NOTES ON A SIMPLE OVERLAPPING GENERATIONS GROWTH MODEL

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1. Model Equations

Time is denoted $t = 1, 2, 3, \ldots$

In each period t, a cohort of size $N_t = \exp(tg_N)$ of young agents is born. These agents are each endowed with one unit of labor when young. Thus, their total endowment of labor for the cohort as a whole is N_t. They have no other endowment. At time $t+1$, these agents are old. We denote their consumption while young and old by c_t^y, c_{t+1}^o.

The agents born at time t have preferences

$$U(c_t^y, c_{t+1}^o) \tag{1}$$

For most examples, we use the specific utility function

$$\log(c_t^y) + \beta \log(c_{t+1}^o)$$

Exercise Note that this utility function has the property that young agents save fraction $\beta/(1+\beta)$ of their wealth regardless of the interest rate R available on saving. To see this point, solve the problem

$$\max_{c^y, c^o} \log(c_t^y) + \beta \log(c_{t+1}^o)$$

subject to the budget constraint

$$c^y + \frac{1}{R} c^o = Wealth$$

for any value of $R > 0$ and observe that the solution is

$$c^y = \frac{1}{1+\beta} Wealth$$

$$c^o = R \frac{\beta}{1+\beta} Wealth$$

We let $\{W_t, R_{kt}\}_{t=1}^{\infty}$ denote the wage rates and rental rates on physical capital at all dates $t = 1, 2, \ldots$. We let $\{R_t\}_{t=1}^{\infty}$ denote the (gross) interest rates on savings in bonds by the young. We assume that the government taxes wage income at rate τ_t^W and capital rental income at rate τ_t^K.

The agents born at t choose consumption c_t^y, c_{t+1}^o, savings in bonds b_{t+1}^y, and holdings of physical capital while old $k_{t+1}^y \geq 0$ and have budget constraints

$$c_t^y = (1-\tau_t^W)W_t - b_{t+1}^y - k_{t+1}^y \tag{2}$$

$$c_{t+1}^o = \left[(1-\tau_{t+1}^K)R_{kt+1} + (1-\delta)\right] k_{t+1}^y + R_t b_{t+1}^y \tag{3}$$

At time $t=1$, there is a cohort of initial old of size $N_0 = 1$, their consumption is denoted by c_1^o. These agents are endowed with the initial stock of physical capital K_1 and the principal and interest due on the initial stock of government bonds $R_0 B_1$. These agents consume the rental income from their capital, the proceeds from selling the undepreciated capital, and the principal and interest due on government bonds

$$c_1^o = \left[(1-\tau_1^K)R_{k1} + (1-\delta)\right] K_1 + R_0 B_1 \tag{4}$$

There is a government that chooses government spending $\{G_t\}_{t=1}^\infty$, tax rates $\{\tau_t^W, \tau_t^K\}_{t=1}^\infty$, and issuances of government bonds $\{B_{t+1}\}_{t=1}^\infty$. The initial principal and interest due on government bonds $R_0 B_1$ is taken as given. The budget constraint for the government is given by

$$B_{t+1} = G_t - \tau_t^W W_t N_t - \tau_t^K R_{kt} K_t + R_{t-1} B_t \tag{5}$$

Output is a Cobb-Douglas function of physical capital and labor

$$Y_t = K_t^\alpha N_t^{1-\alpha} \tag{6}$$

The representative firm's profit maximization problem is given by

$$\max_{k,l} k^\alpha l^{1-\alpha} - R_{kt} k - W_t l$$

where R_{kt} is the rental rate on capital and W_t is the wage rate at t. Standard arguments give that in equilibrium

$$R_{kt} K_t = \alpha Y_t, \quad \textit{and} \quad R_{kt} = \alpha \frac{Y_t}{K_t} \tag{7}$$

$$W_t N_t = (1-\alpha) Y_t, \quad \textit{and} \quad W_t = (1-\alpha)\frac{Y_t}{N_t} \tag{8}$$

The resource constraints for this economy are

$$N_t c_t^y + N_{t-1} c_t^o + G_t + K_{t+1} - (1-\delta)K_t = Y_t \tag{9}$$

The capital and bond market clearing conditions are

$$K_{t+1} = k_{t+1}^y N_t \tag{10}$$

and

$$B_{t+1} = b_{t+1}^y N_t \tag{11}$$

with $N_t = \exp(t g_N)$.

Definition of Equilibrium: Given population $\{N_t\}_{t=0}^{\infty}$, initial capital stock K_1, initial principal and interest due on government bonds R_0B_1, and government policies $\{G_t, \tau_t^W, \tau_t^K, B_{t+1}\}_{t=1}^{\infty}$, an equilibrium is a collection of prices $\{R_t, R_{kt}, W_t\}_{t=1}^{\infty}$ and an allocation $\{c_t^y, c_t^o, K_t, Y_t\}_{t=1}^{\infty}$ together with bond and capital holdings of the young $\{b_{t+1}^y, k_{t+1}^y\}_{t=1}^{\infty}$, such that for all dates $t \geq 1$, the budget constraint for the initial old 4 and for the government 5 are satisfied, output is given as in equation 6 and the resource constraint 9 and the capital and bond market clearing conditions 10 and 11 are satisfied, the rental rate on physical capital and the wage rate are consistent with profit maximization as in equations 7 and 8, and, given prices, the allocation of consumption, bonds, and physical capital for agents born at t, $c_t^y, c_{t+1}^o, b_{t+1}^y, k_{t+1}^y$, maximize these agents' utility 1, subject to the budget constraints 2 and 3.

Comment on National Savings and Investment

Note that aggregate private consumption at t in this model is given by

$$N_t c_t^y + N_{t-1} c_t^o$$

so that national saving, equal to output less private and public consumption, is given by

$$Y_t - (N_t c_t^y + N_{t-1} c_t^o + G_t)$$

From equation 9, we have that aggregate saving equals aggregate investment. This has to hold in our economy simply as an accounting identity (output equals expenditure).

This model does have the property that the saving done by the young does play a special role: the saving of the young purchases all of the physical capital and bonds outstanding in the economy. This is a different concept than savings equals investment. It is that the young need to buy the outstanding stock of assets in the economy. To see this point, note that the budget constraint of the old 3 determines their consumption. The observation that since the aggregate production function is constant returns to scale,

$$Y_t = R_{kt} K_t + W_t L_t$$

together with the resource constraint 9 and the budget constraint for the old 3 and the market clearing conditions 10 and 11 imply that

$$N_t c_t^y + G_t + K_{t+1} = W_t N_t + \tau_t^K R_{kt} K_t - R_{t-1} B_t$$

Now use the government budget constraint 5 to get that

$$B_{t+1} + K_{t+1} = (1 - \tau_t^W) W_t N_t - N_t c_t^y \tag{12}$$

The left hand side is the outstanding stock of bonds and capital carried into period $t+1$ (measured at the end of period t) while the right hand side is the personal saving of the young (their disposable income less their consumption). This equation will play an important role in solving the model below.

2. Solving for Equilibrium

We first solve for equilibrium in which the government issues no bonds, so $B_t = 0$ for all t. We consider government bonds in the subsequent problems. We solve for equilibrium in the following steps.

We first solve the utility maximization problem of the agents born at time t. Observe first that in any equilibrium in which agents hold positive amounts of physical capital $k^y_{t+1} > 0$, we must have that the after tax return to physical capital is equal to the interest rate on bonds, that is

$$R_t = (1 - \tau^K_{t+1})R_{kt+1} + (1 - \delta) \tag{13}$$

Exercise. Show that if this equation were not satisfied, then the agent could increase his or her utility by shifting his or her portfolio between capital k^y_{t+1} and bonds b^y_{t+1}, thus violating the equilibrium condition of utility maximization. Note that while there is a constraint that $k^y_{t+1} \geq 0$, there is no constraint that $b^y_{t+1} \geq 0$

Then observe that we can consolidate the two budget constraints 2 and 3 of the agents born at time t into a single lifetime budget constraint

$$c^y_t + \frac{1}{R_t}c^o_{t+1} = Wealth = (1 - \tau^W_t)W_t \tag{14}$$

Exercise. Show that with log utility as in equation 1, the optimal allocation of consumption is given by

$$c^y_t = \frac{1}{1+\beta}Wealth = \frac{1}{1+\beta}(1 - \tau^W_t)W_t \tag{15}$$

This result gives us two special properties of our model. First, as a result of the assumption of log utility, we have that the agents born at time t consume a constant fraction $1/(1 + \beta)$ of their lifetime wealth. Second, because these agents are endowed only with labor when young, their lifetime wealth does not depend on interest rates. These two features of our model will make it particularly easy to solve, because we can solve for labor income as a function of the current capital stock and, with that solution, we will be able to solve for the savings of the young independently of the interest rate. We would not be able to do this if we used something other than log utility (like CRRA utility with relative risk aversion different than one) or if the agents born at t has an endowment at $t + 1$ (which would make their lifetime wealth a function of the interest rate).

We now solve for the savings of the young in equilibrium. We impose that $B^y_{t+1} = 0$ as required by the bond market clearing condition 11 when the government does not issue bonds. We use the budget constraint 2 together with our solution for consumption when young in equation 15 to get

$$k^y_{t+1} = \frac{\beta}{1+\beta}(1 - \tau^W_t)W_t.$$

Multiplying both sides of this equation by L_t and using equations 6, 8 and 10, we get

$$K_{t+1} = \frac{\beta}{1+\beta}(1-\tau_t^W)(1-\alpha)K_t^\alpha L_t^{1-\alpha} \tag{16}$$

With this equation and an initial value of K_1 together with the exogenous labor supply $N_t = \exp(tg_N)$, we can solve for the entire sequence of equilibrium capital stocks $\{K_{t+1}\}_{t=1}^\infty$.

Once we have the entire sequence of capital stocks, we compute the sequence of output $\{Y_t\}_{t=1}^\infty$ from equation 6, the sequences of rental rates of capital and wages $\{R_{kt}, W_t\}_{t=1}^\infty$ from equations 7 and 8. The sequence of equilibrium interest rates $\{R_t\}_{t=1}^\infty$ is given by equation 13. The sequence of consumption of the young $\{c_t^y\}_{t=1}^\infty$ is given from equation 15. The sequences of capital and bond holdings of the young are given from equations 10 and 11 with $B_t = 0$ for all t. The sequence of consumption of the old $\{c_{t+1}^o\}_{t=1}^\infty$ is given from equations 3 and 4 with capital and bond holdings as specified above. The sequence of government expenditure can then be found from either the resource constraint 9 or the government budget constraint 5. You should verify that if one of these holds, the other does as well.

2.1. **Steady-state.** We now characterize the steady state to which this economy converges.

To solve for steady-state, we continue with the assumption that bonds are equal to zero, $B_t = 0$ for all t. We assume that tax rates are constant over time at $\tau_t^W = \tau^W$ and $\tau_t^K = \tau^K$. Because the population in this economy is growing, we have that the capital stock grows as well. The steady-state in this economy is characterized by a constant capital labor ratio which we solve for from equation 16. Define $k_t = K_t/N_t$ to be the capital labor ratio. If we divide both sides of this equation by N_t, we get

$$k_{t+1}\frac{N_{t+1}}{N_t} = \frac{\beta}{1+\beta}(1-\tau^W)(1-\alpha)k_t^\alpha \tag{17}$$

The steady-state capital output ratio in this economy is then given by the positive constant capital labor ratio that satisfies equation 17

$$k_{ss} = \left[\exp(-g_N)\frac{\beta}{1+\beta}(1-\tau^W)(1-\alpha)\right]^{1/(1-\alpha)} \tag{18}$$

Let $y_t = Y_t/N_t$ denote output per worker and $g_t = G_t/N_t$ denote government spending per worker. We then have

$$y_{ss} = k_{ss}^\alpha$$
$$W_{ss} = (1-\alpha)y_{ss} = (1-\alpha)k_{ss}^\alpha$$
$$R_{kss} = \alpha\frac{y_{ss}}{k_{ss}} = \alpha k_{ss}^{\alpha-1}$$

$$c^y_{ss} = \frac{1}{1+\beta}(1-\tau^W)(1-\alpha)k^\alpha_{ss}$$
$$c^o_{ss} = \exp(g_L)\left[(1-\tau^K)R_{kss} + (1-\delta)\right]k_{ss}$$

and

$$g_{ss} = \left[\tau^W(1-\alpha) + \tau^K\alpha\right]k^\alpha_{ss}$$

The steady-state interest rate is given by

$$R_{ss} = \left[(1-\tau^K)R_{kss} + (1-\delta)\right] = \exp(g_N)\frac{1+\beta}{\beta}\frac{(1-\tau^K)}{(1-\tau^W)}\frac{\alpha}{(1-\alpha)} + (1-\delta)$$

3. Problems

Problem 1: Use equation 17 to show that there is a unique positive value of the capital labor ratio k_{ss} to which the economy converges from any initial positive capital labor ratio k_1. Moreover, show that, in logs, this convergence is linear, i.e.

$$\log k_{t+1} - \log k_{ss} = \alpha\left(\log k_t - \log k_{ss}\right)$$

Problem 2: In this problem, we want to consider whether the competitive equilibrium without government is Pareto Optimal. So, continue with the assumption that $B_t = 0$ and assume that $\tau^K = \tau^W = 0$ and thus $G_t = 0$ for all t as well. This corresponds to a baseline of no government. To simplify the calculations below, assume that $\delta = 1$, so that there is full depreciation of capital from one generation to the next. The steady-state allocation is given as above.

Now imagine that the economy starts as time $t = 1$ with population of the initial old of $N_0 = 1$ and of the young of $N_1 = \exp(g_N)$ and with initial capital stock $K_1 = \exp(g_L)k_{ss}$. As before, we have $N_t = \exp(tg_N)$ for all t. Assume that in period $t = 1$, the government permanently lowers taxes on capital below zero $\tau^K < 0$ (subsidizing capital) and raises taxes on labor $\tau^W > 0$ in a manner that leaves government tax collections unchanged at zero (i.e. so that the government budget constraint 5 holds with $G_t = B_t = 0$ for all t). Specifically set $\tau^K = -\epsilon$ for some $\epsilon > 0$.

Part A: Compute the required value of τ^W in terms of ϵ to balance the government budget.

Part B: Compute the new steady state allocation, using a tilde over a variable to denote the new steady-state value. For each variable, compute the ratio of the new to the old steady-state value. Show that the new steady-state capital labor ratio is smaller than the original steady-state capital labor ratio.

Part C: Compute the change in lifetime utility for agents across steady-states, that is

$$\log(\tilde{c}^y_{ss}) - \log(c^y_{ss}) + \beta\left[\log(\tilde{c}^o_{ss}) - \log(c^o_{ss})\right]$$

Give conditions on parameters such that this change in utility is positive for small values of ϵ.

Part D: Assume that the condition on parameters needed to insure that the difference in utilities across steady-states from this tax policy is positive, give an argument that if the government implemented this tax policy at $t = 1$ in the economy that started at the steady-state level of capital corresponding to zero taxes, then there exists a level of ϵ such that all agents would benefit, including the initial old generation at $t = 1$.

Part E: Assume that

$$\beta > \frac{\alpha}{1-2\alpha}, \alpha \in (0, 1/2).$$

Assume that the economy starts as time $t = 1$ with population of the initial old of $N_0 = 1$ and of the young of $N_1 = \exp(g_N)$ and with initial capital stock $K_1 = \exp(g_N)k_{ss}$. As before, we have $N_t = \exp(tg_N)$ for all t. Assume that in period $t = 1$, the government permanently raises taxes on capital above zero $\tau^K = \epsilon$ and subsidizes labor $\tau^W < 0$ in a manner that leaves government tax collections unchanged at zero (i.e. so that the government budget constraint 5 holds with $G_t = B_t = 0$ for all t).

Show that this policy cannot result in a Pareto improvement over the equilibrium allocation with taxes set to zero.

Problem 3: In this problem, we study the dynamics of our economy when the government has outstanding government bonds. Specifically, assume that the economy starts with capital stock K_1 and initial principal and interest due on its bonds of R_0B_1. We look to solve for the equilibrium path of capital and bonds $\{K_{t+1}, B_{t+1}\}_{t=1}^{\infty}$ and the corresponding capital and bondholdings of the young $\{k_{t+1}^y, b_{t+1}^y\}_{t=1}^{\infty}$.

Assume that the government sets tax and spending policies so that the government's primary surplus is always balanced. Assume that government spending is a constant share of output, $G_t = \bar{g}Y_t$. If

$$G_t = \tau_t^W W_t L_t + \tau_t^K R_{kt} K_t$$

for all t, then, we can set constant taxes $\tau_t^W = \tau^W$ and $\tau_t^K = \tau^K$ that satisfy

$$\bar{g} = \left[\tau^W(1-\alpha) + \tau^K\alpha\right]$$

and have the evolution of the stock of government bonds given from the government budget constraint 5 reduces to

$$\frac{B_{t+1}}{N_{t+1}}\exp(g_N) = R_{t-1}\frac{B_t}{N_t} \tag{19}$$

Note that under these assumptions, the government only raises enough taxes to pay for current government expenditure. It does not raise taxes to pay the interest and principle due on government bonds. We want to explore what happens with a government that acts this way. Does this fiscal policy violate the government budget constraint? What impact does it have on the economy? Specifically, we look to solve our model to see if there exist equilibria corresponding to such government policies and initial conditions.

The bond market clearing condition 11 implies that $b^y_{t+1} = B_{t+1}/N_t$. The budget constraint for young agents 2 together with the solution for equilibrium consumption of the young 15, output 6, and wages 8 and multiplying by N_t implies that

$$\frac{K_{t+1}}{N_{t+1}} \exp(g_N) = \frac{\beta}{1+\beta}(1-\alpha)\left(\frac{K_t}{N_t}\right)^{\alpha} - \exp(g_N)\frac{B_{t+1}}{N_{t+1}} \tag{20}$$

Equation 13 together with equation 7 imply that the interest rate on bonds is given by

$$R_t = (1-\tau^K)\alpha\left(\frac{K_{t+1}}{N_{t+1}}\right)^{\alpha-1} + (1-\delta) \tag{21}$$

Given initial conditions K_1/N_1 and R_0B_1/N_1, we can use equations 19, 20, and 21 to solve for a sequence of capital labor and bond labor ratios and bond interest rates $\{K_{t+1}/N_{t+1}, B_{t+1}/N_{t+1}, R_t\}$. Note that this sequence can be used to construct an equilibrium of this economy with these policies only if the implied sequence of capital stocks $\{K_{t+1}\}$ remains positive for all values of t. Note that $K_{t+1} > 0$ is sufficient to ensure that the interest rate R_t is always a positive number, and this in turn implies that if the initial stock of government debt is positive, then government debt is always positive.

We now consider which types of competitive equilibria we can have. It is clear that we can have equilibria in which the initial stock of outstanding debt $R_0B_1 = 0$ as this is the type of equilibria we solved for above. We now look for equilibria in which $B_{t+1} > 0$ for all t. We first find a steady-state in which B_{t+1}/N_t and K_{t+1}/N_{t+1} are constant over time. We then consider whether the economy converges to this steady-state or not.

Part A: Find a steady-state in which B_{t+1}/N_t and K_{t+1}/N_{t+1} are constant over time. To do so, first show that the interest rate in this steady-state must satisfy $R_{ss} = \exp(g_N)$. Then find the level of K_{t+1}/N_{t+1} consistent with this interest rate. Then solve for the level of debt B_{t+1}/N_{t+1}.

Part B: Assume that the economy starts at time $t = 1$ with the steady-state level of capital you found in part A, i.e., $K_1/N_1 = k_{ss}$, and initial principal and interest on the debt due greater than the steady-state level of debt you found in part A, i.e. $R_0B_1/N_1 > b_{ss}$. Are the sequences of capital stocks, bonds, and interest rates implied by equations 19, 20, and 21 consistent with equilibrium? If not, why not?

Part C: Assume that the economy starts at time $t = 1$ with the steady-state level of capital you found in part A, i.e., $K_1/N_1 = k_{ss}$, and initial principal and interest on the debt due smaller than the steady-state level of debt you found in part A, i.e. $R_0B_1/N_1 < b_{ss}$. Are the sequences of capital stocks, bonds, and interest rates implied by equations 19, 20, and 21 consistent with equilibrium?