:*

$$PSNE \subset S^\infty.$$

Proof.

Suppose not. Then $\exists\ i, T$ such that

$$s_j^* \in S_j^T \text{ and } s_i^* \in S_i^{T+1}. \implies \exists s_i' \in S_i^T \text{ such that } u_i(s_i', s_{-i}) \quad > u_i(s_i^*, s_{-i}) \forall s_{-i} \in S_{-i}^T$$

But we defined T so that $s_i^* \in S_{-i}^T$

$\implies u_i(s_i', s_{-i}^*) > (s_i^*, s_{-i}^*) \Rightarrow\!\Leftarrow$

$\therefore s_i^*$ is a PSNE.

$\square$

Corollary. *If G is solvable by pure strategy iterated strict dominance, then G has at most one Nash equilibrium.*

Proposition. *If G is finite and solvable by pure-strategy iterated strict dominance, then*

$$PSNE \subset S^\infty.$$

Definition (Finite game)**.** *G is finite if $S_i < \infty \forall i$.*

1.0.5 Infinite Game

This game is solvable by iterated strict dominance but the solution is *not* a pure strategy Nash equilibrium.

	0	1	2	3	4	
0	0, 0	0, 1	0, 2	0, 3	0, 4	...
1	1, 0	-1, -1	-1/2, -1/2	-1/3, -1/3	...	...
2	2, 0	-1/2, -1/2	-1/3, -1/3	-1/4, -1/4	...	...
3	3, 0	-1/3, -1/3	-1/4, -1/4	...	...	...
4	4, 0	-1/4, -1/4	$\vdots$	$\ddots$	...	...
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	...

$$S_1^0 = S_2^0 = \mathbb{Z}^+$$
$$S_1^1 = S_2^1 = \{0\}$$
$$\vdots$$
$$S_1^\infty = S_2^\infty = \{0\}$$

2 October 28, 2019

2.1 Outline

Last class: PSNE in Discrete games

1. Check all boxes
2. S^* is a PSNE iff $s_i^* \in BR_i(s_{-i}^*) \forall i$. Also PSNE $\subset S^\infty$.

Today: PSNE in continuous games We can't check all boxes in continuous games, so we have to use strategy (2).

2.1.1 Example: Cournot Competition

Two firms choose q_1 and q_2 simultaneously. The market determines a price $p(q_1 + q_2)$.

$$\begin{aligned}
\mathcal{I} &= \{1, 2\} \\
S_1 = S_2 &= \mathbb{R}^+ \\
u_1(q_1, q_2) &= q_1 \underbrace{(\overbrace{1}^{\text{arbitrary market size}} - (q_1 + q_2))}_{\text{zero marginal cost}} \\
u_2(q_1, q_2) &= q_2(1 - (q_1 + q_2))
\end{aligned}$$

Now find each firm's BR correspondence. We have three options for $BR_2(q_1)$:

$$\begin{aligned}
BR_2(q_1) &\in \begin{cases} 0 \\ \emptyset \\ \frac{\partial u_2}{\partial q_2}(q_1, BR_w(q_1)) = 0 \end{cases} \\
\frac{\partial u_2}{\partial q_2} &= 1 - q_1 - 2q_2|_{q_2 = BR_2(q_1)} \\
&= 0 \\
\implies BR_2(q_1) &= \begin{cases} \frac{1-q_1}{2} & \text{if } q_1 \in [0, 1) \\ 0 & \text{if } q_1 \geq 1 \end{cases} \\
BR_1(q_2) &\in \begin{cases} \frac{1-q_2}{2} & \text{if } q_2 \in [0, 1) \\ 0 & \text{if } q_2 \geq 1 \end{cases}
\end{aligned}$$

Equilibrium requires two conditions:

$$q_1^* \in BR_1(q_2^*) \qquad \text{(Condition 1)}$$
$$q_2^* \in BR_2(q_1^*) \qquad \text{(Condition 2)}$$

We can use these conditions to set up and solve a system of two equations in two unknowns.

$$\begin{aligned}
q_2^* &= \frac{1 - q_1^*}{2} \\
q_1^* &= \frac{1 - q_2^*}{2} \\
q_1^* + 2q_2^* &= 1 \\
2q_1^* + q_2^* &= 1 \\
\implies q_1^* &= q_2^* \\
&= \frac{1}{3}
\end{aligned}$$

We can generalize this to N firms:

$$q_1^* = \frac{1}{N+1}.$$

The model ranges from monopoly to perfect competition for $N \in [0, \infty)$.

2.1.2 Hotelling (1929) Competition on a line

$$\begin{aligned}
u &= \begin{cases} v - p_1 - tx \\ v - t(1-x) - p_2 \end{cases} && \text{(pay } p_i \text{ to buy from firm } i\text{)} \\
v - p_1 - t\hat{x} &= v - p_2 - t(1-\hat{x}) && \text{(in general)} \\
\hat{x} &= \frac{1}{2} + \frac{p_2 - p_1}{2t}
\end{aligned}$$

This yields the solution

$$\begin{aligned}
[0, \hat{x}] &\text{ buys from 1.} \\
(\hat{x}, 1] &\text{ buys from 2.} \\
|p_1 - p_2| &< t \\
v &\geq p_1 + t\hat{x} \\
\mathcal{I} &= \{1, \}2 \\
S_1 = S_2 &= \mathbb{R}^+ \\
u_1(p_1, p_2) &= (p_1 - c)\left(\frac{1}{2} + \frac{p_2 - p_1}{2t}\right) \\
u_2(p_1, p_2) &= (p_2 - c)\left(\frac{1}{2}\frac{p_1 - p_2}{2t}\right)
\end{aligned}$$

Claim. *The unique PSNE of this game is* $p_1^* = p_2^* = c + t$.

Proof.

$$\begin{aligned}
BR_2(p_1) &= \arg\max_{p_2} (p_2 - c)\left(\frac{1}{2} + \frac{p_1 - p_2}{2t}\right) \\
0 &= (p_2 - c)\left(\frac{-1}{2t}\right) + \frac{1}{2} + \frac{p_1 - p_2}{2t}\bigg|_{p_2 = BR_2(p_1)} && \text{(FOC)} \\
BR_2(p_1) &= \frac{c + t + p_1}{2} \\
BR_1(p_2) &= \frac{c + t + p_2}{2} \\
2p_2^* &= c + t + p_1^* \\
2p - 1^* &= c + t + p_2^* \\
\therefore p_1^* = p_2^* &= c + t
\end{aligned}$$

□

2.1.3 Price Competition with Logit Demand

Setup

$$\mathcal{I} = \{1, 2, \ldots, N\}$$
$$S_1 = S_2 = \ldots = \mathbb{R}^+$$
$$u_i(p_1, \ldots, p_N) = (p_i - c)\,\frac{e^{-\alpha p_i}}{\sum_{k=1}^{N} e^{-\alpha p_k}}$$
$$u_{ij} = v - \alpha p_j + \varepsilon_{ij}$$

i is a consumer; j is a firm

$$F(\varepsilon_{ij}) = e^{-e^{-(\varepsilon_{ij}+\gamma)}}$$

Claim. *A PSNE is* $p_1^* = p_2^* = \ldots = p_n^* = c + \frac{1}{\alpha}\left(1 + \frac{1}{N-1}\right)$.

Note that $p \not\to c$ as $n \to \infty$.

Proof. Suppose $p_1 = p_2 = \ldots = p_n = p^*$ is a PSNE. This approach is akin to "guess and check".

$$p^* \in \arg\max_{p_j} (p_j - c) D(p_j, p^*_{-j})$$
$$0 = (p^* - c)\frac{dD_j}{\partial p_j}(p^*, p^*) + D_j(p^*, p^*) \tag{FOC}$$
$$p^* = c - \frac{D_j(p^*, p^*)}{\frac{\partial D_j}{\partial p_j}(p^*, p^*)}$$

This line uses the fact that, in a symmetric equilibrium, $D_j = \frac{1}{n}$.

$$\begin{aligned}
p^* &= c + \frac{\frac{1}{n}}{\frac{\sum_k e^{-\alpha p_k}(-\alpha)e^{-\alpha p_j}\left((1-\alpha)e^{-\alpha p_j}\right)}{\left(\sum_k e^{-\alpha p_k}\right)^2}} \\
&= c + \frac{\frac{-1}{N}}{-\alpha\left(\frac{1}{N}\right) + \alpha\left(\frac{1}{N^n}\right)} \\
p^* &= c + \frac{N}{\alpha(N-1)} \\
&= c + \frac{1}{\alpha}\left(1 + \frac{1}{N-1}\right)
\end{aligned}$$

□

Proposition. *Let G be a two-player game. Suppose $S_i \subset \mathbb{R}$ is compact and u_i is continuous and has increasing differences in (s_i, s_{-i}). Let $\underline{s}_i = \inf S_i^\infty$ and $\overline{s}_i = \sup s_i^\infty$. Then $(\underline{s}_1, \underline{s}_2)$ and $(\overline{s}_1, \overline{s}_2)$ are pure strategy Nash equilibria of G.*

Corollary. *If s^* is the unique PSNE of G, then G is solvable by pure strategy iterated strict dominance.*

Proof.

1. Define an iterative algorithm to find a point a^*.
2. Show a^* is a PSNE.
3. Show $a_i^* = \sup S_i^\infty$.

$$\begin{aligned}
\overline{a}_i^0 &\equiv \sup S_i \\
\overline{a}_i^1 &\equiv \sup BR_i(\overline{a}_{-i}^0) \\
\overline{a}_i^{k+1} &\equiv \sup BR_i(\overline{a}_i^k) \\
\overline{a}_i^\infty &\equiv \lim_{k\to\infty} \overline{a}_i^k
\end{aligned}$$

Lemma. *$\overline{a}_i^\infty$ exists because $a_i^{k+1} \leq \overline{a}_i^k$ and S_i is compact.*

Proof.

$$\sup BR_i(s_{-i}) = \sup \arg\max_{s_i} u_i(s_i, s_{-i})$$

When u has increasing differences, $\sup \arg\max$ is increasing in s_{-i}. $\overline{a}_i^1 \leq \overline{a}_i^0$ by definition, so $\overline{a}_i^2 \leq \overline{a}_i^1$. □

Lemma.

$$(\overline{a}_1^\infty, \overline{a}_2^\infty) \text{ is a PSNE.}$$

Proof.

Given any i, s_i',

$$\begin{aligned}
u_i(\overline{a}_i^k, \overline{a}_{-i}^k) &\geq u_i(s_i', \overline{a}_{-i}^k) \quad \forall i, k \\
\lim_{k\to\infty} u_i(\overline{a}_i^\infty, \overline{a}_{-i}^\infty) &\geq \lim_{k\to\infty} u_i(s_i', \overline{a}_{-i}^k) \\
\implies u_i(\overline{a}_i^\infty, \overline{a}_{-i}^\infty) &\geq u_i(s_i', \overline{a}_{-i}^\infty) \quad \forall i, s_i' \\
\therefore (\overline{a}_i^\infty, \overline{a}_{-i}^\infty) &\text{ is a PSNE.}
\end{aligned}$$

□

Lemma.

$$\overline{a}_i^\infty = \sup s_i^\infty$$

Note that $\overline{a}_i^\infty \leq \sup S_i^\infty$ because $\overline{a}_i^\infty$ is a Nash equilibrium and Nash equilibria are not eliminated by pure strategy iterated strict dominance. Need only to show

$$s_i > \overline{a}_i^\infty \implies s_i \notin S_i^k.$$

Lemma.

$$s_i > \overline{a}_i^k \implies s_i \notin S_i^k.$$

Proof. By induction on k, s_i is strictly dominated by $\overline{a}_i^k$ in the previous step. This proof uses increasing differences. $\square$

$\square$

3 October 30, 2019

3.1 Outline

Last class

1. Nash equilibria in continuous time
   - $s_1 = BR_1(s_2)$
   - $s_2 = BR_2(s_1)$
   - $p^* = c + t$ in Hotelling
   - $p^* = c + \frac{1}{\alpha}\left(\frac{N}{N-1}\right)$ in the Logit model
2. Nash equilibria an Iterated Strict Dominance.
   - If u is continuous, there are N players, u has increasing differences, and S_1 and S_2 are compact, then $\sup S_i^\infty$ and $\inf S_i^\infty$ are Nash equilibria.
   - As a corollary, if G has a unique Nash equilibrium it is solvable by iterated strict dominance.

Corollary. *Suppose $N = 2$, u is continuous, S_1 and S_2 are compact, and u_i has decreasing differences. Then if G has a unique Nash equilibrium it is solvable by iterated strict dominance.*

Proof. Define a new game by

$$\tilde{u}_i(S_1, s_2) = u_i(s_1, -s_2).$$

$\square$

This approach only works for 2-player games.

3.2 Multiple Equilibria

Nash does not imply uniqueness.

3.2.1 Brexit Game

Three choices:

A— No deal

B— Remain

C— Brexit with a deal

Three groups of MPS:

1. $u_1(A) > u_1(B) > u_q(C)$
2. $u_2(B) > u_2(C) > u_2(A)$
3. $u_c(C) > u_3(A) > u_3(B)$

Suppose 1, 2 and 3 simultaneously choose $s_i \in \{A, B, C\}$. The majority wins. If there is a three-way tie, A wins.

Claim. *G has 5 pure strategy Nash equilibria:*

1. *A, A, A*
2. *B, B, B*
3. *C, C, C*
4. *A, B, A*
5. *A, C, C*

3.2.2 Basketball in Grad School

	7.30	8.00
7.30	9, 9	-5, 8
8.00	8, -5	7, 7

Can we figure out which of a pair of Pareto-ranked Nash equilibria is played? Games can also have no pure strategy Nash equilibria: either no equilibria, or a mixed equilibria.

3.3 Mixed Strategies

Let G be a game with strategy space $S_1, \dots, S_n$.

Definition. *A mixed strategy for i is a probability distribution σ_i on S_i.*

Notation

1. σ as a function (measure): $\sigma_1(\text{rock}) = \frac{1}{2}$, etc.
2. σ as a vector: $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$
3. $\frac{1}{2}R + \frac{1}{4}P + \frac{1}{4}S$

Definition. *Σ_i or $\Delta(S_i)$ is the set of probability distributions on S_i.*

$$\Sigma = \Sigma_1 \times \Sigma_2 \times \ldots \times \Sigma_I$$
$$u_i(\sigma_i, \sigma_{-i}) \equiv \sum_{s_i, s_{-i}} u_i(s_i, s_{-i}) \sigma_i(s_i) \sigma_{-i}(s_{-j})$$

Note that $u_i : \Sigma \to \mathbb{R}$.

Definition. *$\sigma^* \in \Sigma$ is a mixed strategy Nash equilibrium of G if*

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*) \quad \forall i, \sigma_i' \in \Sigma_i.$$

3.4 Testing for Nash Equilibrium

Proposition. *σ^* is a Nash equilibrium if and only if*

$$u_i(\sigma^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^* \quad \forall i, s_i \in S_i.$$

3.4.1 Example: Testing for NE in Rock/Paper/Scissors

$$\left(\frac{1}{3}R + \frac{1}{3}P + \frac{1}{3}S; \frac{1}{3}R + \frac{1}{3}P + \frac{1}{3}S;\right) \text{ is a Nash equilibrium.}$$

$$u_i(\sigma^*, \sigma^*) = 0$$
$$u_i(R, \sigma^*) = 0$$
$$u_i(P, \sigma^*) = 0$$
$$u_i(S, \sigma^*) = 0$$

3.5 Finding Mixed Nash Equilibria

Definition.

$$\text{supp}(\sigma_1) \equiv \{s_i \in S_i \mid \sigma_1(s_i) > 0\}$$

Proposition. *If σ^* is a Nash equilibrium and $s_i', s_i'' \in \text{supp}(\sigma_i^*)$, then*

$$u_i(s_i', \sigma_{-i}^*) = u_i\left(s_i'', \sigma_{-i}^*\right).$$

3.6 Algorithm for finding an equilibrium in 2×2 games

Tax Fraud

	don't audit	audit
honest	10, 10	10, 9
cheat	15, 5	-35, 10

Check Player 1's indifference conditions:

$$\begin{aligned} u_1(H, \sigma_2^*) &= 10 \\ u_1(C, \sigma_2^*) &= 15\sigma_2^*(D) + (-35)\sigma_2^*(A) \\ &= 15 - 50\sigma_2^* A \\ u_1(H, \sigma_2^*) &= u_1(C, \sigma_2^*) \\ &= 10 \\ \implies 10 &= 15 - 50\sigma_2^*(A) \\ 50\sigma_2^*(A) &= 5 \\ \implies \sigma_2^*(A) &= \frac{1}{10} \\ \therefore \sigma_2^* &= \frac{1}{10}A + \frac{9}{10}D \end{aligned}$$

Now check Player 2's indifference condition:

$$\begin{aligned} u_2(A, \sigma_1^*) &= 9\sigma_1^*(H) + 10\sigma_1^*(C) \\ u_2(D, \sigma_1^*) &= 10\sigma_1^*(H) + 5\sigma_1^*(C) \\ 10 - \sigma_1^*(H) &= 5 + 5\sigma_1^*(H) \\ \sigma_1^*(H) &= \frac{5}{6} \\ \implies \sigma_1^* &= \frac{5}{6}H + \frac{1}{6}C \end{aligned}$$

3.7 Algorithm for finding Nash Equilibria in 3×3 games

1. Pick all possible support pairs (T_1, T_2) where $T_1 \subset S_1$ and $T_2 \subset S_2$.
2. Solve indifference equations; find what σ^* must be if T_1, T_2 are the supports.
3. Check that σ^* is a Nash equilibrium using the testing calculation.

4 November 4, 2019

4.1 Full Support 3×3 Games

4.1.1 Rock-Paper-Scissors

	R	P	S
R	0, 0	-1, 1	1, -1
P	1, -1	0,0	-1, 1
S	-1, 1	1, -1	0, 0

$$\begin{aligned}
\text{Player 1's indifference} &\implies \sigma_2^* \\
\text{Player 2's indifference} &\implies \sigma_1^*
\end{aligned}$$

Suppose (σ_1^*, σ_2^*) is a full-support Nash equilibrium.

$$\begin{aligned}
u_1(R, \sigma_2^*) &= -q + 1 - p - q \\
&= -p - 2q + 1 \\
u_1(P, \sigma_2^*) &= p - 1 - p - 1 \\
&= 2p + 1 - 1 \\
u_1(S, \sigma_2^*) &= -p + q
\end{aligned}$$

Indifference conditions give us

$$\begin{aligned}
u_1(R, \sigma_2^*) &= u_1(P, \sigma_2^*) \\
u_1(R, \sigma_2^*) &= u_1(S, \sigma_2^*)
\end{aligned}$$

This yields a system of two equations in two unknowns. After a bit of algebra, we get

$$\begin{aligned}
3p + 3q &= 2 \\
3q &= 1 \\
\implies \sigma_2^* &= \frac{1}{3}R + \frac{1}{3}P + \frac{1}{3}S
\end{aligned}$$

4.2 Larger Games

Three approaches

1. Solve $N-1$ equations in $N-1$ unknowns. Look for some simple patterns in the equations.
2. Use iterated dominance to reduce to a simpler game.
3. Make a clever observation.

4.2.1 All-Pay Auction

$i \in \{1, 2\}$ choose bids $b_1, b_2 \in \{0, 1, 2, \ldots, 100\}$; high bid wins \$100 but both bidders pay. Both lose if there is a tie.

Conjecture Suppose (σ^*, σ^*) is a Nash equilibrium. Suppose

$$\text{supp}(\sigma^*) = \{\underline{b}, b^1, \ldots, b^k, \bar{b}\}.$$

Claim.

$$\underline{b} = 0$$

Proof.

$$\begin{aligned} u_1(\underline{b}, \sigma^*) &= 0 - \underline{b} \\ u_1(0, \sigma^*) &= 0 - 0 \\ &= 0 \\ \underline{b} \in BR \implies \underline{b} &= 0. \end{aligned}$$

□

Claim.

$$\text{supp}(\sigma^*) = \{0, 1, 2, \ldots, N\} \textit{ for some } N \textit{ (no holes)}$$

Proof. Suppose not. Then there is a b such that $b \notin \text{supp}(\sigma^*)$ but $b + 1 \in \text{supp}(\sigma^*)$.

$$\begin{aligned} u_1(b+1, \sigma^*) &= \left(\sum_{j=0}^{b} \sigma^*(j) \right) \cdot 100 - (b+1) \\ u_1(b, \sigma^*) &= \left(\sum_{j=0}^{b-1} \sigma^*(j) \right) \cdot 100 - b \\ b \notin \text{supp}(\sigma^*) \implies \sum_{j=0}^{b} \sigma^*(j) &= \sum_{j=0}^{b-1} \sigma^*(j) \end{aligned}$$

□

Now let's construct the Nash equilibrium strategy. We know that it starts with $\underline{b} = 0$ and that there are no holes. Also, we know that all items in the support of a Nash equilibrium must have the same expected utility, and that

$u_1(0) = 0$. This is enough to get our inductive proof started.

$$\begin{aligned}
u_1(0, \sigma^*) &= u_1(1, \sigma^*) \\
0 &= 100(\sigma^*(0)) - 1 \\
\implies \sigma^*(0) &= \frac{1}{100} \\
0 &= 100(\sigma^*(0) + \sigma^*(1)) - 2 \\
0 &= 100\left(\frac{1}{100} + 100\sigma^*(1)\right) - 2 \\
\implies \sigma^*(1) &= \frac{1}{100} \\
&\vdots \\
\sigma^*(99) &= \frac{1}{100} \\
\sum_{0}^{99} \frac{1}{100} &= 1 \\
\therefore \sigma^*(100) &= 0
\end{aligned}$$

4.3 Iterated Dominance

	L	M	R
U	10, 9	10, 6	10, 10
M	-5, 9	15, 10	11, 12
D	-35, 10	10, 7	15, 5

$$u_w(\sigma_1, C) < u_2\left(\sigma_1, \frac{1}{2}L + \frac{1}{2}R\right)$$

After eliminating C, M is dominated by $\frac{3}{4}U + \frac{1}{4}D$.

Definition. *Redefine S^∞ by*

$$\begin{aligned}
S_i^0 &= S_i \\
S_i^{k+1} &= \left\{s_i \in S_i^k \mid \nexists \sigma_i' \in \Delta(S_i^k) \ s.t. \ u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i} \ \forall s_{0i} \in S_{-i}^k\right\} \\
S_i^\infty &= \bigcup_{k=0}^{\infty} S_i^k
\end{aligned}$$

In the previous example,

$$\begin{array}{lclclcl}
S_1^0 &=& \{U, M, D\} & \quad & S_2^0 &=& \{L, M, R\} \\
S_1^1 &=& \{U, M, D\} & \quad & S_2^1 &=& \{L, R\}
\end{array}$$

The idea is to cross out pure strategies that are dominated by mixtures.

Proposition. *a) Let σ^* be a Nash equilibrium of game G. If $\sigma_i^* > 0$, then $s_i \in S_i^\infty$.*

b) Suppose $G = (S, u)$. Define $G^{ISD} = (S^\infty, u)$. If G is finite and σ^* is a Nash equilibrium of G^{ISD}, then σ^* is a Nash equilibrium of G.

4.4 Correlated Equilibrium

	L	R
U	5, 1	0, 0
D	4, 4	1, 5

$$\begin{aligned} (U, L) &\mapsto 5, 1 \\ (D, r) &\mapsto 1, 5 \\ \left(\frac{1}{2}U + \frac{1}{2}D, \frac{1}{2}L + \frac{1}{2}R\right) &\mapsto \frac{5}{2}, \frac{5}{2} \end{aligned}$$

Is there a way to ensure a "good" equilibrium? Consider a randomizing device that chooses $x \in \{1, 2, 3\}$ uniformly at random.

1. If $x = 1$, text "U" to Player 1 and "L" to Player 2.
2. If $x = 2$, text "D" to Player 1 and "L" to Player 2.
3. If $x = 3$, text "D" to Player 1 and "R" to Player 2.

Note that it is an equilibrium to follow the messages, and that the payoff from following the messages is

$$\frac{1}{3}(5 + 4 + 1) = \frac{10}{3} > \frac{5}{2}.$$

Definition. *A correlated equilibrium of G is a distribution $q^* \in \Delta(S_1 \times S_2)$ such that $s_i \in BR_i(q^*(s_{-i} \mid s_i))$ $\forall i, s_i$ with $q_i^*(s_i) > 0$.*

The set of correlated equilibria is convex. It can extend beyond the convex hull of the Nash equilibria.

Proposition. *A distribution $q \in \Delta(S_1 \times S_2)$ is a correlated equilibrium if and only if*

$$\sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i} q_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \overbrace{u_i(s_i', s_{-i})}^{\text{actually play } s_i'} \underbrace{q_i(s_i, s_{-i})}_{\text{when told to play } s_i}.$$

We also need

$$q(s_i, s_{-i}) \geq 0 \quad \forall i, s_i, s_{-i}$$

and

$$\sum_{s_i \in S_i, s_{-i} \in S_{-i}} q(s_i, s_{-i}) = 1.$$

5 November 6, 2019

Theorem. *Every finite normal-form game has a Nash equilibrium.*

Proof. Let Σ be the space of mixed strategy profiles for the game G. Define $r : \Sigma \rightrightarrows \Sigma$ by

$$r(\sigma) = BR_1(\sigma_{-1}) \times BR_2(\sigma_{-2}) \times \ldots \times BR_I(\sigma_{-I}).$$

Note that σ is a Nash equilibrium if and only if $\sigma \in r(\sigma)$. We want to show that $r(\sigma)$ has a fixed point. We're going to use Kakutani's fixed point theorem. To apply Kakutani, we need to check four conditions:

1. Σ is a non-empty, compact, convex subset of $\mathbb{R}^n$.
2. $r(\sigma)$ is non-empty $\forall \sigma$.
3. $r(\sigma)$ is convex $\forall \sigma$.
4. r is upper hemicontinuous.

1. This is obvious; follows from the definition of a probability distribution. $\square$

2. It suffices to show that $BR_i(\sigma_{-i})$ is non-empty for all i and σ_{-i}. $u_i(\sigma_i, \sigma_{-i})$ is a continuous function of σ_i defined on the compact set Σ_i. This implies that $BR(\cdot, \sigma_{-i})$ achieves its maximum. The arg max is the best response correspondence, therefore the best response always exists. $\square$

3. WTS:

$$\sigma' \in r(\sigma), \sigma'' \in r(\sigma), \text{ and } \lambda \in [0,1] \implies \lambda\sigma' + (1-\lambda)\sigma'' \in r(\sigma)$$

It suffices to show that

$$\lambda\sigma_i' + (1-\lambda)\sigma_i'' \in BR_i(\sigma_{-i}) \text{ if } \left\{ \begin{array}{l} \sigma_i' \in BR_i(\sigma_{-i}) \text{ and} \\ \sigma_i'' \in BR_i(\sigma_{-i}) \end{array} \right.$$

This result is obvious from the definition of BR_i; taking mixtures of objects with the same expected utility yields the same expected utility. $\square$

4.

$$\Sigma \text{ compact } \implies r \text{ UHC } \iff \text{graph}(r) \text{ is closed.}$$

WTS:

$$(\sigma^n \hat{\sigma}^n \ ) \, in \, \text{graph}(r), \sigma^n \to \sigma, \hat{\sigma}^n \to \hat{\sigma} \implies (\sigma, \hat{\sigma} \in \text{graph}(r))$$

It suffices to show

$$
\begin{aligned}
\hat{\sigma}_i &\in BR_i(\sigma_{-i}) \quad \forall i \\
u_i(\hat{\sigma}_i, \sigma_{-i}) &= \lim_{n\to\infty} u(\hat{\sigma}_i^n, \sigma_{-i}^n) \\
&\geq \lim_{n\to\infty} u(\sigma_i', \sigma_{-i}^n) \quad \forall \sigma_i' \\
&= u_i(\sigma_i', \sigma_{-i}) \\
\implies \hat{\sigma}_i &\in BR_i(\sigma_{-i})
\end{aligned}
$$

□

□

Proposition (Glicksberg)**.** *If S_i are non-empty, compact subsets of $\mathbb{R}^n$ and the u_i are continuous, then a Nash equilibrium exists.*

Proof. Use a more powerful fixed point theorem. □

5.0.1 Example: Sion Wolfe (1957)

o

$$
\begin{aligned}
\mathcal{I} &= \{1, 2\} \\
S_1 = S_2 &= [0,1] \\
u_1(s_1, s_2) &= \begin{cases} 1 \text{ if } & s_1 > s_2 \quad \text{or} \quad s_1 < s_2 - \frac{1}{2} \\ 0 \text{ if } & s_1 = s_2 \quad \text{or} \quad s_1 = s_2 - \frac{1}{2} \\ -1 \text{ if } & s_i \in \left(s_2 - \frac{1}{2}, s_2\right) \end{cases}
\end{aligned}
$$

Claim. *This game has no pure or mixed Nash equilibria.*

Proof sketch.

Claim. *Given any σ_2, there exists σ_1 with $u_1(\sigma_1, \sigma_2) \geq \frac{3}{7} - \varepsilon$ because one strategy from $\left\{0, \frac{1}{2} - \varepsilon, 1\right\}$ works well.*

Claim. *Given any σ_1, there exists a σ_2 with $u_2(\sigma_1, \sigma_2) \geq \frac{1}{3}$ because one of $\left\{\frac{1}{4}, \frac{1}{2}, 1\right\}$ works well.*

If (σ_1, σ_2) is a Nash equilibrium, then

$$u_1(\sigma_1, \sigma_2) + u_2(\sigma_1, \sigma_2) \geq \frac{3}{7} - \frac{1}{3} = \frac{2}{21} \Rightarrow\!\Leftarrow$$

This is a contradiction since our game is zero-sum. □

What do we know about equilibria in discontinuous games?

Proposition (Dasgupta-Maskin (1986))**.** *Suppose $S_i \subset \mathbb{R}$ is compact. Suppose u_i are all continuous except possibly on a single 1-dimensional subset of $\mathbb{R}^I$. Suppose $\sum_i u_i(s)$ is upper semicontinuous and each $u_i(s_i, s_{-i})$ is weakly lower semicontinuous in s_i. Then G has a mixed Nash equilibrium.*

Proposition (Reny 1999 Econometrica)**.** *In discontinuous games, gives conditions when*

1. *Pure strategy Nash equilibria exist.*
2. *Mixed strategy Nash equilibria exist.*
3. *Symmetric mixed Nash equilibria exist.*

5.1 Number of Equilibria

Let $G^1 = (S, u^1)$ and $G^2 = (S, u^2)$ where S is finite. Define

$$\|G_1 - G_2\| = \left(\sum_{i, S_i, S_{-i}} (u_i^1(s_i, s_{-i}) - u_i^2(s_i, s_{-i}))^2 \right)^{\frac{1}{2}}$$

Definition. *A property holds generically if there exists a set X of games in which the property holds and both*

1. $G \in X \implies \exists \varepsilon > 0$ *such that* $\|G' - G\| < \varepsilon \implies G' \in X$*, and*
2. $G \notin X \implies \forall \varepsilon > 0\ \exists G'$ *such that* $\|G' - G\| < \varepsilon \implies G' \in X$

Proposition. *Generically, finite normal form games have a finite and odd number of Nash equilibria.*

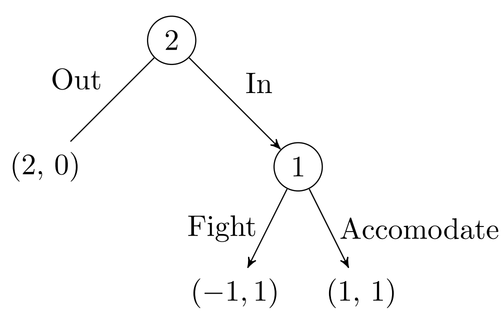
Intuition Nash equilibria are fixed points of $\sigma \mapsto r(\sigma)$. From the graphical proof of Brouwer's theorem, it's clear why there is usually a finite and odd number of fixed points.

6 November 13, 2019

6.1 Extensive Form Games

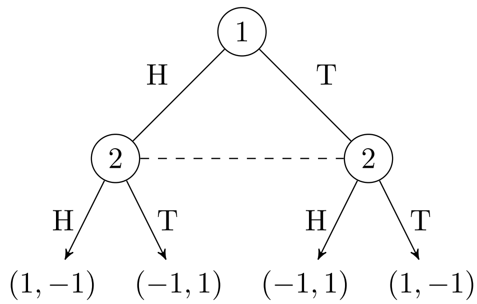
Normal form games don't have an information structure or a timing component.

6.1.1 Entry game



6.1.2 Matching Pennies

$i \in \{1, 2\}$ simultaneously choose $x_i \in \{H, T\}$. Player 1 wins if $x_i = x_j$; player 2 wins if $x_i \neq x_j$.



The dashed line in this extensive form diagram indicates that Player 2 doesn't know what Player 1 did, so he isn't sure exactly which node he's at.

Definition. *A finite extensive form game consists of*

1. *A finite set $\mathcal{I}$ of players.*
2. *A finite set T of nodes that form a tree, with terminal nodes $Z \subset T$ and several functions on the domain $T - Z$:*
   (a) *$i(t)$ denotes the player who moves; $i : T - Z \to \mathcal{I}$*
   (b) *$A(t)$ denotes the available actions*
   (c) *$n(t, a)$ is the successor node function*
3. *Payoff functions $u_i : Z \to \mathbb{R}$*
4. *Information partition $h : T - Z \rightrightarrows T - Z$ such that*
   (a) *$t \in h(t) \forall t$*
   (b) *$t' \in h(t) \implies i(t') = i(t); A(t') = A(t); h(t') = h(t)$*

6.2 Normal Form Analysis

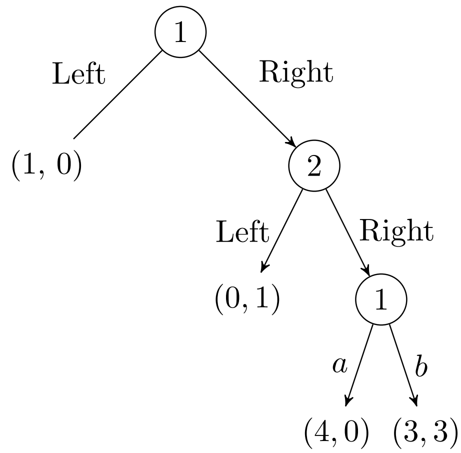
$$H_i = \{T - Z \mid S = h(t) \text{ for some } t \text{ with } i(t) = i\}$$
$$A_i = \bigcup_{i(t)=i} A(t)$$

Definition. *A pure strategy for player i is a function*

$$S : H_i \to A_i \text{ such that } S(h) \in A(h) \quad \forall h \in H_i.$$

We can use all possible $s_i \in S$ to translate from extensive form games to normal form games. However, we lose information on timing during this translation.

6.2.1 Left-Right



Player 1 has 4 pure strategies:

1. $\{L, a\}$
2. $\{L, b\}$
3. $\{R, a\}$
4. $\{R, b\}$

Let's express this as a normal form game:

	L	R
La	1, 0	1, 0
Lb	1, 0	1, 0
Ra	0, 1	4, 0
Rb	0, 1	3, 3

6.3 Nash Equilibrium as a Solution Concept

We often end up with multiple Nash equilibria in the corresponding normal form game. In the entry game, the Nash equilibria are

1. $(Fight, Out)$
2. $(Accommodate, In)$
3. $(\alpha F + (1-\alpha)A, Out)\ \ \forall \alpha \in \left[\frac{1}{2}, 1\right)$

$(Fight, Out)$ and $(\alpha F + (1-\alpha)A, Out)$ are examples of equilibria resulting from non-credible threats. This can result in some perverse outcomes we would like to avoid.

6.3.1 Stackelberg Competition

Player 1 chooses $q_1 \in [0,1]$. Player 2 sees q_1, then chooses $q_2 \in [0,1]$. Cournot payoffs

$$u_i(q_1, q_2) = q_i\,(1 - (q_i + q_j)) \quad i \neq j; i, j \in \{1, 2\}.$$

Claim. *For any $\hat{q} \in [0,1]$, there exists a Nash equilibria with $s_1^* = \hat{q}$.*

Proof. Consider

$$\begin{aligned} s_1^* &= \hat{q} \\ s_2^* &= \begin{cases} \dfrac{1-\hat{q}}{2} & \text{if } q_1 = \hat{q} \\ 1 - q_1 & \text{if } q_1 \neq \hat{q}_2 \end{cases} \end{aligned}$$

To Show that this is a Nash equilibrium, check

$$s_1^* \in BR_1(s_2^*) \text{ and } s_2^* \in BR_2(s_1^*).$$

$$\begin{aligned} u_1(\hat{q}, s_2^*) &= \hat{q}\left(1 - \left(\hat{q} + \frac{1-\hat{q}}{2}\right)\right) \\ &= \hat{q}\left(\frac{1-\hat{q}}{2}\right) \\ &\geq 0 \\ u_1(q', s_2^*) &= q'\,(1 - (q' + (1 - q'))) \\ &= q' \cdot 0 \\ &= 0 \\ u_2(s_1^*, s_2) &= s_2(\hat{q})\,(1 - (\hat{q} + s_2^*(\hat{q}))) \end{aligned}$$

This expression only depends on $s_2(\hat{q})$, and is a best response so long as

$$s_2(\hat{q}) \in \arg\max_{q_2} q_2\,(1 - (\hat{q} - q_2))$$

□

6.4 New Solution Concept: Subgame Perfect Equilibria

The idea is to play a Nash equilibrium at all nodes.

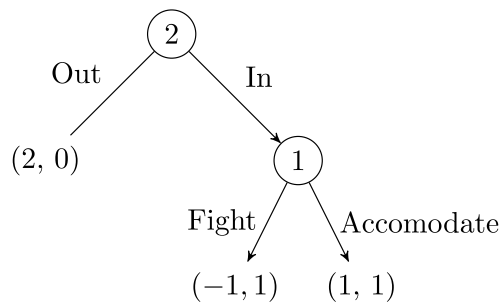
Definition. *Let G be an extensive form game. A subgame G' of G consists of*

1. *A subset T' of the nodes of T consisting of a single non-terminal node t along with all of its successors, and this subset must preserve information sets.*
2. *Functions $a, i, n, u,$ and h as in G.*

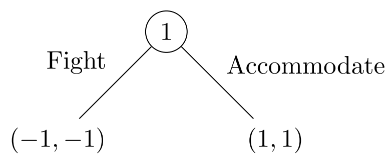
Definition. *A strategy profile s^* is a subgame perfect equilibrium of G if the restriction of s^* to G' is a Nash equilibrium of G' for all subgames G' of G.*

6.4.1 Entry game revisited

The entry game has two subgames; the entire game and the game restricted to Player 1's move.



When we look at the smaller subgame, we have:



In this subgame, $u_1(F) = -1$ and $u_1(A) = 1$). Therefore $s_1^* = A$ is the only possible subgame perfect strategy for player 1. S^* must also be a Nash equilibrium in the whole game. From earlier, (In, A) is the only Nash equilibrium with $s_1^* = A$.

6.4.2 Stackelberg Game

Claim. *The unique subgame perfect equilibrium of this game is* $s_1^* = \frac{1}{2}$, $s_2^* = \frac{1-q_1}{2} \forall q_1$.

Note that this yields $q_1 = \frac{1}{2}$; $q_2 = \frac{1}{4}$.

Proof. Every choice of $q_1 \in [0, 1]$ starts a subgame. s_2^* gives a Nash equilibrium in this subgame if and only if

$$\begin{aligned} & s_2^*(q_1) \in \arg\max_{q_2} q_2 \left(1 - (q_1 - q_2)\right) \\ \iff & s_2^*(q_1) = \frac{1 - q_1}{2} \forall q_1. \end{aligned}$$

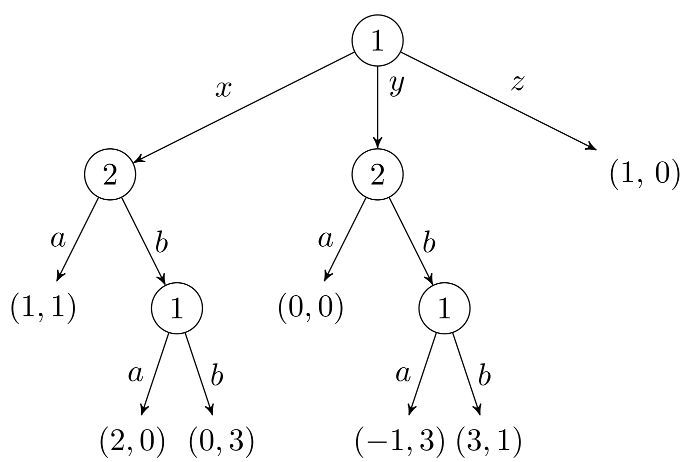
Now look at the whole game. Player 1 is playing a best response if and only if

$$
\begin{aligned}
& q_1 \in \arg\max_{q_1} u_1(q_1, s_2^*(q_1)) \\
\iff & q_1 \in \arg\max_{q_1} q_1(1 - (q_1 + s_2^*(q_1)) \\
\iff & q_1 \in \arg\max_{q_1} q_1 \left(1 - \left(q_1 + \frac{1 - q_1}{q_2}\right)\right) \\
& \in \arg\max_{q_1} q_1 \left(\frac{1 - q_1}{2}\right)
\end{aligned}
$$

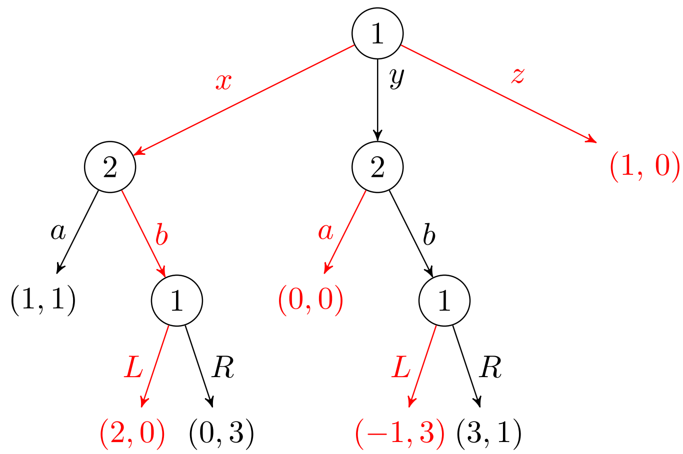
□

7 November 18, 2019

7.1 Backwards Induction



With backward induction, we want to start from the leaves and eliminate irrational play. In this case, we end up with the unique subgame perfect equilibrium (y, b, R). In the diagram below, the play that we rule out is highlighted in red.



Basic algorithm. Start at the bottom and work up.

Definition (Perfect Information)**.** *A game has perfect information if*

$$h(t) = \{t\}\ \forall t \in T - Z.$$

Proposition. *Any finite game with perfect information has a pure strategy subgame perfect equilibrium. Fixing everything about the game except the payoffs, the subgame perfect equilibrium is generically unique.*

Proof Sketch We want to show that a subgame perfect equilibrium always exists and is unique provided

$$u_i(z) \neq u_i(z')\ \forall i, z, z' \in Z.$$

We will prove this by induction on the number of stages of the game.

Let $T_1 = \{t_1\}$ where t_1 is the initial node of the game tree.

$$T_2 = \{t \in T - Z \mid \exists a \text{ with } n(t_1, a) = t\}$$
$$\vdots$$
$$T_{k+1} = \{t \in T - Z \mid \exists a, t' \text{ with } n(t'a) = t \wedge t' \in T_k\}$$

Define the number of stages

$$K(G) = \min\{k \mid T_1 \cup T_2 \cup \ldots \cup T_k = T - Z\}$$

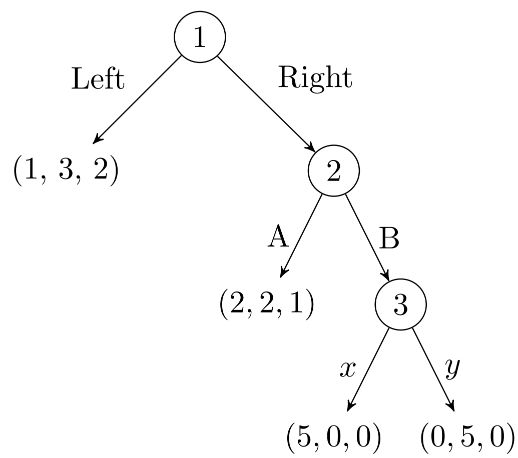
Suppose G has one stage. Then

$$s^* \text{ is an SPE} \iff s^*_{i(t_1)} \in \arg\max_{a \in A(t_1)} u_i(t_1)(n(t_1, a))$$

This is a max over a finite set, so the maximum exists. The max is unique so long as payoffs are different (which happens with probability 1 generically). To finish the proof, we need to show that $k + 1$ stage games have an equilibrium if k stage games have an equilibrium. We can prove this via backward induction; simply solve the leaf and replace the leaves of the new truncated game, G', with the payoffs from rational play at the original leaves of G.

7.2 Non-uniqueness

7.2.1 Uncommon Case



To find the equilibria in this game, we need to try both $3 = x$ and $3 = y$. This gives us two unique subgame perfect equilibria:

1. $(R, A, x$ is a subgame perfect equilibrium.
2. (L, B, y) is a subgame perfect equilibrium.

We also have a number of mixed strategy SPE:

1. $(L, B, \alpha x + (1 - \alpha)y) \quad \alpha \in [0, 1/5]$
2. $(R, B, \alpha x + (1 - \alpha)y) \quad \alpha \in [1/5, 3/5]$
3. $(R, A, \alpha x + (1 - \alpha)y) \quad \alpha \in [3/5, 1]$

The other players can start mixing at 1/5 and 3/5 because they are indifferent.

7.2.2 More common case

	L	M	R
T	3, 1	0, 0	5, 0
M	2, 1	1, 2	3, 1
B	1, 2	0, 1	4, 4

This leads to three equilibria:

$$(T, L) = (3, 1)$$
$$(M, C) = (1, 2)$$
$$\left(\frac{1}{2}T + \frac{1}{2}M, \frac{1}{2}L + \frac{1}{2}C\right) = \begin{pmatrix}\frac{3}{2}\end{pmatrix}$$

Now imagine that the two players play this game, twice. Is there a subgame perfect equilibrium where (B, R) is played at $t = 1$? Yes, if Player 2 can make a credible threat: Play (B, R) at $t = 1$. If Player 1 plays B at $t = 1$, then play (L, T) at $t = 2$. If Player 1 plays T or M at $t = 1$, play (M, C) at $t = 2$.

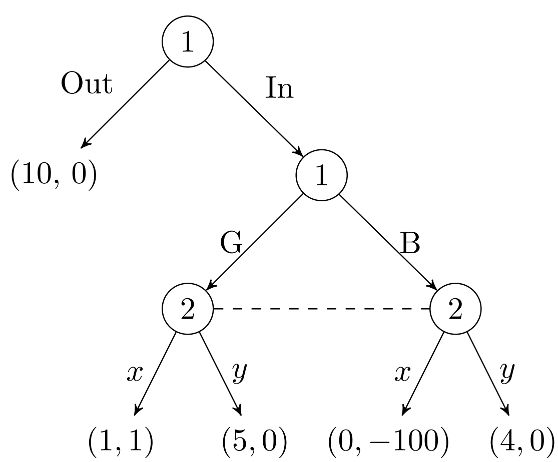
To show this is Nash, check the subgames. For $t = 2$ subgames, we already know that (T, L) and (M, C) are Nash. Now check the full game.

$$u_1(s_1^*, s_2^*) = 4 + 3 = 7$$
$$u_1(s_1', s_2^*) \leq \overbrace{5 + 1}^{\text{deviate } t=1} < 7$$
$$\leq \underbrace{4 + 2}_{\text{deviate } t=2} < 7$$

$u_1(s_1', s_2^*) < u_1(s_1^*, s_2^*)$, so s_1^* is a best response. We can also show that $u_2(s_1^*, s_2') \leq u_2(s_1^*, s_2^*)$.

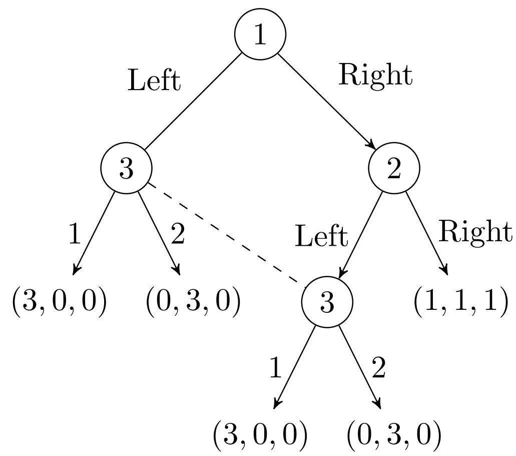
7.3 Critique of Subgame Perfect Equilibrium

1. We might *not* want to impose off-path rationality.



If Player 2 gets a chance to play, G, x is the unique Nash outcome. But, in order for Player 2 to get a chance to play, he must know that Player 1 has done something irrational—so why do we want to impose cascading rationality at places that can't be reached if rationality holds?

2. Subgame perfect equilibria assumes that player know off-path actions.



In this game, there is no equilibrium with (R, R). Consider the strategy

$$(R, R, \alpha(1) + (1-\alpha)(2))$$

This isn't a subgame perfect equilibrium if $\alpha > \frac{1}{3}$ because Player 1 will deviate. This also isn't a subgame perfect equilibrium if $\alpha < \frac{2}{3}$ because Player 2 will deviate.

SCE. Drew's notion of "Self-confirming equilibrium" allows (R, R): player 1 thinks $\alpha < 1/3$; player 2 thinks $\alpha > 1/3$, but neither ever sees any evidence to contradict his beliefs.

3. Games with many stages require a great deal of faith in common knowledge of rationality.

8 November 20, 2019

8.1 Infinite Games

I. At $t = 0, 2, 4, \ldots$ Player 1 offers $(x_i, 1 - x_i)$. Player 2 says yes or no. Yes ends the game.

II. If Player 2 says no, player 2 offers $(1 - x_2, x_2)$. Player 1 says yes or no.

III. The game continues until some player says yes.

Payoffs. If (x, y) is agreed to at time t, payoffs are $\delta^t x$ and $\delta^t y$.

In this case, we can't use backwards induction. Instead, we use a two-step procedure to find an equilibrium.

1. Conjecture a subgame perfect equilibrium. There are three ways to do this:
   (a) Think about credible threats.
   (b) Focus on payoff space.
   (c) Backwards induction on a finite version of the infinite horizon game.
2. Check that the conjecture is correct.

8.2 Checking equilibria in infinite horizon games

Throughout this section, we will focus on a class of games defined by Fudenberg and Tirole: Infinite horizon games with observable actions.

Definition (Infinite horizon games with observed actions)**.** • *At times $t = 0, 1, 2, \ldots$ some subset of players choose actions.*

- *Players observe period t actions before they choose period $t + 1$ actions.*
  - *This keeps the information set simple.*
- *Payoffs are*
$$u_i : A^0 \times A^1 \times \ldots \times A^t \times \ldots \to \mathbb{R}.$$
  - *Often these payoffs take the form*
$$u_i(a^0, a^1, \ldots) = \sum_{t=0}^{\infty} \delta^t u_i(a^t)$$

Definition (Continuity at infinity)**.** *A game G is continuous at infinity if*

$$\lim_{t\to\infty} \underbrace{\sup}_{i,\sigma,\sigma' \text{ s.t. } \sigma(h^t)=\sigma'(h^t)\forall h\in t\leq T} |u_i(\sigma) - u_i(\sigma')| = 0.$$

In words, if two strategies are the same for the first T periods, then their payoffs become arbitrarily close together (e.g. because they are very highly discounted after period T).

8.2.1 Undiscounted War of Attrition

The following game is not continuous at infinity.

$$a_i^t \in \{\text{fight}, \text{out}\} \text{ at } t = 0, 1, 2, \ldots$$

$$\text{If } a^t = \begin{matrix} \{f, f\} & \text{game continues} \\ \{Q, f\} & \text{game ends} \end{matrix}$$

$$u_i(a^0, a^1, \ldots) = \begin{cases} 1 - ct & \text{if } i \text{ wins at } t \\ -ct & \text{if } i \text{ loses at } t \end{cases}$$

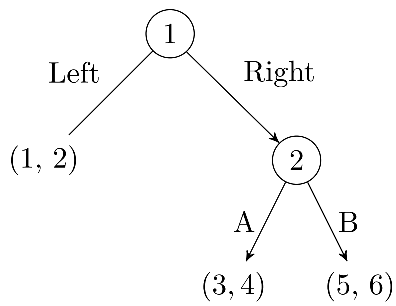
The game is discontinuous at infinity because the winning player always has utility one unit higher than the losing player, so $\limsup \neq 0$.

Definition (Conditional Payoffs)**.** *Define*

$$u(\sigma_1, \sigma_2 \mid x_t) \equiv u(\tilde{\sigma}_1, \tilde{\sigma}_2),$$

where $(\tilde{\sigma}_1, \tilde{\sigma}_2)$ *are strategies that initially play so that* x_t *is reached, then play* (σ_1, σ_2)*.*

Example.



$$\begin{aligned} u_1(L, A) &= 1 \\ u_1(L, A \mid y) &= 3 \\ u_1(L, A \mid y) &\equiv u_1(R, A) \end{aligned}$$

This last equivalence is true because we need to play R to get to the second player's action—then we follow the prescribed strategy, which has an action for this node because strategies are complete contingent plans.

Note.

$$(\sigma_i^*, \sigma_{-i}^*) \text{ is a SPE } \iff u_i(\sigma_i^*, \sigma_{-i}^* \mid x_t) \geq u_i(\sigma_i', \sigma_{-i}^* \mid x_t)\, \forall i, \sigma_i', \text{ and } x_t \text{ that start a subgame}$$

Theorem. *Suppose G is continuous at infinity. Then, σ^* is a subgame perfect equilibrium of G if and only if there does not exist a strategy profile $\hat{sigma}_i$ that differs from σ_i^* only in the play of player i at a single information set h_i^t with*

$$u_i(\hat{\sigma}_i \sigma_{-i}^* \mid h_i^t) > u_i(\sigma_i^*, \sigma_{-i}^* \mid h_i^t).$$

In words, we can check the subgame perfect condition one information set at a time instead of checking complication deviations that involve changing play at multiple nodes simultaneously.

8.2.2 Finding and proving equilibria in the bargaining game

Claim. *The bargaining game has a unique subgame perfect equilibrium. In each period, the proposer (i) chooses* $x_i = \frac{1}{1+\delta}$ *and the responder says*

$$\begin{aligned} \textit{yes if } 1 - x_i &\geq \frac{\delta}{1+\delta} \\ \textit{no if } 1 - x_i &< \frac{\delta}{1+\delta}. \end{aligned}$$

Existence. Case 1—proposal node. Consider h_i^t where i is proposing. Call our claimed strategy s^*. Following s^*,

$$u_i(s_i^*, s_{-i}^* \mid h_i^t) = \delta^t \frac{1}{1+\delta}.$$

If i deviates to an s_i' that asks for a larger share at h_i^t,

$$u_i(s_i', s_{-i}^* \mid h_i^t) = \delta^{t+1} \frac{\delta}{1+\delta} < \frac{\delta^t}{1+\delta}.$$

We compute the payoff $\delta^{t+1}\frac{\delta}{1+\delta}$ by applying our theorem: since we only need to consider deviations at one node, we can assume that our players follow the conjectured strategy at the next node, which leads to a more-discounted version of the payoff we already computed.

If instead i deviates to a strategy s_i' that asks for a smaller share at h_i^t, our player gets a smaller share—obviously this is not optimal.

Case 2—acceptance node. Assume that Player i offered z. What are the conditional payoffs to *yes* and *no*?

$$\begin{aligned} \textit{yes} &\mapsto \delta^t z \\ \textit{no} \text{ and follow } s^* \text{ in future periods } &\mapsto \delta^{t+1} \frac{1}{1+\delta} \end{aligned}$$

If $z \geq \frac{\delta}{1+\delta}$, "yes" is *a* best response. If $z < \frac{\delta}{1+\delta}$, "no" is *the* best response. In both cases, following s_i^* is at least as good as deviating. □

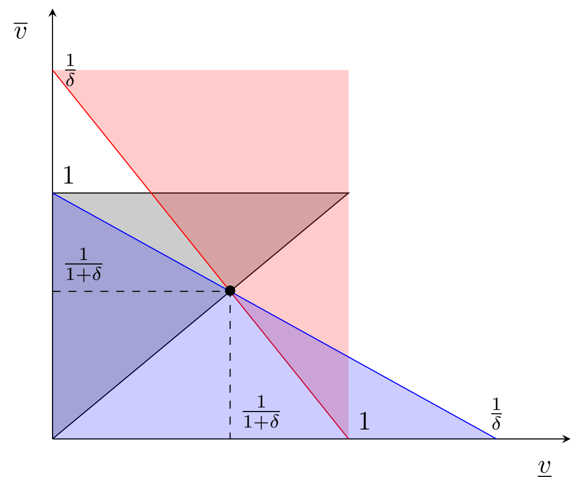
Uniqueness. We will prove uniqueness by focusing on the (very small) payoff space. Let $\underline{v}$ and $\overline{v}$ be the lowest and highest equilibrium payoffs for a proposer. This gives us three conditions on the payoffs.

1. $0 \leq \underline{v} \leq \overline{v} \leq 1$

2. $$\begin{aligned} 1 - \overline{v} &\geq \delta \underline{v} && \text{(or else the responder says no)} \\ \equiv \delta \underline{v} + \overline{v} &\leq 1 \end{aligned}$$

3. $$\underline{v} \geq 1 - \delta \overline{v} \qquad \text{(because responder must accept } \delta\overline{v} + \varepsilon\text{)}$$

We can solve these three inequalities algebraically, but it is more instructive to look at a picture in payoff space.



The gray shaded region comes from our first condition and bounds the payoff space. The red shaded region comes from the condition $\delta\overline{v} + \underline{v} \geq 1$. The blue shaded region comes from $\delta\underline{v} + \overline{v} \leq 1$. These three conditions give a unique intersection, the black dot, which gives the unique payoffs for this game. More formally, conditions 1–3 imply that the equilibrium payoffs are $\frac{1}{1+\delta}$ and $\frac{\delta}{1+\delta}$. $\square$

Note. This bargaining result does not hold when there are three or more players, or if there are simultaneous moves, etc.

9 November 25, 2019

9.1 Repeated Games

Definition (Average payoff)**.** *Let $G = (A, g)$ be a normal form game. Let G^∞ be the infinite horizon extensive form game where*

- *At $t = 0, 1, 2, \ldots$ players play G.*
- *Player observe period t play before choosing period $t + 1$ actions.*
- *Payoffs are*
$$u_i(s_i, s_{-i}) = \sum_{t=0}^{\infty} \delta^t g_i(a_i^t, a_{-i}^t).$$
- *Call $(1 - \delta) \cdot u_i(s_i, s_{-i})$ the average payoff.*

9.1.1 Repeated Prisoner's Dilemma

	C	D
C	1, 1	-1, 2
D	2, -1	0, 0

If the two players play this game 100 times, backwards induction yields the equilibrium (D, D) for each game.

Claim. *In the infinitely repeated Prisoner's Dilemma, if $\delta \geq 1/2$ there is a subgame perfect equilibrium in which the players player (C, C) every period on the equilibrium path.*

Proof. Let

$$s_i^*(h^t) = \{ \; C \text{ if } (C, C) \text{ has been played in every period} D \text{ otherwise}$$

We want to show that this strategy has the "no single deviation" property.

Case 1. Consider a history h^t with $s_i^*(h^t) = D$. Let $\pi_i(h^t)$ be the payoff i has received so far. The payoffs from the two possible actions in this period are

$$\begin{aligned} s_i^* &\mapsto \pi_i(h^t) + \delta^t \cdot 0 + \delta^{t+1} \cdot 0 + \ldots \\ C, \text{ then } s_i^* &\mapsto \pi_i(h^t) + \underbrace{\delta^t \cdot -1}_{\text{not profitable}} + \delta^{t+1} \cdot 0 + \delta^{t+2} \cdot 0 + \ldots \end{aligned}$$

□

Case 2. Consider a history h^t with $s_i^*(h^t) = C$.

$$\begin{aligned} s_i^* &\mapsto \overbrace{1 + \delta + \ldots + \delta^{t+1}}^{\pi_i(h^t)} + \delta^t \cdot 1 + \delta^{t+1} \cdot 1 + \ldots \\ D, \text{ then } s_i^* &\mapsto \underbrace{1 + \delta + \ldots + \delta^{t+1}}_{\pi_i(h^t)} + \delta^2 \cdot 2 + \delta^{t+1} \cdot 0 + \delta^{t+2} \cdot 0 + \ldots \end{aligned}$$

This deviation is not profitable if

$$\begin{aligned} \delta^t + \delta^{t+1} + \ldots &\geq 2\delta^t \\ 1 + \delta + \delta^2 + \ldots &\geq 2 \\ \frac{1}{1-\delta} &\geq 2 \\ \iff \delta &\geq \frac{1}{2} \end{aligned}$$

$\therefore$ For $\delta \geq \frac{1}{2}$, s_i^* is an SPE.

□

□

Definition (Feasible payoffs)**.** *A payoff vector $v \in \mathbb{R}^I$ is feasible in G if there exist action profiles $a^1, a^2, \ldots, a^k$ and weights $w^1, w^2, \ldots, w^k$ such that $w_j \geq 0 \forall j$ and $\sum w_j = 1$ with*

$$v_i = \sum_{j=1}^{K} w^j g_i(a^j) \forall i.$$

Definition (Strictly individually rational)**.** *v_i is strictly individually rational if*

$$v_i \geq \underline{v}_i \equiv \min_{\sigma_{-i} \in \sigma_{-i}} \max_{\sigma_i(\sigma_{-i}) \in \Sigma_i} g_i(\sigma_i, \sigma_{-i}).$$

Proposition. *If v is the average payoff in a Nash equilibrium of G^∞, then v is feasible and weakly individually rational.*

Proof. Feasibility is obvious; v is a weighted average of payoffs $g_i(a_i, a_{-i})$. Suppose i deviates to

$$\hat{s}(h^t) = \arg\max_{a_i \in A_i} g_i(a_i, s^*_{-i}(h^t)).$$

$$u_i(\hat{s}_i, s^*_{-i}) \geq \underline{v}_i + \delta \underline{v}_i + \delta^2 \underline{v}_i + \ldots$$

$$= \frac{1}{1-\delta} \underline{v}_i.$$

The no profitable deviation condition implies

$$u_i(s^*_i, s^*_{-i}) \geq \frac{\underline{v}_i}{1-\delta}$$

□

Theorem (Folk Theorem)**.** *Suppose the feasible, strictly individually rational payoff set is I-dimensional. Then, for any v that is feasible and strictly individually rational, there exists a $\underline{\delta} < 1$ such that $\forall \delta \in [\underline{\delta}, 1)$ there exists a subgame perfect equilibrium σ^* of $G^\infty(\delta)$ with average payoffs v.*

Proof sketch. Directly construct strategies:

1. Pick a sequence $a^1, a^2, a^3, \ldots$ with average payoff v and have players follow it as long as nobody deviates.

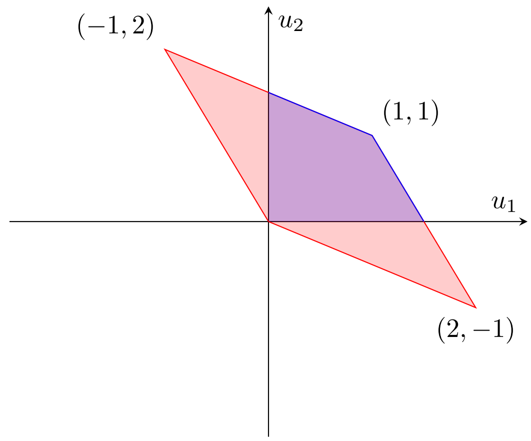
2. If player i deviates at time t, punish i for T periods by playing σ_{-i} that holds i to $\underline{v}_i$.

3. After T periods, reward everyone who carried out the punishment.

□

How can we get away from the Folk Theorem and actually make predictions?

9.1.2 Picture of Feasible and rational payoff sets



In this picture, the red shaded area depicts the feasible payoff et, and the blue area depicts the strictly individually rational set. Note that neither axis is included in the strictly individually rational set.

9.2 Markov Equilibrium

Markov equilibria are an applied approach to getting “normal” play; it rules out weird punishment strategies like the one we used to prove the Folk Theorem.

9.2.1 Common Pool Fishing

Two fishermen choose $x_i^* \in [0, 1]$ at $t = 0, 1, 2, \ldots$

$$\begin{aligned} q_i^t &= \begin{cases} x_1^t \cdot \delta^t & \text{if } x_1^t + x_2^t \leq 1 \\ \frac{x_i^t}{x_1^t + x_{-i}^t} & \text{if } x_1^t + x_2^t \geq 1 \end{cases} \\ S^{t+1} &= (1 + r^t)(S^t - (q_1^t + q_2^t)) \\ \pi_i^t &= P(q_1^t + q^t - 2) \cdot q_i^t \end{aligned}$$

9.2.2 Savings for hyperbolic consumers

$$u_i = v(c^t) + \beta \sum_{s=1}^{\infty} \delta^2 v(c^{t+2}); \quad \beta < 1$$

We can model this as a game between period $0, 1, 2, \ldots$ selves.

Definition (Markov equivalent). *Let G be a multi-stage game with observed actions; $h^t \leftrightarrow (a^0, a^1, \ldots, a^{t-1}, i)$. Period t histories are Markov equivalent if $\forall \{\alpha^t\}$ and $\{\beta^t\}$,*

$$u_i(h_i^t \frown \{\alpha^t\} \mid h_i^t) \geq u_i(h_i^t \frown \{\beta^t\} \mid h_i^t) \iff u_i(h_i^{t'} \frown \{\alpha^t\} \mid h_i^{t'}) \geq u_i(h_i^{t'} \frown \{\beta^t\} \mid h_i^{t'}).$$

For example, two states are Markov equivalent if they have the same number of fish; the idea is that the future looks identical from both information sets.

Definition (Markov)**.** *A strategy s_i is Markov if $s_i(h^t) = s_i(h_i^{t'})$ are Markov equivalent.*

Definition (Markov Perfect Equilibrium)**.** *A strategy profile s^* is a Markov Perfect Equilibrium (MPE) if s^* is a subgame perfect equilibrium and all players use Markov strategies.*

10 November 27, 2019

10.0.1 Fishing game

Same setup as last class; each period, each player chooses an amount of effort to expend catching fish.

$$
\begin{aligned}
q_{it} &= \begin{cases} x_{it}A_t & \text{if } x_1 + x_2 \le 1 \\ \frac{x_i}{x_{it}+x_{-it}} & \text{if } x_1 + x_2 > 1 \end{cases} \\
Q_t &= q_{1t} + q_{2t} \\
P(Q_t) &= 300 - Q_t \\
A_{t+1} &= 2(A_t - Q_t)
\end{aligned}
$$

A non-Markov strategy can be written as

$$S_i(x_{1,0}, x_{2,0}, \dots, x_{1,t-1}, x_{2,t-1}).$$

Markov strategies can only condition on payoff-relevant outcomes, so they must take arguments from a much smaller set:

$$S_i(A_t).$$

Unfortunately, in our fishing example the Markov restriction does *not* lead to a unique equilibrium. For example, for δ large enough one Markov-perfect equilibrium is

$$s_i^* = \begin{cases} \frac{75}{300} & \text{if } A_t = 300 \\ 1 & \text{if } A_t \neq 300. \end{cases}$$

Another Markov-perfect equilibrium (for δ large enough) is

$$s_i^* = \begin{cases} \frac{90}{300} & \text{if } A_t = 300 \\ \frac{60}{240} & \text{if } A_t = 240 \\ 1 & \text{if } A_t \notin \{240, 300\}. \end{cases}$$

To see this is a subgame perfect equilibrium, show there are no profitable single-history deviations. We need to check two cases.

Histories with $A_t = 300$.

$$
\begin{aligned}
s_i^* &\mapsto \underbrace{90 \cdot 120}_{10{,}800} + \delta \underbrace{60 \cdot 180}_{10{,}800} + \delta^2 \cdot 10,800 + \dots \\
\text{Deviate to } 60 &\mapsto \underbrace{60, 150}_{9{,}000} + \delta 10,800 + \delta^2 10,800 + \dots
\end{aligned}
$$

The last deviation to check is a deviation to the static best response to 90, which happens to be 105. We'll check this with a generous upper bound.

$$
\leq 105 \cdot 105 + \delta \underbrace{\frac{1}{2} 75 \cdot 150}_{\text{upper bound}} + \delta^2 \cdot 0 + \delta^3 \cdot 0 + \dots
$$

This will give us a δ for which deviations are unprofitable. □

Proof. Cases with $A_t = 240$ Repeat steps above, same result □

Upshot: Markov doesn't help when there is off-path play.

10.0.2 Consumption Model

Consumer chooses $x_0, x_1, x_2, \dots$

$$
\begin{aligned}
u_1(x_0, \dots, x_t, \dots) &= \log(c_0) + \beta \sum_{t=1}^{\infty} \delta^t \log(c_t) \\
c_0 &= x_0 A_0 \\
c_t &= \overbrace{\underbrace{A_0(1 - x_0)(1 + r)}_{A_1}(1 - x_1)(1 + r)}^{A_2} \dots (1 - x_{t-1})(1 + r) \cdot x_t \\
&= A_0 \left(\prod_{s<t} (1 - x_s) \right) (1 + r)^t x_t
\end{aligned}
$$

We can rewrite this in logs.

$$
u_0 = \log(A_0) + \log(x_0) + \beta \sum_{t=1}^{\infty} \delta^t \left[\log(A_0) + t \log(1 + r) + \sum_{s<t} \log(1 - x_s) + \log x_t \right]
$$

All h_t are Markov equivalent because of log utility.

$$
\begin{aligned}
x^* &\in \arg\max_x u_{i0}(x, x^*, x^*, \ldots) \\
x^* &\in \arg\max_x \log(x) + \beta \sum_{t=1}^{\infty} \delta^t \log(1-x) + \underbrace{\text{constants, etc.}}_{\text{no } x} \\
&= \arg\max \log(x) + \frac{\beta\delta}{1-\delta} \log(1-x) \\
0 &= \frac{1}{x^*} + \frac{\beta\delta}{1-\delta} \cdot \frac{-1}{1-x^*} \qquad \text{(FOC)} \\
x^* &= \frac{1}{1+\frac{\beta}{\delta}1-\delta} \\
&= \frac{1-\delta}{1-\delta+\beta\delta}
\end{aligned}
$$

10.1 Games of Incomplete Information

10.1.1 Public Good Game

	Call	Don't
Call	$1-c_1, 1-c_2$	$1-c_1, 1$
Don't	$1, 1-c_2$	$0, 0$

Example 1

$$
\begin{aligned}
&c_1 \sim U[0,2] \\
&c_2 \sim U[0,2] \\
&c_i \text{ known only to } i
\end{aligned}
$$

Example 2

$$
\begin{aligned}
&c_1 = \frac{1}{3}; \text{ known to both} \\
&c_2 \in \{\underline{c}, \overline{c}\} \text{ known only to 2}
\end{aligned}
$$

Player 1 has prior $\Pr(c = \underline{c}) = p'$.

Example 3

$$
1 \text{ knows } c_1 = \frac{1}{3}, c_2 = \frac{2}{3}
$$
$$
2 \text{ knows } c_1, c_2 \in [0,2] \times [0,2] \text{ with prior on } [0,2] \times [0,2]
$$

Definition (Game of Incomplete Information)**.** *A game with incomplete information*

$$
G = (\Theta, S, p, u)
$$

consists of

1. *A set* $\Theta = \Theta_1 \times \ldots \times \Theta_I$ *where* Θ_i *is the set of possible states of knowledge of* i.

2. $S = S_1 \times \ldots \times S_I$ *gives possible strategies (type independent).*

3. *A probability distribution* p *on the set* Θ.

4. *Payoff functions* $u_i : S \times \Theta \to \mathbb{R}$.

From earlier…

Example 1

$$\begin{aligned} \Theta_1 &= [0,2] \\ \Theta_2 &= [0,2] \\ p &\text{ uniform on } [0,2] \times [0,2] \end{aligned}$$

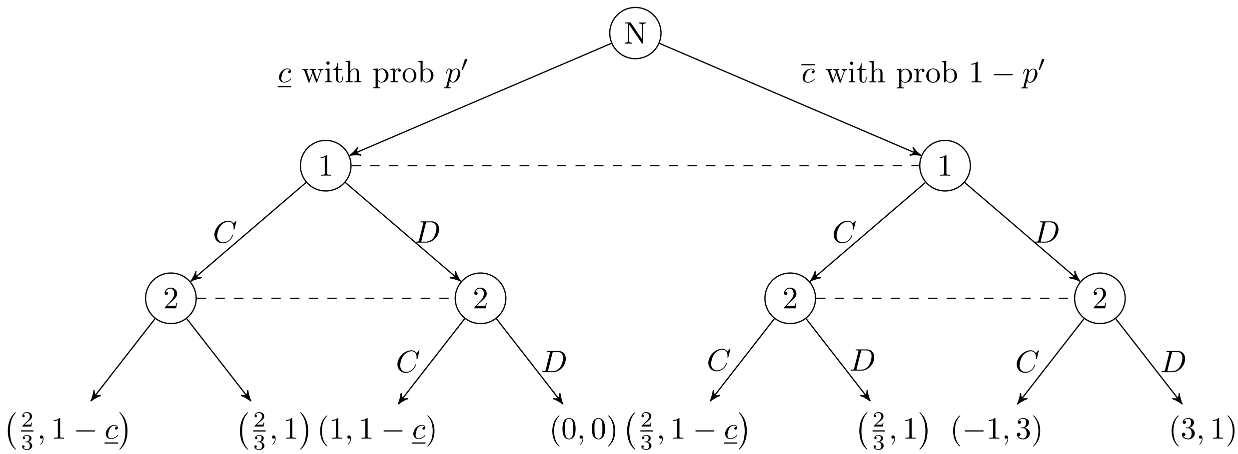
Example 2

$$\begin{aligned} \Theta_1 &= \left\{ \frac{1}{3} \right\} \\ \Theta_2 &= \{\underline{c}, \overline{c}\} \\ p\left(\frac{1}{3}, \underline{c}\right) &= p' \\ p\left(\frac{1}{3}, \overline{c}\right) &= 1 - p' \end{aligned}$$

Example 3

$$\begin{aligned} \Theta_1 &= [0,2] \times [0,2] \\ \Theta_2 &= \{\theta_1\} \end{aligned}$$

10.2 Harsanyi Representation



The Harsanyi representation adds an initial move by Nature. Nature chooses Θ from p and tells player i $\theta_i \in \Theta_i$. The Harsanyi representation gives us a new solution concept: Bayesian Nash Equilibrium (BNE). Bayesian Nash equilibrium is the same as Nash equilibrium in the Harsanyi game.

Definition (Bayesian Strategy). *A Bayesian strategy for player i in game G is*

$$f_i : \Theta_i \to S_i.$$

Let $S_i^{\theta_i}$ be the set of possible Bayesian strategies.

Definition (Bayesian Nash equilibrium). *A profile $(f_1^*, \dots, f_N^*)$ is a Bayesian Nash equilibrium if and only if*

$$f_i^* \in \arg\max_{f_i \in S_i^{\Theta_i}} \sum_{\theta_i, \theta_{-i}} u_i(f_i(\theta_i), f_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) p(\theta_i, \theta_{-i}) \quad \forall i.$$

11 December 2, 2019

11.0.1 Advisor Call/Don't Call

$$\begin{aligned} c_1 &= \frac{1}{3}, \text{ common knowledge} \\ c_2 &= \begin{cases} \underline{c} & \text{with probability } p \\ \bar{c} & \text{with probability } 1-p \end{cases} \\ 0 < \underline{c} &< 1 < \bar{c} \\ p &< \frac{1}{2} \end{aligned}$$

This game has discrete Θ and discrete A; we will make these continuous later in this class and see how the problems become more challenging.

Claim. *The unique Bayesian Nash equilibrium is*

$$\begin{aligned} f_1^* = &\ \textit{call} \\ f_2^* = &\ \textit{don't call}\ \forall c. \end{aligned}$$

Proof via Iterated Conditional Dominance. For $\overline{c}$-type player 2, calling is strictly dominated.

$$\begin{aligned} \mathbb{E}u_2(f_1^*,\ \text{call}; c_1, \overline{c}) &= 1 - \overline{c} \\ &< 0 \\ \mathbb{E}u_x(f_1^*,\ \text{don't}; c_1, \overline{c}) &\geq 0 \\ \implies f_2^* = &\ \text{don't call in any BNE.} \end{aligned}$$

Claim.

$$f_2^*(\overline{c}) = D \implies f_1^* = \ \textit{call}$$

Proof.

$$\begin{aligned} \mathbb{E}u_1\left(\text{call}, f_2^*; c_1, c_2\right) &= \frac{2}{3} \\ \mathbb{E}u_1\left(\text{don't}, f_2^*; c_1, c_2\right) &= p \cdot u_1\left(\text{don't}, f_2^*; c_1, \underline{c}\right) + \underbrace{(1-p)u_1\left(\text{don't}, f_2^*; c_1, \overline{c}\right)}_{0} \\ \therefore &\leq p \end{aligned}$$

$$\begin{aligned} p < \frac{1}{2} \implies \mathbb{E}u_1(\text{call}, \bullet) &> \mathbb{E}u_1(\text{don't}, \bullet) \\ \implies &\quad \text{call is a BR for player 1} \end{aligned}$$

□

Claim.

$$f_1^* = \ \textit{call} \implies f_2^*(\underline{c}) = \ \textit{don't}$$

Proof.

$$\begin{aligned} u_2(f_1^*, \text{call}; c_1, \underline{c}) &= 1 - \underline{c} \\ u_2(f_1^*, \text{don't}; c_1, \underline{c}) &= 1 \end{aligned}$$

□

□

11.0.2 Advisor Call/Don't Call [2]

Suppose

$$\begin{aligned} c_1 &\sim U[0,2], \text{ known to 1} \\ c_2 &\sim U[0,2], \text{ known to 2} \end{aligned}$$

The unique Bayesian Nash equilibrium is

$$\begin{aligned} f_1^*(c_1) &= \begin{cases} \text{call} & \text{if } c_1 \leq \frac{2}{3} \\ \text{don't} & \text{if } c_1 > \frac{2}{3} \end{cases} \\ f_2^*(c_2) &= \begin{cases} \text{call} & \text{if } c_2 \leq \frac{2}{3} \\ \text{don't} & \text{if } c_2 > \frac{2}{3} \end{cases} \end{aligned}$$

In this example Θ is continuous and A is discrete.

Proof. To show this is a Bayesian Nash equilibrium, note that if Player 2 follows f_2^*, from Player 1's perspective Player 2 calls $\frac{1}{3}$ of the time.

$$\begin{aligned} \mathbb{E}_{c_1} u_1(\text{call}, f_2^*; c_1, c_2) &= u_1\left(\text{call}, \frac{1}{3}C + \frac{2}{3}D, c_1\right) \\ \mathbb{E}_{c_2} u_2(\text{call}, f_1^*; c_1, c_2) &= u_2\left(\text{call}, \frac{1}{3}C + \frac{2}{3}D, c_1\right) \\ u_i(\text{call}) &= 1 - c \\ u_i(\text{don't}) &= \frac{1}{3} \\ \text{Calling is a BR} \iff c_1 &\leq \frac{2}{3} \end{aligned}$$

□

To find BNE/Show it is unique...

Observation. If f^* is a Bayesian Nash equilibrium and $f_1^*(\bar{c}_i) = \mathit{call}$, then $f_1^*(c') = \mathit{call}\ \forall c' < \bar{c}_1$.

Proof.

$$\begin{aligned} \mathbb{E}_{u_1}(D, f_2^*; \bar{c}_1, c_2) &= \mathbb{E}_{u_1}(D, f_2^*; c_1', c_2) \\ \mathbb{E}_{u_1}(C, f_2^*; \bar{c}_1, c_2) &< \mathbb{E}_{u_1}(C, f_2^*; c_1', c_2) \forall c_1' < \bar{c}_1 \end{aligned}$$

In equilibrium, $f_1^* = \mathit{call}$ implies

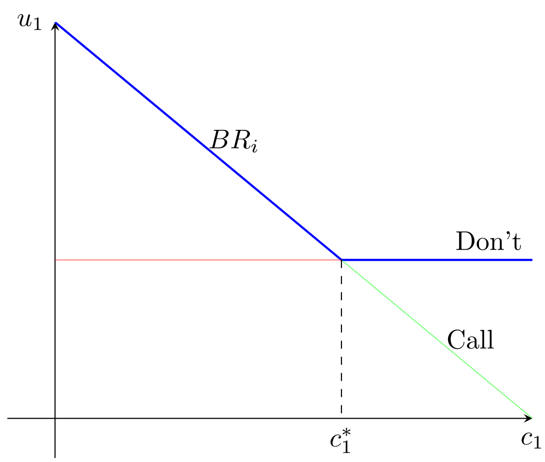
$$\begin{aligned} \mathbb{E}_{u_1}(C, f_2^*; \bar{c}_1, c_2) &\geq \mathbb{E}_{u_1}(D, f_2^*; \bar{c}_1, c_2), \text{ and} \\ \mathbb{E}_{u_1}(C, f_2^*; \bar{c}_1, c_2) &< \mathbb{E}_{u_1}(C, f_2^*; \bar{c}_1, c_2)\ \forall c_1' < \bar{c}_1 \\ \implies f_i^*(c_i) &= \begin{cases} \text{call} & \text{if } c_i \leq c_i^* \\ \text{don't} & \text{if } c_i > c_i^* \end{cases} \end{aligned}$$

Now find c_1^* and c_2^*. Suppose $c_1^* \in (0,1)$ and $c_w^* \in (0,1)$. Note also that $\mathbb{E}_{u_1}$ is continuous in c_1. We can now set up a system of two equations in two unknowns.

$$\mathbb{E}_{u_1}(C, f_2^*; c_1, c_2) = \mathbb{E}_{u_1}(D, f_2^*; c_1, c_2) \text{ at } c^* \tag{1}$$
$$\mathbb{E}_{u_2}(C, f_1^*; c_1, c_2) = \mathbb{E}_{u_2}(D, f_1^*; c_1, c_2) \text{ at } c^* \tag{2}$$

$$\begin{aligned}(1) \text{ and } (2) \implies 1 - c_1^* &= \frac{c_2^*}{2}\\ 1 - c_2^* &= \frac{c_1^*}{2}\\ \therefore c_1^* = c_2^* &= \frac{2}{3}\end{aligned}$$

□



Here the red line shows the payoffs to playing "Don't" and the green line shows the payoffs to playing "Call". At c_1^* the payoffs to both actions are identical; for values of $c_1 < c_1^*$ the best response is to play "Call"; for $c_1 > c_1^*$ the best response is to play "Don't". I've highlighted the best response correspondence in blue. Finding equilibrium strategies essentially amounts to finding the threshold value and then specifying actions above/below that value.

11.0.3 Advisor Call/Don't Call [3]

Θ continuous and A continuous.

$$f_1^*(\theta_1(\in \arg\max_{a_1 \in A_1} \mathbb{E}_{u_1}(a_1, f_2^*; \theta_1, \theta_2) \ \forall \theta_1 \in \Theta_1$$
$$f_2^*(\theta_2) \in \arg\max_{a_2 \in A_2} \mathbb{E}_{u_2}(a_2, f_1^*; \theta_1, \theta_2) \ \forall \theta_2 \in \Theta_2$$

If FOC holds...

$$0 = \frac{\partial}{\partial a_2} \mathbb{E}_{u_2}(a_2, f_1^*; \theta_1, \theta_2)\Big|_{a_2 = f_2^*(\theta_2)} \ \forall \theta_2 \in \Theta_2$$

Three Approaches

1. Realize that $BR_1(f_2^*)$ only depends on a summary statistic $z(f_2^*)$, e.g. $\mathbb{E}(f_2^*(c_2))$. Then we only need to find a fixed point in two dimensions,

$$\begin{aligned} z(f_1^*) &\equiv z_1 \\ z(f_2^*) &\equiv z_2. \end{aligned}$$

2. Guess and check

3. Start solving the system of differential equations and realize that the answer is either easy or can be solved numerically.

11.0.4 Example of Technique [1]—Cournot Competition

$$\begin{aligned} P(q_1 + q_2) &= 2 - (q_1 + q_2) \\ c_i &\sim U[0,1] && \text{(known only to } i\text{)} \end{aligned}$$

Claim. *The unique Bayesian Nash equilibrium is*

$$f_i^*(c_i) = \frac{3}{4} - \frac{c_i}{2} \text{ for } i \in \{1,2\}.$$

Proof.

$$\begin{aligned}
\mathbb{E}_{u_1}(q_1, f_2^*(c_2); c_1, c_2) &= \int_{c_2} q_q(2 - q_1 - f_2^*(c_2) - c_1) dc_2 \\
&= 2q_1 - q_1^2 - c_1 q_1 - c_1 \mathbb{E}\left(f_2^*(c_2)\right) \\
z_1 &\equiv \mathbb{E}\left(f_1^*(c_1)\right) \\
z_2 &\equiv \mathbb{E}\left(f_2^*(c_2)\right) \\
f_1^*(c_1) &= \arg\max_{q_1} 2q_1 - q_1^2 - c_1 q_1 - q_1 \mathbb{E}\left(f_2^*(c_2)\right) \\
0 &= 2 - 2q_1 - c_1 - \mathbb{E}\left(f_2^*(c_2)\right)\Big|_{q_1 = f_1^*(c_1)} \quad \forall c_1 && \text{(FOC)} \\
\iff f_1^*(c_1) &= 2 - c_1 - \frac{\mathbb{E}\left(f_2^*(c_2)\right)}{2} \\
&= \frac{2 - c_1 - z_2}{2} \\
z_1 &= \mathbb{E}\left(f_1^*(c_1)\right) \\
&= \mathbb{E}\left(\frac{2 - c_1 - z_2}{2}\right) \\
&= 1 - \frac{\mathbb{E}(c_1)}{2} - \frac{z_2}{2} \\
z_1 &= \frac{3}{4} - \frac{z_2}{2} \\
z_2 &= \frac{3}{4} - \frac{z_1}{2} && \text{(Player 2's FOC)} \\
\implies z_1 = z_2 &= \frac{1}{2}
\end{aligned}$$

□

11.0.5 Example—First Price Auction

Suppose players 1 and 2 simultaneously choose bids b_1 and b_2. The highest bidder wins and pays their bid.

$$u_1(b_1, b_2; v_1, v_2) = \begin{cases} v_1 - b_1 & \text{if } b_1 > b_2 \\ \frac{1}{2}(v_1 - b_1) & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases}$$

Suppose $v_1, v_2 \sim U[0,1]$, independent, and v_i known only to player i.

Claim. *The unique Bayesian Nash equilibrium is*

$$f_1^*(v_1) = \frac{v_1}{2}; \quad f_2^*(v_2) = \frac{v_2}{2}.$$

Proof. To show this is an equilibrium...

$$\begin{aligned}
\text{Suppose } f_2^*(v_2) &= \frac{v_2}{2} \\
\text{Bidding } b \text{ wins} \iff f_2^*(v_2) &< b \\
\iff \frac{v_2}{2} &< b \\
\iff v_2 &< 2b \\
f_1^*(v_1) &= \arg\max_b (v_1 - b)2b \\
0 &= 2v_1 - 4b\Big|_{b=f_1^*(v_1)} \qquad \text{(FOC)} \\
f_1^*(v_1) &= \frac{1}{2}
\end{aligned}$$

□

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12.0.1 First Price Auction

Nature choose $v_1, v_2 \sim U[0,1]$ and tells player i v_i. Players submit bids b_1 and b_2. Let's revisit our claim from the end of last class.

Claim. *The unique Bayesian Nash equilibrium is*

$$f_1^*(v_1) = \frac{v_1}{2}; \quad f_2^*(v_2) = \frac{v_2}{2}.$$

Let's go through a slightly different proof.

Proof. Let's start by assuming that $f_1^*(v_1)$ is monotone increasing, differentiable, $f_1^* = f_2^*$, and the first order conditions always maximize values. These assumptions make our problem simpler.

$$\begin{aligned}
f_1^*(v_1) &= \arg\max_{b_1} (v_1 - b_1)\mathbb{P}(f_2^*(v_2) < b_1) + \frac{1}{2}(v_1 - b_1)\underbrace{\mathbb{P}(f_2^*(v_2 = b_1))}_{0} \\
&= \arg\max_{b_1} (v_1 - b_1) f_2^{*-1}(b_1) \\
0 &= (v_1 - b_1)\frac{d}{db_1} f_2^{*-1}(b_1) + (-1) f_2^{*-1}(b_1)\Big|_{b_1 = f_1^*(v_1)} \ \forall v_1 \qquad \text{(FOC)} \\
0 &= (v_1 - f_1^*(v_1))\frac{1}{f_2^{*\prime}(f_2^{*-1}(f_1^*(v_1)))} - f_2^{*-1}(f_1^*(v_1))
\end{aligned}$$

Use $f_1^* = f_2^*$ and $f^{*-1}(f^*(v)) = v$.

$$\begin{aligned}
0 &= (v_1 - f_1^*(v_1))\frac{1}{f_1^*(v_1)} - v_1 \ \forall v_1 \\
v_1 &= f_1^*(v_1) + f_1^{*\prime}(v_1) v_1 \ \forall v_1
\end{aligned}$$

Integrate…

$$\frac{1}{2}v_1^2 + k = v_1 f_1^*(v_1)\ \forall v_1$$
$$\implies f_1^*(v_1) = \frac{1}{2}v_1 + \frac{k}{v_1}$$

Only $k = 0$ is a valid solution.

□

12.1 Purifying Mixed Equilibria

	Ignore	Manufacture
Cancel	$0, 0$	$0, -1$
Advertise	$1, 0$	$-1, 3$

The unique Nash equilibrium in this game is

$$\frac{3}{4}Cancel + \frac{1}{4}Advertise; \frac{1}{2}Ignore + \frac{1}{2}Manufacture.$$

Imagine this is only an approximation of the game. Suppose instead that Nature choose $\alpha, \beta \sim U[0,1]$; α is known only to Player 1, and β is known only to Player 2; $\varepsilon > 0$ and small.

	Ignore	Manufacture
Cancel	$\varepsilon\alpha, \varepsilon\beta$	$\varepsilon\alpha, -1$
Advertise	$1, \varepsilon\beta$	$-1, \beta$

How would people play this game?

Claim. *The unique Bayesian Nash equilibrium of the game with incomplete information is*

$$s_1^*(\alpha) = \begin{cases} C \text{ if } \alpha > \alpha^* \\ A \text{ if } \alpha < \alpha^* \end{cases}$$
$$s_2^* = \begin{cases} I \text{ if } \beta > \beta^* \\ M \text{ if } \beta < \beta^* \end{cases}$$
$$\alpha^* = \frac{2+\varepsilon}{8+\varepsilon^2}$$
$$\beta^* = \frac{4-\varepsilon}{8+\varepsilon^2}$$
$$\varepsilon \approx 0 \implies \alpha^* \approx \frac{1}{4};\ \beta^* \approx \frac{1}{2}$$
$$\varepsilon\alpha^* = (1-\beta^*)1 + \beta^* \cdot -1 \qquad \text{(Indifference for Player 1)}$$
$$\varepsilon\beta^* = (1-\alpha^*) \cdot -1 + \alpha^* \cdot 3 \qquad \text{(Indifference for Player 2)}$$

Theorem. *Given $G = (S, u)$, define $G^\varepsilon(\Theta, S, p, u^\varepsilon)$ by*

$$\Theta = \Theta_1 \times \ldots \times \Theta_I; \quad \Theta_i : S \to [-1, 1],$$
$$p \text{ is a probability distribution on } \Theta,$$
$$u_i^\varepsilon(S, \Theta) = u_i(S) + \varepsilon\Theta_i(S).$$

Generically, for all twice-differentiable distributions p, every Nash equilibrium G is the limit $\varepsilon \to 0$ of the distribution of play in a Bayesian Nash equilibrium of G^ε.

12.2 Dynamic Games with Incomplete Information

Simple Signaling Games

0. Nature chooses Θ_1 from Θ_1 with distribution $P(\theta_1)$.
1. Player 1 sees θ_1, chooses $a_1 \in A_1$.
2. Player 2 sees a_1, chooses $a_2 \in A_2$

Payoffs $u_1(a_1, a_2, \theta_1)$ and $u_2(a_1, a_2, \theta_1)$.

12.2.1 Example—Getting an MBA

Nature chooses

$$\theta = \begin{cases} 2 \text{ with probability } \frac{1}{2} \\ 3 \text{ with probability } \frac{1}{2}. \end{cases}$$

The student chooses

$$a \in \{0, 1\}. \qquad \text{(Number of MBAs to get)}$$

The "Labor Market" sees a and chooses a wage $w \in \mathbb{R}$ for the worker.

$$u_1(a, w; \theta) = w - \frac{ca}{\theta}$$
$$u_2(a, w; \theta) = -(w - \theta)^2$$

How do we analyze games like this? Note that Subgame Perfect equilibria is a horrible solution concept. For example, $s_1^*(\theta) = 1 \ \forall\theta)$ is a Subgame Perfect equilibrium, and

$$w^*(a) = \begin{cases} 2.5 & \text{if } a = 1 \\ -c & \text{if } a = 0. \end{cases}$$

12.3 New Solution Concept—Perfect Bayesian Equilibrium

Definition (Perfect Bayesian Equilibrium)**.** *A Perfect Bayesian equilibrium (PBE) in the signaling game is a strategy profile* (s_1, s_2) *along with beliefs* $\mu_2(\theta_1 \mid a)$ *such that*

1. *Strategies are optimal given beliefs:*

$$s_1^*(\theta_1) \in \arg\max u_1(a, s_2^*(a); \theta) \ \forall \theta_1$$
$$s_2^* \in \arg\max \mathbb{E}_{\mu(\theta|a)} u_2(a_1, a_2; \theta)$$

2. *Beliefs are compatible with prior and* s_1^* *via Bayes rule:*

$$\mu_2(\theta \mid a) = \frac{P(\theta) \cdot \mathbb{P}(s_1^*(\theta) = a)}{\sum_{\theta'} P(\theta') \cdot \mathbb{P}(s_1^*(\theta') = a)}.$$

12.3.1 Types of Perfect Bayesian Equilibria

1. Separating—different people take different actions; we learn something about people from their actions.
2. Pooling—everyone takes the same actions and we learn nothing about players.
3. Semi-separating—somewhere in between the first two types; one player ends up mixing.

13 December 9, 2019

13.1 Perfect Bayesian Equilibrium for Job market Signaling

Stage 0. Nature chooses

$$\theta = \begin{cases} \mathbb{P}(\theta = 2) & = & p \\ \mathbb{P}(\theta = 3) & = & 1 - p \end{cases}$$

Stage 1. The worker observes θ and chooses effort

$$e \in \{0, 1\}.$$

Effort comes with a disutility $\frac{c \cdot e}{\theta}$. Education is useless; type θ workers have productivity θ regardless of how much effort they expend.

Stage 2.

$$u_1(e, w, \theta) = w - \frac{c \cdot e}{\theta}$$
$$u_2(e, w, \theta) = -(w - \theta)^2$$

13.1.1 Separating PBE

Claim. *For* $2 \leq c \leq 3$*, a separating Pure Bayesian equilibrium of this game is*

$$s^*(\theta = 2) = 0$$
$$s^*(\theta = 3) = 1$$
$$\mu_2(\theta = 3 \mid a_1 = 0) = 0$$
$$\mu_2(\theta = 3 \mid a_1 = 1) = 1$$
$$w = \begin{cases} 2 \text{ } if \text{ } a_1 = 0 \\ 3 \text{ } if \text{ } a_1 = 1 \end{cases}$$

Proof. We derive beliefs from Bayes' rule. The market behaves optimally. We only need to check that the specified strategies are best responses for both types.

BR for type $\theta =$

$$2 \geq 3 - \frac{c \cdot 1}{2}$$
$$\iff c \geq 2$$

BR of type $\theta = 3$

$$3 - \frac{c \cdot 1}{3} \geq 2$$
$$\iff c \leq 3$$

□

If we had a continuous strategy space instead, there are a lot of separating equilibria.

$$s_1^*(\theta = 2) = 0$$
$$x_1^*(\theta = 3) = \overline{e}$$
$$w = \begin{cases} 2 \text{ if } a_1 \neq \overline{e} \\ 3 \text{ if } a_1 = \overline{e} \end{cases}$$

For a given c, we can solve for the appropriate $\overline{e}$.

13.1.2 Pooling PBE

Claim. *Set* $p = \frac{1}{2}$*. For* $c \leq 1$*, a pooling Perfect Bayesian equilibrium of this games is*

$$s^*(\theta = \{2, 3\}) = 1$$
$$\mu(\theta = 3 \mid a_1 = 0) = 0$$
$$\mu(\theta = 3 \mid a_1 = 1) = \frac{1}{2}$$
$$w = \begin{cases} 2.5 \text{ } if \text{ } a_1 = 1 \\ 2 \text{ } if \text{ } a_1 = 0 \end{cases}$$

Proof. Again, we will use Bayes' rule on path and check the best response correspondence for type $\theta = 2$.

$$\begin{aligned} 2.5 - \frac{c}{2} &\geq 2 \\ \iff c &\leq 1 \end{aligned}$$

$\square$

13.1.3 Semi-pooling PBE

Claim. *For $1 \leq c \leq 2$, a semi-separating equilibrium of the game is*

$$\begin{aligned} \sigma(\theta = 3) &= 1 \\ \sigma(\theta = 2) &= \begin{cases} 1 \text{ with probability } q \\ 0 \text{ with probability } 1 - q \end{cases} \\ \mu(\theta = 3 \mid a_1 = 0) &= 0 \\ \mu(\theta = 3 \mid a_1 = 1) &= \frac{1}{1+q} \\ w &= \begin{cases} 2 \text{ if } a_1 = 0 \\ 2 + \frac{1}{1+q} \text{ if } a_1 = 1 \end{cases} \\ \mu(\theta = 3 \mid a_1 = 1) &= \frac{\frac{1}{2} \cdot 1}{\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot q} \\ &= \frac{1}{1+q} \end{aligned}$$

Proof. We need to check the best response for type $\theta = 2$.

$$\begin{aligned} 2 + \frac{1}{1+1} - \frac{c}{2} &\geq 2 \\ \iff 1 + q &= \frac{2}{c} \\ \iff q &= \frac{2-c}{c} \end{aligned}$$

q generates a well-defined probability for $c \in (1, 2)$. $\square$

13.2 IPO Game

Stage 0. Nature chooses θ;

$$\mathbb{P}(\theta = 1) = \mathbb{P}(\theta = 2) = \frac{1}{2}.$$

Stage 1. The entrepreneur sees θ and chooses (p, q), where

$$\begin{aligned} p \equiv &\text{ price per share,} \\ q \equiv &\text{ fraction of the firm for sale.} \end{aligned}$$

Stage 2. The investor accepts or declines the entrepreneur's offer, denoted $d \in \{y, n\}$. Shares return θ to the investor but only $\frac{\theta}{2}$ to the entrepreneur (perhaps because of a liquidity-constrained entrepreneur, etc.).

$$u_E\left((p,q), d, \theta\right) = \begin{cases} pq + (1-q)\frac{\theta}{2} & \text{if} \quad d = y \\ \frac{\theta}{2} & \text{if} \quad d = n \end{cases}$$

$$u_I\left((p,q), d, \theta\right) = \begin{cases} (\theta - p)\, q & \text{if} \quad d = y \\ 0 & \text{if} \quad d = n \end{cases}$$

Claim. *A separating equilibrium of this game is*

$$s_E^* = \begin{cases} (1,1) & \textit{if } \theta = 1 \\ \left(2, \frac{1}{3}\right) & \textit{if } \theta = 2 \end{cases}$$

$$\mu_I\left(\theta = 1 \mid (p,q)\right) = \begin{cases} 0 \textit{ if } (p,q) = \left(2, \frac{1}{3}\right) \\ 1 \textit{ if } (p,q) = (1,1) \\ 1 \textit{ if } p < 1 \\ 1 \textit{ otherwise} \end{cases}$$

$$s_I^* = \begin{cases} y \textit{ if } (p,q) = (1,1) \\ y \textit{ if } (p,q) = \left(\frac{2}{3}, 1\right) \\ y \textit{ if } p < 1 \\ n \textit{ otherwise} \end{cases}$$

Additional Observations.

1. For any $x \in (1,2)$, there is a separating Perfect Bayesian equilibrium where the high type sells $\frac{1}{3}$ of the firm at price $p = x$.

2. For any $x \in \left(1, \frac{3}{2}\right)$, there is a pooling equilibrium where both types sell the entire company at price $p = x$.

13.3 PBE Refinement: Intuitive Criterion

Consider the pooling equilibrium $\left(\frac{4}{3}, 1\right)$. Type $\theta = 2$ can present the following argument:

1. Everyone sells all their shares at $p = \frac{4}{3}$.

2. Instead, I am willing to sell half of my shares at $p = 2$.

3. If I were a low type, this would yield payoff

$$\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot \frac{1}{2} < \frac{4}{3}.$$

4. But since I am a high type,

$$\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 1 > \frac{4}{3}.$$

Therefore $\left(\frac{4}{3}, 1\right)$ is not "intuitive", so our equilibrium refinement eliminates the pooling equilibrium $\left(\frac{4}{3}, 1\right)$.

Definition. *$BR(T, a_1)$ denotes the set of Player 2's best responses to beliefs with support $T \in \Theta$.*

$$BR(T, a_1) = \bigcup_{\mu \in \Delta(T)} \arg\max_{a_2 \in A_2} \mathbb{E}_\mu \left[u_2\left(a_1, a_2, \theta\right)\right]$$

Definition. *A PBE σ^* of G is said to fail the intuitive criterion if there exist $a_1 \in A, \theta' \in \Theta$, and $J \subset \Theta$ such that*

$$u_1(\sigma^*, \theta) > \max_{a_2 \in BR(\Theta, a_1)} u_1(a_1', a_2, \theta) \quad \forall \theta \in J \tag{1}$$

$$u_1(\sigma^*, \theta') < \min_{a_2 \in BR(\Theta \setminus J, a_1)} u_1(a_1, a_2, \theta'). \tag{2}$$

14 December 11, 2019

14.1 Intuitive Criterion in the IPO Game

The pooling equliibrium $\left(\frac{4}{3}, 1\right)$ fails the intuitive criterion.

$$\begin{aligned} J &= \{1\} \\ \theta' &= 2 \\ a_1 &= \left(2 - \varepsilon, \frac{1}{2}\right) \end{aligned}$$

The equilibrium

$$s_1^*(\theta) = \begin{cases} (1, 1) & \textbf{if } \theta = 1 \\ \left(x, \frac{1}{4}\right) & \textbf{if } \theta = 2; \quad x \in (1, 2) \end{cases}$$

fails the intuitive criterion.

$$\begin{aligned} J &= \{1\} \\ a_1 &= \left(2 - \varepsilon', \frac{1}{4} + \varepsilon\right) \\ \theta' &= 2 \end{aligned}$$

Consider $\theta = 1$.

$$\left(\frac{1}{4} + \varepsilon\right)(2 - \varepsilon') + \left(\frac{3}{4} - \varepsilon\right) \cdot \frac{1}{2} \approx \frac{7}{8} < 1.$$

Now consider $\theta = 2$. In equilibrium,

$$\frac{1}{4}x + \frac{3}{4} < \frac{5}{4}$$

If type $\theta = 2$ plays a_1 instead,

$$\left(\frac{1}{4} + \varepsilon\right)(2 - \varepsilon') + \left(\frac{3}{4} - \varepsilon\right) \approx \frac{5}{4}.$$

In this game, the only Perfect Bayesian Equilibrium that survives the intuitive criterion is

$$s_1^*(\theta) = \begin{cases} (1,1) & \textbf{if } \theta = 1 \\ \left(2, \frac{1}{3}\right) & \textbf{if } \theta = 2. \end{cases}$$

14.2 Cheap Talk Games

Stage 0. Nature chooses a type $\theta \in \Theta$ for Player 1.

Stage 1. Player 1 sees θ and chooses a message $m \in M$.

Stage 2. Player 2 sees m and chooses an action $a_2 \in A_2$.

$$u_1(m, a_2, \theta) = u_1(a_2, \theta)$$
$$u_2(m, a_2, \theta) = u_2(a_2, \theta)$$

If all θ have the same preferences over A_2, then cheap talk doesn't help.

14.2.1 Example

Stage 0. $\theta \in [0,1]$; $\quad \theta \sim U[0,1]$

Stage 1. Player 1 sees θ and then sends message $m \in M$.

Stage 2. Player 2 chooses $x \in [0,1]$.

$$u_1(m, x, \theta) = -\left(x - (\theta + a)\right)^2$$
$$u_2(m, x, \theta) = -(x - \theta)^2$$

Claim. *If $a > 0$, there is no separating Perfect Bayesian equilibrium.*

Proof sketch. Suppose there is a fully separating equilibrium. Then there is an equilibrium strategy $s^*(\theta)$ that is one-to-one. When the government sees $s^*(\theta)$, it believes the type is θ with probability 1. If type θ plays $s^*(\theta + a)$, the government believes θ is $\theta + a$, hence plays $\theta + a$. □

Definition (Babbling equlibrium). *A babbling equilibrium is*

$$s^*(\theta) = m \quad \forall \theta.$$

Claim. *For $a \in \left(0, \frac{1}{4}\right)$, a partially separating equilibrium is*

$$M = \{high, low\}$$

$$s_1^*(\theta) = \begin{cases} low & \text{if } \theta \leq \hat{\theta} \\ high & \text{if } \theta > \hat{\theta} \end{cases}$$

$$f_{low}(\theta) = \begin{cases} \frac{1}{\hat{\theta}} & \text{if } \theta \leq \hat{\theta} \\ 0 & \text{if } \theta > \hat{\theta} \end{cases}$$

$$f_{high}(\theta) = \begin{cases} 0 & \text{if } \theta < \hat{\theta} \\ \frac{1}{1-\hat{\theta}} & \text{if } \theta \geq \hat{\theta} \end{cases}$$

$$\mu_2(\theta \mid m) = \begin{cases} f_{low}(\theta) & \text{if } m = low \\ f_{high}(\theta) & \text{if } m = high \end{cases}$$

$$s_2^*(m) = \begin{cases} \frac{\hat{\theta}}{2} & \text{if } m = low \\ 1 + \frac{\hat{\theta}}{2} & \text{if } m = high \end{cases}$$

Proof. Beliefs are Bayesian and $s_2^*(m)$ is optimal.

$$u_1(\text{low}, s_2^*(m), \theta) = -\left(\frac{\hat{\theta}}{2} - (\theta + a)\right)^2$$

$$u_1(\text{high}, s_2^*(m), \theta) = -\left(\frac{1+\hat{\theta}}{2} - (\theta + a)\right)^2$$

Low is better if $\theta + a$ is closer to $\frac{\hat{\theta}}{2}$ than to $\frac{1+\hat{\theta}}{2}$. This is true if

$$\theta + a \leq \frac{\hat{\theta}}{2} + \frac{1}{4}.$$

Therefore, we have a Perfect Bayesian equilibrium if

$$\theta + a \leq \frac{\hat{\theta}}{2} + \frac{1}{4} \quad \forall \theta \leq \hat{\theta}$$

$$\theta + a \geq \frac{\hat{\theta}}{2} + \frac{1}{4} \quad \forall \theta \geq \hat{\theta}.$$

Indifference equates these payoffs.

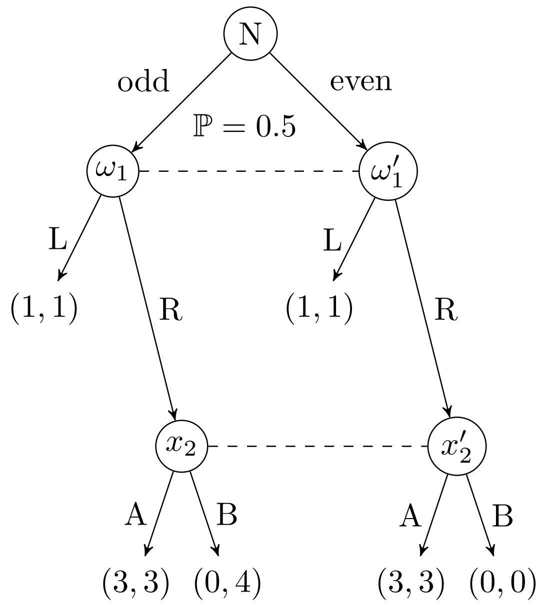
$$\hat{\theta} + a = \frac{\hat{\theta}}{2} + \frac{1}{4} \iff \hat{\theta} = \frac{1}{2} + 2a,$$

which requires $a \in \left[0, \frac{1}{4}\right.$.

□

14.3 Limiations of Perfect Bayesian equilibria

The definition of Perfect Bayesian equilibrium may allow beliefs that are inconsistent with the game tree.



Claim. *A Perfect Bayesian equilibrium in this game is*

$$
\begin{aligned}
s_1^* &= L \\
s_2^* &= B \\
\mu_1(\omega \mid \{\omega, \omega'\}) &= \frac{1}{2} \\
\mu_2(x \mid \{x, x'\}) &= 0.8 \qquad (*)
\end{aligned}
$$

The condition in $(*)$ doesn't make any sense because of the information sets we have. The agent should learn that $\mathbb{E}(x \mid \{x, x'\}) = 0.5$ because of Nature's play and the player's uncertainty, but the equilibrium we've constructed requires $\mu_2(x \mid \{x, x'\}) = 0.8$. Let's introduce an equilibrium refinement to rule out cases like this.

14.4 Sequential Equilibrium

Sequential equilibrium is a small refinement of Perfect Bayesian equilibrium; it is the most commonly accepted solution concept for dynamic Bayesian games.

Definition (Consistent Assessment)**.** *An assessment (σ, μ) of the simple signaling game G is consistent if $(\sigma, \mu) = \lim_{N\to\infty}(\sigma^N, \mu^N)$ for some sequence of assessments (σ^N, μ^N) such that σ^N is totally mixed and μ^N is derived from σ^N by Bayes Rule.*

The "totally mixed" condition is necessary to remove off-path play; once everything is totally mixed, then everything is on path.

Definition (Sequential Equilibrium)**.** *An assessment* (σ, μ) *is a sequential equilibrium if it is consistent and it is a Perfect Bayesian equilibrium.*

Claim. *previous Perfect Bayesian equilibriumk is not a sequential equilibrium.*

Proof. If σ^N is totally mixed, then

$$\begin{aligned}\mu_2^N(x \mid \{x, x'\}) &= \frac{\frac{1}{2}\sigma_1^N(R)}{\frac{1}{2}\sigma_1^N(R)\frac{1}{2}\sigma_1^N(R)} \\ &= \frac{1}{2}.\end{aligned}$$

Hence, for any completely mixed (σ^N, μ^N),

$$\lim_{N\to\infty} \mu_2^N(x \mid \{x, x'\}) = \frac{1}{2}.$$

□

14.4.1 Example of Sequences for Sequential Equilibrium

$$\begin{aligned}\sigma_i^N &\in \overbrace{[0,1]}^{L} \times \overbrace{[0,1]}^{R} \\ \sigma_1^N(x \mid \{x, x'\}) &= \left(\frac{1}{N}, 1 - \frac{1}{N}\right) \\ \mu_1^N(\omega \mid \{\omega, \omega'\}) &= \frac{1}{2} \\ {}_2^N(x \mid \{x, x'\}) &= \frac{1}{2} \\ \sigma_2^N &= \left(1 - \frac{1}{N}, \frac{1}{N}\right) \\ \mu_2^* &= \frac{1}{2} \\ \sigma_2^* &= (1, 0) \\ \sigma_1^* &= (0, 1) \\ \mu_1^* &= \frac{1}{2}\end{aligned}$$