Lecture Notes for Economics 203
Core Economics III
Game Theory, Imperfect Competition, and Other Applications

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Contents

1 Strategic Environments

1.1 Components of a non-cooperative game ("extensive form")

Example Matching Pennies, Version A (MP-A)

Two players, 1 and 2.

Player 1 names heads (H) or tails (T).

Player 2 observes 1's choice, and then names heads (h) or tails (t)

If match, 2 pays \$2 to 1

If no match, 1 pays \$2 to 2

Example Matching Pennies, Version B (MP-B)

Two players, 1 and 2.

Player 1 names heads (H) or tails (T).

Player 2 names heads (h) or tails (t) without having observed 1's choice

If match, 2 pays \$2 to 1

If no match, 1 pays \$2 to 2

1. Who is playing?

I players, labelled $i = 1, 2, ..., I$

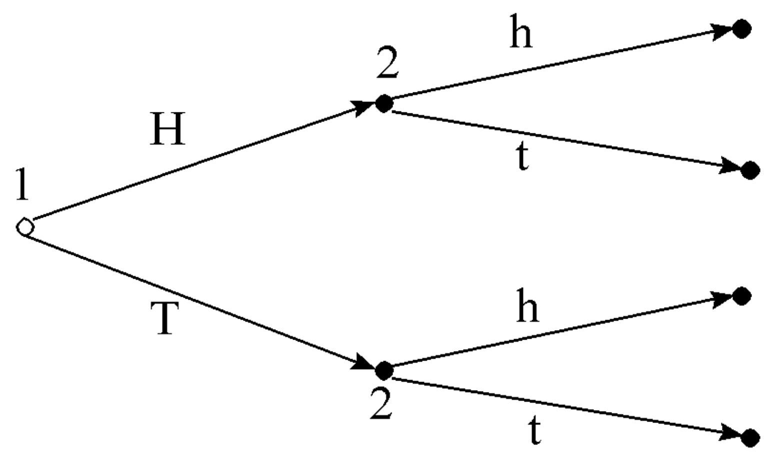
Also (potentially) "nature," N, used to represent uncertainty

MP-A and MP-B: $\{1, 2\}$

2. What is the sequence of events and decisions?

Concept of a game tree: A formal representation of the sequence of events and decisions (related to decision trees)

Example: Game tree for MP-A and MP-B



Components of a game tree:

1. Nodes
2. A mapping from nodes to the set of players
3. Branches
4. A mapping from branches to the set of action labels
5. Precedence (a partial ordering)
6. A probability distribution over branches for all nodes that map to nature, N

Definition: The *root* of a tree is a node with two properties: (i) it has no predecessors, and (ii) it is a predecessor for everything else (equivalently, everything else is a successor).

Definition: A *terminal node* is a node for which there are no successors.

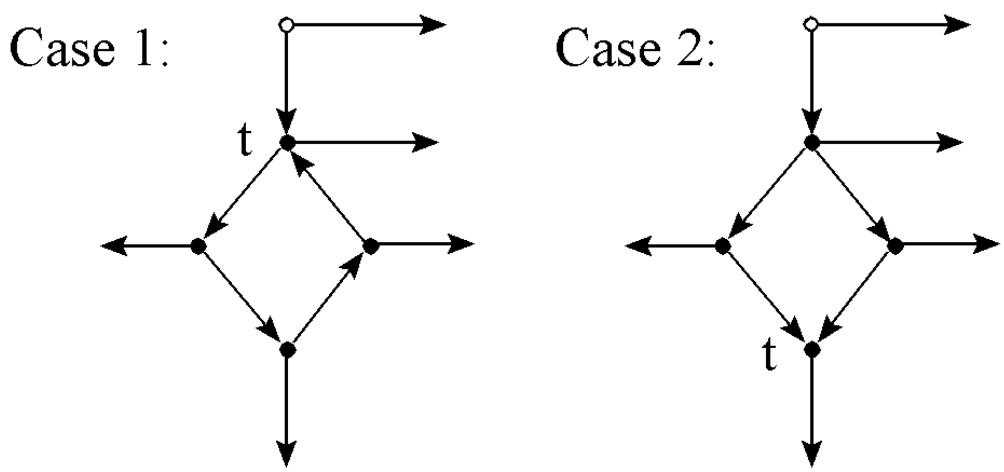
Assumptions:

A1. The number of nodes is finite (relax later).

A2. There exists a unique root.

A3. There is a unique path (following precedence) from the root of the tree to each terminal node.

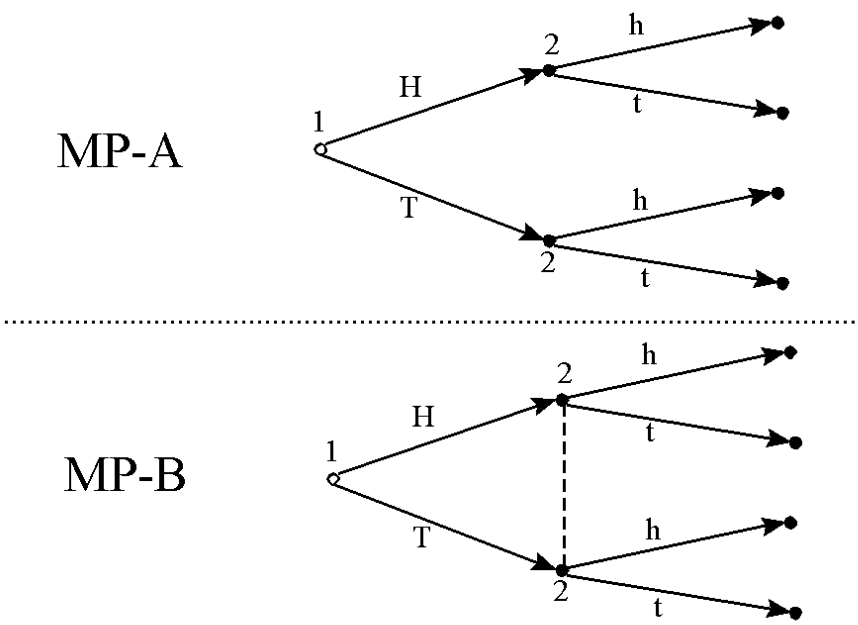
Examples of things A3 rules out:



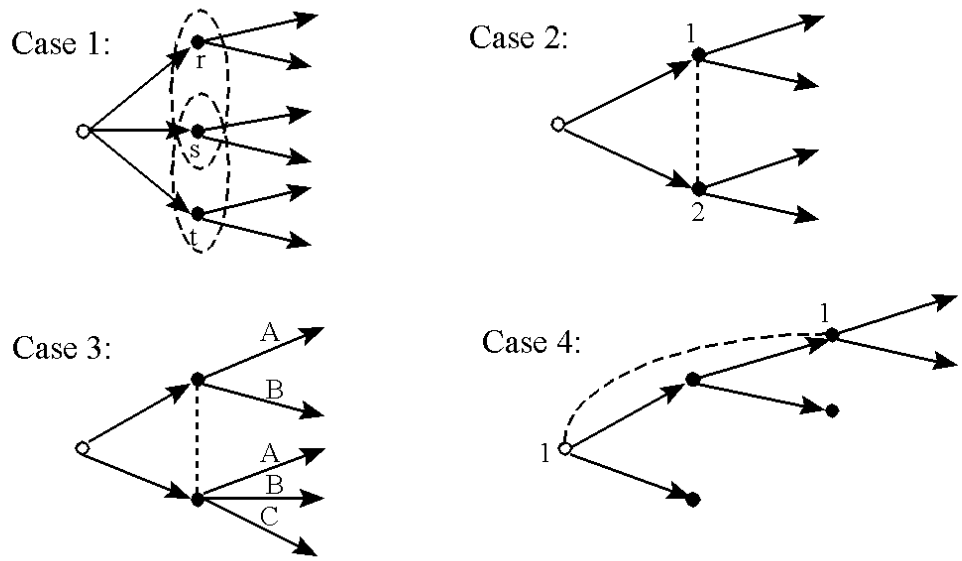
3. What does each player know when making a decision?

Formal representation of information: A partition of the set of nodes (i.e. a set of mutually exclusive and exhaustive subsets), such that (i) the same player makes the decisions at all nodes within any element of the partition, (ii) the same actions are available at all nodes within any element of the partition, and (iii) no element of the partition contains both a node and its predecessor.

Examples:

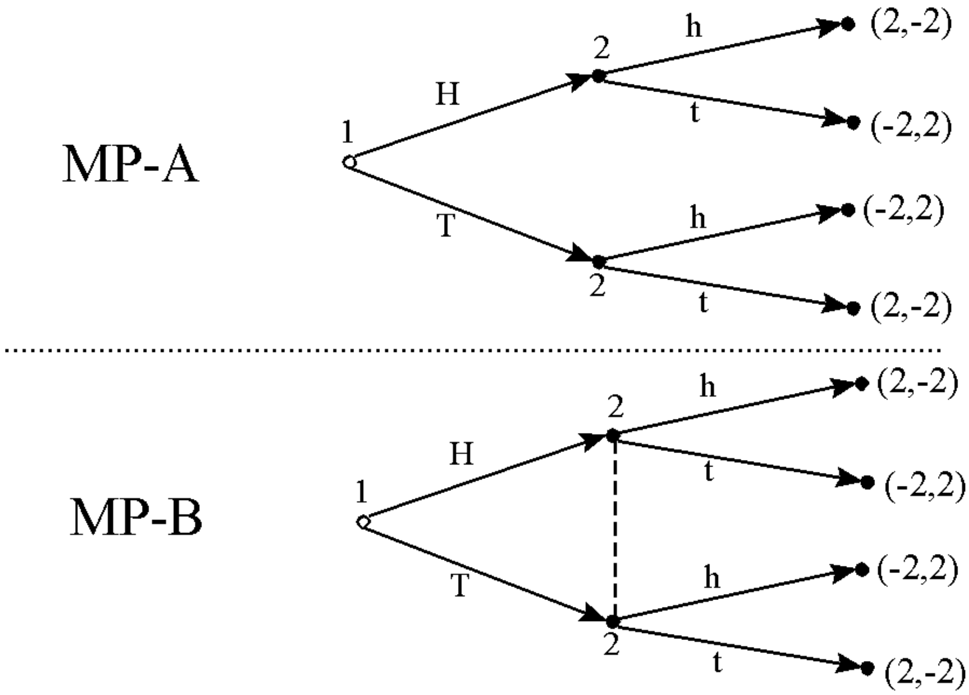


Examples of configurations that are ruled out:



4. How are payoffs determined?

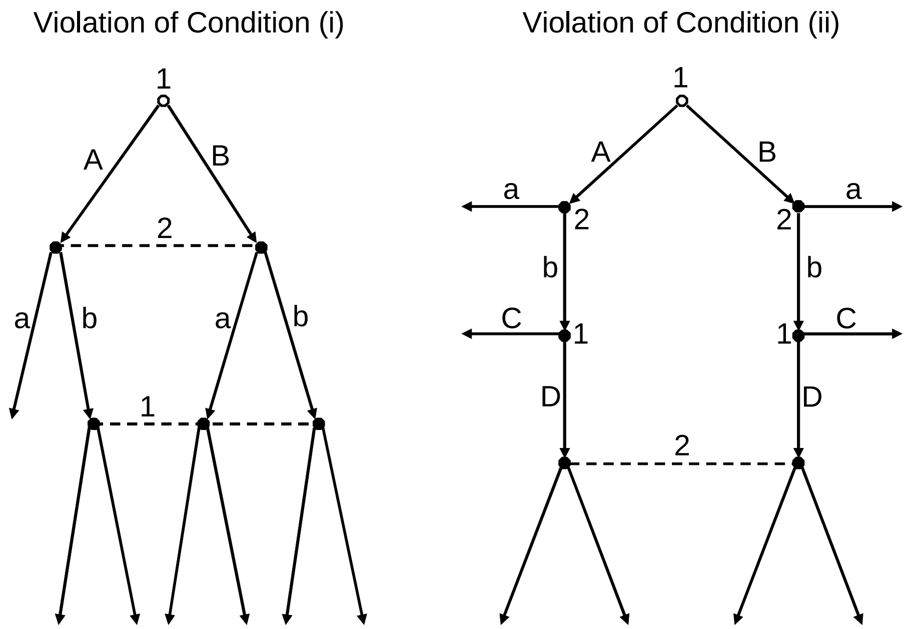
Formal representation of payoffs: A mapping from terminal nodes to a vector of utilities.



1.2 Some special classes of extensive form games

Games of Perfect Recall: Informally, (i) a player never forgets a decision that he or she took in the past, and (ii) a player never forgets information that he or she possessed when making a previous decision (for a formal definition, see e.g. Kreps p. 374).

Examples of what is ruled out:



Games of Perfect Information: Every information set is a singleton.

Examples: MP-A, Tic-Tac-Toe, Chess

1.3 Strategies and the strategic (normal) form

Concept of a strategy: A mapping that assigns a feasible action to all information sets for which the player is the decision-maker (a *complete contingent plan*).

Mathematical structure of a strategy: Let A be the set of action labels associated with all branches in the game. Suppose that there are K_i information sets for which player i is the decision-maker. Then a strategy for player i, denoted s_i, is an element of some *strategy set* $S_i \subseteq A^{K_i}$. That is, s_i is a K_i-dimensional vector, which lists an action for each information set.

Examples:

MP-A: $S_1 = \{H, T\}$, $S_2 = \{(h, h), (h, t), (t, h), (t, t)\}$

MP-B: $S_1 = \{H, T\}$, $S_2 = \{h, t\}$

Some terms and notation: $s = (s_1, s_2, ..., s_I)$ is a *strategy profile.*

$S = \prod_{i=1}^{I} S_i$ is the set of feasible strategy profiles.

$s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_I)$ is a profile of strategies for every player but i

$S_{-i} = \prod_{j \neq i} S_j$ is the set of feasible strategy profiles for every player but i

Payoffs: A strategy profile s induces a probability distribution over terminal nodes. Let $g_i(s)$ denote the associated expected utility of player i, given this probability distribution, and given the mapping from terminal nodes into i's utility. Let $g(s) = (g_1(s), g_2(s), ..., g_I(s))$. Notice that $g : S \to \Re^I$. This is known as the *payoff function.*

A game in strategic (normal) form: A collection of players $\{1, ..., I\}$, a strategy profile set S, and a payoff function g. (We will sometimes write this as (S, g), where the set of players is implicit.)

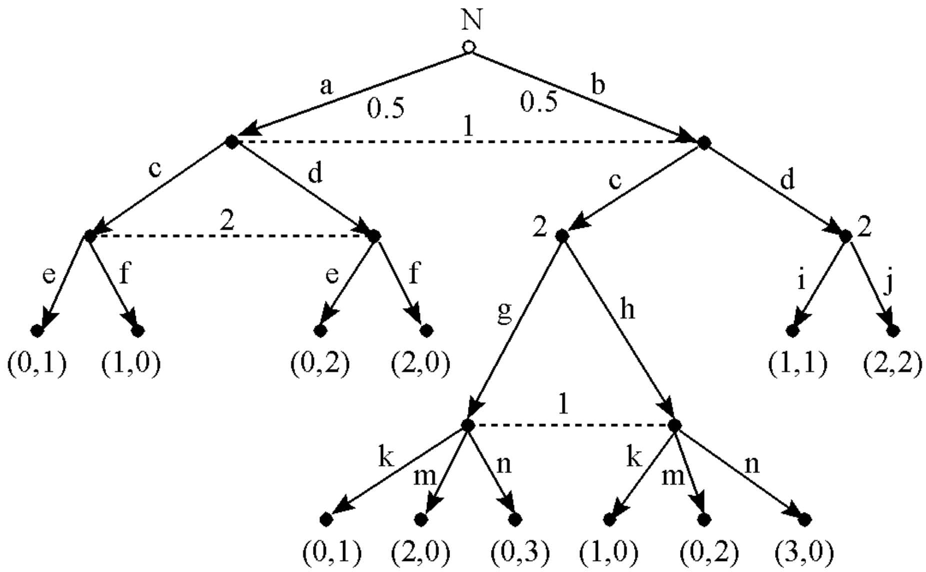
Example 1: MP-A

		Player 2			
		hh	ht	th	tt
Player 1	H	2,-2	2,-2	-2,2	-2,2
	T	-2,2	2,-2	-2,2	2,-2

Example 2: MP-B

		Player 2	
		h	t
Player 1	H	2,-2	-2,2
	T	-2,2	2,-2

Example 3: A game with random events



Player 1: 2 information sets; 2 possible choices at first, 3 possible choices at second; cardinality of strategy set $= 2 \times 3 = 6$. $S_1 = \{(c,k),(c,m),(c,n),(d,k),(d,m),(d,n)\}$

Player 2: 3 information sets; 2 possible choices at each; cardinality of strategy set $= 2^3 = 8$. $S_2 = \{(e,g,i),(e,g,j),(e,h,i),(e,h,j),(f,g,i),(f,g,j),(f,h,i),(f,h,j)\}$.

$S = S_1 \times S_2$. $|S| = 6 \times 8 = 48$ (number of strategy profiles).

Illustrative calculations for the payoff function:

(e,g,i) and (c,k) yields (0,1) with probability 0.5, and (0,1) with probability 0.5. Expected payoff is (0,1).

(e,h,i) and (d,m) yields (0,2) with probability 0.5 and (1,1) with probability 0.5. Expected payoff is (0.5, 1.5).

(f,g,j) and (c,n) yields (1,0) with probability 0.5 and (0,3) with probability 0.5. Expected payoff is (0.5, 1.5).

1.4 Some applications to models of imperfect competition

Setting: Industry consists of 2 firms producing a homogeneous product. Demand is $Q(P)$; inverse demand is $P(Q)$; both are monotonically decreasing. Firm i produces q_i at cost $c_i(q_i)$, with $c_i(0) = 0$.

Simultaneous quantity competition (Cournot): Firms 1 and 2 simultaneously choose quantity q_i. Normal form: $s_i = q_i$, $S_i = \Re_+$, $g_i(s_1, s_2) = P(s_1 + s_2)s_i - c_i(s_i)$.

Simultaneous price competition (Bertrand): Firms 1 and 2 simultaneously choose price p_i, consumers purchase from the firm with the lowest price (splitting equally between the two firms in the event of a tie). Normal form: $s_i = p_i$, $S_i = \Re_+$,

$$g_i(s_1, s_2) = \begin{cases} 0 \text{ if } s_i > s_j \\ s_i Q(s_i) - c_i(Q(s_i)) \text{ if } s_i < s_j \\ \frac{s_i Q(s_i)}{2} - c_i\left(\frac{Q(s_i)}{2}\right) \text{ if } s_i = s_j \end{cases}$$

Sequential quantity competition with imperfect observability: Firm 1 chooses q_1. Firm 2 observes either that $q_1 \leq q^*$, or that $q_1 > q^*$, then firm 2 chooses quantity. Normal form: $s_1 = q_1$, $S_1 = \Re_+$. For player 2, there are two information sets, so consequently $s_2 \in \Re_+^2 = S_2$. For $s_2 = (s_{21}, s_{22})$, s_{21} indicates firm 2's choice if firm 1 selects $s_1 \leq q^*$, while s_{22} indicates firm 2's choice if firm 1 selects $s_1 > q^*$.

$$g_1(s_1, s_2) = \begin{cases} P(s_1 + s_{21})s_1 - c_1(s_1) \text{ if } s_1 \leq q^* \\ P(s_1 + s_{22})s_1 - c_1(s_1) \text{ if } s_1 > q^* \end{cases}$$

$$g_2(s_1, s_2) = \begin{cases} P(s_1 + s_{21})s_{21} - c_2(s_{21}) \text{ if } s_1 \leq q^* \\ P(s_1 + s_{22})s_{22} - c_2(s_{22}) \text{ if } s_1 > q^* \end{cases}$$

Sequential quantity competition (Stackelberg): Firm 1 chooses q_1. Firm 2 observes q_1, and then chooses q_2. Normal form: $s_1 = q_1$, $S_1 = \Re_+$. For firm 2, there is an uncountably infinite number of information sets, so the dimensionality of firm 2's strategy set is uncountably infinite. Firm 2's strategy is a function $s_2 : \Re_+ \rightarrow \Re_+$, and S_2 is a function space.

$$g_1(s_1, s_2) = P(s_1 + s_2(s_1))s_1 - c_1(s_1)$$

$$g_2(s_1, s_2) = P(s_1 + s_2(s_1))s_2(s_1) - c_2(s_2(s_1))$$

2 Strategic Choice in Static Games of Complete Information

Games of Complete Information: Each player knows the payoff received at each terminal node for every other player.

2.1 Dominant strategies

Example: The Prisoners' dilemma

		Player 2	
		Not Fink	Fink
Player 1	Not Fink	-2,-2	-5, -1
	Fink	-1, -5	-4, -4

Definition: s_i is a *dominant strategy* for i iff for all $\widehat{s}_i \in S_i$ (with $\widehat{s}_i \neq s_i$) and $s_{-i} \in S_{-i}$, $g_i(s_i, s_{-i}) > g_i(\widehat{s}_i, s_{-i})$.

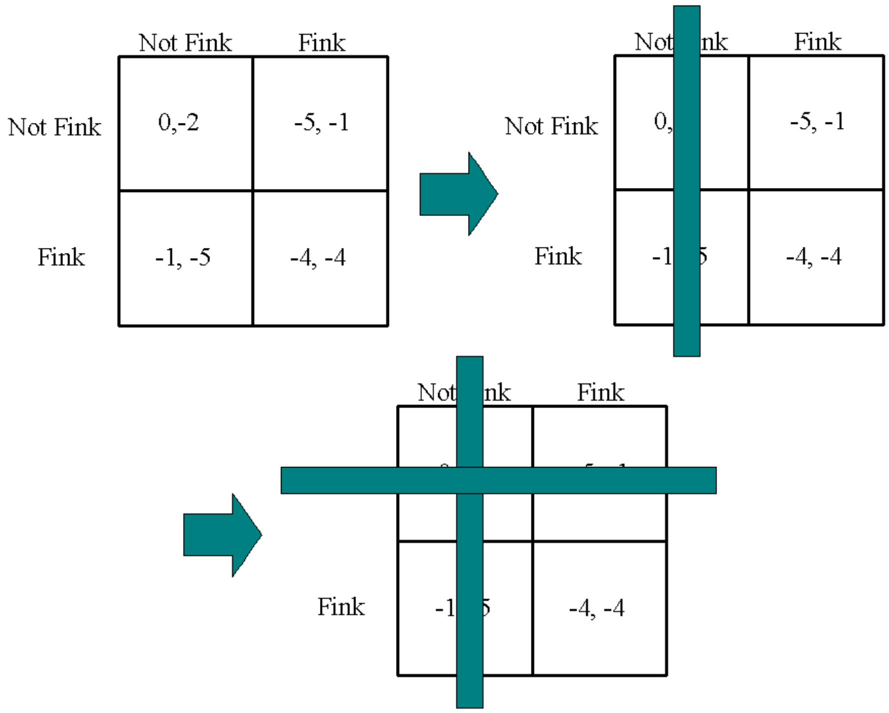
Remark: It is unusual for games to have dominant strategy solutions.

Example: A modified Prisoners' dilemma

		Player 2	
		Not Fink	Fink
Player 1	Not Fink	0,-2	-5, -1
	Fink	-1, -5	-4, -4

2.2 Dominance and iterated dominance

Motivation based on previous example:



Notation: $K_i \equiv |S_i|$. For player i, label the individual strategies s_i^k, $k = 1, ..., K_i$.

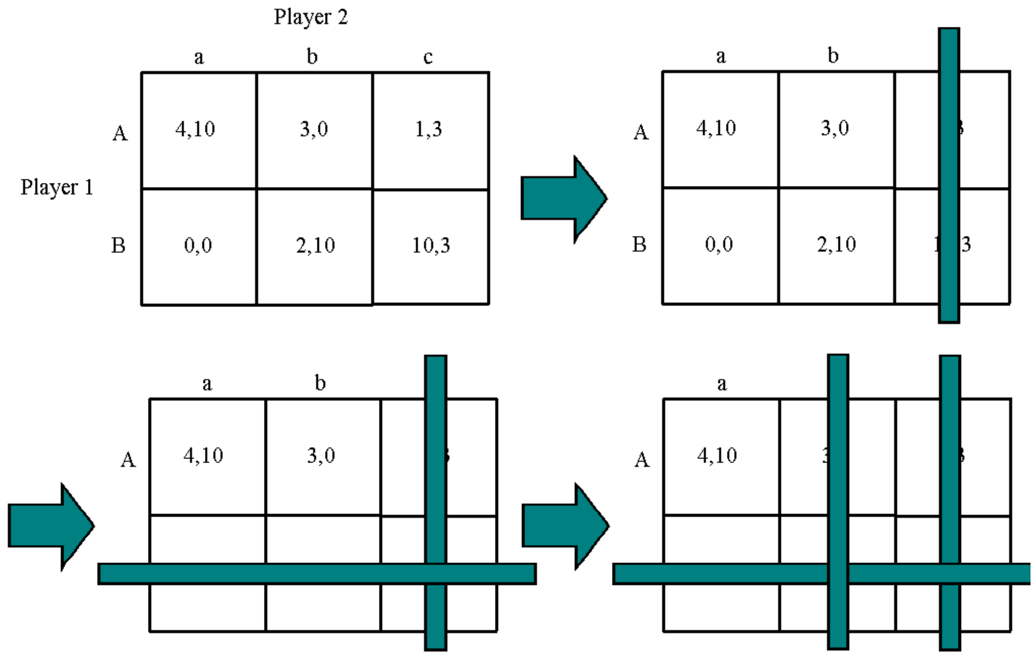
Definition: s_i^k is a *dominated strategy* for player i iff $\exists \rho_m \geq 0$ for $1 \leq m \leq K_i$ with $\sum_{m=1}^{K_i} \rho_m = 1$ such that, for all $s_{-i} \in S_{-i}$,

$$\sum_{m=1}^{K_i} \rho_m g_i(s_i^m, s_{-i}) > g_i(s_i^k, s_{-i}).$$

Procedure: Iteratively delete dominated strategies

Terminology: If iterative deletion of dominated strategies yields a unique outcome (as in the preceding example), we say that the game is *dominance solvable.*

Illustration of the role of randomizations:



Remarks (concerning iterative deletion of dominated strategies):

1. The order of deletion is irrelevant.

2. To justify iterative deletion of dominated strategies, one must assume *common knowledge* of rationality.

3. Most games are not dominance solvable.

Illustration: MP-B. Neither player has a dominated strategy.

Application #1: Linear Cournot model with two identical firms: $c_i(q_i) = cq_i$, $p = a - bQ$.

Payoff functions:

$$g_i(q_i, q_{-i}) = (a - c)q_i - bq_iq_{-i} - bq_i^2$$

First order condition:

$$(a-c) - bq_{-i} - 2bq_i = 0$$

"Best response" functions (same for both firms):

$$\gamma(q_{-i}) = \max\left\{\frac{(a-c)}{2b} - \frac{q_{-i}}{2}, 0\right\}$$

Theorem: The linear Cournot model with two identical firms is dominance solvable. The solution involves $q = \frac{a-c}{3b}$ for each firm.

Lemma: If $q_{-i} \in [q^0, q^1]$, then all $q_i < \gamma(q^1)$, and all $q_i > \gamma(q^0)$ are dominated.

Proof: Take some $q_i < \gamma(q^1)$. Define

$$\begin{aligned} D(q_{-i}) &\equiv g_i\left(\gamma(q^1), q_{-i}\right) - g_i\left(q_i, q_{-i}\right) \\ &= (a-c)\left(\gamma(q^1) - q_i\right) + b\left[q_{-i}\left(q_i - \gamma(q^1)\right) + q_i^2 - \left[\gamma(q^1)\right]^2\right] \end{aligned}$$

Note that $D(q^1) > 0$. Also notice that

$$\frac{dD(q_{-i})}{dq_{-i}} = b\left(q_i - \gamma(q^1)\right) < 0.$$

Consequently, for $q_{-i} < q^1$, $D\left(q_{-i}\right) > 0$. Thus, q_i is dominated by $\gamma(q^1)$.

The argument for $q_i > \gamma(q^0)$ is completely symmetric, and is left as an exercise. Q.E.D.

Remark: The lemma relies only on the submodularity of the payoff function.

Proof of the Theorem: We use the lemma to iteratively delete dominated strategies. This produces a sequence of intervals, as follows:

Strategy set: $[0, \infty)$

First iteration: $[0, \gamma(0)]$ (note: $\gamma(0)$ is the monopoly quantity, $(a-c)/2b$)

Second iteration: $[\gamma^2(0), \gamma(0)]$

Third iteration: $[\gamma^2(0), \gamma^3(0)]$, etc.

The lower bounds of these intervals form a sequence $\gamma^{2t}(0)$.

The upper bounds of these intervals form a sequence $\gamma^{2t+1}(0)$.

Note that

$$\gamma^t(0) = -\left(\frac{a-c}{b}\right)\sum_{j=1}^{t}\left(-\frac{1}{2}\right)^j.$$

This series is absolutely convergent; hence it is also convergent. Thus, the upper and lower bounds converge to a single point. Since all other quantities are ruled out, the game is dominance solvable.

To determine the solution, note that

$$\begin{aligned}
-\sum_{j=1}^{\infty}\left(-\frac{1}{2}\right)^j &= -\sum_{m=1}^{\infty}\left(\left(-\frac{1}{2}\right)^{2m-1}+\left(-\frac{1}{2}\right)^{2m}\right) \\
&= \sum_{m=1}^{\infty}\left(\left(\frac{1}{2}\right)^{2m-1}-\left(\frac{1}{2}\right)^{2m}\right) \\
&= \sum_{m=1}^{\infty}\left(\frac{2}{2^{2m}}-\frac{1}{2^{2m}}\right) \\
&= \sum_{m=1}^{\infty}\left(\frac{1}{4}\right)^m = \frac{1}{3}
\end{aligned}$$

So $\gamma^\infty(0) = \frac{a-c}{3b}$, as claimed. Q.E.D.

Application #2: Linear Cournot oligolpoly with $N > 2$ identical firms.

The lemma continues to apply, except now q_{-i} refers to the *total* quantity of all other firms ($q_{-i} = \sum_{j \neq i} q_j$). Hence, iterative domination proceeds as follows:

Strategy set: $[0, \infty)$

First iteration: $[0, \gamma(0)]$

Second iteration: $[\gamma((N-1)\gamma(0)), \gamma(0)]$

But $(N-1)\gamma(0) \geq \frac{a-c}{b}$ (equal if $N = 3$), so $\gamma\left((N-1)\gamma(0)\right) = 0$. Consequently, nothing happens after the first iteration. All dominance tells us is that each firm will produce not more than the monopoly quantity.

Application #3: Bertrand competition with a discrete price grid (prices are quoted in "pennies") and identical linear costs, $c_i(q_i) = cq_i$.

Price is chosen from $\{0, 1, 2, 3, ...\}$; c is an integer.

First iteration: eliminate 0 (dominated by $p_i = c$)

Second iteration: eliminate 1 (dominated by $p_i = c$)

N^{th} iteration: eliminate N provided that $N < c$.

Conclusion: price can't be lower than unit cost. No other prices are eliminated (when $p_j = c$, firm i earns zero profits for all $p_i \geq c$, so no price at or above marginal cost is dominated).

Application #4: Bertrand competition with unrestricted prices ($p_i \in \Re_+$) and linear costs ($c_i(q_i) = c_i q_i$, $c_i > 0$)

$p = 0$ is dominated by $p = c_i$.

Is any other $p > 0$ dominated? No. Choose any p', $p'' > 0$. Neither price is dominated by the other. Why? Consider $p_{-i} \in (0, \min\{p', p''\})$. Then $g_i(p', p_{-i}) = g_i(p'', p_{-i}) = 0$. Note in particular that prices below cost are not eliminated by domination.

2.3 Weak dominance

Motivating example: Bertrand competition with linear costs. Below-cost prices do not seem plausible.

A simpler motivating example: The "Less Than" game

Two players, simultaneous choices.

Each player picks 0 or 1.

Object: name a number that is lower than the one named by your opponent.

Winner gets \$1, loser gets nothing.

No prizes in the event of ties.

Normal form:

		Player 2	
		0	1
Player 1	0	0,0	1,0
	1	0,1	0,0

Note: neither strategy is dominated.

Definition: s_i^k is a *weakly dominated strategy* for player i iff $\exists \rho_m \geq 0$ for $1 \leq m \leq K_i$ with $\sum_{m=1}^{K_i} \rho_m = 1$ such that, for all $s_{-i} \in S_{-i}$,

$$\sum_{m=1}^{K_i} \rho_m g_i(s_i^m, s_{-i}) \geq g_i(s_i^k, s_{-i}),$$

with strict inequality for some $s_{-i} \in S_{-i}$.

Remark: Dominated strategies, as we have defined them, are sometimes called *strictly dominated strategies*. The term "dominated," when used by itself, will always denote strict domination.

Example: MP-A

		Player 2			
		hh	ht	th	tt
Player 1	H	2,-2	2,-2	-2,2	-2,2
	T	-2,2	2,-2	-2,2	2,-2

ht is (strictly) dominated by th; hh and tt are weakly dominated by th. Thus, all choices for player 2 are dominated (either weakly or strictly), except th. Given that Player 2 will play th, Player 1 is indifferent between playing H and T.

Application #1: Bertrand competition with identical linear costs.

Any $p' < c$ is weakly dominated by $p'' = c$. If $p_{-i} < p'$, both p' and p'' yield the same payoff (zero). But if $p_{-i} \geq p'$, p' yields a strictly negative payoff, while p'' still yields zero.

$p' = c$ is weakly dominated by p'' slightly greater than c. The payoffs associated with p' are zero for all p_{-i}, while the payoffs associated with p'' are always non-negative, and strictly positive for $p_{-i} > p''$.

$p' > p^m$ (the monopoly price, which, for simplicity, we will assume to be unique) is weakly dominated by p^m. If $p_{-i} < p^m$, both yield zero payoffs. If $p_{-i} \in [p^m, p')$, p^m yields strictly positive payoffs, while p' yields a zero payoff. If $p_{-i} \geq p'$, the payoff for p^m strictly exceeds the payoff from p'.

Provided that $Q(p)(p-c)$ is strictly increasing in p on $[c, p^m]$, no other $p' \in (c, p^m]$ is weakly dominated. Choose any such p'. Consider any probability distribution over non-negative prices, with CDF F, that does not place all weight on p'. If $F(p') = 1$, then, for any $p_{-i} > p'$, randomizing according to F produces a strictly lower payoff than p'. If $F(p') < 1$, then, for any $p_{-i} > p$, the payoff from this randomization scheme is bounded above by $F(p_{-i})Q(p_{-i})(p_{-i} - c)$, which is strictly less than $Q(p')(p' - c)$ for p_{-i} sufficiently close to p'. But the latter term is what firm i earns, for such p_{-i}, when choosing p'.

p' yields strictly higher profits if $p_{-i} \in (p', p'')$. Consider $p'' < p'$; p' yields strictly higher profits if $p_{-i} > p'$.

Conclude: weak domination elminates everything but $(c, p^m]$.

Application #2: A "second price" auction.

I players ("bidders").

Each has a valuation, v_i, for some indivisible object that is being auctioned.

Players simultaneously submit sealed bids, p_i.

The highest bid wins the object. In case of ties, the winner is determined randomly (with equal probability for all winning bids).

The winner pays the *second highest* bid, p^s (in the event of a tie, this equals the winning bid).

Payoffs are: $v_i - p^s$ for the winner, 0 for all others.

Theorem: In the second-price auction, every $p' \neq v_i$ is weakly dominated by $p'' = v_i$.

Proof: Let p_{-i} denote the vector of bids by players other than i, and let p^m_{-i} denote the highest bid among these players. There are two cases to consider.

Case 1: $p' < v_i$.

(i) $p^m_{-i} < p'$. Then i's payoff is the same $(v_i - p^m_{-i})$ regardless of whether it bids p' or p''.

(ii) $p^m_{-i} = p'$. Then i's payoff is strictly higher when it bids p'' $(v_i - p^m_{-i} > 0)$ then when it bids $p'(\frac{1}{n}(v_i - p^m_{-i})$, where n is the number of winning bids).

(iii) $p^m_{-i} \in (p', p'')$. Then i's payoff is strictly higher when it bids p'' $(v_i - p^m_{-i} > 0)$ then when it bids p' (0).

(iv) $p^m_{-i} \geq p''$. Then i's payoff is zero regardless of whether it bids p' or p''.

Case 2: $p' > v_i$

(i) $p^m_{-i} < p''$. Then i's payoff is the same $(v_i - p^m_{-i})$ regardless of whether it bids p' or p''.

(ii) $p^m_{-i} = p''$. Then i's payoff is zero regardless of whether it bids p' or p''.

(iii) $p^m_{-i} \in (p'', p']$. Then i's payoff is strictly negative when it bids p', and zero when it bids p''.

(iv) $p^m_{-i} > p'$. Then i's payoff is zero regardless of whether it bids p' or p''. Q.E.D.

Caveat: The theorem does not establish that nothing weakly dominates playing v. However, this is true. (As an exercise, prove this. In particular, show that, for any randomization over bids with CDF F, there is some p^m_{-i} such that bidding v is strictly better.)

Application #3: A simple voting problem.

I players (voters), $I \geq 3$.

J policy options, $j = 1, ..., J$.

Players simultaneously cast votes: they name j. Abstentions are not permitted. $S_i = \{1, ..., J\}$

The policy with the most votes is adopted (in the event of ties, a policy is selected with equal probabilities from the set of policies receiving the most votes)

Payoffs for i are v_i^j when policy j is adopted. We assume that $v_i^j \neq v_i^k$ for all i, j, k.

Observation: Domination does not eliminate any strategies. Imagine that all other players vote for the same policy. Then i's vote cannot affect the outcome; therefore, i's payoff is the same for all $s_i \in S_i$.

Theorem: In the simple voting problem, weak domination eliminates the possibility that i would vote for her least favorite policy $j^{'}(i) \equiv \arg\min_j v_i^j$. With $I > 3$, weak domination does not rule out any other strategy.

Proof: Fix any strategy for voters $k \neq i$. Let π (a J-dimensional probability vector) denote the outcome if i does not vote.

Now imagine that i votes for $j'(i)$. There are three possibilities: (a-i) no change in outcome, (a-ii) $j'(i)$ is added to the set of possible outcomes, or (a-iii) the set of possible outcome shrinks to $j'(i)$. Possibility (a-i) is π, and the other possibilities are strictly worse than π.

Next imagine that i votes for $j''(i) \equiv \arg\max_j v_i^j$. There are three possibilities: (b-i) no change in outcome, (b-ii) $j''(i)$ is added to the set of possible outcomes, or (b-iii) the set of possible outcomes shrinks to $j''(i)$. Possibility (b-i) yields π, and the other possibilities are better than π.

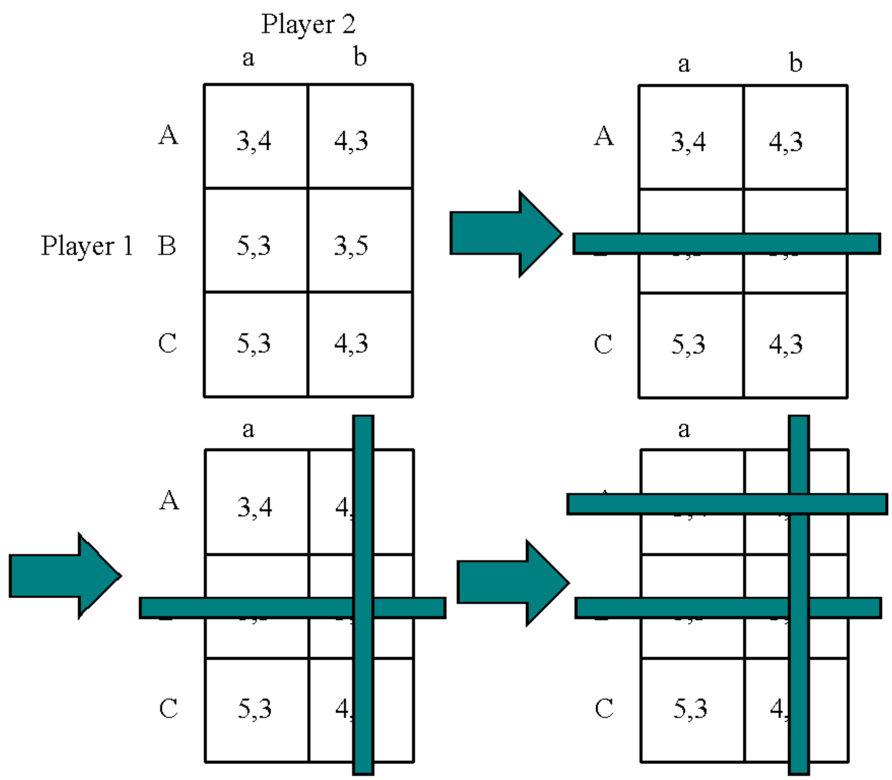
Thus, regardless of what other voters have chosen, the outcome when i votes for $j''(i)$ is at least as attractive to i as when i votes for $j'(i)$. In cases where (a-ii), (a-iii), (b-ii), or (b-iii) occurs, voting for $j''(i)$ is strictly better than voting for $j'(i)$, so voting for $j'(i)$ is weakly dominated.

Now we argue that weak domination does not rule out any other strategy when $I > 3$. First assume that I is odd. Suppose that $J^*_{-i}(s_{-i}) = \{j'(i), k\}$, and that every $j \neq i$ votes either for $j'(i)$ or for k. Then i is strictly better off voting for k than voting for anything else. Next suppose that I is even. Again assume that every $j \neq i$ votes either for $j'(i)$ or for k, but in this case imagine that $j'(i)$ receives one more vote than k. Once again, i is strictly better off voting for k than voting for anything else. Q.E.D.

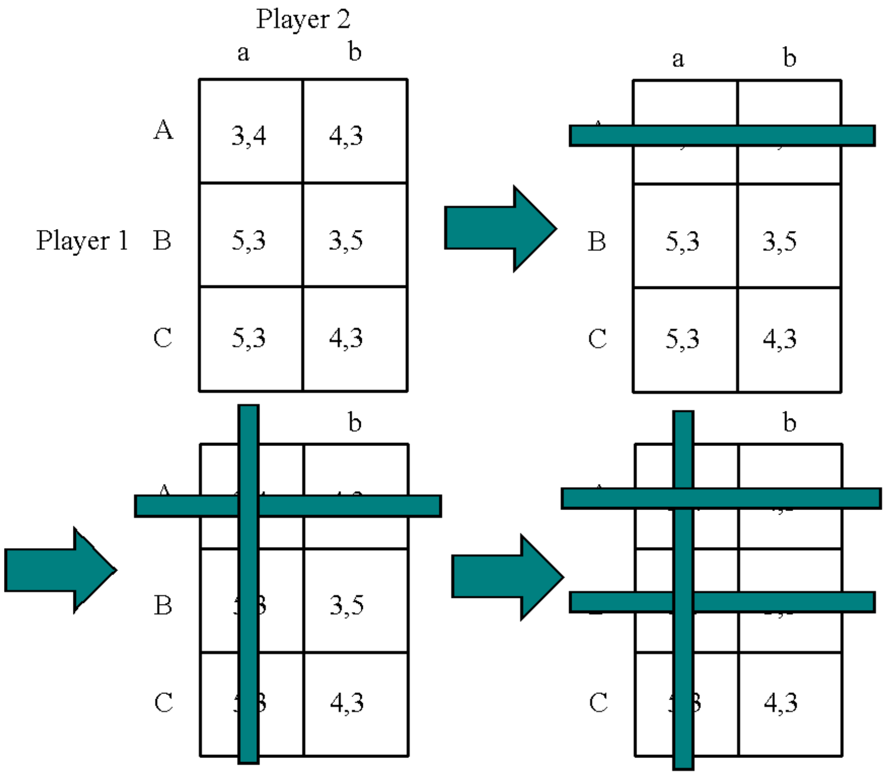
Corollary: If players avoid weakly dominated strategies, and if $J = 2$, then the policy preferred by the majority is victorious.

Suggestion: Why not apply the weak dominance criterion iteratively?

Example (demonstrating that the order of deletion matters):



Alternatively:



2.4 Rationalizability

Definition: (i) A 1-*rationalizable strategy* for player i is an element of S_i that is a best response to some (independent) probability distribution over strategies for other players.

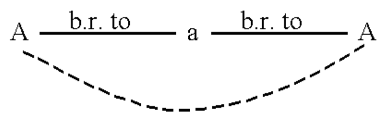
(ii) A k-*rationalizable strategy* for player i is an element of S_i that is a best response to some (independent) probability distribution over $(k-1)$-rationalizable strategies for other players.

(iii) A *rationalizable strategy* is an element of S_i that is k-rationalizable for all k.

An example:

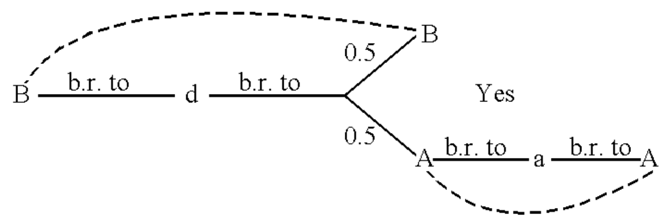
		Player 2: a	b	c	d
Player 1	A	4,10	3,0	1,3	2,6
	B	0,0	2,10	10,3	3,6

Is A rationalizable?



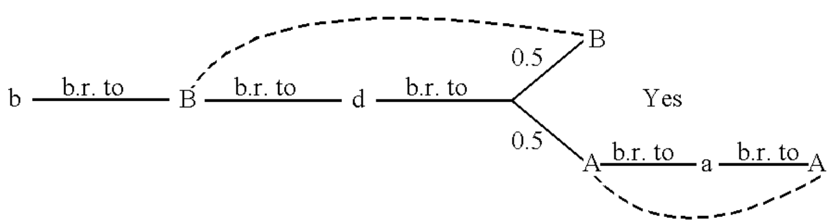
Yes

Is B rationalizable?



Same reasoning implies a and d are rationalizable.

Is b rationalizable?



Is c rationalizable? No, because it isn't a best response to anything.

Remark: For two player games, rationalizable strategies are exactly what is left over after iterative deletion of dominated strategies. This equivalence generalizes to more than 2 players if one does not insist on independence. If one does insist on independence, the set of rationalizable strategies is smaller.

2.5 Pure strategy Nash equlibrium

Definition: $s^* = (s_1^*, ..., s_I^*) \in S$ is a *pure strategy Nash equilibrium* if for all i and $s_i \in S_i$,

$$g_i(s_i^*, s_{-i}^*) \geq g_i(s_i, s_{-i}^*).$$

MP-A: (H, th) and (T, th).

MP-B: No pure strategy Nash equilibria.

Theorem: A finite game of perfect information has a pure strategy Nash equilibrium

Proof: To keep things simple, we'll treat cases with no uncertainty - Nature makes no choices. (As an exercise, try to generalize the proof to include cases with uncertainty.)

Define the *length* L of the game as the longest path (measured in number of branches, following precedence) from a root node to a terminal node. The proof is by induction on the length of the game.

Begin with $L = 1$. This is just a one-person decision problem with a finite number of choices. Every finite set of real numbers contains a maximal element; the corresponding choice is the equililibrium of the one-player game.

Now imagine that the theorem is true for all games of length $L \leq n$. We argue that it must also be true for games of length $n + 1$.

Consider a finite game of perfect information with length $n+1$. This game begins with an initial decision by some player (without loss of generality, player 1). Let T denote the number of potential choices at the root node. Each these choices leads to a successor node. We index these successor nodes $t = 1, ...T$. One can think of each one of these T successor nodes as the root node for a (finite, perfect information) continuation game of of length n or less. By hypothesis, there exists a pure strategy Nash equilibrium for each of these continuation games. Construct an equilibrium for the $n+1$ length game as follows. Within each continuation game, pick a mapping from nodes to actions that corresponds to some equilibrium of that continuation game. Let v_{it} denote the payoff to i for the equilibrium associated with the continuation game emanating from node t. At the root node, let player 1 choose the action that leads to any node $t^* \in \arg\max_t v_{1t}$ (since the set of alternatives is finite, a maximal element exists).

The preceding is plainly a pure strategy Nash equilibrium for the full game. Any deviation by a player $i \neq 1$ has the same effect on i's payoff as a deviation in the continuation game emanating from node t^*. Since, by construction, the strategy profile constitutes an equilibrium in this continuation game, the deviation does not benefit player i. Now consider player 1. Consider some deviation by player 1. For this deviation, imagine that 1's initial choice leads to node $k \in \{1, ..., T\}$. Let u denote 1's payoff from deviating. Since the strategy profile constitutes an equilibrium in the continuation game emanating from node t, we have $u \leq v_{1k} \leq \max_t v_{1t}$, which is 1's payoff in the proposed equilibrium. Hence, the deviation is not in 1's interest. Q.E.D.

Remark: "Zermelo's theorem" suggests an algorithm for finding equilibria: *backward induction.* Procedure: at each penultimate node, identify an optimal choice for the decision-maker. Treat the penultimate node as a terminal node, using the payoffs associated with the aforementioned optimal choice. Repeat until all decisions in the tree are resolved.

Example #1: Chess. Implication of previous theorem: either white always wins, black always wins, or it is always a draw. Solvable by backward induction.

Example #2: The centipede game.

Two players

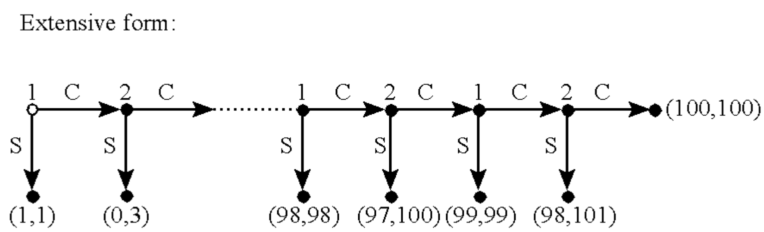
Two piles of money, one for each player. Start with \$1 in each pile.

Players alternate, saying either "stop" (S) or "continue" (C)

As soon as either player says "stop," the game is over

Every time i says "continue," \$1 is taken from i's pile, and \$2 is placed into j's pile.

The game ends automatically when each pile contains \$100.



Implication of backward induction: Say "stop" at every node. The game ends immediately.

How do people actually play? Tend to say "continue" until there is a substantial amount of money in both piles.

Theorem: Suppose $S_1, ..., S_I$ are compact, convex Euclidean sets. Suppose that g_i is continuous in s and quasiconcave in s_i. Then there exists a pure strategy Nash equlibrium.

Terminology:

$X \subset \Re^N$ is *compact* if it is closed (contains all of its limit points) and bounded.

$X \subset \Re^N$ is *convex* if, for all $x, y \in X$ and $\alpha \in [0,1]$, $\alpha x + (1-\alpha)y \in X$.

$f : \Re^N \to \Re$ is *quasiconcave* if, for all $t \in \Re$ and $x, y \in \Re^N$ such that $f(x) \geq t$ and $f(y) \geq t$, $f(\alpha x + (1-\alpha)y) \geq t$ for all $\alpha \in [0,1]$.

A correspondence $F : X \rightrightarrows Y$ is *upper hemicontinuous* if, for any sequences $x^t \to x$ and $y^t \to y$ with $y^t \in F(x^t)$ for all t, $y \in F(x)$.

A correspondence $F : X \rightrightarrows Y$ is *convex valued* if, for all $x \in X$, $F(x)$ is a convex set.

Kakutani Fixed Point Theorem: Suppose that S is compact and convex. Suppose also that $\gamma : S \rightrightarrows S$ is a convex valued, upper hemicontinuous correspondence. Then there exists $s^* \in S$ such that $s^* \in \gamma(s^*)$.

Proof of the Nash existence theorem: Define i's *best response correspondence*, $\gamma_i : S_{-i} \rightrightarrows S_i$, as follows:

$$\gamma_i(s_{-i}) = \{s_i \in S_i \mid s_i \text{ maximizes } g_i(\cdot, s_{-i})\}$$

Define $\gamma : S \rightrightarrows S$ as follows: $\gamma(s) = \gamma_1(s_{-1}) \times \gamma_2(s_{-2}) \times ... \times \gamma_N(s_{-N})$.

Note that a Nash equilibrium is a fixed point of γ; that is, $s \in S$ such that $s \in \gamma(s)$.

Step 1: γ_i is an upper hemicontinuous correspondence. This is an immediate consequence of Berge's maximum theorem. A direct proof follows.

Suppose the claim is false. Then there exists $s_{-i}^t \to s_{-i}^*$ and $s_i^t \in \gamma_i(s_{-i}^t)$ converging to s_i' $\notin \gamma_i(s_{-i}^*)$. It follows that there must exist $s_i^* \in S_i$ with

$$g_i(s_i', s_{-i}^*) < g_i(s_i^*, s_{-i}^*)$$

By continuity of g_i, $g_i(s_i^t, s_{-i}^t)$ converges to $g_i(s_i', s_{-i}^*)$, and $g_i(s_i^*, s_{-i}^t)$ converges to $g_i(s_i^*, s_{-i}^*)$. But then, for t sufficiently large, we must have $g_i(s_i^t, s_{-i}^t) < g_i(s_i^*, s_{-i}^t)$, which implies that $s_i^t \notin \gamma_i(s_{-i}^t)$, a contradiction.

Step 2: γ_i is convex valued.

Suppose that $s_i', s_i'' \in \gamma_i(s_{-i})$. Plainly, $g_i(s_i', s_{-i}) = g_i(s_i'', s_{-i}) \equiv g^*$. Since g_i is quasiconcave in s_i, $g_i(\alpha s_i' + (1-\alpha)s_i'', s_{-i}) \geq g^*$ for $\alpha \in [0,1]$. The previous expression must hold with equality, or it would contradict $s_i', s_i'' \in \gamma_i(s_{-i})$. But then $\alpha s_i' + (1-\alpha)s_i'' \in \gamma_i(s_{-i})$, as claimed.

Step 3: There exists $s^* \in S^*$ such that $s^* \in \gamma(s^*)$.

From steps 1 and 2, it follows that $\gamma : S \Rightarrow S$ is a convex valued, upper hemicontinuous correspondence. By assumption, S is compact and convex. The existence of a fixed point follows from the Kakutani fixed point theorem. Q.E.D.

2.6 Applications of pure strategy Nash equilibrium

2.6.1 Bertrand (price) competition with homogeneous products

Structure: $N \geq 2$ firms simultaneously select price. Customers purchase from the firm with the lowest announced price, dividing equally in the event of ties. Quantity purchased is given by a continuous, strictly decreasing function $Q(P)$.

Theorem: If $c_i(q_i) = cq_i$ (identical linear costs), then there exists an equilibrium in which all output is sold at the price $p = c$, and there does not exist an equilibrium in which output is sold at any other price.

Proof: Suppose $(p_1^*, p_2^*, ..., p_N^*)$ is a Nash equilibrium. Define

$$\underline{p} \equiv \min_i p_i^*$$

Assume, without loss of generality, that $p_1^* = \underline{p}$.

The theorem asserts that $\underline{p} = c$. Suppose that this is false. Then $\underline{p} > c$ ($\underline{p} < c$ is not possible since firm 1 would be losing money, which it could avoid by deviating to $p_1 = c$). There are two cases to consider.

(i) $p_2^* > \underline{p}$. Then firm 2 receives a payoff of 0. By deviating to any $p_2 \in (c, \underline{p})$, firm 2 would earn a positive profit. Consequently, the initial configuration could not have been an equilibrium.

(ii) $p_2^* = \underline{p}$. Then firm 2 receives a payoff $\pi_2 \leq (\underline{p} - c)Q(\underline{p})/2$. If instead firm 2 set $p_2' \in (c, \underline{p})$, it would earn $(p_2' - c)Q(p_2')$. Since Q is continuous, firm 2 can, by choosing p_2' close enough to (but smaller than) $\underline{p}$, obtain a payoff arbitrarily close to $(\underline{p} - c)Q(\underline{p})$, which is greater than the payoff received in the proposed equilibrium.

We have ruled out the possibility that $\underline{p} \neq c$. To conclude the proof, we must demonstrate the existence of an equilibrium with $\underline{p} = c$. Suppose that every firm sets $p_i = c$. Then every firm receives a payoff of zero. Deviating to a lower (below-cost) price clearly cannot yield positive profits. Moreover, a firm cannot make any sales by setting a higher (above-cost) price. Consequently, there is no profitable deviation. Q.E.D.

Henceforth, to keep things simple, order the firms so that $c_1 \leq c_2 \leq ... \leq c_N$, and assume that $Q(p)(p - c_1)$ is increasing in p over the range $[c_1, c_2 + \varepsilon]$ for some $\varepsilon > 0$. Relaxing this assumption is straightforward, but requires attention to some tedious details.

Theorem: Suppose that $c_i(q_i) = c_i q_i$. If $c_1 < c_2$, then no pure strategy Nash equilibrium exists.

Proof: Define $\underline{p}$ as before. One rules out equilibria with $\underline{p} > c_2$ through essentially the same argument as in the last theorem (either firm 1 or firm 2 must have an incentive to undercut the other). If $\underline{p} = c_2$ and $p_i = c_2$ for some $i > 1$, firm 1 could profit by setting a price slightly below $\underline{p}$. If $\underline{p} = c_2$ and $p_i > c_2$ for all $i > 1$, firm 1 could profit by raising its price slightly. If $\underline{p} < c_2$, then firm 1 must be making all sales (otherwise some other firm i could avoid losses by deviating to $p_i = c_i$). This means that firm 1 is quoting a price strictly below any other quoted price. But then firm 1 could increase its profits by quoting a slightly higher price. Q.E.D.

Alternative approach #1: Treat customers as players in the game. When two or more firms qouote the same price, customers are indifferent. As usual, we are free to resolve this indifference in a way that is consistent with equilibrium.

Theorem: Suppose that $c_i(q_i) = c_i q_i$. Order the firms so that $c_1 \leq c_2 \leq ... \leq c_N$. With alternative approach #1:

(i) there does not exist any equilibrium in which output is sold at a price less than c_1 or greater than c_2,

(ii) if $c_1 = c_2$, there exists an equilibrium in which all output is sold at the price $p = c_1$, and

(iii) if $c_1 < c_2$, for all $p \in [c_1, c_2]$ there exists a Nash equilibrium in which all output is sold at the price p. Moreover, in any equilibrium, firm 1 makes all of the sales.

Proof: Define $\underline{p}$ as before.

(i) The argument closely parallels that given in the proof of the first theorem in this section. Briefly: we can't have $\underline{p} < c_1$ because some firm i could then avoid losing money by setting $p_i = c_1$. If $\underline{p} > c_2$ and firm 2 makes no sales, firm 2 could earn positive profits by setting $p_2 \in (c_2, \underline{p})$. If $\underline{p} > c_2$ and firm 2 does make some sales, then firm 1 could increase profits by setting a price slightly below (but sufficiently close to) $\underline{p}$.

(ii) Suppose that $p_1 = p_2 = c_2$, $p_i > c_1$ for $i > 2$, and all customers purchase the good from firm 1. Every firm a payoff of zero. Deviations to prices below c_1 result in negative profits; deviations to prices above c_1 yield zero profits, and deviations to c_1 (by $i > 2$) yield non-positive profits receives (strictly negative when $c_i > c_1$ and i makes strictly positive sales).

(iii) Choose some $p \in [c_1, c_2]$. Suppose that $p_1 = p_2 = p$, $p_i > p$ for $i > 2$, and all customers purchase the good from firm 1. Firm 1 earns a non-negative profit, but cannot earn a greater profit by raising price (as it would sell nothing) or by lowering price (as it already serves the entire market, and as $Q(p)(p - c_1)$ is increasing in p over the range $[c_1, c_2]$). All other firms earn zero, but cannot sell positive quantity unless they set $p_i < c_2$, in which case they would incur losses.

To see that firm 1 must make all sales, note that, for $\underline{p} < c_2$, any other firm i making positive sales could avoid losses by setting $p_i = c_i$. For $\underline{p} = c_2$, if any other firm made positive sales, firm 1 could achieve a higher payoff by setting a price slightly below (but sufficiently close to) $\underline{p}$. Q.E.D

Question: For the case of $c_1 < c_2$, are all of these equlibria equally plausible? Equilibria with $p < c_2$ require some firm to set a price below cost. As we have seen, this is a weakly dominated strategy. We will return to this issue later.

Alternative approach #2: Assume that prices must be quoted in discrete units ("pennies"); that is, the feasible price set is $\{0, 1, 2, ...\}$. Assume also that c_i is a whole number for each i. Finally, return to the assumption that customers split evenly between the lowest-price firms in the event of a tie.

The case of $c_1 = c_2$: (i) For the same reasons as before, there is an equilibrium in which $p_1 = p_2 = c_1$, and all other firms set higher prices.

(ii) There is also an equilibrum in which $p_1 = p_2 = c_1 + 1$; any firm i with $c_i = c_1$ matches this price, while all others set higher prices.

(iii) One can show that all output must be sold at either a price of c_1 or at a price of $c_1 + 1$. (Check: firm 1 would undercut any higher price.)

(iv) Notice that, as the price grid becomes increasingly fine, we obtain convergence to the same answer we obtained with the original model (when price is selected from $\mathbb{R}_+$), and with alternative approach #1.

The case of $c_1 < c_2$. (i) For the same reasons as above, there is an equilibrium in which $p_1 \in \{c_1 + 1, c_1 + 2, ..., c_2\}$, $p_2 = p_1 + 1$, and all other firms charge higher prices.

(ii) One can show that there does not exist an equilibrium for which output is sold at any other price.

(iii) One can also show that firm 1 must sell all output.

(iv) Notice that, as the price grid becomes increasingly fine, we obtain convergence to the same answer we obtained with alternative approach #1. In particular, the prices charged by firms 1 and 2 become identical in the limit, but all customers purchase the good from firm 1.

Some observations concerning Bertrand competition (in a setting with one-shot, simultaneous choices and homogeneous products):

1. The effects of "concentration": $N = 1$ implies monopoly, while $N \geq 2$ implies perfect competition.

2. The properties of the demand curve (e.g. elasticity) are irrelevant.

3. The effects of production costs. Suppose $N = 2$, and that marginal costs are c_1 and c_2, with $c_1 < c_2$. Then changing c_1 has no effect on price.

Competitive Analysis:

Herfindahl-Hirshman Index: $\mathcal{H} = 10,000 \times \sum_{i=1}^{N} \alpha_i^2$, where α_i denotes the market share of firm i

Lerner Index: $\mathcal{L}_i = \frac{p_i - c_i'(q_i)}{p_i}$ (represents firm i's markup over marginal cost, measured as a percent of price), $\mathcal{L} = \sum_{i=1}^{N} \alpha_i \mathcal{L}_i$.

Thought experiment: Start with $c_1 = c_2$. Reduce c_1. What happens to the HHI, to the LI, and to welfare?

Initially: $HHI = 5,000$, $LI = 0$.

After the change, $HHI = 10,000$, $LI > 0$ (depending on which price is selected).

Concentration rises, and markups rise. That sounds bad, but in fact social welfare rises.

Remark: Price competition seems like the right model, but yields various implausible predictions. As we proceed, we will be relaxing three assumptions that are essential for producing the preceding results:

(i) products are homogeneous,

(ii) costs are linear, and

(iii) firms interact only once.

2.6.2 Bertrand (price) competition with differentiated products

1. Non-spatial models

A model with two symmetric firms:

$q_1 = Q(p_1, p_2)$

$q_2 = Q(p_2, p_1)$

Assumptions: Q is continuous, $Q_1 < 0$, $Q_2 > 0$, $|Q_1| \geq |Q_2|$

Constant marginal cost, c, identical for both firms

Solving the model:

Payoff for firm 1: $Q(p_1, p_2)(p_1 - c)$

First order condition: $Q_1(p_1, p_2)(p_1 - c) + Q(p_1, p_2) = 0$

Implication: $p_1 - c = -Q(p_1, p_2)/Q_1(p_1, p_2) > 0$ (provided firm 1 produces strictly positive quantity). In contrast to case with homogeneous products, firms earn strictly positive profits.

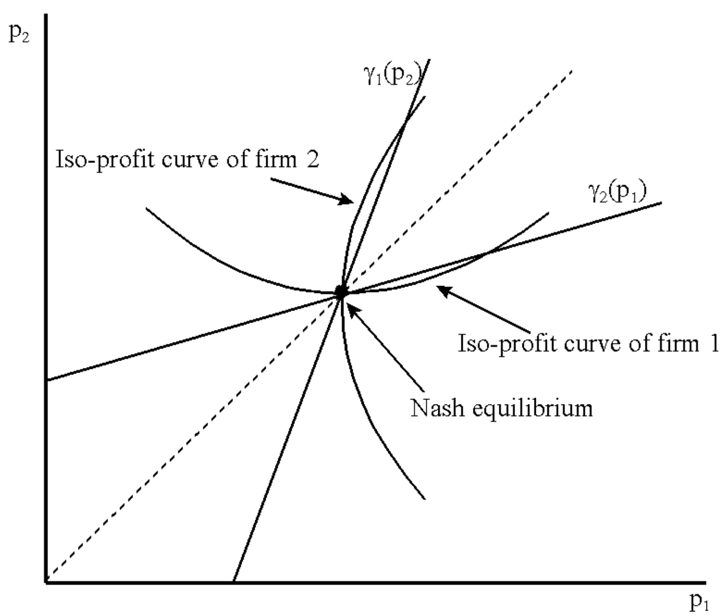
Implicit differentiation yields:

$$\frac{dp_1}{dp_2} = -\frac{Q_2 + (p_1 - c)Q_{12}}{2Q_1 + (p_1 - c)Q_{11}}$$

If the second-derivative terms are small (e.g. linear demand implies $Q_{11} = Q_{12} = 0$), we have

$$0 < \frac{dp_1}{dp_2} < \frac{1}{2}$$

Graphically:



Observation: best response functions are upward sloping – a case of *strategic complements.*

From the perspective of the firms, equilibrium is locally inefficient; one could increase profits for both firms by raising both prices.

2. Spatial models with horizontal differentiation

Structure of the model:

Two firms

Product characteristic is measured by some variable x

Product space is the line segment $[0, 1]$

Firms are endowed with product characteristics: a for firm 1, $1-b$ for firm 2, with $a < 1-b$

Production costs: c per unit

Consumers are indexed by $\theta \in [0, 1]$

Each consumer purchases either one unit, or nothing

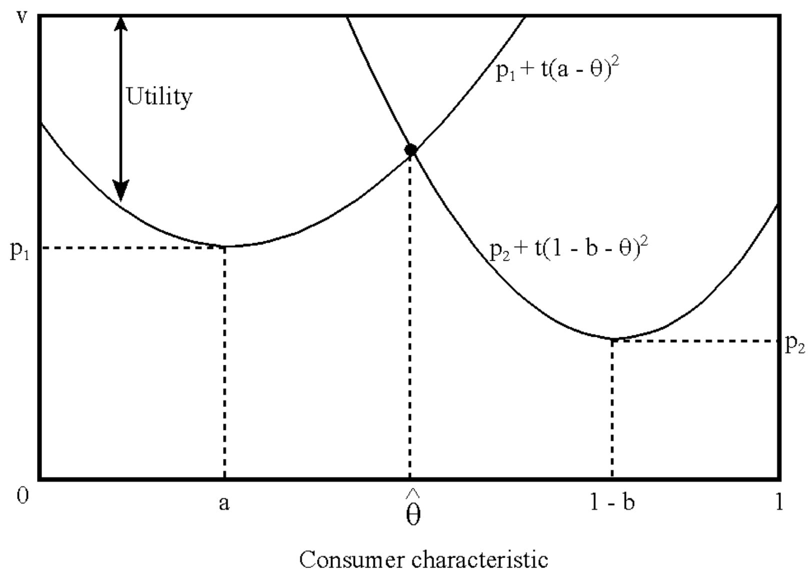
Payoff for a type θ consumer: 0 if no purchase, $v - p - t(x - \theta)^2$ if purchases a type x good at price p.

Population distribution of θ is uniform over $[0, 1]$

Known as a "Hotelling spatial location" model

Solving the model:

Graphical depiction of consumer's problem



For simplicity, we will assume that v is very large so that all customers buy something, and so that $\widehat{\theta}$ is well-defined.

The firms divide up the market as follows: $\theta < \widehat{\theta}$ purchase from firm 1, $\theta > \widehat{\theta}$ purchase from firm 2. So,

$$
\begin{aligned}
Q_1(p_1, p_2) &= \widehat{\theta}(p_1, p_2, a, b, t) \\
Q_2(p_1, p_2) &= 1 - \widehat{\theta}(p_1, p_2, a, b, t)
\end{aligned}
$$

where $\widehat{\theta}(p_1, p_2, a, b, t)$ is given by the solution of

$$p_1 + t(a - \widehat{\theta})^2 = p_2 + t(1 - b - \widehat{\theta})^2$$

Solving, we get:

$$\widehat{\theta} = \frac{p_2 - p_1}{2t(1 - a - b)} + \frac{1 + a - b}{2}$$

It is easy to check that these demand functions satisfy the assumptions used for the non-spatial model, above; moreover, the second partials are zero. Consequently, we have a standard price setting problem with strategic complementarities (upward sloping best response functions).

Finding the equilibrium:

Firm 1's profits: $(p_1 - c)\widehat{\theta}(p_1, p_2, a, b, t)$

First order condition for firm 1:

$$p_1 = c + \frac{1}{2}(p_2 - c) + (1 + a - b)(1 - a - b)\frac{t}{2}$$

Notice: (i) strategic complements, (ii) $p_1 > c$, (iii) price rises with c, (iv) price rises with t (markets are more "insulated")

Combining this with the corresponding equation for firm 2 and solving simultaneously yields:

$$\begin{aligned} p_1^* &= c + t(1 - a - b)\left(1 + \frac{a - b}{3}\right) \\ p_2^* &= c + t(1 - a - b)\left(1 + \frac{b - a}{3}\right) \end{aligned}$$

Notice: (i) equilibrium prices are strictly greater than costs, (ii) equilibrium prices rise with costs and with transportation costs, (iii) for $a = b$, equilibrium prices decline with a (they fall as the firms move closer together).

Exercise: Is this model dominance solvable?

3. Spatial models with vertical differentiation

Structure of the model:

Two firms, L and H, with linear production costs

Firm i produces a good that yields quality v_i, with $v_H > v_L$

Each consumer purchases either one unit of one of these goods, or nothing.

Consumers are characterized by a preference parameter, θ, which indicates the value attached to quality. If a consumer of type θ purchases a good of type i, her utility is given by

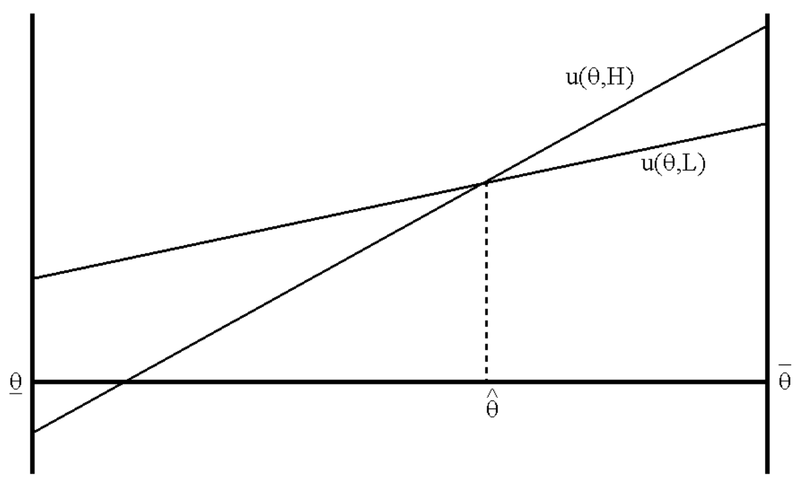
$$u(\theta, i) = \theta v_i - p_i$$

If the consumer purchases nothing, her utility is zero.

θ is distributed uniformly on the interval $[\underline{\theta}, \overline{\theta}]$

Solving the model:

Graphical depiction of the consumer's problem:



The firms divide up the market as follows: $\theta < \widehat{\theta}$ purchase from firm L, $\theta > \widehat{\theta}$ purchase from firm H.

$\widehat{\theta}$ satisfies:

$$\widehat{\theta} v_H - p_H = \widehat{\theta} v_L - p_L,$$

so

$$\widehat{\theta} = \frac{p_H - p_L}{v_H - v_L}$$

Thus, assuming a total population normalized to unity, demands are given as follows:

$$\begin{aligned} Q_H(p_L, p_H) &= \left[\overline{\theta} - \frac{p_H - p_L}{v_H - v_L}\right]\left[\overline{\theta} - \underline{\theta}\right]^{-1} \\ Q_L(p_L, p_H) &= \left[\frac{p_H - p_L}{v_H - v_L} - \underline{\theta}\right]\left[\overline{\theta} - \underline{\theta}\right]^{-1} \end{aligned}$$

Notice that this is a completely standard case of Bertrand price competition with differentiated products and linear demands. It will therefore exhibit strategic complementarities (upward sloping best response functions). Solving for the equilibrium is completely standard, and analogous to the case of horizontal differentiation.

Theorem: Assume that $\overline{\theta} < 2\underline{\theta}$. There exists an equilibium in which firm H makes all of the sales and earns strictly positive profits.

Remark: This result contrasts sharply with the case of horizontal differentiation.

Proof: We claim that the following strategy profile is an equilibrium:

$$p_H^* = c + \underline{\theta}(v_H - v_L) > c$$

$$p_L^* = c$$

Note that a type θ consumer receives a net payoff of $\pi_H = v_H(\theta - \underline{\theta}) + v_L\underline{\theta} - c$ from purchasing firm H's product, and $\pi_L = v_L\theta - c$ from purchasing firm L's product. Moreover, $\pi_H - \pi_L = (\theta - \underline{\theta})(v_H - v_L)$. This expression equals zero for $\theta = \underline{\theta}$, and is strictly greater than zero for all $\theta > \underline{\theta}$. Thus, essentially all consumers (all but a set of measure zero) must purchase from firm H, and we can resolve the indifference of type $\underline{\theta}$ consumers in favor of firm H.

Now we check to see whether this is an equilibrium. Firm L cannot do better by lowering price (since this would require $p_L < c$), or by raising price (since it would sell nothing). Firm H cannot benefit from reducing price (since quantity would be unaffected). We need to check whether firm H benefits from raising price.

For any $p_H \geq p_H^*$, firm H will make sales to all customers with

$$v_L\theta - c \leq v_H\theta - p_H,$$

or

$$\theta \geq \theta^* \equiv \frac{p_H - c}{v_H - v_L}$$

The associated level of profit for firm 1 is

$$\begin{aligned}
\Pi_H(p_H) &= \left(\frac{\overline{\theta} - \theta^*}{\overline{\theta} - \underline{\theta}}\right)(p_H - c) \\
&= \left(\frac{1}{\overline{\theta} - \underline{\theta}}\right)\left(\overline{\theta}(p_H - c) - \frac{(p_H - c)^2}{v_H - v_L}\right)
\end{aligned}$$

Differentiating with respect to p_H, we obtain:

$$\Pi_H'(p_H) = \left(\frac{1}{\overline{\theta} - \underline{\theta}}\right)\left(\overline{\theta} - \frac{2\,(p_H - c)}{v_H - v_L}\right)$$

Evaluating this expression at $p_H = p_H^*$, we obtain:

$$\Pi_H'(p_H^*) = 1 - \left(\frac{\underline{\theta}}{\overline{\theta} - \underline{\theta}}\right)$$

Note that, for $\overline{\theta} - \underline{\theta} < \underline{\theta}$, this expression is strictly negative. Moreover,

$$\Pi_H''(p_H) = -\left(\frac{1}{\overline{\theta} - \underline{\theta}}\right)\left(\frac{2}{v_H - v_L}\right) < 0$$

So $p_H = p_H^*$ maximizes firm 1's profits. Q.E.D.

2.6.3 Cournot (quantity) competition

Basic Cournot model:

$N \geq 2$ firms

Firms simultaneous produce their quantity, q_i for firm i

Production costs are $c_i(q_i)$

Notation: $q \equiv (q_1, ..., q_N)$, $Q \equiv \sum_{i=1}^{N} q_i$, $Q_{-i} \equiv \sum_{j \neq i} q_i$

The quantity Q is taken to market, where it is auctioned off at the market clearing price, $P(Q)$

Payoffs: $g_i(q) = P\left(q_i + \sum_{j \neq i} q_j\right) q_i - c_i(q_i)$

Existence:

$S_i = \Re_+$ Convex, closed, but not bounded. This does not pose a problem in practice. Analytic procedure: impose an artificial bound at some extremely large quantity, find an equilibrium, and show that it remains an equilibrium when one removes the bound.

To apply our earlier theorem, need $g_i(q)$ to be continuous and quasiconcave in q_i. Since P and c_i are usually assumed to be continuous, there is usually no problem with continuity. However, we don't necessarily have quasiconcavity.

Sufficient conditions for quasiconcavity: (i) c_i is convex, and (ii) P is concave. Condition (i) rules out increasing returns to scale, which is not surprising, since it creates difficulties with existence in other settings as well. Condition (ii) is not a conventional property of demand functions; it is satisfied for linear functions, but not for isoelastic ones.

For what follows, we will assume that a pure strategy Nash equilibrum exists.

Properties of Cournot best responses:

Firm i solves:

$$\max_{q_i} P(q_i + Q_{-i})q_i - c_i(q_i)$$

First order condition:

$$P'(Q)q_i + P(Q) - c_i'(q_i) = 0$$

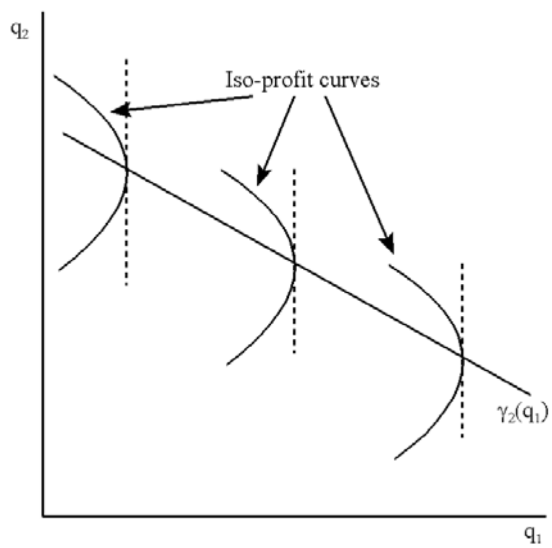
Implicit differentiation yields:

$$\frac{dq_i}{dQ_{-i}} = -\frac{P''(Q)q_i + P'(Q)}{P''(Q)q_i + 2P'(Q) - c_i''(q_i)}$$

Impose sufficient conditions for existence: $c'' \geq 0$, $P'' \leq 0$. Then

$$-1 < \frac{dq_i}{dQ_{-i}} < 0$$

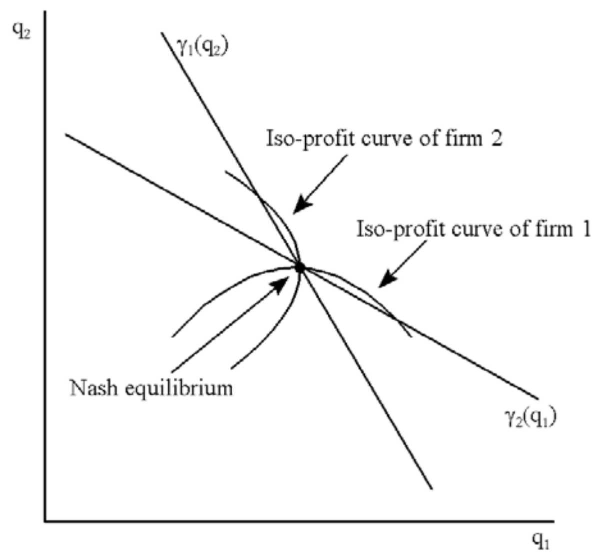
Graphical depiction of best response function for firm 2:



Note that the best response function slopes downwards. This is a case of *strategic substitutes*.

Properties of Nash equilibrium:

Graphical depiction:



Parametric example: $c_i(q_i) = cq_i$, $p = a - bQ$. We have previously solved for the best response function:

$$\gamma_i(Q_{-i}) = \frac{a-c}{2b} - \frac{Q_{-i}}{2}$$

For a symmetric equilibrium, this implies

$$q^* = \frac{a-c}{2b} - \frac{(N-1)q^*}{2},$$

or

$$q^* = \frac{a-c}{b(N+1)}$$

$$p^* = c + \frac{1}{N+1}(a-c)$$

Some remarks:

1. From perspective of firms, equilibrium is locally inefficient. Total quantity exceeds the monopoly quantity.

2. Comparative statics differ from Bertrand because of strategic substitutability (vs. strategic complementarity). Example: assume constant unit costs, c_1 and c_2, and raise c_2. Cournot: best response for firm 2 shifts down. Result: firm 2's quantity falls (2 becomes less aggressive), while firm 1's quantity rises (1 becomes more aggressive). Bertrand: best response for firm 2 shifts up. Result: firm 2's price rises (2 becomes less aggressive), and firm 1's price rises (1 becomes less aggressive as well).

3. Price always exceeds marginal cost. To see this, rewrite the first order condition:

$$P(Q) = c_i'(q_i) - P'(Q)q_i > c_i'(q_i)$$

4. Achieve perfect competition in the limit as the number of firms grows. To illustrate, assume symmetry. The first-order condition becomes:

$$P(Q^*) = c'\left(\frac{Q^*}{N}\right) - \frac{1}{N}P'(Q^*)Q^*$$

   As long as we don't change demand as we add firms, Q^* and $P'(Q^*)$ should remain bounded, which means that price converges to marginal cost. For Cournot, adding firms therefore moves the outcome smoothly from the monopoly case to the competitive case. One can also see this directly in the parametric case considered above.

5. In contrast to Bertrand, the more efficient firm need not produce all of the output. The first order conditions imply that, for any firms i and j,

$$P'(Q)(q_i - q_j) = c_i'(q_i) - c_j'(q_j)$$

   Thus, if firms exhibit non-increasing returns to scale, and if $c_i'(x) < c_j'(x)$ for all x, then (i) $q_i^* > q_j^*$ (the more efficient firm produces more), (ii) $c_i'(q_i^*) < c_j'(q_j^*)$ (the equilibrium violates productive efficiency, in the sense that the lower cost firm produces at lower cost on the margin), and (iii) $p - c_i'(q_i^*) > p - c_j'(q_j^*)$ (the more efficient firm has a larger markup).

Competitive Analysis:

Using the first order condition and the definition of demand elasticity, ε, we have:

$$\mathcal{L}_i = \frac{p - c_i'(q_i)}{p} = \frac{-P'(Q)q_i}{p} = \frac{\alpha_i}{\varepsilon}$$

Using this formula and the definition of the industry Lerner index, we have:

$$\mathcal{L} = \sum_{i=1}^{N} \alpha_i \mathcal{L}_i = \frac{\sum_{i=1}^{N} \alpha_i^2}{\varepsilon} = \frac{\mathcal{H}}{\varepsilon}$$

Notice: (i) The equilibrium markup is positively related to concentration, (ii) The equilibrium markup is negatively related to demand elasticity.

Example: Three firms, constant returns to scale, costs c_i for firm i, constant elasticity of demand. Initial configuration: $c_1 = c_2 = c_3$, $\mathcal{H} = \frac{1}{3}$.

Exercise #1: Remove one firm. Then $\mathcal{H}$ rises, $\mathcal{L}$ rises, p rises, and welfare falls. This is a "bad" increase in concentration.

Exercise #2: Reduce c_1. We know that firm 1 ends up with a market share $\lambda > \frac{1}{3}$, while the other firms split the rest of the market. Thus,

$$\mathcal{H} = \lambda^2 + 2\left(\frac{1-\lambda}{2}\right)^2 = \frac{3\lambda^2 - 2\lambda + 1}{2}$$

This reaches a global minimum at $\lambda = \frac{1}{3}$. Therefore, when firm 1's costs fall, $\mathcal{H}$ rises, $\mathcal{L}$ rises, p falls, and welfare rises. This is a "good" increase in concentration.

Exercise #3: Increase c_1. Same reasoning, but different conclusion: $\mathcal{H}$ rises, $\mathcal{L}$ rises, p rises, and welfare falls. So the relationship between concentration and either price or welfare isn't stable, even when it is driven only by cost changes.

Lesson: a welfare evaluation of a change in concentration depends on the factors that caused it.

2.6.4 Public Goods

Definition: A public good has two characteristics: (i) it displays *non-rivalry* in consumption, and (ii) it is *non-excludable*.

A simple model:

N consumers

Two goods: a public good, g, and a private good, x

Each consumer is endowed with some amount of the private good, z_i; let $Z \equiv \sum_{i=1}^{N} z_i$

The public good is produced from the private good through a convex technology, for which the cost of producing g in terms of x is given by the convex function $c(g)$

A feasible allocation consists of a vector $(g, x_1, x_2, ..., x_N)$ such that $Z = \sum_{i=1}^{N} x_i + c(g)$

Utility for consumer i is given by a concave function $u^i(g, x_i)$

The social optimum:

Imagine that the planner has a Samuelson-Bergson social welfare function of the form

$$W(u^1, ..., u^N) = \sum_{i=1}^{N} \alpha_i u^i$$

Maximize this function over allocations subject to the feasibility constraint by setting up the Lagrangian and differentiating (assuming an interior solution):

$$\alpha_i u_x^i(g, x_i) = \lambda$$

$$\sum_{i=1}^{N} \alpha_i u_g^i(g, x_i) = \lambda c'(g)$$

Using the first expression to solve for $\alpha_i = \lambda / u_x^i$ and substituting this into the second expression yields:

$$\sum_{i=1}^{N} \frac{u_g^i(g, x_i)}{u_x^i(g, x_i)} = c'(g),$$

which we can rewrite in a more familiar form (the Samuelson condition):

$$\sum_{i=1}^{N} MRS^i_{g,x} = MRT_{g,x}$$

A non-cooperative equilibrium:

Game structure: each consumer simultaneously selects a voluntary contribution, t_i, to the public good. Consumer i consumes $z_i - t_i$ of the private good. The level of the public good is $g = c^{-1}\left(\sum_{i=1}^{N} t_i\right)$.

Let $(t_1^*, ..., t_N^*)$ denote some Nash equilibrium. Define $g^* = c^{-1}\left(\sum_{i=1}^{N} t_i^*\right)$, $x_i^* = z_i - t_i^*$, and $T_{-i}^* = \sum_{j \neq i} t_j^*$. Then t_i^* is the solution to:

$$\max_t \; u^i\left(c^{-1}(T_{-i}^* + t), z_i - t\right)$$

Assuming the solution is interior, this implies

$$\frac{u_g^i(g^*, x_i^*)}{u_x^i(g^*, x_i^*)} = c'(g^*)$$

which we can rewrite in a more familiar form:

$$MRS^i_{g,x} = MRT_{g,x}$$

Notice that this diverges from the Samuelson conditions for optimal provision.

Public provision with private contributions:

Structure of government funding for the public good: a tax m_i is levied on each consumer i, and all revenues, $M = \sum_{i=1}^{N} m_i$, are contributed to the public good. Private contributions are allowed. The private contribution game is as described above. When consumers make their decisions concerning contributions, they regard taxes and the government contribution as predetermined.

Claim: provided that the equilibrium is interior, a change in government provision does not change either aggregate provision of the public good or private consumption by any individual. In other words, public contributions to the public good fully crowd out private contributions.

Demonstration: Consumer i chooses t_i to maximize

$$u^i\left(c^{-1}(T_{-i}+t_i+M), z_i-m_i-t_i\right)$$

We can think of each individual as choosing x_i, with the implied transfer $t_i = z_i - m_i - x_i$. Making this change of variables, we see that each consumer chooses x_i to maximize

$$\begin{aligned} & u^i\left(c^{-1}\left(\sum_{i\neq j}(z_j-m_j-x_j)+(z_i-m_i-x_i)+\sum_{i=1}^{N}m_i\right), x_i\right) \\ = \; & u^i\left(c^{-1}\left(Z-\sum_{i\neq j}x_j-x_i\right), x_i\right) \end{aligned}$$

Thus, an interior equilibrium, $(x_1^*, ..., x_N^*)$, satisfies the following first order condition for each i:

$$\begin{aligned} & u_g^i\left(c^{-1}\left(Z-\sum_{i=1}^{N}x_i^*\right), x_i^*\right) \\ = \; & c'\left(c^{-1}\left(Z-\sum_{i=1}^{N}x_i^*\right)\right)u_x^i\left(c^{-1}\left(Z-\sum_{i=1}^{N}x_i^*\right), x_i^*\right) \end{aligned}$$

This determines the equilibrium as a solution to a system of N equations in N unknowns. Notice that the tax variables and government contributions have completely disappeared from this problem, and therefore cannot affect the solution. Implication: provided that $m_i \leq z_i - x_i^*$ (that is, x_i^* is a feasible choice given m_i), the equilibrium does not depend upon government taxes and public provision of the public good.

Redistribution with private contributions:

Observation: the preceding argument remains true even when $\sum_{i=1}^{N} m_i = 0$ and $m_i < 0$ for some i. This is a case of pure redistribution. Consequently, as long as everyone is making voluntary contributions to the public good, redistributional policies affect neither the level of the public good nor the private consumption of any individual.

Qualification: changes in taxes and transfers may have real affects on the equilibrium allocation to the extent they alter the set of consumers making strictly positive transfers to the public good.

2.6.5 Voting

Same model as before:

I players (voters), $I \geq 3$.

J policy options, $j = 1, ..., J$.

Players simultaneously cast votes: they name j. Abstentions are not permitted. $S_i = \{1, ..., J\}$

The policy with the most votes is adopted (in the event of ties, a policy is selected with equal probabilities from the set of policies receiving the most votes)

Payoffs for i are v_i^j when policy j is adopted. We assume that $v_i^j \neq v_i^k$ for all i, j, k.

Pure strategy Nash equilibria:

Claim: For any policy $j \in \{1, ..., J\}$, there is a pure strategy Nash equilibrium in which j is adopted.

Demonstration: Pick any j. Let $s_i = j$ for all i. No player can change the outcome by deviating.

Note: This result remains valid even when all players agree that policy j is the worst possible outcome. It holds even when $J = 2$ (the case for which weak dominance selects the majority-preferred outcome).

2.7 Mixed strategy Nash equilibrium

2.7.1 Motivation

(i) Pure strategy Nash equilibria do not always exist

(ii) In some situations, randomized decision-making seems like a natural solution.

Example: MP-B

		Player 2	
		h	t
Player 1	H	2,-2	-2,2
	T	-2,2	2,-2

Proposed solution: Player 1 and player 2 both flip their coins. Expected payoffs for player 1:

$$\text{For } H : \frac{1}{2}(2) + \frac{1}{2}(-2) = 0$$

$$\text{For } T : \frac{1}{2}(-2) + \frac{1}{2}(2) = 0$$

Consequently, provided that player 2 flips his coin, player 1 can't improve upon flipping her coin. Conversely, provided that player 1 flips her coin, player 2 can't improve upon flipping his coin.

2.7.2 Definitions

There are two approaches to modeling randomized choices.

Approach #1: Randomizations over (normal form) pure strategies ("instructions"). These randomizations are known as *mixed strategies*.

Consider a finite normal form game, $(\{S_i\}_{i\in 1,...,I}, g)$.

Let $K_i \equiv |S_i| < +\infty$.

Define a new game, $(\{\Delta_i\}_{i\in 1,...,I}, \pi)$, as follows:

Let Δ_i be the K_i-dimensional simplex. $\delta_i \in \Delta_i$ is interpreted as a probability vector, where δ_{ik} gives the probability that player i will play her k-th pure strategy, s_i^k.

Example: MP-A. Player 2's strategy set is $\{s_1, s_2, s_3, s_4\} = \{hh, ht, th, tt\}$. A mixed strategy δ_i is a four-dimensional vector $(\delta_{i1}, \delta_{i2}, \delta_{i3}, \delta_{i4})$, where $\sum_{k=1}^{4} \delta_{ik} = 1$, and where $\delta_{ik} \in [0, 1]$ denotes the probability of playing s_k. Note that there are three degrees of freedom for choosing a mixed strategy in this game.

$\Delta \equiv \Delta_1 \times ... \times \Delta_I$

$\delta \in \Delta$ consists of a probability vector for each player, $(\delta_1, ..., \delta_I)$

$\pi : \Delta \to \Re^I$ assigns a payoff to each player for each element of Δ, as follows:

$$\pi_i(\delta) = E_s[g_i(s) \mid \delta]$$

That is, $\pi_i(\delta)$ represents the expected payoff to player i when choices are governed by the probability vectors $(\delta_1, ..., \delta_I)$.

Definition: A *mixed strategy Nash equilibrium* of the game $(\{S_i\}_{i\in 1,\ldots,I}, g)$ is a pure strategy equilibrium of the game $(\{\Delta_i\}_{i\in 1,\ldots,I}, \pi)$.

Approach #2: Randomizations over actions at each information set in the extensive form. These randomizations are known as *behavior strategies*.

Example: MP-A. Player 1 chooses from the set $\{H, T\}$ at only one information set. Therefore, mixed strategies and behavior strategies are equivalent. Player 2 chooses from the set $\{h, t\}$ at two information sets, which we can identify with player 1's preceding action, H or T. A behavior strategy for player 2 consists of a randomization over the set $\{h, t\}$ for the information set associated with H, call it $\delta_{2H} = (\delta_{2Hh}, \delta_{2Ht})$ (where δ_{2Hk} denotes the probability of playing $k \in \{h, t\}$ having observed that player 1 chose H), and another randomization over the set $\{h, t\}$ for the information set associated with T, call it $\delta_{2T} = (\delta_{2Th}, \delta_{2Tt})$ (where δ_{2Tk} denotes the probability of playing $k \in \{h, t\}$ having observed that player 1 chose T). Notice that there are only two degrees of freedom in choosing a behavior strategy for this game (since $\delta_{2Kh} + \delta_{2Kt} = 1$ for $K \in \{H, T\}$). In contrast, a mixed strategy for player 2 consists of a randomization $\delta_2 = (\delta_{2hh}, \delta_{2ht}, \delta_{2th}, \delta_{2tt})$ over the pure strategies hh, ht, th, and tt. Notice that there are three degrees of freedom in choosing a mixed strategy for this game (since $\delta_{2hh} + \delta_{2ht} + \delta_{2th} + \delta_{2tt} = 1$).

Kuhn's Theorem: Consider any game of perfect recall. For any behavior strategy of player i, there is a mixed strategy for i that yields the same distribution over outcomes for any strategies, mixed or behavioral, chosen by other players. Moreover, for any mixed strategy of player i, there is a behavior strategy for i that yields the same distribution over outcomes for any strategies, mixed or behavioral, chosen by other players. In this sense, mixed strategies and behavior strategies are equivalent.

Example: MP-A. Suppose that player 1 chooses with probabilities $(\delta_{1H}, \delta_{1T})$ (this can be viewed as either a mixed or behavior strategy). If player 2 follows a mixed strategy

$(\delta_{2hh}, \delta_{2ht}, \delta_{2th}, \delta_{2tt})$, the probability distribution over outcomes is as follows:

Outcome	Probability
H, h	$\delta_{1H}(\delta_{2hh} + \delta_{2ht})$
H, t	$\delta_{1H}(\delta_{2th} + \delta_{2tt})$
T, h	$\delta_{1T}(\delta_{2hh} + \delta_{2th})$
T, t	$\delta_{1T}(\delta_{2ht} + \delta_{2tt})$

If instead player 2 follows a behavior strategy $(\delta_{2Hh}, \delta_{2Ht}, \delta_{2Th}, \delta_{2Tt})$, the probability distribution over outcomes is as follows:

Outcome	Probability
H, h	$\delta_{1H}\delta_{2Hh}$
H, t	$\delta_{1H}\delta_{2Ht}$
T, h	$\delta_{1T}\delta_{2Th}$
T, t	$\delta_{1T}\delta_{2Tt}$

Beginning with any mixed strategy, one can plainly select a feasible behavior strategy that achieves the same probability distribution over outcomes (use $\delta_{2Hh} = \delta_{2hh} + \delta_{2ht}$ and $\delta_{2Th} = \delta_{2hh} + \delta_{2th}$). Beginning with any behavior strategy one can also select a feasible mixed strategy that achieves the same probability distribution over outcomes. To see this, begin by selecting some arbitrary value $\delta^* \in [0, 1]$ for δ_{2hh}. To equate all of the terms in the preceding tables, we must then have $\delta_{2ht} = \delta_{2Hh} - \delta^*$, $\delta_{2th} = \delta_{2Th} - \delta^*$, and $\delta_{2tt} = 1 - \delta_{2Th} - \delta_{2Hh} + \delta^*$. This is a feasible mixed strategy provided that all three of these implied probabilities are non-negative. This requires $\delta^* \leq \min\{\delta_{2Hh}, \delta_{2Th}\}$ and $\delta^* \geq \delta_{2Th} + \delta_{2Hh} - 1$. But since $\delta_{2Th} + \delta_{2Hh} - 1 \leq \min\{\delta_{2Hh}, \delta_{2Th}\}$, it is always possible to select some $\delta^* \in [0, 1]$ that satisfies these conditions.

2.7.3 Existence

Theorem: Every finite game has a mixed strategy equlibrium.

Proof: Δ_i is compact and convex. π_i is continuous and quasiconcave in δ_i, because it is linear in probabilities:

$$E_s\left[g_i(s) \mid \delta\right] = \sum_{k=1}^{K_i} \delta_{ik} E_{s_{-i}}\left[g_i\left(s_i^k, s_{-i}\right) \mid \delta_{-i}\right]$$

Consequently, we can imply the existence theorem for pure strategy Nash equilibria. Q.E.D.

2.8 Applications of mixed strategy Nash equilibria

How do we solve for mixed strategy equilibria?

Helpful observation: In any mixed strategy equlibrium, a player must be indifferent between all pure strategies to which he attaches strictly positive probability. To see why, inspect the previous formula.

Illustration with MP-B: If player 1 places positive probability on both choices, then player 2 randomizes to make player 1 indifferent:

$$2\delta_{2h} - 2\delta_{2t} = -2\delta_{2h} + 2\delta_{2t}$$

So

$$2\delta_{2h} - 2(1 - \delta_{2h}) = -2\delta_{2h} + 2(1 - \delta_{2h})$$

Solving, we obtain $\delta_{2h} = \frac{1}{2} = \delta_{2t}$. The game is symmetric, so a similar argument implies $\delta_{1H} = \frac{1}{2} = \delta_{1T}$.

2.8.1 An auction problem

Rules of the auction:

I bidders for an object

The object has a common value v to all bidders

All bidders simultaneously submit sealed bids, b_i

The highest bidder wins (in the event of a tie, the winner is selected at random from the highest bidders)

Bidders pay their bids to the auctioneer regardless of whether they win.

Payoffs: $v - b_i$ if player i wins the object, and $-b_i$ if i does not win

Equilibrium analysis:

There do not exist any pure strategy Nash equilibria (easy to check)

Consider symmetric mixed strategy equilibria. Let $F(x)$ denote the CDF for the bid.

Implication of the mixed strategy indifferent condition:

$$F(x)^{I-1}(v-x) - (1 - F(x)^{I-1})x = C$$

where C, the expected payoff, is not yet known. This implies:

$$F(x) = \left[\frac{C+x}{v}\right]^{\frac{1}{I-1}}$$

Using $F(v) = 1$, we infer that $C = 0$ (implying that bidders compete away all of the surplus and end up with an expected payoff of zero). Thus,

$$F(x) = \left[\frac{x}{v}\right]^{\frac{1}{I-1}}$$

From this, it follows that:

$$E(x) = \int_0^v dF(x)xdx = \frac{v}{I}$$

Consequently, for total expected revenue, we have $IE(x) = v$.

Comment: with further work, one can establish uniqueness.

2.8.2 Bertrand competition with capacity constraints

Structure of the model:

2 firms, each with fixed capacity K

Cost structure:

$$c_i(q_i) = \begin{cases} 0 \text{ if } q_i \leq K \\ \infty \text{ if } q_i > K \end{cases}$$

Single buyer with reservation value v.

Demand structure:

$$Q(p) = \begin{cases} Q \text{ if } p \leq v \\ 0 \text{ if } p > v \end{cases}$$

The firms compete by simultaneously announcing prices. The buyer makes purchases (if any) from the firm quoting the lowest price, and splits purchases equally between the two firms in the event of a tie. If the low-price firm has insufficient capacity to satisfy demand, the buyer purchases the residual from the high-price firm.

Assume: (i) $2K > Q$ (otherwise $p = v$ is a pure strategy Nash equilibrium), and (ii) $K < Q$ (otherwise $p = 0$ is a Nash equilibrium).

Pure strategy Nash equilibria:

Claim: there does not exist a pure strategy Nash equilibrium.

Payoffs:

$$g_i(p_i, p_j) = \begin{cases} 0 \text{ if } p_i > v \\ (Q-K)p_i \text{ if } v \geq p_i > p_j \\ Qp_i/2 \text{ if } v \geq p_i = p_j \\ Kp_i \text{ if } v \geq p_i < p_j \end{cases}$$

We argue as follows:

(i) We can rule out any strategy profile with $p_1 \neq p_2$ in the standard way, as the firm with the lower price could increase profits by slightly raising price.

(ii) We can rule out any strategy profile with $p_1 = p_2 > 0$ in the standard way, as either firm could increase profits by slightly undercutting the other.

(iii) We can rule out any strategy profile with $p_1 = p_2 = 0$, on the grounds that either firm could earn positive (and therefore higher) profits by setting $p = v$, in which case its payoff would be $(Q - K)v > 0$.

Some properties of mixed strategy Nash equilibria:

We will look for symmetric equilibria, wherein each firm chooses some probability distribution on $[0, v]$. We will use F to denote the CDF for this distribution.

We claim that any symmetric equilibrium mixed strategy (F) must have the following two properties:

Property #1: F is an atomless distribution ($F(x)$ is continuous)

The payoff from playing any price p (given that the other firm's choices are dictated by F) is given by

$$\pi_i(p, F) = \left(\lim_{q \uparrow p} F(q)\right)(Q - K)p + \left(F(p) - \lim_{q \uparrow p} F(q)\right)\frac{Qp}{2} + (1 - F(p))\, Kp$$

Suppose there is a probability atom at $p' > 0$. Compare the payoff to a firm from playing p' to the payoff from playing $p' - \varepsilon$ for some small ε. This creates a shift in probability weight from the second term to the third term in the preceding expression. The magnitude of this shift remains bounded away from zero even when ε is arbitrarily small. Since the payoff associated with the third term is higher, expected payoff must rise for sufficiently small ε, even though price is lower (the impact of the latter effect becomes small on the same order as ε). There is also a shift in probability weight from the first term to the second, but this reinforces the increase in payoff.

Next suppose there is a probability atom at $p' = 0$. Compare the payoff to a firm from playing p' to the payoff from any positive $p \leq v$; the former profit is zero, while the latter is strictly positive.

One implication of property #1: The probability of a tie is zero. This allows us to simplify the expression for payoffs:

$$\pi_i(p, F) = F(p)(Q - K)p + (1 - F(p))Kp$$

Property #2: $\text{supp}(F) = [a, v]$, where $v > a > 0$.

Significance: there are no "flat" portions of F, except for an initial segment starting at zero.

Suppose on the contrary that there is some interval $[b, d]$ such that $F(p) > 0$ and $F'(p) = 0$ for $p \in [b, d]$. Then over this range,

$$\begin{aligned} \frac{\partial \pi_i(p, F)}{\partial p} &= K + F(p)(Q - 2K) \\ &\geq K + (Q - 2K) \\ &= Q - K > 0 \end{aligned}$$

Consequently, by playing $p = d$, the firm would achieve strictly higher profits than by playing a price in $\text{supp}(F)$ sufficiently close to b. We know that $a > 0$ because of the indifference condition (since $\pi_i(0, F) = 0 < \pi_i(v, F)$).

Solving for the mixed strategy Nash equilibrium:

From the mixed strategy indifference condition, we know that $\forall p \in [a, v]$,

$$F(p)(Q - K)p + (1 - F(p))Kp = C$$

(where C is the expected payoff, still to be determined), or

$$F(p) = \frac{C - Kp}{(Q - 2K)p}$$

Next we solve for C by setting $F(v) = 1$:

$$F(v) = \frac{C - Kv}{(Q - 2K)v} = 1$$

which implies $C = (Q - K)v$. So

$$F(p) = \left(\frac{K}{2K - Q}\right) - \frac{(Q - K)v}{(2K - Q)p}$$

Finally, we find a by using $F(a) = 0$:

$$a = \left(\frac{Q}{K} - 1\right) v \in (0, v)$$

Properties of the mixed strategy Nash equilibrium:

Pure strategy equilibria exist when K lies at the boundaries of the interval $[\frac{Q}{2}, Q]$. As K approaches these boundaries, the mixed strategy equilibrium converges to the pure strategy equilibrium.

(i) As $K \downarrow Q/2$, $a \uparrow v$, which means that the support of F converges to v. In the limit where $K = Q/2$, there is a pure strategy equilibrium with $p = v$.

(ii) As $K \uparrow Q$, $a \downarrow 0$, which means that the support of F converges to $[0, v]$. However, for any $p > 0$, $F(p)$ converges to unity. Consequently, all of the probability is mass is converging towards 0. In the limit where $K = Q$, there is a pure strategy equilibrium with $p = 0$.

2.9 Normal form refinements of Nash equilibrium

(i) The voting problem

(ii) The Bertrand problem with homogeneous products and differing linear costs (firms pricing below cost)

2.9.1 Application of weak dominance

Criterion: Discard all Nash equilibria that involve any player selecting a weakly dominated strategy with positive probability.

Application #1: The simple voting problem.

Eliminates any equilibrium in which an individual votes for his least favorite outcome.

Application #2: Bertrand pricing, constant unit costs, $p \in \Re_+$ (either method of resolving consumer indifference).

Every pure strategy Nash equilibrium involves at least one player selecting a weakly dominated strategy, $p_i \leq c_i$.

Application #3: Bertrand pricing, constant unit costs, $p \in \{0, 1, 2, ...\}$ (and $c_i \in \{0, 1, 2, ...\}$ for all i, with $c_1 \leq c_2 \leq c_3...$)

The weak dominance criterion rules out every equilibrium in which any player selects $p_i \leq c_i$. The outcome differs according to whether $c_1 < c_2$. There are two cases to consider:

(i) $c_1 = c_2$. Then the weak dominance criterion selects the equilibrium with $p_1 = p_2 = c_1 + 1$. Note: as the price grid becomes finer, the equilibrium price converges to c_1. Consequently, it is natural to accept $p_1 = p_2 = c_1$ as a solution when $p \in \Re_+$, even though, as a technical matter, both firms play weakly dominated strategies.

(ii) $c_1 < c_2$. Then the weak dominance criterion selects the equilibrium with $p_1 = c_2$, and $p_2 = c_2 + 1$. Note: as the price grid becomes finer, the equilibrium price for firms 1 and 2 both converge to c_2. Consequently, it is natural to accept $p_1 = p_2 = c_2$ as the solution when $p \in \Re_+$ even though, as a techical matter, firm 2's strategy is weakly dominated.

2.9.2 Trembling hand perfection

Alternative perspective on motivating problems: There is always some risk that another player will make a "mistake."

Formalization: Consider a finite normal form game $(\{S_i\}_{i \in 1,...,I}, g)$; as before, let K_i denote the cardinality of player i's pure strategy set.

Let $\Delta_i(\varepsilon)$ be the collection of vectors $(\delta_{i1}, ..., \delta_{iK_i})$ such that $\delta_{ik} \in [\varepsilon, 1]$ for all $k \in \{1, ..., K_i\}$, and $\sum_{k=1}^{K_i} \delta_{ik} = 1$. This is the set of probability distributions over player i's pure strategies with the property that everything is played with at least probability ε.

Let $\Delta(\varepsilon) = \Delta_1(\varepsilon) \times ... \times \Delta_I(\varepsilon)$

An $\varepsilon-$*constrained mixed strategy equilibrium* for the game $(\{S_i\}_{i\in 1,...,I}, g)$ is a pure strategy Nash equlibrium for the game $(\{\Delta_i(\varepsilon)\}_{i\in 1,...,I}, \pi)$ (where π is the expected payoff function, exactly as before).

Definition #1: A *trembling hand perfect equilibrium* is any limit of ε-constrained equilibria as ε goes to zero.

A strategy δ_i is *totally mixed* if $\delta_{ik} > 0$ for all $k \in \{1, ..., K_i\}$.

Definition #2: A *trembling hand perfect equilibrium* is a mixed strategy Nash equilibrium, δ, with the following property: there exists some sequence of totally mixed strategy profiles $\delta^t \to \delta$, such that, for all i and t, $\pi_i(\delta_i, \delta^t_{-i}) \geq \pi_i(s_i, \delta^t_{-i})$ for all $s_i \in S_i$.

Some theorems:

(i) Definition #1 and definition #2 are equivalent.

(ii) In a trembling-hand perfect equilibrium, no player selects a weakly dominated strategy with positive probability. For two-player games, an equilibrium is trembling-hand perfect if and only if no player selects a weakly dominated strategy with positive probability. (For more than two players, the set of trembling-hand perfect equilibria is potentially smaller, unless one allows for correlations between the trembles of different players, in which case the two sets of equilibria are once again the same.)

Remark: Trembling-hand perfection can be extended to games with infinite strategy sets by taking ε to be a lower bound on density, rather than probability. The second definition must then be adjusted as follows: as $\delta^t \to \delta$, each player i has some sequence of best responses converging to δ_i.

Application: Bertrand competition with homogenous products and identical linear costs, c.

We know that, in any pure strategy Nash equilibrium, all output is sold at the price $p = c$. We also know that this price is weakly dominated. However, this equilibrium is trembling-hand perfect, in the sense just described. Similarly, when costs differ across firms, $p_1 = p_2 = c_2$ is trembling hand perfect, even though $p_2 = c_2$ is weakly dominated.

2.10 Arguments for and against Nash equilibrium

2.10.1 Possible justifications for Nash equilibrium

1. Under certain circumstances, choices may converge to a Nash equilibrium.

2. In Nash equilibrium, players have "rational expectations."

3. Any deterministic theory of strategic behavior must have the property that agents play Nash equilibria.

4. Nash equilibria are "focal."

5. Nash equilibria are self-enforcing agreements.

6. Experimental evidence.

2.10.2 Possible criticisms of Nash equilibrium

1. A misguided objection: The "Nash assumption" – opponents choices are fixed, and would not change if a player made an out-of-equilibrium choice – is not valid.

An alternative: conjectural variations equilibrium.

Two player illustration for a game (S_1, S_2, g_1, g_2).

Definition: A *conjectural variations equilibrium* consists of $s^* \in S$ and, for each player i, a function $r_i^* : S_i \to S_j$ (where $r_i^*(s_i)$ represent i's conjecture of what j will play if i chooses s_i) such that

(i) s_i^* maximizes $g_i(s_i, r_i^*(s_i))$ over $s_i \in S_i$, and

(ii) $r_i^*(s_i^*) = s_j^*$.

Key points:

(i) "Nash assumption" is that each player takes the strategies of the other players as fixed, not their actions.

(ii) If a particular model (such as Cournot or Bertrand) does not allow for responses, this is a problem with the model, and not with the equilibrium concept.

(iii) The conjectural variations concept does not coherently depict strategic choice even in single-decision games, as it is impossible for both players to move first.

2. Nash equilibrium is too inclusive (see section 2.9, above, and the material in sections 4 and 5, below).

3. Nash equlibrium is insufficiently inclusive. It may be possible to construct self-enforcing agreements that are not Nash equilibria, by introducing sources of correlation between choices (see the following section).

2.11 Correlated equilibrium

Example:

		P 2	
		a	b
P 1	A	9, 9	6, 10
	B	10, 6	0, 0

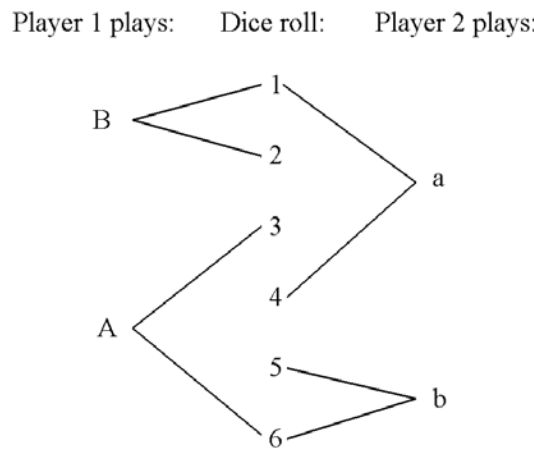
Pure strategy Nash equilibria: (A,b) with payoffs (6,10), and (B,a) with payoffs (10,6)

Mixed strategy Nash equilibrium: $\left(\frac{6}{7}, \frac{1}{7}\right)$ for each player, with expected payoff ≈ 8.57.

One potential agreement: Appoint a third party to flip a coin. The party announces "heads" or "tails." If heads, the players play (A, b). If tails, the players play (B, a). Expected payoff is 8. Note: can choose the parameters differently so that this dominates the mixed strategy equilibrium (change the 9s to 8s).

Another potential agreement: Appoint a third party to roll a six-sided dice. Neither player observes the outcome. The third party tells $P1$ whether the number falls into the set $\{1, 2\}$ or $\{3, 4, 5, 6\}$. He tells $P2$ whether the number falls into the set $\{1, 2, 3, 4\}$ or $\{5, 6\}$. Players can condition their choices on the information received from the third party.

Select actions as follows:



Check to see that this is self-enforcing (since the game is symmetric, we need only check for player 1):

If player 1 is told that the dice roll is in the set $\{1, 2\}$, she believes that 2 will play a with certainty. B is a best response to a.

If player 1 is told that the dice roll is in the set $\{3, 4, 5, 6\}$, she believes that 2 will play both a and b with probability 0.5. In that case, A yields an expected payoff of 7.5,while B yields an expected payoff of 5. A is the best response.

Note: expected payoff here is $8\frac{1}{3}$, which is still not quite as good as the mixed strategy equilibrium, but we'll see in a moment that we can do even better.

Definition: Consider a finite game $\left(\{S_i\}_{i=1}^I, g\right)$. δ^* (a probability distribution over S) is a correlated equilibrium iff $\forall i$ and for all s_i chosen with strictly positive probability, s_i solves

$$\max_{s_i' \in S_i} E_{s_{-i}} \left[g_i(s_i', s_{-i}) \mid s_i, \delta \right]$$

Return to example:

Translating the previous equilibrium into these terms:

P 1 \ P 2	a	b
A	1/3	1/3
B	1/3	0

Are there other correlated equilibria? Consider the following family of distributions:

P 1 \ P 2	a	b
A	γ	$(1-\gamma)/2$
B	$(1-\gamma)/2$	0

Conditions for this to be a correlated equilibrium (since it's symmetric, we only have to check player 1):

If player 1 is told to play B, she thinks that player 2 will select a with certainty. B is a best response.

If player 1 is told to play A, she thinks that player 2 will select a with probability $\frac{2\gamma}{1+\gamma}$, and that player 2 will select b with probability $\frac{1-\gamma}{1+\gamma}$. A is a best response provided that

$$9\left(\frac{2\gamma}{1+\gamma}\right)+6\left(\frac{1-\gamma}{1+\gamma}\right)\geq 10\left(\frac{2\gamma}{1+\gamma}\right)+0\left(\frac{1-\gamma}{1+\gamma}\right)$$

This holds provided that $\gamma \in [0, \frac{3}{4}]$.

$\gamma = 0$ corresponds to the very first correlated agreement we considered. The most attractive equilibrium is the one for which $\gamma = \frac{3}{4}$. The associated expected payoff is $8\frac{3}{4}$, which is strictly higher than the payoff available from any mixed or pure strategy Nash equilibrium.

3 Strategic Choice in Static Games with Incomplete Information

A game is said to have *incomplete information* when some players do not know the payoffs of other players.

Applications include: competition between firms with private information about costs and technology, auctions where each potential buyer may attach a different valuation to the item, negotiations with uncertainty about the other party's preferences or objectives, and so forth.

Example #1: Friend or Foe

Two players

Player 1's preferences are common knowledge

Player 2 may be either a "friend" or a "foe," but player 1 doesn't know which.

The normal forms are as follows:

"Friend"

		P2	
		h	t
P1	H	3, 1	0, 0
	T	2, 1	1, 0

"Foe"

		P2	
		h	t
P1	H	3, 0	0, 1
	T	2, 0	1, 1

3.1 Bayesian equilbrium

In general, dealing with such situations would seem to require us to specify beliefs about others' payoffs, beliefs about others' beliefs about others' payoff, etc.

Harsanyi's innovation: Model games of incomplete information as games of imperfect information in which nature begins by selecting payoffs. Players observe their own payoffs, but don't necessarily observe payoffs selected for others. As usual, we regard the probability decision governing nature's decision as common knowledge. Look for a Nash equilibrium of this extended game (sometimes known as a *Bayesian game*). This is called a *Bayesian Nash equilibrium.*

A general formulation: Each player has a payoff function $g_i(s, \theta_i)$ for $s \in S$ and $\theta_i \in \Theta_i$, where θ_i is sometimes referred to as player i's "type." Define $\Theta = \Theta_1 \times ... \times \Theta_I$. For simplicity, assume Θ is a Euclidean set. Let the CDF $F(\theta)$ (for $\theta \in \Theta$) describe the joint distribution of the players' types. Each player knows her own value of θ_i, and F is common knowledge, but i does not know θ_j for any $j \neq i$.

We can describe this as a *Bayesian game* characterized by the vector $\left(\{S_i\}_{i=1}^{I}, g, \Theta, F\right)$

Alternatively, we can describe this as a more conventional game (the "extended game"), as follows.

Play unfolds as follows: (i) nature selects types, (ii) players observe their own types, but do not observe the types of any other players, (iii) players simultaneously select $s_i \in S_i$.

Think of a pure strategy for player i as a mapping $\sigma_i : \Theta_i \to S_i$. For any particular θ_i, $\sigma_i(\theta_i)$ represents the choice that type i would make upon learning that she is type θ_i. Let Σ_i denote the set of possible mappings from Θ_i to S_i. We will think of this as i's strategy set. Let $\Sigma = \Sigma_1 \times ... \times \Sigma_I$.

Given F and g, we can, for any $\sigma \in \Sigma$, compute an expected payoff for player i, $u_i(\sigma)$, as follows:

$$u_i(\sigma) = E_\theta \left[g_i \left(\sigma_1(\theta_1), ..., \sigma_I(\theta_I), \theta_i \right) \right]$$

Let $u(\sigma) = (u_1(\sigma), ..., u_I(\sigma))$.

Definition: A (pure strategy) *Bayesian Nash equlibrium* of the Bayesian game $\left(\{S_i\}_{i=1}^I, g, \Theta, F \right)$ is a (pure strategy) Nash equilibrium of the game $\left(\{\Sigma_i\}_{i=1}^I, u \right)$.

For the following, suppose that Θ_i is a finite set for each i (the extension to infinite sets is straightforward).

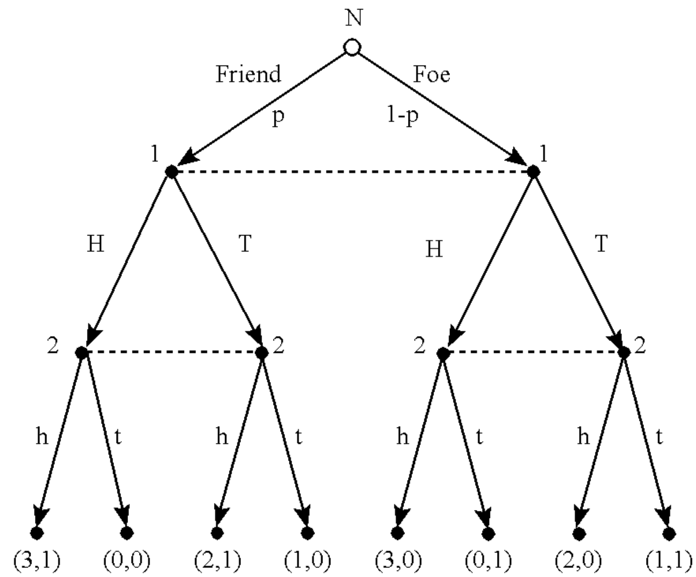
Equivalent definition: A collection of decision rules $(\sigma_1, ..., \sigma_I)$ is a (pure strategy) *Bayesian Nash equilibrium* for the Bayesian game $\left(\{S_i\}_{i=1}^I, g, \Theta, F \right)$ if and only if, for all i and all $\theta_i \in \Theta_i$ occuring with positive probability, $\sigma_i(\theta_i)$ solves

$$\max_{s_i \in S_i} E_{\theta_{-i}} \left[g_i \left(s_i, \sigma_{-i}(\theta_{-i}), \theta_i \right) \mid \theta_i \right]$$

Example #1, revisited: Let p denote the probability that player 2 is a friend; $1-p$ is the probability that player 2 is a foe.

1. Illustration of the first definition

Extensive form of extended game:



Normal form for extended game:

P2

P1		hh	ht	th	tt
	H	3, p	3p, 1	3-3p, 0	0, 1-p
	T	2, p	1+p, 1	2-p, 0	1, 1-p

Nash equilibrium: (H, ht) if $p \geq \frac{1}{2}$, (T, ht) if $p \leq \frac{1}{2}$.

2. Illustration of the second definition

Decision rules, for the case of $p > \frac{1}{2}$:

$$\sigma_1 = H$$

$$\sigma_2(\theta_2) = \begin{cases} h \text{ if } \theta_2 = friend \\ \\ t \text{ if } \theta_2 = foe \end{cases}$$

Check: 2's choices are optimal for each realization of type. Given 2's decision rule, 1's payoffs are $3p$ from H, and $1 + p$ from T; with $p > \frac{1}{2}$, H is optimal.

For $p < \frac{1}{2}$, $\sigma_1 = T$, same decision rule for 2. For $p = \frac{1}{2}$, 1 could choose H or T, or randomize between them; 2's decision rule is unchanged.

3.2 Application: Externalities

Problem: How should the government deal with externalities (including public goods) when it does not have information about consumers' preferences? In general, agents will have incentives either to understate or exaggerate certain aspects of their preferences. How can the government elicit correct information?

The model: 2 consumers, $i = 1, 2$

Consumption bundles $x_i \in B_i$ (budget sets)

x_i may be a vector (e.g. private consumption and contribution to a public good).

Utility: $u_i(x_1, x_2, \theta_i) + T_i$, where T_i is a monetary transfer, and θ_i is a taste parameter known only to i (can be a scalar or a vector)

Let θ_i^T be the true value of i's taste parameter

The taste parameters are drawn from some probability distribution with CDF F, which we take for simplicity to be the same for both individuals. F is common knowledge.

Government policy: Consumers will be asked to name their types. They are not constrained to tell the truth. Let θ_i^A be i's announced type. The government's policy maps these announcements to allocations. Consumers take the government's policy mapping as given when choosing announcements. The announcement game is a game of incomplete information.

Government's objective: $\max u_1 + u_2$ subject to $x_i \in B_i$, $i = 1, 2$.

Note: The government actually maximizes $u_1+u_2+T_1+T_2$, but $T_1+T_2 = 0$ is an additional constraint.

Let $x_i^*(\theta_1, \theta_2)$ denote the optimal solution as a function of type. That is, $x_i^*(\theta_1, \theta_2)$ solves

$$\max_{x_1, x_2} u_1(x_1, x_2, \theta_1) + u_2(x_1, x_2, \theta_2)$$

Remarks on approach: The government's problem is known as *mechanism design* problem.

We will study *direct revelation mechanisms*, in which reported types are mapped to outcomes on the assumption that the parties have revealed their information truthfully.

To justify our focus on direct revelation mechanisms, we invoke a theorem known as the *revelation principle*. This theorem states that, in searching for an optimal mechanism within a much broader class, the designer can, without loss, restrict attention to direct revelation mechanisms for which truth-telling is an optimal strategy for each party.

A possible tax/subsidy scheme: Attempt to implement the first-best outcome.

$$\begin{aligned}
x_i^P(\theta_1^A,\theta_2^A) &= x_i^*(\theta_1^A,\theta_2^A) \\
T_1(\theta_1^A,\theta_2^A) &= \int u_2\left(x_1^*\left(\theta_1^A,\theta_2^T\right), x_2^*\left(\theta_1^A,\theta_2^T\right), \theta_2^T\right) dF(\theta_2^T) + K_1\left(\theta_2^A\right) \\
&\equiv t_1\left(\theta_1^A\right) + K_1\left(\theta_2^A\right)
\end{aligned}$$

The expression for T_2 is symmetric.

The idea: tax captures the expected externality which 1's choice has on 2.

Given this scheme, what will the consumers choose? This is a game of incomplete information.

Claim: Announcing the truth (that is, $\theta_i^A\left(\theta_i^T\right) = \theta_i^T$) is a Bayesian Nash equilibrium.

Proof: Consider consumer 1's decision (the argument is identical for player 2). Imagine that consumer 2 follows the decision rule $\theta_2^A\left(\theta_2^T\right) = \theta_2^T$. Then consumer 1's payoff is given (as a function of 1's announced type, θ_1^A) as follows:

$$\begin{aligned}
&\int \left[u_1\left(x_1^*(\theta_1^A,\theta_2^T), x_2^*(\theta_1^A,\theta_2^T), \theta_1^T\right) + t_1(\theta_1^A) + K_1(\theta_2^T)\right] dF(\theta_2^T) \\
= &\int u_1\left(x_1^*(\theta_1^A,\theta_2^T), x_2^*(\theta_1^A,\theta_2^T), \theta_1^T\right) dF(\theta_2^T) + t_1(\theta_1^A) + \int K_1(\theta_2^T) dF(\theta_2^T)
\end{aligned}$$

$$= \int \left[u_1\left(x_1^*(\theta_1^A,\theta_2^T), x_2^*(\theta_1^A,\theta_2^T), \theta_1^T\right) + u_2\left(x_1^*\left(\theta_1^A,\theta_2^T\right), x_2^*\left(\theta_1^A,\theta_2^T\right), \theta_2^T\right)\right] dF(\theta_2^T)$$
$$+ \int K_1(\theta_2^T) dF(\theta_2^T)$$

Now imagine maximizing this expression over θ_1^A. The second term does not depend upon θ_1^A, so we can ignore it. Consider the integrand of the first term. By definition, $x_1^*(\theta_1^T,\theta_2^T)$ and $x_2^*(\theta_1^T,\theta_2^T)$ maximize $u_1(x_1,x_2,\theta_1^T)+u_2(x_1,x_2,\theta_2^T)$. So clearly, consumer 1 does at least as well by announcing θ_1^T, thereby producing $x_1^*(\theta_1^T,\theta_2^T)$ and $x_2^*(\theta_1^T,\theta_2^T)$ for each θ_2^T, than by announcing some other θ_1^A, thereby producing $x_1^*(\theta_1^A,\theta_2^T)$ and $x_2^*(\theta_1^A,\theta_2^T)$ for each θ_2^T. Q.E.D.

Implication: For this mechanism, there exists a Bayesian Nash equilibrium that achieves the first-best outcome.

A potential problem: The budget may not balance.

A solution: Let $K_1(\theta_2^A) = -t_2(\theta_2^A)$, and $K_2(\theta_1^A) = -t_1(\theta_1^A)$. Then

$$\begin{aligned} T_1(\theta_1^A,\theta_2^A) + T_2(\theta_1^A,\theta_2^A) &= \left[t_1(\theta_1^A) - t_2(\theta_2^A)\right] + \left[t_2(\theta_2^A) - t_1(\theta_1^A)\right] \\ &= 0 \end{aligned}$$

Consequently, the budget balances for all realizations of preferences.

Reservations concerning this mechanism:

1. There may be other Bayesian Nash equilibria.

2. Common knowledge of F is restrictive.

3. More generally, Nash equilibrium may be unappealing.

An alternative tax/subsidy scheme: Attempt to achieve the first best through a mechanism that has a dominant strategy solution (in which case uniqueness is assured, common knowledge of distributions is unimportant, and the solution concept is difficult to challenge):

$$
\begin{aligned}
x_i^P(\theta_1^A,\theta_2^A) &= x_i^*(\theta_1^A,\theta_2^A) \\
T_1(\theta_1^A,\theta_2^A) &= u_2\left(x_1^*\left(\theta_1^A,\theta_2^A\right), x_2^*\left(\theta_1^A,\theta_2^A\right), \theta_2^A\right) + K_1\left(\theta_2^A\right) \\
&= t_1\left(\theta_1^A,\theta_2^A\right) + K_1\left(\theta_2^A\right)
\end{aligned}
$$

The expression for T_2 is symmetric.

The idea: tax captures the externality which 1's choice has on 2, assuming 2's announcement is correct.

Given this scheme, what will the consumers choose? This is another game of incomplete information.

Claim: Announcing the truth (that is, $\theta_i^A\left(\theta_i^T\right) = \theta_i^T$) is a (weakly) dominant strategy.

Proof: Since the problem is symmetric, we can consider consumer 1's decision. Let G denote the CDF for θ_2^A (since 2 may not be telling the truth, G may bear practically no relation to F). Given any realization of type, θ_1^T, consumer 1's expected payoff is given by

$$
\begin{aligned}
& \int \left[u_1\left(x_1^*(\theta_1^A,\theta_2^A), x_2^*(\theta_1^A,\theta_2^A), \theta_1^T\right) + t_1(\theta_1^A,\theta_2^A) + K_1(\theta_2^A)\right] dG(\theta_2^A) \\
= & \int \left[u_1\left(x_1^*(\theta_1^A,\theta_2^A), x_2^*(\theta_1^A,\theta_2^A), \theta_1^T\right) + u_2\left(x_1^*\left(\theta_1^A,\theta_2^A\right), x_2^*\left(\theta_1^A,\theta_2^A\right), \theta_2^A\right)\right] dG(\theta_2^A) \\
& + \int K_1(\theta_2^A) dG(\theta_2^A)
\end{aligned}
$$

By definition, $x_1^*(\theta_1^T,\theta_2^A)$ and $x_2^*(\theta_1^T,\theta_2^A)$ maximize $u_1(x_1,x_2,\theta_1^T)+u_2(x_1,x_2,\theta_2^A)$. So clearly, consumer 1 does at least as well by announcing θ_1^T, thereby producing $x_1^*(\theta_1^T,\theta_2^A)$ and $x_2^*(\theta_1^T,\theta_2^A)$ for each θ_2^A, than by announcing some other θ_1^A, thereby producing $x_1^*(\theta_1^A,\theta_2^A)$ and $x_2^*(\theta_1^A,\theta_2^A)$ for each θ_2^A. Q.E.D.

A potential problem: As before, the budget may not balance. In this case, we cannot use the same trick as before, because t_1 depends on both θ_1^A and θ_2^A.

A partial solution: Let

$$\begin{aligned} K_1(\theta_2^A) &= -E_{\theta_1^T} t_2(\theta_1^T, \theta_2^A) \\ &= -\int u_1\left(x_1^*\left(\theta_1^T, \theta_2^A\right), x_2^*\left(\theta_1^T, \theta_2^A\right), \theta_1^T\right) dF(\theta_1^T) \end{aligned}$$

Then the expected government deficit, conditional upon truth-telling, is

$$E_{\theta_1^T, \theta_2^T}\left[T_1(\theta_1^T, \theta_2^T) + T_2(\theta_1^T, \theta_2^T)\right] = 0$$

Consequently, the budget balances in expectation in equilibrium, but not necessarily for all realizations of preferences, and not outside of equilibrium.

Comments: (i) This is known as a *Groves mechanism.* (ii) One can prove that there does not exist a dominant strategy mechanism that balances the budget for all realizations of preferences.

3.3 Application: Auctions

A simple bidding problem:

Seller offers a single indivisible object

N bidders, $i = 1, ..., N$

Each bidder i has a reservation valuation v_i

Valuations are private information

v_i is the realization of a random variable with CDF F on $[v_\ell, v_h]$

F is the same for all bidders (symmetry)

For some purposes, we will assume that F is uniform on $[0, \alpha]$

Realizations are independent across bidders

This is a case of *independent private valuations*

We will consider outcomes of four different types of auctions, making comparisons across auctions to determine the most profitable method of selling the good.

1. Second-price sealed bid auction

Bidders simultaneously submit sealed bids, b_i (for i); highest bid wins, but pays the second highest bid. A winner is chosen from among the high bidders with equal probability in the event of a tie.

We have already analyzed this model with complete information. We found that $b_i = v_i$ weakly dominated all other choices.

With incomplete information, the analysis is unchanged.

How much revenue does the seller receive?

The seller always receives the second highest bid, which equals the second highest valuation. Consequently, expected revenue equals the expected second order statistic of the set of valuations, $v_{(2)}$.

$$E\left[v_{(2)}\right] = \int_{v_\ell}^{v_h} vN(N-1)f(v)\left[F(v)\right]^{N-2}\left[1-F(v)\right]dv$$

For the special case where F is uniform on $[0, \alpha]$, we have

$$\begin{aligned} E\left[v_{(2)}\right] &= N(N-1)\int_0^\alpha v\frac{1}{\alpha}\left(\frac{v}{\alpha}\right)^{N-2}\left[1-\frac{v}{\alpha}\right]dv \\ &= \alpha\left(\frac{N-1}{N+1}\right) \end{aligned}$$

2. First-price sealed bid auction

Bidders simultaneously submit sealed bids, b_i (for i); highest bidder wins, and pays her own bid. A winner is chosen from among the high bidders with equal probability in the event of a tie.

Payoffs are $v_i - b_i$ for the winner, and 0 for all other bidders.

To construct a pure strategy Bayesian Nash equilibrium, we must identify decision rules, $\sigma_i(v_i)$, mapping valuations, v_i, to bids, b_i.

We will confine attention to symmetric equilibria characterized by a single decision rule, $\sigma(v)$.

Assume at the outset that $\sigma(v)$ is strictly increasing in v. (Later, we will have to verify that this property is in fact satisfied by the rule that we derive).

In that case, the probability of winning upon submitting the bid b is $[F(\sigma^{-1}(b))]^{N-1}$.

The associated expected surplus is given by

$$\pi(v,b) = (v-b)\left[F(\sigma^{-1}(b))\right]^{N-1}$$

In equilibrium, the expected payoff for a bidder with valuation v is

$$\Pi(v) = \pi(v,\sigma(v))$$

Notice that

$$\frac{d\Pi(v)}{dv} = \frac{\partial\pi(v,\sigma(v))}{\partial v} + \frac{\partial\pi(v,\sigma(v))}{\partial\sigma} \times \frac{d\sigma(v)}{dv}$$

By the Envelope Theorem, $\frac{\partial\pi(v,\sigma(v))}{\partial\sigma} = 0$. Consequently,

$$\begin{aligned} \frac{d\Pi(v)}{dv} &= \left[F(\sigma^{-1}(\sigma(v)))\right]^{N-1} \\ &= [F(v)]^{N-1} \end{aligned}$$

Moreover, notice that

$$\begin{aligned}\Pi(v) &= (v-\sigma(v))\left[F(\sigma^{-1}(\sigma(v)))\right]^{N-1} \\ &= (v-\sigma(v))\left[F(v)\right]^{N-1}\end{aligned}$$

Taking the previous two expressions and substituting them into the identity

$$\int_{v_\ell}^{v} \frac{d\Pi(w)}{dw} dw = \Pi(v) - \Pi(v_\ell)$$

we obtain

$$\int_{v_\ell}^{v} \left[F(w)\right]^{N-1} dw = (v-\sigma(v))\left[F(v)\right]^{N-1} - 0$$

Rearranging provides an explicit solution for $\sigma(v)$:

$$\sigma(v) = v - \int_{v_\ell}^{v} \left[\frac{F(w)}{F(v)}\right]^{N-1} dw$$

Notice: (i) This is increasing in v (as we assumed at the outset). (ii) Each customer bids less than his or her actual valuation. (iii) $\lim_{N\to\infty}\sigma(v) = v$, (iv) in general, the highest bid will not be equal to the second highest valuation, so realized revenues will differ from those collected in a second-price auction.

The next step is to compute expected revenue. In a first-price auction, the seller receives the highest bid. Consequently, expected revenue equals the expected value of the first order statistic of the set of bids:

$$E\left[b_{(1)}\right] = \int_{v_l}^{v_h} \sigma(v) N f(v)\left[F(v)\right]^{N-1} dv$$

We will specialize to the case of the uniform distribution, which implies

$$\begin{aligned}\sigma(v) &= v - \int_{0}^{v} \left[\frac{\alpha w}{\alpha v}\right]^{N-1} dw \\ &= v\left[\frac{N-1}{N}\right]\end{aligned}$$

Consequently, for this case, we have

$$\begin{aligned}
E\left[b_{(1)}\right] &= E\left[v_{(1)}\right]\left(\frac{N-1}{N}\right) \\
&= \left(\frac{N-1}{N}\right)\int_0^\alpha vNf(v)\left[F(v)\right]^{N-1}dv \\
&= \left(\frac{N-1}{N}\right)\int_0^\alpha vN\left(\frac{1}{\alpha}\right)\left[\frac{v}{\alpha}\right]^{N-1}dv \\
&= \left(\frac{N-1}{N}\right)\alpha\left(\frac{N}{N+1}\right) \\
&= \alpha\left(\frac{N-1}{N+1}\right)
\end{aligned}$$

Notice: expected revenues are exactly the same as for the second-price auctions. This is a special case of the *revenue equivalence theorem*, which states that first and second-price sealed bid auctions always generate the same expected revenues in auctions with independent private valuations, irrespective of the distribution function. This equivalence holds despite the fact that, for any given realization of valuations, realized revenues will differ across these two auctions.

General proof of revenue equivalence theorem: Consider any auction structure and an associated Bayesian Nash equilibrium $\sigma(v)$. Let $p(b)$ denote the equilibrium probability of winning as a function of the bid, and let $t(b)$ denote the expected payment to the auctioneer, given the bid. The expected payoff to a bidder with valuation v is

$$\Pi(v) = vp(\sigma(v)) - t(\sigma(v))$$

Using the Envelope Theorem exactly as we did earlier, we have

$$\frac{d\Pi(v)}{dv} = p(\sigma(v))$$

So

$$\begin{aligned}
\Pi(v) &= \Pi(v_\ell) + \int_{v_\ell}^{v}\frac{d\Pi(w)}{dw}dw \\
&= \Pi(v_\ell) + \int_{v_\ell}^{v} p(\sigma(w))dw
\end{aligned}$$

But then, using the definition of $\Pi(v)$, we have

$$t(\sigma(v)) = vp(\sigma(v)) - \int_{v_\ell}^{v} p(\sigma(w))dw - \Pi(v_\ell)$$

Expected revenue for the auctioneer is then

$$NE_v\left[t(\sigma(v))\right] = NE_v\left[vp(\sigma(v)) - \int_{v_\ell}^{v} p(\sigma(w))dw - \Pi(v_\ell)\right]$$

But for both first and second price auctions, we have shown that bids are strictly increasing in valuations, so $\Pi(v_\ell) = 0$ and

$$p(\sigma(v)) = [F(v)]^{N-1}$$

Thus, expected revenue is the same for both types of auctions. Q.E.D.

3. English (ascending price) auction: Posted price of the good is slowly increased. Bidders drop out, until there is only one left, who purchases the good for the price at which the second-to-last bidder dropped out. (If the last bidders drop out simultaneously, a winner is selected from among them at random.)

Weakly dominant strategy for each bidder: stay in until the posted price exceeds your valuation.

Implication: the good is sold for a price equal to the second highest valuation, exactly as in a second-price sealed bid auction.

4. Dutch (descending price) auction: Posted price of the good is slowly decreased, until a bidder claims the object at the posted price. (In the event of a tie, the winner is selected at random from those attempting to claim the object at the posted price.)

A strategy is completely described by a decision rule that specifies a price at which to claim the object.

The mapping from these strategy profiles into payoffs is the same as for the first-price auction.

Thus, the descending price auction is strategically equivalent to the first-price auction, and should produce the same outcome.

Remarks:

(i) All four varieties of auctions produce the same revenue for auctions with independent private valuations.

(ii) Revenue equivalence does not necessarily hold once one relaxes the assumption of independent private valuations.

(iii) Another natural question is whether there is any other kind of auction that yields greater revenue. This is another *mechanism design* question, and it can be addressed once again by invoking the revelation priciple. In general, one can improve upon the payoff that the seller obtains from the first price auction by specifying a reserve price in excess of the seller's valuation.

4 Dynamic Games and the Problem of Credibility

Example #1: A simple model of entry into an industry.

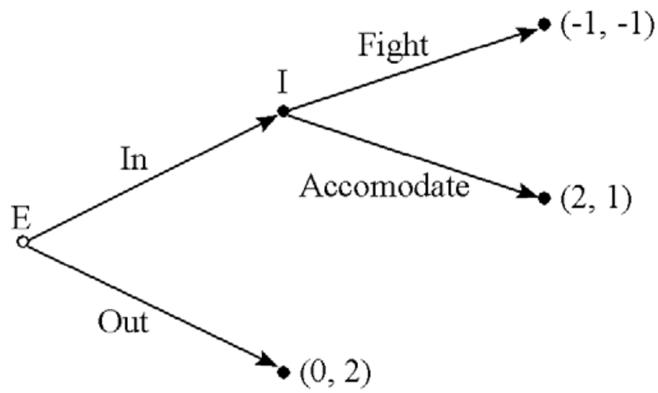
Two firms: an incumbent (I), and a potential entrant (E)

Entrant moves first – selects either "In" or "Out"

If the entrant chooses "Out," the game ends

If the entrant chooses "In", the incumbent must choose to "$Fight$" or to "*Accomodate*"

The extensive form, including resulting payoffs, is as follows:



The normal form is as follows:

		Incumbent	
		Fight	Accomodate
Entrant	Out	0, 2	0, 2
	In	-1, -1	2, 1

There are two Nash equilibria: (In, *Accomodate*), and (Out, $Fight$)

Only (In, *Accomodate*) makes sense. For (Out, $Fight$), the entrant is discouraged from entering by a non-credible threat.

Note: in this example, weak dominance eliminates (Out, $Fight$), but the problem here is more general.

General problem: impose conditions to guarantee that players anticipate sensible things "off the equilibrium path."

4.1 Subgame perfection

Notation:

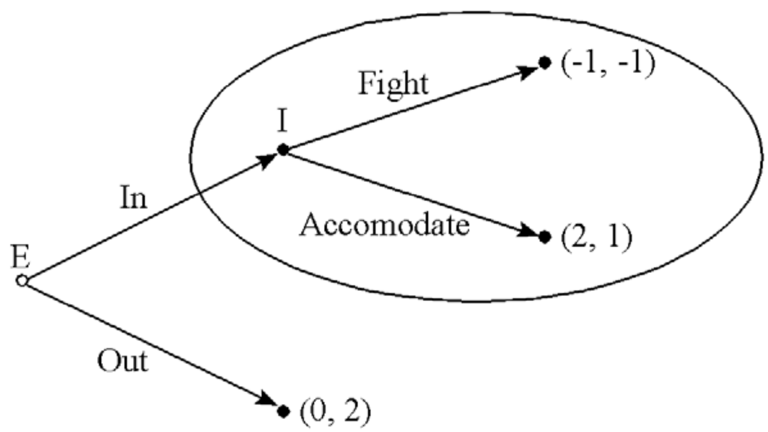
t denotes a node in the game

$h(t)$ denotes the information set (element of the information partition) containing t

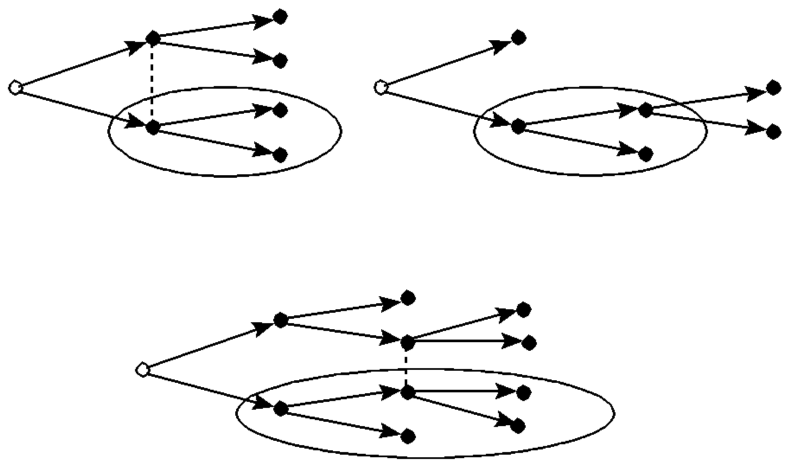
$S(t)$ denotes the successors to the node t

Definition: A *proper subgame* of an extensive form game is $\{t\} \cup S(t)$ (along with the associated mappings from information sets to players, from branches to action labels, and from terminal nodes to payoffs) such that $h(t) = \{t\}$ and $\forall\, t' \in S(t)$, $h(t') \subseteq S(t)$.

An illustration of a proper subgame:



Some examples of collections of nodes that do not form proper subgames:

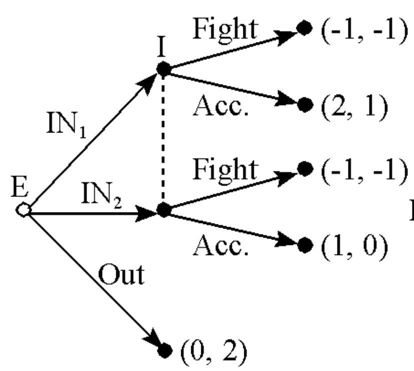


Definition: (Due to Selten) Consider a Nash equilibrium in behavior strategies, δ^*. This equilibrium is *subgame perfect* iff for every proper subgame, the restriction of δ^* to that subgame forms a Nash equilibrium in behavior strategies.

Observation: For finite games, one can solve for subgame perfect Nash equilibria ($SPNE$) by using backward induction on the subgames of the extensive form.

Return to example #1: There is one proper subgame (shown in preceding illustration). The only equilibrium of this subgame involves the Incumbent choosing to "*Accomodate*." Consequently, (In, *Accomodate*) is subgame perfect, whereas (Out, $Fight$) is not.

Example #2: Similar to previous game, except that E can enter with one of two different production technologies, IN_1 or IN_2. The payoffs from IN_1 are as in the previous example; the payoffs from IN_2 are shown below:



		Incumbent	
		Fight	Accomodate
Entrant	IN$_1$	-1, -1	2, 1
	IN$_2$	-1, -1	1, 0
	Out	0, 2	0, 2

Nash equilibria: (IN_1, *Accomodate*), (Out, $Fight$).

Since there are no proper subgames, both equilibria are subgame perfect. However, (Out, $Fight$) is still implausible (all we have done is to add an inferior entry technology).

The problem: We need a theory of sensible choices at each information set, and not just in proper subgames.

4.2 Perfect Bayesian equilibrium

What is missing from subgame perfection? Beliefs at each information set. If we had a description of initial beliefs at an information set, we could ask whether choices are reasonable given those beliefs.

Notation:

X is the set of all decision nodes in the game

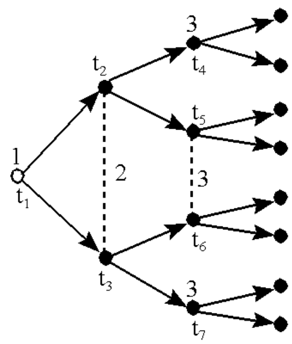
H is the set of information sets (the information partition)

$h(t)$ is the information set containing node $t \in X$

$\varphi(h)$ denotes the player who makes the decision at information set h

Definition: A *system of beliefs* is a mapping $\mu : X \to [0, 1]$ such that, $\forall\, h \in H$, $\sum_{t \in h} \mu(t) = 1$.

Illustration:



$X = \{t_1, t_2, t_3, t_4, t_5, t_6, t_7\}$

$H = \{\{t_1\}, \{t_2, t_3\}, \{t_4\}, \{t_5, t_6\}, \{t_7\}\}$

h	$\{t_1\}$	$\{t_2, t_3\}$	$\{t_4\}$	$\{t_5, t_6\}$	$\{t_7\}$
$\varphi(h)$	1	2	3	3	3

A system of beliefs:

t	t_1	t_2	t_3	t_4	t_5	t_6	t_7
$\mu(t)$	1	λ	$1-\lambda$	1	ρ	$1-\rho$	1

Definition: A behavior strategy profile δ is *sequentially rational* given a system of beliefs μ if for all h the actions for $\varphi(h)$ at $h \cup [\cup_{t \in h} S(t)]$ are optimal starting from h given an initial probability over h governed by μ, and given that other players will adhere to δ_{-i}.

Definition: (δ^*, μ^*) is a *weak perfect Bayesian equilibrium* $(WPBE)$ iff

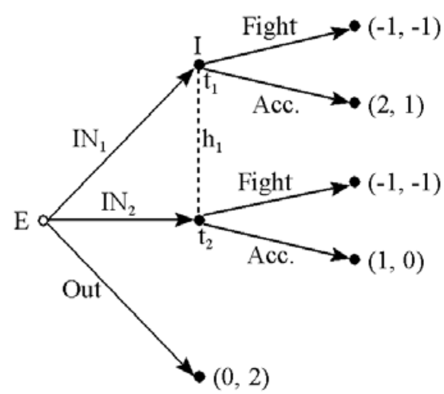
(i) the behavior strategy profile δ^* is sequentially rational given μ^*, and

(ii) where possible, μ^* is computed from δ^* using Bayes rule. That is, for any information set h with $\text{Prob}(h \mid \delta^*) > 0$ and any $t \in h$,

$$\mu^*(t) = \frac{\text{Prob}(t \mid \delta^*)}{\text{Prob}(h \mid \delta^*)}$$

Remark: $WPBE$ places minimal possible restrictions on out-of-equilibrium beliefs (it merely requires that they exist).

Return to example #2: Consider the Nash equilibrium (Out, $Fight$). Can we supplement this with a system of beliefs for which the strategy profile is sequentially rational?



System of beliefs consists of a single parameter, λ, such that $\mu(t_1) = \lambda$ and $\mu(t_2) = 1 - \lambda$.

For the proposed equilibrium, $\text{Prob}(h_1 \mid \delta^*) = 0$, so we cannot compute λ from Bayes rule. We are therefore allowed to pick any value of $\lambda \in [0, 1]$.

Note, however, that for all $\lambda \in [0, 1]$, I's optimal choice at h_1 is "*Accomodate*." Hence, the equilibrium strategies are not sequentially rational for any beliefs. This is not a weak perfect Bayesian equilibrium.

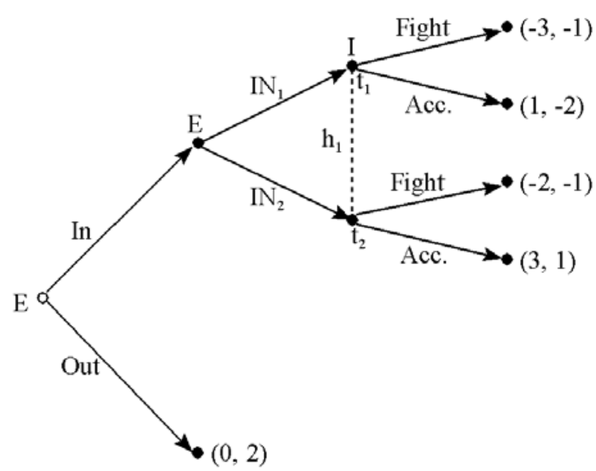
Now consider the Nash equilbrium $(IN_1, \textit{Accomodate})$. For this equilibrium, $\text{Prob}(h_1 \mid \delta^*) = 1$, so we can compute λ from Bayes rule:

$$\mu(t_1) = \text{Prob}(t_1 \mid h_1) = \frac{\text{Prob}(t_1 \mid \delta^*)}{\text{Prob}(h_1 \mid \delta^*)} = \frac{1}{1} = 1$$

Likewise, $\mu(t_2) = 0$. Note that *Accomodate* is optimal given these beliefs. This is a weak perfect Bayesian equilibrium.

Notes: (i) A $SPNE$ need not be a $WPBE$. (ii) A $WPBE$ need not be a $SPNE$.

Example #3: Another variant of the entry game



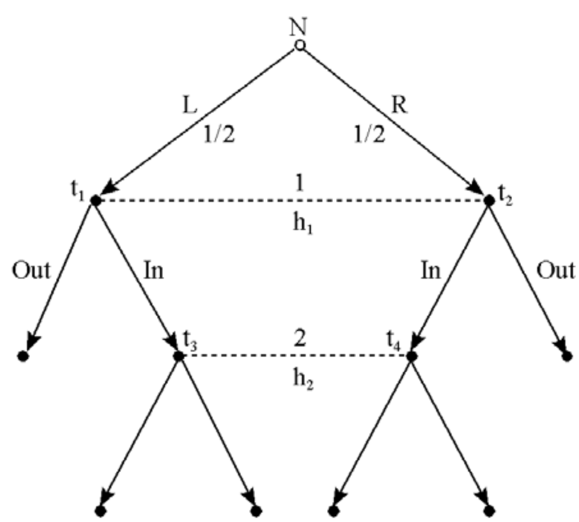
A $WPBE$: E plays (Out, IN_2 if IN), I plays $Fight$ (if h_1 is reached), I's beliefs at h_1: $\mu(t_1) = 1$, $\mu(t_2) = 0$.

This equilibrium is not, however, subgame perfect. There is one proper subgame, and (IN_2, *Accomodate*) is the equilibrium for this subgame. Consequently, E should play In at the root node. This is a $SPNE$, and it is also a $WPBE$.

The first $WPBE$ is not reasonable. If E plays In, then E plainly prefers IN_2. Yet if I observes that E has played In, I infers that E has played IN_1. These beliefs seem inconsistent with E's incentives. $WPBE$ does not rule them out, however, because they are off the equilibrium path.

Remark: One can strengthen $WPBE$ in a variety of ways. For example, one can define a *perfect Bayesian equilibrium* (PBE) as a $WPBE$ that is also a $WPBE$ in all proper subgames. This assures subgame perfect. However, it still doesn't appear to place sufficient restrictions on out-of-equilibrium beliefs.

Example #4: Another case for which $WPBE$ allows peculiar beliefs:



Suppose we have an equilibrium in which 1 played *Out* at h_1. Then, to make this a $WPBE$, we are allowed to pick any beliefs for 2 at the information set h_2, e.g. including $\mu(t_3) = 0.9$, and $\mu(t_4) = 0.1$. But it is clear that the only sensible belief is $\mu(t_3) = \mu(t_4) = 0.5$ (nature picks L and R with equal probabilities, and 1's choice cannot depend on N's choice).

4.3 Sequential equilibrium

Proposed criterion for restricting beliefs: Out-of-equilibrium beliefs must be consistent with some strategies that actually reach all information sets with strictly positive probability, and aren't very different from the equilibrium strategies.

Definition: A behavior strategy profile δ is *strictly mixed* if every action at every information set is selected with strictly positive probability.

Note: For a strictly mixed behavior strategy profile δ, every information set is reached with strictly positive probability. Consequently, one can completely infer a system of beliefs, μ, from δ using Bayes rule. Let M denote the mapping from strictly mixed behavior strategies to systems of belief.

Definition: (δ, μ) is *consistent* iff there exists a sequence of strictly mixed behavior strategy profiles $\delta_n \to \delta$ such that $\mu_n \to \mu$, where $\mu_n = M(\delta_n)$.

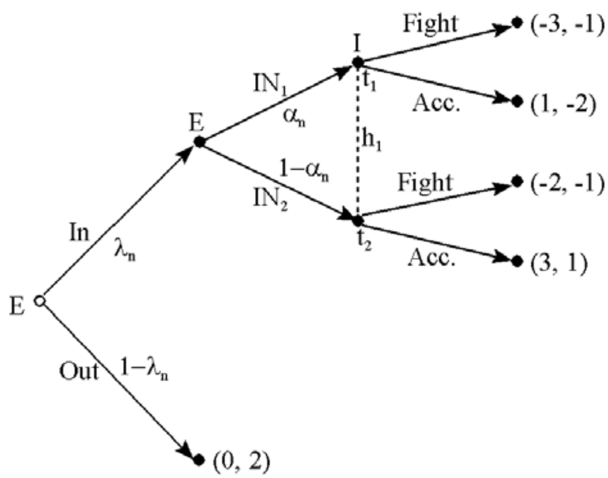
Definition: (δ^*, μ^*) is a *sequential equilibrium* (SE) iff it is sequentially rational and consistent.

Remark: Compared to $WPBE$, SE places an additional restriction on beliefs; hence $SE \subseteq WPBE$. It is also possible to show that a sequential equilibrium is subgame perfect; since $SE \subseteq WPBE$, an SE is also a PBE. Thus, we have

$$SE \subseteq PBE \subseteq \left\{ \begin{array}{c} WPBE \\ SPNE \end{array} \right\} \subseteq NE$$

Return to example #3: Recall the undesirable equilibrium: E plays $(Out,\ IN_2 \text{ if } IN)$, I plays $Fight$ (if h_1 is reached), I's beliefs at h_1: $\mu(t_1) = 1$, $\mu(t_2) = 0$.

Are these beliefs consistent? Consider a sequence of completely mixed behavior strategies, as illustrated below, with $\lambda_n \to 0$ and $\alpha_n \to 0$ (so that the sequence converges to the equilibrium strategy for E).

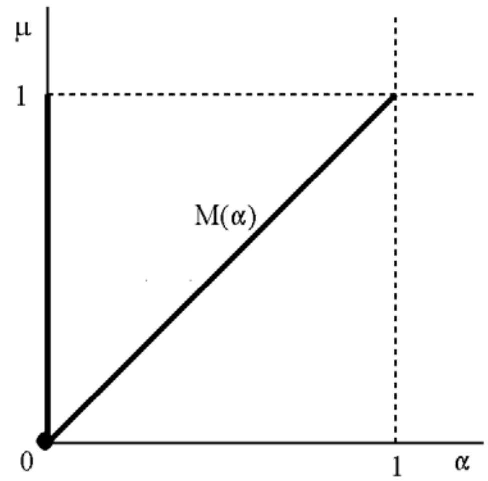


Apply Bayes rule for any element of this sequence:

$$\text{Prob}(t_1 \mid h_1) = \frac{\lambda_n \alpha_n}{\lambda_n} = \alpha_n$$

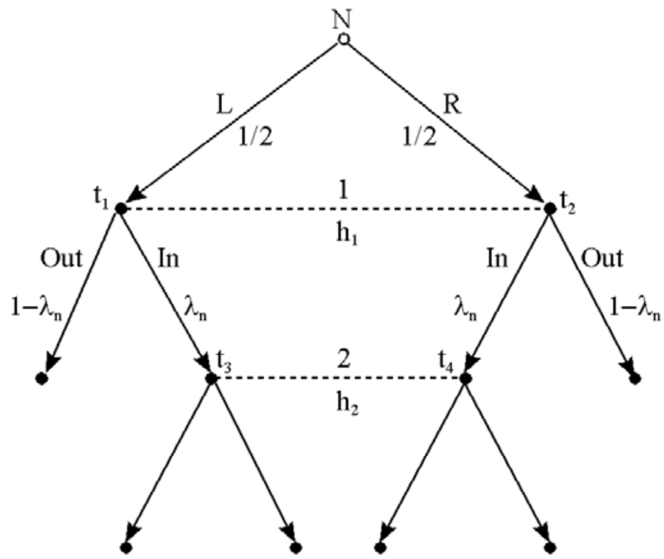
Plainly, this converges to 0 as $n \to \infty$. But then, to satisfy consistency, we must have $\mu^*(t_1) = 0$, $\mu^*(t_2) = 1$. The proposed beliefs violate this condition. Hence, the equilibrium is not sequential.

Graphical interpretation:



Return to example #4: Recall the problemmatic beliefs: player 1 selects "*Out*," and player 2 makes some inference $\mu(t_3) \neq 0.5$.

Are these beliefs consistent? Consider a sequence of completely mixed behavior strategies, as illustrated below, with $\lambda_n \to 0$ (so that the sequence converges to the equilibrium strategy for 1).

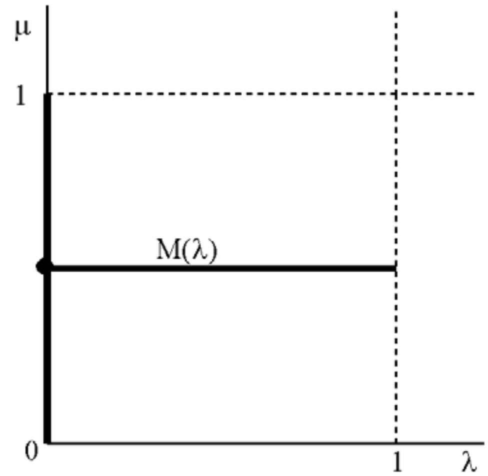


Apply Bayes rule for any element of this sequence:

$$\text{Prob}(t_3 \mid h_2) = \frac{0.5\lambda_n}{0.5\lambda_n + 0.5\lambda_n} = 0.5$$

Plainly, this converges to 0.5 as $n \to \infty$. But then, to satisfy consistency, we must have $\mu^*(t_3) = \mu^*(t_4) = 0.5$. The proposed beliefs violate this condition. Hence, the associated equilibrium cannot be sequential.

Graphical interpretation:



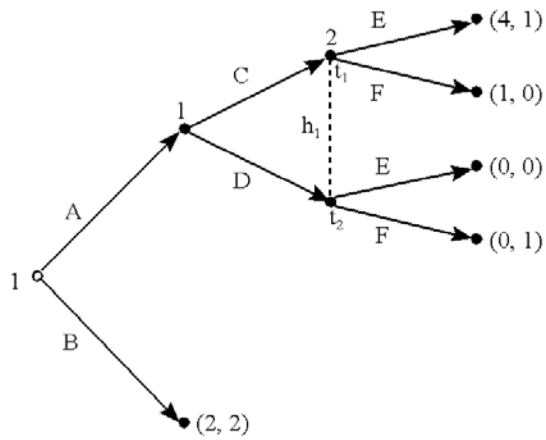
In each case, consistency requires us to take a continuous selection from the correspondence $M(\delta)$ (more generally, we insist on a selection that makes this correspondence lower hemicontinuous).

4.4 Trembling-hand perfection (extensive form)

Motivation: If players make all conceivable mistakes with strictly positive probability, then all information sets will be reached with positive probability, and there won't be any need to think about beliefs off the equilibrium path.

Problem: Normal form trembling hand perfect equilibria need not be subgame perfect. One can demonstrate this in two player games, where normal form trembling hand perfection is equivalent to selecting Nash equilibria in which no player chooses a weakly dominated strategy. Avoiding weakly dominated strategies in the normal form is not sufficient to guarantee subgame perfection.

Example #5:



		P2	
		E	F
P1	A,C	4,1	1,0
	A,D	0,0	0,1
	B,C	2,2	2,2
	B,D	2,2	2,2

Three Nash equilibria: $((A, C), E)$, $((B, C), F)$, and $((B, D), F)$.

There is one proper subgame, and only one Nash equilibrium for this subgame: (C, E). Consequently, $((A, C), E)$ is the only subgame perfect equilibrium.

Neither player has a weakly dominated strategy in any of the three Nash equilibria. Consequently, all three equilibria are normal-form trembling-hand perfect.

Why $((B, C), F)$, and $((B, D), F)$ survive: trembles can place more weight on A, D than on A, C. This is peculiar since, having played A, it is obviously better for player 1 to select C rather than D.

Note: If one first eliminates dominated strategies (A, D) and then looks for equilibria in strategies that are not weakly dominated, one isolates $((A, C), E)$.

Extensive form trembling-hand perfection: The idea is to have players tremble independently among all actions at each information set.

Formalization: The *agent normal form* of an extensive form game is the normal form that one would obtain if each player selected a different agent to make her decisions at every information set, and if all of the player's agents acted independently, with the object of maximizing the original player's payoff.

For the game just considered, the agent normal form is as follows, where player 1a picks rows, player 1b picks columns, and player 2 picks boxes:

E

	C	D
A	4,4,1	0,0,0
B	2,2,2	2,2,2

F

	C	D
A	1,1,0	0,0,1
B	2,2,2	2,2,2

Definition: A Nash equilibrium is *extensive form trembling-hand perfect* $(EFTHP)$ iff it is normal form trembling-hand perfect in the agent normal form of the same game.

For the game just considered, the only extensive form trembling-hand perfect equilibrium is $((A, C), E)$. One can see this directly: if player 1a plays A with positive probability, then player 1b will necessarily prefer C to D.

Relation between $EFTHP$ **and** SE**:** (i) Both use trembles in the extensive form to derive implications for information sets off the equilibrium path

(ii) SE is easier to compute, as it requires one only to think about best responses to the limiting strategies, and not along the entire sequence of strategies

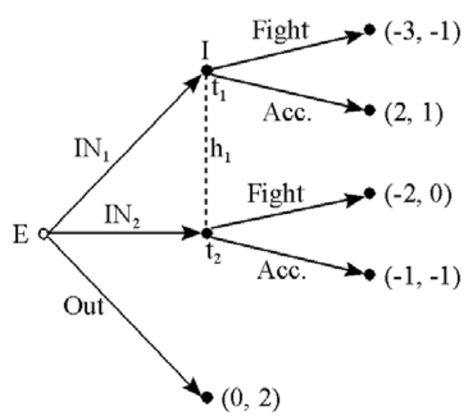
(iii) SE entails an explicit treatment of beliefs, whereas $EFTHP$ does not

(iv) $EFTHP \subseteq SE$

(v) For generic finite games, $EFTHP = SE$

4.5 Motivation for additional refinements

Example #6: Modify example #2 as follows:



Can verify that (Out, $Fight$) and (IN_1, *Accomodate*) are both still Nash equilibria.

Both are also sequential equilibria.

This is obvious for the case of (IN_1, *Accomodate*), since beliefs are implied by the strategies.

For (Out, $Fight$), consider beliefs of the form $\mu(t_1) = 0.25$, $\mu(t_2) = 0.75$. It is easy to check that $Fight$ is optimal given these beliefs. Moreover, these beliefs are consistent with the equilibrium strategy. To see this, consider the following sequence of strategies for E:

$$\begin{aligned}
\text{Prob}(Out) &= 1 - \lambda_n \\
\text{Prob}(IN_1) &= 0.25\lambda_n \\
\text{Prob}(IN_2) &= 0.75\lambda_n
\end{aligned}$$

Note that, as required,

$$\text{Prob}(t_1 \mid h_1) = \frac{0.25\lambda_n}{0.25\lambda_n + 0.75\lambda_n} = 0.25$$

This is not a reasonable equilibrium. E can assure himself of 0 by playing Out. E can get at most -1 by playing IN_2. Therefore, he would never play IN_2. On the other hand, E might get as much as $+2$ by playing IN_1, which exceeds what he could get by playing Out. This means that E could conceivably justify playing IN_1. Thus, if I finds herself at information set h_1, she should conclude that E has played IN_1, with the expectation that I will play *Accomodate*. Given this conclusion, it is in fact optimal for I to play *Accomodate*.

This is an example of a "forward induction" argument. We will see more of this when we come to dynamic games of incomplete information.

5 Dynamic Games with Complete Information: Applications

5.1 Leader-follower problems

5.1.1 Sequential quantity competition (the Stackelberg model)

Framework

Industry consists of two firms producing a homogeneous product

Firm 1 chooses q_1, then firm 2 chooses q_2 having observed q_1

Price is given by $P(q_1 + q_2)$

Costs for firm i are given by $c_i(q_i)$

Profits for firm i are given by $\pi_i(q_1, q_2) = P(q_1 + q_2)q_i - c_i(q_i)$

Strategy sets

Firm 1 selects $q_1 \in \Re_+$

Firm 2 selects $Q_2 : \Re_+ \to \Re_+$

Payoff mapping $g_i(q_1, Q_2) = \pi_i(q_1, Q_2(q_1))$

Nash equilibria

A (pure strategy) Nash equilibrium consists of (q_1^*, Q_2^*) such that:

(i) q_1^* solves $\max_{q_1} g_1(q_1, Q_2^*)$

(ii) Q_2^* solves $\max_{Q_2} g_2(q_1^*, Q_2)$

These conditions are equivalent to:

(i)$'$ q_1^* solves $\max_{q_1} \pi_1(q_1, Q_2^*(q_1))$

(ii)′ $Q_2^*(q_1^*)$ solves $\max_{q_2} \pi_2(q_1^*, q_2)$

Condition (ii)′ is in turn equivalent to

(ii)″ $Q_2^*(q_1^*) \in \gamma_2(q_1^*)$, where γ_2 is the Cournot best response correspondence

Note that condition (ii)″ only restricts the value of the function Q_2^* evaluated at the point q_1^*. To sustain a Nash equilibrium, one is free to select any value for $Q_2^*(q_1)$ when $q_1 \neq q_1^*$. In particular, one can use:

$$Q_2^*(q_1) = \begin{cases} \gamma_2(q_1^*) \text{ when } q_1 = q_1^* \\ +\infty \text{ otherwise} \end{cases}$$

In this way, one can sustain as an outcome almost any point along firm 2's best response function. However, firm 2's responses to quantities other than q_1^* are not credible. This is because we have not insisted on subgame perfection.

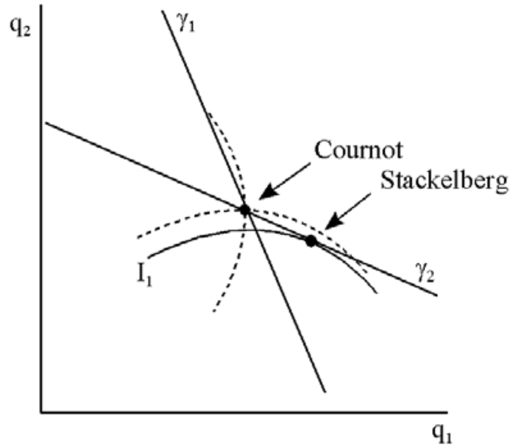
Subgame perfect Nash equilibrium

Every choice of q_1 defines a separate subgame

An equilibrium in any such subgame entails firm 2 making a best response to the choice of firm 1, q_1, that defines the subgame. That is, firm 2 must select $\gamma_2(q_1)$ in response to every value of q_1. In other words, subgame perfection requires $Q_2^*(q_1) = \gamma_2(q_1)$.

To have a Nash equilibrium, condition (i) (or equivalently condition (i)′) must still hold. Consequently, q_1^* solves $\max_{q_1} \pi_1(q_1, \gamma_2(q_1))$.

Graphically:



This is known as the *Stackelberg equilibrium*.

Compared to the Cournot equilibrium, firm 1 produces more (is more aggressive), and firm 2 produces less (is less aggressive).

Firm 1 (the leader) does (i) better than in a simultaneous move setting, and (ii) better than firm 2.

Firm 2 (the follower) does worse than in a simultaneous move setting.

This game gives rise to a *first mover advantage.*

5.1.2 Sequential price competition

Framework

Industry consists of two firms

Consider the cases of homogeneous and heterogeneous products separately

Firm 1 chooses p_1, then firm 2 chooses p_2 having observed p_1

Sales for firm i are given by $Q_i(p_1, p_2)$

Costs for firm i are given by cq_i

Profits for firm i are given by $\pi_i(p_1, p_2) = Q_i(p_1, p_2)\,(p_i - c)$

Strategy sets

Firm 1 selects $p_1 \in \Re_+$

Firm 2 selects $P_2 : \Re_+ \rightarrow \Re_+$

Payoff mapping $g_i(p_1, P_2) = \pi_i(p_1, P_2(p_1))$

Remark: We will move directly to subgame perfect equilibria, skipping Nash equilibria involving non-credible out-of-equilibrium responses.

Homogeneous products, version 1: When indifferent, consumers split equally between the firms.

Notice that firm 2's best response correspondence, $\gamma_2(p_1)$, is not well defined for all p_1. This implies that there are always subgames in which Nash equilibrium do not exist. Consequently, subgame perfect equilibria do not exist.

Homogeneous products, version 2: When indifferent, consumers resolve indifference in favor of firm 2. Assume also that profits are strictly increasing in price over $[c, p^m]$, where p^m is the monopoly price.

For this case, $P_2^*(p_1) = \gamma_2(p_1) = \max\{c, \min\{p_1, p^m\}\}$

Note that $g_1(p_1, P_2^*) = 0$ for all $p_1 \geq c$. Consequently, firm 1 is completely indifferent with respect to all prices not below costs.

Implication: any (p_1, P_2^*) with $p_1 \geq c$ is a subgame perfect equilibrium. Consequently, the equilibrium price can be any $p \in [c, p^m]$.

Note: it is not possible to eliminate any of these outcomes through dominance arguments.

Question: what would the equilibria be with a discretized price space?

Compared to the Bertrand equilibrium, firm 1 sets a (weakly) higher price (is less aggressive), and firm 2 sets a (weakly) higher price (is less aggressive).

Firm 1 (the leader) does (i) the same as in a simultaneous move setting, and (ii) (weakly) worse than firm 2.

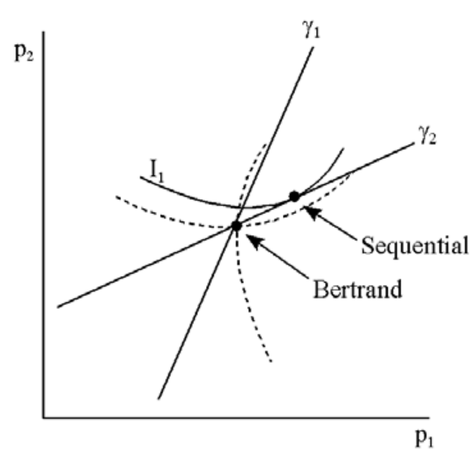
Firm 2 (the follower) does (weakly) better than in a simultaneous move setting.

Here, there is a *second mover advantage*.

Heterogeneous products

The analysis is the same as with Stackelberg, except with Bertrand best responses instead of Cournot.

Graphically:



Compared to the Bertrand equilibrium, firm 1 sets a higher price (is less aggressive), and firm 2 sets a higher price (is less aggressive).

Assuming symmetry, firm 1 (the leader) does (i) better than in a simultaneous move setting, and (ii) worse than firm 2 (firm 1 sets the higher price, and firm 2 makes a best response)

Firm 2 (the follower) does better than in a simultaneous move setting.

Again, there is a *second mover advantage.*

5.1.3 Lessons concerning pure strategy equilibria in leader-follower games

1. The leader always does (weakly) better than in a simultaneous move setting. (Note that this is not necessarily true for mixed strategy equilibria. MP-B is an example.)

2. Whether firm 2 does better or worse than in a simultaneous move settings depends on whether the model exhibits strategic substitutes or strategic complements.

3. Whether there is a first or second mover advantage (in terms of which player does better in symmetric models) is related to whether the model exhibits strategic substitutes or strategic complements.

5.2 Price competition with endogenous capacity constraints

Framework

Industry consists of 2 firms producing a homogeneous good

Demand: $Q(p)$, downward sloping

Inverse demand: $P(q)$

Competition unfolds in two stages

Stage 1: Firms simultaneously select capacities, K_i, at a cost per unit of c.

Stage 2: Having observed each others' capacities, firms select prices simultaneously. Customers attempt to purchase the good from the firm naming the lowest price. However, firm i can only sell quantity up to capacity K_i. Consumers who are unable to purchase the good from the low-price firm may purchase it (if available) from the high-price firm.

Note: With downward sloping demand, consumers implicitly differ in their valuations of the good (or they have different valuations for different units). Consequently, the quantity sold by the high-price firm will depend upon which consumers are turned away from the low-price firm.

Assumption: Consumers are rationed to maximize total surplus (efficient rationing). High value consumers purchase from the low-price firm (they are the most eager), and low value consumers purchase from the high-price firm. Thus:

$$\text{If } p_i > p_j \Longrightarrow \begin{cases} q_j = \min\{K_j, Q(p_j)\} \\ q_i = \min\{K_i, \max\{Q(p_i) - K_j, 0\}\} \end{cases}$$

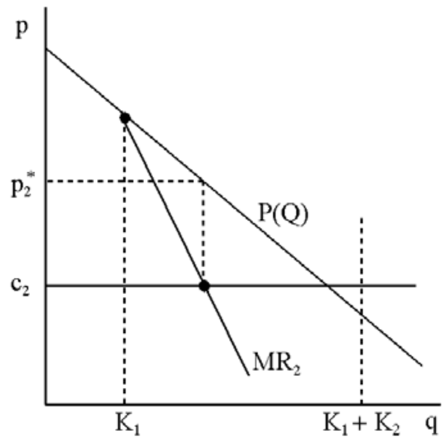
$$\text{If } p_i = p_j \Longrightarrow q_i = \min\left\{K_i, \max\left\{\frac{Q(p_i)}{2}, Q(p_i) - K_j\right\}\right\}$$

In other words, if there is a tie, each firm has a "claim" on half of the market, but can also serve any customers turned away by its rival.

Let c_2 denote the unit cost of production in stage 2.

Review

If either firm has chosen $K_i < Q(c_2)$ in stage 1, $p = c_2$ is no longer an equilibrium for stage 2. Graphically (for the case of downward sloping demand):



When capacity constraints are binding, pure strategy Nash equilibria frequently do not exist. The reason: $g_i(p_1, p_2)$ is discontinuous at $p_1 = p_2 > c_2$.

However, in such cases, there are generally mixed strategy equilibria. Consequently, allowing for either pure or mixed strategy equilibria in subgames, we can solve for subgame perfect Nash equilibria in the two-stage game with endogenous capacity.

Analytic preliminaries

Henceforth, for notational simplicity, assume $c_2 = 0$. This is just a normalization.

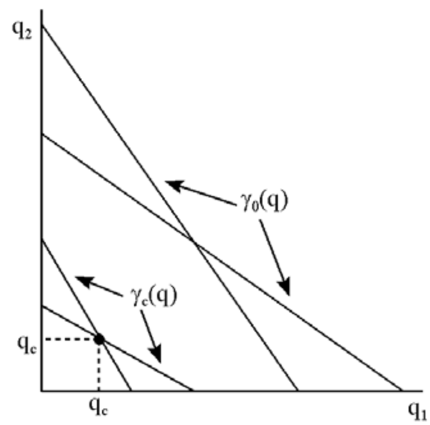
Some definitions:

$\gamma_c(q)$ denotes the Cournot best response function with constant unit costs c, i.e. $\arg\max_{q'} \left(P(q+q') - c\right) q'$

Let q_c denote the Cournot equilibrium quantity with constant unit costs c.

$\gamma_0(q)$ denotes the Cournot best response with zero unit costs, i.e. $\arg\max_{q'} P(q+q')q'$

Graphically:

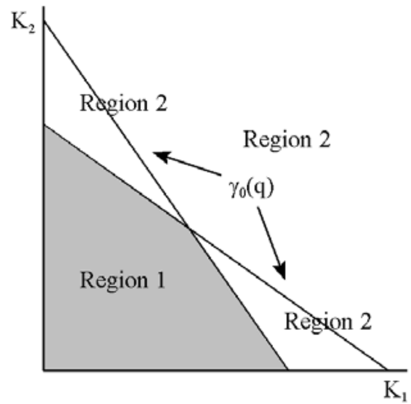


Theorem: (q_c, q_c) followed by $(P(2q_c), P(2q_c))$ occurs on the equilibrium path of a subgame perfect Nash equilibrium. Furthermore, it is the unique SPNE outcome of this game.

Interpretation: If capacity is a not easily changed in the short run but price is, the outcome will be Cournot, not Bertrand.

Partial demonstration: We will demonstrate the existence of an equilibrium with the indicated outcome under certain conditions, and discuss the existence of such an equilibrium more generally. We will not prove uniqueness (see Kreps and Scheinkman for the complete argument).

Divide the capacity space into the following two regions.



In region 1, both firms have chosen capacity less than or equal to its zero-cost best response quantity. In region 2, at least one firm has chosen capacity greater than their zero-cost best response quantities.

Each point in the non-negative quadrant of the capacity plane defines a distinct subgame. We need to know what happens in each of those subgames.

Claim: For any subgame following capacity choices (K_1, K_2) in region 1, there is an equilibrium for which both firms name price $P(K_1 + K_2)$ with probability 1, and each sells exactly K_i (in fact, this is the unique equilibrium).

Proof of the claim: Suppose that firm j selects $p_j = p^* \equiv P(K_1 + K_2)$. We will show that $p_i = p^*$ is firm i's best repsonse.

Selecting $p_i = p^*$, firm i sells K_i and earns profits of $K_i p^*$. For any $p_i < p^*$, firm i's quantity would also be equal to its capacity, and its profits would be $K_i p_i < K_i p^*$. Consequently, $p_i < p^*$ is less profitable that $p_i = p^*$.

Now consider firm i's best choice among prices satisfying $p_i \geq p^*$. Firm i's profits are then given by

$$[Q(p_i) - K_j]p_i$$

Consider the following change of variables: $q_i \equiv Q(p_i) - K_j$. Note that $p_i = Q^{-1}(q_i + K_j) = P(q_i + K_j)$. Substituting this into the expression for i's profits, we have

$$q_i P(q_i + K_j)$$

Moreover, substituting into the constraint $p_i \geq p^*$ yields $P(q_i + K_i) \geq P(K_i + K_j)$, or $q_i \leq K_i$. Thus, firm i's best choice solves

$$\max_{K_i \geq q_i \geq 0} q_i P(q_i + K_j)$$

Ignoring the constraint that $K_i \geq q_i$, we know that firm i's best deviation would entail $q_i = \gamma_0(K_j)$. But, since we are in region 1, we also know that $K_i \leq \gamma_0(K_j)$. Thus, the best choice for firm i is $q_i = K_i$. Changing variables back to prices, we see that the best choice for i satisfies $p_i = P(K_i + K_j)$, which is what we set out to prove.

Remark: This argument, by itself, proves the theorem for a class of games. Consider what happens when c gets large:

(i) γ_c shifts toward the origin while γ_0 stays put.

(ii) We can rule out larger and larger sets of possible first stage choices through dominance. Specifically, let $\pi^m \equiv \max_q qP(q)$ (monopoly profits assuming zero costs). Plainly, firm i will never choose any K_i such that $cK_i > \pi^m$. Consequently, we can rule out all $K_i > \pi^m/c$. As c rises, this rules out more capacity choices.

When c is sufficiently large, all of region 2 is ruled out by dominance. Consequently, we know that the game is played entirely within region 1. But we have just seen that, for region 1, any pair of choices (K_1, K_2) yields an equilibrium with price $P(K_1 + K_2)$, and each firm sells exactly K_i. In other words, once we substitute for the second stage outcome, the first stage is just Cournot. This game is said to have an *exact Cournot reduced form*; its outcome must be the Cournot quantities, (q_c, q_c).

Of course, for lower values of c, one must explicitly consider the possibility of ending up in region 2. The analysis is more difficult.

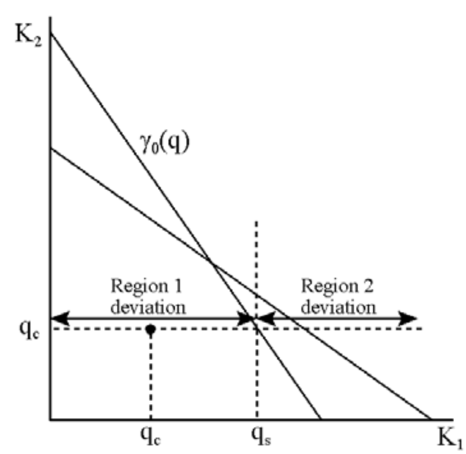
Discussion of region 2: One can demonstrate the following:

(i) For all capacity pairs in region 2, there exists a mixed strategy equilibrium (possibly pure)

(ii) Expected profits for firm i are decreasing in K_i within region 2.

(iii) Expected profits change continuously as one moves across the boundary from regime 1 to regime 2.

The first half of the theorem (existence of a Cournot-equivalent SPNE) follows directly from these properties. Graphically:



We already know that deviations within region 1 ($q_1 \leq q_s$) do not make firm 1 better off. Since expected profits for firm 1 are decreasing in K_1 within region 2, and since expected profits change continuously as one moves across the boundary from regime 1 to regime 2, any $q_1 > q_s$ yields strictly lower profits than q_s. Therefore, q_c is the optimal response to q_c.

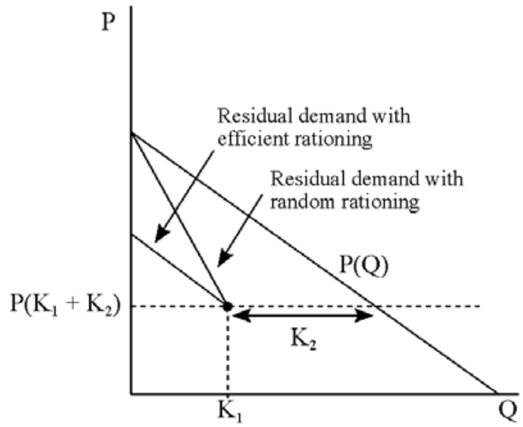
Note: This game (unlike the one with high c after deleting dominated strategies) does not have an exact Cournot reduced form. Choices of capacity in region 2 do not yield reduced form Cournot payoffs. However, the game nevertheless gives rise to the Cournot outcome.

Remarks concerning the theorem

1. It is somewhat fragile with respect to the rationing rule.

Consider the case of random rationing (high value consumers are no more likely than anyone else to buy from the low-price firm). This is considered in Davidson and Deneckere, *RAND Journal of Economics*, 1986.

The residual demand curve is steeper because, even with a higher price, a firm still attracts some of the high value customers who are not as discouraged by the high price. Graphically:



Steeper residual demand implies that there is a greater benefit to raising price. This can destroy the original equilibrium

However, even with this modification, the model yields: (i) equilibria with strictly positive profits, and (ii) downward sloping first-stage reduced form reaction functions (that is, strategic substitutabilities). Thus, the model retains the essential features of the Cournot solution.

2. It is somewhat fragile with respect to observability.

Suppose that firm cannot observe each others' capacities. This is equivalent to saying that they choose capacity and price simultaneously (even though capacity is a longer-run decision). In that case, one can show that the equilibria yield zero profits, like Bertrand.

3. Caution: although this is a justification for studying Cournot equilibria, the justification is specific to this particular model. This is not a general justification for Cournot.

5.3 Price competition with endogenous product characteristics

Framework:

Two firms

Product characteristic is measured by some variable x

Product space is the line segment $[0, 1]$

Product characteristics: a for firm 1, $1 - b$ for firm 2

Consumers are indexed by $\theta \in [0, 1]$

Each consumer purchases either one unit, or nothing

Payoff for a type θ consumer: 0 if no purchase, $v - p - t(x - \theta)^2$ if purchases a type x good at price p.

Population distribution of θ is uniform over $[0, 1]$

Timing of decisions:

Stage 1: The firms simultaneously select product characteristics a and b (by convention, we can always label the firms so that $a \leq 1 - b$).

Stage 2: Having observed a and b, the firms simultaneously select prices. Firms produce output to meet demand at a cost of c per unit.

Analysis:

Each pair (a, b) defines a separate subgame.

We have previously solved for equilibrium conditional upon fixed (a, b). This is the equilibrium that will prevail in each subgame.

From our previous analysis, we know that, for any given vector (p_1, p_2, a, b), firm 1's profits are given by

$$\pi^1(p_1, p_2, a, b) = (p_1 - c)\widehat{\theta}(p_1, p_2, a, b, t)$$

where

$$\widehat{\theta}(p_1, p_2, a, b, t) = \frac{p_1 - p_2}{2t(1-a-b)} + \frac{1+a-b}{2}$$

Moreover, we know that, in equilibrium,

$$\begin{aligned} p_1^* &= c + t(1-a-b)\left(1 + \frac{a-b}{3}\right) \\ p_2^* &= c + t(1-a-b)\left(1 + \frac{b-a}{3}\right) \end{aligned}$$

Now let's consider firm 1's optimal choice of a in stage 1. Note that

$$\frac{d\pi^1}{da} = \frac{\partial \pi^1}{\partial p_1}\frac{dp_1^*}{da} + \frac{\partial \pi^1}{\partial p_2}\frac{dp_2^*}{da} + \frac{\partial \pi^1}{\partial a}$$

Firm 1's first order condition tells us that $\frac{\partial \pi^1}{\partial p_1} = 0$ when evaluated at p_1^*. One can evaluate the last two terms in the preceding expression using the formulas reproduced above. After some algebra, one obtains:

$$\frac{d\pi^1}{da} = -\left(p_1^* - c\right)\left[\frac{3a+b+1}{6(1-a-b)}\right] < 0$$

With firm 1 to the left of firm 2 ($a < 1-b$), this implies that firm 1 wishes to reduce a (move further to the left). Consequently, firm 1's optimal choice in stage 1 is to set $a = 0$, regardless of b. A symmetric argument implies that firm 2 sets $b = 1$. Thus, the equilibrium involves the maximum possible product differentiation.

Welfare analysis: It is easy to check that $(a, b) = (\frac{1}{4}, \frac{3}{4})$ maximizes social surplus. Conclude that competition produces excessive product differentiation.

Remarks:

(i) Price competition plays a central role in producing this result. Price competition becomes less severe when the firms are positioned further away from each other. Firms endogenously select a highly differentiated configuration to minimize price competition. The associated prediction is that firms will seek niches in markets.

Illustration of the role of price competition: Suppose that the government regulates the industry, setting a price p such that $v - p > t$ (all buyers are willing to travel the length of the interval to purchase the good). This is a pure location problem. There is a unique equilibrium: $a = 1 - b = \frac{1}{2}$ (no product differentiation).

(ii) The differentiation result depends upon functional form assumptions.

Exercise: Solve the two-stage model for the case where transportation costs are given by $t\,|x - \theta|$.

5.4 Entry

5.4.1 Homogeneous goods

Demand $Q(p)$, inverse demand $P(q)$

There is a very large number of identical potential firms.

Production costs for each firm: $c(q)$

Entry with sunk costs

Stage 1: Each potential firm decides whether it is *in* the industry, or *out* of the industry. If it chooses to be *in* the industry, it pays a cost $K > 0$.

Stage 2: Having observed the number of actual competitors, firms play some market game (e.g. Cournot or Bertrand).

From the perspective of stage 2, the entry cost is known as a *sunk cost*. The distinguishing features of a sunk cost are: (1) it is incurred before the commercial success or failure of the enterprise is established, and (2) it is non-recoverable.

SPNE: This game has a finite number of stages, so we solve it by backward induction. Let $\pi(N)$ denote the (expected) profits to each firm in the equilibrium that prevails with N active firms (assume uniqueness for simplicity).

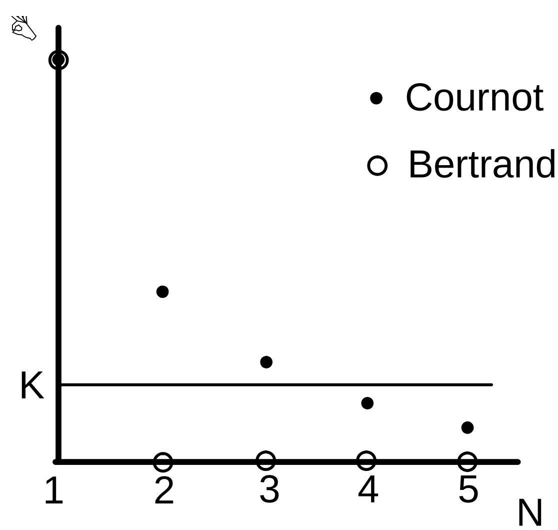
Now consider the first stage. SPNE requires N^e firms to elect in, where

$$\pi(N^e) \geq K$$

and

$$\pi(N^e + 1) \leq K$$

Solving for SPNE graphically in the case of Cournot and Bertrand:



Remarks

(i) $N^e = 1$ for Bertrand, which gives rise to the monopoly outcome for all $K > 0$.

(ii) As long as $\pi(N)$ is decreasing in N, a decline in K (weakly) increases the number of firms.

(iii) As long as $\pi(N) > 0$ for all N, $\lim_{K \to 0} N^e = +\infty$.

This provides a foundation for various results concerning convergence to competitive equilibrium.

Example: Cournot. Recall the condition for aggregate equilibrium quantity in the symmetric case:

$$P(Q) = c'\left(\frac{Q}{N^e}\right) - \frac{1}{N^e}P'(Q)Q$$

The second term vanishes as N goes to infinity, so price converges to marginal cost.

Similarly, one can hold K fixed and let the market get large: $Q(p) = \alpha Q^0(p)$, with $\alpha \to \infty$.

Note: The convergence result does not hold for Bertrand competition.

(iv) Suppose that π is also a function of some parameter θ, $\pi(N, \theta)$. Suppose that π is strictly increasing in θ for all N. Then an increase in θ weakly increases N^e.

Property (iv), though rather obvious, leads to some surprising results.

Suppose that θ indexes demand – that an increase in θ shifts the demand curve out without changing the elasticity of demand. For a fixed N, this will generally increase π. Thus, N^e will rise with θ. But then, it might well be that price falls. For an example, consider the Cournot game. The equilibrium price-cost margin is $\frac{1}{N\varepsilon}$. This does not change with θ. However, larger θ increases profits and induces more entry. With larger N^e, the equilibrium price-cost margin (and therefore the equilibrium price) is lower. Thus, we have a situation where an increase in demand causes a decrease in price.

Suppose that θ indexes "conduct" – e.g. $\theta = 0$ corresponds to Bertrand, while $\theta = 1$ corresponds to Cournot (higher values of θ are less competitive in the sense that they lead to higher equilibrium profits). A less competitive environment – higher θ – leads to greater entry (higher N^e). With less competitive conduct, one can thereby end up with more competitive solutions (compare the two-stage Cournot and Bertrand outcomes above).

Welfare analysis

Is the level of equilibrium entry efficient? Clearly, it's not efficient in the first-best sense if the continuation equilibrium involves a wedge between price and marginal cost. Is it second-best efficient? That is, if an omniscient planner could specify the number of firms, but could not prescribe prices, how would her choices compare with the equilibrium outcome? (Arguably, this is what merger policy entails.)

The planner's problem:

$$\max_N W(N) \equiv \int_0^{Nq(N)} P(s)ds - Nc(q(N)) - NK$$

Let N^* denote the solution. To keep things simple, as an approximation we take N to be a continuous variable.

Theorem: If, for all N

(i) $\frac{d}{dN}[Nq(N)] > 0$ (aggregate quantity rises with N)

(ii) $\frac{d}{dN}q(N) < 0$ (per-firm quantity falls with N), and

(iii) $P(Nq(N)) - c'(q(N)) > 0$ (price exceeds marginal cost), then

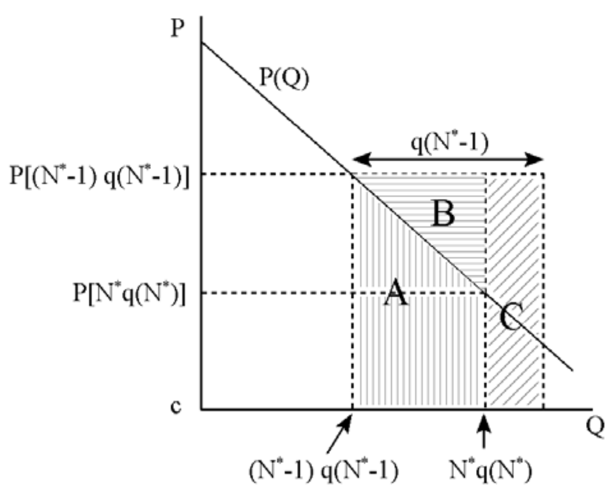
$N^e > N^*$ (entry is socially excessive).

Proof:

$$
\begin{aligned}
W'(N) &= P\left[Nq(N)\right]\left[Nq'(N)+q(N)\right]-c(q(N)) \\
&\quad -Nc'(q(N))q'(N)-K \\
&= \left[\pi(N)-K\right]+N\left[P(Nq(N))-c'(q(N))\right]q'(N) \\
&< \pi(N)-K
\end{aligned}
$$

So if $W'(N^*)=0$, $\pi(N^*)>K$. Since $\pi(N)$ is decreasing in N, and since $\pi(N^e)=K$, $N^e>N^*$. Q.E.D.

Remark: If firms are treated as indivisible, then $N^e \geq N^*-1$. Graphically (normalizing marginal costs at zero for simplicity):



Area A is the gross-of-K social gain from adding the N^*-th firm. Obviously, $K<A<A+B+C$. But $A+B+C=\pi(N^*-1)$, so $\pi(N^*-1)>K$. Consequently, there must be at least N^*-1 firms in equilibrium.

Intuition: Entry is associated with an externality. Firms steal business from each other.

The cost of entry, K, is a social cost, but not all of the gains are social gains – much is just a redistribution of surplus.

Entry with fixed costs

Same setting, but now assume the cost K is incurred in stage 2, and only if the firm produces a strictly positive quantity. (If the entrant actually has to pay K in stage 1, this investment is recoverable.)

Here, the cost of entry is a *fixed* cost of production (it is only incurred if the firm decides to produce something).

Consider the following special case: Bertrand competition with linear variable costs, $c(q) = cq$ (so that total costs are given by $cq + K$ for $q > 0$, and zero for $q = 0$). Assume also that there is some quantity that a monopolist could produce profitably (i.e. the demand curve lies somewhere above the average cost curve).

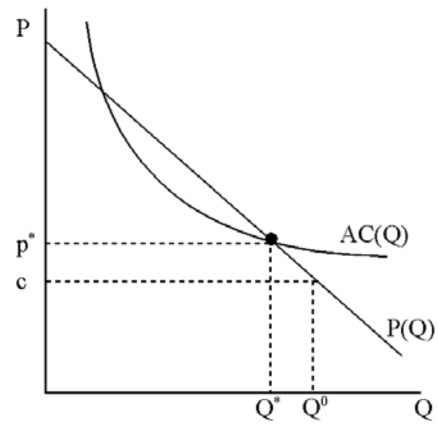
Claim: In equilibrium, at least two firms name

$$p^* = \min \left\{ p \mid p \geq \left[\frac{K + cQ(p)}{Q(p)} \right] \right\}$$

Only one firm produces.

Reason: One cannot have an equilibrium in which any output is sold at a price above p^*; otherwise, a firm could enter, set a lower price, and earn strictly positive profits. Plainly, one cannot have an equilibrium in which output is sold at a lower price, since the seller would be losing money. Finally, if two firms set this price and one subsequently produces all of the output, then both earn zero profits, and no firm can earn strictly positive profits by deviating to any other price.

Graphically:



Interpretation: There is a single firm. It breaks even, setting prices equal to average costs. This is called a *contestable market* (Baumol, Panzar, and Willig).

Comparison with outcome in the presence of sunk entry costs: In both instances, only one firm is active. However, with fixed costs, the active firm earns zero profits. With sunk costs, the active firm earns monopoly profits.

Source of the difference: sinking entry costs alerts the other firm to the presence of a competitor, and the firm can react to entry. With fixed costs, entry is "hit and run" – competitors have no opportunity to react to entry. Entry is more difficult in the case of sunk costs because prices are strategic complements – the ability to react to entry means that the entrant faces a more aggressively competitive environment.

Welfare analysis

Consider the following second-best planning problem:

$$\max_{q,N} W(q,N) \text{ subject to } \pi(q,N) \geq KN$$

Here, the planner can control both the number of firms and their outputs/prices. However, the problem is second-best because the planner must guarantee each firm non-negative profits.

Solution: As long as $P(Q) > c$, it is socially beneficial to increase quantity (marginal social benefits exceed marginal social costs). In a first-best exercise, one would increase quantity to Q^0, but this is ruled out in the second-best problem. The non-negative profits constraint binds at Q^*; hence Q^* is the second-best optimum.

Conclusion: Equilibrium in the entry problem with fixed costs (and no sunk costs) achieves the second-best outcome. The result extends to more complicated settings (e.g. multiproduct firms).

5.4.2 Horizontally differentiated products

Framework:

Product characteristic is measured by some variable x

Product space is the line segment $[0, 1]$

Consumers are indexed by $\theta \in [0, 1]$

Each consumer purchases either one unit, or nothing

Payoff for a type θ consumer: 0 if no purchase, $v - p - t(x - \theta)^2$ if purchases a type x good at price p.

Population distribution of θ is uniform over $[0, 1]$

Potentially infinite number of firms

Timing of decisions:

Stage 1: Firms choose either to stay *out*, or to *enter* with some particular product characteristic, x. Entrants incur the sunk cost K.

Stage 2: Having observed the set of entrants and associated product characteristics, the operating firms simultaneously select prices. Firms produce output to meet demand at a cost of c per unit.

Question: What happens in the limit as K goes to zero? Without product differentiation, we know that we obtain the monopoly outcome for all $K > 0$. Does this change with product differentiation?

Theorem: Consider any interval $[\theta_1, \theta_2]$. Let $r(\theta_1, \theta_2)$ denote the (subgame perfect Nash) equlibrium probability that at least one firm enters with a product characteristic in this interval ($x \in [\theta_1, \theta_2]$). Then $\lim_{K \to 0} r(\theta_1, \theta_2) = 1$.

Remarks:

(i) The theorem applies to all possible equilibria, both pure and mixed (which is why the result is stated in terms of probability). Note that existence is guaranteed.

(ii) The theorem implies that, in the limit, firms blanket the product space. This in turn implies that price converges to c (check the formulas for equilibrium price in the two-firm case – as a approaches $1 - b$, equilibrium price declines to c).

Proof of the theorem: Assume that the theorem is false. Then, for some θ_1, θ_2, and $\varepsilon > 0$, there exists a sequence $K_n \to 0$ such that $\lim_{n \to \infty} r(\theta_1, \theta_2) < 1 - \varepsilon$. For any n, consider a firm that chooses not to enter in equilibrium (thereby earning a profit of zero). Consider the following strategy as a potential deviation:

$$\text{Stage 1: Enter with } x' = \frac{\theta_1 + \theta_2}{2}$$

$$\text{Stage 2: In all subgames, set } p' = c + \frac{t}{2}\left(\frac{\theta_2 - \theta_1}{2}\right)^2$$

We will calculate a lower bound on the payoff resulting from adoption of this strategy.

If at least one other firm enters with product type in $[\theta_1, \theta_2]$, then sales and variable profits for the deviating entrant are non-negative. This occurs with probability no greater than $1 - \varepsilon$.

With probability of at least ε, no other firm enters with product type in $[\theta_1, \theta_2]$. For all such cases, the entrant does at least as well as when other firms enter with product characteristics of both θ_1 and θ_2, in each instance charging $p = c$. We calculate the entrant's variable profits for this worst-case scenario. The entrant will make sales to all customers in the interval $[x' - z, x' + z]$, where z is given by the solution to

$$v - c - t\left(\theta_1 - (x' - z)\right)^2 = v - p' - tz^2$$

Some algebra reveals that

$$z = \frac{\theta_2 - \theta_1}{8}$$

The entrant's variable profits are then given by

$$2z(p' - c) = \frac{t}{32}\left(\theta_2 - \theta_1\right)^3 > 0$$

Consequently, the entrant's expected profits (before observing others' entry and product characteristic decisions) are bounded below by $\varepsilon \frac{t}{32}\left(\theta_2 - \theta_1\right)^3$. For K sufficiently small, the entrant's net expected profits, $\varepsilon \frac{t}{32}\left(\theta_2 - \theta_1\right)^3 - K$, are strictly positive. Since the entrant has a profitable deviation, the system could not have been in equilibrium – a contradiction. Q.E.D.

5.4.3 Vertically differentiated products

Framework:

Product differentiated according to quality, $v \in [v_L, v_H]$

Each consumer purchases either one unit of one of these goods, or nothing.

Consumers are characterized by a preference parameter, θ, which indicates the value attached to quality. If a consumer of type θ purchases a good of quality v at price p, her utility is given by

$$u(\theta, v, p) = \theta v - p$$

If the consumer purchases nothing, her utility is zero.

θ is distributed uniformly on the interval $[\underline{\theta}, \overline{\theta}]$

Potentially infinite number of firms

Timing of decisions:

Stage 1: Firms choose either to stay *out*, or to *enter* with some particular product quality, v. Entrants incur the sunk cost K.

Stage 2: Having observed the set of entrants and associated product qualities, the operating firms simultaneously select prices. Firms produce output to meet demand at a cost of c per unit (costs are invariant with respect to quality).

Question: What happens in the limit as K goes to zero? Without product differentiation, we know that we obtain the monopoly outcome for all $K > 0$. With horizontal product differentiation, one obtains convergence to the competitive outcome. What happens with vertical differentation?

Theorem: Assume that $\overline{\theta} < 2\underline{\theta}$. For all $K > 0$ there exists an $SPNE$ wherein a single firm enters with quality v_H, and earns monopoly profits.

Remarks:

(i) This result coincides with the outcome for no product differentiation, and contrasts with the outcome for horizontal product differentiation. Here, one does not obtain convergence to perfect competition in the limit.

(ii) One can also demonstrate that this is the unique outcome for this game.

Proof:

For convenience, designate the single firm that enters in equilibrium as firm 1. Consider any subgame in which firm 1 enters with $v_1 = v_H$, and some other firm (for convenience, "firm 2") enters with $v_L \leq v_2 \leq v_H$. We claim that there exists a continuation equilibrium in which firm 2 earns zero profits. For $v_2 = v_H$, this follows directly from the usual Bertrand argument (in equilibrium, $p = c$, and neither firm earns variable profits). For $v_2 < v_H$, the claim follows from the theorem presented in section 2.6.2.

Now we prove the theorem by constructing a $SPNE$ with the desired properties. Strategies are as follows. In the first stage, firm 1 enters with quality v_H and all other firms stay out. In the second stage, if firm 1 is the only active firm, it sets the monopoly price (conditional on the quality of its product). If firm 1 is active along with one other firm, and if firm 1's quality is v_H, then the firms play the continuation equilibrium described in the previous paragraph. For any other configuration of entrants and qualities, select some arbitrary continuation equilibrium.

We claim that this is, in fact, a $SPNE$. By construction, strategies give rise to an equilibrium in the second stage for every first stage outcome. Thus, we need only check that first stage choices are optimal. Since firm 1 earns monopoly profits in equilibrium, it is obvious that firm 1 does not have a profitable deviation, regardless of how one selects the continuation outcomes in subgames where $v_1 \neq v_H$. Given firm 1's strategy and the continuation equilibria described above, any other firm would, after entering, earn zero variable profits, irrespective of its product quality. Since

$K > 0$, this implies that the entrant would receive negative profits. Thus, entry is not a profitable deviation. Q.E.D.

How does this generalize?

Two dimensions of generalization: (i) unit production costs are $c(v)$, with $c'(v) > 0$, and (ii) $\overline{\theta} - \underline{\theta}$ may be large.

Consider the following hypothetical: All products are offered, with $p(v) = c(v)$ for all v. What would consumers purchase?

Possibility #1: All consumers purchase the good with quality v_H. (Note: $c(v) = c$, the case considered above, assures this). Then, as $K \to 0$, the number of firms in equilibrium has a finite bound, and firms earn positive profits.

Possibility #2: Consumers with higher values of θ would purchase goods with higher v. Provided $c''(v) > 0$, one can show that, for the hypothetical, the set of purchased qualities will be some interval $[v', v'']$. The resulting behavior then resembles the outcome obtained with horizontal product differentiation. As $K \to 0$, firms blanket the interval, and the outcome converges to perfect competition.

5.4.4 Non-spatially differentiated products

Motivation: One can also think about endogenous entry in models with non-spatially differentiated products. One advantage: in spatial models, there are two dimensions to product variety: the number of products, and the type of products. With appropriate non-spatial models, one can summarize variety simply by the number of products. This makes it easier to compare equilibrium diversity with optimal diversity.

Intellectual history: Edward Chamberlin, *The Theory of Monopolistic Competition*, 1933.

Defining characteristics of monopolistic competition:

1. Products are distinct, and each firm faces a downward sloping demand curve.

2. The decisions of any given firm have negligible effects on any other firm.

3. Free entry, with zero profits.

A formalization:

Vector of N differentiated commodities, x (where N is very large)

Numeraire good, y

Representative consumer with utility function

$$u(x, y) = y + g\left(\sum_{i=1}^{N} f(x_i)\right)$$

Maximizing subject to the budget constraint $y + \sum_{i=1}^{N} p_i x_i = I$ yields the following first order conditions:

$$g'\left(\sum_{j=1}^{N} f(x_j)\right) f'(x_i) = p_i$$

The first-order condition functions as an inverse demand function.

Assume that firms have "U-shaped" production cost function $c(x_i)$. Profits are given by:

$$p_i x_i - c(x_i) = g'\left(\sum_{j=1}^{N} f(x_j)\right) f'(x_i) x_i - c(x_i)$$

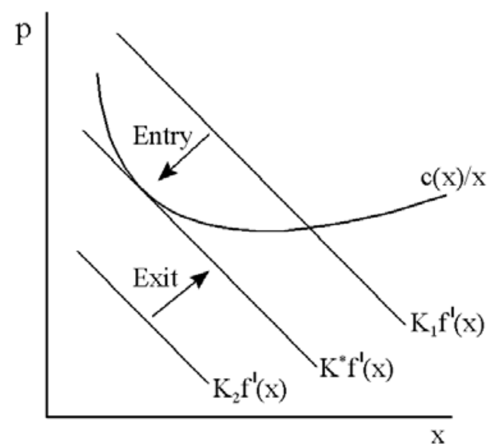
Definition: A *monopolistically competitive equilibrium* (MCE) consists of (N^*, x^*) (interpreted as N^* firms each producing x^*) such that the following two conditions hold:

(i) $g'\left(N^* f(x^*)\right) f'(x^*) x^* - c(x^*) = 0$

(ii) x^* maximizes $g'\left(N^* f(x^*)\right) f'(x) x - c(x)$

Note: For condition (ii), the firm takes the value of $g'(N^*f(x^*)) \equiv K$ as a fixed parameter, acting as if it faces the inverse demand function $Kf'(x) = p$. Justification: firms are small relative to the market, so their individual effects on K are negligible, even though their collective effects on K are substantial. For this reason, a monopolistically competitive equilibrium is *not* a Nash equilibrium – it does not fit into the framework of equilibrium analysis developed in this course.

Graphical depiction of MCE



Remarks on product variety:

Chamberlin's work led to the conventional view that monopolistic competition is inefficient because firms hold excess capacity, or equivalently that it is inefficient because there is too much variety. Reason: firms produce at an inefficiently low scale.

That argument is generally incorrect. It may be socially desirable to produce a good even if there is insufficient demand to produce it at minimally efficient scale.

The relation between equilibrium variety and optimal variety is dictated by two factors:

(i) When a product is added, the revenues generated fall short of incremental consumer surplus because the firm can't perfectly price discriminate. This creates a bias toward too little entry.

(ii) Firms don't take into account the effect of introducting a product on the profits of others. If the goods are substitutes, this creates a bias toward too much entry (we have already seen this in a setting with homogeneous products). If the goods are complements, it creates a bias toward too little entry (e.g. nuts and bolts).

If goods are complements, these effects are reinforcing, and there is too little variety relative to the social optimum. If goods are substitutes, the effect can go in either direction.

Caveat: specific conclusions here depend upon the notion of social optimality that is used. A natural second-best problem requires the planner to consider only allocations in which each producer breaks even. The first-best problem would ignore this constraint.

5.5 Entry deterrence

Theme: Actions taken by an incumbent firm prior to the arrival of an entrant may alter competitive conditions in a way that makes entry less attractive. The potential to discourage entrants may distort incumbents' choices away from the decisions they would make either if they were not facing entry, or if entry were certain. For this to occur, decisions taken prior to entry must have lasting effects. Such effects will exist if decisions represent long-term commitments.

Observation: Certain activities, such as predation and limit pricing, do not commit the firm to future actions. Consequently, they cannot act as entry deterents in the sense considered here. One needs alternative theories (e.g. reputation in models of incomplete information) to explain why certain actions can deter entry even without commitment.

5.5.1 Product differentiation

Product selection

Motivation: Choice of product characteristics may represent commitments over some reasonably long time frame. Choice of product characteristics may therefore deter entry.

Model: Consider the standard Hotelling spatial location model with two firms (an incumbent and an entrant), product characteristics on the unit interval, quadratic transportation costs, and a uniform distribution of consumers.

Suppose decisions take place as follows:

Stage 1: The incumbent selects product characteristic $x_I \in [0, \frac{1}{2}]$

Stage 2: Having observed x_I, the entrant selects either *in* or *out*; if *in*, the entrant must select product characteristic x_E, and incurs an entry cost K

Stage 3: Having observed entry decisions and product characteristics, the incumbent and the entrant simultaneously choose prices.

Let $\Pi_i(x_I, x_E)$ denote reduced form profits (having solved for the stage 3 equilibrium, as we did earlier in the course).

Observations:

(1) If the incumbent were alone, it would maximize profits by setting $x_I = \frac{1}{2}$.

(2) If the entrant enters, it chooses $x_E = 1$.

(3) $x_I = \frac{1}{2}$ also minimizes $\Pi_E(x_I, 1)$ for $x_I \in [0, \frac{1}{2}]$.

Conclusions:

(1) If $\Pi_E(\frac{1}{2}, 1) > K$, the incumbent cannot deter entry. It will set $x_I = 0$.

(2) If $\Pi_E(\frac{1}{2}, 1) < K$, the incumbent will set $x_I = \frac{1}{2}$, thereby preempting entry.

Notice that the incumbent acts differently than it would if entry occurred with certainty. However, it acts the same as it would if no-entry occurred with certainty. This is an example of a situation in which entry is *blockaded*.

Product proliferation

Motivation: Proliferation of products by an incumbent may deter entry. (See Schmalensee's analysis of product proliferation by General Mills and Kellogg in the Ready to Eat Breakfast Cereal market).

Model: Two possible goods, $i = A, B$. Substitutes, but not perfect substitutes.

Two firms, incumbent (I) and entrant (E).

Firms can potentially produce perfectly substitutable versions of each good.

Stage 1: The incumbent selects products: $p_I \in \{A, B, AB, N\}$ (N for none). Incurs setup cost K_i if it chooses to produce product i.

Stage 2: Having observed p_I, the entrant selects either *in* or *out*; if *in*, the entrant must select products $p_E \in \{A, B, AB, N\}$, and incurs and entry cost K_i if it chooses to produce product i.

Stage 3: Having observed entry decisions and product characteristics, the incumbent and the entrant simultaneously choose prices (Bertrand). Production costs are c_i per unit for good i (same for both firms).

Notation (reduced form variable profits):

Note: if both firms produce any good, profits earned from sales of that good are zero

π_i^0: profits earned from sales of good i when only good i is produced

π_i^c: profits earned from sales of good i when good j is produced competitively ($p = c$)

π_i^d: profits earned from sales of good i when a different firm produces each good

π^m: profits earned from sales of both goods when the same firm produces both goods

Assumptions:

1) $\pi^m > \pi_i^0$ (the goods are not perfect substitutes)

2) $\pi_i^0 > \pi_i^d > \pi_i^c$ (the goods are substitutable to some degree)

3) $\pi_A^0 - K_A > \max\{\pi_B^0 - K_B,\ \pi^m - K_A - K_B\} > 0$ (without the possibility of entry, I would only produce product A)

4) $\pi_i^d - K_i > 0$ (entry is not blockaded)

5) $\pi^m - K_A - K_B > \max_i \left\{\pi_i^d - K_i\right\}$ (a differentiated duopoly is less attractive than a multiproduct monopoly)

Solution:

Stage 2:

If $p_I = A$, then $p_E = B$; I earns $\pi_A^d - K_A$

If $p_I = B$, then $p_E = A$; I earns $\pi_B^d - K_B$

If $p_I = AB$, then $p_E = N$; I earns $\pi^m - K_A - K_B$

If $p_I = N$, then $p_E = A$; I earns 0

Stage 1: By assumption 5, firm I chooses $p_I = AB$.

Implication:

If entry was not possible, incumbent would produce only one product. If entry was certain, incumbent would produce at most one product. Here, the incumbent chooses to proliferate products because this represents a commitment that deters entry.

Remark: One can also make this point in the context of a spatial location model. The incumbent has an incentive to crowd the product space to eliminate niches.

Issue: Even though product characteristics may be somewhat fixed in the short-run, it may be possible to withdraw products quickly and at relatively low cost. Moreover, the incentives for withdrawal are greater for an incumbent than for an entrant, because the incumbent captures some of the lost business through other imperfectly substitutable products. Consequently, product proliferation may not be a credible commitment.

This issue is taken up in the paper by Ken Judd. We can study it in the context of the preceding model by adding a stage 2.5 (between stages 2 and 3), in which the incumbent and the entrant are both allowed to withdraw products at a cost W_i for product i.

Exercise: show that, as long as W_i is not too large, the only subgame perfect equilibrium involves I producing product A, and E producing product B (assuming $\pi_A^d - K_A > \pi_B^d - K_B$).

Conclusion: product proliferation only deters entry if it is a firm commitment, and cannot be reversed at low cost.

5.5.2 Capacity

Motivation: Can incumbent firms deter entry by holding extra capacity? In particular, will they hold excess (unused) capacity? Example: Alcoa decision (1945).

Model:

Two firms: an incumbent (firm 1) and a potential entrant (firm 2) producing a homogeneous product.

Each firm will make investments in capacity, k_i. Each unit of capacity for firm i costs r_i.

Ultimately, firms will also choose quantities (Cournot), subject to the constraint that $q_i \leq k_i$. Production costs w_i per unit.

Inverse demand function: $p = a - bQ$.

Sequence of decisions:

Stage 1: The incumbent makes an irreversible investment in capacity, k_1^0.

Stage 2: The entrant chooses *in* or *out*. If *in*, the entrant pays the cost K.

Stage 3: The firms simultaneously choose capacity, k_i, and quantity, q_i, subject to these constraints: $k_1 \geq \max\{k_1^0, q_1\}$, and $k_2 \geq q_2$.

Central questions: (1) Does the fact that the incumbent can put k_1^0 in place and pay for it in advance change anything? (2) Will the incumbent install capacity that it does not ultimately use?

Intuition: By installing capacity in advance, a portion of what would be the marginal cost in stage 3, $q_1 c_1$, is converted to a sunk cost, thereby reducing the marginal cost that the incumbent perceives in stage 3. This makes the incumbent play more aggressively. Since quantities are strategic substitutes, it also makes the entrant play less aggressively in stage 3, which benefits the incumbent at the expense of the entrant. Thus, the incumbent has an incentive to invest in capacity preemptively. When the effect on the entrant is sufficiently strong, preemptive investment may deter entry.

Analysis of the model:

We solve for subgame perfect equilibria recursively, beginning in stage 3 and working back to the start of the game.

Stage 3: What are the within-stage best response functions?

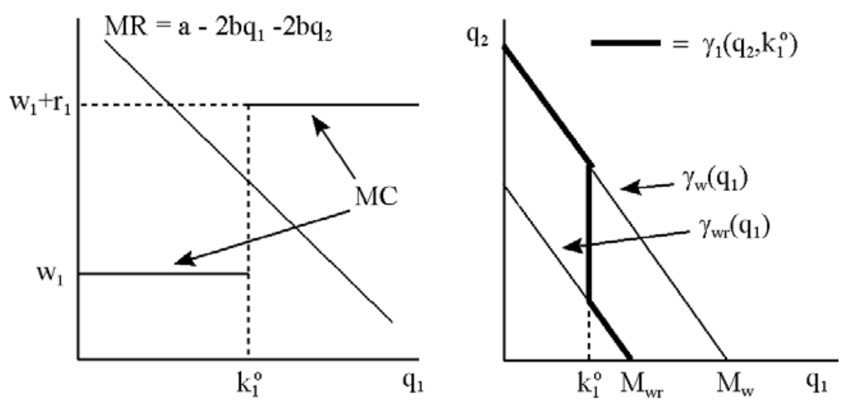
Observation: the best response function for each firm depends only on the quantity of the other firm, and not on the other firm's capacity.

Consider firm 1. It acts as if it is a Cournot competitor to firm 2, with discontinuous marginal costs. In particular, marginal costs are w_1 for $q_1 < k_1^0$, and $w_1 + r_1$ for $q_1 > k_1^0$.

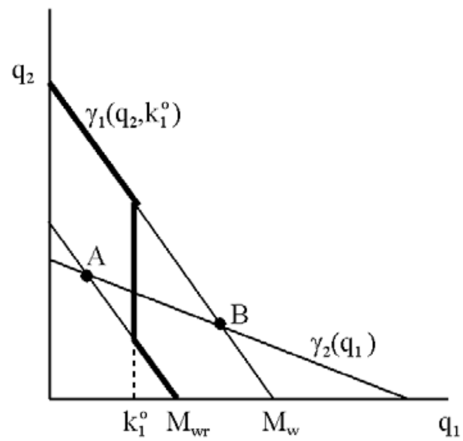
Let $\gamma_w(q)$ denote the best response function with marginal cost w_1

Let $\gamma_{wr}(q)$ denote the best response function with marginal cost $w_1 + r_1$

Graphically, obtain $\gamma_1(q, k_1^0)$ as follows:



Consider firm 2. Clearly, this firm would always choose $q_2 = k_2$. Consequently, it acts as if it is a Cournot competitor to firm 1, with marginal cost $w_2 + r_2$, and best response function $\gamma_2(q_1)$. Putting these together:



Let

$$A \equiv (q_L, \gamma_2(q_L))$$

$$B \equiv (q_H, \gamma_2(q_H))$$

Then the outcome is:

$$A \text{ if } k_1^0 < q_L$$

$$B \text{ if } k_1^0 > q_H$$

$$\left(k_1^0, \gamma_2(k_1^0)\right) \text{ if } k_1^0 \in (q_L, q_H)$$

As one varies k_1^0, one maps out all of the points on $\gamma_2(q_1)$ between A and B.

Let $q_i^*(k_1^0)$ be the equilibrium quantity for firm i given k_1^0, and let $\pi_i^*(k_1^0)$ denote associated profits.

Notice that $q_2^*(k_1^0)$ and $\pi_2^*(k_1^0)$ are both decreasing in k_1^0.

Stage 2: Firm 2 enters if $\pi_2^*(k_1^0) > K$, remains *out* if $\pi_2^*(k_1^0) < K$, and is indifferent for the case of equality.

Stage 1: There are three cases to consider.

Case (i): $\pi_2^*(M_{wr}) < K$. The monopolist achieves the overall monopoly outcome by installing capacity M_{wr}. Entry is blockaded. There is no unused capacity.

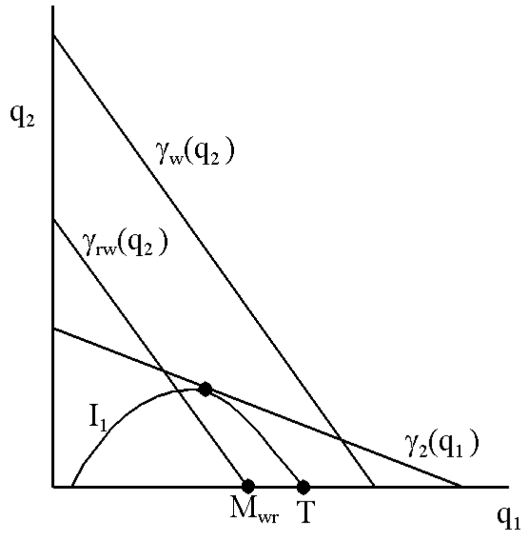
Case (ii): $\pi_2^*(q_H) > K$. This means that it is impossible to deter entry. The incumbent selects k_1^0 to achieve its most-preferred point on the segment $\overline{AB}$. There is no unused capacity.

Case (iii): $K \in (\pi_2^*(q_H), \pi_2^*(M_{wr}))$.

Define Z as the solution to $\pi_2^*(Z) = K$. To deter entry, the incumbent must set $k_1^0 \geq Z$. Since $Z > M_{wr}$, and since the profit function is concave in quantity, the optimal entry deterrence strategy is to set $k_1^0 = Z$. Entry is deterred, and the incumbent ends up producing Z (since $Z < M_w$); there is no unused capacity.

The incumbent compares the profits earned from this optimal deterrence strategy with the profits earned from the optimal accommodation strategy, which involves setting k_1^0 to achieve its most-preferred point on the segment $\overline{AB}$.

Note that K affects the profits earned from optimal deterrence, but does not affect the profits from optimal accommodation. Larger K makes deterrence relatively more attractive. Graphically:



If K is very large, then $Z < M_{wr}$, and entry is blockaded. For intermediate values of K, $Z \in (M_{wr}, T)$, and the incumbent holds extra capacity to deter entry. For small values of K, either Z does not exist (deterrence is impossible), or $Z > T$, so the incumbent accommodates entry.

Conclusions:

(1) The incumbent may deter entry by building more capacity that it would either if entry were impossible, or if it occurred with certainty. Intuition: an investment in capacity lowers costs, thereby making the incumbent more aggressive if faced with entry.

(2) The incumbent does not, however, build unused (excess) capacity. Intuition: quantities are strategic substitutes. We know that the incumbent will exhaust capacity if entry occurs. If entry does not occur, rival's quantity is lower (0), and therefore the incumbent's optimal choice of quantity is at least as great.

Observation: Manipulation of capacity by an entrant may also enhance the feasibility of entry. Intuition: by limiting its own capacity, the entrant reduces the threat to the incumbent, and thereby reduces the incumbent's incentives to drive it from the market.

A simple model: There is an incumbent I with essentially unlimited capacity, and a potential entrant E that must build capacity to enter.

The firms produce a single homogeneous good with production costs c_i, $i = I, E$.

The incumbent is more efficient: $c_I < c_E$.

Demand is given by $Q(p)$

Timing of decisions:

Stage 1: The entrant chooses *in* or *out*; if *in*, it pays a setup cost K, names its capacity, k_E, and a price, p_E. Capacity is costless.

Stage 2: The incumbent names a price p_I. In the event of ties, consumers resolve their indifference in favor of the incumbent.

Solve the model by backward recursion.

Stage 2 solution: Let

$$\Pi^u(p_E) \equiv \max_{p \leq p_E} (p - c_I) Q(p)$$

$$\Pi^A(p_E, k_E) \equiv \max_{p \geq p_E} (p - c_I) [Q(p) - k_E]$$

If $\Pi^u(p_E) > \Pi^A(p_E, k_E)$, the incumbent undercuts the entrant, setting $\arg\max_{p \leq p_E} (p - c_I) Q(p)$, and the entrant sells nothing.

If $\Pi^u(p_E) < \Pi^A(p_E, k_E)$, the incumbent sets $\arg\max_{p \geq p_E} (p - c_I) [Q(p) - k_E]$, which is a higher price than the entrant's, and the entrant sells k_E.

Stage 1 solution:

To make positive sales, the entrant must select (p_E, k_E) subject to the constraint that $\Pi^u(p_E) \leq \Pi^A(p_E, k_E)$.

Note that a low value of k_E increases $\Pi^A(p_E, k_E)$, while a low value of p_E reduces $\Pi^u(p_E)$. Thus, the entrant must restrict capacity and set a relatively low price. Note that the constraint is always satisfied for k_E and p_E sufficiently small, so the constraint set is non-empty.

The entrant solves $\max_{p_E,k_E}(p_E - c_E)k_E$ subject to $\Pi^u(p_E) \leq \Pi^A(p_E, k_E)$. If K is sufficiently small, entry occurs.

Conclusion: By limiting capacity, even an inefficient entry can gain a foothold in an industry.

Intuition: Prices are strategic complements. Capacity makes the entrant play aggressively, inducing the incumbent to play aggressively, which reduces the entrant's profits. Consequently, the entrant weakens itself (by establishing less capacity) in order to induce the incumbent to be less aggressive.

5.5.3 Vertical contracts

Motivation: Recent high-profile antitrust actions raise important questions about the potential anticompetitive effects of exclusionary relationships between entities operating at different levels of a production process.

Examples of vertical practices:

1. Exclusive dealing: Contractual relation between two firms in a vertical chain, wherein one firm agrees to buy only from, or sell only to the other.

2. Tying: A firm is said to engage in tying when it makes the sale (or price) of one of its products conditional upon the purchase of some other product. Includes requirements contracts and bundling.

The "Chicago view": There is only one monopoly rent to be had. If a firm extracts rents through prices, it can't also extract rent through exclusivity.

Model:

Three parties: a buyer B, a seller S, and an entrant E.

B purchases some good from S and/or E

B's demand is given by $x(p)$ (derived from quasi-linear utility so that consumer surplus is valid)

$z(p)$ denotes the level of consumer surplus at price p

Per unit production costs for firm i are c_i, with $c_S > c_E$

The entrant also pays a fixed setup cost F if it decides to produce

Assume that the entrant is more efficient: $(c_S - c_E)x(c_S) > F$

Sequence of decisions:

Stage 1: S offers an exclusive contract to B, involving a payment t to B in return for a commitment to exclusivity.

Stage 2: B decides whether to accept the offer.

Stage 3: E decides whether to enter the market. Prior to entry, contracts with E are impossible.

Stage 4: S and (possibly) E set prices, and B determines purchases. Renegotiation of any contract between S and B is not allowed.

Solution:

Stage 4: If E has not entered or if B and S have an exclusive deal, S sets the monopoly price p^*; B receives $z(p^*) + t$, S receives $\pi^* - t$ (where $t = 0$ if there is no exclusive arrangement). E receives 0 if it has not entered, and $-F$ if it has entered.

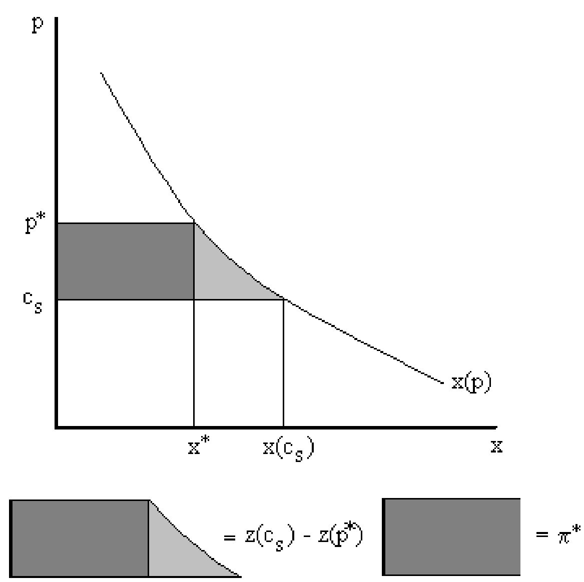
If E has entered and if B and S do not have an exclusive deal, E and S set $p = c_S$, and E makes all sales; B receives $z(c_S)$, E receives $x(c_S)(c_S - c_E) - F > 0$, and S earns 0.

Stage 3: E enters iff B and S do not have an exclusive deal.

Stage 2: B will accept the offered contract only if $z(p^*) + t \geq z(c_S)$.

Stage 1: S is willing to offer the contract only if $\pi^* - t \geq 0$.

Conclusion: the parties can enter an exclusive arrangement only if $\pi^* \geq z(c_S) - z(p^*)$. But this condition is never satisfied, so exclusion never occcurs.



Question: How general is the Chicago argument?

A model of exclusive dealing

Intuition: With more than one buyer, the decision of one buyer to enter into an exclusive arrangement with the seller confers negative externalities on the other buyers, which the first buyer ignores. Reciprocal externalities lead to suboptimal decisions on the part of the buyers.

Model:

Exactly the same as the preceding model, except that there are two identical buyers, B_1 and B_2.

Assume that the buyers' demand are independent of each others' purchases (in particular, they are not competitors in a downstream market).

Assume that the entrant is more efficient at serving both buyers: $2(c_S - c_E)x(c_S) > F$.

However, assume that the entrant is less efficient at serving only one buyer: $(c_S - c_E)x(c_S) < F$.

In stage 1, the seller makes separate (and potentially different) offers to B_1 and B_2.

Solution:

Stage 4: If E has not entered, or if E has entered and both buyers have signed exclusive deals, S sets the monopoly price p^*; B_i receives $z(p^*) + t$, S receives $2(\pi^* - t)$ (where $t = 0$ if there is no exclusive arrangement). E receives 0 if it has not entered and $-F$ if it has.

If E has entered, buyer i has signed an exclusive deal, and buyer j has not, the S charges i the monopoly price p^* earning $\pi^* - t$, B_i receives $z(p^*) + t$, E makes all sales to B_j at the price c_S, earning $(c_S - c_E)x(c_S) - F$, and B_j receives $z(c_S)$.

If E has entered and if neither buyer has signed an exclusive deal, E and S set $p = c_S$, and E makes all sales; B_i receives $z(c_S)$, E receives $2x(c_S)(c_S - c_E) - F > 0$, and S earns 0.

Stage 3: E enters iff neither buyer has signed an exclusive deal.

Stage 2: If $t_i > z(c_S) - z(p^*)$, B_i accepts the exclusive offer.

If $t_i = z(c_S) - z(p^*)$, B_i accepts the exclusive offer if she expects j to accept, and is indifferent if she expects j to reject.

If $t_i \in (0, z(c_S) - z(p^*))$, B_i accepts the exclusive offer if and only if she expects j to accept.

If $t_i = 0$, B_i rejects the exclusive offer if she expects j to reject, and is indifferent is she expects j to accept.

Note: for some (t_1, t_2), there are multiple continuation equilibria. Any equilibrium involves some selection from these.

Let $T \equiv \{(t_1, t_2) \mid$ at least one buyer accepts in the continuation equlibrium$\}$

Stage 1: The best exclusive strategy for S is to pick $(t_1, t_2) \in T$ to minimize $t_1 + t_2$. For simplicity, imagine that we have picked the continuation equlibria so that the minimum exists. If $t^M \equiv \min_{(t_1,t_2)\in T} t_1 + t_2 > 2\pi^*$, then no exclusion occurs. If $t^M \leq 2\pi^*$, then there is an exclusive equilibrium; with strict inequality, one necessarily gets exclusion.

Claim: If $2\pi^* > z(c_S) - z(p^*)$, then all equilibria involve exclusion.

Reason: For all $\varepsilon > 0$, $(z(c_S) - z(p^*) + \varepsilon, 0) \in T$. The resulting payoff for S is $2\pi^* - (z(c_S) - z(p^*) + \varepsilon)$. Under the preceding condition, this is strictly positive for small ε. This is equivalent to $t^M < 2\pi^*$.

Notes: (i) The value of t_i is indeterminant. In fact, for any $t \in [0, \min\{2\pi^*, z(c_S) - z(p^*)\}]$, there is an equlibrium in which S pays t (in total) for exclusivity.

(ii) In an exclusive equilibrium for which $t_i < z(c_S) - z(p^*)$ for $i = 1, 2$, there is also a non-exclusive continuation equilibrium for period 2 onward. The non-exclusive outcome might be considered more plausible, in that the buyers might coordinate their choices.

A modified model:

Assume that S approaches B_1 and B_2 sequentially, with B_1 responding to S's offer before S makes an offer to B_2.

Motivation: exclusive contracts are often of significant duration, and staggered.

Claim: Provided that $2\pi^* > z(c_S) - z(p^*)$, the only subgame perfect equilibrium involves S purchasing exclusivity from B_1 for nothing, thereby excluding E at no cost.

Demonstration: Consider the possible outcomes when S approaches B_2.

If S has signed an exclusive deal with B_1, then B_1 receives a payoff of $z(p^*) + t_1$.

If S has not signed an exclusive deal with B_1, then S will successfully offer $z(c_S) - z(p^*)$ in return for exclusivity (while B_2 is indifferent about accepting, one must resolve B_2's choice in favor of accepting if S is to have a best choice in this subgame). As a result, B_1's payoff will be $z(p^*)$.

Now consider the outcome when S approaches B_1.

B_1 knows that, if it accepts an offer, its payoff will be $z(p^*) + t_1$, and if it rejects, it payoff will be $z(p^*)$. Hence, it is willing to accept for all $t \geq 0$. For S's optimum to be well defined, we must resolve B_1's indifference in favor of acceptance when $t = 0$.

A model of tying

Two markets, X and Y

Production:

Two firms, I (the incumbent) and E the potential entrant.

Firm 1 is a monopolist for X, and produces output in this market at unit cost c_X

Each firm can produce a product for market Y.

Firm i produces output in the Y market at unit cost c_Y^i.

Demand:

Each consumer buys either 0 or 1 unit of X, and values this unit at w (same for all consumers).

I and E produce differentiated products for market Y (Y_I and Y_E)

Each consumer buys either 0 or 1 unit of Y, and attaches a value v_i to this unit if it is Y_i (provided that he is not also consuming Y_j)

Consumers differ in their valuations of the Y products, and there is some distribution F of (v_I, v_E) across the population. We normalize the total population to unity.

Utility is given by the sum of valuations for the consumed goods, minus the total price paid.

Sequence of decisions:

Stage 1: I decides whether to tie X and Y_I together.

Stage 2: E chooses either *in* or *out*; if *in*, it pays a setup cost K.

Stage 3: Bertrand competition. Firms name prices simultaneously (if both are present). Firm E, if present, names price p_E. If firm I has not tied its products together, it names a price q for X, and a price p_I for Y_I. If I has tied its products together, it names a price r for the bundle.

Solution:

Stage 3: There are three cases to consider.

(i) I has not tied and E has entered.

A consumer buys Y_i rather than Y_j or nothing iff $v_i - p_i > \max\{v_j - p_j, 0\}$. Let

$$G_i(p_I, p_E) = \{(v_I, v_E) \mid p_i < \min\{p_j + v_i - v_j, v_i\}\}$$

and

$$y_i(p_I, p_E) = \int_{G_i(p_I, p_E)} dF$$

$y_i(p_I, p_E)$ is the demand function for Y_i.

Let $\gamma_i(p_j)$ be firm i's Bertrand best response function:

$$\max_{p_i} (p_i - c_Y^i) y_i(p_I, p_E)$$

Assume $0 < \gamma_i' < 1$ (prices are strategic complements)

Continuation equilibrium:

I sets $q = w$.

I and E set p_I^* and p_E^* such that $p_i^* = \gamma_i(p_j^*)$, $i = I, E$.

(ii) I has tied and E has entered.

A consumer buys the bundle from I, rather than Y_E or nothing, iff $w + v_I - r > \max\{v_E - p_E, 0\}$, or equivalently.

$$r - w < \min\{p_E + v_I - v_E, v_I\}$$

Similarly, a consumer buys the good from E, rather than the bundle or nothing, if $v_E - p_E > \max\{w + v_1 - r, 0\}$, or equivalently

$$p_E < \min\{r - w + v_E - v_I, v_E\}$$

Without tying, the corresponding expression was

$$p_i < \min\{p_j + v_i - v_j, v_i\}$$

Notice that $r - w$ has simply taken the place of p_I.

This implies that demand for Y_i is given by $y_i(r - w, p_E)$, where y_i is the same function as before.

Firm I chooses r to solve:

$$\max_r (r - c_X - c_Y^I) y_i(r - w, p_E)$$

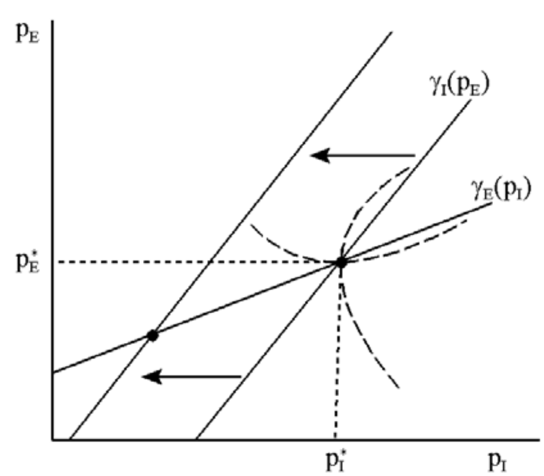
Make the following change of variables: $p_I \equiv r - w$. Then I's problem is

$$\max_{p_I} \left(p_I - c_Y^i + (w - c_X)\right) y_I(p_I, p_E)$$

Note that this is the same as in the case without a tie, except for the presence of the $w - c_X$ term.

Intuitively, the margin from selling Y_I is effectively larger because Y_I is tied to the sales of a profitable. Same effect as reducing the costs of Y_I.

Effect: Tie makes I more aggressive in market Y pricing.



Note that both I's profits and E's profits are lower than if I does not tie.

(iii) E has not entered. Regardless of whether I has tied, I sets the monopoly price.

Stage 2: E enters if anticipated profits are sufficient to cover entry costs, K. Notice that anticipated profits are strictly higher if I has not tied. Consequently, there are circumstances under which E will not enter if I has tied, but will enter if I has not tied.

Stage 1: There are several possibilities to consider.

(i) E will not enter whether or not I ties. Then I is indifferent between tying and not tying.

(ii) E will enter whether or not I ties. Then I will not tie.

(iii) E enters if I does not tie, but does not enter if I does tie. Then I will tie, and this will deter entry.

Note: I must be able to commit to a tie. Otherwise, when E entered, I would always abandon the tie.

5.6 Sequential bargaining

Environment:

Two parties, 1 and 2, involved in a negotation

Object is to split a pie of size 1 into shares x (for 1) and $1-x$ (for 2)

Sequence of decisions:

Starting with party 1, the parties take turns making offers

An offer consists of $x \in [0, 1]$

After each offer, the non-offering parties either *accepts* or *rejects*

If she *accepts*, the pie is divided as per the offer, and the game ends

If she *rejects*, the game continues

The game may continue forever (infinite horizon version), or end after a finite number of rounds (finite horizon version)

Payoffs:

Player i discounts the future at the rate $\delta_i \in (0, 1)$

If the parties expect division x to be implemented in the t-th round, party 1 receives a payoff of $\delta_1^t x$, and party 2 receives a payoff of $\delta_2^t(1-x)$

If no agreement is ever reached, payoffs are zero

Solution of the finite horizon version:

Solve for $SPNE$ by backward recursion, beginning with final period, T.

Without loss of generality, assume that party 1 makes the offer in T.

Period T: 1 proposes $x = 1$ and 2 accepts.

Period $T-1$: Given the continuation equilibrium (which gives 1 a payoff of unity one period hence), 1 will not accept any $x < \delta_1$. Consequently, 2 proposes δ_1 and 1 accepts.

Period $T-2$: Given the continuation equilibrium (which gives 2 a payoff of $1-\delta_1$ one period hence), 2 will not accept any $1-x < \delta_2(1-\delta_1)$. Consequently, 1 proposes $1-\delta_2(1-\delta_1)$, and 2 accepts.

Exercise: Solve for the $SPNE$ of a T-period bargaining model first for T even, and then for T odd. Show that the payoffs from these equilibria converge to a common limit as $T \to \infty$. Solve for the limiting payoffs.

Solution of the infinite horizon version:

Theorem: Consider the following strategies:

When making an offer, player i always demand the share $(1-\delta_j)/(1-\delta_i\delta_j)$.

When receiving an offer, player i accepts any share greater than $\delta_i(1-\delta_j)/(1-\delta_i\delta_j)$, and rejects any smaller share.

These strategies constitute a $SPNE$. Moreover, it is the unique $SPNE$ outcome for this game.

Remarks:

(i) Uniqueness is surprising, given other results on infinite horizon games (next section)

(ii) In equilibrium, there is never any delay in reaching a settlement

(iii) When discount factors are equal, the proposer receives $(1-\delta)/(1-\delta^2) = 1/(1+\delta) > \frac{1}{2}$. Note that, as $\delta \rightarrow 1$ (the parties become increasingly patient), equilibrium shares converge to $\frac{1}{2}$.

Proof that the strategies consistitute a $SPNE$:

Using a standard dynamic programming argument, it suffices to evaluate 1-period deviations.

First, imagine that player i is making the offer. If i demands any share less than $(1-\delta_j)/(1-\delta_i\delta_j)$, the offer will be accepted, and i's payoff will be lower than in the equilibrium. If i demands any share greater than $(1-\delta_j)/(1-\delta_i\delta_j)$, the offer will be rejected, and in the next period i will receive $1-(1-\delta_i)/(1-\delta_i\delta_j)$. Discounted to the period in which i makes the offer, this is equivalent to

$$\delta_i\left[1-\frac{1-\delta_i}{1-\delta_i\delta_j}\right] = \delta_i^2\frac{1-\delta_j}{1-\delta_i\delta_j} < \frac{1-\delta_j}{1-\delta_i\delta_j}$$

Thus, the deviation yields a lower payoff for i.

Second, imagine that player i is receiving the offer. If i rejects an offer, i will receive a payoff of $(1-\delta_j)/(1-\delta_i\delta_j)$ in the following period. Thus, it is optimal for i to accept offers of at least $\delta_i(1-\delta_j)/(1-\delta_i\delta_j)$, and to reject lower offers.

Proof that the equilibrium outcome is unique:

Let $v_L(i,j)$ and $v_H(i,j)$ denote, respectively, the lowest and highest $SPNE$ continuation payoffs to player i when player j makes the current offer. Since the infinite horizon problem is stationary, these functions do not depend on the period t.

Strategy of proof: we will show that, for each (i,j) (including $i=j$), $v_L(i,j)=v_H(i,j)$, and we will solve for this common value.

We proceed in a series of steps.

(i) For $i \neq j$,

$$v_L(i,i) \geq 1-\delta_j v_H(j,j)$$

Reason: j gets no more than $v_H(j,j)$ in the continuation, and will therefore accept $\delta_j v_H(j,j)$ today.

(ii) For $i \neq j$,

$$v_H(j,i) \leq \delta_j v_H(j,j)$$

This follows from (i) and $v_L(i,i)+v_H(j,i) \leq 1$. Intuitively, j gets no more than $v_H(j,j)$ in the continuation. Player i therefore does not need to offer j more than $\delta_j v_H(j,j)$ today.

(iii) For $i \neq j$,

$$v_H(i,i) \leq \max\left\{1-\delta_j v_L(j,j), \delta_i v_H(i,j)\right\}$$

Reason: At this point, we don't know whether $v_H(i,i)$ is achieved by making an offer than it accepted, or one that is rejected.

Assume for the moment that it is achieved by making an offer that is accepted. Since player j gets a discounted continuation payoff of at least $\delta_j v_L(j,j)$, j will reject any x such that $1-x \leq \delta_j v_L(j,j)$. Thus, for this case, $v_H(i,i) \leq 1-\delta_j v_L(j,j)$.

Now assume that $v_H(i,i)$ is achieved by making an offer that is rejected. Then the best i can hope for is the best outcome for i when j makes the offer one period hence: $\delta_i v_H(i,j)$.

(iv) Combining (ii) and (iii), we have

$$v_H(i,i) \leq \max\left\{1-\delta_j v_L(j,j), \delta_i^2 v_H(i,i)\right\}$$

(v) We claim that

$$\max\left\{1-\delta_j v_L(j,j), \delta_i^2 v_H(i,i)\right\} = 1-\delta_j v_L(j,j)$$

Assume not. Then, by (iv), $v_H(i,i) \leq \delta_i^2 v_H(i,i)$, which implies $v_H(i,i) \leq 0$. But, since neither δ_j or $v_L(j,j)$ can exceed unity, we then have $1-\delta_j v_L(j,j) \geq \delta_i^2 v_H(i,i)$, a contradiction.

(vi) Combining (i), (iv), and (v), we have

$$v_L(i,i) \geq 1-\delta_j v_H(j,j) \geq 1-\delta_j\left(1-\delta_i v_L(i,i)\right)$$

Rearranging yields

$$v_L(i,i) \geq \frac{1-\delta_j}{1-\delta_j\delta_i}$$

Again combining (i), (iv), and (v), we have

$$v_H(i,i) \leq 1-\delta_j v_L(j,j) \leq 1-\delta_j\left(1-\delta_i v_H(i,i)\right)$$

Rearranging yields

$$v_H(i,i) \leq \frac{1-\delta_j}{1-\delta_j\delta_i}$$

But since $v_L(i,i) \leq v_H(i,i)$, we have $v_L(i,i) = v_H(i,i) = (1-\delta_j)/(1-\delta_i\delta_j)$.

(vii) Combining (ii) and (vi) yields

$$v_H(i,j) \leq \delta_i v_H(i,i) \leq \delta_i \left(\frac{1-\delta_j}{1-\delta_j\delta_i}\right)$$

Next note that

$$v_L(i,j) \geq \delta_i v_L(i,i)$$

This follows because i can always refuse j's offer, receiving at least $v_L(i,i)$ in the following period. Combining this with (vi) yields

$$v_L(i,j) \geq \delta_i \left(\frac{1-\delta_j}{1-\delta_j\delta_i}\right)$$

Since $v_L(i,j) \leq v_H(i,j)$, we have $v_L(i,j) = v_H(i,j) = \delta_i(1-\delta_j)/(1-\delta_i\delta_j)$. Q.E.D.

6 Repeated Games with Complete Information

Motivating example: The Prisoners' dilemma, again.

Note: payoffs here have been increased by 5 to avoid negative numbers

		Player 2	
		Not Fink	Fink
Player 1	Not Fink	3, 3	0, 4
	Fink	4, 0	1, 1

Dominant strategy solution is (Fink, Fink)

Cooperation may seem more plausible. One reason: participants may expect to interact more in the future. If you "Fink" today, your opponent may retaliate in the future.

Simple device for getting at these issues: imagine that the game is repeated.

Two important cases: (i) potentially infinite repetitions, (ii) finite repetitions.

6.1 Infinitely repeated games

6.1.1 Some preliminaries

Terminology: The game played in each period is called the *stage game*. The dynamic game formed by infinite repetitions of a stage game is called a *supergame*.

Observations:

(i) Even if the stage game is finite, the associated supergame is not.

(ii) There are no terminal nodes. How do we assign payoffs?

Evaluating payoffs:

What we need: a mapping from strategy profiles into expected payoffs. (For finite games, this is given by the composition of the mapping from strategy profiles into distributions over terminal nodes, with the mapping from terminal nodes to payoffs.)

For repeated games, one can assume that payoffs are distributed immediately after each play of the stage game. Then any strategy profile maps to a distribution of paths through the game tree, and any path through the game tree maps to a sequence of payoffs for each player i, $v_i = (v_i(1), v_i(2), ...)$.

Remaining issue: We need a mapping from strategy profiles to scalar payoffs for each player. How do we get from payoff sequences to scalar payoffs?

General answer: assume that players have utility functions mapping sequences of payoffs into utility: $u_i(v_i)$

Some specific answers:

(i) Use discounted payoffs: $u_i(v_i) = \sum_{t=1}^{\infty} \delta^{t-1} v_i(t)$. Note: the discount factor may reflect both time preference and a probability of continuation.

(ii) For the case of no discounting, we can use the average payoff criterion:

$$u_i(v_i) = \lim_{T\to\infty} \left(\frac{1}{T}\right) \sum_{t=1}^{T} v_i(t)$$

Another possibility for the case of no discounting: the *overtaking criterion.*

Strategies:

The nature of strategies will depend upon assumptions about what is observed each time the game is played.

For the time being, we will assume that, each time the stage game is played, all players can observe all previous choices.

With this assumption, each sequence of choices up to (but not including) period t corresponds to a separate information set (for each player) in period t. Consequently, we proceed as follows.

Let $a_i(t)$ denote the action taken by player i in period t

Let $a(t) = (a_1(t), ..., a_I(t))$ be the profile of actions chosen in period t

A t-history, $h(t)$, is a sequence of action profiles $(a(1), ..., a(t-1))$, summarizing everything that has occurred prior to period t.

Since, by assumption, $h(t)$ is observed by all players, there is, for each player, a one-to-one correspondence between t-histories and period t information sets.

Consequently, a strategy is a mapping from all values of $t \in \{1, 2, ...\}$ and all possible $t-$histories to period t actions (for period 1, the set of 1-histories is degenerate).

6.1.2 Nash equilibria with no discounting

The prisoners' dilemma

Illustration of a t-history: $((NF, NF), (F, NF), (F, F), ..., (NF, F))$

Example of a strategy: $\sigma_i^N(t, h(t)) \equiv F$ for all t, $h(t)$.

Note: this repeats the equilibrium of the stage game in every period.

Claim: If players use the average payoff criterion, $\left(\sigma_1^N, \sigma_2^N\right)$ is a Nash equilibrium.

Demonstration: Check to see whether a player can gain by deviating from this strategy, given that his opponent plays this strategy.

If the player sticks to the strategy, the sequence of payoffs will be $v_i(t) = 1$ for all t. The average payoff is 1.

If player i deviates to any other strategy while j sticks to σ_j^N, the sequence of payoffs for i contains zeros and one, so the average payoff cannot exceed 1.

Conclusion: Repeating the equilibrium of the stage game is a Nash equilibrium for the supergame.

Remark: The same proposition obviously holds with discounting, and without discounting using the overtaking criterion (a sequence of ones always beats a sequence of ones and zeros).

Exercise: Prove that this point is completely general (it holds for all stage games).

Question: Can we get anything other than repetitions of the stage game equilibrium?

Another possible strategy:

$$\text{In period 1, play } NF$$

$$\text{In period } t > 1\text{, play } \begin{cases} F \text{ if } h(t) \text{ contains an } F \\ \\ NF \text{ otherwise} \end{cases}$$

With these strategies, the game would unfold as follows: Both players would play NF forever. If any player ever deviated from this path, then subsequently both players would play F forever.

Let's imagine that both players select this strategy.

Claim: If players use the average payoff criterion to evaluate payoffs, this is a Nash equilibrium.

Demonstration: Check to see whether a player can gain by deviating from this strategy, given that his opponent plays this strategy.

If the player sticks to the strategy, the sequence of payoffs will be $v_i(t) = 3$ for all t. The average payoff is 3.

Now consider a deviation to some other strategy. Let t' be the first period t for which this strategy dictates playing F when $h(t)$ does not contain an F (if there is no such t', then the deviation also generates an average payoff of 3). If the player deviates to this strategy, the sequence of payoffs will be

$$v_i(t) \begin{cases} = 3 \text{ for } t < t' \\ = 4 \text{ for } t = t' \\ \leq 1 \text{ for } t > t' \end{cases}$$

(For $t > t'$, this follows because the opponents strategy will always dictate playing F). The associated average payoff is not larger than 1, and therefore certainly less than 3. Consequently, this is a Nash equilibrium.

Remarks: (i) This example demonstrates that cooperation is possible. Cooperation is sustained through the threat of punishment.

(ii) We cannot necessarily claim that the players will cooperate in this way, because there are many other Nash equilibria.

Illustration: For all t, let $h^*(t)$ be the t-history such that (i) $a_1(t') = F$ for $t' < t$ odd, and $a_1(t') = NF$ for $t' < t$ even; (ii) $a_2(t') = NF$ for all $t' < t$. In words: player 1 has alternated between NF and F, while player 2 has always chosen NF.

Consider the following strategies:

$$\sigma_1(t, h(t)) = \begin{cases} NF \text{ if } h(t) = h^*(t) \text{ and } t \text{ is even} \\ F \text{ otherwise} \end{cases}$$

$$\sigma_2(t, h(t)) = \begin{cases} NF \text{ if } h(t) = h^*(t) \\ \\ F \text{ otherwise} \end{cases}$$

In other words, player 1 alternates between NF and F, while player 2 always plays NF (the result is $h^*(\infty)$). However, if either deviates from this path, both subsequently play F forever.

Claim: When players use the average payoff criterion, there is also a Nash equilibrium wherein both players select the preceding strategy.

Demonstration: As long as no player deviates, player 1's payoffs alternate between 3 and 4, while 2's payoffs alternate between 3 and 0. Average payoffs are 3.5 for player 1, and 1.5 for player 2.

Now imagine that player i considers deviating to some other strategy. If this deviation has any effect on the path of outcomes (and hence on payoffs), there must be some period t' in which the actions taken diverge from $h^*(\infty)$. Assuming that player j sticks with its equilibrium strategy, $v_i(t) \leq 1$ for $t > t'$. Consequently, player i's average payoff is 1. This is less than the payoff received by both players in equilibrium.

Question: What other outcomes are consistent with equilibrium?

We will proceed by the process of elimination.

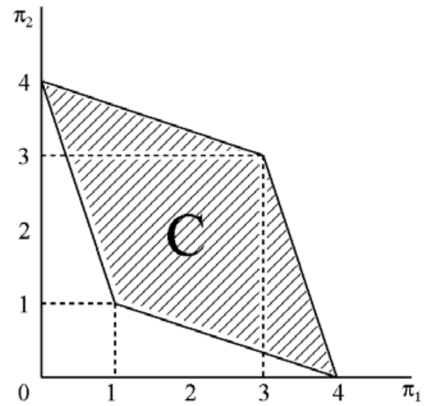
Some definitions

First, we identify the set of feasible payoffs (including things that can be achieved through arbitrary randomizations).

Let $C = \{w \mid w$ is in the convex hull of payoff vectors from pure strategy profiles in the stage game$\}$

Remark: C is potentially larger than the set of payoffs achievable through mixed strategies in the stage game, since we allow for correlations.

For our example (the prisoners' dilemma):



Can anything in C occur in equilibrium as an average payoff? No. Each player can assure himself of a payoff of at least unity each period by playing F all of the time. Therefore, we know that no player can get a payoff smaller than unity.

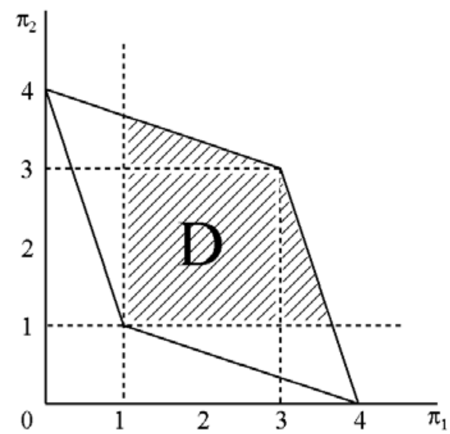
More generally, we define player i's *minmax* payoff:

$$\pi_i^m = \min_{(\delta_1,\ldots,\delta_{i-1},\delta_{i+1},\ldots,\delta_I)} \max_{\delta_i} \pi_i(\delta)$$

This is the average payoff that player i can assure himself, simply by making a best response to what everyone else is supposed to do (according to their strategies) in every period. Player i cannot receive an average payoff less than π_i^m in equilibrium.

Consequently, define $D = \{w \mid w \in C \text{ and } w_i \geq \pi_i^m \text{ for all } i\}$

For our example (the prisoners' dilemma):



D is called the set of *feasible and individually rational payoffs*.

Finally, define $E = \{w \mid \text{there is a Nash equilibrium with average payoff vector } w\}$

Question: How does E compare with D? It's reasonably clear that E isn't larger, but can it be smaller?

The folk theorem

The folk theorem: Consider a supergame formed by repeating a finite stage game an infinite number of times. Suppose that players use the average payoff criterion to evaluate outcomes. Then $E = D$.

Sketch of proof: The proof consists of three steps.

Step 1: $E \subseteq D$. We have already covered this.

Step 2: Consider any sequence of actions $h'(\infty)$ yielding average payoffs $w \in D$. Then there exists a pure strategy Nash equilibrium for the supergame where this sequence of actions is taken on the equilibrium path. We show this by construction.

Consider the following strategies. If play through period $t-1$ has conformed to $h'(t)$, players continue to follow $h'(\infty)$ in period t. If play has not conformed to $h'(t)$, inspect the actual history $h(t)$ to find the first lone deviator (in other words, ignore any period in which there are multiple deviators). If no lone deviator exists in any period prior to t, then revert to following $h'(\infty)$ in period t. If the first lone deviator is i, then all $j \neq i$ play

$$\delta_{-i}^m = \arg \min_{(\delta_1,\ldots,\delta_{i-1},\delta_{i+1},\ldots,\delta_I)} \max_{\delta_i} \pi_i(\delta)$$

while i plays some arbitrarily assigned action.

It is easy to check that this is a Nash equilibrium. If all players choose their equilibrium strategies, the outcome is $h'(\infty)$, and the average payoff for i is $w_i \geq \pi_i^m$ (since $w \in D$ by assumption). If player i deviates, then, assuming all others play their equilibrium strategies, i will be the first deviator, and subsequently can do not better than π_i^m in any period. This means that i's average payoff will be no greater than π_i^m. Thus, the deviation does not benefit i.

Step 3: For all $w \in D$, there exists a sequence of actions yielding average payoffs of w.

Idea: alternate actions to produce the same frequencies as the randomization. This is easy if the randomization involves rational frequencies. If it involves irrational frequencies, one varies the frequency in the sequence to achieve the right frequency in the limit.

Interpretation of the folk theorem: (i) Anything can happen. Comparative statics are problematic.

(ii) The inability to write binding contracts is not very damaging. Anything attainable through a contract is also obtainable through a self-enforcing agreement, at least with no discounting. The equilibrium that gets played is determined by a process of negotiation. It is natural to expect players to settle on some self-enforcing agreement that achieves the efficient frontier. The precise location may depend upon bargaining strengths.

6.1.3 Nash equilibria with discounting

Now imagine that players evaluate payoffs according the utility function $u_i(v_i) = \sum_{t=1}^{\infty} \delta^{t-1} v_i(t)$. For simplicity, take the rate of discounting $\delta \in (0,1)$ to be common for all players.

Remark: We can think of δ as the product of a pure rate of time preference, ρ, and a continuation probability, λ (measuring the probability of continuing the game in period $t+1$, conditional upon having reached t): $\delta = \rho\lambda$. In particular, assume that, if the game ends, subsequent payoffs are zero (this is just a normalization). Let T be the realized horizon of the game. Then expected payoffs are

$$
\begin{aligned}
& \sum_{k=1}^{\infty} \left[prob(T=k) \sum_{t=1}^{k} \rho^{t-1} v_i(t) \right] \\
= & \sum_{k=1}^{\infty} \left[\lambda^{k-1}(1-\lambda) \sum_{t=1}^{k} \rho^{t-1} v_i(t) \right] \\
= & \sum_{k=1}^{\infty} \sum_{t=1}^{k} \left[\lambda^{k-1}(1-\lambda) \rho^{t-1} v_i(t) \right] \\
= & \sum_{t=1}^{\infty} \sum_{k=t}^{\infty} \left[\lambda^{k-1}(1-\lambda) \rho^{t-1} v_i(t) \right] \\
= & \sum_{t=1}^{\infty} \left[\rho^{t-1} v_i(t) \, (1-\lambda) \, \lambda^{t-1} \sum_{k=t}^{\infty} \lambda^{k-t} \right] \\
= & \sum_{t=1}^{\infty} (\lambda\rho)^{t-1} \, v_i(t)
\end{aligned}
$$

The magnitude of δ in any context will depend upon factors such as the frequency of interaction, detection lags, and interest rates.

The prisoners' dilemma

Imagine again that both players use the following strategies:

$$\text{In period 1, play } NF$$

$$\text{In period } t > 1\text{, play } \begin{cases} F \text{ if } h(t) \text{ contains an } F \\ \\ NF \text{ otherwise} \end{cases}$$

If players discount payoffs at the rate δ, is this a Nash equilibrium?

If the player sticks to the strategy, the sequence of payoffs will be $v_i(t) = 3$ for all t. The discounted payoff is

$$\sum_{t=1}^{\infty} 3\delta^{t-1} = \frac{3}{1-\delta}$$

Now consider a deviation to some other strategy. Without loss of generality, imagine that player i deviates to F in period 1. Player i knows that j will play F in all subsequent periods (since this is dictated by j's strategy). Consequently, it is optimal for i to play F in all subsequent periods, having deviated in the first. Thus,

$$v_i(t) \begin{cases} = 4 \text{ for } t = 1 \\ \\ = 1 \text{ for } t > 1 \end{cases}$$

Player i's discounted payoffs are

$$4 + \sum_{t=2}^{\infty} \delta^{t-1} = 4 + \frac{\delta}{1-\delta}$$

Player i therefore finds the best deviation unprofitable when

$$\frac{3}{1-\delta} \geq 4 + \frac{\delta}{1-\delta}$$

This is equivalent to

$$\delta \geq \frac{1}{3}$$

Thus, the strategies still constitute an equilibrium provided that the players do not discount the future too much.

This is also true for the other equilibrium considered above.

The folk theorem

Analysis of prisoners' dilemma suggests that it becomes possible to sustain cooperative outcomes as δ gets larger.

A natural conjecture: it is possible to sustain all feasible, individually rational cooperative outcomes as $\delta \to 1$.

Problem: as $\delta \to 1$, discounted payoffs become unbounded. To discuss what occurs in the limit, we need to renormalize payoffs.

Renormalize payoffs as follows:

$$u_i(v_i) = (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} v_i(t)$$

Notice we can now think of utility as a weighted average of the single period payoffs,

$$u_i(v_i) = \sum_{t=1}^{\infty} \mu_t v_i(t),$$

where $\mu_t = (1-\delta)\delta^{t-1}$, and $\sum_{t=1}^{\infty} \mu_t = 1$.

With this normalization, the set of feasible discounted payoffs is C.

Moreover, as $\delta \to 1$, this converges to the average payoff criterion.

In this setting, the folk theorem needs to be restated slightly: for any $w \in D$ with $w_i > \pi_i^m$ for all i, there exists $\delta^* < 1$ such that for all $\delta \in (\delta^*, 1)$, w is the payoff vector for some Nash equilibrium.

The proof is very similar to that of the original folk theorem.

6.1.4 Subgame perfect Nash equilibria

The equilibria constructed to establish the folk theorem may not be subgame perfect. Punishing a player through a minmax strategy profile may not be credible, since the punishers may suffer. Can we sustain cooperation in $SPNE$?

The prisoners' dilemma

Claim: All of the Nash equilibria considered above for the repeated prisoners' dilemma are $SPNE$.

Demonstration: All Nash equilibrium strategies necessarily consistute Nash equilibria in all subgames that are reached along the equilibrium path. For any subgame off the equilibrium path, the prescribed strategies are $\left(\sigma_1^N, \sigma_2^N\right)$. All subgames are identical to the original game, and $\left(\sigma_1^N, \sigma_2^N\right)$ is a Nash equilibrium for the original game. Thus, we have a Nash equilibrium in every subgame.

Nash reversion

Generalization: The players can attempt to support cooperation by using repetitions of a stage-game equilibrium as a punishment. These punishments are known as *Nash reversion* (the players attempt to cooperate, but revert to a static Nash outcome if someone deviates).

Formally, consider some arbitrary stage game, as well as the supergame consisting of the infinitely repeated stage game. Assume that the stage game has at least one Nash equilibrium. For any particular stage-game Nash equilibrium, a^*, consider the following strategies:

$$\sigma_i^N(t, h(t)) = a_i^* \text{ for all } t,\ h(t)$$

Now imagine that we want to support some particular outcome, $h^*(\infty)$, as an equilibrium path. Let's try to do this with the following strategies: If play through period $t-1$

has conformed to $h^*(t)$, players continue to follow $h^*(\infty)$ in period t. If play has not conformed to $h^*(t)$, then players use σ^N.

Claim: If the aforementioned strategies constitute a Nash equilibrium, then the equilibrium is subgame perfect.

Demonstration: We use precisely the same argument as for the prisoners' dilemma. All Nash equilibrium strategies necessarily constitute Nash equilibria in all subgames that are reached along the equilibrium path. For any subgame off the equilibrium path, the prescribed strategy profile is σ^N. All subgames are identical to the original game, and σ^N is a Nash equilibrium of the original game. Thus, we have a Nash equilibrium in every subgame.

Remark: Since σ^N is an equilibrium of the supergame irrespective of whether the players use discounted payoffs, average payoffs, or the overtaking criterion, this claim is equally valid for all methods of evaluating payoffs.

Implication: When one uses Nash reversion to punish deviations, it is particularly simple to build $SPNE$ and check subgame perfection: one simply makes sure that the equilibrum is Nash (equivalently, that no player has an incentive to deviate from a prescribed choice on the equilibrium path).

The folk theorem

For some games, Nash reversion is as severe as minmax punishments.

Examples:

(i) The prisoners' dilemma

(ii) Bertrand competition (minmax payoffs and Nash payoffs are both zero)

However, Nash reversion is frequently much less severe than minmax punishments.

Example: Cournot competition (minmax payoffs are 0, while Nash profits are strictly positive)

Question: Does the validity of the folk theorem depend, in general, on the ability to use non-credible punishments (at least for stage games with the property that Nash payoffs exceed minmax)?

Answer: Subject to some technical conditions, one can prove versions of the folk theorem (with and without discounting) for $SPNE$. The proofs are considerably more difficult.

Implication: If the stage-game Nash equilibrium payoffs exceed minmax payoffs, then, for δ sufficiently close to unity, there exist more severe punishments than Nash reversion.

Example:

		Player 2	
		Not Fink	Fink
Player 1	Not Fink	3, 3	0, 4
	Fink	4, 1	1, 0

This game has only one Nash equilibrium: (F, NF). Note that this gives player 1 the maximum possible payoffs. It is therefore impossible to force player 1 to do anything through Nash reversion.

Exercise: For this example, construct a $SPNE$ in which (NF, NF) is chosen on the equilibrium path. Either use the average payoff criterion, or assume an appropriate value for δ.

We will see another explicit example of punishments that are more severe than Nash reversion when we analyze the dynamic Cournot model.

6.1.5 A short list of other topics

1. Repeated games with imperfect observability of actions.
2. Repeated games with incomplete information (reputation)
3. Heterogeneous horizons (models with overlapping generations, or both short and long-lived players)
4. Renegotiation

6.2 Finitely repeated games

One might think that finitely repeated games get to look a lot like infinitely repeated games when the horizon is sufficiently long. This is correct for Nash equilibria (where credibility is not required), but not for subgame perfect equilibria.

Theorem: Consider any finitely repeated game. Suppose that there is a unique Nash equilibrium for the stage game. Then there is also a unique $SPNE$ for the repeated game, consisting of the repeated stage game equilibrium

Proof: By induction (with T denoting the number of repetitions).

For $T = 1$, the repeated game is the stage game, which has a unique Nash equilibrium

Now assume the theorem is true for $T - 1$. Consider the T-times repeated game. All subgames beginning in the second period simply consist of the of the $(T - 1)$-times repeated game, which, by assumption, has a unique $SPNE$. Thus, in a $SPNE$, actions taken in the first period have no effect on choices in subsequent periods. In equilibrium first period choices must therefore be mutual best responses for the stage game. This means that the first period choices must be the Nash equilibrium choices for the stage game. Q.E.D.

Remarks:

(i) It is often said that a finitely repeated game "unravels" from the end, much like the centipede game.

(ii) Cooperation may be possible when the stage game has multiple Nash equilibria.

Example:

		Player 2: A	B	C
Player 1	a	4, 4	0, 0	0, 5
	b	0, 0	3, 3	0, 0
	c	5, 0	0, 0	1, 1

There are two Nash equilibria: (b, B) and (c, C). (a, A) is Pareto superior, but it is not a Nash equilibrium.

Imagine that the game is played twice in succession without discounting.

Strategies: Play a (A) in the first period. If the outcome in the first period was (a, A), play b (B) in the second period; otherwise, play c (C).

This is plainly a Nash equilibrium: any other strategy yields a gain of at most 1 unit in the first period, and a loss of at least 2 in the second period.

It is also a $SPNE$, since it prescribes a Nash equilibrium in every proper subgame.

Remark: There are folk theorems for finite horizon games formed by repetitions of stage games that possess multiple equilibria.

6.3 Applications

6.3.1 The repeated Bertrand model

Stage game: $N \geq 2$ firms simultaneously select price. Customers purchase from the firm with the lowest announced price, dividing equally in the event of ties. Quantity purchased is given by a continuous, strictly decreasing function $Q(P)$. Firms produce with constant marginal cost c. Let

$$\pi(p) \equiv (p - c)Q(p)$$

We assume that $\pi(p)$ is increasing in p on $[c, p^m]$ (where p^m is the monopoly price).

Observation: (i) Nash reversion involves setting $p = c$, which generates 0 profits. This is also the minmax profit level. Thus, Nash reversion generates the most severe possible punishment. Anything that can be sustained as an equilibrium outcome can be sustained using Nash reversion as punishments. Therefore, we can, without loss of generality, confine attention to equilibria that make use of Nash reversion.

(ii) The static Bertrand solution is unique. Thus, we know that no cooperation can be sustained in $SPNE$ for finite repetitions. Henceforth, we focus on infinite repetitions.

Analysis of equilibria:

Consider the following $h(\infty)$: both firms select some price $p^* \in [c, p^m]$ in every period.

Assuming that players discount future utility, when can we sustain this path as the outcome of a $SPNE$?

Given the preceding observation, we answer this question by determining the conditions under which this outcome can be supported as a Nash equilibrium using Nash reversion.

In equilibrium each firm receives a payoff of

$$\sum_{t=1}^{\infty} \frac{\pi(p^*)}{N} \delta^{t-1}$$

If a firm deviates to a price higher than p, it obviously earns nothing. If it deviates to a price below p, it will earn nothing in subsequent periods (since price will be driven to marginal cost), and its current period profits are bounded above by $\pi(p^*)$. Thus, no firm has an incentive to deviate provided that

$$\sum_{t=1}^{\infty} \frac{\pi(p^*)}{N} \delta^{t-1} \geq \pi(p^*)$$

This is equivalent to

$$\frac{1}{1-\delta} \geq N$$

which in turn implies

$$\delta \geq \frac{N-1}{N}$$

Provided discounting is not too great, cooperation is possible.

Implications:

(i) Cooperation becomes more difficult with more firms. For $N = 2$, cooperation is sustainable iff $\delta \geq \frac{1}{2}$. As $N \to \infty$, the threshhold discount factor converges to unity.

(ii) The equilibrium condition is independent of π, and therefore independent of p^* (the price we are attempting to sustain). For the Bertrand game, either everything is sustainable, or nothing is sustainable.

(iii) There is no longer a sharp discontinuity between one firm and two, as in the static Bertrand model. However, given (ii), there is still a sharp discontinuity between some N and $N+1$, where the best cooperative equilibrium shifts from monopoly to perfect competition.

6.3.2 The repeated Cournot model

Stage game: $N = 2$ firms simultaneously select quantities. The market clearing price is given by $P(Q) = a - bQ$. Firms produce with constant marginal cost c.

Let $Q^m = \frac{a-c}{2b}$ denote monopoly quantity, and let $\pi^m = \frac{(a-c)^2}{4b}$ denote monopoly profits.

Let $q^c = \frac{a-c}{3b}$ denote Cournot duopoly quantity, and let $\pi^c = \frac{(a-c)^2}{9b}$ denote Cournot duopoly profits (both per firm).

Observation: If the static Cournot equilibrium is unique, we know that it is impossible to sustain cooperation in $SPNE$ for finitely repeated games. Henceforth we focus on infinitely repeated games.

Analysis of equilibria using Nash reversion:

Consider the following $h(\infty)$: each firm sets $\frac{Q^m}{2}$ in every period.

Assuming that players discount future utility, when can we sustain this path as the outcome of a $SPNE$, using Nash reversion?

In equilibrium each firm receives a payoff of

$$\sum_{t=1}^{\infty} \frac{\pi^m}{2} \delta^{t-1} = \frac{1}{1-\delta} \frac{(a-c)^2}{8b}$$

Imagine instead that the firm makes a static best response to $\frac{Q^m}{2}$ (this is its best possible deviation). Best response profits given that the rival plays $\frac{Q^m}{2}$ are $\frac{9}{64}\frac{(a-c)^2}{b}$. In every subsequent period (after the deviation occurs), the deviating firm earns the static Cournot profits, π^c. The deviation therefore yields discounted profits of

$$\frac{9}{64}\frac{(a-c)^2}{b}+\frac{\delta}{1-\delta}\frac{(a-c)^2}{9b}$$

The proposed strategies therefore form an equilibrium iff

$$\frac{9}{64}\frac{(a-c)^2}{b}+\frac{\delta}{1-\delta}\frac{(a-c)^2}{9b}\leq\frac{1}{1-\delta}\frac{(a-c)^2}{8b}$$

Rearranging this expression yields

$$\delta\geq\frac{9}{17}>\frac{1}{2}$$

Implication: Using Nash reversion, it is easier to get cooperation with Bertrand than with Cournot. In the static setting, Bertrand is more competitive. Consequently, Nash reversion punishments are more severe.

Exercise: We know that Cournot profits decline with the number of firms. This means that, for the repeated Cournot game, Nash reversion punishments become more severe with more firms. Does the preceeding "implication" mean that, for Cournot, cooperation is easier to sustain with more firms? If not, why not?

Remark: The preceeding concerns the sustainability of the monopoly outcome. One can perform a similar calculation for other quantities. In contrast to the Bertrand model, it turns out that it is easier to sustain less cooperative outcomes (that is, the threshhold discount factors are lower). Indeed, for Cournot, it is possible to sustain some degree of cooperation (profits in excess of π^c) for all $\delta>0$. This is a consequence of the envelope theorem: as one reduces quantities starting at the Cournot equilibrium, the improvement in profits is first-order, but the change in the difference between profits and best-deviation profits is second order.

Exercise: For the linear Cournot model, solve for the most profitable symmetric equilibrium sustained by Nash reversion, as a function of the discount factor, δ.

Alternative punishments

Motivation: From the folk theorem, it is obvious that more severe punishments may be available than Nash reversion. In principle, the associated strategies could be extremely complex, which would make them difficult to analyze.

Under some circumstances, however, it is possible to characterize the most severe punishments within large classes of strategies, and to show that the associated strategies have a relatively simple "stick and carrot" structure. We illustrate using the Cournot model.

Definitions: A symmetric stick-and-carrot equilibrium for the repeated Cournot model is characterized by two levels of quantity, q^L and q^H, with $q^L < q^H$.

Let $g_i(q_i, q_j)$ denote firm i's profits when it produces q_i and j produces q_j. Assume we have chosen q^L and q^H so that $g_i(q^L, q^L) > g_i(q^H, q^H)$.

We define a stick-and-carrot strategy, $\sigma^{sc}(t, h(t))$, by induction on t:

(i) $\sigma^{sc}(1, h(1)) = q^L$ (start by playing q^L).

(ii) Having defined $\sigma^{sc}(t-1, h(t-1))$ for all feasible histories $h(t-1)$, we define $\sigma^{sc}(t, h(t))$ as follows. If $q_i(t-1) = \sigma^{sc}(t-1, h(t-1))$ for $i = 1, 2$, then $\sigma^{sc}(t, (h(t-1), q(t-1))) = q^L$. Otherwise, $\sigma^{sc}(t, (h(t-1), q(t-1))) = q^H$.

In words, the choice between q^L and q^H is always determined by play in the previous period. If firms have played their prescribed choices in the previous period, then they play q^L. If one or both deviated in the previous period, they play q^H.

When both players select stick-and-carrot strategies, play evolves as follows. On the equilibrium path, (q^L, q^L) is played every period. If a firm deviates in a single period $t-1$, both players play (q^H, q^H) in the following period as a punishment, after which they return to (q^L, q^L) forever. Notice that, if a player deviates, this strategy requires the player to participate in its own punishment in the following period by playing q^H. If it refuses and instead deviates in the punishment period, the punishment is prolonged. If, on the other hand, it cooperates in its punishment, the punishment period ends and cooperation is restored. Thus, there is both a "stick" (a one-period punishment) and a "carrot" (a reward for participating in the punishment). Use of the carrot can lead players to willingly participate in a very severe one-period punishment.

Analysis of equilibrium: We now derive the conditions under which $(\sigma^{sc}, \sigma^{sc})$ is a $SPNE$ for the infinitely repeated Cournot game. Using the standard dynamic programming argument, it suffices to check single-period deviations.

Given the stationary structure of the game and of the equilibrium, there are only two deviations to check: from $\sigma^{sc}(t, h(t)) = q^L$, and from $\sigma^{sc}(t, h(t)) = q^H$.

From q^L we have:

$$g_i\left(\gamma_i(q^L), q^L\right) - g_i\left(q^L, q^L\right) \leq \delta\left[g_i\left(q^L, q^L\right) - g_i\left(q^H, q^H\right)\right]$$

From q^H we have:

$$g_i\left(\gamma_i(q^H), q^H\right) - g_i\left(q^H, q^H\right) \leq \delta\left[g_i\left(q^L, q^L\right) - g_i\left(q^H, q^H\right)\right]$$

Specialize to the case where the "carrot" is the monopoly outcome, and the "stick" is the competitive outcome (price equal to marginal cost). That is, $q^L = \frac{a-c}{4b}$ and $q^H = \frac{a-c}{2b}$. Then these expressions can be rewritten as

$$\frac{(a-c)^2}{16b} \leq \delta \frac{(a-c)^2}{8b}$$

$$\frac{(a-c)^2}{64b} \leq \delta \frac{(a-c)^2}{8b}$$

Notice that the second expression is redundant. The first simplifies to $\delta \geq \frac{1}{2}$.

Implications: Since $\frac{1}{2} < \frac{9}{17}$, these strategies allow the firms to sustain the monopoly outcome at lower discount rates than with Nash reversion. Indeed, they can now achieve the monopoly outcome for the same range of discount factors as with the infinitely repeated Bertrand model.

Remark: The stick used here yields zero profits for a single period. One can also use more severe sticks that yield negative profits for a single period. Under some conditions, this allows one to construct punishments that yield zero discounted payoffs. The firms are willing to take losses in the short-term because they expect to earn positive profits in subsequent periods.

6.3.3 Cooperation with cyclical demand

Motivation: There is some evidence indicating that oligopoly prices tend to be countercyclical (oligopolists are more prone to enter price wars when demand is strong). If one thinks in terms of conventional supply and demand curves, this is counterintuitive. Note: the evidence is controversial.

Insight: The ability to sustain cooperation depends generally on the importance of the future relative to the present (we saw this with respect to the role of δ). When the present looms large relative to the future, cooperation is more difficult to sustain. This is what occurs during booms.

Model:

Demand is random. Each period, one of two states, H or L, is realized. The states are equally probable, and realizations are independent across periods. Demand for state i is $Q_i(p)$, with $Q_H(p) > Q_L(p)$ for all p.

N firms acting Bertrand competitors.

Production costs are linear with unit cost c.

Notation:

Let $\pi_k(p)$ represents industry profits in state k with price p:

$$\pi_k(p) \equiv (p-c)Q_k(p)$$

Let π_k^m denote industry monopoly profits in state k:

$$\pi_k^m = \max_p \pi_k(p)$$

Equilibrium analysis:

Conditions for equlibrium:

Consider any stationary, symmetric equilibrium path such that both firms select the price p_H in state H and p_L in state L.

Construct equilibrium strategies using Nash reversion (here, these are the most severe possible subgame perfect punishments since they yield zero profits)

Given the stationary structure of the problem and the usual dynamic programming argument, we need only check to see whether the firms have incentives to make one period deviations in each state.

For state H:

$$\left(\frac{N-1}{N}\right)\pi_H(p_H) \leq \left(\frac{\delta}{1-\delta}\right)\left[\frac{1}{2}\pi_H(p_H) + \frac{1}{2}\pi_L(p_L)\right]\frac{1}{N}$$

For state L:

$$\left(\frac{N-1}{N}\right)\pi_L(p_L) \leq \left(\frac{\delta}{1-\delta}\right)\left[\frac{1}{2}\pi_H(p_H) + \frac{1}{2}\pi_L(p_L)\right]\frac{1}{N}$$

Note that the right-hand sides of these expressions are the same, since the future looks the same irrespective of the current demand state. For any given price, the left-hand side is greater for the high demand state. Therefore, a given price is more difficult to sustain for the high demand state than for the low demand state.

Specialized paramateric assumptions:

Before going further, we will simplify the model by making some parametric assumptions:

(i) $Q = \theta - p$

(ii) $\theta \in \{\theta_L, \theta_H\} = \{1, 2\}$

(iii) $c = 0$

(iv) $N = 2$

Under these assumptions, $\pi_k(p) = p(\theta_k - p)$, $\pi_L^m = \frac{1}{4}$, $p_L^m = \frac{1}{2}$, $\pi_H^m = 1$, and $p_H^m = 1$.

Question: When do we get the full monopoly solution, p_L^m in state L, and p_H^m in state H?

Look back at the constraints. If the constraint is satisfied for monopoly in state H, then it is also satisfied for monopoly in state L. Therefore, we need only check the constraint for state H. Substituting, we have

$$1 \leq \left(\frac{\delta}{1-\delta}\right)\left[\left(\frac{1}{2} \times 1\right) + \left(\frac{1}{2} \times \frac{1}{4}\right)\right]$$

This is equivalent to $\delta \geq \frac{8}{13}$.

Note: Since $\frac{8}{13} > \frac{1}{2}$, it is more difficult to sustain full monopoly here than in the Bertrand model with time-invariant demand.

Question: What happens for lower δ?

We know p_H^m becomes unsustainable for state H. However, the constraint for p_L^m in the low state holds with strict inequality at $\delta = \frac{8}{13}$. Consequently, one would still expect it to hold for slightly smaller δ.

Proceed as follows: Assume that, for some $\delta < \frac{8}{13}$, p_L^m is sustainable. Calculate the highest level of sustainable profits, π_H, for state H. If $\pi_H \geq \pi_L^m$, then the initial assumption is valid, and we have an equilibrium.

To compute the highest level of sustainable profits, π_H, for state H under the aforementioned assumption, we substitute into the equilibrium constraint:

$$\pi_H \leq \left(\frac{\delta}{1-\delta}\right)\left[\frac{1}{2}\pi_H + \left(\frac{1}{2} \times \frac{1}{4}\right)\right]$$

For the highest sustainable level of state H profits, this constraint holds with equality. Rearranging yields

$$\pi_H^\delta = \frac{\delta}{8 - 12\delta}$$

One can check the following:

$$\text{For } \delta = \frac{8}{13},\ \ \pi_H^\delta = 1 = \pi_H^m$$

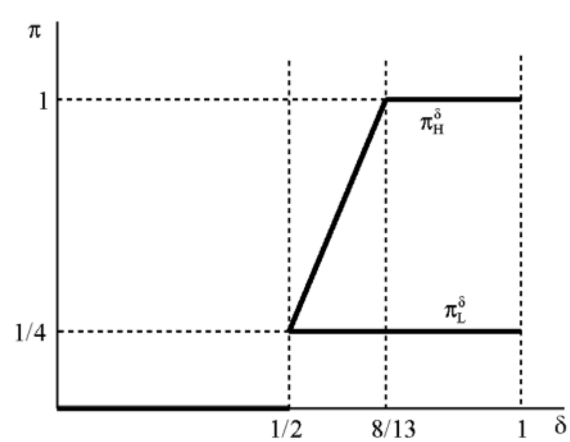
$$\text{For } \delta = \frac{1}{2},\ \pi_H^\delta = \frac{1}{4} = \pi_L^m$$

Thus, as long as $\delta \in \left[\frac{1}{2}, \frac{8}{13}\right]$, the assumption that π_L^m is sustainable is valid.

Exercise: Verify that, when $\delta < \frac{1}{2}$, the only $SPNE$ outcome involves repetitions of the static Bertrand outcome (price equal to marginal cost). As in the standard repeated Bertrand model, no cooperation is sustainable for discount factors below $\frac{1}{2}$.

Properties of equilibrium:

(i) π_H^δ (sustainable profits in the high demand state) is increasing in δ. Graphically:



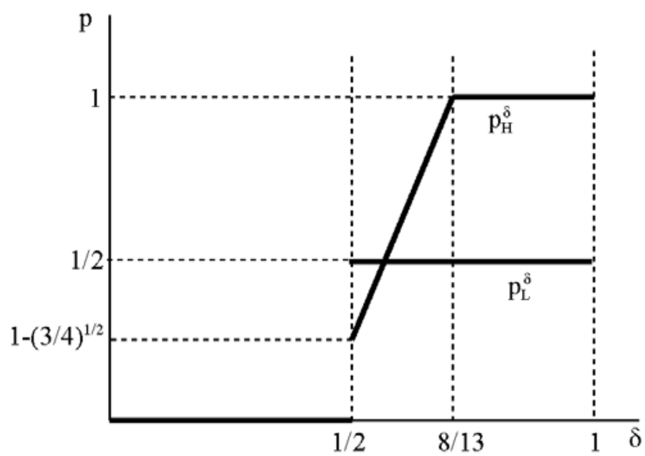
(ii) Comparison of prices in the two states

For $\delta \geq \frac{8}{13}$, $p_H = p_H^m > p_L^m = p_L$. Prices move pro-cyclically (higher in booms).

For $\delta = \frac{1}{2}$, $\pi_H = \pi_L^m$. To achieve the same profits in the high demand state as in the low demand state, prices must be lower in the high demand state. Therefore, prices move counter-cyclically.

To find the best cooperative price in state H for $\delta = \frac{1}{2}$, we set $p_H(2 - p_H) = \pi_L^m = \frac{1}{4}$, which yields $p_H = 1 - \left(\frac{3}{4}\right)^{1/2}$.

Graphically, the most cooperative sustainable prices look like this:



Conclusion: There is a range of discount factors over which the best sustainable price moves countercyclically.

6.3.4 Multimarket contact

Motivation: In certain circles, there is a concern that large, conglomerate enterprises are anticompetitive.

Corwin Edwards: When firms come into contact with each other across many separate markets (geographic or otherwise), opportunistic behavior in any market is likely to be met with retaliation in many markets, and this may blunt the edge of competition.

Is this reasoning correct from the perspective of formal game theory?

(i) It is correct that, with multimarket contact, deviations may lead to more severe punishments involving larger numbers of markets. However,

(ii) Knowing this, if a firm were to deviate from a cooperative agreement, it would deviate in all markets. Consequently, it is not obvious that multimarket contact does anything more than increase the scale of the problem.

It turns out that multimarket contact can facilitate cooperation, but not for the reasons suggested by Edwards.

The central insight:

Notation:

i denotes firm

k denotes market

G_{ik} denotes the net gain to firm i from deviating in market k for the current period, for a particular equilibrium

π_{ik}^c denotes the discounted payoff from continuation (next period forward) for firm i in market k, assuming no deviation from the equilibrium in the current period.

π^p_{ik} denotes the discounted "punishment" payoff from continuation (next period forward) for firm i in market k, assuming that i deviates from the equilibrium in the current period.

Equilibrium conditions when markets are separate:

For each i and k,

$$G_{ik} + \delta \pi^p_{ik} \leq \delta \pi^c_{ik}$$

Note that there are $N \times K$ such constraints.

Equilibrium conditions when markets are linked strategically:

For each i,

$$\sum_k G_{ik} + \delta \sum_k \pi^p_{ik} \leq \delta \sum_k \pi^c_{ik}$$

Notice that there are N constraints.

Implication: Multimarket contact pools incentive constraints across markets. This may enlarge the set of outcomes that satisfies the incentive constraints.

For example, the set $\{(x, y) \mid x \leq 4 \text{ and } y \leq 4\}$ is strictly smaller than the set $\{x, y \mid x + y \leq 8\}$.

As it turns out, pooling incentive constraints strictly expands the set of sustainable outcomes, and in particular improves upon the best cooperative outcome, in a number of different circumstances. We will study one of them.

Illustration: Slack enforcement power in one market

Idea: When there is more enforcement power than needed to achieve full cooperation in one market, the extra enforcement power can be used in another market where full cooperation is not achievable. We will consider two examples.

Example #1: Differing numbers of firms in each market.

Suppose firms produce homogeneous goods in each market and compete by naming prices (Bertrand)

Imagine that there are two markets. There are N firms in market 1 and $N+1$ firms in market 2. Moreover,

$$\frac{N-1}{N} < \delta < \frac{N}{N+1}$$

From our analysis of the infinitely repeated Bertrand problem, we know that the monopoly price is sustainable for market 1:

$$\sum_{t=1}^{\infty} \frac{\pi(p_1^m)}{N} \delta^{t-1} > \pi(p_1^m)$$

However, no cooperative price $p_2 > c$ is sustainable for market 2:

$$\sum_{t=1}^{\infty} \frac{\pi(p_2)}{N+1} \delta^{t-1} < \pi(p_2)$$

Thus, if single-market firms operate in both markets, market 1 will be monopolized, while market 2 will be competitive.

Now suppose that N conglomerate firms operate in both markets, and that one single-market firm operates in market 2. Let $1-\alpha$ denote the share of market 2 served by the single-market firm. We will attempt to sustain a cooperative arrangement wherein the N conglomerate firms divide the remaining share (α) equally. The incentive constraint for the single-market firm is:

$$\sum_{t=1}^{\infty} (1-\alpha)\pi(p_2)\delta^{t-1} \geq \pi(p_2)$$

This is equivalent to the requirement that $\alpha \leq \delta$. Thus, the conglomerate firms must cede at least the share $1-\alpha$ to the single-market firm to deter the single-market firm from deviating.

For the conglomerate firms, the incentive constraint becomes

$$\sum_{t=1}^{\infty}\left[\frac{\pi(p_1^m)}{N}+\alpha\frac{\pi(p_2)}{N}\right]\delta^{t-1}\geq\pi(p_1^m)+\pi(p_2)$$

We will attempt to sustain a cooperative arrangement that cedes as little market share to the single-market firm as possible ($\alpha=\delta$). Making this substitution and rearranging, we obtain (after some algebra):

$$\pi(p_2)\leq\pi(p_1^m)\left(\frac{\delta-\frac{N-1}{N}}{\frac{N}{N+1}-\delta}\right)\left(\frac{N}{N+1}\right)$$

Under our assumptions (cooperation is sustainable in market 1 but not in market 2), the RHS of this inequality is strictly positive. Thus, through multimarket contact, one can always sustain $p_2>c$ in market 2 without sacrificing profits in market 1. If δ is sufficiently close to $\frac{N}{N+1}$ (cooperation in market 2 is almost sustainable in isolation), one can achieve monopoly profits in market 2. Note that the conglomerate firms always cede a larger market share to the single-market share to sustain cooperation.

Example #2: Cyclical demand

Consider the model analyzed in the preceding section. Imagine that there are two such markets, and that the same firms operate in both markets. Suppose moreover that the demand shocks in these markets are perfectly negatively correlated (so that there is always a market in state H and a market in stage L). Since the markets are symmetric, this means that there is really only one demand state. Pooling incentive constraints across markets, we obtain:

$$\begin{aligned}&\left(\frac{N-1}{N}\right)[\pi_H(p_H)+\pi_L(p_L)]\\ \leq\;&\left(\frac{\delta}{1-\delta}\right)\left[\frac{1}{2}\pi_H(p_H)+\frac{1}{2}\pi_L(p_L)\right]\frac{2}{N}\end{aligned}$$

After cancellation, one obtains

$$N-1\leq\frac{\delta}{1-\delta}$$

This is equivalent to $\delta \geq \frac{N-1}{N}$, which is exactly the same as for the simple repeated Bertrand model. For example, with $N = 2$, we obtain full cooperation in both states for all $\delta \geq \frac{1}{2}$.

Remark: As the correlation between the demand shocks rises, the gain to multimarket contact declines. When the shocks are perfectly positively correlated, there is no gain. This implies that the potential harm from multimarket contact is greater when the markets are less closely related.

6.3.5 Price wars

Motivation: Price wars appear to occur in practice. However, in the standard model, one only obtains price wars off the equilibrium path. This means they happen with probability zero.

Insight: One can generate price wars on the equilibrium path by considering repeated games in which actions are not perfectly observable. To enforce cooperation, the players must punish outcomes that are correlated with deviations. But sometimes those outcomes occur even without deviations, setting off a price war. In that case, the punishments must be chosen very carefully to assure that the consequences of the occasional war do not outweigh the benefits of cooperation.

Model:

$N = 2$ firms produce a homogeneous good with identical unit costs c

The firms compete by naming prices (Bertrand); indifferent consumers divide equally between the firms.

Firms do not observe each others' price choices, even well after the fact.

Demand in each period is either "high" or "low"

Low demand states occur with probability α. Consumers purchase nothing.

High demand states occur with probability $1 - \alpha$. Consumer purchase $Q(p)$.

Realizations of demand are independent across periods.

The firms cannot directly observe the state of demand, even well after the fact.

A firm only observes its own price and the quantity that it sells.

Let p^m denote the solution to $\max_p (p - c)Q(p) \equiv \pi^m$.

Analysis of equilibrium:

Object: sustain p^m

Problem: if a firm ends up with zero sales, there are two explanations: (i) demand is low, and (ii) its competitor has deviated. It cannot tell the difference.

To sustain p^m, the equilibrium must punish deviations. The only alternative is to enter a punishment phase (price war) any time a firm has zero sales.

Key difference from previous models: the trigger for a price war occurs with strictly positive probability in equilibrium.

Because price wars will actually occur, the firms want the consequences of these wars to be no more severe than absolutely necessary to sustain cooperation. We no longer use *grim strategies* that involve punishing forever.

"Trigger price" strategies:

Charge p^m initially.

As long as firm i has played p^m and made positive sales (or played $p > p^m$) in all previous periods, it continues to play p^m

If, in any period $t-1$, firm i either deviated to $p < p^m$ or made zero sales, the game enters a punishment phase in period t.

In the punishment phase, both firms charge $p = c$ for T periods.

When the punishment phase is over, the strategies reinitialize, treating the first non-punishment period as if it were the first period of the game.

Value functions:

Let V^c denote the expected present value of payoffs from the current period forward when play is not in a punishment phase.

Let V^p denote the expected present value of payoffs from the current period forward in the first period of a punishment phase.

These valuations are related as follows:

$$V^c = (1-\alpha)\left[\frac{\pi^m}{2} + \delta V^c\right] + \alpha\left[0 + \delta V^p\right]$$

$$V^p = \sum_{s=0}^{T-1}(\delta^s \times 0) + \delta^T V^c$$

Substituting in the first expression for V^p using the second expression yields:

$$V^c = (1-\alpha)\left[\frac{\pi^m}{2} + \delta V^c\right] + \alpha\delta^{T+1}V^c$$

Next we solve for V^c:

$$V^c = \frac{(1-\alpha)\frac{\pi^m}{2}}{1-\delta(1-\alpha)-\alpha\delta^{T+1}}$$

Deviations:

Now we evaluate the profitability of a deviation. Plainly, there is no incentive to deviate during a punishment phase (the prescribed actions constitute the equilibrium for the stage game). We need only check the desirability of deviating outside of punishment phases. The best possible deviation is to slightly undercut p^m. The expected present value of the resulting profits is given by

$$\begin{aligned} V^d &= (1-\alpha)\left(\pi^m + \delta V^p\right) + \alpha\left(0 + \delta V^p\right) \\ &= (1-\alpha)\pi^m + \delta^{T+1} V^c \end{aligned}$$

Deviations are unprofitable as long as $V^c \geq V^d$. This requires

$$V^c \geq (1-\alpha)\pi^m + \delta^{T+1} V^c$$

This is equivalent to:

$$V^c \geq \frac{\pi^m(1-\alpha)}{1-\delta^{T+1}}$$

Now we substitute the expression for V^c derived above. The π^m term cancels – as in the standard repeated Bertrand model, the feasibility of cooperation is all or nothing. Rearranging terms yields the equilibrium condition:

$$2\delta(1-\alpha) + (2\alpha - 1)\delta^{T+1} \geq 1$$

The length of the punishment period:

Note that the equlibrium does not hold for $T = 0$ (the left hand side reduces to δ)

An increase in T reduces the absolute value of the second term on the LHS. This can increase the value of the LHS only if $2\alpha - 1$ is negative. In that case, the value of the LHS remains bounded below $2\delta(1-\alpha)$. Consequently, the equilibrium holds for some $T > 0$ if and only if

(i) $2\alpha - 1 < 0$, and

(ii) $2\delta(1-\alpha) > 1$

Condition (ii) implies condition (i), so we only need to check (ii). When (ii) is satisfied, cooperation is possible. The best cooperative equilibrium involves the least severe punishments consistent with incentive compatibility. This requies us to pick the smallest value of T satisfying the equilibrium condition.

Note that (ii) can be rewritten as

(ii)′ $\delta > \frac{1}{2(1-\alpha)}$

For the special case of $\alpha = 0$, this gives $\delta > \frac{1}{2}$, which is the correct answer for the Bertrand model when there is no observability problem.

Conclusions:

(i) Price wars are observed in equilibrium.

(ii) Price wars are set off by declines in demand. (Note that this contrast with the model of cyclical demand, in which prices fall when demand is high. The key difference concerns observability.)

When an equilibrium price war occurs, everyone believes correctly that no one deviated. It may seem odd to enter a punishment phase under those circumstances. However, if the firms didn't punish this non-deviation, the incentives to comply with the cooperative agreement would vanish.

(iii) Equilibrium price wars are transitory.

(iv) Imperfect observability makes cooperation more difficult (it raises the threshhold value of δ).

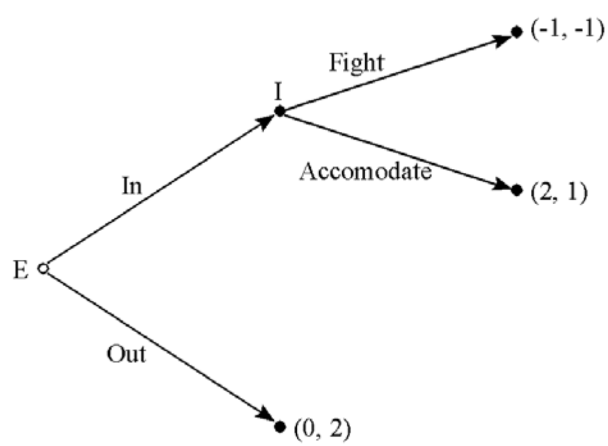
7 Strategic Choice in Dynamic Games with Incomplete Information

Games of incomplete information become particularly interesting in dynamic settings where the actions of players can reveal something about their types. Knowing this, players have incentives to tailor their actions to manipulate the inferences of others. Naturally, others anticipate this manipulation, and attempt to make inferences subject to the knowledge that they are being manipulated.

7.1 Reputation

7.1.1 Reputation with complete information

We will illustrate some ideas using a variant of the simply entry game considered earlier.



We know that if this game is played only once, the only plausible outcome ($SPNE$) is (In, Accommodate). Yet, in practice, an incumbent might have a reason to play fight if confronted with entry, in order to establish a reputation for toughness.

We say that an individual has a reputation if they are expected to behave a certain way in the current environment because they have behaved the same way in similar environments in the past when playing against others.

To model reputation, it is therefore natural to consider a setting in which (1) the game is repeated, and (2) the incumbent encounters *different* opponents each period. This is an example of repeated game with both long-lived and short-lived players.

Infinite repetitions: Imagine that the incumbent in the above game plays this game repeatedly against a sequence of different opponents. These repetitions continue forever. The incumbent discounts the future at the rate δ. The opponents care only about their payoffs in the current period.

Clearly, this repeated game has a subgame perfect equilibrium in which all opponents enter and the incumbent always accommodates entry. However, there may be other equilibria (even though opponents all have one-period horizons).

Proposed equilibrium:

Strategy for opponents: If either all opponents have stayed out in the past, or if the incumbent has never accommodated entry in the past, then play Out; otherwise, play In.

Strategy for incumbent: Imagine that the current opponent enters. If either all opponents have stayed out in the past or if the incumbent has never accommodated entry in the past, then play $Fight$; otherwise, play Acc.

Check that this is a $SPNE$:

First check the choices for the opponents.

(1) Suppose that either all opponents have stayed out in the past, or that the incumbent has never accommodated entry in the past. Then the current opponent expects the

incumbent to respond to entry by playing $Fight$. Consequently, the opponent's best choice is to stay Out.

(2) Now suppose that some opponent has entered in the past and that the incumbent has accommodated entry. Then the current opponent expects the incumbent to accommodate entry in the current period. Consequently, the opponent's best choice is to play In.

Next we check the choices for the incumbent.

(1) Suppose that either all opponents have stayed out in the past, or that the incumbent has never accommodated entry in the past. Imagine that the current opponent plays In. If the incumbent plays the choice prescribed by her equilibrium strategy ($Fight$), she receives a payoff of -1 in the current period, and a payoff of 2 in all subsequent periods (since all subsequent opponents will stay out). Her discounted payoff is therefore $-1 + \frac{2\delta}{1-\delta}$. If the incumbent instead plays *Acc*, she receives a payoff of 1 in the current period and in all future periods (since, according to the strategies, entry will occur and she will accommodate). Her discounted payoff is therefore $\frac{1}{1-\delta}$. Thus, it is optimal for the incumbent to play $Fight$ provided that

$$\frac{2\delta}{1-\delta} - 1 \geq \frac{1}{1-\delta}$$

This requires $\delta \geq \frac{2}{3}$.

(2) Suppose that some opponent has entered in the past and that the incumbent has accommodated entry. Imagine that the current opponent plays In. Regardless of what the incumbent does, all future opponents will play In. Consequently, the incumbent's best choice is to accommodate entry.

Remarks:

(1) The incumbent benefits from a "reputation" for toughness. She fights anyone who enters to sustain this reputation. If she ever fails to fight, she loses this reputation. This provides her with the incentive to fight. However,

(2) In equilibrium, the incumbent never does anything to create or to maintain this reputation. She is simply endowed with it.

(3) There are also $SPNE$ in which the incumbent does not benefit from a reputation.

(4) The ability to sustain a reputation vanishes when the horizon is finite (standard unravelling argument).

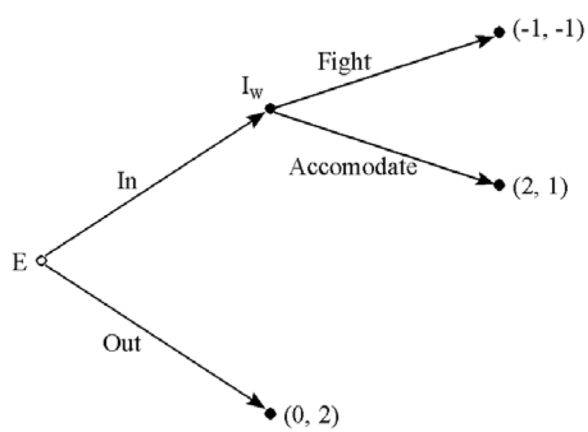
An alternative approach to modeling reputation: Beliefs about types in games of incomplete information.

7.1.2 Reputation with incomplete information

A one-shot game:

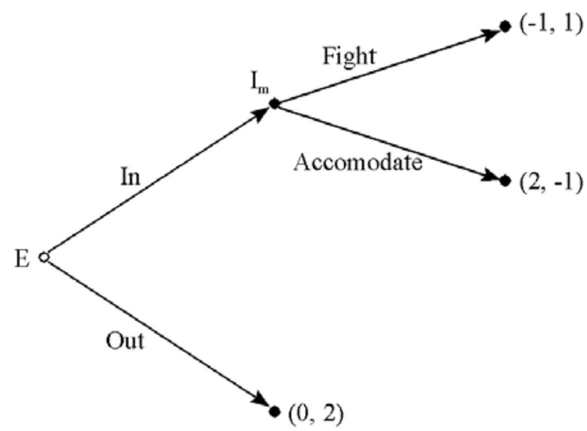
Two types of incumbents: "Macho" (M) or "Wimpy" (W)

If the incumbent is known to be wimpy, the game is:



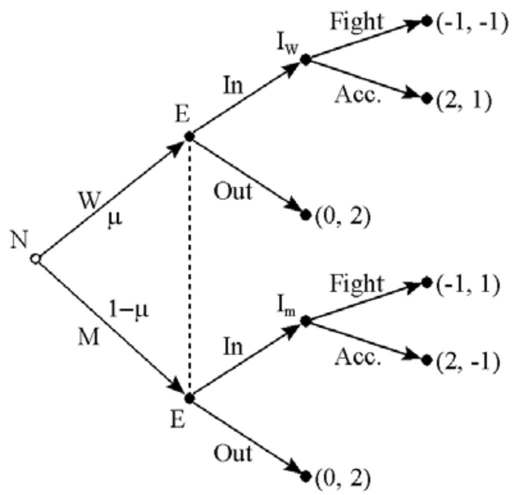
The unique $SPNE$ is (In, Acc)

If the incumbent is known to be macho, the game is:

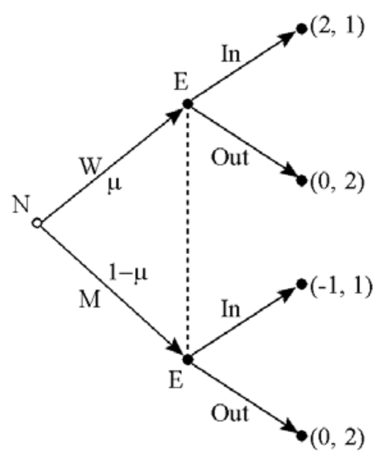


The unique $SPNE$ is $(Out, Fight)$

Now assume that the entrant does not know the incumbent's type. He assumes that the incumbent is wimpy with probability μ and macho with probability $1 - \mu$. The game is:



The requirement of subgame perfection allows us to reduce this tree to:



This is a simple single-person decision problem. E's payoff from In is $3\mu - 1$. E's payoff from Out is 0.

Thus, E enters if $\mu > \frac{1}{3}$, does not enter if $\mu < \frac{1}{3}$, and is indifferent if $\mu = \frac{1}{3}$.

Introducing reputation: Imagine that the incumbent has been operating in the industry for some time. The entrant will look back on the incumbent's past behavior to make inferences about whether the incumbent is macho or wimpy. Knowing this, the incumbent may behave in a way designed to mislead potential entrants. This compounds the entrant's inference problem.

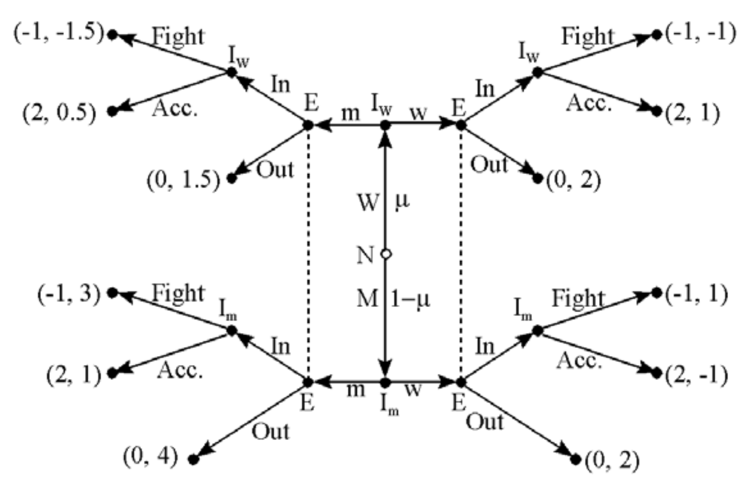
One could model this in a setting with (finitely) repeated interaction. To keep things simple, we will use a more highly stylized model.

A simple representation: Suppose that, prior to playing the preceding game (stage 2), the incumbent has the opportunity to raid the market of another firm (stage 1). In stage 1, the incumbent can choose either to raid (called m for "acting macho"), or not to raid (called w for "acting wimpy"). (One can think of this as playing a similar game against some other opponent.)

In stage 1, payoffs are as follows: I_m receives 2 from m and 0 from w, while I_w receives $-\frac{1}{2}$ from m and 0 from w

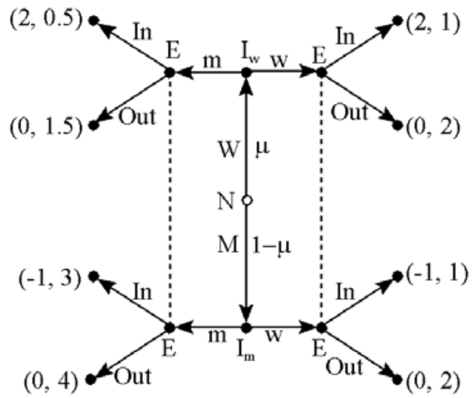
Payoffs from the two stages are additive.

Extensive form:



Sequential equilibria

We will solve for the sequential equilibria of this game. Since sequential equilibria are subgame perfect, we can replace any proper subgame possessing a unique equilibrium with the payoffs from that equilibrium. Here, that means substituting for the incumbent's final decisions, as follows:



We will work with the reduced game and abstract from the incumbent's final decisions (they are always implicit)

The solution will differ according to whether μ is greater or less than $\frac{1}{3}$.

Solution for $\mu < \frac{1}{3}$:

Claim #1: There is a sequential equilibrium in which both I_w and I_m play m. E enters if the incumbent has played w, but does not enter if the incumbent has played m. Beliefs off the equilibrium path: if w is observed, E believes he faces I_w with probability 1.

Demonstration: First we verify that all actions are optimal, given beliefs and other players' strategies

I_w: receives 1.5 from m and 1 from w, so m is optimal

I_m: receives 4 from m and 1 from w, so m is optimal

E if m: receives $3\mu - 1$ from In, and 0 from Out; with $\mu < \frac{1}{3}$, Out is optimal

E if w: given beliefs, receives 2 from In and 0 from Out, so In is optimal

Beliefs are consistent: Let I_w play w with probability ε, and let I_m play w with probability ε^2. $\Pr(I = I_w \mid w) = \frac{\mu\varepsilon}{\mu\varepsilon + (1-\mu)\varepsilon^2}$, which converges to unity as $\varepsilon \to 0$.

Claim #2: There are no other sequential equilibria.

First notice that I_m's lowest possible payoff is 3 if it plays m, and its highest possible payoff is 2 if it plays w. Thus, I_m must play m with certainty in all sequential equilbria.

Next we argue that there cannot be an equilibrium in which I_w plays w with certainty. In that case, m would indicate that the incumbent was I_m with probability 1, so E would not enter upon observing m, and w would indicate that the incumbant was I_w with probability 1, so the E would enter upon observing w. But then, by playing m rather than w, I_w would increase her payoffs from 1 to 1.5.

Finally we argue that there cannot be an equilibrium in which I_w mixes between m and w. I_w would be willing to mix only if she were indifferent. Upon observing w, E would infer that the incumbent was I_w with probability 1 (since I_m makes this choice with probability zero). E would therefore choose In, which means that I_w would receive a payoff of 1. Upon observing m, E would infer that the incumbent was I_w with probability less $\lambda < \mu < \frac{1}{3}$, and would therefore choose Out, which means that I_w would receive a payoff of 1.5. Thus, I_w could not be indifferent between w and m.

Conclusion: There is a unique sequential equilibrium. It has the property that I_w imitates I_m in order to deter entry. I_w succeeds because E fears that the incumbent might be macho upon observing m. In other words, I_w acts macho to disguise her true type.

Solution for $\mu > \frac{1}{3}$:

Claim #1: There is no pure strategy sequential equilibrium.

We have already shown that I_m must play m.

Can I_w play m with probability 1? No. Upon observing m, E would infer that the incumbent is I_w with probability $\mu > \frac{1}{3}$; we have already shown that, for such inferences, E enters. Thus, if I_w plays m, she receives a payoff of 0.5. On the other hand, if I_w plays w, she receives a payoff no smaller than 1. Consequently, I_w would have an incentive to deviate to w.

Can I_w play w with probability 1? No. Upon observing m, E would infer that the incumbent is I_m with probability 1 and not enter. Upon observing w, E would infer that the incumbent is I_w with probability 1 and enter. Thus, I_w would receive a payoff of 1.5 from playing m, and a payoff of 1 from playing w. I_w would therefore deviate to w.

Constructing the (unique) mixed strategy equilibrium:

We know that I_m must play m with probability 1.

If I_w is mixing between m and w, then we also know that, upon observing w, E must infer that the incumbent is I_w, and consequently E will enter.

That leaves two decisions: I_w's choice between m and w, and E's choice between In and *Out* conditional upon having observed m.

To have a MSE, I_w must randomize between m and w to make E indifferent between In and Out conditional on observing m. Moreover, E must randomize between In and Out conditional upon observing m to make I_w indifferent between m and w.

Let E randomize between In and Out with equal probabilities (conditional upon observing m). Then I_w's expected payoff from playing m is 1. If I_w plays w, E will enter and I_w's payoff will also be 1. Thus, this strategy for E makes I_w indifferent between m and w, as required.

Next, let I_w randomize between m and w with probabilities φ and $1-\varphi$, respectively, where

$$\varphi \equiv \frac{1-\mu}{2\mu}$$

Note that, for $\mu \in [\frac{1}{3}, 1]$, $\varphi \in [0, 1]$, as required of a probability.

Having observed m, E's posterior belief is:

$$\begin{aligned} \Pr(I = I_w \mid m) &= \frac{\varphi\mu}{(1-\mu)+\varphi\mu} \\ &= \frac{1}{3} \end{aligned}$$

With this posterior, E is indifferent between entering and not entering, as required.

Interpretation of the MSE:

(i) When $\mu \geq \frac{1}{3}$, wimps deter entry in some fraction of cases by randomly imitating macho players.

(ii) Deterrence is not complete.

(iii) Macho incumbents are hurt by this imitation.

(iv) The probability of imitation declines as μ rises, but the effect on macho types does not change with μ.

7.2 Cooperation in finite horizon games

Idea: Agents may be able to sustain cooperation if, by cooperating, they can constructively mislead others about their objectives.

Model: Stage game (generalized prisoners' dilemma):

		Player 2	
		C	N
Player 1	C	1, 1	-a, b
	N	b, -a	0, 0

Assume $a > 0$, $b > 1$.

Finite repetitions, no discounting

Imagine that player 2 is either "sane" with probability $1 - \theta$, or "crazy" with probability θ.

A sane player optimizes subject to beliefs.

A crazy player behaves mechanically as follows: Plays C as long as player 1 has never played N in the past. Otherwise, play N.

If the game is played once, player 1 and the sane player 2 will both choose N.

If the game is played more than once, the possibility that player 2 might be crazy obviously gives player 1 some incentive to play C in early rounds. Less obviously, it also gives player 2 an incentive to play C, since 2 may wish to give 1 the impression that 2 is crazy.

Two repetitions

Second period: Both player 1 and the sane player 2 will plainly choose N regardless of what has previously transpired, and regardless of player 1's beliefs about player 2's type. The crazy player 2 picks N if player 1 chose N in the first period, and C if player 1 chose C in the first period.

First period: The sane player 2 knows that player 1 will choose N in the second period regardless of what player 2 chooses in the first period. Therefore, the sane player 2 will choose N. The crazy player 2 will, of course, choose C.

Player 1 knows that, in the first period, the sane player 2 will choose N, and that the crazy player 2 will choose C. Player 1 also knows the continuation for the second period. If player 1 chooses C, her total expected payoff is $\theta(1+b)-(1-\theta)a = \theta(1+b+a)-a$. If player 1 chooses N, her total expected payoff is θb. Thus, 1 plays C provided that $\theta(1+b+a)-a > \theta b$, or $\theta > \frac{a}{1+a}$. Player 1 plays N when this inequality is reversed, and is indifferent when equality holds.

Three repetitions

Beliefs: Let μ denote player 1's probability assessment that player 2 is crazy, assessed at the beginning of the second round of play, conditional upon the outcome of the first round.

Continuation equilibria when player 1 has chosen C in round 1:

In this case, the continuation game is the same as the two-repetition game considered above, with μ replacing θ.

If $\mu > \frac{a}{1+a}$, the continuation is as described in the "two-repetitions" section above, with player 1 selecting C in the second round.

If $\mu < \frac{a}{1+a}$, the continuation is as described in the "two-repetitions" section above, with player 1 selecting N in the second round.

If $\mu = \frac{a}{1+a}$, the continuation is as described in the "two-repetitions" section above, with player 1 potentially randomizing between C and N in the second round.

Continuation equilibria when player 1 has chosen N in round 1

In this case, the continuation game differs from the two-repetition game considered above, since the "crazy" player 2 will select N in both periods, no matter what else happens. This means that the third-round outcome must be (N, N) no matter what has happened in the second round; consequently, the outcome will also be (N, N) in the second round.

An equilibrium with full cooperation in round 1

Under what conditions is it an equilibrium for both players choose C in period 1?

Observe that the equilibrium would involve no revelation of information in the first round (since both types of player 2 play C).

Given that no information is revealed in the first round (in equilibrium), what will happen in the continuation game?

With no information revealed in the first round, the continuation will be as described above, with $\mu = \theta$.

Suppose that $\theta > \frac{a}{1+a}$. Then player 1 selects C in the second round, and receives a continuation payoff of $\theta(1 + b + a) - a$, while the sane player 2 selects N in the second round and receives a continuation payoff of b (see above). Thus, player 1's total equilibrium payoff is $1 + \theta(1 + b + a) - a$, while the sane player 2's total equilibrium payoff is $1 + b$.

Now let's check to see whether it is optimal for both players to select C in the first period (assuming that the other will do so).

Player 1: If player 1 chooses N, her total payoff is b (given that continuation payoffs are zero when player 1 choses N in round 1). Thus, player 1 is willing to play C in round 1 provided that $1 + \theta(1 + b + a) - a \geq b$. This simplifies to $\theta \geq \frac{a+b-1}{a+b+1} = \theta^*$. Note that, under our assumptions, $\theta^* \in (0, 1)$.

Player 2: If the sane player 2 chooses N, his total payoff is b (in any sequential equilibrium, continuation payoffs are zero when the sane player 2 has revealed his type, since $\mu = 0 < \frac{a}{1+a}$). Thus, player 2 is willing to play C in round 1 provided that $1 + b \geq b$, which is always satisfied.

Conclude: Under appropriate parametric assumptions, there is a sequential equilibrium in which both of the players play C in the first round.

Remarks: (1) In the preceding equilibrium, the sane player 2 imitates the crazy player 2 because this maintains the possibility in player 1's mind that player 2 is crazy, and thereby induces player 1 to play C in round 2 (which in turn permits player 2 to earn a payoff of b in round 2).

(2) Note that we obtain cooperation in the first round, possible cooperation (or partial cooperation) in the second round, and no cooperation in the last round. This is consistent with the way people play the repeated prisoners' dilemma: cooperation occurs in early rounds, and collapses in later rounds.

(3) This is only one equilibrium, and we need to make some restrictive parametric assumptions to generate it. Cooperation is only one possible outcome.

Large numbers of repetitions:

Assume that the game is played T times. We will show that, for large T, one *necessarily* obtains cooperation in *almost every* period.

Theorem: In any sequential equilibrium, the number of stages where one player or the other plays N is bounded above by a constant that depends on θ, but is independent of T.

Proof: The proof proceeds in several steps.

(1) If the sane player 2's type ever becomes known to player 1 prior to some period t, then both players must select N in round t and in all succeeding rounds. This follows from induction on the number of stages left in the game. It is obviously true for $t = T$. Now assume that it is true for some arbitrary $t \leq T$. In round $t-1$, both players know that their actions cannot affect outcomes in subsequent rounds. Consequently, they must make static best responses in round $t-1$, which means that they both play N.

(2) If both players have selected C in rounds 1 through $t-1$, and if the sane player 2 selects N in round t, then both players must select N in all succeeding rounds. Since the choice of N in round t reveals player 2's type as sane, this follows directly from step (1).

(3) If player 1 plays N in round $t-1$, then player 1 and both types of player 2 all select N in all subsequent rounds.

Suppose that player 1 chooses N in round $t-1$.

Claim: In all rounds $t' > t-1$, both the crazy player 2 and the sane player 2 will play N. For the crazy player 2, this is obvious. If the sane player 2 chooses C in round t', he will reveal himself to be sane and, by step (1), both players will select N in all subsequent periods. If player 2 instead chooses N in round t', he receives a strictly higher round t' payoff (irrespective of what player 1 chooses), and the continuation can be no worse (since he can choose N in every subsequent period). Consequently, player 2 chooses N, as claimed.

Now consider player 1. Based on the preceding claim, this player knows that both the crazy player 2 and the sane player 2 will choose N in all rounds $t' > t-1$. Consequently, in any round $t'' > t-1$, player 1 knows that the subsequent actions of player 2 (whether sane or crazy) are independent of player 1's round t'' action. Thus, player 1 must make a static best response to player 2's expected choice in round t'', which means that she chooses N.

(4) Define $M^* \equiv \frac{b+(1-\theta)a}{\theta}$. If neither player has selected N in any round up to and including t' where $t' < T - M^*$, then player 1 must select C in round $t'+1$.

Suppose on the contrary that there was an equlibrium in which, with strictly positive probability, player 1 selects N in round $t'+1$. Then, by step (3), she receives at most b from that point forward (since the continuation is for both players to select N in all subsequent rounds).

Imagine instead that she deviates to the following continuation strategy: play C until player 2 plays N, and thereafter play N. Since no information about player 2's type has been revealed through round t', her total payoff from that point forward is no worse than $\theta(T-t')-(1-\theta)a > \theta M^* -(1-\theta)a = b$. Consequently, this is a profitable deviation, which contradicts our initial supposition.

(5) If neither player has selected N in any round up to and including t' where $t' < T-M^*-1$, then player 2 must select C in round $t'+1$.

Suppose on the contrary that there was an equlibrium in which, with strictly positive probability, player 2 selects N in round $t'+1$. Then, by steps (2) and (4), his continuation payoff is b (player 1 will play C in round $t'+1$ and N thereafter).

Imagine instead that player 2 deviates to the following continuation strategy: play C in round $t'+1$ and N in all subsequent rounds. Then, by step (4), he receives a payoff of 1 in round $t'+1$ and a payoff of b in round $t'+2$ (since $t'+1 < T - M^*$, player 1 selects C in round $t' + 2$). By step (2), he receives a payoff of 0 thereafter. Thus, his total payoff is $1 + b > b$. This is a profitable deviation, which contradicts our initial supposition.

We conclude that neither player selects N with positive prbability in any round $t < T - M^*$. Note that M^* is independent of T. Q.E.D.

Remarks:

(1) Though we have ruled out equililibria in which either player selects N with positive probability in any round $t < T - M^*$, we have not proven that there exists an equilibrium in which the players select C in these rounds. However, combining our theorem with a general result on the existence of sequential equilibria establishes this point.

(2) The structure of the cooperative equilibria is difficult to characterize. Generally, it involves players cooperating early on, with cooperation breaking down probabilistically (because players use mixed strategies) as the end of the game approaches. This describes experimental outcomes reasonably accurately.

(3) Note that, for any θ, no matter how small, the fraction of rounds in which we necessarily obtain cooperation goes to unity as the number of rounds becomes large. In this sense, players almost always cooperate with long horizons, even if craziness is only a remote possibility.

(4) The preceding analysis presupposes a particular form of craziness. If one allows for all conceivable forms of craziness, one obtains folk-like theorems (anything can happen with sufficiently long finite horizons and arbitrarilly small probabilities of craziness provided that one does not restrict the form of craziness).

7.3 Signaling

The model:

There is a single worker and two potential employers

The worker's productivity (marginal revenue product) is given by θ, and is the same for both firms.

For simplicity, $\theta \in \{\theta_L, \theta_H\}$ (two possible types, with $\theta_H > \theta_L > 0$)

θ is known to the worker, but not to the firms

Both firms have the same prior beliefs: the worker is type θ_L with probability λ

Eventually, the firms will bid for the worker's services. However, before entering the job market, the worker has the opportunity to obtain education.

Education is costly, but does nothing for the worker's level of productivity.

The cost of obtaining the level of education e is given by a function $c(e, \theta)$

Assume: $c_1 > 0$ (education is always costly on the margin) and $c_{12} < 0$ (education is less costly on the margin for more productive workers)

Firms observe educational attainment before making wage offers.

Workers have a reservation wage, 0. The level is a normalization; the more important assumption here is that all workers have the same outside opportunities.

Formally, the game proceeds as follows:

Stage 1: Nature selects θ

Stage 2: Having observed θ, the worker selects e

Stage 3: Having observed e but not θ, the firms simultaneously select wage offers, w_i $(i = 1, 2)$

Stage 4: The worker accepts at most one job offer.

Payoffs: The worker receives $u(w, e, \theta) = w - c(e, \theta)$ if she accepts an offer (and $-c(e, \theta)$ otherwise), the winning firm receives $\theta - w$, and the losing firm receives 0.

7.3.1 Weak Perfect Bayesian Equilibria

Where we are headed: There are WPBE for this game in which high quality workers receive education while lower quality workers do not, and the market pays educated workers more despite the fact that education does not contribute to productivity. Education instead serves as a signal of productivity.

Simplifying assumption: We will confine attention to WPBE in which the two firms have the same beliefs, contingent upon the observed level of e.

Solution by backward recursion:

Stage 4: Provided that the highest wage offer is non-negative, the worker accepts it. In the event of a tie, the worker could accept the offer from either firm.

Stage 3: Let $\mu(e)$ denote the beliefs of the firms (specifically, the probability that the worker is of type θ_L) conditional upon having observed e. The expected productivity of the worker is then $\mu(e)\theta_L + (1 - \mu(e))\theta_H$. This represents the value of the worker to each firm. Stage 3 involves Bertrand bidding between the firms. Consequently, we know from earlier results that the resulting equilibrium price is

$$w(e) = \mu(e)\theta_L + (1 - \mu(e))\theta_H$$

Stage 2: $w(e)$ describes the tradeoff between wages and education facing the worker. To determine the worker's best choice, we have to study her preferences in greater detail.

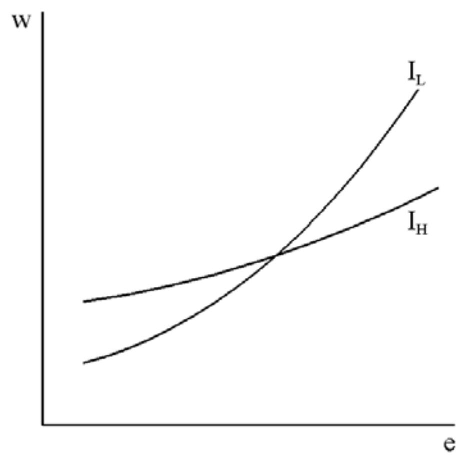
Implicitly differentiating $u(w, e, \theta) = C$, we obtain:

$$\left.\frac{dw}{de}\right|_{u=C} = c_1(e, \theta) > 0$$

Thus, indifference curves slope upward. Moreover,

$$\frac{d}{d\theta}\left(\left.\frac{dw}{de}\right|_{u=C}\right) = c_{12}(e,\theta) < 0$$

Thus, indifference curves are flatter for higher productivity workers. Graphically:



These indifference curves exhibit a characteristic known as the *Spence-Mirrlees single crossing property.*

For any $w(e)$, we can find the worker's optimal choice by selecting the point of tangency with an indifference curve.

Question: Where does $w(e)$ (equivalently, $\mu(e)$) come from? It is implied by (and must be consistent with) the worker's choice on the equilibrium path. For a $WPBE$, it can be anything between θ_L and θ_H off the equilibrium path (that is, for values of e not chosen by the worker). This is because, for such e, one can choose $w(e)$ arbitrarily.

This flexibility in selecting the function $w(e)$ gives rise to a variety of different kinds of equilibria.

Categories of equilibria:

1. Separating equilibria. Different types choose different education levels.

2. Pooling equilibria. Different types choose the same education level.

3. Hybrids. Some members of a different type pool with members of another type, while some separate.

We will study separating and pooling equilibria in greater detail.

Separating equilibria:

Let $e(\theta)$ denote the worker's strategy

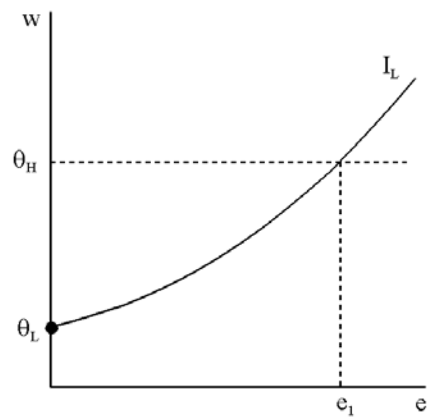
Claim #1: In any separating equilibrium, $w(e(\theta_i)) = \theta_i$ for $i = H, L$. (The worker is paid her marginal product.)

Proof: In a $WPBE$, beliefs are derived from Bayes rule where possible. Type θ_L workers choose $e(\theta_L)$ with probability one, while type θ_H workers choose it with probability zero. Therefore, when $e(\theta_L)$ is observed, the firms must believe $\mu(e(\theta_L)) = 1$, which implies $w(e(\theta_L)) = \theta_L$. A similar argument holds for $e(\theta_H)$.

Claim #2: In any separating equilibrium, $e(\theta_L) = 0$.

Proof: Suppose not. By claim #1, type θ_L workers are paid θ_L. Suppose they chose $e = 0$. Since $\mu(0) \in [0, 1]$, $w(0) \geq \theta_L$. This deviation eliminates costly education without reducing pay, so it is beneficial.

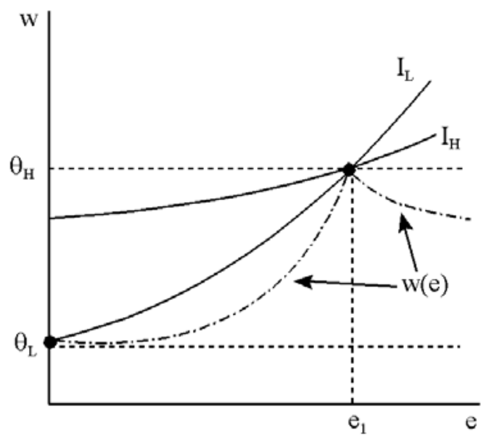
From claim #2, it follows that type θ_L workers receive a payoff of $u(\theta_L, 0, \theta_L)$. This places them on an indifference curve as follows:



Now we can see how to construct a separating equilibrium. By claim #1, θ_H workers must receive an allocation somewhere on the dashed horizontal line at θ_H. If this allocation was to the left to I_L, then the θ_L workers would imitate the θ_H workers (by picking the level of education, $e(\theta_H)$, that the θ_H workers select in equilibrium).

Suppose then that we select the level of education defined by the intersection of I_L and the horizontal line at θ_H. On the graph, this corresponds to e_1. Formally, $e(\theta_H) = e_1$, where e_1 is defined by $u(\theta_L, 0, \theta_L) = u(\theta_H, e_1, \theta_L)$.

Now imagine that the wage schedule $w(e)$ is as follows:



Given this wage function, type θ_L is happy to select $e(\theta_L) = 0$, and type θ_H is happy to select $e(\theta_H) = e_1$. If the firms observe $e \in \{0, e_1\}$, they make the correct inference about the worker's type given the worker's strategy, and set the wage equal to marginal product.

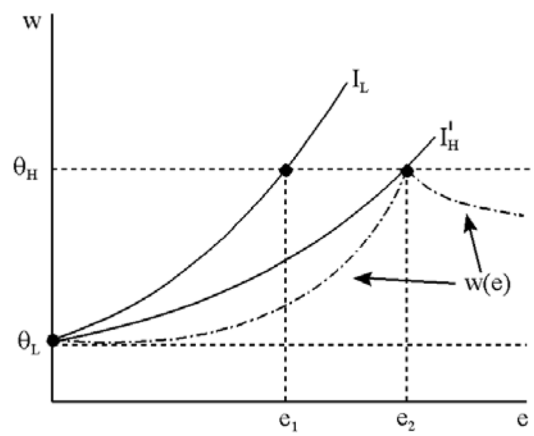
Every other value of e occurs with zero probability in equilibrium. Consequently, for $WPBE$, we are free to select an arbitrary value for $\mu(e)$, provided that $\mu(e) \in [0, 1]$. This is equivalent to selecting an arbitrary function $w(e)$ such that (i) $w(e) \in [\theta_L, \theta_H]$, (ii) $w(0) = \theta_L$, and (iii) $w(e_1) = \theta_H$. The wage schedule in the diagram satisfies these properties. Another wage schedule that would suffice to sustain this outcome: $w(e) = \theta_L$ for $e < e_1$, and $w(e) = \theta_H$ for $e \geq e_1$ (which corresponds to $\mu(e) = 1$ for $e < e_1$ and $\mu(e) = 0$ for $e \geq e_1$).

Remarks: (i) In this equilibrium, education acts as a signal of productivity, and is correlated with wages. However, it does not add to productivity.

(ii) The critical assumption here is that education is less costly for more highly productive workers (single crossing).

Question: Are there other WPB separating equilibria?

Let I'_H be the type θ_H indifference curve through the type θ_L equilibrium allocation. Let e_2 denote the intersection between I'_H and the horizontal line at θ_H. Formally, e_2 satisfies $u(\theta_L, 0, \theta_H) = u(\theta_H, e_2, \theta_H)$. Graphically:



Thus, there is a WPB separating equilibrium in which type θ_H workers choose e_2.

In fact, there is a WPB separating equilibrium for every choice of $e(\theta_H) \in [e_1, e_2]$.

One cannot, however, construct a separating equilibrium for $e(\theta_H) \notin [e_1, e_2]$.

Remark: The first separating equilibrium considered above weakly Pareto dominates all of the others. In some sense, it seems like the most natural, and certainly least wasteful, result. We will return to this point later.

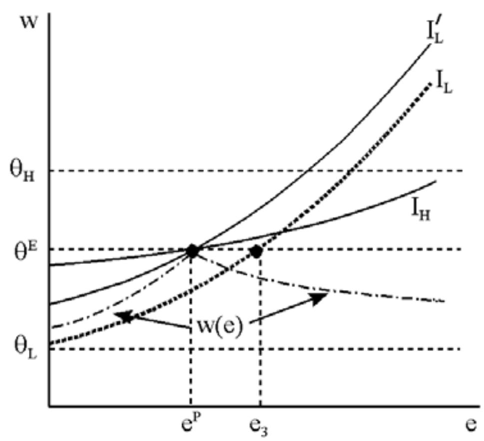
Pooling equilibria:

In a pooling equilibrium, every worker chooses the same education level, e^P, with probability 1 ($e(\theta_L) = e(\theta_H) = e^P$). Therefore, in a $WPBE$, the employer must, upon seeing this education level, hold beliefs $\mu(e^P) = \lambda$. It follows that $w(e^P) = \lambda\theta_L + (1-\lambda)\theta_H \equiv \theta^E$.

Let e_3 be defined by $u(\theta_L, 0, \theta_L) = u(\theta^E, e_3, \theta_L)$; it is given by the intersection of a horizontal line at θ^E and the type θ_L indifference curve through the point $(e, w) = (0, \theta_L)$

Claim: For every $e^P \in [0, e_3]$, there is a pooling equilibrium in which both types of workers choose e^P with probability one.

Illustrate construction graphically:



Notice that $w(e^P) = \theta^E$ as required. For all other values of e, we are free to choose any value for $w(e) \in [\theta_L, \theta_H]$. In the graph, we have made these choices so that it is optimal for both type of workers to select e^P. Another equilibrium wage schedule that would suffice to sustain this outcome: $w(e) = \theta_L$ for $e < e^P$, and $w(e) = \theta^E$ for $e \geq e^P$ (which corresponds to $\mu(e) = 1$ for $e < e^P$ and $\mu(e) = \lambda$ for $e \geq e^P$).

Notice that one cannot have a pooling equilibrium with $e^P > e_3$. Type θ_L workers would then prefer $(w, e) = (w(0), 0)$ to (θ^E, e^P) (since $w(0) \geq \theta_L$), which means that they would deviate to $e = 0$.

Remark: Pooling equilibria with strictly positive levels of education are extremely inefficient. Education accomplishes nothing – it neither adds to productivity nor differentiates the workers, as in a separating equilibrium. Nevertheless, workers still incur the costs of obtaining education because there are substantial wage penalties for the uneducated.

7.3.2 Equilibrium refinements

How do we resolve the vast multiplicity of $WPBE$? The problem comes from having too much latitude in selecting the wage schedule $w(e)$. $WPBE$ only ties it down at educational levels that are chosen in equilibrium. Elsewhere, one can choose any schedule satisfying the restriction that $w(e) \in [\theta_L, \theta_H]$.

Other standard refinements, such as sequential equilibrium, are not helpful, even though some of these outcomes are not very plausible.

For signaling games, the literature has developed a number of "forward induction" refinements.

Equilibrium dominance

Definition: Consider some $WPBE$ in which type θ_L receives a payoff of u_L and type θ_H receives a payoff of u_H. This equilibrium satisfies the *equilibrium dominance* condition iff $\mu(e') = 0$ for all e' satisfying the following conditions:

(i) e' is not chosen by either type in equilibrium

(ii) $u(w, e', \theta_L) < u_L$ for all $w \in [\theta_L, \theta_H]$, and

(iii) $u(w, e', \theta_H) \geq u_H$ for some $w \in [\theta_L, \theta_H]$.

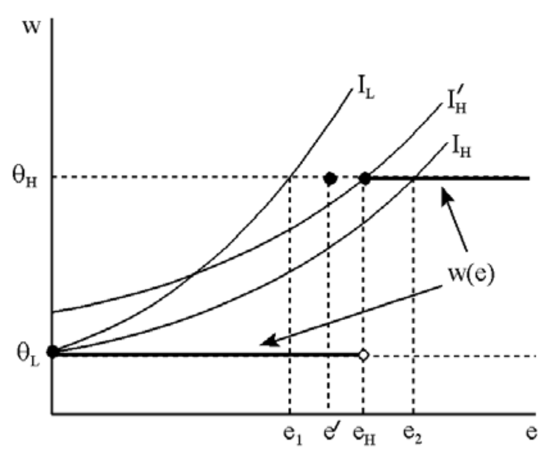
Condition (ii) is the equilibrium dominance condition. It says that e' is not as good as the equilibrium outcome for θ_L regardless of the outcome that e' generates. Note that this differs from the usual notion of dominance.

Condition (iii) states that there is some conceivable continuation outcome following e' for which the choice of e' improves upon the equilibrium outcome for θ_H. Note that, to check this condition, it suffices to consider $w = \theta_H$.

Intuition: If we observe an e' satisfying these conditions, we can rule out the possibility that a type θ_L might have made this choice, but we can't rule out the possibility that a type θ_H might have made this choice. Hence we should infer that the deviator is a type θ_H. This is sometimes known as the *intuitive criterion*.

Claim: The most efficient separating equilibrium (with $e(\theta_H) = e_1$) satisfies equilibrium dominance. No other $WPBE$ satisfies equilibrium dominance. In other words, the equilibrium dominance criterion reduces the $WPBE$ set to a single outcome.

Demonstration: (1) First, consider any inefficient separating equilibrium. Graphically:



In this separating equilibrium, $e(\theta_L) = 0$ and $e(\theta_H) = e_H$. Consider some $e' \in (e_1, e_H)$. Let's check the three conditions.

(i) e' is not chosen in equilibrium.

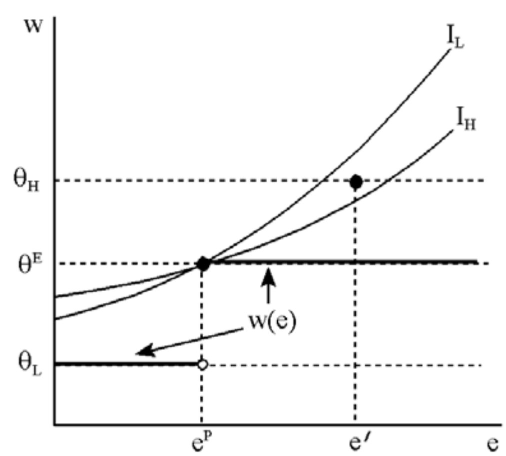
(ii) By construction, $u_L = u(\theta_L, 0, \theta_L) > u(w, e', \theta_L)$ for all $w \in [\theta_L, \theta_H]$.

(iii) By construction, $u_H = u(\theta_H, e_H, \theta_H) < u(w, e', \theta_H)$ for $w = \theta_H$.

Consequently, $\mu(e') = 0$, which rules out any $w(e') < \theta_H$. Since any wage schedule supporting the preceding allocation must have $w(e') < \theta_H$ (since type θ_H would otherwise choose e'), any such equilibrium does not satisfy equilibrium dominance.

Intuition: A type θ_H worker should be able to go to an employer and make the following speech. "I have obtained the level of education e', and I am a θ_H. You should believe me when I tell you this. If indeed I was a θ_H, it would be in my interests obtain e' and to tell you this, assuming that you would believe me. However, if I was a θ_L, it would not be in my interests to obtain e' and to tell you this regardless of what you would then believe. Consequently my claim is credible."

(2) Next, consider any pooling equilibrium. Graphically:



In this pooling equilibrium, $e(\theta_L) = e^P$ and $e(\theta_H) = e^P$. Consider some e', as shown. Let's check the three conditions.

(i) e' is not chosen in equilibrium.

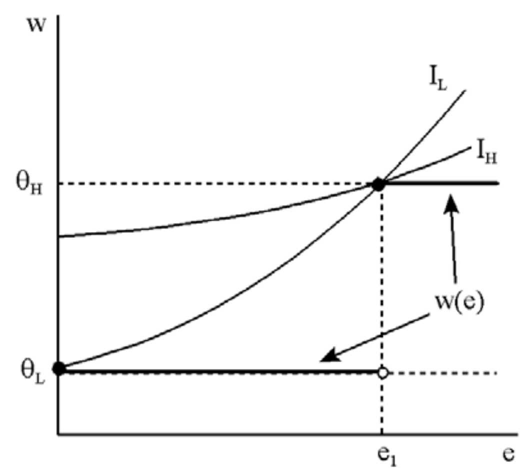
(ii) By construction, $u_L = u(\theta_L, 0, \theta_L) > u(w, e', \theta_L)$ for all $w \in [\theta_L, \theta_H]$.

(iii) By construction, $u_H = u(\theta_H, e_H, \theta_H) < u(w, e', \theta_H)$ for $w = \theta_H$.

Consequently, $\mu(e') = 0$, which rules out any $w(e') < \theta_H$. Since any wage schedule supporting the preceding allocation must have $w(e') < \theta_H$ (since type θ_H would otherwise choose e'), any such equilibrium does not satisfy equilibrium dominance.

Intuition: A type θ_H worker can make the same speech as in the last instance.

(3) Finally consider the efficient separating equilibrium. Graphically:



Consider any e' not chosen in equilibrium.

If $e' < e_1$, then $u(\theta_H, e', \theta_L) > u(\theta_L, 0, \theta_L) = u_L$, so condition (ii) cannot hold.

If $e' > e_1$, then $\nexists w \in [\theta_L, \theta_H]$ such that $u(w, e', \theta_H) > u(\theta_H, e_1, \theta_H) = u_H$, so condition (iii) cannot hold.

Thus, the equilibrium dominance condition does not restrict beliefs for any out-of-equilibrium action.

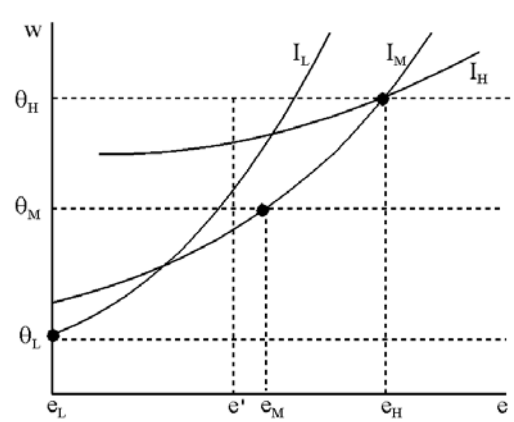
Intuition: There is nothing to be gained from increasing the level of education. For any $e' < e_1$, a type θ_H worker might attempt to convince the employer of his type by making a speech similar to the one above: "I have obtained the level of education e', and I am a θ_H. You should believe me when I tell you this. If indeed I was a θ_H, it would be in my interests obtain e' and to tell you this, assuming that would believe me." Unfortunately, the employer would respond: "Yes, but if you were a type θ_L worker, it would also be in your interests to obtain e' and to tell me this, assuming that I would believe you. Consequently your speech is not credible."

Limitations of equilibrium dominance:

Though equilibrium dominance is remarkably powerful in this setting, the argument turns out to be less general than one might hope.

Illustration: Consider a signaling model with three types of workers, θ_L, θ_M, and θ_H (for low, medium, and high).

Graphical representation of an inefficient separating equilibrium:



θ_M could separate from θ_L at a lower level of education, e'. Will equilibrium dominance eliminate the inefficient equilibrium?

For condition (ii), we need: $u(w, e', \theta_L) < u(\theta_L, 0, \theta_L) = u_L$ for all $w \in [\theta_L, \theta_H]$.

The same argument as before tells us that this is true for $w \in [\theta_L, \theta_M]$. However, there's no reason it has to be true for $w \in [\theta_M, \theta_H]$, and indeed it isn't true in the picture.

Intuition: θ_M makes the speech: "I'm playing e', and I'm a θ_M. You should believe me because, if you do, it's in my interests to make this speech. Moreover, if I'm a θ_L, it's not in my interests to make this speech."

The employer reacts: "I'm not so sure you're a θ_M. Let's imagine that you're a θ_H. As a θ_H, you might be hoping that, by making this speech, I'm going to infer that you're a θ_H. That wouldn't be an unreasonable thing for you to hope since, if that were your inference, then any θ_H would have an incentive to make this speech. So I can't rule out that you're a θ_H, and neither can my competitor (the other employer). By making this speech, you might be hoping that we bid your wage up to θ_H. But in that case I can't rule out the possibility that your a θ_L either. A θ_L who thought that the speech would generate the inference of θ_H would certainly have an incentive to make the speech. So the speech doesn't prove anything."

Stronger refinements:

A variety of alternative criteria have been developed. We will cover the $D1$ criterion, which is generally easy to use and often reduces the equilibrium set significantly.

The $D1$ criterion: Consider some equilibrium in which θ_k receives an equilibrium payoff of u_k. Consider some e' that is not chosen in equilibrium by any type. For each k, define $w_k(e')$ from $u(w_k(e'), e', \theta_k) = u_k$. Now suppose that, for some j and k, $w_j(e') < w_k(e')$. Then $\mu_k(e') = 0$.

In words, the $D1$ criterion states that, if one ever observed e', one would assume that this choice was made by the type of individual most inclined to make it. This is very strong. We are not simply saying that $\frac{\mu_j(e')}{\mu_k(e')} > \frac{\lambda_j}{\lambda_k}$ where λ_i denotes population proportion (so the inequality implies that posteriors place more weight on j relative to k than priors) – we're saying that k is ruled out altogether.

Application: Return to the preceding diagram of an inefficient separating equilibrium with three types. As drawn, $w_M(e') < w_L(e') < w_H(e')$. Consequently, applying the $D1$ criterion, $\mu_M(e') = 1$, and $\mu_L(e') = \mu_H(e') = 0$. Consequently, $w(e') = \theta_M$. But then type θ_M would choose e' instead of e_M, destroying the equilibrium.

Provided that the single crossing property is satisfying and that the is no upper bound on e, one can show that the $D1$ criterion isolates the most efficient separating equilibrium in this class of models, regardless of how many types there are.

Criticisms of forward induction refinements:

1. In some circumstances, pooling equilibria may be the most plausible outcomes.

Return to the case of two types. What happens as $\lambda \to 0$ (so that the population consists almost entirely of highly productive workers)?

The separating equilibrium does not change. It does not depend on the population fractions.

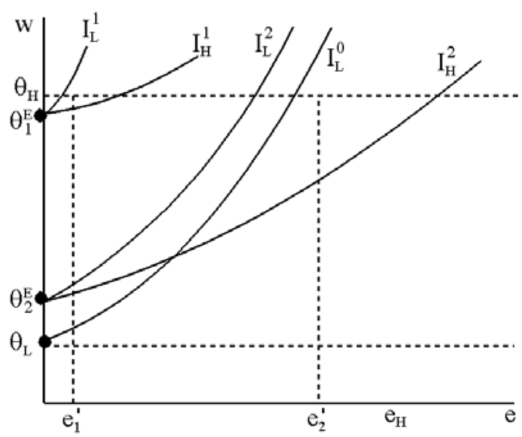
The efficient pooling equilibrium converges to $(e, w) = (0, \theta_H)$

Consequently, when λ is small, pooling equilibria Pareto dominate all separating equilibria. Moreover, the separating equilibrium seems increasingly silly: if the chances of being a θ_L are one in a billion, would we expect everyone to nevertheless obtain large amounts of costly education merely to prove that they are not that one?

This suggests that there might be something wrong with the argument for the intuitive criterion.

One possibility: Suppose that the speech ("I'm a θ_H...") is credible to the employer. Then it is in the interests of *every* θ_H to make the speech. Therefore, if a worker fails to make the speech, one can infer that she is a type θ_L. The employer should then be willing to offer only θ_L to anyone failing to make the speech.

This observation does not affect the inefficient separating equilibria, since the θ_L workers are paid θ_L anyway. However, it does affect the pooling equilibrium. If λ is sufficiently small, then the θ_L workers will prefer to make the speech rather than receive θ_L. Graphically:



Imagine that λ is small (average productivity is θ_1^E in the diagram). Then, for any $(w, e) = (\theta_H, e')$ preferred to $(\theta_1^E, 0)$ by type θ_H, (θ_H, e') is also preferred to $(\theta_L, 0)$ by θ_L (which is what type θ_L would end up with if they didn't choose e' and make the speech, assuming the speech is credible). This is illustrated for the point e_1' in the diagram.

So, for this case, the assumption that the speech is credible implies that it is not credible.

Does this reasoning always rescue the pooling equilibrium? No. Imagine that λ is large (average productivity is θ_2^E in the diagram). Note that type θ_H prefers (θ_H, e_2') to $(\theta_2^E, 0)$, but that θ_L prefers $(\theta_L, 0)$ to (θ_H, e_2'). Thus, type θ_H would have an incentive to make the speech assuming that it is credible, and type θ_L would not have an incentive to make the speech even assuming that the employers would then infer that she is a θ_L. Thus, the speech is credible.

Conclude: One still eliminates the pooling equilibria when the most efficient separating equilibrium is better for type θ_H. However, the pooling equilibrium survives when it Pareto dominates the most efficient separating equilibrium.

2. If we think that players actually may make mistakes with some small probability, then all information sets are reached with positive probability, and there is no room for refinements of beliefs.

The forward induction reasoning breaks down. Suppose that θ_H makes her speech, "I am a θ_H and I have chosen e' because..." The employer might simply say, "How do I know you're not just a θ_L who made a mistake, and now you're trying to make the most of it?"

Pushing this line of reasoning brings us back to trembling-hand perfection and sequential equilibria. These refinements don't help us with signaling problems unless we know enough to place structure on the mistake-generation process.

7.4 Pure communication (cheap talk)

Question: To what extent is it possible for agents to communicate information credibly through costless messages?

Model:

Two agents: a sender and a receiver.

θ denote private information known to the sender but not to the receiver (we will also refer to this as the sender's "type").

This private information affects the payoff for both the sender and the receiver.

Assume that θ is distributed uniformly on the interval $[0, 1]$

The receiver must take some action $a \in \Re$. This action affects both the sender and receiver.

The receiver's payoffs are $u_R(a, \theta) = -(a - \theta)^2$.

Note that this is optimized at $a = \theta$. In other words, the receiver would like to match the action to the sender's type, taking the "appropriate" action for each type. (In an employment situtation, one could think of this as shorthand for paying the appropriate wage and assigning skill-appropriate tasks).

The sender's payoffs are $u_S(a, \theta) = -(a - \theta - c)^2$

Note that this is optimized at $a = \theta + c$. In other words, each sender wants the receiver to believe that he is a higher type than he actually is. (In an employment situation, the worker might want the employer to think that he is more qualified than he actually is, but he doesn't want to be thought of as too qualified lest the employer assign him to a job that the can't handle.)

In this setting, can the sender credibly communicate anything about his private information?

Imagine that choices are made as follows:

Stage 1: Nature chooses θ and reveals the realization to the sender.

Stage 2: The sender chooses some message $m \in M$. This choice is payoff-irrelevant (hence the name "cheap talk").

Stage 3: Having observed m, the receiver selects a.

Remark: We obviously can't have a perfectly informative equilibrium. If each type could credibly announce its type ($m = \theta$), then each type θ would imitate type $\theta + c$. So how much information can the parties communicate credibly?

A "babbling" equilibrium:

Consider the following strategies and beliefs. For every $m \in M$ the receiver beliefs about θ are uniform over $[0, 1]$, and the receiver responds to every $m \in M$ by setting $a = \frac{1}{2}$. All sender-types send the same message, m^*.

Given the receiver's responses, the sender's choice is optimal (since the message is irrelevant). The receiver's responses are optimal given beliefs. The beliefs are derived from equilibrium strategies through application of Bayes rule (trivially) when $m = m^*$. Hence this is always a $WPBE$. It is also easy to check that it is a sequential equilibrium.

A two-message equilibrium:

Are there also equilibria in which the senders bifurcate into two groups, differentiated by two different messages?

Let's try to construct such an equilibrium. We will refer to the two messages as m_1 ("grunt") and m_2 ("snort").

Let $a(m)$ denote the receiver's response to the message m. If the equilibrium conveys information, it must be the case that $a(m_1) \neq a(m_2)$. Without loss of generality, assume that $a(m_1) < a(m_2)$.

Claim #1: A two-message equilibrium has the property that the population divides into two segments, with $\theta \in [0, \theta^*)$ choosing m_1, $\theta \in (\theta^*, 1]$ choosing m_2, and θ^* indifferent (and therefore choosing either message).

Demonstration: Note that

$$\begin{aligned}\Delta &\equiv u_S(a(m_2), \theta) - u_S(a(m_1), \theta) \\ &= -(a(m_2) - \theta - c)^2 + (a(m_1) - \theta - c)^2\end{aligned}$$

From this it follows that

$$\frac{d\Delta}{d\theta} = 2(a(m_2) - a(m_1)) > 0$$

Thus, if type θ' prefers $a(m_2)$ to $a(m_1)$, then type $\theta'' > \theta'$ must also prefer $a(m_2)$ to $a(m_1)$. It follows that the types must divide into two segments, with all types in the lower segment weakly preferring $a(m_1)$, and all types in the upper segment weakly preferring $a(m_2)$. By continuity, the type on the boundary between the two segments must be indifferent between $a(m_1)$ and $a(m_2)$. This establishes the claim.

Claim #2: $a(m_1) = \frac{\theta^*}{2}$, and $a(m_2) = \frac{\theta^*+1}{2}$.

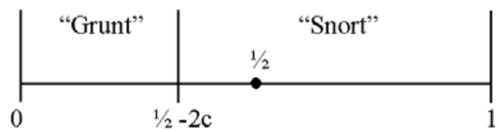
Demonstration: Since the receiver has quadratic preferences, she always sets $a = E(\theta \mid m)$. The preceding expressions are simply the conditional expectations for $\theta \in [0, \theta^*]$, and $\theta \in [\theta^*, 1]$, respectively (given the uniform distribution).

Solving for the equilibrium: We use the fact that θ^* is indifferent between the two messages: $u_S(a(m_1), \theta^*) = u_S(a(m_2), \theta^*)$. Given our functional assumptions and claim #2, this is equivalent to

$$(\theta^* + c) - \frac{\theta^*}{2} = \frac{\theta^* + 1}{2} - (\theta^* + c)$$

This simplifies to $\theta^* = \frac{1}{2} - 2c$.

Graphically:



Remarks: (1) Provided $c < \frac{1}{4}$, this is an equilibrium. Note that there is also a babbling equilibrium.

(2) It is easy to show that one can turn this into a $WPBE$ when $|M| > 2$; simply have the receiver believe that $\theta = 0$ when unchosen messages are selected (one can check that this is also sequential).

(3) Notice that the equilibrium is asymmetric: the upper segment is larger than the lower segment. Thus, communication is more informative in the "lower quality" range.

A three message equilibrium:

Is it possible to have more informative communication? Next we see whether it is possible to divide the population into three groups.

Using precisely the same arguments as before, one can show that:

(1) The population divides into three segments, with $\theta \in [0, \theta_1)$ choosing m_1 ("grunt"), $\theta \in (\theta_1, \theta_2)$ choosing m_2 ("snort"), $\theta \in (\theta_2, 1]$ choosing m_3 ("shriek"), θ_1 indifferent

between m_1 and m_2 (and therefore choosing either message), and θ_2 indifferent between m_2 and m_3 (and therefore choosing either message).

(2) $a(m_1) = \frac{\theta_1}{2}$, $a(m_2) = \frac{\theta_1+\theta_2}{2}$, and $a(m_3) = \frac{\theta_2+1}{2}$ (these are just the conditional expectations of types within each segment).

Equilibrium here must satisfy two conditions instead of one.

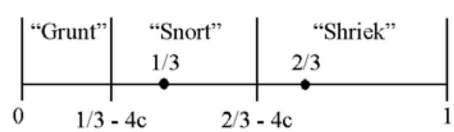
First, we have the indifference condition for θ_1: $u_S(a(m_1), \theta_1) = u_S(a(m_2), \theta_1)$. We rewrite this as:

$$(\theta_1 + c) - \frac{\theta_1}{2} = \frac{\theta_1 + \theta_2}{2} - (\theta_1 + c)$$

Second, we have the indifference condition for θ_2: $u_S(a(m_2), \theta_2) = u_S(a(m_3), \theta_2)$. We rewrite this as:

$$(\theta_2 + c) - \frac{\theta_1 + \theta_2}{2} = \frac{1 + \theta_2}{2} - (\theta_2 + c)$$

These are two linear equations in two unknowns. Solving, we have $\theta_1 = \frac{1}{3} - 4c$, and $\theta_2 = \frac{2}{3} - 4c$. Graphically:



Remarks: (1) Provided $c < \frac{1}{12}$, this is an equilibrium. Note that there is also a two-message equilibrium and a babbling equilibrium.

(2) As above, it is easy to show that one can turn this into a $WPBE$ (and a sequential equilibrium) when $|M| > 3$.

(3) Notice again that the higher segments are larger, so that communication continues to be more informative in the lower quality range.

Arbitrary numbers of messages:

What is the limit on the informativeness of equilibrium communication?

Using precisely the same arguments as before, one can show that:

(1) The population divides into consecutive segments, with $\theta \in [0, \theta_1)$ choosing m_1, $\theta \in (\theta_t, \theta_{t+1})$ choosing m_{t+1}, $\theta \in (\theta_{T-1}, 1]$ choosing m_T, and all types on the boundaries between segments indifferent between the messages assigned to those segments.

(2) $a(m_1) = \frac{\theta_1}{2}$, $a(m_{t+1}) = \frac{\theta_{t+1}+\theta_t}{2}$, and $a(m_T) = \frac{\theta_{T-1}+1}{2}$ (these are just the conditional expectations of types within each segment).

Using $u_S(a(m_1), \theta_1) = u_S(a(m_2), \theta_1)$ implies, as before:

$$(\theta_1 + c) - \frac{\theta_1}{2} = \frac{\theta_1 + \theta_2}{2} - (\theta_1 + c)$$

We can rewrite this as

$$(\theta_2 - \theta_1) = \theta_1 + 4c$$

Similarly, using $u_S(a(m_t), \theta_t) = u_S(a(m_{t+1}), \theta_t)$ implies (after simplification):

$$(\theta_{t+1} - \theta_t) = (\theta_t - \theta_{t-1}) + 4c$$

Notice that this is a second-order difference equation in θ. Solving it, we obtain:

$$\theta_t = t\theta_1 + 4c \sum_{k=1}^{t-1} k$$

We are free to pick θ_1. Any choice of θ_1 implies a sequence of boundary points, θ_t, between successive segments.

If, for some choice of θ_1, there exists a T such that $\theta_T = 1$, then this partition corresponds to an equilibrium with T segments (messages).

Claim: There exists an equilibrium with T distinct segments (messages) if and only if

$$4c \sum_{k=1}^{T-1} k \leq 1$$

Demonstration: When this condition holds, we simply set

$$\theta_1 = \frac{1}{T}\left(1 - 4c \sum_{k=1}^{T-1} k\right)$$

When this condition fails to hold, then $\theta_T > 1$ for every possible choice of θ_1.

Remarks:

(1) For any given value of c, there exists a T^* such that there exist equilibria with T distinct segments iff $T \leq T^*$ (including $T = 1$, the case of babbling). In other words, there can be up to (and including) T^* informative messages. T^* provides a bound on the informativeness of language.

(2) Note that $\theta_t - \theta_{t-1} = \theta_1 + (t-1)4c$. Thus, the segments become wider as one moves to higher values of θ. Once again, communication is more informative at the lower end of the quality spectrum.

(3) As before, it is easy to show that one can turn this into a $WPBE$ or a SE if there are unused messages.