Econ203B HW3

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1 Question 2

The optimal asymptotic weight matrix takes the form

$$\Omega = (\mathbb{E}[ZZ'U^2])^{-1}.$$

Homeskedasticity implies

$$\Omega = \frac{1}{\sigma^2}(\mathbb{E}[ZZ'])^{-1}.$$

The optimal weighting matrix should converge in probability to $\frac{1}{\sigma^2}(\mathbb{E}[ZZ'])^{-1}$, so the asymptotic variance is

$$\Sigma_0 = \sigma^2(\mathbb{E}[XZ'](\mathbb{E}[ZZ'])^{-1}\mathbb{E}[ZX'])^{-1}.$$

2SLS is equivalent to IV with the weighting matrix

$$\Omega = \left(\frac{1}{n}\sum_{i=1}^{n} Z_i Z_i'\right)^{-1},$$

which converges in probability to

$$\Omega = \sigma^2(\mathbb{E}[XZ'](\mathbb{E}[ZZ'])^{-1}\mathbb{E}[ZX'])^{-1}.$$

This is the same as the optimal weighting matrix, so 2SLS is efficient.

2 Question 5

a)

Dep. Variable:	lwage	**R-squared:**	0.228
Model:	OLS	**Adj. R-squared:**	0.227
Method:	Least Squares	**F-statistic:**	221.4
Date:	Mon, 22 Feb 2021	**Prob (F-statistic):**	9.75e-167
Time:	00:17:18	**Log-Likelihood:**	-1436.5
No. Observations:	3010	**AIC:**	2883.
Df Residuals:	3005	**BIC:**	2913.
Df Model:	4		

	coef	**std err**	**t**	**P>|t|**	**[0.025**	**0.975]**
const	3.3152	0.666	4.974	0.000	2.008	4.622
educ	0.0423	0.003	15.281	0.000	0.037	0.048
black	-0.2327	0.017	-13.310	0.000	-0.267	-0.198
age	0.1334	0.047	2.850	0.004	0.042	0.225
agesquared	-0.0016	0.001	-2.016	0.044	-0.003	-4.5e-05

Omnibus:	38.147	**Durbin-Watson:**	1.779
Prob(Omnibus):	0.000	**Jarque-Bera (JB):**	43.474
Skew:	-0.222	**Prob(JB):**	3.63e-10
Kurtosis:	3.387	**Cond. No.**	7.71e+04

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.
[2] The condition number is large, 7.71e+04. This might indicate that there are strong multicollinearity or other numerical problems.

b)

Dep. Variable:	lwage	**R-squared:**	-0.345
Model:	IV2SLS	**Adj. R-squared:**	-0.347
Method:	Two Stage Least Squares	**F-statistic:**	103.3
		Prob (F-statistic):	1.48e-82
Date:	Mon, 22 Feb 2021		
Time:	00:17:18		
No. Observations:	3010		
Df Residuals:	3005		

	coef	**std err**	**t**	**P>|t|**	**[0.025**	**0.975]**
const	3.4235	0.880	3.891	0.000	1.698	5.149
educ	0.1729	0.028	6.239	0.000	0.119	0.227
black	-0.0092	0.052	-0.175	0.861	-0.112	0.094
age	-0.0021	0.068	-0.031	0.976	-0.136	0.131
agesquared	0.0008	0.001	0.639	0.523	-0.002	0.003

Omnibus:	12.417	**Durbin-Watson:**	1.833
Prob(Omnibus):	0.002	**Jarque-Bera (JB):**	12.971
Skew:	-0.127	**Prob(JB):**	0.00153
Kurtosis:	3.198	**Cond. No.**	7.71e+04

c) Yes, this estimator is efficient, since we have the same number of instruments and explanatory variables. In this case the weighting matrix doesn't matter.

d)

Dep. Variable:	lwage	**R-squared:**	-0.474
Model:	IV2SLS	**Adj. R-squared:**	-0.476
Method:	Two Stage Least Squares	**F-statistic:**	96.69
		Prob (F-statistic):	1.71e-77
Date:	Mon, 22 Feb 2021		
Time:	00:17:18		
No. Observations:	3010		
Df Residuals:	3005		

	coef	**std err**	**t**	**P> |t|**	**[0.025**	**0.975]**
const	3.4350	0.921	3.730	0.000	1.629	5.241
educ	0.1868	0.028	6.707	0.000	0.132	0.241
black	0.0146	0.053	0.275	0.783	-0.089	0.119
age	-0.0165	0.071	-0.233	0.816	-0.155	0.122
agesquared	0.0010	0.001	0.822	0.411	-0.001	0.003

Omnibus:	8.927	**Durbin-Watson:**	1.830
Prob(Omnibus):	0.012	**Jarque-Bera (JB):**	9.183
Skew:	-0.106	**Prob(JB):**	0.0101
Kurtosis:	3.167	**Cond. No.**	7.71e+04

e) No, the first IV estimator isn't "efficient" in the normal (non-Econ) sense of the word, because we have additional information (another instrument) that we aren't using.

```python
import numpy as np
import pandas as pd
import statsmodels.api as sm
import scipy.io
from statsmodels.sandbox.regression.gmm import IV2SLS


def load_matlab_data(filename='Schooling.mat'):
    matlab_data = scipy.io.loadmat(filename)
    df = pd.DataFrame(
        columns=[data_field for data_field in matlab_data.keys() if data_field[0] != '_']
        )
    df = df.drop('ans', axis='columns')
    df = df.drop('None', axis='columns')
    for column in df.columns:
        print(column)
        df[column] = matlab_data[column].flatten()
    return df


def run_OLS(dataframe, filename=False):
    X = df[['educ', 'black', 'age', 'agesquared']]
    Y = df[['lwage']]
    X = sm.add_constant(X)
    lm = sm.OLS(Y, X)
    results = lm.fit()
    print(results.summary())
    if filename:
        with open(filename, 'w') as text_file:
            print(f'{results.summary().as_latex()}', file=text_file)


def run_IV(dataframe, instruments, filename=False):
    X = df[['educ', 'black', 'age', 'agesquared']]
    Z = df[instruments]
    Y = df[['lwage']]
    X = sm.add_constant(X)
    Z = sm.add_constant(Z)
    iv = IV2SLS(Y, X, Z)
    results = iv.fit()
    print(results.summary())
    if filename:
        with open(filename, 'w') as text_file:
            print(f'{results.summary().as_latex()}', file=text_file)


if __name__ == '__main__':
    df = load_matlab_data()
    run_OLS(dataframe=df, filename='ols.tex')
    run_IV(dataframe=df, instruments=['nearc4', 'black', 'age', 'agesquared'],
           filename='iv1.tex')
    run_IV(dataframe=df, instruments=['nearc2', 'nearc4', 'black', 'age', 'agesquared'],
           filename='iv2.tex')
```

3 Question 6

a) We will need assumptions IV1–4.

$$\begin{aligned}
\hat{\beta}_{1n} &= \arg\min \|\frac{1}{\sqrt{n}}\sum_{1=1}^{n}(Y_i - X_i'b)Z_{1i}\|^2 \\
&= \arg\min \mathbb{Y}'\mathbb{Z}_1\mathbb{Z}_1'Y - 2b'\mathbb{X}'\mathbb{Z}_1\mathbb{Z}_1'\mathbb{Y} + b'\mathbb{X}'\mathbb{Z}_1\mathbb{Z}_1'\mathbb{X}b \\
\hat{\beta}_{1n} &= (\mathbb{Z}_1'\mathbb{X})^{-1}\mathbb{Z}_1'\mathbb{Y} \qquad \text{(FOC)} \\
&= \beta_0 + (\mathbb{Z}_1'\mathbb{X})^{-1}\mathbb{Z}_1'\mathbb{U} \\
(\mathbb{Z}_1'\mathbb{X})^{-1}\mathbb{Z}_1'\mathbb{U} &= \left(\frac{1}{n}\mathbb{Z}_1'\mathbb{X}\right)^{-1}\frac{1}{n}\mathbb{Z}_1'\mathbb{U} \\
&\xrightarrow{p} 0 \\
\sqrt{n}(\hat{\beta}_{1n} - \beta_0) &= \sqrt{n}(\beta_0 + (\mathbb{Z}_1'\mathbb{X})^{-1}\mathbb{Z}_1'\mathbb{U} - \beta_0) \\
&= \left(\frac{1}{n}\mathbb{Z}_1'\mathbb{X}\right)^{-1}\frac{1}{\sqrt{n}}\mathbb{Z}_1'\mathbb{U} \\
\frac{1}{\sqrt{n}}\mathbb{Z}_1'\mathbb{U} &\xrightarrow{d} \mathcal{N}(0, \mathbb{E}[ZZ'U^2]) \\
\frac{1}{n}\mathbb{Z}_1'\mathbb{X} &\xrightarrow{p} \mathbb{E}[Z_1X'] \\
\implies \sqrt{n}(\hat{\beta}_{1n} - \beta_0) &\xrightarrow{d} \mathcal{N}(0, (\mathbb{E}[Z_1'X])^{-1}\mathbb{E}[Z_1Z_1'U^2](\mathbb{E}[XZ_1'])^{-1})
\end{aligned}$$

b)

$$\hat{U_1 i} = U_i = X_i'(\hat{\beta}_{1n} - \beta_0)$$

Define

$$\begin{aligned}
e_{z1} &= \begin{bmatrix} I_{d_{z1}} \\ 0 \end{bmatrix} \\
e_{z2} &= \begin{bmatrix} 0 \\ I_{d_{z2}} \end{bmatrix} \\
\implies \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(Y_i - X_i'\hat{\beta}_{1n})Z_{2i} &= \frac{1}{n}\sum_{i=1}^{n}\hat{U}_{1i}e_{z2}'Z_i \\
&= e_{z2}'\frac{1}{\sqrt{n}}\sum_{i=1}^{n}U_iZ_i - e_{z2}'\frac{1}{n}\sum_{i=1}^{n}Z_iX_i'\sqrt{n}(\hat{\beta}_{1n} - \beta_0)
\end{aligned}$$

Using the result from part a,

$$\begin{aligned}
\sqrt{n}(\hat{\beta}_{1n} - \beta_0) &= \left(\frac{1}{n}\mathbb{Z}_1'\mathbb{X}\right)^{-1}\frac{1}{\sqrt{n}}\mathbb{Z}_1'\mathbb{U} \\
&= \left(\frac{1}{n}\sum_{i=1}^{n}Z_{1i}X_i'\right)^{-1}e_{1z}'\frac{1}{\sqrt{n}}\sum_{i=1}^{n}U_iZ_i \\
\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(Y_i - X_i'\hat{\beta}_{1n})Z_{2i} &\xrightarrow{d} \mathcal{N}(0, (e_{z2}' - e_{z2}'\mathbb{E}[ZX'](\mathbb{E}[Z_1X'])^{-1})\mathbb{E}[ZZ'U^2](e_{z2}' - e_{z2}'\mathbb{E}[ZX'](\mathbb{E}[Z_1X'])^{-1})')
\end{aligned}$$

c) We can just use the sample analog of the asymptotic variance from the last question,

$$(e_{z2}' - e_{z2}'\frac{1}{n}\sum_{i=1}^{n}Z_iX_i'\left(\left(\frac{1}{n}\sum_{i=1}^{n}Z_{1i}X_i'\right)\right)^{-1}e_{1z}')\left[\frac{1}{n}\sum_{i=1}^{n}Z_iZ_i'\hat{U}_{i1}^2\right](e_{z2}' - e_{z2}'\frac{1}{n}\sum_{i=1}^{n}Z_iX_i'\left(\left(\frac{1}{n}\sum_{i=1}^{n}Z_{1i}X_i'\right)\right)^{-1}e_{1z}')'$$

d) To prevent my answer from running off the page, call the estimator from part (c) $\hat{V}_n$. Then we can use the test

$$\phi_n = \mathbb{1}\left\{\left\|\hat{V}_n^{-1/2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(Y_i - X_i'\hat{\beta}_{1n})Z_{2i}\right\|^2 > c_{1-\alpha}\right\}$$

4 Question 8

a)

$$\begin{aligned}
\hat{\beta}_n &= \left\{\mathbb{X}_n'\mathbb{Z}_n(\mathbb{Z}_n'\mathbb{Z}_n)^{-1}\mathbb{Z}_n'\mathbb{X}_n\right\}\mathbb{X}_n'\mathbb{Z}_n(\mathbb{Z}_n\mathbb{Z}_n)^{-1}\mathbb{Z}_n\mathbb{Y}_n \\
&= \left(\sum_{i=1}^{n} X_iZ_i\left(\sum_{i=1}^{n} Z_i^2\right)\sum_{i=1}^{n} Z_iX_i\right)^{-1}\sum_{i=1}^{n} X_iZ_i\left(\sum_{i=1}^{n} Z_i^2\right)\sum_{i=1}^{n} Z_iY_i \\
&= \frac{\sum_{i=1}^{n} Z_iY_i}{\sum_{i=1}^{n} X_iZ_i} \\
&= \beta_0 + \frac{\sum_{i=1}^{n} Z_iU_i}{\sum_{i=1}^{n} Z_iX_i} \\
\mathbb{P}(|\hat{\beta}_n - \beta_0| > \varepsilon) &= \mathbb{P}\left(\left|\beta_0 + \frac{\sum_{i=1}^{n} Z_iU_i}{\sum_{i=1}^{n} Z_iX_i} - \beta_0\right| > \varepsilon\right) \\
&= \mathbb{P}\left(\left|\frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_iU_i}{\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_iX_i}\right| > \varepsilon\right) \\
&= \mathbb{P}\left(\left|\frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_iU_i}{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(Z_iX_i - \frac{\pi}{\sqrt{n}}) + \pi}\right| > \varepsilon\right) \\
\lim_{n\to\infty}\mathbb{P}(|\hat{\beta}_n - \beta_0| > \varepsilon) &= \lim_{n\to\infty}\mathbb{P}\left(\left|\frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_iU_i}{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(Z_iX_i - \frac{\pi}{\sqrt{n}+\pi})}\right| > \varepsilon\right) \qquad \text{(CLT)} \\
&= \mathbb{P}\left(\left|\frac{N_1}{N_2 + \pi}\right| > \varepsilon\right) \\
&\neq 0
\end{aligned}$$

$\hat{\beta}_n$ is not consistent for β_0.

b)

$$\begin{aligned}
&\left\{\left|\frac{N_1}{N_2 + \pi}\right| > \varepsilon\right\} \subseteq (\{|N_1| > M_\varepsilon\} \cup \{|N_2 + \pi| \le M\}) \\
\implies \lim_{n\to\infty}\mathbb{P}(|\hat{\beta}_n - \beta_0| > \varepsilon) &= \mathbb{P}\left(\left|\frac{N_1}{N_2 + \pi}\right| > \varepsilon\right) \\
&\le \mathbb{P}\left(\{|N_1| > M\varepsilon\} \cup \{|N_2 + \pi| \le M\}\right) \\
&= \mathbb{P}(|N_1| > \varepsilon M) + \mathbb{P}(|N_2 + \pi| \le M) - \mathbb{P}(\{|N_1| > M\varepsilon\} \cap \{|N_2 + \pi| \le M\}) \\
&\le \mathbb{P}(|N_1| > \varepsilon M) + \mathbb{P}(|N_2 + \pi| \le M)
\end{aligned}$$

c)

$$
\begin{aligned}
\lim_{n\to\infty} \mathbb{P}(|\hat{\beta}_n - \beta_0| > \varepsilon) &\leq \mathbb{P}\left(|N_1| > \varepsilon\frac{\pi}{1+\varepsilon}\right) + \mathbb{P}\left(|N_2 + \pi| \leq \frac{\pi}{1+\varepsilon}\right) \\
&= 1 - \mathbb{P}\left(-\frac{\varepsilon\pi}{1+\varepsilon} \leq N_1 \leq \frac{\varepsilon\pi}{1+\varepsilon}\right) + \mathbb{P}\left(-\frac{\pi}{1+\varepsilon} - \pi \leq N_2 \leq \frac{\pi}{1+\varepsilon} - \pi\right) \\
&= 1 - \left(\Phi\left(\frac{\varepsilon\pi}{1+\varepsilon}\right) - \Phi\left(-\frac{\varepsilon\pi}{1+\varepsilon}\right)\right) + \left(\Phi\left(-\frac{\varepsilon\pi}{1+\varepsilon}\right) - \Phi\left(\frac{-2\pi - \pi\varepsilon}{1+\varepsilon}\right)\right) \\
&= 2\Phi\left(-\frac{\varepsilon\pi}{1+\varepsilon}\right) + 1 - \Phi\left(\frac{\varepsilon\pi}{1+\varepsilon}\right) - \Phi\left(\frac{-2\pi - \pi\varepsilon}{1+\varepsilon}\right) \\
&\leq 3\Phi\left(-\frac{\varepsilon\pi}{1+\varepsilon}\right)
\end{aligned}
$$

d)

$$
\begin{aligned}
3\Phi\left(-\frac{0.1\pi}{1+0.1}\right) &< 0.1 \\
\implies \Phi\left(-\frac{1}{11}\pi\right) &< \frac{1}{30} \\
\implies \pi &> -11\Phi^{-1}\left(\frac{1}{30}\right)
\end{aligned}
$$

We can use the test

$$
\nVdash\{\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_i X_i + 11\Phi^{-1}\left(\frac{1}{30}\right) > c_{1-\alpha}\}
$$

5 Question 9

We can estimate β_0, β_1 from the moment conditions

$$\mathbb{E}\left[(Y - \beta_0 - D\beta_1)\begin{pmatrix}1\\Z\end{pmatrix}\right] = 0.$$

Writing out each condition,

$$\begin{aligned}
0 &= \mathbb{E}[Y] - \beta_0 - \beta_1\mathbb{E}[D] \\
\implies \beta_0 &= \mathbb{E}[Y] - \beta_1\mathbb{E}[D] && \text{(first condition)} \\
0 &= \mathbb{E}[YZ] - \beta_0\mathbb{E}[Z] - \beta_1\mathbb{E}[DZ] \\
&= \mathbb{E}[YZ] - (\mathbb{E}[Y] - \beta_1\mathbb{E}[D])\mathbb{E}[Z] - \beta_1\mathbb{E}[DZ] \\
&= \mathbb{E}[YZ] - \mathbb{E}[Y]\mathbb{E}[Z] - \beta_1(\mathbb{E}[DZ] - \mathbb{E}[D]\mathbb{E}[Z]) \\
\implies \beta_1 &= \frac{\mathrm{Cov}(Z, Y)}{\mathrm{Cov}(Z, D)} \\
&= \frac{\mathbb{E}[Y \mid Z = 1] - \mathbb{E}[Y \mid Z = 0]}{\mathbb{E}[D \mid Z = 1] - \mathbb{E}[D \mid Z = 0]}
\end{aligned}$$

Independence implies

$$\begin{aligned}
\mathbb{E}[D \mid Z = 1] - \mathbb{E}[D \mid Z = 0] &= \mathbb{E}[D_i(1) - D_i(0)] \\
\mathbb{E}[Y \mid Z = 1] - \mathbb{E}[Y \mid Z = 0] &= \mathbb{E}[(Y_i(1) - Y_i(0))(D_i(1) - D_i(0))] \\
\implies \beta_1 &= \frac{\mathbb{E}[(Y_i(1) - Y_i(0))(D_i(1) - D_i(0))]}{\mathbb{E}[D_i(1) - D_i(0)]}
\end{aligned}$$

Now, using the assumption $\mathbb{P}(D_i(0) = 0) = 1$,

$$\begin{aligned}
\mathbb{E}[(Y_i - Y_i(0))(D_i(1) - D_i(0))] &= \mathbb{E}[Y_i(1) - Y_i(0) \mid D_i(1) = 1]\mathbb{P}(D_i(1) = 1) \\
\mathbb{E}[D_i(1) - D_i(0)] &= \mathbb{P}(D_i(1) = 1) \\
\implies \beta_1 &= \mathbb{E}[Y_i(1) - Y_i(0) \mid D_i(1) = 1]
\end{aligned}$$

6 Question 10

a)

$$\begin{aligned}
\hat{\beta}_n^{IV} &= (\mathbb{X}_n'\mathbb{Z}_n\hat{\Omega}_n\mathbb{Z}_n'\mathbb{X}_n)^{-1}\mathbb{X}_n'\mathbb{Z}_n\hat{\Omega}_n\mathbb{Z}_n'\mathbb{Y}_n + o_p(1) \\
&= \beta_0 + (Z_n'X_n)^{-1}\mathbb{Z}_n'e_n + o_p(1) \\
\sqrt{n}\{\hat{\beta}_n^{IV} - \beta_0\} &= \left(\frac{1}{n}\sum_{i=1}^n Z_iX_i'\right)^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^n Z_i\varepsilon_i + o_p(1) \\
&= \mathbb{E}[ZX']^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^n Z_i\varepsilon_i + o_p(1)
\end{aligned}$$

Repeating with $\hat{\beta}^{OLS}$,

$$\begin{aligned}
\hat{\beta}_n^{OLS} &= (\mathbb{X}_n'\mathbb{X}_n)^{-1}\mathbb{X}_n'\mathbb{Y}_n + o_p(1) \\
&= (\mathbb{X}_n'\mathbb{X}_n)^{-1}\mathbb{X}_n'(X_n\beta_0 + e_n) + o_p(1) \\
&= \beta_0 + (\mathbb{X}_n'\mathbb{X}_n)^{-1}\mathbb{X}_n'e_n + o_p(1) \\
\sqrt{n}\{\hat{\beta}_n^{OLS} - \beta_0\} &= \left(\frac{1}{n}\sum_{i=1}^{n} X_iX_i'\right)^{-1}\frac{1}{\sqrt{n}}X_i\varepsilon_i + o_p(1) \\
&= \mathbb{E}[XX']^{-1}\frac{1}{\sqrt{n}}X_i\varepsilon_i + o_p(1)
\end{aligned}$$

Putting these two results together,

$$\begin{aligned}
\begin{pmatrix} \sqrt{n}\{\hat{\beta}_n^{OLS} - \beta_0\} \\ \sqrt{n}\{\hat{\beta}_n^{IV} - \beta_0\} \end{pmatrix} &= \begin{pmatrix} \beta_0 + (\mathbb{X}_n'\mathbb{X}_n)^{-1}\mathbb{X}_n'e_n + o_p(1) \\ \mathbb{E}[ZX']^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^n Z_i\varepsilon_i + o_p(1) \end{pmatrix} \\
&= \begin{pmatrix} \beta_0 + (\mathbb{X}_n'\mathbb{X}_n)^{-1}\mathbb{X}_n'e_n \\ \mathbb{E}[ZX']^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^n Z_i\varepsilon_i \end{pmatrix} + \begin{pmatrix} o_p(1) \\ o_p(1) \end{pmatrix} \\
&= \begin{bmatrix} \mathbb{E}[XX']^{-1} & 0 \\ 0 & \mathbb{E}[XX']^{-1} \end{bmatrix} \begin{pmatrix} \frac{1}{\sqrt{n}}\sum_{i=1}^n X_i\varepsilon_i \\ \frac{1}{\sqrt{n}}\sum_{i=1}^n Z_i\varepsilon_i \end{pmatrix} + o_p(1)
\end{aligned}$$

b)

$$\begin{aligned}
T_n &= \|\Omega^{-1/2}\sqrt{n}(\hat{\beta}_n^{OLS} - \hat{\beta}_n^{IV})\|^2 \\
&= \|\Omega^{-1/2}(\sqrt{n}(\hat{\beta}_n^{OLS} - \beta_0) - \sqrt{n}(\hat{\beta}_n^{IV} - \beta_0))\|^2 \\
\begin{pmatrix} \frac{1}{\sqrt{n}}\sum_{i=1}^n X_i\varepsilon_i \\ \frac{1}{\sqrt{n}}\sum_{i=1}^n Z_i\varepsilon_i \end{pmatrix} &\xrightarrow{d} \mathcal{N}(0, \Sigma) \\
\Sigma &= \begin{bmatrix} \mathbb{E}[\varepsilon^2 XX'] & \mathbb{E}[\varepsilon^2 XZ'] \\ \mathbb{E}[\varepsilon^2 ZX'] & \mathbb{E}[\varepsilon^2 ZZ'] \end{bmatrix}
\end{aligned}$$

To save space on the page, define

$$\begin{aligned}
A &\equiv \begin{bmatrix} \mathbb{E}[XX']^{-1} & 0 \\ 0 & \mathbb{E}[XX']^{-1} \end{bmatrix} \\
r &\equiv [I_d, -I_d] \\
\sqrt{n}(\hat{\beta}_n^{OLS} - \beta_0) - \sqrt{n}(\hat{\beta}_n^{IV} - \beta_0) &= rA\begin{pmatrix} \frac{1}{\sqrt{n}}\sum_{i=1}^n X_i\varepsilon_i \\ \frac{1}{\sqrt{n}}\sum_{i=1}^n Z_i\varepsilon_i \end{pmatrix} + o_p(1) \\
&\xrightarrow{d} \mathcal{N}(0, rA\Sigma A'r') \qquad \text{(CMT)} \\
rA\Sigma A'r'^{-1/2}(\sqrt{n}(\hat{\beta}_n^{OLS} - \beta_0) - \sqrt{n}(\hat{\beta}_n^{IV} - \beta_0)) &\xrightarrow{d} \mathcal{N}(0, I_d) \\
T_n &\equiv \|rA\Sigma A'r'^{-1/2}\sqrt{n}(\hat{\beta}_n^{OLS} - \hat{\beta}_n^{IV})\|^2 \\
&\xrightarrow{d} \chi_d^2
\end{aligned}$$

c) We can use our test statistic from the last part of the question,

$$\phi_n = \mathbb{1}\{T_n > c_{1-\alpha}\}.$$

d)

$$
\begin{aligned}
\hat{\beta}_n^{OLS} - \hat{\beta}_n^{IV} &= (\mathbb{X}_n'\mathbb{X}_n)^{-1}\mathbb{X}_n' e_n - (Z_n' X_n)^{-1} Z_n' e_n + o_p(1) \\
&= (\mathbb{X}_n'\mathbb{X}_n)^{-1}\mathbb{X}_n' e_n + o_p(1) \\
\mathbb{E}[X\varepsilon] &\neq 0 \implies \\
\hat{\beta}_n^{OLS} - \hat{\beta}_n^{IV} &\xrightarrow{p} \mathbb{E}[XX']^{-1}\mathbb{E}[X\varepsilon] \\
&\neq 0
\end{aligned}
$$

Since the proposed test multiplies this term by $\sqrt{n}$, this value becomes arbitrarily large as $n \to \infty$.