Chris Ackerman Econ 202B

Midterm ; W2021

$t \in \{0, 1, 2, \ldots\}$

s_0 fixed, S finite

$s^t = (s_0, s_1, \ldots, s_t)$

$i \in \{1, 2, \ldots, I\}$

utility $u_i(c)$, beliefs $\pi_{it}(s^t)$,

endowment $y_{it}(s^t)$; common $\beta \in (0,1)$

AD assets in zero net supply; also

long lived asset with dividend $d_t(s^t)$,

positive supply $\bar{k} > 0$.

We introduce short-selling constraints:

$$a_{it+1}(s^t, s) \geq 0$$

$$k_{it}(s^t) \geq 0$$

① Agent's problem

utility

$$\max_{\{c_t(s^t),\, a_{t+1}(s^t, s),\, k_{it}(s^t)\}} \sum_{t=0} \sum_{s^t \in S^t} \beta^t \pi_t(s^t)\, u_i(c_{it}(s^t))$$

$$\text{s.t.} \quad c_t(s^t) + \sum_{s_{t+1}} Q_{t+1}(s_{t+1} | s^t)\, a_{it+1}(s^t, s_{t+1}) + p_t(s^t) k_t(s^t)$$

$$\leq y_t(s^t) + a_t(s^t) + d_t(s^t) k_{t-1}(s^{t-1})$$

where $p_t(s^t)$ is the price to purchase the long-lived asset @ time t, history s^t

$$\lim_{T \to \infty} \sum_{s^{T+1}} \left[\prod_{t=0}^{T} Q_{t+1}(s_{t+1} | s^t)\, a_{T+1}(s^{T+1}) + p_t(s^t) k_t(s^t) \right] = 0$$

$$a_{t+1}(s^t, s) \geq 0 \quad ; \quad k_t(s^t) \geq 0$$

$$a_0(s_0),\ k_0(s_0) \quad \text{given}$$

② An equilibrium is an allocation

$\{c, a, k\} = \{c_t(s^t), a_t(s^t), k_t(s^t) : t \geq 0, s^t \in S^t\}$ and

a price system

$(Q, p) : \{(Q_{t+1}(s_{t+1}|s^t), p_t(s^t)) : t \geq 0, s^t \in S^t, s_{t+1} \in S\}$

such that

① given the price system the allocation solves the agents problem

② markets clear; $c_t(s^t) = y_t(s^t)$, $a_t(s^t) = 0$,

$k_t(s^t) = \bar{k} \quad \forall t \geq 0, \; s^t \in S^t$

③

$$\mathcal{L} = \sum_{t \geq 0} \sum_{s^t} \Big\{ \beta^t \pi_t(s^t) u_i(c_{it}(s^t)) +$$

$$\lambda_t(s^t) \Big(y_t(s^t) + a_t(s^t) + d_t(s^t) k_{t-1}(s^{t-1}) - c_t(s^t)$$
$$- \sum_{s_{t+1}} Q_t(s^t, s_{t+1}) a_{t+1}(s^t, s_{t+1}) - p_t(s^t) k_t(s^t) \Big) \Big\}$$

$$\lambda_t(s^t) \Big(y_t(s^t) + a_t(s^t) + d_t(s^t) k_{t-1}(s^{t-1}) - c_t(s^t)$$

$$- \sum_{s_{t+1}} Q_t(s^t, s_{t+1}) a_{t+1}(s^t, s_{t+1}) - p_t(s^t) k_t(s^t) \Big) = 0$$

FOC wrt consumption:

$$\beta^t \pi_{it}(s^t) u'(c_{it}(s^t)) = \lambda_{it}(s^t)$$

FOC wrt Arrow-Debreu securities

$$\lambda_{it}(s^t) Q_{t+1}(s_{t+1} | s^t) = \lambda_{it+1}(s^{t+1})$$

FOC wrt long-lived asset

$$\lambda_{it}(s^t) p_t(s^t) = \lambda_{it+1}(s^t, s_{t+1}) \left[p_{t+1}(s^{t+1}) + d_{t+1}(s^{t+1}) \right]$$

④ From the assumption that Arrow securities are in zero net supply,

$$\sum_i \sum_t \sum_{s^t} a_{it+1}(s^t, s) = 0$$

The no short-selling constraint implies that $a_{it+1}(s^t, s) \geq 0 \;\forall\, i, t, s^t, s.$

Thus,

$$\sum_i \sum_t \sum_{s^t} a_{it+1}(s^t, s) = 0 \iff$$

$$a_{it+1}(s^t, s) = 0 \;\forall\, i, t, s^t, s,$$

since this is the only way to clear the market for Arrow securities without allowing negative asset positions (short selling).

⑤ Let agent 1 achieve the maximum for this expectation and drop the i index

$$p_t(s^t) = \mathbb{E}\left[\beta \frac{u'(c_{t+1}(s^t,s))}{u'(c_t(s^t))}\left(d_{t+1}(s^t,s)+p_{t+1}(s^t,s)\right)\right]$$

Suppose that this equation doesn't hold,

$$p_t(s^t) < \mathbb{E}\left[\beta \frac{u'(c_{t+1}(s^t,s))}{u'(c_t(s^t))}\left(d_{t+1}(s^t,s)+p_{t+1}(s^t,s)\right)\right]$$

Agent 1 would want to buy ϵ shares of the long-lived asset and sell

$\epsilon\left[p_{t+1}(s^{t+1})+d_{t+1}(s^{t+1})\right]$ shares of the Arrow security for all s_{t+1}. This would increase agent 1's consumption in (t,s^t) by

$$\epsilon\left[\sum_{s_{t+1}} Q_{t+1}(s_{t+1}|s^t)\left(p_{t+1}(s^{t+1})+d_{t+1}(s^{t+1})\right)-p_t(s^t)\right]$$

This implies that agent 1 has infinite demand for the asset at (t,s^t) and markets don't clear.

At any price below this, agent 9 would have infinite demand and thus bid the price up. At any price above this, nobody would want to buy the asset, and since $\bar{k}>0$ the market would not clear.

(6) We have already shown that $a_{it+1}(s^t, s) = 0 \;\forall\, i, t, s^t, s$. Thus we can simplify the budget constraint for agent i to

$$c_{it}(s^t) + p_t(s^t) k_{it}(s^t) = y_{it}(s^t) + [p_t(s^t) + d_t(s^t)] k_{it-1}(s^{t-1})$$

$k_i \leq \bar{k} \;\forall i$

From (5), we know that $p_t(s^t)$ is a finite number and $d_t(s^t)$ is a finite number, let $p_t(s^t) < P \;\forall s^t$, and $d_t(s^t) < D \;\forall s^t$

$\bar{k} \to 0 \implies \forall \varepsilon > 0,\ \bar{k} < \varepsilon$. Substitute these inequalities into our budget constraint

$$c_{it}(s^t) + \underbrace{P\varepsilon}_{\to 0} = y_{it}(s^t) + \underbrace{(P+D)\varepsilon}_{\to 0}$$

$$\implies c_{it}(s^t) = y_{it}(s^t) \quad \forall i, t, s^t$$

(4) $d(L)=0$; $d(H)=1$

$\pi_1(H|H)=\frac{1}{2}+\omega$ $\quad\quad$ $\pi_2(H|L)=\frac{1}{2}+\omega$

$\pi_1(H|L)=\frac{1}{2}$ $\quad\quad$ $\pi_2(H|H)=\frac{1}{2}$

In state H, agent 1 is more optimistic about the prospects of the long lived asset; $\mathbb{E}_1(d|H)=\frac{1}{2}+\omega > \mathbb{E}_2(d|H)=\frac{1}{2}$

In state L, agent 2 is more optimistic about the long-lived asset. The reasoning is the same as before but now we take expectations conditioned on L.

⑧ $p_t(s) = \mathbb{E}\left[\beta \frac{u_i'(c_{i,t+1}(s^t,s))}{u_i'(c_{i,t}(s^t))}\left(d_{t+1}(s^t,s) + p_{t+1}(s^t,s)\right)\right.$

mark $\mathbb{E} = 1/2 + \omega$ for $s \in \{L, H\}$

$$p_t(s) = \left[\frac{\beta u_i'(\bar{y})}{u_i'(\bar{y})}\left(\left[\frac{1}{2}+\omega\right]\cdot 1 + p_{t+1}(s^t,s)\right)\right.$$

conjecture $p_t = s = \frac{\beta(1/2+\omega)}{1-\beta} \quad \forall s$

$\Downarrow$

$$\frac{\beta(1/2+\omega)}{1-\beta} = \beta\left(\frac{1}{2}+\omega + \frac{\beta(1/2+\omega)}{1-\beta}\right)$$

$$\frac{1/2+\omega}{1-\beta} = \frac{1}{2}+\omega + \frac{\beta(1/2+\omega)}{1-\beta}$$

$$1/2+\omega = (1/2+\omega)(1-\beta) + \beta(1/2+\omega)$$

$$1/2+\omega = 1/2+\omega - \beta(1/2+\omega) + \beta(1/2+\omega)$$

$$\underline{0 = 0}$$

(9) Agent 1 buys the asset in the high state and sells it in the low state. Agent 2 takes the opposite side of these trades.

(10)

$$p_i(s) = \frac{\beta u_i'(c_{it+1}(s^t,s))}{u_i'(c_{it}(s^t))}\left(d_{t+1}(s^t,s)\right)$$

This equation is the same as what we had earlier, but it removes the resale value of the asset, $p_{t+1}(s^t,s)$. Since we're removing something that is strictly positive from p $\forall i, s$, $p_i(s) < p$.

The asset pricing equation consists of a fundamental value component and an asset component. This quote says that investors are being speculative whenever they value an asset above its fundamental value, which occurs whenever the bubble component is strictly positive.

(11) In this case, agent 1 holds the asset in all states, since this allows agent 2 to "gamble". The more appropriate term here is insurance, where the more risk tolerant agent has more exposure to fluctuations in y and c.

Heterogeneous beliefs aren't generating this phenomenon; agents trade securities either due to heterogeneous beliefs <u>or</u> heterogeneous risk tolerances.