Chapter 12

Diamond-Mortensen-Pissarides

In this chapter we study an elementary version of today's leading modeling framework for discussing equilibrium unemployment, which is based on the work of Diamond (1982) and Mortensen and Pissarides (1994). Compared to McCall (1970), this model is much less focused on studying workers' decisions to accept and reject job offers. Instead, it focuses on firms' decisions to create jobs in response to labor market conditions, and on whether the wage setting mechanism will give adequate incentives for firms to create these jobs.

12.1 Setup

Time is discrete and infinite, $t \in \{0, 1, 2, \cdots\}$. There is a measure one continuum of workers. We assume that each worker is risk-neutral, infinitely lived, with discount factor $\beta \in (0, 1)$. A worker can be in either one of two states: unemployed, which we denote by "U", or employed, which we denote by "E". When employed, the worker is paid the wage w, to be determined in equilibrium. When unemployed, he enjoys benefits of value $z > 0$. One can interpret z either as the value of leisure, or as the unemployment benefit.

Each firm can employ one worker to produce output $y > z$. To hire a worker, a firm must post a vacancy at a cost c per period. There is a large number of firms that can freely enter the market and create vacancies. After a job is filled, separation may occur with probability $\delta \in (0, 1)$, in which case the worker goes back to be unemployment in the following period and the firm has to post a new vacancy if it wants to hire a new worker.

Matching process. We assume that the meeting process between workers and firms imperfect. Every period, only a fraction of vacant firms and unemployed workers can meet. Moreover, the meeting process is bilateral, i.e., firms and worker will meet in pairs and not in a centralized labor market. Both are relevant and realistic features of the labor market. The first feature will imply a positive level of unemployment. The second feature means that the firm and the worker will bargain over the wage.

To specify the matching process between workers and firms, the literature follows either one of two approaches. The first approach insists on microfoundations. It is explicit about the process through which workers locate firms. A simple example would be to assume that workers draw one firm at random with positive probability or, as in the previous chapter, send multiple applications to randomly chosen firms. The second approach, which we will follow here, does not explicitly specify the meeting process but instead postulates that the aggregate number of meetings is given by some abstract function of the number of unemployed workers and the number of vacant firms. The premise is that it would be too complex to describe the heterogenous search processes of each individual workers and firms in the economy. Instead, in order to analyze aggregate labor market outcomes, it is sufficient to note that these individual search processes must add up, statistically, to some aggregate production function of matches between workers and firms. There is a caveat, of course: this aggregate production function of matches is the result of agents' optimal behavior, and so it is presumably not invariant to changes in policy parameters – see Lagos (2000) for a detailed argument.

Formally, we assume that, every period, the aggregate number of matches between workers and firms is given by the function $M(u, v)$ which takes as input the number of unemployed workers $u \in [0, 1]$ and the number of vacancies posted $v \geq 0$. We impose the following assumptions on the matching function $M(u, v)$.

- Imperfect matching: $M(u, v) \leq \min(u, v)$, with a strict inequality if $u > 0$ and $v > 0$.

This first assumption means that matching is imperfect, in the sense that the actual number of matches that occurs is less than the efficient number of matches, $\min(u, v)$. Note for future reference that this assumption implies that $M(0, v) = M(u, 0) = 0$.

- Increasing in (u, v).

This second assumption simply means that there are more matches when the market grows larger, either because there are more unemployed workers or because there are more vacancies.

- Constant returns to scale $M(\lambda u, \lambda v) = \lambda M(u, v)$, $\forall \lambda > 0$

This third assumption states that when the market doubles in size, then the number of matches also doubles in size. This rules out in particular "thick market externalities", i.e., the possibility that doubling the size of the market more than doubles the number of matches, as in the original work of Diamond (1982). There are various reasons why researchers have decided to abstract from such thick-market externalities by assuming constant returns to scale. First, it is well known since Diamond (1982) that these externalities can create multiple equilibria. While this is interesting, a commonly held view is that multiplicity is somewhat mechanical and ultimately distracting: it is not central to the question at hand. Putting aside these theoretical considerations, a more substantive justification for applied research is empirical: Petrongolo and Pissarides (2001) survey empirical studies of the relationship between job creation, $M(u, v)$, unemployment, u, and vacancies, v. and they argue that constant returns is a reasonable assumption.[1]

- Concave in (u, v)

Finally, the last assumption of concavity means that the matching process exhibit congestion. When the number of unemployed worker increases, the marginal increase in the number of matches decrease.

[1] In other markets, such as financial markets, there is a strong presumption that thick market externalities are important and should be modelled explicitely. See [add reference here, Pagano?]

Assume that all vacancies have, at each time, the same matching probability. Then, by the Law of Large Numbers, $v \times$ vacancy match probability $= M(u,v)$, so that

$$\text{vacancy match probability} = \text{job filling rate} = \frac{M(u,v)}{v} = M\left[1, \left(\frac{v}{u}\right)^{-1}\right] = q(\theta)$$

where used the fact that the matching function has constant returns to scale, and where $\theta \equiv \frac{v}{u}$ is the vacancy-to-unemployed workers in the market, what is commonly referred to as "labor market tightness". By our maintained assuptions that $M(u,v)$ is concave, it follows that $q(\theta)$ is decreasing in θ.[2] By our maintained assumption that $M(u,0) = 0$, it also follows that $\lim_{\theta\to\infty} q(\theta) = 0$.

Likewise, the matching probabilities for a unemployed worker is

$$\text{worker match probability} = \text{job finding rate} = \frac{M(u,v)}{u} = \frac{M(u,v)}{v}\frac{v}{u} = \theta q(\theta).$$

Using the same argument as above, one can show that $\theta q(\theta)$ increasing and concave in θ and that $\lim_{\theta\to 0} \theta q(\theta) = 0$.

12.2 Search-and-bargaining equilibrium

Unemployment dynamics. We first examine the equilibrium dynamics of unemployment, taking as given the market tightness, θ. Each period, a fraction δ of employed workers transition to the unemployed state, and a fraction $\theta q(\theta)$ of the unemployed workers transition to the employment state. Hence, the law of motion for the unemployment rate is

$$u(t+1) = [1 - \theta q(\theta)]\, u(t) + \delta\, [1 - u(t)]\,.$$

In steady state the number of hires is equal to the numer of separations:

$$\theta q(\theta)\, u = \delta\,(1-u)\,,$$

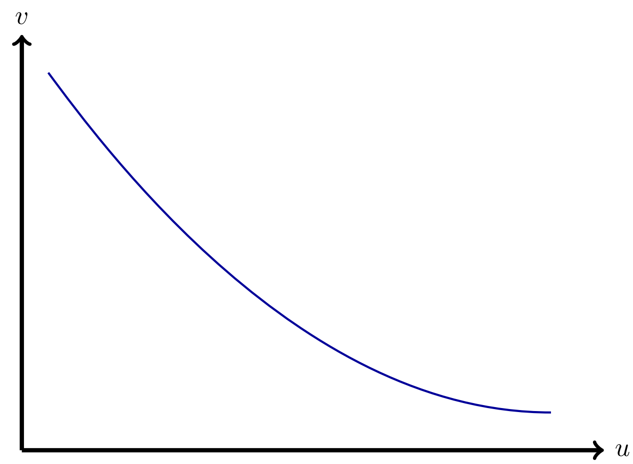
which implies that the steady state unemployment rate is:

$$u = \frac{\delta}{\theta q(\theta) + \delta}. \tag{12.2.1}$$

One sees that this function is increasing in δ and decreasing in θ. We have therefore derived a relationship that explains how the separation rate δ links together the unemployment rate and the number of vacancies, $M(u,v) + \delta u = \delta$. The inverse relationship between u and v is usually referred to by empiricists as the "Beveridge Curve". Figure 12.1 illustrates.

[2] Although $q(\theta)$ is often drawn as a convex function, it does not necessarily have this property. Check form example $M(u,v) = \lambda \min\{u,v\}$ for some $\lambda < 1$.

Figure 12.1: The Beveridge Curve



Value functions. We let V_U and V_E denote a worker value in the unemployment ("U") and employment ("E") states. Let Π_V the value for a firm of posting a vacancy in the current period, and posting optimally thereafter. Let Π_F denote the value of a firm with a filled job. Assuming that workers and firms always find it optimal to start a relationship upon matching, these values satisfy the following Bellman equations:

$$V_U = z + \beta \left[\theta q(\theta) V_E + (1 - \theta q(\theta)) V_U\right] \tag{12.2.2}$$

$$V_E = w + \beta \left[\delta V_U + (1 - \delta) V_E\right] \tag{12.2.3}$$

$$\Pi_V = -c + \beta \left[q(\theta) \Pi_F + (1 - q(\theta)) \Pi_V\right] \tag{12.2.4}$$

$$\Pi_F = y - w + \beta \left[\delta \Pi_V + (1 - \delta) \Pi_F\right], \tag{12.2.5}$$

For simplicity we assume here that $\Pi_V \geq 0$, i.e. firms always find it weakly optiomal to create a vacancy, something that will be true in the equilibrium we consider

Wages. To understand wage determination, let us imagine that a worker and a firm match and set some wage $\tilde{w}$. We use the "tilde" notation, $\tilde{w}$, to highlights that, in a bilateral match, a worker and a firm may consider setting any wage, not necessarily the wage w set by other workers and firms. Of course, in equilibrium, we will have $w = \tilde{w}$. The value to the worker and the firm of setting the wage $\tilde{w}$ for the course of their relationship is:

$$\begin{aligned} V_E(\tilde{w}) &= \tilde{w} + \beta \left[\delta V_U + (1 - \delta) V_E\right] \\ \Pi_F(\tilde{w}) &= y - \tilde{w} + \beta \left[\delta \Pi_V + (1 - \delta) \Pi_F\right] \end{aligned}$$

Hence, the net value to the worker and the firm of starting the relationship with wage $\tilde{w}$ are:

$$
\begin{aligned}
V_E(\tilde{w}) - V_U &= \frac{\tilde{w} - (1-\beta)V_U}{1-\beta(1-\delta)} \\
\Pi_F(\tilde{w}) - \Pi_V &= \frac{y - \tilde{w} - (1-\beta)\Pi_V}{1-\beta(1-\delta)}.
\end{aligned}
$$

The expressions are intuitive. Consider for example the net value to the worker, $V_E(\tilde{w}) - V_U$. Over the course of the relationship, the worker receives the wage $\tilde{w}$ every period but incurs an opportunity cost: he could have remained unemployed and continued to search for a job, which has a per-period (annuity) value of $(1-\beta)V_U$. The net utility is obtained by calculating the present value of $\tilde{w}-(1-\beta)V_U$, with the adjusted discount factor, $\beta(1-\delta)$, the time discount factor multiplied by the probability of continuing the relationship next period. Put differently, if we let τ denote the random time at which a newly created job is destroyed, then the present value factor is

$$
\frac{1}{1-\beta(1-\delta)} = \mathbb{E}\left[\sum_{t=0}^{\tau} \beta^t\right] = \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \mathbb{I}_{\tau \geq t}\right] = \sum_{t=0}^{\infty} \Pr\left[\tau \geq t\right] \beta^t = \sum_{t=0}^{\infty} \left[\beta(1-\delta)\right]^t = \frac{1}{1-\beta(1-\delta)},
$$

the discounted length of the relationship. The net value to the firm, $\Pi_F(\tilde{w}) - \Pi_F$, has a similar interpretation.

Clearly, the worker and the firm will find it optimal to start the relationship if it makes them better off than continuing their search: $V_E(\tilde{w}) - V_U \geq 0$ and $\Pi_F(\tilde{w}) - \Pi_V^+ \geq 0$. We have:

Lemma 12.2.1. *There exists some wage $\tilde{w}$ such that $V_E(\tilde{w}) - V_U \geq 0$ and $\Pi_F(\tilde{w}) - \Pi_V^+ \geq 0$ if and only if $\Sigma = V_E(\tilde{w}) - V_U + \Pi_F(\tilde{w}) - \Pi_V = \frac{y-(1-\beta)(V_U+\Pi_V)}{1-\beta(1-\delta)} \geq 0$.*

Proof. The "only if" part is obvious. For the "if" part, suppose that $\Sigma \geq 0$. Then, the wage $\tilde{w} = (1-\beta)V_U$ is such that $V_E(\tilde{w}) - V_U = 0$ and $\Pi_F(\tilde{w}) - \Pi_V = \Sigma \geq 0$. □

In the Lemma, Σ represents the surplus: the sum of the worker's and the firm's net utilities. Note that it does not depend on $\tilde{w}$, since the wage is just a transfer from the firm to the worker, and so it cancels out when the net utilities are added together. The Lemma states that the firm and the worker find it optimal to form a relationship if and only if the output per period, y, is greater than the annuitized outside value of search, $(1-\beta)(V_U + \Pi_V)$.

The formulas for $V_E(\tilde{w}) - V_U$ and $\Pi_F(\tilde{w}) - \Pi_V^+$ reveal that there are many wages that both parties would be willing to accept. Namely, the lowest wage that the worker would be willing to accept is the wage $\underline{w}$ such that $V_E(\underline{w}) - V_U = 0$ and $\Pi_F(\underline{w}) - \Pi_V = \Sigma$, that is, $\underline{w} = (1-\beta)V_U$. Likewise, the highest wage that a firm would be willing to pay is $\bar{w} = y - (1-\beta)\Pi_V$. All in all, we conclude that the firm and the worker may agree on any wage $w \in [\underline{w}, \bar{w}]$, what is commonly refered to as the bargaining set.

To choose among these many possible wages in the bargaining set, $[\underline{w}, \bar{w}]$, we need to specify a wage setting mechanism. It is common to assume that the firm and the worker bargain over the wage, so that the equilibrium wage solves the equation:

$$V_E(w) - V_U = \phi\left[V_E + \Pi_F - V_U - \Pi_V\right] = \phi\Sigma \tag{12.2.6}$$

where $\phi \in [0,1]$ represents the share of the surplus extracted by the worker, which is commonly interpreted as its bargaining power. Note that the equation implies that $\Pi_F(w) - \Pi_V^+ = (1-\phi)\Sigma$, that is, that the firm receives the complementary fraction $1-\phi$ of the surplus.

The game theoretic foundations of this formula go back to the axiomatic theory of bargaining due to Nash (1950), and the strategic theory of Rubinstein (1982). Other wage setting mechanisms can also be considered, for example assuming competitive search as in Section 12.4 below. One must keep in mind here that the choice of the wage setting mechanism far from innocuous, and can lead to very distorted market outcomes. This makes sense: from the Diamond paradox we studied in Section 11.2, we already know that, even if the market is fully decentralized, we can obtain monopoly outcomes by assuming that workers makes take-it-or-leave-it offers to firm. The recent literature has shown quantitatively that one can obtain very distorted outcomes in other cases too. For example, Hall (2005) argues that rational decision making in bilateral meeting only only constrains the wage to lie inside the barganing set, $w \in [\underline{w}, \bar{w}]$. He then goes on to show quantitatively that this constraint is consistent with wages remaining sticky over the business cycle.

Free entry. Since we assume that there is a large number of firm who can create jobs, it follows that firms will keep entering in the search market and create vacancies as long as the value of a vacancy is positive, until it is driven down to zero. Therefore, the free entry condition is:

$$\Pi_V = 0, \tag{12.2.7}$$

where we assume for simplicity that some vacancy are created in equilibrium.

Definition 12.2.1. A search-and-matching equilibrium consists of $(V_E, V_U, \Pi_F, \Pi_V, w, \theta, u)$ that solve the system of equations (12.2.1) and (12.2.2) to (12.2.7).

Notice that the system of equations is "recursive": we can first solve for values and market tightness, and then solve for distributions. Here, the distribution is admittedly very simple (it is summarized by just one number, u!) but it turns out that the "recursivity" property holds much more generally, and the recent literature has leveraged the property to forumate and solve much richer versions of the original model: for example, version of the model with rich aggregate and individual dynamics, as well as workers and firm heterogeneity (see for example Menzio and Shi, 2011).

We start by deriving an expression for the surplus, Σ, using the Bellman equations and the free entry condition, which implies that $\Pi_V^+ = 0$. We add (12.2.5) and (12.2.3), we subtract (12.2.2) and use $\Pi_V^+ = 0$. We obtain:

$$\Sigma = y - z + \beta(1-\delta)\Pi_F + \beta(1-\delta-\theta q(\theta))(V_E - V_U).$$

Combing with the surplus sharing equations (12.2.6):

$$\Sigma = y - z - \beta\left[1 - \delta - \theta q(\theta)\phi\right]\Sigma.$$

Rearranging:

$$\Sigma = \frac{y - z}{1 - \beta + \beta\left[\delta + \theta q(\theta)\phi\right]}.$$

Next we characterize the labor market tightness in an equilibrium. From Bellman equation (12.2.4) and using the suprlus sharing equation, we obtain that:

$$\Pi_V = -c + \beta q(\theta)(1 - \phi)\Sigma.$$

Combining this with the the expression of surplus, and with the free entry condition, we obtain an equation that characterizes θ:

$$c = (y - z)\frac{\beta q(\theta)(1 - \phi)}{1 - \beta\left[1 - \delta - \theta q(\theta)\phi\right]}. \tag{12.2.8}$$

Figure 12.2 illustrates how this equation pins down θ. When $\theta = 0$, the RHS equals

$$(y - z)\frac{\beta q(0)(1 - \phi)}{1 - \beta(1 - \delta)} = (1 - \phi)\beta\sum_{t=0}^{\infty}\left[\beta(1 - \delta)\right]^t(y - z),$$

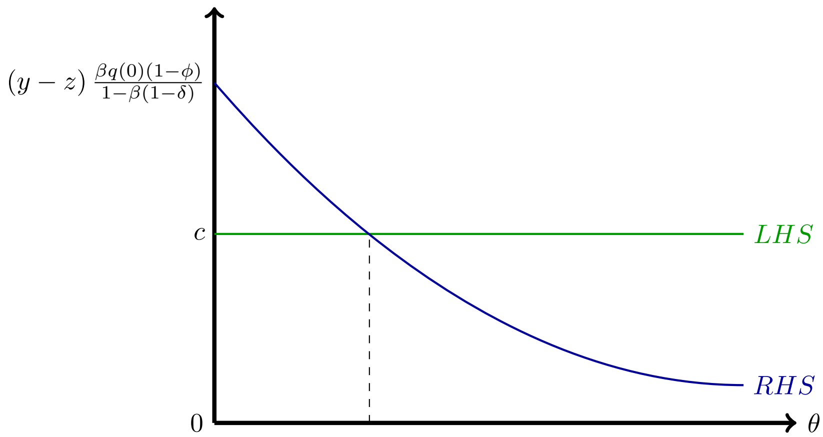
which is the value of a vacancy when it is filled with probability $q(0)$ next period. When θ goes to infinity, the RHS goes to zero, as a vacancy that will never be filled has no value. As long as $c < (y - z)\frac{\beta(1-\phi)}{1-\beta(1-\delta)}$, there is a unique equilibrium with entry. Otherwise, there is no equilibrium with entry.

The equilibrium value of θ is decreasing in (c, z, δ, ϕ) and increasing in (y, β). Notice that when the firm's share of surplus is very low ($\phi \to 1$), the equilibrium θ is zero. This because wages are set after the vacancy cost is sunk. If $\phi \simeq 1$, then $\Pi_F(w) \simeq \Pi_V = 0$, meaning that the firm expects that it will have to pay a wage that is approximately equal to y, the output per worker. Hence the firm does not enter because it expects that it won't the vacancy cost, which has to be incurred for at least one period. This is a version of the famous hold-up problem.

12.3 The planner's problem

Many macroeconomic policies seek to improve the functioning of labor market and reduce unemployment. But recall the quote of Stigler (1961) at the beginning of Chapter 11: it would certainly be suboptimal to fully eliminate the unemployment created by search friction. What is, then, the optimal level of unemployment? Should the government subsidize the creation of vacancy to reduce u? To answer this and other related questions, we conduct the following thought experiment: we imagine that a benevolent planner can decide how many vacancies firms should create, and whether workers and firm should form matches, but is otherwise constrained by *the same matching technology*

Figure 12.2: Equilibrium labor market tightness



as workers and firms. We ask: under what conditions does the planner's solution coincides with the search-and-bargaining equilibrium?

The planner solves for a constrained efficient allocation, i.e. an allocation that maximizes utilitarian welfare, subject to the constraints imposed by the search and matching technology:

$$\max_{\{N(t),V(t),t\geq 1\}} \sum_{t=0}^{\infty} \beta^t \left[N(t)y + (1-N(t))\,z - cV(t)\right]$$

subject to

$$N(t) \leq (1-\delta)\,N(t-1) + M\left[1-N(t-1), V(t-1)\right]$$
$$N(0) \text{ given.}$$

Here $N(t) = 1-u(t)$ denotes the number of filled jobs and $V(t)$ the numbers of vacancies created at time t. The planner faces the "job creation constraint". It says that the number of jobs that the planner can create in any time period must be less than the net number of non-destroyed job inherited from the previous period, plus the number of matches between unemployed workers and vacancies.

One may wonder why is utilitarian welfare the right criterion. It is the equally weighted sum of agents' utility. Why is equal weight the right thing to do? What if we want to give more weight to the unemployed? It turns out that, because workers are risk neutral, utilitarian welfare is the criterion that the planner will want to maximize to find any Pareto optimal allocation, even if it wants to put more weight on some group of workers. Namely, all Pareto allocations will be characterized by the same path of $N(t)$ and $v(t)$. They will differ only in terms of how aggregate output is distributed to agents.

To solve the above inter-temporal maximization problem, we set up the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[N(t)y + (1-N(t))\, z - cV(t)\right] + \sum_{t=1}^{\infty} \beta^t \lambda(t) \left[(1-\delta)\, N(t-1) + M\left(1-N(t-1), V(t-1)\right) - N(t)\right].$$

Note that we are dealing with a nice concave problem with a bounded state variable, so the following first order conditions will be sufficient:

$$\begin{array}{lcl} [V(t)] & : & c \geq \beta\lambda(t+1)\frac{\partial M}{\partial V}\left(1-N(t), V(t)\right) \text{ with " = " if } V(t) > 0 \\ [N(t)] & : & \lambda(t) = y - z + \beta\left(1-\delta\right)\lambda(t+1) - \beta\lambda(t+1)\frac{\partial M}{\partial N}\left(1-N(t), V(t)\right) \end{array} \tag{12.3.1}$$

together with a transversality condition for the multiplier

$$\lim_{t\to\infty} \beta^t \lambda(t) = 0. \tag{12.3.2}$$

The first-order condition with respect to $V(t)$ compares the cost of posting a vacancy with the discounted benefit. This discounted benefit is equal to the discount factor β, multiplied by the multiplier on the job creation constraint λ, multiplied by the marginal amount of matches created by increasing vacancies.

The first-order condition with respect to $N(t)$ gives us an equation for the multiplier on the job formation constraint, and looks very much like the Bellman equations we stated before. It represents the marginal social value of creating a new job. This marginal value equals to $y-z$, the output net of leisure, plus the expected discounted value of a new match next period (the expectation comes about because the new match may be destroyed with probability δ). There is a negative term, however. This negative term arises because, by creating a new match today, the planner reduces the size of the unemployment pool so removes some unemployed worker from the matching function. As a result, less match will be created next period. This is an opportunity cost for the planner, just as the option value of continuing search is an opportunity cost for a worker and a firm who contemplate forming a match.

Evaluating these conditions at steady state ($N(t) = N$, $V(t) = V$, $\lambda(t) = \lambda$, $\forall t$) and re-arranging (eliminating λ) we obtain:

$$c = (y-z)\,\frac{\beta\frac{\partial M}{\partial V}\left(1-N, V\right)}{1-\beta\left(1-\delta\right) + \beta\frac{\partial M}{\partial N}\left(1-N, V\right)}. \tag{12.3.3}$$

By comparing this equation with equation (12.2.8) that characterizes labor market tightness in the search equilibrium, it is easy to see that, when $\frac{\partial M}{\partial V} = q(\theta)(1-\phi)$ and $\frac{\partial M}{\partial N} = \theta q(\theta)\phi$, the two equations are identical. Equivalently, the following restrictions on the surplus sharing rule will lead to the search-and-bargaining equilibrium and the social optimum to coincide:

$$1-\phi = \frac{V}{M}\frac{\partial M}{\partial V}$$
$$\phi = \frac{u}{M}\frac{\partial M}{\partial N}.$$

This is called the "Hosios" condition, famously derived in Hosios (1990), and it says that firms should appropriate a share of the surplus that is equal to the elasticity of the matching function with respect to vacancies. This condition aligns the the firm's private value of creating a vacancy with its social value. The private value depends on the share of the surplus appropriated, $1-\phi$, and on a firm average contribution to the matching process, $\frac{M}{V}$. The social value, on the other hand, depends on the marginal contribution to the matching process, $\frac{\partial M}{\partial V}$. Note that there can be too little or too much vacancy creation in an equilibrium relative to the corresponding social optimum. Too little vacancy creation makes sense. If firms have very low share of the surplus, they have very little incentive to invest in vacancies: this is a symptom of a hold up problem. Investment is done ex-ante, but at the time of negotiation with a prospective hire the vacancy cost is sunk and not share with the worker via bargaining. Too many vacancies arise when the firm appropriates more than its marginal contribution to the matching process.

12.4 Competitive Search Equilibrium

The analysis of the planning problem shows that the equilibrium and the social optimum only coincide in a "knife-edge" case: when the bargaining weights of workers and firms happen to be equal to the elasticities of the matching function with respect to the measure of unemployed workers and the measure of vacancies. In what follows, we study a wage-setting mechanism, different from ex-post bargaining, that delivers efficiency. The equilibrium concept and efficiency result is originally due to Moen (1994). It is now used in a variety of models and plays a prominent role in both the theoretical and applied literature.

To intuitively understand how a competitive search equilibrium works and why it delivers efficiency, recall the results of the two previous sections. We showed that inefficiency arises because of a hold-up problem: firms have to incur the vacancy cost before negotiating a wage with workers. In a competitive search equilibrium, by contrast, wages are set before the search process starts. Namely, it is assumed that firms commit to wages at the same time they post vacancies, and that workers direct their search towards the wages posted by firms. Firms who post higher wage fill their vacancies faster because they attract more workers.

Firms. Let us start with the problem of a firm. If the firm chooses to post the wage w this period, it anticipates that workers will flow in the market until the vacancy-to-unemployment ratio, or market tightness, is equal to some $\Theta(w)$. The function $\Theta(w)$ represents the firm's rational expectations about the equilibrium relationship between posted wages to market tightness; it is defined for any wage w, and not only for the wage that will be posted in equilibrium. For now we take this function as given, but will determine it later in equilibrium.

Let $\Pi_V(w)$ denote the value of posting a wage w this period, and behaving optimally thereafter.

Let $\Pi_F(w)$ denote the value of a filled job at wage w. These values solve the Bellman equations:

$$\Pi_V(w) = -c + \beta\left[q\left(\Theta(w)\right)\Pi_F(w) + \left(1 - q\left(\Theta(w)\right)\right)\Pi_V^*\right] \tag{12.4.1}$$

$$\Pi_F(w) = y - w + \beta\left[(1-\delta)\,\Pi_F(w) + \delta\Pi_V^*\right], \tag{12.4.2}$$

where Π_V^* is the maximized value of a vacant firm. This maximized value appears because, in keeping with the Bellman principle, we can find an optimal wage-posting policy by studying the value of one-period deviations, whereby the firm posts a wage w this period but behaves optimally forever after. In the Bellman equation (12.4.1), a vacant firm posting wage w incurs the vacancy cost c. With probability $q\left(\Theta(w)\right)$, it meets an unemployed worker and become filled at wage w, with a continuation value $\Pi_F(w)$. With probability $1 - q\left(\Theta(w)\right)$ the vacancy is not filled, and the continuation value is Π_V^*, the maximized value of a vacant firm. From the firm's Bellman equations we solve for its value functions in terms of the parameters and the maximized value:

$$\Pi_F(w) = \frac{y - w + \beta\delta\Pi_V^*}{1-\beta(1-\delta)}.$$

$$\Pi_V(w) = -c + \beta q\left(\Theta(w)\right)\frac{y - w - (1-\beta)\Pi_V^*}{1-\beta(1-\delta)} + \beta\Pi_V^*.$$

Workers. Let us now turn to workers. Let $V_U(w,\theta)$ denote the value of going, for this period, to a market for jobs paying wage w, with tightness θ, and behaving optimally thereafter. Let $V_E(w)$ denote the value of being employed at wage w. The Bellman equations for the workers are:

$$V_U(w,\theta) = z + \beta\left[\theta q(\theta)\,V_E(w) + \left(1 - \theta q(\theta)\right)V_U^*\right] \tag{12.4.3}$$

$$V_E(w) = w + \beta\left[(1-\delta)\,V_E(w) + \delta V_U^*\right], \tag{12.4.4}$$

where V_U^* is the maximized value of an unemployed worker. As before, the maximized value V_U^* appears because we are calculating the value of a one-period deviation: this period the worker goes to the market for jobs paying wage w, but he behaves optimally forever after. Solving for the value functions in terms of the parameters and the maximized value:

$$V_E(w) = \frac{w + \beta\delta V_U^*}{1-\beta(1-\delta)}$$

$$V_U(w,\theta) = z + \beta\theta q(\theta)\frac{w - (1-\beta)V_U^*}{1-\beta(1-\delta)} + \beta V_U^*.$$

Tightness function. Recall that the tightness function, $\Theta(w)$, represents a firm's rational expectations about the labor-market tightness it would face if it posted the wage w. If we were studying a wage-posting game between a finite number of firms, these rational expectations would be pinned down by Subgame Perfection: that is, $\Theta(w)$ would be market tightness faced by the firm in the subgame where it posts some wage w, but other firms post the equilibrium wage (see, for example, Burdett, Shi, and Wright, 2001). Here, although we are not studying an explicit game, we can nevertheless capture the spirit of subgame perfection by imposing the following equilibrium conditions for the tightness function, $\Theta(w)$:

$$\Theta(w) = \begin{cases} 0 & \text{if } V_U(w,0) > V_U^* \\ \in [0,+\infty) & \text{if } V_U(w,\Theta(w)) = V_U^* \\ +\infty & \text{if } V_U(w,\infty) < V_U^*. \end{cases} \tag{12.4.5}$$

If the firm post wage w, then there are three possibilities. First, if the wage w is very high, or if the outside option V_U^* is very low, then the firm will attract all workers. In that case, the vacancy to unemployment ratio is $\Theta(w) = 0$, and a worker must find it strictly optimal to come to this market even when $\theta = 0$, that is, $V_U(w,0) > V_U^*$. Second, if the wage is in an intermediate range, then workers will flow in the market until they are indifferent between going to this market and receive the value $V_U(w,\Theta(w))$, or other markets, and receive the value V_U^*. That is, the market tightness $\Theta(w)$ must solve the equation $V_U(w,\Theta(w)) = V_U^*$. Third, if the wage is very low, then the firm will attract no worker, $\Theta(w) = \infty$. Workers must find it strictly sub-optimal to come to this market even when $\theta = \infty$, that is, $V_U(w,\infty) < V_U^*$.

We are now in a position to define an equilibrium:

Definition 12.4.1. A competitive search equilibrium consists of $(\Pi_V^*, \Pi_F^*, V_U^*, V_E^*, w^*, \theta^*, \Theta(w))$ such that

1. $\Pi_V^*, \Pi_F^*, V_U^*, V_E^*, w^*$ solve the Bellman equations (12.4.1) to (12.4.4) with $\theta^* = \Theta(w^*)$;
2. $\Theta(w)$ satisfies condition (12.4.5); and
3. $\Pi_V(w^*) = 0 \geq \Pi_V(w)$, $\forall w$.

What you see here is that one equation has changed relative to the case of bargaining: condition 3 now requires firms to choose wage optimally in face of labor market frictions.

Proposition 12.4.1. *The competitive search equilibrium is efficient.*

Proof. We solve for the equilibrium in which $v > 0$.

A firm chooses w to maximize $\Pi_V(w)$, taking as given the market tightness function $\Theta(w)$. Or equivalently, the firms chooses both w and θ to maximize profits, but subject to the condition that $\theta = \Theta(w)$, or equivalently that $V_U(w,\theta) = V_U^*$:

$$\max_{w,\theta} \Pi_V(w) = -c + \beta q(\theta) \frac{y - w - (1-\beta)\Pi_V^*}{1 - \beta(1-\delta)} + \beta \Pi_V^*.$$

subject to

$$V_U(w,\theta) = z + \beta\theta q(\theta) \frac{w - (1-\beta)V_U^*}{1 - \beta(1-\delta)} + \beta V_U^* = V_U^*.$$

It is as if each firm were choosing market tightness θ in its own submarket, subject to the constraint that the submarket delivers the right amount of utility, V_U^*, to unemployed workers. A firm does not literally choose θ of course. It is the mechanics of competitive search that leads to this result. Re-arranging the constraint:

$$\beta\theta q\left(\theta\right)\frac{w-(1-\beta)V_U^*}{1-\beta\left(1-\delta\right)}=(1-\beta)V_U^*-z\Longleftrightarrow q(\theta)w=\frac{1-\beta(1-\delta)}{\beta\theta}\left[(1-\beta)V_U^*-z\right]+q(\theta)(1-\beta)V_U^*.$$

Substitute into the objective:

$$\begin{aligned}\Pi_V(w)&=-c+\frac{\beta q\left(\theta\right)}{1-\beta\left(1-\delta\right)}\left[y-\left(1-\beta\right)\left(V_U^*+\Pi_V^*\right)\right]-\frac{1}{\theta}\left[(1-\beta)V_U^*-z\right]\\&=-c+\frac{\beta M(\frac{1}{\theta},1)}{1-\beta\left(1-\delta\right)}\left[y-\left(1-\beta\right)\left(V_U^*+\Pi_V^*\right)\right]-\frac{1}{\theta}\left[(1-\beta)V_U^*-z\right].\end{aligned}$$

The problem is concave in $\frac{1}{\theta}$ and therefore the first order condition is sufficient. Using $\Pi_V^*=0$, the first order condition is

$$0=\frac{\beta\frac{\partial M}{\partial u}}{1-\beta\left(1-\delta\right)}\left[y-(1-\beta)V_U^*\right]-\left[(1-\beta)V_U^*-z\right],$$

which can be re-written as

$$(1-\beta)V_U^*\left[1-\beta\left(1-\delta\right)+\beta\frac{\partial M}{\partial u}\right]=\beta\frac{\partial M}{\partial u}y+\left[1-\beta\left(1-\delta\right)\right]z.$$

Let ϕ solve $V_E-V_U=\phi\Sigma$, then solving equilibrium equations: (Liyan's note: this step to me is not so obvious, took me a while to verify)

$$(1-\beta)V_U^*=z+\beta\frac{M}{u}\phi\Sigma=z+\frac{\beta\frac{M}{u}\phi(y-z)}{1-\beta(1-\delta)+\frac{M}{u}\phi}=\frac{\beta\frac{M}{u}\phi y+\left[1-\beta(1-\delta)\right]z}{1-\beta(1-\delta)+\frac{M}{u}\phi}.$$

Combing with the first order condition, ϕ must satisfy the following condition:

$$\phi=\frac{u}{M}\frac{\partial M}{\partial u}.$$

In this last step of the proof we solve for the share of surplus appropriated by the worker in equilibrium. For this we note that any given pricing mechanism will be associated with some surplus sharing parameter ϕ. In the basic DMP model we leave this ϕ unspecified. Here we see that it arises from the optimization problem of firms. We solve for ϕ by noting that, given the unknown ϕ, we can solve for V_U exactly as in the DMP model. Then we compare our formula with that coming from the first order condition of the firm and "identify unknown coefficients". We find this way that the ϕ coming out of the competitive search model exactly coincides with the one that delivers the Hosios condition. Hence, competitive search delivers efficiency.

□