Problem Set 4. Econ 202B, 2021

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1 Problems

1.1 A representative agent economy

Time is discrete and the horizon infinite. Every period, a stochastic event $s \in \{L, H\}$ (L is for "low" and H is for "high") is drown according to the probability distribution $\{\pi(L), \pi(H)\}$, independently from other periods. There is one single representative agent with CRRA utility and discount factor $\beta \in (0, 1)$. We take β to be small enough so that all long lived assets we consider in this problem set have finite prices. The aggregate endowment at time (t, s^t) is denoted by $y_t(s^t)$ and satisfies:

$$
\begin{aligned}
y_{t+1}(s^{t+1}) &= y_t(s^t)g(s_{t+1}) \\
y_0(s_0) &= 1.
\end{aligned}
$$

where $g(H) > g(L)$.

1. State the representative agent's problem in the model in which complete markets open at time zero only. Define a competitive equilibrium with time zero markets. Calculate the equilibrium time-zero price of consumption at time (t, s^t), $q_{0t}(s^t)$.

2. State the representative agent's problem in the model in which complete markets for one-step ahead Arrow Debreu securities open at each time. Define a competitive equilibrium with sequential markets.

   (a) Calculate the equilibrium price of one-period-ahead arrow securities, $Q_{t+1}(s_{t+1} \mid s^t)$.

   (b) Calculate the stochastic discount factor, $M_{t+1}(s_{t+1} \mid s_t)$.

   (c) Calculate the "risk-neutral" probabilities of state $s \in \{L, H\}$. How do they differ from the physical probabilities, and why?

   (d) Calculate the time (t, s^t) price of a one-period zero coupon bond, i.e., a security paying off 1 unit of good at time $t + 1$ regardless of s_{t+1}.

   (e) Calculate the time (t, s^t) price of a n-period zero coupon bond, i.e., a security paying off 1 unit of good at time $t + n$ regardless of $(s_{t+1}, s_{t+1}, \ldots, s_{t+n})$.

1.2 Corporate bond spread

Time is discrete and runs from $t = 0$ to $t = \infty$. Every period, one state is drawn, $s \in \{L, H\}$. The process s_t is a Markov chain with transition probability matrix,

$$\Pi = \begin{bmatrix} 1-q & q \\ q & 1-q \end{bmatrix},$$

where we assume that $q < \frac{1}{2}$. The initial condition is $s_0 = L$. There is a representative agent with CRRA utility:

$$\sum_{t \geq 0} \beta^t \sum_{s^t} \pi_{0t}(s^t \mid s_0) \frac{c_t(s^t)^{1-\gamma} - 1}{1-\gamma}.$$

The aggregate endowment, $y_t(s^t)$, follows

$$y_{t+1}(s^{t+1}) = g(s_{t+1}) y_t(s^t).$$

Assume that $g(L) < g(H)$ and, to ensure bounded present values, $\beta \max\{g(H)^{1-\gamma}, g(L)^{1-\gamma}\} < 1$. Suppose that, every period, agents can trade a complete set of Arrow securities.

1. Define the agent's problem.

2. Define an equilibrium.

3. Derive a formula for the price of a risk-less one-period bond with face value 1.

4. Consider a firm who issues a one-period corporate bond with face value normalized to 1. The firm operates for just a period, and its profits (after paying workers but before paying bond holders) is $\Pi(s)$, with $\Pi(L) < 1 < \Pi(H)$. Bond holders are paid the face value of profits are large enough to do so. Otherwise, the firm defaults and the payoff of bond holders is equal to the firm's profits. Derive a formula for the price of the corporate bond.

5. Using the previous question explain the determinants of the price difference between the risk-less bond and corporate bond. Is the price difference larger in good times or in bad times? Explain why.

1.3 Heterogeneous beliefs

This problem is based on a paper by Sandroni (ECMA, 2000). You will study a competitive equilibrium in which agents have heterogeneous beliefs and time-separable utility function with identical discount factor. Time is discrete $t \in \{0, 1, 2, \ldots\}$. Every period a stochastic event s is drawn from a finite set S, independently from earlier period and according to the distribution $\{\pi_0(s)\}_{s \in S}$. The initial event, s_0, is fixed. As in class, we denote the history of events up to time t by $s^t = (s_0, s_1, \ldots, s_t)$.

The aggregate endowment at time zero is normalized to $y(s_0) = 1$ and it evolves over time according to:

$$y_{t+1}(s^t, s_t) = y_t(s^t) g(s_{t+1})$$

where $s \mapsto g(s)$ is a deterministic function such that

$$\mathbb{E}\left[\log(g)(s)\right] = \sum_s \pi_0(s) \log\left[g(s)\right] > 0.$$

There are two types of households, indexed by $j \in \{1, 2\}$, with a measure one of each and with heterogenous preferences and beliefs. The endowment of a household of type j at time (t, s^t) is denoted by $y_{jt}(s^t)$. Household of type j the aggregate endowment intertemporal utility is:

$$\sum_{t \geq 0, s^t \in S^t} \beta^t \pi_{jt}(s^t) u\left[c_{jt}(s^t)\right],$$

where

$$u_j(c) = \begin{cases} \frac{c^{1-\gamma_j}}{1-\gamma_j} & \text{if } \gamma_j \neq 1 \\ \log(c) & \text{if } \gamma_j = 1 \end{cases}$$

and $\gamma_j > 0$. Lastly, we assume that the subjective belief of the household is that the state of the world is drawn according to the distribution $\{\pi_j(s)\}_{s \in S}$, independently across periods.

1. Use the Law of Large Numbers to show that $\lim_{t \to \infty} y_t(s^t) = \infty$, π_0-almost surely.

2. The coefficient of relative risk aversion (RRA) is defined as $RRA = -\frac{cu''(c)}{u'(c)}$. Calculate the RRA coefficient of household j.

3. The elasticity of intertemporal substitution (EIS) is defined as $\frac{d\log(c_{t+1}/c_t)}{d\log(R_t)}$, where $\{c_t\}$ is the consumption path of a household in a deterministic economy when the interest

rate between t and $t+1$ is R_t. Calculate the EIS of household j. How is it related to the RRA coefficient?

4. Assume markets open at time zero and let $\hat{c}_{jt}(s^t) = c_{jt}(s^t)/y_t(s^t)$ denote the consumption share of households of type j. Show that, in an equilibrium, there exists some $\rho > 0$ such that the consumption share solves

$$\hat{c}_{1t}(s^t) + \delta_t(s^t)\left[\hat{c}_{1t}(s^t)\right]^{\gamma_1/\gamma_2} = 1 \text{ where } \delta_t(s^t) = \rho\left[y_t(s^t)\right]^{\frac{\gamma_1}{\gamma_2}-1} \times \left[\frac{\pi_{2t}(s^t)}{\pi_{1t}(s^t)}\right]^{1/\gamma_2}.$$

5. Suppose there is no uncertainty. Under what condition on the parameters do we have that $\lim_{t\to\infty} \hat{c}_{1t} = 0$? Why does agent 1 find it optimal to see his/her consumption share vanish as $t \to \infty$? Is it a violation of the "history independence" property of Pareto efficient allocation we discussed in class? Why?

6. Suppose that households have homogenous beliefs, $\pi_1(s) = \pi_2(s)$ for all $s \in S$. Under which conditions on the parameters do we have: $\lim_{t\to\infty} \hat{c}_{1t}(s^t) = 0$, π_0-almost surely? Why does agent 1 find it optimal to see his/her consumption share vanish as $t \to \infty$? Is it a violation of the "history independence" property of Pareto efficient allocation we discussed in class? Why?

7. Suppose that households have homogenous relative risk aversion, $\gamma_1 = \gamma_2$.

   (a) Under which condition over beliefs do we have that $\lim_{t\to\infty} \hat{c}_{1t}(s_t) = 0$, π_0-almost surely? Why does agent 1 find it optimal to see his/her consumption share vanish as $t \to \infty$? For this it is helpful to note that a measure of distance of household's 1 belief to the truth is the relative entropy of his/her probability distribution relative to π_0, $\mathbb{E}\left[\log\left(\frac{\pi_0(s)}{\pi_1(s)}\right)\right] = \sum_s \pi_0(s)\log\left(\frac{\pi_0(s)}{\pi_1(s)}\right)$, which is positive and zero if and only if $\pi_0 = \pi_1$.

   (b) What does a household of type 1 believe about its asymptotic consumption share?

   (c) Is it possible to find $\pi_1 \neq \pi_2$ such that none of the consumption share converge to either one or zero? For these parameters, what is the asymptotic behavior of the consumption share? If you are stuck with this question, show me simulation of the consumption share process on Matlab and offer conjectures.

8. Derive the general condition on parameter for $\lim_{t\to\infty} \hat{c}_{1t}(s^t) = 0$, π_0-almost surely.

2 Solutions

2.1 A representative agent economy

1. The problem for the agent with time-zero markets, given prices, is

$$\begin{aligned}
&\max_{\{c_t(s^t)\}_{t\geq 0, s^t\in S^t}} \sum_{t\geq 0}\sum_{s^t\in S^t} \beta^t \pi_t(s^t) u(c_t(s^t)) \\
&\text{s.t.} \sum_{t\geq 0}\sum_{s^t\in S^t} q_{0t}(c_t(s^t) - y_t(s^t)) \leq 0, \\
&\qquad\qquad c_t(s^t) \geq 0, \text{ for each } t \geq 0, s^t \in S^t.
\end{aligned}$$

Define an allocation for the agent as a sequence for consumption $\{c_t(s^t)|t \geq 0, s^t \in S^t\}$.
Define a price system as a sequence $\{q_{0t}(s^t) \geq 0|t \geq 0, s^t \in S^t\}$.
An equilibrium in time-zero markets consists of a price system and allocation , such that the allocation solves the agent's problem (as defined above) given the price system and the market clears (in this case, $c_t(s^t) = y_t(s^t)$ for each t and s^t).

To calculate the equilibrium prices, we use the first-order condition of the agent

$$\beta^t \pi_t(s^t) u'(c_t(s^t)) = \lambda q_{0t},$$

where λ is the Lagrange multiplier on the time-zero budget constraint. Still, the prices in this equation depend on endogenous objects (consumption). Using the market clearing condition ($c_t(s^t) = y_t(s^t)$), we can write

$$\beta^t \pi_t(s^t) u'(y_t(s^t)) = q_{0t},$$

where I ignore λ since the price system is only identified up to a constant of proportionality. Since utility is assumed to be CRRA, we can simplify this further to

$$\beta^t \pi_t(s^t) y_t(s^t)^{-\gamma} = q_{0t},$$

with γ being the coefficient of relative risk aversion.

2. The agent's problem with sequential markets is

$$\max_{\{c_t(s^t), a_{t+1}(s^{t+1})\}_{t\geq 0, s^t \in S^t, s^{t+1}\in S^{t+1}}} \sum_{t\geq 0} \beta^t \pi_t(s^t) u(c_t(s^t))$$
$$\text{s.t. } c_t(s^t) + Q_{t+1} a_{t+1}(s^{t+1}) \leq y_t(s^t) + a_t(s^t), \text{ for each } t, s^t,$$
$$c_t(s^t) \geq 0, \text{ for each } t, s^t,$$
$$\text{No Ponzi Game condition, and}$$
$$a_0(s_0) = 0.$$

Define an allocation for the agent as a consumption sequence $\{c_t(s^t) | t \geq 0, s^t \in S^t\}$ and an asset holding sequence $\{a_t(s^t) | t \geq 0, s^t \in S^t\}$. Define a price system as a sequence $\{Q_{t+1}(s^{t+1}) \geq 0 | t \geq 0, s^{t+1} \in S^{t+1}\}$.

An equilibrium in sequential markets consists of a price system and allocation, such that the allocation solves the agent's problem (as defined above) given the price system and the market clears (in this case, $c_t(s^t) = y_t(s^t)$ **and** $a_t(s^t) = 0$ for each t and s^t).

(a) From the first-order condition with respect to Arrow securities,

$$\lambda_{t+1}(s^t, s_{t+1}) = Q_{t+1}(s_{t+1}|s^t)\lambda_t(s^t),$$

where $\lambda_t(s^t)$ is the lagrange multiplier in the sequential budget constraint at time t, history s^t (not to be confused with last question's λ).

From the first-order condition with respect to consumption,

$$\beta^t \pi_t(s^t) u'(c_t(s^t)) = \lambda_t(s^t).$$

Now, we can use this expression for $\lambda_t(s^t)$ in the previous equation for $Q_{t+1}(s_{t+1}|s^t)$.

$$Q_{t+1}(s_{t+1}|s^t) = \frac{\beta^{t+1}\pi_{t+1}(s^{t+1}) u'(c_{t+1}(s^{t+1}))}{\beta^t \pi_t(s^t) u'(c_t(s^t))}.$$

Simplifying terms, using the fact that states are distributed independently from other periods, and replacing $c_t(s^t) = y_t(s^t)$ because in equilibrium, that market must clear.

$$Q_{t+1}(s_{t+1}|s^t) = \frac{\beta\pi(s_{t+1}) u'(y_{t+1}(s^{t+1}))}{u'(y_t(s^t))}.$$

Since utility is CRRA,

$$Q_{t+1}(s_{t+1}|s^t) = \beta\pi(s_{t+1}) \left(\frac{y_{t+1}(s^{t+1})}{y_t(s^t)}\right)^{-\gamma} = \beta\pi(s_{t+1}) g(s_{t+1})^{-\gamma}.$$

(b) The stochastic discount factor is defined as

$$M_{t+1}(s_{t+1}|s^t) = \frac{Q_{t+1}(s_{t+1}|s^t)}{\pi(s^{t+1}|s^t)}.$$

In this case, this evaluates to

$$M_{t+1}(s_{t+1}|s^t) = \beta g(s_{t+1})^{-\gamma}.$$

(c) The risk neutral probabilities are

$$\tilde{\pi}_{t+1}(s_{t+1}|s^t) = \frac{\pi(s_{t+1})g(s_{t+1})^{-\gamma}}{\sum_{s'} \pi(s')g(s')^{-\gamma}} = \pi(s_{t+1})\frac{g(s_{t+1})^{-\gamma}}{\mathbb{E}_{s'\sim\pi}\left[g(s')^{-\gamma}\right]}.$$

As we can see, the risk-neutral probabilities equal the actual (physical) probabilities times an adjustment factor. That factor depends on the relative size of the growth rate of the particular future state under consideration and an *average* or expected growth rate in the future.

In particular, if the growth rate was the same under any state, the risk-neutral and physical probabilities would coincide; since the marginal utility of consumption would be the same in each state. The same argument holds when there is no risk aversion, $\gamma = 0$. When we are in a state with a higher growth rate, consumption will be larger which means the marginal utility will be smaller. This implies that the agent will not value payoffs in that state as much or, in terms of risk-neutral probabilities, that the agent will act *as if* he believed that state to be less likely than it actually is.

Practice question. Here is a question to get you to think a bit more. Suppose there is one state, s^* such that $g(s^*) = \mathbb{E}[g(s)]$. This is a state where the growth rate is the same as the expected growth rate. Is the risk-neutral probability of that state higher or lower than its physical probability? Why?

(d) A one-period bond is equivalent to a portfolio consisting of one Arrow-Debreu security for each future state in the next period. The price of such portfolio is $\sum_{s_{t+1}} Q(s_{t+1}|s^t) = \beta\mathbb{E}\left[g(s_{t+1})^{-\gamma}\right].$

(e) In order to obtain a payoff of 1 at time $t+n$ if the history up to time $t+n-1$ was s^{t+n-1}, we can buy, at time $t+n-1$ one unit of each Arrow-Debreu securities for next period, that is $\sum_{s_{t+n}} Q(s_{t+n}|s^{t+n-1}) = \beta\mathbb{E}\left[g(s)^{-\gamma}\right].$ Notice that this doesn't depend on the state or history at time $t+n-1$. In order to be able to buy those securities at time $t+n-1$, we need to have those resources which could be obtained by buying, at time $t+n-2$, a quantity $\beta\mathbb{E}\left[g(s)^{-\gamma}\right]$ of Arrow-Debreu securities for each state s_{t+n-1}. That would require us to spend $\beta\mathbb{E}\left[g(s)^{-\gamma}\right]\sum_{s_{t+n-1}} Q(s_{t+n-1}|s^{t+n-2}) = \left[\beta\mathbb{E}\left[g(s)^{-\gamma}\right]\right]^2$ in Arrow-Debreu securities at time $t+n-2$. Iterating in that way, at time t, we need to spend $\left[\beta\mathbb{E}\left[g(s)^{-\gamma}\right]\right]^n$ in order to be able to get 1 unit of consumption at time $t+n$.

Notice that this is saying that we can replicate an n-period risk-free zero-coupon bond by buying a sequence of one-period risk-free zero-coupon bonds. You should think on how this fact relates to the price of Arrow-Debreu securities not being a function of the history.

2.2 Corporate bond spread

1. The agent's problem

$$\max_{\{c_t(s^t),a_t(s^t)\}} \sum_{t\geq 0} \beta^t \sum_{s^t} \pi_{0t}\left(s^t|s_0\right) \frac{c_t\left(s^t\right)^{1-\gamma}-1}{1-\gamma}$$

subject to

$$c_t\left(s^t\right)+\sum_{s_{t+1}} Q_{t+1}\left(s_{t+1}|s^t\right) a_{t+1}\left(s^t,s_{t+1}\right)=y_t\left(s^t\right)+a_t\left(s^t\right)$$

$$\lim_{T\to\infty}\sum_{s^{T+1}}\left[\Pi_{t=0}^T Q_{t+1}\left(s_{t+1}|s^t\right) a_{T+1}\left(s^{T+1}\right)\right]=0$$

$$a_0\left(s_0\right) \text{ given.}$$

2. An equilibrium consists of an allocation $\{c,a\}=\{c_t(s^t),a_t(s^t):t\geq 0,s^t\in S^t\}$ and a price system $Q=\{Q_{t+1}(s_{t+1}|s^t):t\geq 0,s^t\in S^t,s_{t+1}\in S\}$ such that

   - given the price system, the allocation solves the agent's problem;
   - markets clear: $c_t(s^t)=y_t(s^t)$ and $a_t(s^t)=0$, $\forall t\geq 0$, $s^t\in S^t$.

3. From the agent's problem, we have

$$Q_{t+1}\left(s_{t+1}|s^t\right)=\beta\frac{\pi_{0,t+1}\left(s^{t+1}|s_0\right)}{\pi_{0t}\left(s^t|s_0\right)}\left(\frac{c_{t+1}\left(s^{t+1}\right)}{c_t\left(s^t\right)}\right)^{-\gamma}.$$

Since the process s_t is a Markov chain, $\frac{\pi_{0,t+1}\left(s^{t+1}|s_0\right)}{\pi_{0t}(s^t|s_0)}=\pi\left(s_{t+1}|s_t\right)$. Using the market clearing condition $c_t(s^t)=y_t(s^t)$, the equilibrium price of the Arrow securities is

$$Q_{t+1}\left(s_{t+1}|s^t\right)=\beta\pi\left(s_{t+1}|s_t\right)g\left(s_{t+1}\right)^{-\gamma}.$$

Let $p_t^r(s^t)$ denote the price of the risk-less one period bond with face value 1.

$$p_t^r\left(s^t\right)=\sum_{s_{t+1}}Q_{t+1}\left(s_{t+1}|s^t\right)=\beta\sum_{s_{t+1}}\pi\left(s_{t+1}|s_t\right)g\left(s_{t+1}\right)^{-\gamma}.$$

Therefore the price of risk-less bond only depends on the current state s_t, i.e. $p_t^r(s^t)=p^r(s_t)$.

4. Let $p_t^c(s^t)$ denote the price of the corporate bond.

$$p_t^c\left(s^t\right) = \sum_{s_{t+1}} Q_{t+1}\left(s_{t+1}|s^t\right) \min\left\{\Pi\left(s_{t+1}\right), 1\right\} = \beta \sum_{s_{t+1}} \pi\left(s_{t+1}|s_t\right) g\left(s_{t+1}\right)^{-\gamma} \min\left\{\Pi\left(s_{t+1}\right), 1\right\}.$$

Similarly the price of the corporate bond only depends on the current state s_t, i.e. $p_t^c(s^t) = p^c(s_t)$.

5. The price difference between the risk-less bond and corporate bond is

$$p^r(s) - p^c(s) = \beta \sum_{s'} \pi\left(s'|s\right) g\left(s'\right)^{-\gamma} \max\left\{1 - \Pi\left(s'\right), 0\right\} = \beta\pi\left(L|s\right) g\left(L\right)^{-\gamma} \left(1 - \Pi\left(L\right)\right),$$

which is determined by the level of loss upon default and the probability of default. Since

$$\pi\left(L|L\right) = 1 - q > \pi\left(L|H\right) = q,$$

the price difference is larger in bad times:

$$p^r(L) - p^c(L) > p^r(H) - p^c(H).$$

This is because the level of loss upon default is the same, while the probability of default will be higher if the current state is low, as the low state persists.

2.3 Heterogeneous beliefs

1. Take log of the aggregate endowment:

$$\log y_t(s^t) = \log y(s_0) + \sum_{k=1}^{t} \log g(s_k) = \sum_{k=1}^{t} \log g(s_k).$$

Every period s is drawn independently from distribution $\{\pi_0(s)\}_{s\in S}$, therefore $\log g(s_1)$, $\log g(s_2)$, $\cdots$ are iid. By the strong law of large number,

$$\frac{\log y_t(s^t)}{t} = \frac{\sum_{k=1}^{t} \log g(s_k)}{t} \xrightarrow{a.s.} \mathbb{E}\left[\log g(s)\right] > 0.$$

Thus $\lim_{t\to\infty} y_t(s^t) = \infty$, π_0-almost surely.

2. The derivatives of utility $u_j'(c) = c^{-\gamma_j}$ and $u_i''(c) = -\gamma_j c^{-\gamma_j - 1}$,

$$RRA_j = -\frac{cu''(c)}{u'(c)} = \gamma_j.$$

3. In a deterministic economy, households of type j's problem:

$$\begin{aligned} \max_{\{c_{jt}\}_{t=0}^{\infty}} \quad & \beta^t u(c_{jt}) \\ \text{s.t.} \qquad & c_{jt} + \frac{1}{R_t} b_{t+1} = y_{jt} + b_t,\ \forall t. \end{aligned}$$

The interest rate is

$$R_t = \frac{u'(c_{jt})}{\beta u'(c_{jt+1})} = \frac{1}{\beta}\left(\frac{c_{jt+1}}{c_{jt}}\right)^{\gamma_j}.$$

Plug the expression of interest rate into EIS:

$$EIS_j = \frac{d\log\left(c_{jt+1}/c_{jt}\right)}{d\log R_t} = \frac{d\log\left(c_{jt+1}/c_{jt}\right)}{\gamma_j d\log\left(c_{jt+1}/c_{jt}\right)} = \frac{1}{\gamma_j}.$$

That is, EIS is the inverse of RRA.

4. Households of type j's problem:

$$\begin{aligned} \max_{c_j} \quad & \sum_{t\geq 0, s^t\in S^t} \beta^t \pi_{jt}(s^t) u\left[c_{jt}(s^t)\right] \\ \text{s.t.} \quad & \sum_{t\geq 0, s^t\in S^t} q_{0t}(s^t)\left(c_{jt}(s^t) - y_{jt}(s^t)\right) = 0. \end{aligned}$$

The Lagrangian

$$\mathcal{L} = \sum_{t\geq 0, s^t\in S^t} \left\{\beta^t \pi_{jt}(s^t) u\left[c_{jt}(s^t)\right] + \mu_j q_{0t}(s^t)\left(y_{jt}(s^t) - c_{jt}(s^t)\right)\right\}.$$

Take FOC w.r.t. $c_{jt}(s^t)$:

$$\beta^t \pi_{jt}(s^t) c_{jt}(s^t)^{-\gamma_j} = \mu_j q_{0t}(s^t), \ \forall j, t, s^t.$$

Take the ratio of the FOC of agent 2 over the FOC of agent 1:

$$\frac{\pi_{2t}(s^t) c_{2t}(s^t)^{-\gamma_2}}{\pi_{1t}(s^t) c_{1t}(s^t)^{-\gamma_1}} = \frac{\mu_2}{\mu_1}.$$

Transform consumption into consumption shares:

$$\frac{\pi_{2t}(s^t) \hat{c}_{2t}(s^t)^{-\gamma_2}}{\pi_{1t}(s^t) \hat{c}_{1t}(s^t)^{-\gamma_1}} y_t(s^t)^{\gamma_1 - \gamma_2} = \frac{\mu_2}{\mu_1}.$$

Solve for $\hat{c}_{2t}(s^t)$ as a function of $\hat{c}_{1t}(s^t)$:

$$\hat{c}_{2t}(s^t) = \left[\frac{\mu_1}{\mu_2}\right]^{\frac{1}{\gamma_2}} \left[y_t(s^t)\right]^{\frac{\gamma_1}{\gamma_2} - 1} \left[\frac{\pi_{2t}(s^t)}{\pi_{1t}(s^t)}\right]^{\frac{1}{\gamma_2}} \hat{c}_{1t}(s^t)^{\frac{\gamma_1}{\gamma_2}}.$$

The market clearing condition:

$$\hat{c}_{1t}(s^t) + \hat{c}_{2t}(s^t) = \hat{c}_{1t}(s^t) + \left[\frac{\mu_1}{\mu_2}\right]^{\frac{1}{\gamma_2}} \left[y_t(s^t)\right]^{\frac{\gamma_1}{\gamma_2} - 1} \left[\frac{\pi_{2t}(s^t)}{\pi_{1t}(s^t)}\right]^{\frac{1}{\gamma_2}} \hat{c}_{1t}(s^t)^{\frac{\gamma_1}{\gamma_2}} = 1.$$

Therefore $\rho = \left[\frac{\mu_1}{\mu_2}\right]^{\frac{1}{\gamma_2}}$ exists and is a positive constant such that consumption share solves

$$\hat{c}_{1t}(s^t) + \rho \left[y_t(s^t)\right]^{\frac{\gamma_1}{\gamma_2} - 1} \left[\frac{\pi_{2t}(s^t)}{\pi_{1t}(s^t)}\right]^{\frac{1}{\gamma_2}} \hat{c}_{1t}(s^t)^{\frac{\gamma_1}{\gamma_2}} = 1. \tag{1}$$

5. When $\pi_1(s) = \pi_2(s)$, $\forall s \in S$, we have $\frac{\pi_{2t}(s^t)}{\pi_{1t}(s^t)} = 1$. Equation (1) can be simplified to

$$\hat{c}_{1t}(s^t) + \rho \left[y_t(s^t)\right]^{\frac{\gamma_1}{\gamma_2} - 1} \hat{c}_{1t}(s^t)^{\frac{\gamma_1}{\gamma_2}} = 1.$$

From part 1, we have shown that $\lim_{t \to \infty} y_t(s^t) = \infty$, π_0-almost surely. When $\gamma_1 > \gamma_2$, we would have $\lim_{t \to \infty} \hat{c}_{1t}(s^t) = 0$, π_0-almost surely.

Interpretation: the households with higher risk aversion γ_j would have lower EIS. They are less willing to substitute inter-temporally, so they frontload consumption. This result does not rely on there being uncertainty in the model. The same result would hold even if the aggregate endowment is deterministically growing.

6. When $\gamma_1 = \gamma_2$, equation (1) can be simplified to

$$\hat{c}_{1t}(s^t) + \rho \left[\frac{\pi_{2t}(s^t)}{\pi_{1t}(s^t)}\right]^{\frac{1}{\gamma_2}} \hat{c}_{1t}(s^t) = 1.$$

(a) Take log of $\frac{\pi_{2t}(s^t)}{\pi_{1t}(s^t)}$:

$$\log\left(\frac{\pi_{2t}(s^t)}{\pi_{1t}(s^t)}\right) = \sum_{k=1}^{t} \log\left(\frac{\pi_2(s_k)}{\pi_1(s_k)}\right).$$

For $\lim_{t\to\infty} \hat{c}_{1t}(s^t) = 0$, π_0-almost surely, we would need to have $\lim_{t\to\infty} \frac{\pi_{2t}(s^t)}{\pi_{1t}(s^t)} = \infty$, π_0-almost surely. Since $\log\left(\frac{\pi_2(s_1)}{\pi_1(s_1)}\right)$, $\log\left(\frac{\pi_2(s_2)}{\pi_1(s_2)}\right)$, $\cdots$ are iid, by the strong law of large number,

$$\frac{\log\left(\frac{\pi_{2t}(s^t)}{\pi_{1t}(s^t)}\right)}{t} = \frac{\sum_{k=1}^{t}\log\left(\frac{\pi_2(s_k)}{\pi_1(s_k)}\right)}{t} \xrightarrow{a.s.} \mathbb{E}\left[\log\left(\frac{\pi_2(s)}{\pi_1(s)}\right)\right].$$

When $\mathbb{E}\left[\log\left(\frac{\pi_2(s)}{\pi_1(s)}\right)\right] > 0$, $\lim_{t\to\infty} \frac{\pi_{2t}(s^t)}{\pi_{1t}(s^t)} = \infty$, π_0-almost surely.

$$\mathbb{E}\left[\log\left(\frac{\pi_2(s)}{\pi_1(s)}\right)\right] = \mathbb{E}\left[\log\left(\frac{\pi_0(s)}{\pi_1(s)}\right)\right] - \mathbb{E}\left[\log\left(\frac{\pi_0(s)}{\pi_2(s)}\right)\right] > 0.$$

It means that household 1's belief is further away from the truth than household 2. Their consumption share decreases to zero over time as their belief played against them.

(b) Households of type 1 believe that their asymptotic consumption share is 1. They believe that the true distribution $\pi_0 = \pi_1$, which implies that $\lim_{t\to\infty} \hat{c}_{1t}(s^t) = 1$, π_1-almost surely.

(c) Even when households have different beliefs about the distribution, it might still be that

$$\mathbb{E}\left[\log\left(\frac{\pi_2(s)}{\pi_1(s)}\right)\right] = 0.$$

The log of the ratio of subjective probabilities, $\log\left(\frac{\pi_{2t}(s^t)}{\pi_{1t}(s^t)}\right)$, follows a random walk, where the increment $\log\left(\frac{\pi_2(s)}{\pi_1(s)}\right)$ is a mean zero random variable, drawn iid each time. The ratio of subjective probabilities will fluctuate indefinitely between zero and infinity. This implies that the consumption share will fluctuate indefinitely between zero and one.

7. Reorganize equation (1):

$$\hat{c}_{1t}(s^t) + \rho\left[\left[y_t(s^t)\right]^{\gamma_1-\gamma_2} \frac{\pi_{2t}(s^t)}{\pi_{1t}(s^t)}\right]^{\frac{1}{\gamma_2}} \hat{c}_{1t}(s^t)^{\frac{\gamma_1}{\gamma_2}} = 1.$$

Take log of $[y_t(s^t)]^{\gamma_1-\gamma_2} \frac{\pi_{2t}(s^t)}{\pi_{1t}(s^t)}$:

$$\log\left(\left[y_t(s^t)\right]^{\gamma_1-\gamma_2} \frac{\pi_{2t}(s^t)}{\pi_{1t}(s^t)}\right) = \sum_{k=1}^{t}\left((\gamma_1-\gamma_2)\log g(s_k) + \frac{\pi_2(s_k)}{\pi_1(s_k)}\right).$$

The conditions to ensure $\lim_{t\to\infty}[y_t(s^t)]^{\gamma_1-\gamma_2} \frac{\pi_{2t}(s^t)}{\pi_{1t}(s^t)} = \infty$, π_0-almost surely is

$$(\gamma_1-\gamma_2)\,\mathbb{E}\left[\log g(s)\right] + \mathbb{E}\left[\log\left(\frac{\pi_2(s)}{\pi_1(s)}\right)\right] > 0.$$