Chapter 1

Complete Markets

In this section we introduce an infinite-horizon, discrete-time, stochastic endowment economy. Most of the work at the intersection of macroeconomics and finance is based on this standard model. Agents are risk averse, receive random endowments, and make state-contingent consumption plans. They can transfer wealth across times and states using financial markets. I will assume that markets are complete: as will be clear below, this means that markets allow agents to make all possible (or relevant) wealth transfers, subject to budget feasibility. In later sections and chapters we will discuss incomplete markets.

An important technical note is that, because of the infinite horizon, all optimization and fixed-point problems need to be solved in infinite dimensional vector spaces. Although this does not change much the basic economics of these problems, the mathematics become much more involved. I do not address these important technical issues in this chapter, but I will make some notes so that you are aware of them.

1.1 The economic environment

We start with a description of the stochastic environment, as well as of agents' endowments and preferences.

Information. Time is discrete and infinite, $t \in \{0, 1, 2, \cdots\}$. Each period, a stochastic event s is drawn from a finite set S. The initial event, s_0, is fixed. The history of events up to time t, or time-t history, is denoted by $s^t = (s_0, s_1, \ldots, s_t)$. The set of all possible time-t histories is

$$S^t = \{s_0\} \times \underbrace{S \times \cdots \times S}_{t \text{ times}},$$

To visualise the set of histories, it is always useful to picture an event tree, as in Figure 1.1. In this figure, it is assumed that there are two possible events in each period, $S = \{U, D\}$, where "U" stands for "up" and "D" for down. The figure shows all possible histories for $t = \{0, 1, 2\}$.

Having described the set of possible events, we now specify the probability distribution over those events. Namely, we let the probability of observing a history of events $s^t \in S^t$ be $\pi_{0t}(s^t)$. Given that s_0 is given, this is to be interpreted as the probability conditional on the the first draw, s_0. Each

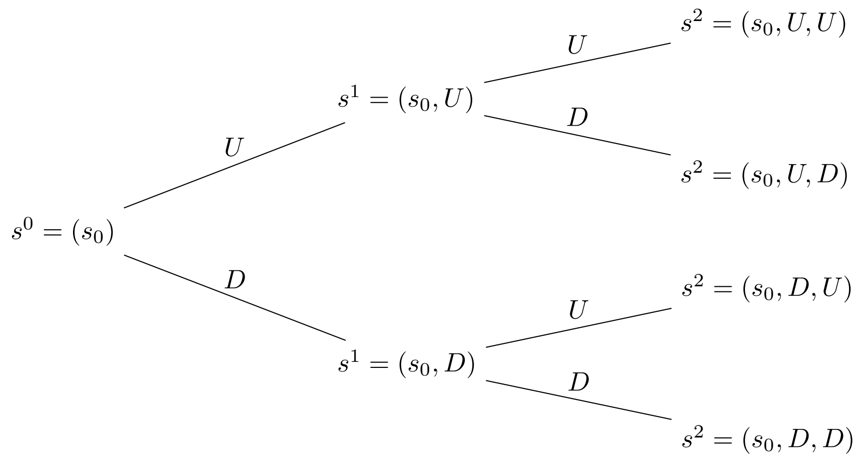

Figure 1.1: An event tree for $t \in \{0, 1, 2\}$ and $S = \{U, D\}$.

function $\pi_{0t}(s^t)$ must be positive, and must satisfy

$$\sum_{s^t \in S^t} \pi_{0t}(s^t) = 1,$$

for all times t. This means that the probabilities of all possible time-t histories must add up to one. The functions must also satisfy the consistency condition:

$$\sum_{s' \in S} \pi_{0t+1}(s^t, s') = \pi_{0t}(s^t) \tag{1.1.1}$$

for all (t, s^t). This ensures that, for all $k \geq 0$, the functions $\pi_{0t+k}(s^{t+k})$ assign the same probability to any given history s^t. By an application of Bayes' rule, we also obtain the conditional probability:

Question: The consistency condition says that this property must be true for $k = 1$. Can you show it is true for $k \geq 2$ as well?

$$\pi_t(s^{t+k}|s^t) = \frac{\pi_{0t+k}(s^{t+k})}{\pi_{0t}(s^t)}.$$

Notice that the consistency condition (1.1.1) above ensures that the conditional probabilities add up to one.[1]

Endowment and preference. There are I types of agents, indexed by $i \in \{1, 2, \cdots, I\}$, with a measure one of each. At any time t and after any history $s^t \in S^t$, agent i has endowment $y_{it}(s^t)$ and consumption $c_{it}(s^t)$. We denote the state contingent stream of endowments by $y_i = \{y_{it}(s^t) : t \geq 0, s^t \in S^t\}$, and the state contingent stream of consumption by $c_i = \{c_{it}(s^t) : t \geq 0, s^t \in S^t\}$.

In what follows, we will restrict y_i and c_i to be bounded, i.e. to belong to the space ℓ_∞ of sequences $x = \{x_t(s^t) : t \geq 0, s^t \in S^t\}$ such that $\sup_{t,s^t} |x_t(s^t)| < \infty$. This is not a very strong assumption in principle: after all, at any point in time, the world is finite. In an infinite-horizon growing economy, however, this assumption would need to be modified: for example, a natural modification would be

[1] For more on the construction of a probability measure on a sequence space, see for example Section 2, in Billingsley (1995). Condition (1.1.1) above solves consistency problem mentioned by Billingsley at the bottom of page 28.

to require that deviations from some appropriately defined trend are bounded.

The agent ranks consumption plan according to a time-discounted expected utility:

$$U_i(c_i) = \sum_{t \geq 0} \beta^t \sum_{s^t \in S^t} \pi_t(s^t) u_i \left[c_{it}(s^t) \right],$$

where the discount factor $\beta \in (0,1)$. The utility function $u_i(c)$ is strictly increasing, twice continuously differentiable, and strictly concave, i.e., $u_i(c) \in \mathcal{C}^2$, $u_i'(c) > 0$, and $u''(c) < 0$. The utility function satisfies the Inada condition $\lim_{c \to 0} u_i'(c) = +\infty$. While standard, the specification of preferences make important restrictions: namely, preferences are time separable, with identical discount factor, and homogenous beliefs. Some of these restrictions turn out to be crucial for some of the results described below. They will be relaxed in later chapters and in some homework assignments.[2]

1.2 Allocations

The outcome of market interactions is an allocation of resources: who consumes what, at every time, after every history.

Definition 1.2.1 (Allocation)**.** An allocation $\{c_i\}_{i \in I}$ specifies, for each agent $i = 1, 2, ..., I$, a consumption plan, $c_i = \{c_{it}(s^t) \geq 0 \, : \, t \geq 0, s^t \in S^t\}$ in ℓ_∞.

Of course, allocations are restricted by the amount of resources available in the economy.

Definition 1.2.2 (Feasible Allocation)**.** An allocation $\{c_i\}_{i \in I}$ is feasible if

$$\sum_{i=1}^{I} \left[c_{it}(s^t) - y_{it}(s^t) \right] = 0, \quad \forall t \geq 0, s^t \in S^t. \tag{1.2.1}$$

That is, an allocation is feasible if the consumptions of all agents add up to the aggregate endowment, at all dates t and after all histories s^t. An important question for what follows will be whether market outcomes are efficient. As is standard, our notion of efficiency is Pareto optimality.

Definition 1.2.3 (Pareto Optimal Allocation)**.** A feasible allocation $\{c_i^\star\}_{i \in I}$ is Pareto Optimal if there is no other feasible allocation $\{c_i\}_{i \in I}$ such that $U_i(c_i) \geq U_i(c_i^\star)$ with a strict inequality for some $i \in I$.

A Pareto problem. To solve for Pareto Optimal allocations we consider the following family of optimization problems. We maximize the utility of one of the agent, say $i = 1$, with respect to all feasible allocations, and subject to delivering to other agents some specified promised utilities $\{\underline{U}_i\}_{i \in I, i \neq 1}$ where $\underline{U}_i \geq u_i(0)/(1-\beta)$. Precisely, we consider the Pareto problem

$$\max U_1(c_1) \tag{1.2.2}$$

with respect to feasible allocations $\{c_i\}_{i \in I}$, and subject to

$$U_i(c_i) \geq \underline{U}_i \tag{1.2.3}$$

[2]See Section 3.2.2 and Problem 4.19 for non-separable utility, and Problem 4.1 for heterogenous beliefs.

for all $i \neq 1$. By varying the level of utilities $\underline{U}_i \geq u_i(0)/(1-\beta)$ promised to all $i \neq 1$, we obtain the set of all Pareto Optimal allocations. Formally:

Proposition 1.2.1. *A feasible allocation $\{c_i^\star\}_{i\in I}$ is Pareto Optimal if and only if it solves (1.2.2) for some collection of promised utilities $\{\underline{U}_i\}_{i\in I, i\neq 1}$.*

To prove the "only if" part, one sets $\underline{U}_i = U_i(c_i^\star)$ and use the definition of Pareto optimality to show that there cannot exist an allocation that achieves a higher value than $\{c_i^\star\}_{i\in I}$ in the optimization program (1.2.2). To prove the "if" part, we proceed by contradiction. Namely, consider some $\{c_i^\star\}_{i\in I}$ solving the program (1.2.2) and suppose that it is not Pareto optimal. That is, there exists some allocation $\{\hat{c}_i\}_{i\in I}$ such that $U_i(\hat{c}_i) \geq U_i(c_i^\star)$ with a strict inequality for some $i \in I$. If the inequality occurs for $j = 1$, then it means that $\{\hat{c}_i\}_{i\in I}$ achieves a higher value in the optimization program (1.2.2), and we have reached the contradiction. Suppose otherwise that the inequality does not occur for $j = 1$ but for some $j \neq 1$. Then $U(\hat{c}_j) > \underline{U}_j \geq u_j(0)/(1-\beta)$, and so $\hat{c}_{jt}(s^t) > 0$ for some (t, s^t) such that $\pi_{0t}(s^t) > 0$. Now since $j \neq 1$ achieves a strictly higher utility than his promise $\underline{U}_j$, we can transfer some of his consumption to $j = 1$ without violating the constraint (1.2.3). Formally, let $\alpha_j \geq 0$ such that $U_j(\alpha_j\, \hat{c}_j) = \underline{U}_j$. Clearly, since $U(\hat{c}_j) > \underline{U}_j$ and $\hat{c}_{jt}(s^t) > 0$ for some (t, s^t), $\alpha_j < 1$. Now consider the allocation $c_1 = \hat{c}_1 + (1-\alpha_j)\,\hat{c}_j$, $c_j = \alpha_j\,\hat{c}_j$, and $c_i = \hat{c}_i$ for all other $i \in I$. The allocation $\{c_i\}_{i\in I}$ is feasible, satisfies all the constraints (1.2.3), and achieves a higher value than $\{c_i^\star\}_{i\in I}$ in the the optimization program (1.2.2), so we have reached a contradiction.

The Lagrangian Solving the optimization problem (1.2.2) turns out to be equivalent to maximize the Lagrangian:

$$\mathcal{L} = \sum_{t\geq 0}\sum_{s^t}\sum_{i\in I}\lambda_i\beta^t\pi_t(s^t)u_i\left[c_{it}(s^t)\right] + \sum_{t\geq 0}\sum_{s^t\in S^t}\theta_t(s^t)\sum_{i\in I}\left[y_{it}(s^t) - c_{it}(s^t)\right]. \tag{1.2.4}$$

For reference on the mathematics of solving optimization problems in infinite-dimensional spaces, see for example Luenberger (1969) Section 8.3 and 8.4. In the Lagrangian, we set λ_1 to one. The multiplier on the constraint (1.2.3) is λ_i and the multiplier on the resource constraint (1.2.1) is $\theta_t(s^t)$. The first-order condition with respect to $c_{it}(s^t)$ is:

$$\beta^t\pi_t(s^t)u_i'\left[c_{it}(s^t)\right] = \frac{\theta_t(s^t)}{\lambda_i}.$$

Taking the ratio of the first-order condition of $i \neq 1$ and $i = 1$, we obtain:

$$\frac{u'\left[c_{it}(s^t)\right]}{u'\left[c_{1t}(s^t)\right]} = \frac{\lambda_1}{\lambda_i} \Rightarrow c_{it}(s^t) = (u_i')^{-1}\left\{\frac{\lambda_1}{\lambda_i}u_1'\left[c_{1t}(s^t)\right]\right\}. \tag{1.2.5}$$

In the equation $(u')^{-1}$ is the inverse of marginal utility. For example, if $u(c) = c^{1-\gamma}/(1-\gamma)$, then $u'(c) = c^{-\gamma}$ and $(u')^{-1}(x) = x^{-1/\gamma}$. Now plugging back into the resource constraint (1.2.1) we obtain, for each t and s^t, a one-equation-in-one-unknown problem:

$$\sum_{i\in I}(u_i')^{-1}\left\{\frac{\lambda_1}{\lambda_i}u_1'\left[c_{1t}(s^t)\right]\right\} = \sum_{i\in I}y_{it}(s^t).$$

By the Intermediate Value Theorem, it immediately follows that, given the multipliers $\{\lambda_i\}_{i\in I}$, this equation has a unique solution: indeed, the left-hand side is strictly increasing, and given Inada condition it is zero when $c_{1t}(s^t) = 0$, and infinity as $c_{1t}(s^t) \to \infty$. This pins down the consumption of every agent as a function of the aggregate endowment and of the multipliers $\{\lambda_i\}_{i\in I}$.

Full insurance and history independence. Notice that Pareto optimal consumption allocation have two striking properties:

- The consumption allocation provides *full insurance*, in the following sense. Holding the aggregate endowment constant, an individual agent's consumption is not affected by fluctuations in his/her individual endowment.

- The consumption allocation is *history independent*: it only depends on the aggregate endowment at time t, but not on the particular history of aggregate endowment before time t. If the same time-t endowment is reached from different histories, then the consumption allocation is the same.

A social planning problem. Now consider the alternative "social planning" optimization program. We give weight $\lambda_i \geq 0$ to agents $i \in I$, with a strictly positive weight for at least some agent, and we maximize:

$$\sum_{i\in I} \lambda_i U_i(c_i), \tag{1.2.6}$$

with respect to feasible allocations $\{c_i\}_{i\in I}$. This leads to:

Question: can you prove this proposition?

Proposition 1.2.2. *A feasible allocation $\{c_i^\star\}_{i\in I}$ is Pareto Optimal if and only if it solves (1.2.6) for some collection of weights $\{\lambda_i\}_{i\in I}$ such that $\lambda_i \geq 0$ for all $i \in I$, with at least one strict inequality.*

Intuitively, the equivalence follows intuitively because the social planning program shares the same Lagrangian as the Pareto problem (1.2.2).

1.3 Equilibrium in time-zero markets

Time zero trading. We consider first a market setting in which all trades occur at time zero, as envisioned in the last chapter of Debreu (1959). Namely, at time zero, an agent can buy or sell contingent claims to consumption: contracts that guarantee the delivery of consumption at time t, after history s^t, for all possible times and histories.[3] Although these contingent claims bear some resemblance to forward contracts, it is fair to say that this market structure is not too realistic: in practice, agents trade many different kinds of financial securities, over multiple time periods. We will show in the next section that, under some conditions, equilibria with a more realistic market structure are essentially the same as equilibria with time-zero markets.

The price of one unit of consumption to be delivered at time t after history s^t is $q_{0t}(s^t)$. We take time-zero consumption to be the numéraire: we normalize its price to $q_{00}(s_0) = 1$. Hence, agent i

[3]In a multi-good environment, this market would be extended to allow agents to trade contingent claims to every different type of consumption goods.

faces the inter-temporal budget constraint:

$$\sum_{t\geq 0}\sum_{s^t\in S^t} q_{0t}(s^t)c_{it}(s^t) = \sum_{t\geq 0}\sum_{s^t\in S^t} q_{0t}(s^t)y_{it}(s^t). \tag{1.3.1}$$

The left- and right-side of the budget constraint are to be interpreted as present values: the present value of the consumption plan, and the present value of the endowment stream. Hence the price $q_{0t}(s^t)$ can be interpreted as a present value discount factor for consumption at time t after history s^t. For example, in a deterministic environment with constant interest rate, we would expect $q_{0t}(s^t)$ to be of the form $1/(1+r)^t$. In the stochastic environment studied here, we do not place such a priori restriction on the present value discount factor. Later we will find that, in equilibrium, this present value discount factor depends in natural way on the time discount factor, the probabilities of history s^t, on the scarcity of goods after history s^t, and on agents' risk-preferences.

Notice that the inter-temporal budget constraint involves two infinite sums, and so may be ill-defined: for example the present value of the agent's consumption plan or endowment stream could be infinite. To deal with this issue, we assume that prices are absolutely summable: they belong to the space of ℓ_1 of sequences such that $\sum_{t\geq 0}\sum_{s^t\in S^t} q_{0t}(s^t) < \infty$. Since consumption plans and endowment streams are bounded, this ensures that present values are finite. With this in mind, we have the following three definitions.

Definition 1.3.1 (Price System in Time-zero Markets)**.** A price system for the model with time-zero market is some sequence $q = \{q_{0t}(s^t) \geq 0 : t \geq 0, s^t \in S^t\} \geq 0$ in ℓ_1.

Definition 1.3.2 (Agent's Problem in Time-zero Markets)**.** The agent's problem is to maximize $U_i(c_i)$ with respect to $c_i \in \ell_\infty$, and subject to the inter-temporal budget constraint (1.3.1).

Definition 1.3.3 (Time-zero Markets Equilibrium)**.** A competitive equilibrium with time-zero markets consists of a feasible allocation $\{c_i\}_{i\in I}$ and a price system q such that, $\forall i$, c_i solves agent i's problem given q.

In the definition, we require the allocation to be feasible, which ensures market clearing. The existence of equilibrium is a non-trivial matter first established by Bewley (1972). To solve for equilibrium, we first set up the Lagrangian corresponding to agent's i problem.

$$\mathcal{L} = \sum_{t\geq 0}\sum_{s^t\in S^t}\left\{\beta^t\pi_t(s^t)u_i\left[c_{it}(s^t)\right] + \mu_i q_{0t}(s^t)\left[y_{it}(s^t) - c_{it}(s^t)\right]\right\},$$

where μ_i is the Lagrange multiplier for agent i's budget constraint.

The first order condition with respect to $c_{it}(s^t)$:

$$\beta^t\pi_{0t}(s^t)u_i'\left[c_{it}(s^t)\right] = \mu_i q_{0t}(s^t),\ \forall i, t, s^t. \tag{1.3.2}$$

Taking the ratio of the first-order condition for time t history s^t, and of the first-order condition for time zero, we obtain:

$$q_t(s^t) = \beta^t\pi_{0t}(s^t)\frac{u_i'\left[c_{it}(s^t)\right]}{u_i'\left[c_{i0}(s_0)\right]}, \tag{1.3.3}$$

where we used the normalization $q_{00}(s_0) = 1$. This equation shows that, for every agent, the price of consumption at time t after history s^t (relative to the price of time zero consumption) is equated to the marginal rate of substitution between consumption at time t after history s^t, and time-zero consumption. This is the same condition you learned about in all micro classes. The only difference is that, in the present dynamic model, goods are indexed by time and history.

Taking the ratio of the first order condition for agent i over the one for agent 1 we obtain

$$\frac{u_i'\left[c_{it}(s^t)\right]}{u_1'\left[c_{1t}(s^t)\right]} = \frac{\mu_i}{\mu_1}.$$

First welfare theorem. Using Proposition 1.2.2 and the first-order condition 1.2.5, we immediately obtain the first-welfare Theorem:

Question: Can you prove this result without taking first-order conditions?

Proposition 1.3.1 (First Welfare Theorem)**.** *An equilibrium allocation is Pareto Optimal.*

An algorithm to solve for equilibrium and a celebrated existence proof. Now using the same manipulations as in Section 1.2, we use the first-order condition to solve for $c_{it}(s^t)$ as a function of $c_{1t}(s^t)$ and find

$$c_{it}(s^t) = \left(u_i'\right)^{-1}\left\{\frac{\mu_i}{\mu_1}u_1'\left[c_{1t}(s^t)\right]\right\}. \tag{1.3.4}$$

The market clearing condition can be written as

$$\sum_i \left(u_i'\right)^{-1}\left\{\frac{\mu_i}{\mu_1}u_1'\left[c_{1t}(s^t)\right]\right\} = \sum_i y_{it}(s^t) = y_t(s^t). \tag{1.3.5}$$

Equation (1.3.5) also implies a simple algorithm to solve for equilibrium, using the multiplier ratios as unknown. Namely, given any collection of multiplier ratios, one can solve for the consumption allocation by first solving for the consumption $c_{1t}(s^t)$ of $i = 1$ using equation (1.3.5), and then for the consumptions of $i \neq 1$ using equation (1.3.4). If one define the price $q_{0t}(s^t)$ using (1.3.2), one obtains that, by construction, the first-order conditions of all agents are satisfied. What needs to be checked is that the budget constraints of all agents hold. Since market clear, by Walras Law we only have to check $I-1$ inter-temporal budget constraints. The $I-1$ budget constraints thus constitute a system of $I-1$ equations for the $I-1$ unknown multiplier ratios $(\mu_2/\mu_1, \mu_3/\mu_1, \ldots, \mu_I/\mu_1)$, which can be solved numerically (and, sometimes, by hand) to find an equilibrium. Notice that this algorithm is a massive improvement over an alternative procedure that would take the prices as unknowns. Indeed, with prices as unknowns, solving for equilibrium would require finding infinitely many prices that clear infinitely many markets. In the proposed algorithm, by contrast, there are only $I-1$ unknown.

The algorithm is also the basis of a celebrated method of proof of equilibrium existence due to Negishi (1960): instead of trying to equilibrate markets by finding equilibrium prices, one tries to find multiplier ratios (or, equivalently, weights for the social planning problem 1.2.6) such that agents' inter-temporal budget constraints hold. Negishi developed his proof for a finite dimensional commodity space, as in the original proof of Arrow and Debreu (1954). But, as explained above, it became especially useful one or two decades later, when researchers became interested in studying equilibria in infinite dimensional commodity spaces (see for example Magill, 1981; Mas-Colell and Zame, 1991).

It is interesting to note that, in the construction of equilibrium above, we never trace out the entire demand curve. That is, we do not solve for the solution of the agent's problem given all admissible price system $q \in \ell_1$. Instead, we use the Pareto-optimality property of equilibrium allocation in order to directly solve for agents' demands at *equilibrium prices*. For some application, it is quite important and interesting to study the demand at any price explicitly – for example if one wants to study optimal portfolio choice, optimal consumption over the life-cycle, etc... For this, one usually makes restrictions on the price system, and use dynamic programming methods for optimization.[4]

Example 1.3.1 (Constant aggregate endowment)**.** Assume that the aggregate endowment remains constant over time, $y_t(s^t) = y$. Given history independence, this means that the consumption of every agent remains constant over time: there is full insurance. Notice that this is true even if individual endowment are stochastic (see the exercise in Section 4.5). Hence, we can write agent i consumption as $c_{it}(s^t) = \alpha_i y$, for some collection of $\alpha_i \geq 0$ such that $\sum_{i=1}^I \alpha_i = 1$. From equation (1.3.3), we obtain that $q_{0t}(s^t) = \beta^t \pi_{0t}(s^t)$, i.e., the price of consumption in period t after history s^t is equal to the discount factor raised to the power t, multiplied by the probability of history s^t. Plugging this into the inter-temporal budget constraint of agent i gives:

$$\alpha_i y \sum_{t \geq 0} \sum_{s^t \in S^t} \beta^t \pi_{0t}(s^t) = \sum_{t \geq 0} \sum_{s^t \in S^t} \beta^t \pi_{0t}(s^t) y_{it}(s^t) \quad \Rightarrow \quad \alpha_i = (1-\beta) \sum_{t \geq 0} \beta^t \mathbb{E}\left[\frac{y_{it}(s^t)}{y}\right].$$

That is, the consumption share α_i of agent i corresponds to her appropriately "discounted" relative endowment. If the agent has a larger relative endowment, his consumption share is larger. And if she receives larger endowment earlier in time, then she also receives a larger consumption share.

Example 1.3.2 (CRRA Utility)**.** The utility function is

$$u(c) = \begin{cases} \frac{c^{1-\gamma}-1}{1-\gamma} & \text{if } \gamma \neq 1 \\ \log(c) & \text{if } \gamma = 1. \end{cases}$$

We have $u'(c) = c^{-\gamma}$ and $(u')^{-1}(x) = (x)^{-\frac{1}{\gamma}}$. The market clearing condition becomes

$$\sum_{i=1}^{I} \left(\frac{\mu_i}{\mu_1}\right)^{-\frac{1}{\gamma}} c_{1t}(s^t) = y_t(s^t).$$

We solve for consumption of agent 1:

$$c_{1t}(s^t) = \frac{\mu_1^{-\frac{1}{\gamma}}}{\sum_{j=1}^{I} \mu_j^{-\frac{1}{\gamma}}} y_t(s^t).$$

[4] See Problem 4.18 and 4.17.

By symmetry, for all agents $i \in \{1, \ldots, I\}$:

$$c_{it}(s^t) = \alpha_i y_t(s^t) \text{ where } \alpha_i \equiv \frac{\mu_i^{-\frac{1}{\gamma}}}{\sum_{j=1}^I \mu_j^{-\frac{1}{\gamma}}}$$

is the consumption share of agent i. The formula makes it clear that the consumption share of agents are constant across time and history. If μ_i is larger, agent i is less wealthy and therefore consumes less. Substituting consumption into (1.3.3), we obtain:

$$q_{0,t}(s^t) = \beta^t \pi_{0t}(s^t) \left(\frac{y_t(s^t)}{y_0(s_0)} \right)^{-\gamma}.$$

Notice that we have here a strong aggregation results: despite heterogeneity in endowment, consumption is priced "as if" there were only a single representative agent in the economy, endowed with $y_t(s^t)$ every period.

Substituting price and consumption into the time zero budget constraint (1.3.1):

$$\sum_{t \geq 0} \sum_{s^t \in S^t} \beta^t \pi(s^t | s_0) \left(\frac{y_t(s^t)}{y_0(s_0)} \right)^{-\gamma} \left[\alpha_i y_t(s^t) - y_{it}(s^t) \right] = 0.$$

We obtain the consumption share of agent i:

$$\alpha_i = \frac{\sum_{t \geq 0} \sum_{s^t \in S^t} \beta^t \pi_{0t}(s^t) y_t(s^t)^{-\gamma} y_{it}(s^t)}{\sum_{t \geq 0} \sum_{s^t \in S^t} \beta^t \pi_{0t}(s^t) y_t(s^t)^{1-\gamma}}.$$

1.4 Equilibrium with sequential markets

The market setting with time zero market is arguably completely at odd with reality: agents do not trade just once contract to deliver consumption contingent on any possible future history. Instead, they optimize dynamically over time: every period, agents choose their consumption and purchase asset portfolios. In this section we consider such a dynamic market setting. We show that equilibrium outcomes turn out to be identical as the one obtained with time-zero markets.

1.4.1 The market setting

We consider the same specification of information, preferences and endowments as before, but with a different market setting. We assume that agents are able to trade every period, both consumption good and financial securities.

Every period, they choose consumption and trade a complete set of "Arrow securities". An Arrow security is a financial asset that pays off 1 unit of consumption good if the state next period is some $s \in S$, and zero unit of consumption good otherwise. The set of Arrow securities is assumed to be complete in the sense that there exists a security for every possible realization of the state next period.

1.4.1.1 The price system

At time t after history s^t, the price of the Arrow security that pays 1 if event $s_{t+1} \in S$ realizes and 0 otherwise is denoted by $Q_{t+1}(s_{t+1} \mid s^t)$.

Definition 1.4.1 (Price System in Sequential Markets)**.** A price system is some $Q = \{Q_{t+1}(s_{t+1}|s^t) \geq 0 \, : \, t \geq 0, s^t \in S^t, s_{t+1} \in S\}$ such that

$$\sum_{t\geq 0} \sum_{s^t \in S^t} \left[Q_t(s_t \mid s^{t-1}) \times Q_{t-1}(s_{t-1} \mid s^{t-2}) \times \ldots \times Q_1(s_1 \mid s_0)\right] < \infty. \tag{1.4.1}$$

In the definition, (1.4.1) is analogous to the condition that $q \in \ell_1$ in the previous section. It ensures that the present values of bounded consumption streams are finite. To see why this is the case, notice that, according to the price system Q, the time-zero present value of a unit of consumption to be delivered at time t after history s^t is:

$$Q_t(s_t \mid s^{t-1}) \times Q_{t-1}(s_{t-1} \mid s^{t-2}) \times \ldots \times Q_1(s_1 \mid s_0).$$

Indeed, the present discount value factor is the answer to the following question: how much consumption good do I need to invest at time zero in order to end up with one unit of consumption at time t after history s^t? The answer to this question is obtained by backward induction. Namely, at time $t-1$ after history s^{t-1}, the agent needs to purchase one unit of Arrow security paying off if the state at time t is s_t. This costs $Q_t(s_t \mid s^{t-1})$. Thus, at time $t-2$ after history s^{t-2}, the agent needs to purchase $Q_{t-1}(s_{t-1} \mid s^{t-2})$ Arrow securities paying off if the state at time $t-1$ is s_{t-1}. This costs $Q_t(s_t \mid s^{t-1}) \times Q_{t-1}(s_{t-1} \mid s^{t-2})$. Continuing on backward until time zero, one obtains that the present value factor is exactly given as above.

1.4.1.2 The agent's problem

When the market opens at time t after history s^t, agent i buys $c_{it}(s^t)$ consumption goods as well as a portfolio of Arrow securities. We let $a_{it+1}(s^t, s_{t+1})$ denote number of Arrow securities in the portfolio paying off one unit of consumption good if next-period state is s_{t+1}, and zero otherwise. Hence, the sequential budget constraint of the agent is:

$$c_{it}\left(s^t\right) + \sum_{s_{t+1}\in S} Q_{t+1}\left(s_{t+1}|s^t\right) a_{it+1}\left(s^t, s_{t+1}\right) = y_{it}\left(s^t\right) + a_{it}\left(s^t\right). \tag{1.4.2}$$

The first term on the left-side is the consumption. The second term is the value of the portfolio of Arrow securities acquired at time t after history s^t. On the right-side, the first term is the endowment, and the second term is the payoff of Arrow securities acquired last period. We denote the sequential trading plan in Arrow security by $a_i = \{a_{it}(s^t) : t \geq 0, s^t \in S^t\}$. The initial security position of the agent is taken to be $a_{i0}(s_0) = 0$.

Constraints on debt accumulation. It is very important to realize that the sequential budget constraint does not, in and of itself, impose any meaningful constraint on the consumption possibilities of the agent. This is because it fails to limit the amount of debt issued by the agent. In fact, an agent who is only faced with (1.4.2) would have incentives to run "Ponzi Scheme": he would always

Question: can you show that, if we only impose (1.4.2), then *any* consumption path would be budget feasible?

Note: For this reason some researchers like to view (1.4.2) as an accounting identity, and call it the "flow find it optimal to consume in excess of its endowment every period by issuing more debt. The debt would grow indefinitely.

What is, then, the missing constraint? Intuitively, a creditor would want to limit the amount of debt issued by the agent. To capture this idea, researchers have followed either one of two approaches. Some authors put a hard borrowing constraint that limits the amount of debt issued by the agent every period: for example, they assume that $a_{it}(s^t) \geq -B$ for some $B \geq 0$. See for example Section 8.8.4 and 8.8.5 in Ljungqvist and Sargent (2018), who introduce the notion of a natural debt limit. Other authors, such as Magill and Quinzii (1994), impose a limit on the rate of growth of the debt issued by the agent, what is often termed a "No Ponzi Game" constraint. I will follow the later approach here and impose:

$$\liminf_{T\to\infty} \sum_{s^T \succeq s^t} \left[Q_T(s_T \mid s^{T-1}) \times Q_{T-1}(s_{T-1} \mid s^{T-2}) \times \ldots \times Q_{t+1}(s_{t+1} \mid s^t) \right] a_T(s^T) \geq 0, \quad (1.4.3)$$

for all times t and histories $s^t \in S^t$. In the sum above $s^T \succeq s^t$ means that history s^T follows s^t (or, more formally, that the first t elements of history s^T are equal to s^t). Also, we use a limit inferior ("$\liminf$") instead of a limit because, in general, the limit is not guaranteed to exist while the limit inferior always is.

In words, the above constraint imposes that, following any history s^t, the asymptotic present value of the debt is positive. For instance, in a deterministic model with constant interest rate, the constraint would take the form $\lim_{T\to\infty} \frac{a_T}{(1+r)^{T-t}} \geq 0$. This means that, asymptotically, the debt cannot grow faster than the interest rate. It is a weaker constraint than a fixed borrowing limit, because it allows debt to grow indefinitely but at a rate that is less than r. It turns out that the two approaches, borrowing limits or No Ponzi Game constraint, are essentially equivalent. Namely, in many cases, as long as the borrowing limit is large enough, the households' optimal plan is the same if he is subject to a borrowing limit, or if he is subject to a No Ponzi Game constraint. The intuition is that, when a household goes over its borrowing limit, its income is not sufficient to cover the interest rate payments, and so it has to roll over ever growing amounts of debts: the debt then grows so fast that the no Ponzi Scheme condition violated.

The transversality condition. The "No Ponzi Game" constraint limits the growth rate of borrowing. It turns out that optimality implies another condition, called the "Transversality condition", which limits the growth rate of saving:

$$\limsup_{T\to\infty} \sum_{s^T \succeq s^t} \left[Q_T(s_T \mid s^{T-1}) \times Q_{T-1}(s_{T-1} \mid s^{T-2}) \times \ldots \times Q_{t+1}(s_{t+1} \mid s^t) \right] a_T(s^T) \leq 0. \quad (1.4.4)$$

It is important to keep in mind that this is not a constraint but an optimality condition. It roughly states that the asymptotic present value of the agents' wealth must be non-positive. The intuition why this is optimal is the following. If the asymptotic present value of wealth were strictly positive, the agent would be saving too much: the agent could consume more today and keep the rest of his consumption plan the same. In Appendix A we provide a proof of the transversality condition in the present environment based on Kamihigashi (2002, 2003).

Combining (1.4.3) and (1.4.4), we obtain the condition:

$$\lim_{T\to\infty} \sum_{s^T \succeq s^t} \left[Q_T(s_T \,|\, s^{T-1}) \times Q_{T-1}(s_{T-1} \,|\, s^{T-2}) \times \ldots \times Q_{t+1}(s_{t+1} \,|\, s^t) \right] a_T(s^T) = 0. \tag{1.4.5}$$

In what follow, to simplify the exposition, I impose (1.4.5) directly to the agent as a constraint – but you should keep in mind that only the "No-Ponzi" side of the equality is truly constraint, while the "Transversality" side is an optimality condition.

Definition 1.4.2 (Budget feasible plans)**.** A consumption and asset holding plan $\{c_i, a_i\}$ is budget feasible for agent i given Q if it satisfies the sequential budget constraint (1.4.2), the No Ponzi Game constraint combined with the Transversality condition, (1.4.5), and the initial condition $a_{i0}(s_0) = 0$.

Definition 1.4.3 (Agent's Problem in Sequential Markets)**.** The agent's problem is to maximize $U_i(c_i)$, with respect to a budget feasible $\{c_i, a_i\}$.

1.4.1.3 The definition of equilibrium

We now are ready to define a competitive equilibrium in sequential markets.

Definition 1.4.4 (Allocation)**.** An allocation $\{c_i, a_i\}_{i\in I}$ specifies, for each agent $i = 1, 2, ..., I$, a consumption plan and a sequential trading plan.

Definition 1.4.5 (Feasible Allocation)**.** An allocation is feasible if

$$\sum_i c_{it}(s^t) = \sum_i y_{it}(s^t) \text{ and } \sum_i a_{it}(s^t) = 0,\ \forall t \geq 0, s^t \in S^t.$$

Definition 1.4.6 (Equilibriun in Sequential Markets)**.** An equilibrium in sequential markets consists of a feasible allocation $\{c_i, a_i\}_{i\in I}$ and a price system Q such that, $\forall i$, $\{c_i, a_i\}$ solves agent i's problem given Q.

1.5 Equivalence between sequential and time-zero markets

In this Section we present an important result: trading in time zero market is essentially equivalent to trading in *complete* sequential market. The result follows because consumption possibilities are the same in both settings. Namely, given appropriate relationship between the price systems in both settings, q and Q, we show in Proposition 1.5.1 that the budget sets are equivalent: any consumption plan that is budget feasible in time-zero market is also feasible in sequential markets, and vice versa. In Corollary 1.5.1, we show that this implies that optimal consumption plans and equilibria are the same as well.

1.5.1 The main results

We first state the main results, and prove them in Section 1.5.2 and 1.5.3 below. First:

Proposition 1.5.1. *Let* $q_{0t}(s^t) \equiv Q_t(s_t \,|\, s^{t-1}) \times Q_{t-1}(s_{t-1} \,|\, s^{t-2}) \times \ldots \times Q_1(s_1 \,|\, s_0)$. *Then, the consumption and asset holding plan* $\{c_i, a_i\}$ *is budget feasible for agent* i *in sequential markets given*

Q if and only if: (i) c_i is budget feasible in time-zero markets given q, and (ii) a_i satisfies

$$a_{it}(s^t) = \sum_{k \geq 0} \sum_{s^{t+k} \succeq s^t} \frac{q_{0t+k}(s^{t+k})}{q_{0t}(s^t)} \left[c_{it+k}(s^{t+k}) - y_{it+k}(s^{t+k}) \right], \tag{1.5.1}$$

for all times $t \geq 0$ and histories $s^t \in S^t$.

For intuition, here is an interpretation of the "if" part. Consider any consumption plan satisfying the time-zero market intertemporal budget constraint. Then, the agent can attain the same consumption plan with sequential markets. To do so, at time t and after history s^t, his financial wealth and his continuation endowment stream must be enough to finance his continuation consumption plan. Given that the budget constraint binds, this means that his financial wealth, $a_{it}(s^t)$, must be exactly equal to the present value of his continuation consumption plan, net of the present value of his continuation endowment stream. A corollary of the Proposition is:

Corollary 1.5.1. *Let $q_{0t}(s^t) \equiv Q_t(s_t \mid s^{t-1}) \times Q_{t-1}(s_{t-1} \mid s^{t-2}) \times \ldots \times Q_1(s_1 \mid s_0)$. If $\{c_i\}_{i=1}^I$ and q is an equilibrium in time-zero market, then $\{c_i, a_i\}_{i=1}^I$ and Q is an equilibrium in sequential markets. Conversely, if $\{c_i, a_i\}_{i=1}^I$ and Q is an equilibrium in sequential markets, then $\{c_i\}_{i=1}^I$ and q is an equilibrium in time-zero market.*

Given the relationship between q and Q stated in the Corollary, we know from Proposition 1.5.1 that consumption possibilities are the same in time zero markets and in sequential markets. Hence, a consumption plan is optimal in time-zero market given q if and only if it is optimal in sequential markets given Q. This implies that, if a consumption allocation and a price system q satisfies equilibrium conditions in time-zero market, the same consumption allocation and the corresponding price system Q is the basis of an equilibrium in sequential markets.

1.5.2 Proof of Proposition 1.5.1

Proof of the "if" part. Consider some consumption plan that is budget feasible in time-zero markets given q and define $a_{it}(s^t)$ as in (1.5.1). Note that $a_{it}(s^t)$ is indeed well defined, because the sequence $q_{0t+k}(s^{t+k})$ is in ℓ_1 and because both consumption and income are in ℓ_∞. We seek to show that $\{c_i, a_i\}$ is budget feasible in sequential markets given Q. That is, we need to show that: (i) initial wealth is zero, $a_{i0}(s_0) = 0$; (ii) the sequential budget constraint (1.4.2) holds at each time $t \geq 0$ and after every history $s^t \in S^t$; and (iii) the No Ponzi Game constraint combined with the Transversality condition, (1.4.5), holds at each time $t \geq 0$ and after every history $s^t \in S^t$.

For (i), we evaluate equation (1.5.1) at $t = 0$. We obtain that $a_{i0}(s_0)$ is the difference between the present value of consumption and the present value of income, which is equal to zero since the consumption plan satisfies the intertemporal budget constraint. Hence, $a_{i0}(s_0) = 0$.

For (ii), we calculate the left-side of the sequential budget constraint (1.4.2) at time t and after history s^t:

$$
\begin{aligned}
& c_{it}\left(s^t\right)+\sum_{s_{t+1}\in S} Q_{t+1}\left(s_{t+1}|s^t\right) a_{it+1}\left(s^t,s_{t+1}\right) \\
= & c_{it}\left(s^t\right)+\sum_{s_{t+1}\in S}\frac{q_{0t+1}\left(s^{t+1}\right)}{q_{0t}\left(s^t\right)}\left\{\sum_{k\geq 0}\sum_{s^{t+1+k}\succeq s^{t+1}}\frac{q_{0t+1+k}\left(s^{t+1+k}\right)}{q_{0t+1}\left(s^{t+1}\right)}\left[c_{it+1+k}\left(s^{t+1+k}\right)-y_{it+1+k}\left(s^{t+1+k}\right)\right]\right\} \\
= & y_{it}\left(s^t\right)+c_{it}\left(s^t\right)-y_{it}(s^t)+\sum_{k\geq 0}\sum_{s^{t+1+k}\succeq s^t}\frac{q_{0t+1+k}\left(s^{t+1+k}\right)}{q_{0t}\left(s^t\right)}\left[c_{it+1+k}\left(s^{t+1+k}\right)-y_{it+1+k}\left(s^{t+1+k}\right)\right] \\
= & y_{it}\left(s^t\right)+c_{it}\left(s^t\right)-y_{it}(s^t)+\sum_{k\geq 1}\sum_{s^{t+k}\succeq s^t}\frac{q_{0t+k}\left(s^{t+k}\right)}{q_{0t}\left(s^t\right)}\left[c_{it+k}\left(s^{t+k}\right)-y_{it+k}\left(s^{t+k}\right)\right] \\
= & y_{it}\left(s^t\right)+\sum_{k\geq 0}\sum_{s^{t+k}\succeq s^t}\frac{q_{0t+k}\left(s^{t+k}\right)}{q_{0t}\left(s^t\right)}\left[c_{it+k}\left(s^{t+k}\right)-y_{it+k}\left(s^{t+k}\right)\right] \\
= & y_{it}\left(s^t\right)+a_{it}\left(s^t\right).
\end{aligned}
$$

In the above, the second equality follows from plugging in the definition of $a_{it+1}(s^t, s_{t+1})$ in equation (1.5.1). The third line follows by adding and subtracting $y_{it}(s^t)$, and by noting that summing over s_{t+1} and then $s^{t+1+k}\succeq s^{t+1}$ is the same as summing directly over $s^{t+1+k}\succeq s^t$. The fourth line follows by changing the index of summation. The fifth line follows by collecting terms. The sixth line follows from the definition of $a_{it}(s^t)$, in equation (1.5.1).

For (iii), we use the definition of $a_{it}(s^t)$.

$$
\begin{aligned}
a_{it}(s^t) = & \sum_{k=0}^{T-t-1}\sum_{s^{t+k}\succeq s^t}\frac{q_{0t+k}(s^{t+k})}{q_{0t}(s^t)}\left[c_{it+k}(s^{t+k})-y_{it+k}(s^{t+k})\right] \\
& +\sum_{k=T-t}^{\infty}\sum_{s^{t+k}\succeq s^t}\frac{q_{0t+k}(s^{t+k})}{q_{0t}(s^t)}\left[c_{it+k}(s^{t+k})-y_{it+k}(s^{t+k})\right] \\
= & \sum_{k=0}^{T-t-1}\sum_{s^{t+k}\succeq s^t}\frac{q_{0t+k}(s^{t+k})}{q_{0t}(s^t)}\left[c_{it+k}(s^{t+k})-y_{it+k}(s^{t+k})\right] \\
& +\sum_{s^T\succeq s^t}\frac{q_{0T}(s^T)}{q_{0t}(s^t)}\sum_{k=0}^{\infty}\sum_{s^{T+k}\succeq s^T}\frac{q_{0T+k}(s^{T+k})}{q_{0T}(s^T)}\left[c_{iT+k}(s^{T+k})-y_{iT+k}(s^{T+k})\right] \\
= & \sum_{k=0}^{T-t-1}\sum_{s^{t+k}\succeq s^t}\frac{q_{0t+k}(s^{t+k})}{q_{0t}(s^t)}\left[c_{it+k}(s^{t+k})-y_{it+k}(s^{t+k})\right]+\sum_{s^T\succeq s^t}\frac{q_{0T}(s^T)}{q_{0t}(s^t)}a_{iT}(s^T),
\end{aligned}
$$

where: the first equality follows by breaking the infinite sum defining $a_{it}(s^t)$ into two part; the second equality follows after changing the summation index in the second summation; and the third equality follows from the definition of $a_{iT}(s^T)$. Since

$$
\lim_{T\to\infty}\sum_{k=0}^{T-t-1}\sum_{s^{t+k}\succeq s^t}\frac{q_{0t+k}(s^{t+k})}{q_{0t}(s^t)}\left[c_{it+k}(s^{t+k})-y_{it+k}(s^{t+k})\right]=a_{it}(s^t),
$$

it must be that $\lim_{T\to\infty}\sum_{s^T\succeq s^t} q_{0T}(s^T)a_{iT}(s^T)=0$.

Proof of the "only if" part. Consider the sequential budget constraint at time $t+k$ after history s^{t+k}, shown in equation (1.4.2). Multiply both sides of the equation by $\frac{q_{0t+k}(s^{t+k})}{q_{0t}(s^t)}$. We obtain a "discounted" sequential budget constraint, i.e., the sequential budget constraint in terms of time t prices:

$$\begin{aligned}&\frac{q_{0t+k}(s^{t+k})}{q_{0t}(s^t)}c_{it+k}(s^{t+k})+\sum_{s_{t+k+1}\in S}\frac{q_{0t+k+1}(s^{t+k+1})}{q_{0t}(s^t)}a_{it+k+1}(s^{t+k+1})\\ =&\frac{q_{0t+k}(s^{t+k})}{q_{0t}(s^t)}y_{it+k}(s^{t+k})+\frac{q_{0t+k}(s^{t+k})}{q_{0t}(s^t)}a_{it+k}(s^{t+k}).\end{aligned}$$

Now add up the sequential budget constraint for all $k=0,1,\ldots,T-t-1$ and all histories $s^{t+k}\succeq s^t$. We obtain:

$$\begin{aligned}&\sum_{k=0}^{T-t-1}\sum_{s^{t+k}\succeq s^t}\frac{q_{0t+k}(s^{t+k})}{q_{0t}(s^t)}c_{it+k}(s^{t+k})+\sum_{k=0}^{T-t-1}\sum_{s^{t+k+1}\succeq s^t}\frac{q_{0t+k+1}(s^{t+k+1})}{q_{0t}(s^t)}a_{it+k+1}(s^{t+k+1})\\ =&\sum_{k=0}^{T-t-1}\sum_{s^{t+k}\succeq s^t}\frac{q_{0t+k}(s^{t+k})}{q_{0t}(s^t)}y_{it+k}(s^{t+k})+\sum_{k=0}^{T-t-1}\sum_{s^{t+k}\succeq s^t}\frac{q_{0t+k}(s^{t+k})}{q_{0t}(s^t)}a_{it+k}(s^{t+k}).\end{aligned}$$

Notice that the second terms on the left-side can be written $\sum_{k=1}^{T-t}\sum_{s^{t+k}\succeq s^t}\frac{q_{0t+k}(s^{t+k})}{q_{0t}(s^t)}a_{it+k}(s^{t+k})$. The second term on the right-side is almost the same with one difference: the time index runs from $k=0$ to $k=T-t-1$, and not from $k=1$ to T. Hence, all the terms cancel out, except for the last one (corresponding to time T on the left-side) and the first one (corresponding to time t on the right-side). Taken together, we simplify the above and obtain:

$$\begin{aligned}&\sum_{k=0}^{T-t-1}\sum_{s^{t+k}\succeq s^t}\frac{q_{0t+k}(s^{t+k})}{q_{0t}(s^t)}c_{it+k}(s^{t+k})+\sum_{s^T\succeq s^t}\frac{q_{0T}(s^T)}{q_{0t}(s^t)}a_{iT}(s^T)\\ =&\sum_{t=0}^{T-t-1}\sum_{s^{t+k}\succeq s^t}\frac{q_{0t+k}(s^{t+k})}{q_{0t}(s^t)}y_{it+k}(s^{t+k})+a_{it}(s^t).\end{aligned}$$

Letting T go to infinity and using the No Ponzi Game constraint combined with the Transversality condition, we obtain that $a_{it}(s^t)$ satisfies the formula (1.5.1). Evaluating this formula at $t=0$, we obtain that the consumption plan satisfies the time-zero intertemporal budget constraint (1.3.1).

1.5.3 Proof of Corollary 1.5.1

Consider an equilibrium in time zero market $\{c_i\}_{i=1}^I$ and q. Define $Q_{t+1}(s_{t+1}\mid s^t)=q_{0t+1}(s^{t+1})/q_{0t}(s^t)$ and $a_{it}(s^t)$ as in Proposition 1.5.1. One immediately sees that the feasibility of c_i implies that of a_i (i.e. the market for Arrow securities clear). So all we need to show is that c_i and a_i are optimal for the agent's problem in sequential markets. Consider any $\hat{c}_i$ and $\hat{a}_i$ which are budget feasible for the agent's problem with sequential markets. By Proposition 1.5.1, $\hat{c}_i$ is also budget feasible for the agent's problem with time-zero markets, given the price system q. By assumption, c_i is optimal for the agent's problem with time-zero market. Hence, we obtain that $U(c_i) \geq U(\hat{c}_i)$, i.e., c_i and a_i are optimal for the agent's problem in sequential markets. This implies that $\{c_i, a_i\}_{i=1}^I$ and Q is an equilibrium in sequential markets. The proof of the converse is analogous and is thus omitted.

1.6 An application: Ricardian equivalence

In this section, we use the logic of Proposition 1.5.1 and Corollary 1.5.1 to study the impact of government tax and debt policy on aggregate economic outcomes. Specifically, we consider a government who needs to finance some exogenous stream of expenditures in an economy with complete markets. The government can levy lump-sum taxes and issue debt. We establish the Ricardian equivalence: we show that the manner in which the stream of expenditures is financed does not matter for aggregate outcomes. For example, if the government cuts taxes and increases debt in the current period, and repay the debt with future taxes, then equilibrium outcomes remain the same.

1.6.1 The economic environment

The government. To see this point, consider the same economy as before but add a government with some exogenous per-capita expenditure plan $g = \{g_t(s^t) : t \geq 0, s^t \in S^t\}$. For simplicity, we assume that government expenditures are thrown away. The government chooses the size of per-capita lump-sum taxes, $\tau = \{\tau_t(s^t) : t \geq 0, s^t \in S^t\}$, and of (state contingent) per-capita debt issuance, $B \equiv \{B_t(s^t) : t \geq 0, s^t \in S^t\}$, subject to the sequential budget constraint and to a No Ponzi game constraint/Transversality condition:

$$g_t\left(s^t\right) + B_t\left(s^t\right) = \sum_{s_{t+1} \in S} Q_{t+1}\left(s_{t+1} | s^t\right) B_{t+1}\left(s^t, s_{t+1}\right) + \tau_t\left(s^t\right) \tag{1.6.1}$$

$$\lim_{T \to \infty} \sum_{s^T \succeq s^t} \frac{q_{0T}\left(s^T\right)}{q_{0t}(s^t)} B_T\left(s^T\right) = 0, \tag{1.6.2}$$

for all times $t \geq 0$ and every history $s^t \in S^t$, where $q_{0t}(s^t) = Q_t(s_t \,|\, s^{t-1}) \times Q_{t-1}(s_{t-1} \,|\, s^{t-2}) \ldots \times Q_1(s_1 \,|\, s_0)$. For simplicity, we assume that, at time $t = 0$, the government starts with no debt, $B_0(s_0) = 0$. At this stage, we do not take a stand on the objective of the government: we just take expenditure as given and impose budget feasibility for the debt and tax policy. The optimal taxation literature considers the problem of choosing a particular debt and tax policy to maximize some government objective, subject to the constraint that economic outcomes constitute a competitive equilibrium.

One may argue that for some reason the No Ponzi game constraint does not apply to the government, because of some special ability to roll over or issue debt. While it is quite unclear why this could be true, one can see that, in fact, it is possible to relax the No Ponzi game constraint of the government and show that it has to hold in an equilibrium. Indeed, as will become clear, the feasibility constraint for Arrow securities show that government debt is held by agents. Hence, the sum of agents' Transversality conditions implies the no Ponzi Game constraint of the government. See the arguments outlined in Sims (1994), and Coşar and Green (2016). What this mean is the following. If the government debt grows too fast, in the sense of violating the No Ponzi Game conditions, then the Transversality condition of agents would be violated. But remember that the Transversality Condition is an *optimality* condition, not a constraint. Hence, a violation of the transversality condition means that agents *are not willing to hold* the debt of the government. Put differently, what ultimately constraint the debt accumulation of the government is the demand of the private sector.

Agents. Agent $i \in \{1, \ldots, I\}$ maximizes

$$\sum_{t \geq 0} \sum_{s^t \in S^t} \beta^t \pi_{0t}(s^t) u_i \left[c_{it}(s^t) \right],$$

with respect to a consumption and asset holdings plan, $(c_i, a_i) = \{c_{it}(s^t), a_{it}(s^t) : t \geq 0, s^t \in S^t\}$, subject to

$$c_{it}\left(s^t\right) + \sum_{s_{t+1} \in S} Q_{t+1}\left(s_{t+1}|s^t\right) a_{i,t+1}\left(s^t, s_{t+1}\right) + \tau_t\left(s^t\right) = y_{it}\left(s^t\right) + a_{it}\left(s^t\right) \tag{1.6.3}$$

$$\lim_{T \to \infty} \sum_{s^T \succeq s^t 0} \frac{q_{0T}(s^T)}{q_{0t}(s^t)} a_{iT}(s^T) = 0, \tag{1.6.4}$$

for all times $t \geq 0$ and histories $s^t \in S^t$, and $a_{i0}(s_0) = 0$.

Definition 1.6.1. An allocation is a collection $\{c_i, a_i\}_{i=1}^I$ of consumption and asset holding plans.

Definition 1.6.2. An allocation is feasible if:

$$\sum_{i=1}^I \left[c_{it}(s^t) + g_t(s^t) \right] = \sum_{i=1}^I y_{it}(s^t)$$

$$\sum_{i=1}^I a_{it}(s^t) = I \times B_t(s^t).$$

Notice that the definition of feasibility is modified to account for government expenditure and debt. Namely, the consumption of private agents must add up to the aggregate endowment *net of* government expenditure. The Arrow securities holding of private agents must add up to government debt issuance – because $B_t(s^t)$ is per capita issuance, total issuance is $I \times B_t(s^t)$. We then define:

Definition 1.6.3. Given a stream of government expenditures, g, a competitive equilibrium consists of a debt and tax policy, $\{\tau, B\}$, a feasible allocation $\{c_i, a_i\}_{i \in I}$, and price system Q such that

- $\{B, \tau\}$ satisfies the government sequential budget constraint (1.6.1) and (1.6.2), given g and Q.
- For all $i \in \{1, \ldots, I\}$, $\{c_i, a_i\}$ solves agent i's problem given Q;

In the definition, taxes and debt are equilibrium objects: this is because they must satisfy the the government budget constraint, which depends on endogenous prices.

As stated, the definition seems to allow for multiple equilibria characterized by different debt and tax policies, $\{\tau, B\}$. That is, one may imagine that different debt and tax policies induce different asset prices and different consumption choices. We will see shortly that this is not the case: as it turns out, the particular choice of tax and debt policy is irrelevant.

1.6.2 Sequential vs. time zero budget sets

Just as in Section 1.5, the budget set defined by the combination of sequential budget constraints and No Ponzi Game constraint/Transversality condition creates the same consumption possibilities as a budget set defined by a single time-zero inter-temporal budget constraint. For the government, the time-zero inter-temporal budget constraint equates the present value of per capita expenditures to the present value of per capita lump sum taxes:

$$\sum_{t\geq 0}\sum_{s^t\in S^t} q_{0t}(s^t)g_t(s^t) = \sum_{t\geq 0}\sum_{s^t\in S^t} q_{0t}(s^t)\tau_t(s^t). \tag{1.6.5}$$

For an agent, the time-zero inter-temporal budget constraint is:

$$\sum_{t\geq 0}\sum_{s^t\in S^t} q_{0t}(s^t)\left[c_{it}(s^t)+\tau_t(s^t)\right] = \sum_{t\geq 0}\sum_{s^t\in S^t} q_{0t}(s^t)y_{it}(s^t). \tag{1.6.6}$$

It is the same constraint as before, except for the fact that the agents' expenditure include the payment of a lump sum tax, $\tau_t(s^t)$. With these definitions in mind, I state two equivalence results, one for the government budget set and one for private agents' budget sets. I do not state the proof because it follows from the exact same steps as Proposition 1.5.1.

Corollary 1.6.1. *Let* $q_{0t}(s^t) \equiv Q_t(s_t\,|\,s^{t-1}) \times Q_{t-1}(s_{t-1}\,|\,s^{t-2}) \times \ldots \times Q_1(s_1\,|\,s_0)$. *Then, the consumption and asset holding plan* $\{c_i, a_i\}$ *is budget feasible for agent* i *in sequential markets given* Q *if and only if it is budget feasible in time-zero markets given* q *and*

$$a_{it}(s^t) = \sum_{k\geq 0}\sum_{s^{t+k}\succeq s^t} \frac{q_{0t+k}(s^{t+k})}{q_{0t}(s^t)}\left[c_{it+k}(s^{t+k})+\tau_{t+k}(s^{t+k})-y_{it+k}(s^{t+k})\right], \tag{1.6.7}$$

for all times $t \geq 0$ *and histories* $s^t \in S^t$.

Corollary 1.6.2. *Let* $q_{0t}(s^t) \equiv Q_t(s_t\,|\,s^{t-1}) \times Q_{t-1}(s_{t-1}\,|\,s^{t-2}) \times \ldots \times Q_1(s_1\,|\,s_0)$. *Then, a policy* $\{g, \tau, B\}$ *is budget feasible for the government in sequential markets given* Q *if and only if it is budget feasible in time-zero markets given* q *and*

$$B_t(s^t) = \sum_{k\geq 0}\sum_{s^{t+k}\succeq s^t} \frac{q_{0t+k}(s^{t+k})}{q_{0t}(s^t)}\left[\tau_{t+k}(s^{t+k})-g_{t+k}(s^{t+k})\right], \tag{1.6.8}$$

for all times $t \geq 0$ *and histories* $s^t \in S^t$.

1.6.3 The main result

The main proposition for this section is that equilibria do not depend on the timing of taxes and debt. To see how this result obtains, follow the logic of Corollary 1.6.1 and 1.6.2. From 1.6.1 , we know that the agent chooses his consumption plan "as if" he were facing the single time-zero intertemporal budget constraint (1.6.6). Notice in particular that the households only care about the present value of lump-sum taxes he will pay to the government – the precise timing of taxes does not matter. Now, look at the the government inter-temporal budget constraint (1.6.2): it implies that the present value of lump sum taxes must be equal to the present value of expenditures. Put differently, if we substitute the government time zero intertemporal budget constraint in the agent time zero intertemporal budget constraint, we obtain:

$$\sum_{t\geq 0}\sum_{s^t\in S^t} q_{0t}(s^t)\left[c_{it}(s^t)+g_t(s^t)\right] = \sum_{t\geq 0}\sum_{s^t\in S^t} q_{0t}(s^t)y_{it}(s^t). \tag{1.6.9}$$

Hence, government policy only constrains agents' consumption choice via the present value of its per capita expenditure. The details of public finance are irrelevant. This means that if the government changes its stream of taxes to τ', or if it changes its debt policy to B', but keeps its expenditure the same, then the agent's consumption remain optimal. The asset holdings must change however, and are determined by (1.6.7). For example, if the government reduces $\tau_t(s^t)$ and increases $\tau_{t+k}(s^{t+k})$, then it must increase the amount of debt it issues, $B_{t+1}(s^t, s_{t+1})$. One sees from (1.6.7) that the asset holdings $a_{it+1}(s^t, s_{t+1})$ of the agent, $a_{it+1}(s^t, s_{t+1})$, must increase as well. Indeed, the agent saves more because he anticipates the future increase in lump sum taxes at time $t+k$. Formally, we obtain:

Proposition 1.6.1. *Consider an equilibrium $\{\tau, B, c_i, a_i, Q\}$ and let*

$$q_{0t}(s^t) \equiv Q_t(s_t \,|\, s^{t-1}) \times Q_{t-1}(s_{t-1} \,|\, s^{t-2}) \times \ldots \times Q_1(s_1 \,|\, s_0).$$

Consider any stream of taxes $\hat{\tau}$ such that:

$$\sum_{t\geq 0}\sum_{s^t\in S^t} q_{0t}(s^t)\hat{\tau}_t(s^t) = \sum_{t\geq 0}\sum_{s^t\in S^t} q_{0t}(s^t)\tau_t(s^t). \tag{1.6.10}$$

Then $\{\hat{\tau}, \hat{B}, c_i, \hat{a}_i, Q\}$ is an equilibrium, where $\hat{B}$ and $\hat{a}$ are given by (1.6.7) *and* (1.6.8) *given $\hat{\tau}$.*

Proof. Combined with Corollary 1.6.2, (1.6.10) implies that the government sequential budget constraint and No Ponzi Game constraint/Transversality condition hold. Next, we need to show that $\{c_i, \hat{a}_i\}$ is optimal given Q and $\hat{\tau}$. Now combine corollary 1.6.1 with the observation that the present value of taxes equals the present value of government expenditure. This implies that a consumption plan c_i' is part of a budget feasible plan $\{c_i', a_i'\}$ if and only if it satisfies the time zero intertemporal budget constraint (1.6.9). Hence, a consumption plan c_i' is part of an optimal budget feasible plan if and only if it maximizes the agent intertemporal utility subject to (1.6.9). □

In particular, notice that it is budget feasible for the government to issue no debt at all, $B_t(s^t) = 0$, and set per-capita taxes equal to per-capita expenditure, $\tau_t(s^t) = g_t(s^t)$. Thus, one can solve for equilibrium price and consumption allocations by assuming there is no government and reducing every agent's endowment by $g_t(s^t)$.

Abel (2018) discusses the logic of the Ricardian Equivalence proposition in details: he places it a broader literature context, and he explains when the proposition continues or fails to hold when some of the main assumptions are relaxed. He emphasize that the logic of the Ricardian equivalence is that, when there is a tax cut financed by an increase in government bonds, the government bonds that households purchase do not constitute any "net wealth" to them, since government budget balance implies that any tax cut in the current period results in a corresponding increase in some future tax liability. One may guess that the argument relies on assuming that consumers have finite lives. For example, a finitely lived consumer may anticipate that he/she may die before having to pay taxes. In a celebrated paper, Barro (1974) showed that finite lives do not matter if agents are altruistic, in the sense that they care about the *inter-temporal utility* of their offspring. In that case, agents in the same extended family are effectively linked together by the same inter-temporal budget constraint: when there is a tax cut, an agent anticipates the tax liability it creates to its offspring, and saves accordingly. Notice that, in this argument, there is an important distinction between caring about the monetary value of bequest left to offsprings, and caring about the inter-temporal utility of offsprings. If an agent care about the inter-temporal utility, then its bequest calculation takes into account both the monetary value of the bequest, and the future tax liability: the agents finds it optimal to save more and increase the bequest, so as to offset the future tax liability of the offsprings.

Economists often express skepticisms about Ricardian proposition because of its strong and unrealistic assumptions such as finite lives, lump sum taxes, or perfect asset markets. But it could very well be that, even though its asssuptions are not exactly true, the underlying economic forces are so powerfull that the Ricardian proposition holds approximately. Empiricially, it is not so easy to build a convincing case against the Ricardian proposition, as explained by Bernheim (1987) (who is, as a matter of fact, a skeptic). For example, even if there appears to be a relationship between taxes and economic activity, this could simply be because taxes are distortionary, not because of some economic response that would run counter the economic forces highlighted by the Ricardian proposition. It is also hard to find changes in taxes and debt policy that would be truly exogenous. For example shocks that change the aggregate economic environment may alter the equilibrium decisions of households and firms and, at the same time, lead to a change in taxe and debt policy. While this would create a statistical relationship between aggregate outcomes and government policy, this would not be evidence that the Ricardian equivalence does not hold. To state this in the formalism of the Proposition, one can always specify a tax and debt policy $\{\tau, B\}$ that is statistically related to equilibrium outcomes. This is not evidence of a causal relationship between $\{\tau, B\}$ and equilibrium outcomes: indeed, a change in $\{\tau, B\}$ would keep outcomes the same. Vice versa, we could imagine a model in which the Ricardian proposition does not hold, and in which there are no statistical relationship between $\{\tau, B\}$ and equilibrium outcomes. This would be the case for example, if agents find that debt is net wealth but they perfectly anticipate fluctuations in $\{\tau, B\}$, in which case their consumption could be perfectly smooth even though $\{\tau, B\}$ is not. In a recent paper, Geerolf and Grjebine (2019) study the economic effects of changes in property taxes in many countries. They argue that property taxes are close to be lump-sum, in the sense that they do not distort much behavior. They use a narrative approach, that is, they use news record to precisely understand what motivates each change in property tax, and select those that appear to be exogenous to concurrent aggregate economic conditions. Their empirical results reject the Ricardian equivalence.