Chapter 3

Asset Pricing Puzzles

In the first quarter, you have learned that the stochastic growth model has realistic predictions for the cyclical behavior of quantities, such as output, investment, consumption, or hours. This observation is often viewed as a major achievement of the "Real Business Cycle" research agenda. Very early on in this agenda, researchers explored whether this or related models also had realistic predictions for asset prices. However, starting with the work of Hansen and Singleton (1982) and Merha and Prescott (1985), it quickly became clear that standard models had puzzlingly bad quantitative predictions about asset prices.

Why is that so problematic? First, many decisions in macroeconomics, such as investment, consumption, etc..., are partly based on asset prices. It would be worrisome that, in our quantitative models, quantities are right but prices are off: it would suggest that agents in the model make empirically realistic decisions but for the wrong reasons. Second, one would like to use asset prices to inform us about the nature and the magnitude of the "fundamental" shocks that hit the macroeconomy. This requires to write a model and derive the equilibrium mapping from these shocks to asset prices. But if this mapping does not generate realistic asset prices, it cannot be trusted to inform us about the shocks.

In this Chapter we will first explain how researchers came to the conclusion that standard macroeconomic models had bad quantitative predictions about asset prices. We will then discuss various solutions proposed in the literature.

3.1 The risk-premium and the risk-free puzzle

The literature has focused on the quantitative predictions of standard macroeconomic models for two returns: the average excess return on equity (Mehra and Prescott, 1985), and the risk-free rate (Weil, 1989). To derive these predictions in a simple way, define the one-period return on asset n:

$$r_{t+1}^{(n)} \equiv \frac{p_{t+1}^{(n)} + d_{t+1}^{(n)}}{p_t^{(n)}} - 1,$$

where we omit the dependence of random variables on the history s^t to simplify notations. Dividing through both sides by $p_t^{(n)}$, the asset pricing equation (2.1.2) can be re-written:

$$\mathbb{E}_t\left[M_{t+1}\left(1+r_{t+1}^{(n)}\right)\right]=1,$$

where M_{t+1} is the stochastic discount factor. Now letting r_t^s and r_t^b be, respectively, the returns on the S&P 500 and the return on a risk free short-term debt, we have

$$\mathbb{E}_t\left[M_{t+1}\left(1+r_{t+1}^s\right)\right]=1 \tag{3.1.1}$$

$$\mathbb{E}_t\left[M_{t+1}\left(1+r_{t+1}^b\right)\right]=1. \tag{3.1.2}$$

Taking the difference equations (3.1.1) and (3.1.2), we obtain:

$$\mathbb{E}_t\left[M_{t+1}\left(r_{t+1}^s-r_{t+1}^b\right)\right]=0. \tag{3.1.3}$$

Economically, "taking the difference" means studying the return of a leveraged investment in the stock market: short-selling one dollar worth of bonds, and investing the proceeds in the S&500. The payoff of this leveraged investment is $r_{t+1}^s-r_{t+1}^b$. Clearly, this investment has zero cost at time t. In the asset pricing equation, this means that the expected present discounted value of the payoff, $\mathbb{E}_t\left[M_{t+1}\left(r_{t+1}^s-r_{t+1}^b\right)\right]$, is equal to the cost of this leveraged investment, which is zero.

After taking unconditional expectations in (3.1.3) and using the Law of Iterated Expectations, we obtain:

$$\mathbb{E}\left[M_{t+1}\left(r_{t+1}^s-r_{t+1}^b\right)\right]=0.$$

Now recall that, for any pair (x,y) of random variables, $\mathrm{cov}(x,y)=\mathbb{E}[xy]-\mathbb{E}[x]\,\mathbb{E}[y]$. Using this identity in the above equation yields:

$$\mathbb{E}\left[M_{t+1}\right]\mathbb{E}\left[r_{t+1}^s-r_{t+1}^b\right]+\mathrm{cov}\left(M_{t+1},r_{t+1}^s-r_{t+1}^b\right)=0.$$

Hence, the average excess return on the S&P500, or "equity premium", is

$$\mathbb{E}\left[r_{t+1}^s-r_{t+1}^b\right]=-\mathrm{cov}\left(\frac{M_{t+1}}{\mathbb{E}\left[M_{t+1}\right]},r_{t+1}^s-r_{t+1}^b\right). \tag{3.1.4}$$

One sees that the volatility of $r_{t+1}^s-r_{t+1}^b$ matters for equity premium but only to the extent that $r_{t+1}^s-r_{t+1}^b$ is negatively correlated with M_{t+1}. In the data, the stock market is pro-cyclical - returns are high when aggregate output is high, but the stochastic discount factor is countercyclical - the marginal utility of consumption is low when aggregate output is high. This suggests that the expected equity premium will be positive in a broad class of standard macroeconomic models. Put differently, the theory has good qualitative predictions. But, as we will see below, the quantitative predictions are off: when we we put numbers in the equation, we obtain an equity premium that is much smaller than what is observed in the data.

3.1.1 The equity premium puzzle

The puzzle arises under the standard assumption that agents have CRRA utility, which implies that $M_{t+1} = \beta \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}$. For simplicity, we will assume that consumption and asset returns follow the processes:

$$\begin{aligned}\frac{c_{t+1}}{c_t} =& g_{t+1} = \bar{g} \exp\left(\varepsilon_{g,t+1} - \frac{\sigma_g^2}{2}\right) \\ 1 + r_{t+1}^i =& \left(1 + \bar{r}^i\right) \exp\left(\varepsilon_{i,t+1} - \frac{\sigma_i^2}{2}\right),\end{aligned}$$

where the vector of shocks, $(\varepsilon_{g,t}, \varepsilon_{s,t}, \varepsilon_{b,t})$ is independently and identically distributed over time according to a (multivariate) normal distribution. In particular, we assume that $\varepsilon_{g,t+1} \sim \mathcal{N}(0, \sigma_g^2)$ and $\varepsilon_{i,t+1} \sim \mathcal{N}(0, \sigma_i^2)$, for $i \in \{s, b\}$. Notice that, although we assume that the shocks are not serially correlated (i.e., shocks at distinct time periods are assumed to be independent), they may be contemporaneously correlated.

In the exponentials above, we subtract half of the variance $\sigma^2/2$ so that the expectation of the exponential is equal to one. Indeed, recall that, if $x \sim N(\mu, \sigma^2)$, then $\mathbb{E}[\exp(x)] = \exp(\mu + \frac{1}{2}\sigma^2)$. This implies in turns that $\mathbb{E}[g_{t+1}] = \bar{g}$ and $\mathbb{E}\left[r_{t+1}^i\right] = \bar{r}_i$. We assume that the shocks $\varepsilon_{g,t+1}$ and $\varepsilon_{i,t+1}$, $i \in \{s, b\}$ are independently and identically distributed across time. This is an empirically reasonable first approximation a business cycle frequencies. One may wonder why it makes sense to take the stochastic process for returns as given in the analysis. After all, returns depend on prices and prices are endogenous: hence, the right way to analyze the equity premium is to start from a primitive specification of the economic environment, including preferences, endowment, etc..., solve for prices, calculate returns, and see how they compare to the one we measured in the data over the last century. This is, in fact, the approach taken by Merha and Prescott (1985) in their original work. But taking the stochastic process for returns as given does make sense, to answer the following question: does the asset pricing equation $\mathbb{E}\left[\beta\left(c_{t+1}/c_t\right)^{-\gamma}\right)\left(1 + r_{t+1}^i\right)\big]$ holds for equity and bonds, given a realistic stochastic process for consumption growth, and for the return on equities and bonds? If it does not, then this means that <u>any</u> equilibrium model in which this asset pricing equation must hold will have a hard time to explain simultaneously consumption growth, equity returns, and bond returns.

The argument works as follows. First, we note that the asset pricing equations (3.1.1) and (3.1.2) impose strong restrictions on the parameters governing the stochastic processes for stock returns, bond returns and consumption growth. To see this, we substitute in (3.1.1) and (3.1.2) the above formula

Data moment	Symbol	Value
Equity Premium	$\bar{r}^s - \bar{r}^b$	5.5%
Risk free rate	$\bar{r}^b$	2%
Covariance between log consumption growth and log market return	$\mathrm{cov}(\varepsilon_g, \varepsilon_s)$	0.0035
Covariance between log consumption growth and log bond return	$\mathrm{cov}(\varepsilon_g, \varepsilon_b)$	-0.00011
Variance of log consumption growth	$\mathrm{var}\,(\varepsilon_g)$	0.0012
Average log consumption growth	$\log \bar{g}$	0.021

Table 3.1: This table is based on monthly data about S&P500 real return and real dividend, one year real bond return rate, and consumption from 1971 to 2013, posted on Robert Shiller website.

for g_{t+1} and $1 + r_{i,t+1}$, $i \in \{s, b\}$, We obtain:

$$\begin{aligned}
1 =& \mathbb{E}\left[M_{t+1}\left(1 + r^i_{t+1}\right)\right] \\
=& \mathbb{E}\left[\beta\left(1+\bar{r}^i\right)\bar{g}^{-\gamma}\exp\left(-\gamma\varepsilon_{g,t+1} + \frac{1}{2}\gamma\sigma_g^2 + \varepsilon_{i,t+1} - \frac{1}{2}\sigma_i^2\right)\right] \\
=& \beta\left(1+\bar{r}^i\right)\bar{g}^{-\gamma}\exp\left[\frac{1}{2}\gamma^2\sigma_g^2 + \frac{1}{2}\sigma_i^2 - \gamma\mathrm{cov}\left(\varepsilon_{g,t+1}, \varepsilon_{i,t+1}\right) + \frac{1}{2}\gamma\sigma_g^2 - \frac{1}{2}\sigma_i^2\right] \\
=& \beta\left(1+\bar{r}^i\right)\bar{g}^{-\gamma}\exp\left[\frac{1}{2}\gamma(\gamma+1)\sigma_g^2 - \gamma\mathrm{cov}\left(\varepsilon_{g,t+1}, \varepsilon_{i,t+1}\right)\right].
\end{aligned}$$

We then compute the ratio $\mathbb{E}\left[M_{t+1}\left(1 + r^s_{t+1}\right)\right] / \mathbb{E}\left[M_{t+1}\left(1 + r^b_{t+1}\right)\right]$, which gives us

$$1 + \bar{r}^s = \left(1 + \bar{r}^b\right)\exp\left[\gamma\mathrm{cov}\left(\varepsilon_{gt+1}, \varepsilon_{st+1} - \varepsilon_{bt+1}\right)\right].$$

We obtain the following expression for the equity risk premium

$$\bar{r}^s - \bar{r}^b = \left(1 + \bar{r}^b\right)\left\{\exp\left[\gamma\mathrm{cov}\left(\varepsilon_{gt+1}, \varepsilon_{st+1} - \varepsilon_{bt+1}\right)\right] - 1\right\}. \tag{3.1.5}$$

This is one key restrictions imposed by the asset pricing equation on the parameters governing the stochastic processes for stock returns, bond returns and consumption growth. The economics of these restrictions is that the expected returns on the stock and the bond markets compensate the representative consumer for risk. In turns, the risk that the representative consumer feels is a function of risk aversion, γ, and of the covariance between consumption growth and the excess return, $\mathrm{cov}\left(\varepsilon_{gt+1}, \varepsilon_{st+1} - \varepsilon_{bt+1}\right)$. As we noted earlier, positive covariance means that the investment tends to pay a lot in good times, when extra consumption is not very valuable because marginal utility is low, and pay little in bad times, when extra consumption is very valuable because marginal utility is high.

To measure the covariances empirically, I use some data posted by Robert Shiller on his website

http://www.econ.yale.edu/~shiller/data/chapt26.xlsx

The online spreadsheet has monthly data from 1871 to 2013 about the stock price index, the dividend payments on the index, the one year bond return, the consumer price index, and aggregate consumption. We combine these data to generate a time series for the *real* log return on the stock market and on treasuries, and on log aggregate consumption growth. In Table 3.1, we report the average real log return on stock and bond, and the covariances between these log returns with log consumption growth. On the left panel of Figure 3.1, we calculate the risk-premium based on the right-side of equation (3.1.5), for various values of the coefficient of relative risk aversion, γ. The result is shown on the left panel of Figure 3.1.

One sees that the relationship between the risk premium and the coefficient of risk aversion is approximately linear. This is because the covariances shown in Table 3.1 are very small, so that (3.1.5) admits the first-order approximation:

$$\bar{r}^s - \bar{r}^b \simeq \gamma \times \left(1 + \bar{r}^b\right) \operatorname{cov}\left(\varepsilon_{gt+1}, \varepsilon_{st+1} - \varepsilon_{bt+1}\right) = \gamma \times 0.0037$$

According to this approximation, one would need a relative risk aversion of about $\gamma = 13.5$ to reach a risk premium of about 5.5%. The consensus in the profession is that this is not a reasonable value. It is too high. To see why, consider the following lottery for a consumer with CRRA utility:

$$c = \begin{cases} w\left(1+g\right) & \text{with probability } \frac{1}{2}, \\ w\left(1-g\right) & \text{with probability } \frac{1}{2}. \end{cases}$$

Compute the certainty equivalent c_E of this lottery, i.e., the deterministic amount of wealth that would give the consumer an identical utility.

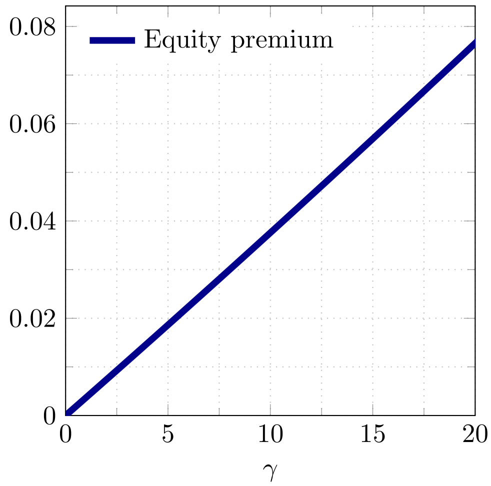
$$\frac{c_E^{1-\gamma}}{1-\gamma} = \frac{1}{2}\frac{\left[w\left(1+g\right)\right]^{1-\gamma}}{1-\gamma} + \frac{1}{2}\frac{\left[w\left(1-g\right)\right]^{1-\gamma}}{1-\gamma}.$$

Re-arranging:

$$\frac{c_E}{w} = \left[\frac{1}{2}\left(1+g\right)^{1-\gamma} + \frac{1}{2}\left(1-g\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}.$$

With $g = 50\%$ and $\gamma = 13.5$, we obtain $c_E/w = 0.53$. It implies that the consumer is willing to give up 47% of his/her wealth upfront, in order to avoid the risk of loosing 50% with probability only 0.5. That is almost as much as the maximum loss he/she may incur with the lottery!

Going beyond introspection, empiricists have made numerous attempts to estimate risk aversion. For example, a large literature uses laboratory experiment in which human subjects chooses amongst lotteries with various levels of riskiness (see, among many others Holt and Laury, 2002). Another literature have thought to exploit natural experiments provided by TV game shows which effetively offer candidates to choose between different lotteries, for example whether to stop the game and earn a low payoff for sure, or continue the game and have the chance or earning much larger payoff (see, among many others Bombardini and Trebbi, 2012). Finally, another strand of the literature is based on real-life market decisions (see, among others Chetty, 2006; Cohen and Einav, 2007). In the bulk of this literature, estimates of relative risk aversion are rather low.



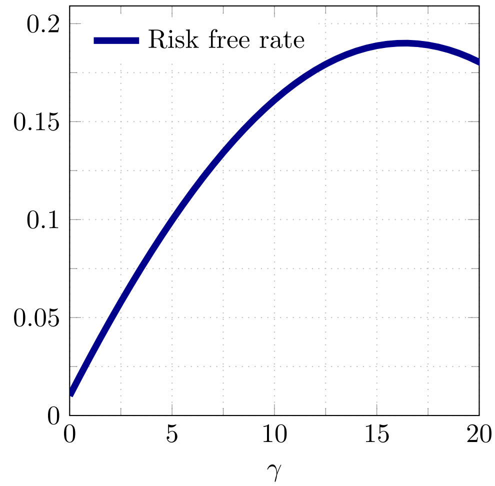


Figure 3.1: The equity premium (left panel) calculated according to equation (3.1.5), and the risk-free rate (right panel) calculated according to equation 3.1.6, for various values of the relative risk aversion coefficient, γ.

3.1.2 The risk-free rate puzzle

Even if one is willing to go along with a calibrated value of $\gamma = 13.5$ or so, one would run into another quantitative puzzle: the risk free rate puzzle, explained in Weil (1989). To see this, consider the equation for the risk-free return:

$$\beta \left(1 + \bar{r}^b\right) \bar{g}^{-\gamma} \exp\left[\frac{1}{2}\gamma(\gamma+1)\sigma_g^2 - \gamma \mathrm{cov}\left(\varepsilon_{g,t+1}, \varepsilon_{b,t+1}\right)\right] = 1,$$

which can be rearranged as:

$$1 + \bar{r}^b = \beta^{-1}\bar{g}^{\gamma} \exp\left[-\frac{1}{2}\gamma(\gamma+1)\sigma_g^2 + \gamma \mathrm{cov}\left(\varepsilon_{g,t+1}, \varepsilon_{b,t+1}\right)\right]. \tag{3.1.6}$$

One sees that the coefficient γ has three distinct effects on the risk free rate.

Let us start with the last term $\gamma \mathrm{cov}\left(\varepsilon_{g,t+1}, \varepsilon_{b,t+1}\right)$. It has the same interpretation as in the risk-premium equation equation (3.1.5): it raises the return if the covariance with consumption growth is positive and vice versa. Here, according to Table 3.1, the covariance is negative so this terms leads to lower risk-free rate.

The middle term, $-\frac{1}{2}\gamma(\gamma+1)\sigma_g^2$, is a function of the variability of consumption growth. It is typically interpreted as a "precautionary motive" term. Precisely, the representative agent has a precautionary demand for saving instruments, which would help him to smooth stochastic fluctuations in her consumption. In equilibrium however, all agents cannot engage in precautionary saving at the same time. Agents must be content with their stochastically fluctuating consumption. This means that the return on all saving instruments has to fall sufficiently so the agents do not find it optimal to engage in precautionary saving. Hence, this term tends to lower the returns on all assets, in particular the risk-free rate.

The first term, $\bar{g}^{\gamma}$, arises because consumption is growing over time. The agent anticipates that she will consume a lot more in the future than today, and so she has incentive to borrow against the future to smooth her consumption across times. But, just as before, in equilibrium all agent cannot simultaneously borrow against the future. Agents must be content with their growing consumption. This means that the borrowing rate has to go up so that agents do not find it optimal to engage in borrowing. Hence, this first term tends to increase the return on all assets, in particular the risk-free rate.

Which effect dominates is, at the end of the day, a quantitative matter. The second precautionary saving effect dominates for large enough γ because it is in order γ^2, while all the other effects are in order γ. However, for reasonable values of γ, it is the third effect that dominates. This is illustrated in the right panel of Figure 3.1. One sees that, for values of γ around 13.5, the risk free rate is very large, around 15%.

3.2 Resolutions of the puzzles

The equity premium and risk free rate puzzles have been central topics of research in macroeconomics and finance for the last 30 years. To solve the puzzles, researchers have taken one of three different approaches. First, researchers have relaxed the basic assumptions made about the consumption growth and return processes. Second, some researchers have relaxed the assumption of Constant Relative Risk Aversion that is typically made on preferences. Finally, researchers have relaxed the assumption of complete markets. We discuss examples of these three approaches in Sections 3.2.1, 3.2.2 and 3.2.2 below. For more on the topic, the reader is referred to Kocherlakota (1996).

3.2.1 Rare disasters

In this section, we study a resolution of the puzzle based on relaxing assumptions about the consumption growth and return processes, the so called " rare disasters" approach of Rietz (1988) and Barro (2006). The basic idea is that the last 100 years of growth and returns in the US are not representative of what a consumer may expect going forward. In particular, consumers may rationally expect that his consumption and return processes could be subject to rare disasters: periods of very low growth and return. International evidence does suggest that disaster occur. They are associated with hyperinflation, period of political instability, wars, etc...

To see how rare disasters help with both the equity premium and the risk-free rate, consider the following process for log consumption growth:

$$\frac{c_{t+1}}{c_t} = g_{t+1} = \begin{cases} \bar{g}\exp\left(\varepsilon_{g,t+1} - \frac{1}{2}\sigma_g^2\right) & \text{with probability } 1 - p_D \\ g_D \ll \bar{g} & \text{with probability } p_D. \end{cases}$$

That is, in normal times, with probability $1 - p_D$, consumption growth is drawn from the same distribution as before. In disaster times, with probability p_D, consumption growth is much lower, and equal to $g_D \ll \bar{g}$. The idea of Rietz (1988) and Barro (2006) is that, in the last 100 years in the US, we did not see any realization of the disaster state. But agents have the full probability distribution in mind, including the possibility of disasters, when they make their investment decisions.

Next, we turn to the stochastic process for stock and bond returns. For stock, it is

$$1 + r^s_{t+1} = \begin{cases} (1+\bar{r}^s)\exp\left(\varepsilon_{s,t+1} - \frac{1}{2}\sigma_s^2\right) & \text{with probability } 1-p_D \\ g_D & \text{with probability } p_D. \end{cases}$$

For bonds, it is:

$$1 + r^b_{t+1} = \begin{cases} \left(1+\bar{r}^b\right)\exp\left(\varepsilon_{b,t+1} - \frac{1}{2}\sigma_b^2\right) & \text{with probability } 1-p_D \\ g_D & \text{with probability } p_D q_D \\ 1 & \text{with probability } p_D(1-q_D). \end{cases}$$

In the specification above, $\bar{r}^s$ and $\bar{r}^b$ represent the average return of stock and bond observed in normal times. The goal is then to show that, in the presence of disaster risk, the asset pricing equation puts new restrictions on these normal time average returns. Notice the assumption made on the return on stocks and bonds. Stocks are maximally exposed to disaster risk: their return is the same as the growth rate of the economy. Bonds are less exposed. With some probability, q_D, bond have a return of 0%. With the complementary probability, $1-q_D$, the return is much lower, equal to g_D. This is a key assumption, because it means that bonds are relatively good hedge against disaster risk. This creates an extra demand for bonds, which increases their equilibrium price and lowers their returns, relative to stocks. This increases the equity premium and lowers the risk free rate at the same time. To see these effects at play in equations, we develop $\mathbb{E}\left[M_{t+1}\left(1+r^s_{t+1}\right)\right] = 1$:

$$\beta\left(1-p_D\right)\left(1+\bar{r}^s\right)\bar{g}^{-\gamma}\exp\left[\frac{1}{2}\gamma(\gamma+1)\sigma_g^2 - \gamma\mathrm{cov}\left(\varepsilon_{g,t+1},\varepsilon_{s,t+1}\right)\right] + \beta p_D g_D^{1-\gamma} = 1.$$

The first term is the same as before, but multiplied by the probability $1-p_D$ of a normal time period. The second term is new, and it account for the possibility of disaster. Proceeding in the same way, we obtain for the risk-free rate:

$$\beta\left(1-p_D\right)\left(1+\bar{r}^b\right)\bar{g}^{-\gamma}\exp\left[\frac{1}{2}\gamma(\gamma+1)\sigma_g^2 - \gamma\mathrm{cov}\left(\varepsilon_{g,t+1},\varepsilon_{b,t+1}\right)\right] + \beta p_D\left[q_D g_D^{1-\gamma} + \left(1-q_D\right)g_D^{-\gamma}\right] = 1.$$

Taking the difference we obtain a formula for the equity premium in “normal times” as follows:

$$\begin{aligned}\bar{r}^s - \bar{r}^b = &\left(1+\bar{r}^b\right)\left\{\exp\left[\gamma\mathrm{cov}\left(\varepsilon_{g,t+1},\varepsilon_{s,t+1}-\varepsilon_{b,t+1}\right)\right]-1\right\} \\ &+ \frac{p_D\left(1-q_D\right)}{1-p_D}\left(\frac{g_D}{\bar{g}}\right)^{-\gamma}\left(1-g_D\right)\exp\left[-\frac{1}{2}\gamma(\gamma+1)\sigma_g^2 + \gamma\mathrm{cov}\left(\varepsilon_{g,t+1},\varepsilon_{s,t+1}\right)\right].\end{aligned}$$

We also have the following expression for the average risk-free rate in “normal times”:

$$\begin{aligned}1+\bar{r}^b = &\frac{\bar{g}^{\gamma}}{\left(1-p_D\right)\beta}\exp\left[-\frac{1}{2}\gamma(\gamma+1)\sigma_g^2 + \gamma\mathrm{cov}\left(\varepsilon_{g,t+1},\varepsilon_{b,t+1}\right)\right] \\ &- \frac{p_D}{1-p_D}\left(\frac{\bar{g}_D}{\bar{g}}\right)^{-\gamma}\left[1-q_D\left(1-g_D\right)\right]\exp\left[-\frac{1}{2}\gamma(\gamma+1)\sigma_g^2 + \gamma\mathrm{cov}\left(\varepsilon_{g,t+1},\varepsilon_{b,t+1}\right)\right].\end{aligned}$$

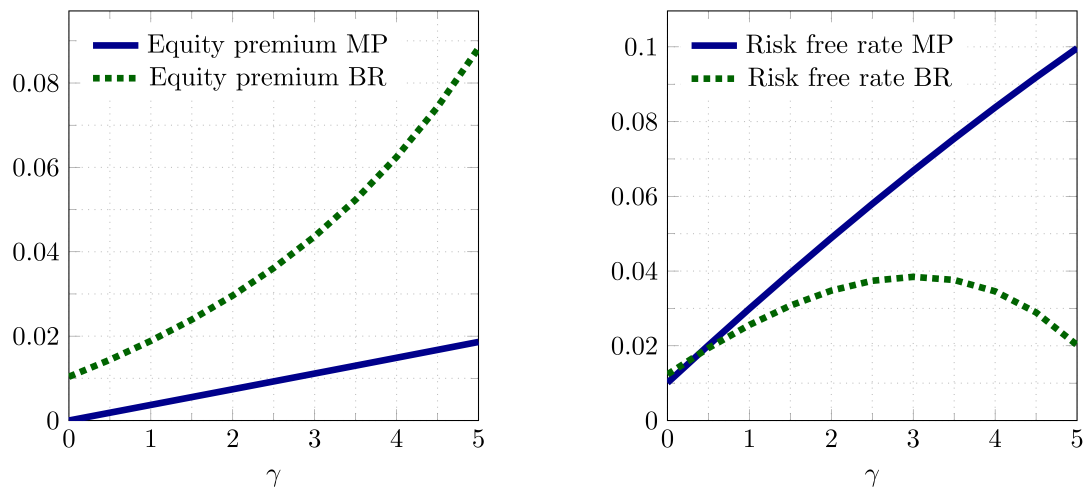


Figure 3.2: The risk premium and the risk free rate with (dashed lines, with BR for "Barrow Rietz") and without (plain lines, with MP for "Merha Prescott") disaster risk, given different values of relative risk aversion. The disaster risk parameters are taken as in Barro (2006): $p_D = 1.7\%$, $g_D = 60\%$, and $q_D = 40\%$. The rest of the parameters are as in Figure 3.1.

For both the equity risk premium and risk-free rate, the first component is identical to the one in the model without disaster risk, in equation (3.1.5) and (3.1.6). But now we have a second component, which reflects the pricing of disaster risk. Notice in particular that the second term in the risk-premium equation is positive only if $q_D < 1$, i.e., only if bonds are differentially exposed to disaster risk.

For his quantitative analysis, Barro (2006) obtains parameters for disaster risk by analyzing economic disasters in a broad cross-section of countries, in the 20^{th} century. For OECD countries, the main disasters were the two World Wars and the great Depression. But is is clear that other disasters also occurred non-OECD countries: for example, in Latin America, contractions of more than 15% of GDP occurred 11 times during the time period under consideration (see Table I in Barro, 2006). All in all, Barro favorite parameter are $p_D = 1.7\%$, $g_D = 60\%$, and $q_D = 40\%$. Figure 3.2 shows the risk premium and the risk free rate with and without disaster risk, for various values of γ. One sees that one can obtain large risk premium and low risk free rate for values of γ in the order of 2 or 3.

3.2.2 Generalized Expected Utility

3.2.2.1 Relative Risk Aversion and Elasticity of Intertemporal Substitution

We have argued that, in a growing economy, interest rates must be higher. This is because agents expect an upward slopping endowment stream, but they prefer a flat consumption stream. Hence they want to borrow against their future endowment to consume more today. However, this cannot happen in equilibrium for all agents at the same time: agents must be content consuming their current endowment. Hence, the rate on all borrowing instruments must increase to discourage borrowing. With CRRA utility, this effect is captured by the term $\bar{g}^{\gamma}$ in equation (3.1.6) for the risk-free rate. In particular, we see that the effect is stronger for larger γ. The reason is that, with CRRA utility, γ not only captures agents' desire to smooth consumption *across states*, but it also captures agents' desire to smooth *across times*. This is true in general of any time- and state–separable intertemporal utility function: additive separability means that agents have similar desires to smooth across state and times.

This observation is often made formally by noting that, with CRRA utility, the relative risk aversion parameter, γ, is inversely related to the Elasticity of Intertemporal Substitution (EIS). To see this, consider a deterministic consumption-saving problem with CRRA utility.

$$\sum_{t\geq 0}\beta^t \frac{c_t^{1-\gamma}-1}{1-\gamma}$$

with respect to $\{c_t, a_t : t \geq 0\}$ and subject to $c_t + \frac{a_{t+1}}{1+r_{t+1}} = y_t + a_t$, a No-Ponzi Game constraint, and $a_0 = 0$. Combining the first-order condition with respect to asset and with respect to consumption, we obtain

$$\beta(1+r_{t+1})\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} = 1 \Rightarrow \frac{d\log\left(\frac{c_{t+1}}{c_t}\right)}{d\log\left(1+r_{t+1}\right)} = \frac{1}{\gamma}.$$

This equation means that, in response to an increase in r_{t+1}, agents will change their saving and borrowing behavior so as to increase the growth rate of consumption between t and $t+1$ – they substitute consumption inter-temporally by either saving more or borrowing less than before. The size of the response depends on $1/\gamma$, which for this reason is termed the elasticity of intertemporal substitution, or EIS. If the response is very large, then clearly this means that agents do not have a strong desire to smooth consumption across times.

3.2.2.2 SDF with Generalized Expected Utility

The EIS and the RRA determine different aspects of agents' behavior. Namely, the EIS governs an agent's willingness to substitute consumption inter-temporally, while RRA governs an agent's willingness to take risk. Hence, there is no fundamental reason why EIS and RRA should be linked together in one single parameter, as is the case with CRRA utility. This observation has lead researchers to use a different utility function in which IES and RRA are separate parameters.

As it turns out, in order to clearly separate EIS from RRA in the utility function, one needs to depart from time separable expected utility and consider more general function of the consumption plan, $U(\{c_t(s^t) : t \geq 0, s^t \in S^t\}$, defined as follows. Given the consumption plan, the continuation utility at time t and history s^t is defined as

$$V_t\left(s^t\right) \equiv U\left(\left\{c_{t+k}\left(s^{t+k}\right), k \geq 0, s^{t+k} \succeq s^t\right\}\right). \tag{3.2.1}$$

The continuation utility is calculated as follows, via the recursion:

$$V_t\left(s^t\right) = W\left\{c_t\left(s^t\right), \mu\left[V_{t+1}\left(s^{t+1}\right)|s^t\right]\right\},$$

a formulation due to Kreps and Porteus (1978). In the recursion, the function μ is a "certainty equivalent" operator. It maps the collection of continuation utility at time $t+1$, $V_{t+1}(s^t, s_{t+1})$, into there certainty equivalent at time t. The function $W(c, \mu)$ is an "aggregator": it gives the time-t

continuation utility as a function of the current consumption, $c_t(s^t)$, and the certainty equivalent of the time $t+1$ continuation utility, $V_{t+1}(s^t, s_{t+1})$. Notice that time separable utility is a special case, with $\mu\left[V_{t+1}\left(s^{t+1}\right) \mid s^t\right] = \mathbb{E}\left[V_{t+1}\left(s^{t+1}\right)|s^t\right]$, and $W(c,\mu) = (1-\beta)u(c) + \beta\mu$.

The common specification used in macro-finance is due to Epstein and Zin (1989), where W takes the form:

$$W(c,\mu) = \begin{cases} \left[(1-\beta)\,c^{1-\eta} + \beta\mu^{1-\eta}\right]^{\frac{1}{1-\eta}}, & \text{if } \eta \neq 1 \\ c^{(1-\beta)}\mu^{\beta}, & \text{if } \eta = 1 \end{cases} \tag{3.2.2}$$

and

$$\mu\left[V_{t+1}\left(s^{t+1}\right)|s^t\right] = \begin{cases} \left[\sum_{s_{t+1}} \pi(s_{t+1}|s^t)V_{t+1}(s^{t+1})^{1-\gamma}\right]^{\frac{1}{1-\gamma}}, & \text{if } \gamma \neq 1 \\ \exp\left(\sum_{s_{t+1}} \pi(s_{t+1}|s^t)\log\left(V_{t+1}(s^{t+1})\right)\right), & \text{if } \gamma = 1. \end{cases} \tag{3.2.3}$$

In this setting, γ is the RRA, while $\frac{1}{\eta}$ is the EIS. If $\gamma = \eta$, we obtain as a special case standard CRRA preferences.

To see why γ is the RRA, imagine that there are two periods, $t \in \{0,1\}$, and that consumption is random and only occur at $t = 1$. Then, the Epstein Zin utility at time zero is

$$\left(\mathbb{E}\left[c_2^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}},$$

is the certainty equivalent of a CRRA utility with coefficient γ. To see why $\frac{1}{\eta}$ is elasticity of intertemporal substitution, consider a non-stochastic environment. In this case, the Epstein-Zin utility can be written as:

$$V_t = \left[(1-\beta)\,c_t^{1-\eta} + \beta V_{t+1}^{1-\eta}\right]^{\frac{1}{1-\eta}}.$$

Re-arranging and iterating forward:

$$V_t^{1-\eta} = (1-\beta)\,c_t^{1-\eta} + \beta V_{t+1}^{1-\eta} = (1-\beta)\sum_{k=0}^{\infty}\beta^k c_{t+k}^{1-\eta},$$

which is the usual time separable utility with constant EIS equal to $1/\eta$.

Asset pricing: the general formula. To study asset pricing in this context, we appeal to the equivalence between time zero and sequential trading, that we studied in Section 1.5. This result only relied on comparing budget sets, and so it continues to apply with the generalized expected utility we consider here. We use our usual notations. We let $q_{0t}(s^t)$ denote the time-zero price of consumption good to be delivered at time t after history s^t, and we let $Q_{t+1}(s_{t+1} \mid s^t)$ denote the price of a one-period ahead Arrow security. Assuming that there is one representative agent with preferences given by (3.2.2) and (3.2.3), we have that:

$$Q_{t+1}\left(s_{t+1}|s^t\right) = \frac{q_{0,t+1}\left(s^{t+1}\right)}{q_{0,t}\left(s^t\right)} = \frac{\dfrac{\partial V_0(s_0)}{\partial c_{t+1}\left(s^{t+1}\right)}}{\dfrac{\partial V_0(s_0)}{\partial c_t\left(s^t\right)}}. \tag{3.2.4}$$

The notation "$\partial V_0(s_0)/\partial c_t(s^t)$" is a short hand for the partial derivative of the time-zero intertemporal utility, defined in equation (3.2.1), with respect to $c_t(s^t)$. Given the representative agent assumption, the partial derivatives are evaluated at the aggregate endowment. Calculating the partial derivative is harder than with time separable utility: we cannot simply differentiate the period utility with respect to $c_t(s^t)$. Instead, we need to apply the chain rule. Fortunately, since preferences have a recursive representation, the chain rule can also be applied recursively. Namely, for all $t \geq 1$:

$$\frac{\partial V_0(s_0)}{\partial c_t\left(s^t\right)} = \frac{\partial W_0(s_0)}{\partial V_1(s^1)}\frac{\partial V_1(s^1)}{\partial c_t(s^t)}.$$

In the above we use the notation "$\partial W_0(s_0)/\partial V_1(s^1)$" as a short hand for the partial derivative of $W\left(c_0\left(s^0\right), \mu\left[V_1\left(s'\right)|s_0\right]\right)$ with respect to $V_1(s^1)$. Iterating forward gives:

$$\frac{\partial V_0(s_0)}{\partial c_t\left(s^t\right)} = \frac{\partial W_0(s_0)}{\partial V_1(s^1)} \times \frac{\partial W_1(s^1)}{\partial V_2(s^2)} \times \ldots \times \frac{\partial W_{t-1}(s^{t-1})}{\partial V_t(s^t)}\frac{\partial V_t(s^t)}{\partial c_t(s^t)}.$$

We then directly taking derivate in (3.2.2) and (3.2.3) to obtain:

$$\frac{\partial W_t\left(s^t\right)}{\partial c_t\left(s^t\right)} = (1-\beta)c_t(s^t)^{-\eta}V_t(s^t)^{\eta},$$

and

$$\frac{\partial W_t\left(s^t\right)}{\partial V_{t+1}\left(s^{t+1}\right)} = \beta\left[\sum_{s_{t+1}}\pi(s_{t+1}|s^t)V_{t+1}(s^{t+1})^{1-\gamma}\right]^{\frac{\gamma-\eta}{1-\gamma}} V_t(s^t)^{\eta}\pi(s_{t+1}|s^t)V_{t+1}(s^{t+1})^{-\gamma}.$$

Plugging all this back in to equation (3.2.4) gives:

$$\begin{aligned} Q_{t+1}\left(s_{t+1}|s^t\right) &= \frac{\dfrac{\partial W_t\left(s^t\right)}{\partial V_{t+1}\left(s^{t+1}\right)} \cdot \dfrac{\partial V_{t+1}\left(s^{t+1}\right)}{\partial c_{t+1}\left(s^{t+1}\right)}}{\dfrac{\partial V_t(s^t)}{\partial c_t\left(s^t\right)}} \\ &= \beta\pi(s_{t+1}|s^t)\left[\frac{c_{t+1}(s^{t+1})}{c_t(s^t)}\right]^{-\eta}\left[\frac{V_{t+1}(s^{t+1})}{\left[\sum_{s_{t+1}}\pi(s_{t+1}|s^t)V_{t+1}(s^{t+1})^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right]^{\eta-\gamma}. \end{aligned}$$

Notice that, when $\eta = \gamma$, we recover the usual CRRA formula. When $\eta \neq \gamma$, the usual CRRA formula is multiplied by a new term, the continuation utility raised to the power $\eta - \gamma$, and normalized by its certainty equivalent.

3.2.2.3 IID growth case of Kocherlakota (1990)

The pricing formula becomes particularly simple when the growth rate of the endowment is i.i.d. In keeping with the log-normal specification I have used so far, I assume that:

$$y_{t+1} = y_t g_{t+1},$$

where the growth g_{t+1} rates are i.i.d. and follows

$$g_{t+1} = \bar{g} \exp\left(\varepsilon_{t+1} - \frac{1}{2}\sigma_g^2\right), \ \varepsilon_{t+1} \sim N(0,1).$$

We guess and verify that, in this case, the continuation value is proportional to the level of the endowment, i.e., $V_t = v y_t(s^t)$, for some constant v to be determined. If the guess is right, then:

$$\begin{aligned} V_t = v y_t &= \left\{(1-\beta)\, y_t^{1-\eta} + \beta \left\{\mathbb{E}_t\left[(v y_t g_{t+1})^{1-\gamma}\right]\right\}^{\frac{1-\eta}{1-\gamma}}\right\}^{\frac{1}{1-\eta}}. \\ &= y_t \left\{1 - \beta + \beta v^{1-\eta} \bar{g}^{1-\eta} \left[\mathbb{E}_t\left[\exp\left((1-\gamma)\varepsilon_{t+1} - \frac{1-\gamma}{2}\sigma_g^2\right)\right]\right]^{\frac{1-\eta}{1-\gamma}}\right\}^{\frac{1}{1-\eta}} \\ &= y_t \left[1 - \beta + \beta v^{1-\eta} \bar{g}^{1-\eta} \exp\left(-\frac{1}{2}\gamma(1-\eta)\sigma_g^2\right)\right]^{\frac{1}{1-\eta}}. \end{aligned}$$

Hence the constant v must solve:

$$v = \left[1 - \beta + \beta v^{1-\eta} \bar{g}^{1-\eta} \exp\left(-\frac{1}{2}\gamma(1-\eta)\sigma_g^2\right)\right]^{\frac{1}{1-\eta}}.$$

One can show that this equation has a unique solution. Having solved for v, the stochastic discount factor is

$$\begin{aligned} M_{t+1} &= \beta g_{t+1}^{-\eta} \left[\frac{v y_t g_{t+1}}{\mathbb{E}_t\left[(v y_t g_{t+1})^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right]^{\eta-\gamma} \\ &= \beta \bar{g}^{-\eta} \exp\left(-\eta\varepsilon_{t+1} + \eta\frac{\sigma_g^2}{2}\right) \left[\frac{\exp(\varepsilon_{t+1})}{\mathbb{E}\left[\exp((1-\gamma)\varepsilon_{t+1})\right]^{\frac{1}{1-\gamma}}}\right]^{\eta-\gamma} \\ &= \beta \bar{g}^{-\eta} \exp(-\gamma\varepsilon_{t+1}) \exp\left(\eta\frac{\sigma_g^2}{2} - (\eta-\gamma)(1-\gamma)\frac{\sigma_g^2}{2}\right). \end{aligned}$$

One sees that the preference specification allows to separate the effect of consumption growth on the risk free rate: now the growth rate of consumption, $\bar{g}$, is raised to a power $-\eta$ instead of -γ. Given that the only stochastic part of the pricing kernel is $\exp(-\gamma\varepsilon_{t+1})$ is the same, the present model genetates the same equity premium as the CRRA model.

3.2.2.4 The almost-IID growth, or "long-run risk", case of Bansal and Yaron (2004)

To obtain a large equity premium at reasonable levels for the RRA, Bansal and Yaron (2004) assume that consumption growth is not i.i.d: instead $\log c_{t+1}/c_t = g_{t+1}$ follows the process:

$$g_{t+1} = \log \bar{g} + x_t + \varepsilon_{g,t+1} - \frac{\sigma_g^2}{2}$$
$$x_{t+1} = \rho x_t + \varphi \varepsilon_{g,t+1}.$$

To understand the manner in which this specification differs from the i.i.d one of Kocherlakota (1990), consider the dynamic effect of a shock $\varepsilon_{g,t+1} = 1$ at time $t+1$, what is commonly referred to as an "impulse response". This shock increases the growth rate of consumption by one init contemporaneously, just as in the i.i.d. case. But, in contrast with the i.i.d. case, the shock also has subsequent impact through the term x_t. Namely, at time $t+2$, consumption growth increases by φ, at $t+3$ by $\rho\varphi$, at $t+4$ by $\rho^2\varphi$ and so on. Thus, the size of the response after $t+1$ is determined by φ, and the persistence of the response by ρ.

In their paper Bansal and Yaron assume that the shock induces a small but quite persistent response, namely, they set φ is close to zero and ρ is close to one. For illustration, in this section we set $\rho = 0.95$ and $\varphi = 0.1$, and we set $\sigma_g^2 = 0.0011$, and otherwise keep all parameters the same as in Section 3.1. Notice that the variance of the shock, σ_g^2, is now set slightly below the variance of consumption growth shown in Table 3.1: this is because consumption growth is not i.i.d. and so variance of the innovation of the shock to g_t is less than the unconditional variance of g_t.

How reasonable is the consumption growth process assumed by Bansal and Yaron? Following earlier authors, they argue that there is a lot of statistical uncertainty surrounding the observed consumption growth process: it is hard to tell apart the i.i.d. from the almost-i.i.d. specification. To see this point more clearly, the bar plot of Figure 3.3 shows the sample auto-correlation of consumption growth (white bars) plus and minus two standard errors (dashed lines above and below the x-axis). The sample auto-correlations are small and the standard errors are large: the picture suggests that the i.i.d. assumption is indeed quite reasonable. But, by the same token, a small auto-correlation is reasonable as well. The picture shows the theoretical auto-correlation if the growth rate process is almost i.i.d. with $\rho = 0.95$ and $\varphi = 0.1$. These theoretical auto-correlations do fall within the standard error bands.

3.2.2.5 Preferences for early vs. late resolution of uncertainty

3.2.3 Incomplete markets

The third approach to solve the equity premium and risk-free rate puzzles is to change the market structure and assume that asset markets are incomplete. The intuition is that incomplete market will make it harder for agent to smooth their consumption. This will make the stochastic discount factor more volatile and may create a larger equity premium. This intuition is incomplete, of course: one also needs to generate a negative correlation between the discount factor and the excess returns. To understand under which conditions this is going to be the case, we adopt the model of Constantinides and Duffie (1996).

There is a measure one continuum of measure one of households with CRRA utility. Households

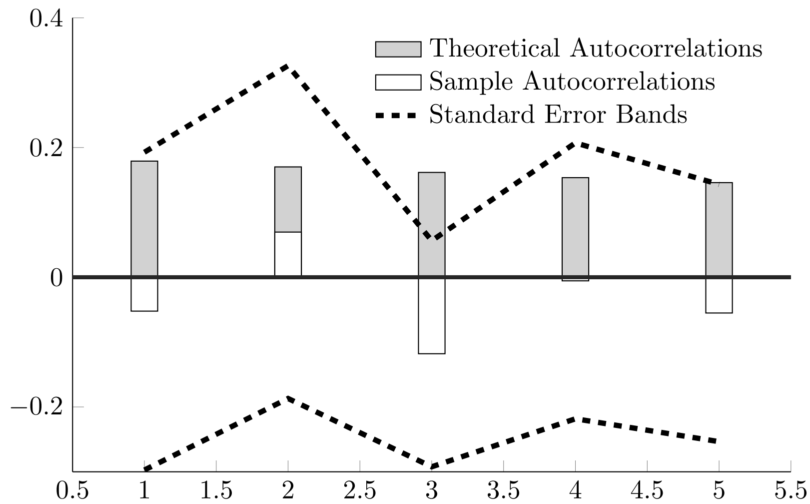


Figure 3.3: The empirical and theoretical autocorrelations, $\text{cov}(g_t, g_{t-k})/\text{var}(g_t)$, of the growth rate process when $\rho = 0.95$ and $\varphi = 0.1$.

face aggregate as well as idiosyncratic labor income risk risk. Every period an aggregate state s is drawn from a finite set S, independently from earlier periods and according to the distribution $\{\pi(s)\}_{s\in S}$; for each household an idiosyncratic state z is drawn from a finite set Z, independently from earlier periods and from the aggregate state, according to the distribution $\{\pi(z)\}_{z\in Z}$. The endowment of an individual agent follows the process:

$$y_{t+1}\left(s^{t+1}, z^{t+1}\right) = y_t\left(s^t, z^t\right) g\left(s_{t+1}\right) h\left(s_{t+1}, z_{t+1}\right).$$

We impose the "adding up" condition:

$$\sum_{z\in Z} \pi(z) h\left(s, z\right) = 1,\ \forall s \in S,$$

which ensures that the growth rate of the aggregate endowment is $g(s_t)$. Namely, at any time t, the Law of Large number implies that the fraction of agent with a history of idiosyncratic shocks z^t is $\pi(z^t)$. Hence, the aggregate endowment can be written:

$$Y_t(s^t) = \sum_{z^t\in Z^t} \pi(z^t) y_t\left(s^t, z^t\right) g\left(s_{t+1}\right) h\left(s_{t+1}, z_{t+1}\right).$$

To verify that $g(s_t)$ is indeed the growth rate of the aggregate endowment notice that:

$$\begin{aligned}
\frac{Y_{t+1}(s^{t+1})}{Y_t(s^t)} &= \frac{\sum_{z^t\in Z^t}\sum_{z_{t+1}\in Z} \pi(z^t)\pi(z_{t+1}) y_t(s^t, z^t) g\left(s_{t+1}\right) h\left(s_{t+1}, z_{t+1}\right)}{\sum_{z^t\in Z^t} \pi(z^t) y_t(s^t, z^t)} \\
&= \sum_{z_{t+1}\in Z} \pi(z_{t+1}) g\left(s_{t+1}\right) h\left(s_{t+1}, z_{t+1}\right) = g(s_{t+1})
\end{aligned}$$

because of the above adding up condition.

Every period agents trade Arrow securities with payoff conditional only on the realization of aggregate state $s_{t+1} \in S$. Let $Q_{t+1}(s_{t+1}|s^t)$ be the price at (t, s^t) of Arrow securities that pay off one unit of consumption in the next period conditional on the realization of state s_{t+1}.

Markets are incomplete since there are no Arrow securities that pay off conditional on idiosyncratic risk histories, z^t: that is, idiosyncratic risk cannot be hedged. In reality, some idiosyncratic risk is clearly insurable. Recent work by Heathcote, Storesletten, and Violante (2014), in a different context, extends the methods of Constantinides and Duffie to analyze economies with partial insurance against idiosyncratic risk.

The problem of an agent is

$$\max \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \sum_{z^t \in Z^t} \beta^t \pi\left(s^t\right) \pi\left(z^t\right) \frac{c_t\left(s^t, z^t\right)^{1-\gamma}}{1-\gamma}$$

with respect to a consumption and asset holding plan, (c, a), and subject to the sequential budget constraint

$$c_t\left(s^t, z^t\right) + \sum_{s_{t+1} \in S} Q_{t+1}\left(s_{t+1}|s^t\right) a_{t+1}\left(s^t, z^t, s_{t+1}\right) = y_t\left(s^t, z^t\right) + a_t\left(s^{t-1}, z^{t-1}, s_t\right),$$

a No Ponzi Game constraint, and the initial condition $a_0(s_0) = 0$.

Definition 3.2.1. An equilibrium consists of a consumption and investment plan

$$(c, a) = \left\{c_t\left(s^t, z^t\right), a_{t+1}\left(s^t, z^t, s_{t+1}\right) : t \geq 0, s^t \in S^t, z^t \in Z^t\right\}$$

and and price system

$$Q = \left\{Q_{t+1}\left(s_{t+1} \,|\, s^t\right) : t \geq 0, s^t \in S^t\right\}$$

such that, (c, a) solves agents' problem given Q and markets clear:

$$\text{Goods market: } \sum_{z^t} \pi\left(z^t\right) c_t\left(s^t, z^t\right) = \sum_{z^t} \pi\left(z^t\right) y_t\left(s^t, z^t\right) = Y_t\left(s^t\right), \ \forall t \geq 0, s^t \in S^t$$
$$\text{Asset markets: } \sum_{z^t} \pi\left(z^t\right) a_{t+1}\left(z^t, s^t, s_{t+1}\right) = 0, \ \forall t \geq 0, s^{t+1} \in S^{t+1}.$$

In general, solving incomplete markets model analytically can be very difficult, especially when there is aggregate risk – it often requires to use numerical methods. The present model turns out to be very easy to solve, because of the following property:

Proposition 3.2.1. *There exists an equilibrium with no trade in which every agent consumes her endowment in each period:*

$$c_t(s^t, z^t) = y_t(s^t, z^t), \ \forall t \geq 0, s^t \in S^t, z^t \in Z^t.$$

Proof. Suppose that the equilibrium is such that agents consume their endowment every period, i.e.

$c_t(s^t, z^t) = y_t(s^t, z^t)$ for all t and s^t. Then, taking the first-order conditions of the agents' maximization problem, evaluated at $c_t(s^t, z^t) = y_t(s^t, z^t)$, we obtain:

$$\begin{aligned} \left[\text{with respect to } c_t(s^t, z^t)\right] &: \; \beta^t \pi\left(s^t\right) \pi\left(z^t\right) \left[y_t\left(s^t, z^t\right)\right]^{-\gamma} = \lambda_t\left(s^t, z^t\right) \\ \left[\text{with respect to } a_t(s^t, z^t, s_{t+1})\right] &: \; \lambda_t\left(s^t, z^t\right) Q_{t+1}\left(s_{t+1} | s^t\right) = \sum_{z_{t+1}} \lambda_{t+1}\left(s^{t+1}, z^t, z_{t+1}\right), \end{aligned}$$

where the λ's are the Lagrange multipliers for the sequential budget constraints. In particular, the second equation implies that:

$$Q_{t+1}(s_{t+1} \mid s^t) = \frac{\sum_{z_{t+1}} \lambda_{t+1}\left(s^{t+1}, z^t, z_{t+1}\right)}{\lambda_t(s^t, z^t)}.$$

This equation says that the price of Arrow security, on the left side, has to be equal to agents' marginal rate of substitution between aggregate states s^{t+1} and s^t, on the right side, assuming that agents consume their endowment. Clearly this cannot be true in general. Indeed, the left-side only depends on the aggregate history, s^{t+1}, while the right-side also depends on the idiosyncratic history z^t. In other words, agents who consume their endowment may disagree about the price of one-step ahead Arrow security, depending on their idiosyncratic history. But everything works out nicely in the particular case studied here. Indeed, replacing the expression for the Lagrange multiplier, we obtain:

$$\begin{aligned} Q_{t+1}\left(s_{t+1} \mid s^t\right) &= \sum_{z_{t+1}} \frac{\beta^{t+1} \pi\left(s^{t+1}\right) \pi\left(z^{t+1}\right) \left[y_{t+1}\left(z^{t+1}, s^{t+1}\right)\right]^{-\gamma}}{\beta^t \pi\left(s^t\right) \pi\left(z^t\right) \left[y_t\left(z^t, s^t\right)\right]^{-\gamma}} \\ &= \beta \pi\left(s_{t+1}\right) g\left(s_{t+1}\right)^{-\gamma} \sum_{z_{t+1}} \pi(z_{t+1}) \left[h\left(s_{t+1}, z_{t+1}\right)\right]^{-\gamma}. \end{aligned}$$

Hence, the right-side turns out not to depend on the idiosyncratic history z^t, so that agents do agree about the price of Arrow security. Conversely, if the price of one-step ahead Arrow security satisfies the above equation, one can verify that agents indeed find it optimal to consume their endowment every period. □

This result relies on two key assumptions homothetic preferences, and iid endowment growth rates. It implies that agents have the same *expected* intertemporal marginal rate of substitution, $\mathbb{E}\left[\beta(c_{t+1}/c_t)^{-\gamma} \mid s_{t+1}, s_t\right]$, conditional on the realization of today and tomorrow's aggregate state. So, all agents agree on the price of Arrow securities that payoff conditional on the aggregate state.

The no-trade result is very convenient but is clearly counterfactual. But it turns out that the insights are much more general: they continues to hold in economies in which agents do trade assets, as shown by Kruger and Lustig (2010). With this in mind, I move on to study the asset pricing implications of the model. Note that $H\left(s_{t+1}\right) \equiv \sum_{z_{t+1}} \pi(z_{t+1}) \left[h\left(s_{t+1}, z_{t+1}\right)\right]^{-\gamma}$ represents the state price adjustment due to non-insurable income risk and depends on whether idiosyncratic labor income risk is pro-cyclical, counter-cyclical, or a-cyclical.

If $h(s,z) = h(z)$, i.e. idiosyncratic labor income risk is a-cyclical,

$$M_{t+1}\left(s_{t+1}|s^t\right) = \beta g\left(s_{t+1}\right)^{-\gamma} \sum_{z_{t+1}} \pi(z_{t+1}) h\left(z_{t+1}\right)^{-\gamma} > \beta g\left(s_{t+1}\right)^{-\gamma}.$$

In the above, the last inequality follows from Jensen's inequality: we have imposed earlier the adding up condition that $\sum \pi(z)h(z) = 1$, and the function $h \mapsto h^{-\gamma}$ is convex. One sees that the level of the SDF changes: this is because households have more volatile consumption and so they have a precautionary demand for saving instruments. In equilibrium, all households cannot simultaneously increase their savings, so the the return on all saving instruments has to fall. This effect may explain the risk-free rate puzzle. However, while the level of the SDF changes, its correlation with stock excess returns remains the same, and so the equity premium remains the same. Indeed, recall the formula (3.1.4) for the equity premium we derived before:

$$\mathbb{E}\left[r^s_{t+1} - r^b_{t+1}\right] = -\text{cov}\left(\frac{M_{t+1}}{\mathbb{E}\left[M_{t+1}\right]}, r^s_{t+1} - r^b_{t+1}\right).$$

As we argued above, when labor income risk is a-cyclical, then the SDF is scaled up in all aggregate states by the *same* constant, then the equity premium will not change.

To increase the equity premium, incomplete market must scale the households' SDF up by more in bad times than in good times. Incomplete market must make households marginal utility larger, on average, conditional on a bad time than conditional on good time. This can be achieved by departing from the assumption that idiosyncratic labor income risk is a-cyclical.

Namely, to scale the SDF up by more in bad times than in good times, we need

$$\sum_{z_{t+1}} \pi(z_{t+1}) \left[h\left(s_{t+1}, z_{t+1}\right)\right]^{-\gamma}$$

to be larger in bad times than in good times. Since $h \mapsto h^{-\gamma}$ is convex, this means that we need countercyclical labor income risk: more variance in bad times and less variance in good times. Notice that the same precautionary saving effect as above is at play, but differentially across states. Namely, if tomorrow is a good time, then households anticipate that their consumption growth will have less idiosyncratic volatility. Households have smaller precautionary motive. All else equal this lowers the price of saving instruments that pay conditional on a good time. Symmetrically, the same effect increases the price of saving instruments that pay conditional on a bad time.

Storesletten, Telmer, and Yaron (2004) have shown that labor income risk is indeed countercyclical using data from the Panel Study of Income Dynamics (PSID). Recently, Guvenen, Ozkan, and Song (2014) have studied income risk using social security data – which is comprehensive over 35 years, and allows for direct non-parametric measurements. They show that labor income *volatility* is a-cyclical, but that *skewness* is countercyclical. The asset pricing implication of this evidence has been recently explored by Schmidt (2013).