1 Notes on the OLG Model

1.1 Introduction

The overlapping generations (OLG) model, introduced by Sameulson (1958), is a dynamic economic model with many interesting properties. It contains agents who are born at different dates and have finite lifetimes, even though the economy goes on forever. This induces a natural heterogeneity across individuals at a point in time, as well as nontrivial life-cycle considerations for a given individual across time. These features of the model can also generate differences from models where there is a finite set of time periods and agents, or from models where there is an infinite number of time periods but agents live forever. In particular, competitive equilibria in the OLG model may not to be Pareto optimal. A closely related feature of the model is that it has a role for fiat money. This means we can use OLG models to address a variety of substantive issues in monetary economics.

1.2 The Basic Model

Suppose that $t = 1, 2, \ldots$, and that at every date t there is born a new generation G_t of individuals who live for two periods. More realistic (longer) lifetimes can be studied, but two periods is the simplest case where the generations overlap. There is also a generation G_0 around at $t = 1$ who only live for one period, called the "initial old." For now, every generation consists of a $[0, 1]$ continuum of homogeneous agents. Let c_{t1} and c_{t2} denote consumption of an individual from G_t, $t \geq 1$, in the 1st and 2nd periods of life, and let e_1 and e_2 denote his (time-invariant) endowments in the 1st and 2nd periods of life. His utility function $u(c_{t1}, c_{t2})$ is strictly increasing and quasi-concave. Members of generation G_0 consume only c_{02} and are endowed with only e_2.

One can define a *Walrasian Competitive Equilibrium* (WCE) for this economy as follows. Let p_t be the price of a unit of the consumption good at date t. Now clearly every member of generation G_0 simply consumes his endowment, $c_{02} = e_2$. For all $t \geq 1$, every member of G_t maximizes $u(c_{t1}, c_{t2})$ subject to

$$p_t c_{t1} + p_{t+1} c_{t2} = p_t e_1 + p_{t+1} e_2 \tag{1}$$

and $c_{tj} \geq 0$.[1] We always write budget constraints with strict equality because u is strictly increasing. Then a WCE is a sequence of prices and allocations $\{p_t, c_{t1}, c_{t2}\}$ such that: $c_{02} = e_2$; given $\{p_t\}$, (c_{t1}, c_{t2}) solves the maximization problem of G_t for all $t \geq 1$; and markets clear in the sense that for all t

$$c_{t1} + c_{t-1,2} = e_1 + e_2. \tag{2}$$

One can also define a *Recursive Competitive Equilibrium* (RCE) as follows. Let s_t denote savings or loans by a member of G_t at t, and R_t the gross (principal plus interest) return on savings between t and $t+1$. Then for all $t \geq 1$, every member of G_t maximizes $u(c_{t1}, c_{t2})$ subject to

$$c_{t1} = e_1 - s_t \tag{3}$$

$$c_{t2} = e_2 + R_t s_t \tag{4}$$

and $c_{tj} \geq 0$.[2] A RCE is a sequence $\{R_t, c_{t1}, c_{t2}\}$ such that: $c_{02} = e_2$; given

[1] We could write (1) more generally as $\sum_j p_j c_{tj} = \sum_j p_j e_{tj}$, where c_{tj} is the consumption and e_{tj} is the endowment in period j of an agent born at t, but it is obvious from the specification of preferences and endowments that c_{tj} and e_{tj} are nonzero only at $j = t$ and $j = t+1$. This means that we do not have to worry if the (otherwise infinite) summations converge when we write the individual budget constraint. However, if we try to price a point in the commodity space with $c_{tj} > 0$ for infinitely many j, convergence becomes an issue (see below).

[2] As in the previous footnote, we could write the budget constraints as

$$c_{tj} = e_{tj} - s_{tj} + R_{j-1} s_{tj-1}$$

$\{R_t\}$, (c_{t1}, c_{t2}) solves the maximization problem of G_t for all $t \geq 1$; and (2) holds for all t.

An immediate result is that WCE and RCE are equivalent concepts: that is, the sets of WCE and RCE allocations are the same. To see this, note that the only difference in the two definitions is in the budget constraints; but, by eliminating s_t from (3) and (4) and setting $R_t = p_t/p_{t+1}$, these two recursive constraints are equivalent to (1). This is a very general result: versions of this equivalence between WCE and RCE hold in much more complicated models, including those where different agents have different (possibly infinite) life times, and even including those where there is uncertainty.

The next result is that the only equilibrium allocation here is autarchy: $(c_{t1}, c_{t2}) = (e_1, e_2)$ for all t. To verify this, first note that homogeneity implies no trade within a generation.[3] Then note that in any equilibrium $c_{02} = e_2$, and combined with market clearing this implies $c_{11} = e_1$. Then the budget constraints imply $c_{12} = e_2$, and so on, so that $(c_{t1}, c_{t2}) = (e_1, e_2)$ for all t. From the first order condition from the individual maximization problem,

for every date j, but the specification of preferences and endowments means that only (3) and (4) are relevant. Generally, if agents live N periods there will be N relevant constraints. Also, if utility is additively separable, we can write the individual problem as a dynamic program with state variable s_{t-1}, control s_t, and value functions defined recursively via Bellman's equation

$$V_t(s_{t-1}) = \max_{s_t}\{U(e_{tj} - s_{tj} + R_{j-1}s_{tj-1}) + \beta V_{t+1}(s_t)\},$$

and the terminal condition $V_{T+1}(s_{T-1}) = 0$, where T is the date at which the individual dies. This should help to bring out the connection between RCE in the OLG model and in the neoclassical growth model.

[3]With homogeneity every member of G_t chooses the same s_t, and since members of G_t are the only ones in the credit market at t, equilibrium requires $s_t = 0$. In the next section we provide an example with heterogeneity where there are within-generation loans in equilibrium.

the (constant) RCE interest rate is given by $R_t = \mu(e_1, e_2)$ for all t, where μ is the marginal rate of substitution function

$$\mu(c_1, c_2) = \frac{u_1(c_1, c_2)}{u_2(c_1, c_2)}. \tag{5}$$

Similarly, the WCE price sequence is given by $p_t/p_{t+1} = \mu(e_1, e_2)$, where we normalize $p_1 = 1$ by appropriate choice of numeraire.

An interesting property of the OLG model is that equilibria may not be Pareto optimal. For example, suppose $(e_1, e_2) = (1, 0)$ and $u(c_{t1}, c_{t2}) = c_{t1} + c_{t2}$, for all $t \geq 1$ (this example may seem special because the indifference curves are linear, but it will be clear below that the point is general). Then the autarchy allocation is Pareto dominated by $(c_{t1}, c_{t2}) = (0, 1)$ for all t, since members of G_0 are strictly better off and members of G_t are no worse off. How is this possible, given that the first welfare theorem says competitive equilibria are always efficient? Evidently the first welfare theorem does not apply in the OLG model, and it worth seeing why.

Generally, the standard proof (for finite economies) that competitive equilibria are efficient proceeds as follows. Suppose by way of contradiction that an equilibrium allocation is dominated by some alternative allocation (c_{hg}), where h indexes households and g indexes goods. Then $\sum_g p_g(c_{hg} - e_{hg}) \geq 0$ for every h, with strict inequality for some h (because a bundle that is as least as good must cost at least as much and a bundle that is strictly better must cost strictly more). Adding over households, we have $\sum_h \sum_g p_g(c_{hg} - e_{hg}) > 0$. Interchanging the summations, we have $\sum_g p_g \sum_h (c_{hg} - e_{hg}) > 0$, which implies $\sum_h (c_{hg} - e_{hg}) > 0$ for some good g. This last inequality violates feasibility. In the OLG model, however, because the sets of goods and agents are infinite we *cannot* in general interchange the summations, and the argument breaks down.[4]

[4] In fact, when the model is not finite, expressions like $\sum_g p_g(c_{hg} - e_{hg}) \geq 0$ may not

In the above argument, the interpretation of $\sum_h \sum_g p_g(c_{hg} - e_{hg}) > 0$ is that people are spending more than their combined income, and the interpretation of $\sum_h(c_{hg} - e_{hg}) > 0$ is that people are consuming more than their combined endowment of good g. In the OLG model, if equilibrium is dominated by an alternative allocation then it is true that people would have to spend more than their income to buy it, but this does not mean that they would have to consume more than is feasible. Consider again the example with $(e_1, e_2) = (1, 0)$ and linear utility, for which $p_t = 1$ for all t in equilibrium. Given the alternative allocation $(c_{t1}, c_{t2}) = (0, 1)$, if we sum the analogue of $p_g(c_{hg} - e_{hg})$ first over goods and then over agents we get 1, but if we sum first over agents and then over goods we get 0; the alternative allocation does cost too much but it does not violate feasibility.

This is not to say that equilibrium in the OLG model is always inefficient. We now show that, with strictly convex indifference curves, the unique equilibrium allocation is Pareto optimal if and only if the marginal rate of substitution at the endowment point is bigger than unity, $\mu(e_1, e_2) \geq 1$. To simplify the presentation, we assume $e_1 > 0$, and take for granted that we must treat everyone within a generation the same. First, suppose $\mu(e_1, e_2) < 1$. To show autarchy is inefficient, consider the alternative allocation $(c_{t1}, c_{t2}) = (e_1 - \varepsilon, e_2 + \varepsilon)$ for all $t \geq 1$ and $c_{02} = e_2 + \varepsilon$, where $\varepsilon \in (0, e_1]$. Clearly this is

be well-defined. This is why in infinite dimensional economies we often define budget constraints by means of a continuous linear functional v that maps the commodity space into the real numbers (following Debreu 1958). Thus, $v(c)$ is the value of any bundle c. The proof in the text goes through in infinite dimensional economies if we replace summations like $\sum_g p_g(c_{hg} - e_{hg})$ with expressions like $v(c_h - e_h)$. We can always represent v as an inner product − that is, there always exists a p such that $v(c) = \sum_g p_g c_g$ for all c − if commodity space is finite; but this is not true in general. In the OLG model, however, it makes sense to define a competitive equilibrium in terms of price sequences even though the number of goods is infinite because no agent consumes or is endowed with more than a finite subset of these goods.

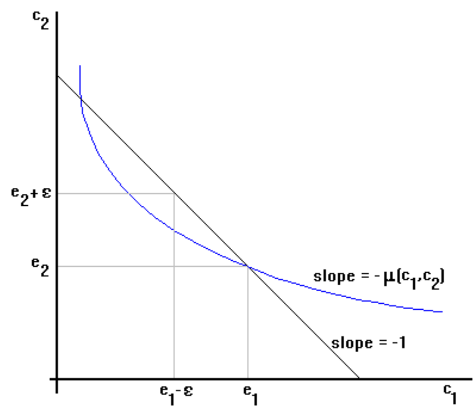

Figure 1: Inefficient Equilibrium

feasible and every member of G_0 is better off. As shown in Figure 1, as long as $\mu < 1$ there is some $\varepsilon > 0$ such that every member of G_t is better off for $t \geq 1$.

Now consider the case where $\mu(e_1, e_2) \geq 1$, and suppose the autarchy allocation is inefficient. Then there is some alternative allocation (c_{t1}, c_{t2}) that $u(c_{t1}, c_{t2}) \geq u(e_1, e_2)$ for all agents, with strict inequality for some agent. Let t be the first date at which the allocation differs from autarchy. We cannot have the old at t consume less than e_2, since then they would be worse off, and so we must have $e_1 - c_{t1} = \varepsilon_t > 0$. Then we must have $c_{t2} - e_2$ not only positive, but bigger than ε_t, because $\mu \geq 1$, and so feasibility dictates $e_1 - c_{t+1,1} = \varepsilon_{t+1} > \varepsilon_t$. Continuing in this fashion we get an increasing sequence ε_t of transfers from the young to the old. This sequence cannot converge (why?); hence, there is some date T such that $\varepsilon_T > e_1$, which is infeasible.

This establishes that equilibrium is Pareto optimal if and only if $\mu(e_1, e_2) \geq$

1. In fact, nothing in the argument depends on (e_1, e_2) denoting the endowment or the the equilibrium allocation. Hence, we can conclude that *any* time-invariant allocation (c_1, c_2) is Pareto optimal if and only if $\mu(c_1, c_2) \geq 1$. Moreover, the first part of the proof applies even if the allocation is not time-invariant: as long as $\mu(c_{t1}, c_{t2}) \geq 1$ for all t, all individuals prefer $(c_{t1} - \varepsilon, c_{t2} + \varepsilon)$ for some $\varepsilon > 0$, and this is feasible as long as c_{t1} is bounded away from 0.

The inefficiency of equilibria when $\mu < 1$ can be explained as follows. The rate at which society can trade c_1 for c_2 is unity, since we can freely transfer from the young to the old. If we set the interest rate to $R_t = 1$, then $\mu < 1$ implies individuals want to save – but saving is inconsistent with equilibrium. To see why, note that if G_1 saves then $c_{11} < e_1$, which implies by virtue of (2) that $c_{02} > e_2$, and this violates the budget constraint of the initial old. Hence there is no scope for savings in equilibrium, even though savings is both feasible and desirable from the point of veiw of soctiety. In the next section we describe a way to allow the initial old to consume more than e_2, and therefore the young to consume less that e_1, so that they can save, in equilibria with money.

It is worth remarking at this stage that the potential inefficiency of competitive equilibrium in the model has nothing to do with the fact that agents overlap in a way that prevents them from getting together at date $t = 1$ and exploiting all possible gains from trade. That is, the result is not due to the fact that at date t the only feasible trades are between G_{t-1} and G_t. This follows directly from the result that the set of RCE and WCE allocations are the same and the observation that the WCE concept makes no reference whatsoever to the dynamics or the generational structure. In WCE, it is *as if* there are complete markets that convene at date $t = 1$, but equilibrium still may be inefficient.

We close this section with an extension of the model that allows population to change. Let n_t be the size of generation t, and assume $n_t = \gamma n_{t-1}$ for all t, for $\gamma > 0$. Now feasibility (market clearing) requires $n_t c_{t1} + n_{t-1} c_{t-1,2} = n_t e_1 + n_{t-1} e_2$, or

$$\gamma c_{t1} + c_{t-1,2} = \gamma e_1 + e_2, \tag{6}$$

for all t, which generalizes (2). Again the unique equilibrium is autarchy, but now the techniques used above imply that it is Pareto optimal if and only if $\mu(e_1, e_2) \geq \gamma$.

With population growth, society can transfer resources across time at rate γ, by taking one unit each from the young agents and distributing the proceeds to the old agents. In equilibrium, zero savings requires $R_t = \mu(e_1, e_2)$. If $\mu(e_1, e_2) < \gamma$ then at rate γ agents would like to save, but in equilibrium they cannot. The case $\mu \geq \gamma$ is sometimes referred to as the "Classical case" (because classical results, like the optimality of competitive equilibrium, hold), and the case $\mu < \gamma$ is referred to as the "Samuelson case" (presumably because he thought it up).

1.3 Money

Into the model described above, we now introduce a constant amount M of *fiat money*, held in period 1 by the initial old G_0. By definition, fiat money is an object that has no intrinsic (consumption) value, but could potentially have exchange value. Let q_t be the value of money at date t. If $q_1 > 0$ then the initial old can consume $c_{02} = e_2 + q_1 M > e_2$ without violating their budget constraint. Let every other agent in G_t, $t \geq 1$, solve the same problem as before, maximizing u subject to (1) and $c_{t1}, c_{t2} \geq 0$. Then a WCE with money is sequence of prices and quantities $\{p_t, q_t, c_{t1}, c_{t2}\}$ such that: $c_{02} = e_2 + q_1 M$; given $\{p_t, q_t\}$, (c_{t1}, c_{t2}) solves the maximization problem of G_t for all $t \geq 1$;

and the market clearing condition (2) holds for all t.[5]

It is more illuminating to discuss money in a recursive setting. In this case G_0 still consumes $c_{02} = e_2 + q_1 M$; but for all $t \geq 1$, G_t maximizes $u(c_{t1}, c_{t2})$ subject to

$$\begin{aligned} c_{t1} &= e_1 - q_t m_t - s_t && (7) \\ c_{t2} &= e_2 + q_{t+1} m_t + R_t s_t, && (8) \end{aligned}$$

and $m_t, c_{t1}, c_{t2} \geq 0$, where m_t is interpreted as savings in terms of dollars or the *demand for money*. If $q_t > 0$ we can combine (7) and (8) into

$$(c_{t2} - e_2) + R_t(c_{t1} - e_1) = q_t m_t \left(\frac{q_{t+1}}{q_t} - R_t \right).$$

Now $\frac{q_{t+1}}{q_t} > R_t$ implies $m_t = \infty$ and $\frac{q_{t+1}}{q_t} < R_t$ implies $m_t = 0$, neither of which are consistent with a monetary equilibrium (which will be seen to require $m_t = M$ for all t). Hence, a monetary equilibrium requires the "arbitrage condition" $\frac{q_{t+1}}{q_t} = R_t$.

Given $\frac{q_{t+1}}{q_t} = R_t$, individuals do not care whether they save in terms of money or loans, only about total savings, $q_t m_t + s_t$. Moreover, assuming homogeneous individuals in each generation, there are no loans in equilibrium: $s_t = 0$. We will see below that if we relax the homogeneity assumption there can be within-generation lending as well as savings in terms of money, but for now we set $s_t = 0$ and write (7) and (8) as

$$\begin{aligned} c_{t1} &= e_1 - q_t m_t && (9) \\ c_{t2} &= e_2 + q_{t+1} m_t + R_t s_t. && (10) \end{aligned}$$

[5] In this definition, the market clearing conditions are in terms of the consumption good at each date. One can also interpret money as being traded in equilibrium, with old agents supplying M units at every date and young agents demanding M units at every date, so that the money market also clears. It is easier to understand this in the recursive version of equilibrium to be defined next.

Then a RCE with money is sequence $\{q_t, c_{t1}, c_{t2}\}$ such that: $c_{02} = e_2 + q_1 M$; given $\{q_t\}$, (c_{t1}, c_{t2}) solves the maximization problem of G_t for all $t \geq 1$; and (2) holds for all t. Alternatively, as long as $q_t > 0$, we can replace (2) by $m_t = M$, since we know (by Walras' law) that the goods market clears if and only if the money market clears.[6]

This is the framework within which we will usually discuss monetary issues in what follows. Given q_t, the nominal price level is $P_t = 1/q_t$ for all t. If $q_t = 0$ ($P_t = \infty$) for all t we have a nonmonetary equilibrium, which is equivalent to the unique equilibrium in the model without money. It is a desirable property of a monetary theory that nonmonetary equilibria exist as special cases, since a prerequisite for fiat currency to have value is that individuals believe that it will. We are interested now in monetary equilibrium, which requires $q_t > 0$ for some t. Notice that if $q_t > 0$ for some t then $q_t > 0$ for all t. This follows because $q_t = 0$ implies the demand for money is 0 at $t-1$: who would give up resources for something that cannot be used in either consumption or trade? If the demand for money at $t-1$ is 0 then $q_{t-1} = 0$, and so on.

With $q_t > 0$, we can insert (9) and (10) into the utility function and differentiate to get the first order condition for an interior solution,

$$\mu(e_1 - q_t m_t, e_2 + q_{t+1} m_t) = \frac{q_{t+1}}{q_t}, \tag{11}$$

where $\mu(c_1, c_2)$ is the marginal rate of substitution function. The solution m_t to (11) gives the *money demand function* as long as $0 < m_t < e_1/q_t$; the first inequality is true by definition in a monetary equilibrium, and the second we can guarantee by assuming $\mu(c_1, c_2) \to \infty$ as $c_1 \to 0$.

If we use the equilibrium condition $m_t = M$, (11) implicitly defines a difference equation, $q_{t+1} = f(q_t)$. A monetary equilibrium is completely

[6] Even though $s_t = 0$, we can still price loans in a monetary equilibrium using $R_t = \frac{q_{t+1}}{q_t}$.

characterized by a bounded sequence $\{q_t\}$ satisfying this difference equation, a special case of which is a *steady state*, or a solution to $q = f(q)$. We require q_t bounded since otherwise $q_t M$ eventually exceeds $e_1 + e_2$, which means that at some date the agents holding the money will buy more than existing resources, and this violates feasibility. Notice that we do not have any intial condition for q_t; we can set q_1 arbitrarily, and the resulting path given by $q_{t+1} = f(q_t)$ will be an equilibrium as long as it remains bounded.

In general, the equation $q_{t+1} = f(q_t)$ satisfies $f(0) = 0$ and $f'(0) = \mu(e_1, e_2)$. We claim that $f'(0) \geq 1$ implies that there is no solution to $f(q) = q$ while $f'(0) = \mu(e_1, e_2) < 1$ implies that there is exactly one solution. To see this, note that solutions to $f(q) = q$ satisfy $T(q) = 0$, where

$$T(q) = -u_1(e_1 - qM, e_2 + qM) + u_2(e_1 - qM, e_2 + qM).$$

Since $T' < 0$, there cannot be more than one solution to $T(q) = 0$. Since $T(q) < 0$ for q near $\frac{e_1}{M}$, there exists a solution if and only if $T(0) > 0$, which holds if and only if $\mu(e_1, e_2) < 1$. Hence, a monetary steady state exists if and only if $\mu(e_1, e_2) < 1$, which is exactly the condition for the nonmonetary equilibrium being inefficient.

We leave as an exercise verification of the following generalization of these results. Suppose there is population growth at rate γ; then $\mu \geq \gamma$ implies the nonmonetary equilibrium is Pareto optimal and no monetary steady state exists; $\mu < \gamma$ implies the nonmonetary equilibrium is not Pareto optimal and a monetary steady state exists.

Consider an example with the *log-linear* utility function, $u(c_1, c_2) = \log(c_1) + \beta \log(c_2)$. This allows us to solve (11) explicitly for the money demand function,

$$m_t = m(q_t, q_{t+1}) = \frac{\beta e_1 q_{t+1} - e_2 q_t}{q_t q_{t+1}(1 + \beta)}, \tag{12}$$

which satisfies $m_t > 0$ if $q_{t+1}/q_t > e_2/\beta e_1$. If $q_{t+1}/q_t \leq e_2/\beta e_1$ then money demand is zero. Notice that $e_2/\beta e_1 = \mu$ is exactly the marginal rate of substitution at the endowment point.

Consider first the special case where $e_2 = 0$; then q_{t+1} drops out of (12) and so $m_t = M$ implies

$$q_t = \frac{\beta e_1}{(1+\beta)M} \equiv q^*$$

for all t. Hence, in this special case there exists a unique monetary equilibrium and it is a steady state. The nominal price level,

$$P_t = \frac{1+\beta}{\beta e_1} M \equiv P^*,$$

is proportional to M, as predicted by the "quantity theory." Real balances, denoted $S_t = q_t M$, are constant at $S_t = S^* = \beta e_1/(1+\beta)$.

The case $e_2 = 0$ is somewhat special, because with log-linear utility it implies the demand for real balances is a constant fraction of e_1 regardless of next period's price level, and it is because of this that there always exists a unique monetary equilibrium. If $e_2 > 0$, then (11) can be rearranged as

$$q_{t+1} = f(q_t) = \frac{e_2 q_t}{\beta e_1 - (1+\beta)M q_t}. \tag{13}$$

An equilibrium is given by any bounded solution to (13).

In this case, $f'(q) > 0$, $f''(q) > 0$, and $f(q) \to \infty$ as $q \to \beta e_1/(1+\beta)M$. As always, if $f'(0) = \mu < 1$ then there is a unique monetary steady state with

$$q_t = \frac{\beta e_1 - e_2}{(1+\beta)M} \equiv q^*,$$

as shown in Figure 2; if $f'(0) = \mu \geq 1$ then there does not exist a steady state with $q > 0$.

We now consider monetary equilibria that are not steady states. Suppose $\mu < 1$, so that a monetary steady state q^* exists, as shown in Figure 2 (if there is no monetary steady state then there are no monetary equilibria). For

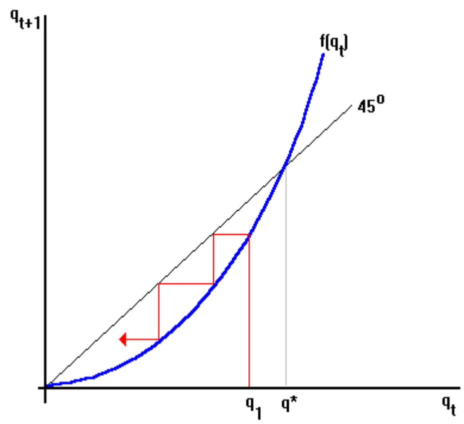

Figure 2: Dynamic Monetary Equilibria

any initial $q_1 > q^*$, (13) implies $q_t \to \infty$; hence, sequences beginning with a value of money in excess of q^* cannot be equilibria. But for any initial $q_1 \in (0, q^*)$, (13) implies $q_t \to 0$ monotonically. Hence, there exist a continuum of bounded solutions to (13), indexed by $q_1 \in (0, q^*)$, and therefore a continuum of monetary equilibria. Counter to the quantity theory, these nonstationary equilibria are characterized by inflation, even though the money supply is constant, since $P_t \to \infty$ as $q_t \to 0$. Inflation in this case is due to self-fulfilling expectations that the value of money will fall. In the limit, the value of money goes to 0 and economy ends up in autarchy.

To close this section, we indicate what happens when we relax the restriction that agents within a generation are homogeneous. This allows for the possibility of private borrowing and lending (or what used to be called "inside money"). Proceeding by way of example, suppose that all agents have log-linear utility with $\beta = 1$, but that endowments differ in the following way: the fraction α_1 of the agents have $e_1 = 1$ and $e_2 = 0$, while the fraction $\alpha_2 = 1 - \alpha_1$ have $e_1 = 0$ and $e_2 = 1$. Let s_{1t} and s_{2t} denote the (real) savings of the two types from generation G_t.

Consider nonmonetary equilibria. The first order condition

$$\mu(e_1 - s_t, e_2 + R_t s_t) = \frac{e_2 + R_t s_t}{e_1 - s_t} = R_t$$

implies savings for the two types are given by $s_{1t} = 1/2$ and $s_{2t} = -1/2R_t$. The equilibrium condition that the loan market clears is $0 = \alpha_1 s_{1t} + \alpha_2 s_{2t}$, which implies $R_t = \alpha_2/\alpha_1$ for all t. From our general welfare results we know that $\alpha_2 \geq \alpha_1$ implies $\mu = R \geq 1$ and the equilibrium is Pareto optimal, while $\alpha_2 < \alpha_1$ implies $\mu = R < 1$ and is not. In the latter case, there exists a Pareto optimal monetary steady state with $R = 1$, $s = 1/2$, and $qM = \alpha_1 - 1/2$. Hence, we see that with heterogeneity the model admits equilibrium where money and private loans coexist, although only if they bear the same rate of return.

1.4 Changes in the Money Supply

In this section we relax the assumption that the money supply is constant, by allowing it to change according to $M_{t+1} = zM_t$. We assume for now that the new is money distributed to members of G_t at date $t+1$ as a lump sum transfer, τ_t (other ways of introducing new money are also considered below). An agent born at $t \geq 1$ maximizes $u(c_{t1}, c_{t2})$ subject to

$$\begin{aligned} c_{t1} &= e_1 - q_t m_t && (14) \\ c_{t2} &= e_2 + q_{t+1}(m_t + \tau_t). && (15) \end{aligned}$$

Taking the first order condition for an interior solution, and then inserting the equilibrium conditions $m_t = M_t$ and $\tau_t = (z-1)M_t$, we have

$$\mu(e_1 - q_t M_t, e_2 + q_{t+1} M_{t+1}) = \frac{q_{t+1}}{q_t}.$$

Equivalently, in terms of real balances $S_t = q_t M_t$, we have

$$\mu(e_1 - S_t, e_2 + S_{t+1}) = \frac{S_{t+1}}{zS_t} \tag{16}$$

In the log-linear case, the generalization of (13) is

$$q_{t+1} = \frac{e_2 q_t}{\beta e_1 - (z+\beta) M_t q_t}. \tag{17}$$

Since M_t is changing with t, there does not exist constant q steady state; but we can look for the natural generalization in which the quantity theory implies $P_t = \Phi M_t$ for some $\Phi > 0$ for all t. Inserting $1/q_t = \Phi M_t$ into (17), we find

$$\Phi = \frac{z+\beta}{\beta e_1 - z e_2}.$$

Thus, $\Phi > 0$ if and only if $z\mu < 1$, where $\mu = e_2/\beta e_1$. This generalizes the condition for a monetary steady state derived above with M constant, $\mu < 1$.

To look for equilibria that do not satisfy the quantity theory, it is convenient to work with real balances S_t rather than the value of money q_t. Multiplying both sides of (17) by M_{t+1} and simplifying, we have

$$S_{t+1} = \frac{z e_2 S_t}{\beta e_1 - (z+\beta) S_t} \equiv f(q_t), \tag{18}$$

generalizing the function f defined above. In particular, there exists a steady state $S^* = f(S^*)$, where

$$S_t = \frac{\beta e_1 - z e_2}{z+\beta} \equiv S^*,$$

if and only if $z\mu < 1$. Of course, S^* corresponds the quantity theory equilibrium derived above. Also, for any initial $S_1 \in (0, S^*)$, the sequence S_t is an equilibrium where $S_t \to 0$. In such an equilibrium P_t increases faster than

M_t.[7]

Returning to the quantity theory equilibrium, where $S_t = S^*$, we have

$$\begin{aligned} c_1^* &= \frac{z(e_1 + e_2)}{z + \beta} \\ c_2^* &= \frac{\beta(e_1 + e_2)}{z + \beta}. \end{aligned}$$

This implies that for every generation G_t, $t \geq 1$, utility is given by

$$u^* = \log(z) - (1 + \beta)\log(z + \beta) + u_0,$$

where u_0 is a constant that does not depend on z. Maximizing u^* with respect to z yields $z = 1$, which means that the money supply is constant. For generation G_0, $c_{02} = e_2 + q_1 M_1$, which is decreasing in z because q_1 is decreasing in z. We conclude that any $z \leq 1$ is Pareto optimal, because increasing z makes the initial old worse off and decreasing z makes every generation $t \geq 1$ worse off, and that any $z > 1$ is not optimal and can be Pareto dominated by $z = 1$.

The result that $z = 1$ is Pareto optimal does not depend on the specific utility function used in the example. In Figure 3 we depict the situation in the (c_1, c_2) plane. A steady state monetary equilibrium is descirbed by two properties. First, (c_1, c_2) must be feasible, so that it must lie on the line with slope -1 through (e_1, e_2). Second, it must solve the individual's utilty maximization problem, so that the slope of the indifference curve at (c_1, c_2) is q_{t+1}/q_t. Since $q_{t+1}/q_t = S_{t+1}/S_t z = 1/z$, the slope of the indifference curve

[7]More generally, with population growth at rate γ the equilibrium conditions are $n_t m_t = M_t$ and $n_t \tau_t = (z - 1)M_t$. Inserting these into (17), noting that $n_t S_t = q_t M_t$, and rearranging, we have

$$S_{t+1} = \frac{z e_2 S_t}{\gamma\beta e_1 - \gamma(z + \beta)S_t},$$

which generalizes (18). Steady state and dynamic monetary equilibria exist if and only if $z\mu < \gamma$.

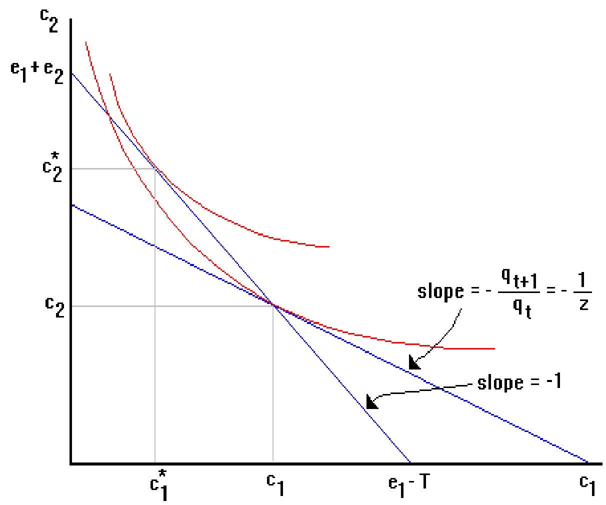

Figure 3: Optimality of Constant Money Supply

is $1/z$. As shown, (c_1^*, c_2^*), which is the equilibrium allocation when $z = 1$, is Pareto optimal, and whenever $z \neq 1$, the equilibrium (c_1, c_2) is dominated by (c_1^*, c_2^*). Hence, every generation G_t for $t \geq 1$ prefers $z = 1$ to $z \neq 1$.[8]

The next thing to do is to consider a version of the model with *storage* – that is, to introduce a technology for converting k units of the good at t into xk units of the good at $t+1$, $x > 0$. Allowing for the possibility of storage means that agents have more than one way to save. With storage, the budget constraints become

$$c_{t1} \quad = \quad e_1 - q_t m_t - k_t \tag{19}$$

[8]This conclusion holds without alteration or qualification for the case with population growth: for any γ, equilibrium if Pareto optinal if $z = 1$ (and *not*, as one may have guessed, if $z = \gamma$).

$$c_{t2} \quad = \quad e_2 + q_{t+1}(m_t + \tau_t) + xk_t. \tag{20}$$

We also impose nonnegativity constraints on k_t, m_t and c_{tj}.

We first look for nonmonetary equilibria, where $q_t = 0$ for all t. In this case, still using the log-linear utility function, the first order condition for an interior maximizer k_t is

$$\frac{1}{e_1 - k_t} = \frac{\beta x}{e_2 + xk_t},$$

which implies

$$k_t = \frac{\beta x e_1 - e_2}{(1+\beta)x} \equiv k^*.$$

If $x > \mu = e_2/\beta e_1$ then $k^* > 0$ and the equilibrium involves positive storage; if $x \leq \mu$ then storage is 0 and $c_{tj} = e_j$.

We now look for monetary equilibria. A necessary condition for a monetary equilibrium to exist is $xz \leq 1$. To see this, note that individuals will only hold money if it yields a return at least as great as storage, $q_{t+1}/q_t \geq x$; but this implies $S_{t+1}/S_t \geq xz$, and if $xz > 1$ then S_t becomes unbounded. Hence we require $xz \leq 1$. Given $xz \leq 1$, we now look for a quantity theory equilibrium with $k_t = 0$ and $S_t = S^*$. As in the case without storage, this implies

$$S^* = \frac{\beta e_1 - ze_2}{z + \beta},$$

which is positive if and only if $z\mu < 1$. Hence, for this monetary equilibrium to exist we need two things: $z\mu < 1$, which means that agents want to save, and $xz \leq 1$ which means that individuals are willing to save by holding money rather than simply storing goods.[9]

[9]In addition to the quantity theory equilibrium, when $z\mu < 1$ and $xz \leq 1$ there also exist monetary equilibria that do not satisfy the quantity equation, although the presence of storage entails different restrictions. In particular, there equilibria where $q_{t+1}/q_t = x$ and agents save in terms of both money and storage at every t. This implies $S_{t+1} = zxS_t$, so that $S_t \to 0$ and asymtoptically real balances vanish. For any $S_1 \in (0, k^*)$, we get such

In terms of welfare, when $x > 1$ the nonmonetary equilibrium is Pareto optimal and the steady state monetary equilibrium is optimal if it exists (since it only exists if $zx \leq 1$, which means $z < 1$, which is optimal by the argument given for the model without storage). If $x \leq 1$, however, equilibrium with $k > 0$ is nonoptimal. To see this, note for that any generation such $k_t > 0$ it is feasible to increase c_1 without decreasing c_2 by eliminating storage and transferring xk from the young to the old at every date thereafter. Finally, notice that when $z = 1$ the monetary equilibrium exists if and only if the nonmonetary equilibrium is nonoptimal.

We now consider some alternative ways to augment the money supply, assuming for simplicity that $x = 0$ (no storage). The analysis to this point has assumed that new money is given away as a lump sum transfer to the old. An alternative policy is to give old agents transfers that are proportional to their money holdings. In this case, the second period budget constraint is $c_{t2} = e_2 + q_{t+1}m_t z$ rather than (15). The first order condition (for a general utility function) is $\mu = zq_{t+1}/q_t$, or, by virtue of the equilibrium conditions,

$$\mu(e_1 - S_t, e_2 + S_{t+1}) = \frac{S_{t+1}}{S_t}, \tag{21}$$

which should be compared to (16). Notice that z does not appear in (21); therefore the set of equilibrium paths for S_t (or for any other real variable) does not depend on z under this policy.[10]

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an equilibrium if for all $t \geq 1$ we set $S_{t+1} = zxS_t$ and set k_t to satisfy the first order condition

$$\frac{1}{e_1 - S_t - k_t} = \frac{\beta x}{e_2 + S_{t+1} + xk_t}.$$

It is easy to check that total savings, $k_t + S_t$, differs from k^* unless $z = 1$, although $k_t \to k^*$ as $t \to \infty$. Finally, in the borderline case where $zx = 1$, for any $S_0 \in (0, k^*)$ there is an equilibrium with $S_t = S_0$ and $k_t = k^* - S_0$ for all t.

[10] It is sometimes said that monetary policy is "superneutral" if changes in the growth rate of M have no real effects, by extension of the idea that monetary policy is "neutral"

Another alternative way to introduce new money is to not give it away but to use it to purchase goods. In this case, the equilibrium condition becomes

$$\mu(e_1 - S_t, e_2 + \frac{S_{t+1}}{z}) = \frac{S_{t+1}}{zS_t}, \tag{22}$$

which should be compared to (16) and (21). As long as $z\mu < 1$, there is a quantity theory equilibrium in which $q_t m_t = S^*$ and $q_t/q_{t+1} = z$ for all t. We can use this model to present a simple public finance analysis of the inefficiency of the inflation tax relative to lump sum taxation.

As shown in Figure 4, in the quantity theory equilibrium, the real revenue raised by printing money is g, since $c_{t1} + c_{t2} = c_{t1} + c_{t-1,2} = e_1 + e_2 - g$. From the picture, it should be clear that if we set $M_t = M_0$ for all t and raise the same revenue by lump sum taxes, there is an equilibrium in which $q_t/q_{t+1} = 1$ and utility is higher for every generation. Of course, if the government does not have recourse to lump sum taxation, then the optimal mix of distorting taxes may well involve some inflation; the point here is simply that the inflation tax is distortionary. The more general point of this section is not only that increasing the money supply may matter, but exactly how one increases it matters a lot.

Finally, to close this section note that the total revenue raised for the government by the inflation tax is a function of the rate of monetary expansion, $g = (z-1)M_t q_t = (\ z-1)S$. For example, the log-linear utility function can be solved explicitly for $S^* = (\beta e_1 - ze_2)/(1+\beta)$, which implies

$$g = (z-1)S^* = \frac{(\beta e_1 - e_2)(z-1) - e_2(z-1)^2}{1+\beta}.$$

Notice that revenue first rises and then falls with z, and $g = 0$ at both $z = 1$ and $z = \bar{z} = \beta e_1/e_2$. Also, g is maximized at $\bar{z}/2$. Hence, we have a classic "Laffer Curve" for the inflation tax.

when changes in the level of M have no real effects. With proportional transfers, monetary policy is superneutral.

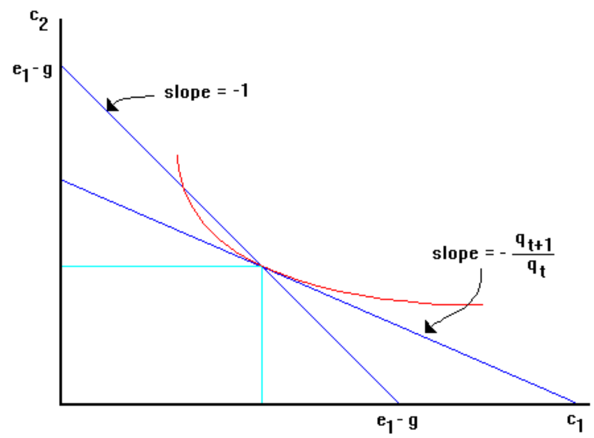

Figure 4: The Inefficiency of the Inflation Tax

1.5 Fiscal Policy

Here we return to nonmonetary economies and consider taxes, transfers and government debt. Let T_1 and T_2 be time-invariant lump-sum taxes (or transfers if negative) on young and old agents and let g denote government consumption. Assuming constant population, for all t, the government budget constraint is given by

$$T_1 + T_2 + \delta_t B_t = g + B_{t-1}, \tag{23}$$

where B_t is the quantity of pure discount securities issued at t and δ_t is their price; that is, each security is a claim to one unit of the consumption good at $t + 1$ and sells this period for δ_t. Condition (23) says that bond sales plus tax revenue has to pay for government consumption plus payments on the outstanding debt.

For all $t \geq 1$, G_t maximizes $u(c_{t1}, c_{t2})$ subject to

$$c_{t1} = e_1 - T_1 - \delta_t b_t \tag{24}$$

$$c_{t2} = e_2 - T_2 + b_t, \tag{25}$$

where b_t denotes demand for bonds by the young members of G_t. Of course, members of G_0 simply consume $c_{02} = e_2 - T_2$. If we assume the government balances the budget every period, then $b_t = B_t = 0$ and $T_1 = -T_2 = T$ for all t (that is, T is simply transferred from the young to the old). In this case, equilibrium entails $c_{t1} = e_j - T$, $c_{t2} = e_2 + T$, and $\delta_t = 1/\mu$, where $\mu = \mu(e_1 - T, e_2 + T)$. By choosing T we can implement any feasible allocation we want. In particular, if $\mu(e_1, e_2) < 1$ and autarchy is inefficient, we can implement the Pareto optimal allocation achieved in the monetary steady state simply by setting $T = S^*$.

This policy is essentially an unfunded social security system that transfers income to old people by taxing young people. In an economy with storage, an alternative would be a fully funded social security system: the government could tax people T when young, store the revenue, and pay them Tx when old, where x is the return on storage. If $x \leq 1$, the fully funded system is nonoptimal and dominated by the tax$-$transfer scheme described above, for the same reason that storage was dominated in the monetary economy in the previous section with $x \leq 1$. More generally, if $x \leq \gamma$, where γ is the growth rate, then a funded system is inefficient.