Chapter 8

The Overlapping Generations Model

In this section we will discuss the second major workhorse model of modern macroeconomics, the Overlapping Generations (OLG) model, due to Allais (1947), Samuelson (1958) and Diamond (1965). The structure of this section will be as follows: we will first present a basic pure exchange version of the OLG model, show how to analyze it and contrast its properties with those of a pure exchange economy with infinitely lived agents. The basic differences are that in the OLG model

- competitive equilibria may be Pareto suboptimal
- (outside) money may have positive value
- there may exist a continuum of equilibria

We will demonstrate these properties in detail via examples. We will then discuss the Ricardian Equivalence hypothesis (the notion that, given a stream of government spending the financing method of the government -taxes or budget deficits- does not influence macroeconomic aggregates) for both the infinitely lived agent model as well as the OLG model. Finally we will introduce production into the OLG model to discuss the notion of dynamic inefficiency. The first part of this section will be based on Kehoe (1989), Geanakoplos (1989), the second section on Barro (1974) and the third section on Diamond (1965). Other good sources of information include Blanchard and Fischer (1989), chapter 3, Sargent and Ljungquist, chapter 8 and Azariadis, chapter 11 and 12.

8.1 A Simple Pure Exchange Overlapping Generations Model

Let's start by repeating the infinitely lived agent model to which we will compare the OLG model. Suppose there are I individuals that live forever. There is one nonstorable consumption good in each period. Individuals order consumption allocations according to

$$u_i(c_i) = \sum_{t=0}^{\infty} \beta_i^t U(c_t^i)$$

Agents have deterministic endowment streams $e^i = \{e_t^i\}_{t=0}^{\infty}$. Trade takes place at period 0. The standard definition of an Arrow-Debreu equilibrium goes like this:

Definition 84 *A competitive equilibrium are prices* $\{p_t\}_{t=0}^{\infty}$ *and allocations* $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i\in I}$ *such that*

1. *Given* $\{p_t\}_{t=0}^{\infty}$, *for all* $i \in I$, $\{\hat{c}_t^i\}_{t=0}^{\infty}$ *solves* $\max_{c^i \geq 0} u_i(c_i)$ *subject to*

$$\sum_{t=0}^{\infty} p_t(c_t^i - e_t^i) \leq 0$$

2.

$$\sum_{i\in I} \hat{c}_t^i = \sum_{i\in I} e_t^i \text{ for all } t$$

What are the main shortcomings of this model that have lead to the development of the OLG model? The first criticism is that individuals apparently do not live forever, so that a model with finitely lived agents is needed. We will see later that we can give the infinitely lived agent model an interpretation in which individuals lived only for a finite number of periods, but, by having an altruistic bequest motive, act so as to maximize the utility of the entire dynasty, which in effect makes the planning horizon of the agent infinite. So infinite lives in itself are not as unsatisfactory as it may seem. But if people live forever, they don't undergo a life cycle with low-income youth, high income middle ages and retirement where labor income drops to zero. In the infinitely lived agent model every period is like the next (which makes it so useful since this stationarity renders dynamic programming techniques easily applicable). So in order to analyze issues like social security, the effect of taxes on retirement decisions, the distributive effects of taxes vs. government deficits, the effects of life-cycle saving on capital accumulation one needs a model in which agents experience a life cycle and in which people of different ages live at the same time in the economy. This is why the OLG model is a very useful tool for applied policy analysis. Because of its interesting (some say, pathological) theoretical properties, it is also an area of intense study among economic theorists.

8.1.1 Basic Setup of the Model

Let us describe the model formally now. Time is discrete, $t = 1, 2, 3, \ldots$ and the economy (but not its people) lives forever. In each period there is a single, nonstorable consumption good. In each time period a new generation (of measure 1) is born, which we index by its date of birth. People live for two periods and then die. By (e_t^t, e_{t+1}^t) we denote generation t's endowment of the consumption good in the first and second period of their live and by (c_t^t, c_{t+1}^t) we denote the consumption allocation of generation t. Hence in time t there are two generations alive, one old generation $t-1$ that has endowment e_t^{t-1} and consumption c_t^{t-1} and one young generation t that has endowment e_t^t and consumption c_t^t. In addition, in period 1 there is an initial old generation 0 that has endowment e_1^0 and consumes c_1^0. In some of our applications we will endow the initial generation with an amount of outside money[1] m. We will NOT assume $m \geq 0$. If $m \geq 0$, then m can be interpreted straightforwardly as fiat money, if $m < 0$ one should envision the initial old people having borrowed from some institution (which is, however, outside the model) and m is the amount to be repaid.

In the next Table 1 we demonstrate the demographic structure of the economy. Note that there are both an infinite number of periods as well as well as an infinite number of agents in this economy. This "double infinity" has been cited to be the major source of the theoretical peculiarities of the OLG model (prominently by Karl Shell).

[1]Money that is, on net, an asset of the private economy, is "outside money". This includes fiat currency issued by the government. In contrast, inside money (such as bank deposits) is both an asset as well as a liability of the private sector (in the case of deposits an asset of the deposit holder, a liability to the bank).

Table 1

		Time				
G		1	2	…	t	$t+1$
e	0	(c_1^0, e_1^0)				
n	1	(c_1^1, e_1^1)	(c_2^1, e_2^1)			
e	$\vdots$			$\ddots$		
r	$t-1$				(c_t^{t-1}, e_t^{t-1})	
a	t				(c_t^t, e_t^t)	(c_{t+1}^t, e_{t+1}^t)
t.	$t+1$					$(c_{t+1}^{t+1}, e_{t+1}^{t+1})$

Preferences of individuals are assumed to be representable by an additively separable utility function of the form

$$u_t(c) = U(c_t^t) + \beta U(c_{t+1}^t)$$

and the preferences of the initial old generation is representable by

$$u_0(c) = U(c_1^0)$$

We shall assume that U is strictly increasing, strictly concave and twice continuously differentiable. This completes the description of the economy. Note that we can easily represent this economy in our formal Arrow-Debreu language from Chapter 7 since it is a standard pure exchange economy with infinite number of agents and the peculiar preference and endowment structure $e_s^t = 0$ for all $s \neq t, t+1$ and $u_t(c)$ only depending on c_t^t, c_{t+1}^t. You should complete the formal representation as a useful homework exercise.

The following definitions are straightforward

Definition 85 *An allocation is a sequence $c_1^0, \{c_t^t, c_{t+1}^t\}_{t=1}^{\infty}$. An allocation is feasible if $c_t^{t-1}, c_t^t \geq 0$ for all $t \geq 1$ and*

$$c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t \text{ for all } t \geq 1$$

An allocation $c_1^0, \{(c_t^t, c_{t+1}^t)\}_{t=1}^{\infty}$ is Pareto optimal if it is feasible and if there is no other feasible allocation $\hat{c}_0^1, \{(\hat{c}_t^t, \hat{c}_{t+1}^t)\}_{t=1}^{\infty}$ such that

$$\begin{aligned} u_t(\hat{c}_t^t, \hat{c}_{t+1}^t) &\geq u_t(c_t^t, c_{t+1}^t) \text{ for all } t \geq 1 \\ u_0(\hat{c}_1^0) &\geq u_0(c_1^0) \end{aligned}$$

with strict inequality for at least one $t \geq 0$.

We now define an equilibrium for this economy in two different ways, depending on the market structure. Let p_t be the price of one unit of the consumption good at period t. In the presence of money (i.e. $m \neq 0$) we will take money to be the numeraire. This is important since we can only normalize the price of one commoditiy to 1, so with money no further normalizations are admissible. Of course, without money we are free to normalize the price of one other commodity. Keep this in mind for later. We now have the following

Definition 86 *Given m, an Arrow-Debreu equilibrium is an allocation $\hat{c}_1^0, \{(\hat{c}_t^t, \hat{c}_{t+1}^t)\}_{t=1}^{\infty}$ and prices $\{p_t\}_{t=1}^{\infty}$ such that*

1. *Given $\{p_t\}_{t=1}^{\infty}$, for each $t \geq 1$, $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ solves*

$$\max_{(c_t^t, c_{t+1}^t) \geq 0} u_t(c_t^t, c_{t+1}^t) \tag{8.1}$$

$$s.t.\ p_t c_t^t + p_{t+1} c_{t+1}^t \leq p_t e_t^t + p_{t+1} e_{t+1}^t \tag{8.2}$$

2. *Given p_1, $\hat{c}_1^0$ solves*

$$\max_{c_1^0} u_0(c_1^0)$$

$$s.t.\ p_1 c_1^0 \leq p_1 e_1^0 + m \tag{8.3}$$

3. *For all $t \geq 1$ (Resource Balance or goods market clearing)*

$$c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t \text{ for all } t \geq 1$$

As usual within the Arrow-Debreu framework, trading takes place in a hypothetical centralized market place at period 0 (even though the generations are not born yet).[2] There is an alternative definition of equilibrium that assumes sequential trading. Let r_{t+1} be the interest rate from period t to period $t+1$ and s_t^t be the savings of generation t from period t to period $t+1$. We will look at a slightly different form of assets in this section.

[2]When naming this definition after Arrow-Debreu I make reference to the market *structure* that is envisioned under this definition of equilibrium. Others, including Geanakoplos, refer to a particular *model* when talking about Arrow-Debreu, the standard general equilibrium model encountered in micro with finite number of simultaneously living agents. I hope this does not cause any confusion.

Previously we dealt with one-period IOU's that had price q_t in period t and paid out one unit of the consumption good in $t+1$ (so-called zero bonds). Now we consider assets that cost one unit of consumption in period t and deliver $1+r_{t+1}$ units tomorrow. Equilibria with these two different assets are obviously equivalent to each other, but the latter specification is easier to interpret if the asset at hand is fiat money.

We define a Sequential Markets (SM) equilibrium as follows:

Definition 87 *Given m, a sequential markets equilibrium is an allocation $\hat{c}_1^0, \{(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)\}_{t=1}^{\infty}$ and interest rates $\{r_t\}_{t=1}^{\infty}$ such that*

1. *Given $\{r_t\}_{t=1}^{\infty}$ for each $t \geq 1$, $(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)$ solves*

$$\max_{(c_t^t, c_{t+1}^t) \geq 0, s_t^t} u_t(c_t^t, c_{t+1}^t)$$
$$s.t. \;\; c_t^t + s_t^t \leq e_t^t \tag{8.4}$$
$$c_{t+1}^t \leq e_{t+1}^t + (1+r_{t+1})s_t^t \tag{8.5}$$

2. *Given r_1, $\hat{c}_1^0$ solves*

$$\max_{c_1^0} u_0(c_1^0)$$
$$s.t. \;\; c_1^0 \;\; \leq \;\; e_1^0 + (1+r_1)m$$

3. *For all $t \geq 1$ (Resource Balance or goods market clearing)*

$$\hat{c}_t^{t-1} + \hat{c}_t^t = e_t^{t-1} + e_t^t \textit{ for all } t \geq 1 \tag{8.6}$$

In this interpretation trade takes place sequentially in spot markets for consumption goods that open in each period. In addition there is an asset market through which individuals do their saving. Remember that when we wrote down the sequential formulation of equilibrium for an infinitely lived consumer model we had to add a shortsale constraint on borrowing (i.e. $s_t \geq -A$) in order to prevent Ponzi schemes, the continuous rolling over of higher and higher debt. This is not necessary in the OLG model as people live for a finite (two) number of periods (and we, as usual, assume perfect enforceability of contracts)

Given that the period utility function U is strictly increasing, the budget constraints (8.4) and (8.5) hold with equality. Take budget constraint (8.5) for generation t and (8.4) for generation $t+1$ and sum them up to obtain

$$c_{t+1}^{t} + c_{t+1}^{t+1} + s_{t+1}^{t+1} = e_{t+1}^{t} + e_{t+1}^{t+1} + (1+r_{t+1})s_t^t$$

Now use equation (8.6) to obtain

$$s_{t+1}^{t+1} = (1+r_{t+1})s_t^t$$

Doing the same manipulations for generation 0 and 1 gives

$$s_1^1 = (1+r_1)m$$

and hence, using repeated substitution one obtains

$$s_t^t = \Pi_{\tau=1}^{t}(1+r_\tau)m \tag{8.7}$$

This is the market clearing condition for the asset market: the amount of saving (in terms of the period t consumption good) has to equal the value of the outside supply of assets, $\Pi_{\tau=1}^{t}(1+r_\tau)m$. Strictly speaking one should include condition (8.7) in the definition of equilibrium. By Walras' law however, either the asset market or the good market equilibrium condition is redundant.

There is an obvious sense in which equilibria for the Arrow-Debreu economy (with trading at period 0) are equivalent to equilibria for the sequential markets economy. For $r_{t+1} > -1$ combine (8.4) and (8.5) into

$$c_t^t + \frac{c_{t+1}^t}{1+r_{t+1}} = e_t^t + \frac{e_{t+1}^t}{1+r_{t+1}}$$

Divide (8.2) by $p_t > 0$ to obtain

$$c_t^t + \frac{p_{t+1}}{p_t}c_{t+1}^t = e_t^t + \frac{p_{t+1}}{p_t}e_{t+1}^t$$

Furthermore divide (8.3) by $p_1 > 0$ to obtain

$$c_1^0 \leq e_1^0 + \frac{m}{p_1}$$

We then can straightforwardly prove the following proposition

Proposition 88 *Let allocation* $\hat{c}_1^0, \{(\hat{c}_t^t, \hat{c}_{t+1}^t)\}_{t=1}^{\infty}$ *and prices* $\{p_t\}_{t=1}^{\infty}$ *constitute an Arrow-Debreu equilibrium with* $p_t > 0$ *for all* $t \geq 1$*. Then there exists a corresponding sequential market equilibrium with allocations* $\tilde{c}_1^0, \{(\tilde{c}_t^t, \tilde{c}_{t+1}^t, \tilde{s}_t^t)\}_{t=1}^{\infty}$ *and interest rates* $\{r_t\}_{t=1}^{\infty}$*with*

$$\begin{aligned} \tilde{c}_t^{t-1} &= \hat{c}_t^{t-1} \text{ for all } t \geq 1 \\ \tilde{c}_t^t &= \hat{c}_t^t \text{ for all } t \geq 1 \end{aligned}$$

Furthermore, let allocation $\hat{c}_1^0, \{(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)\}_{t=1}^{\infty}$ *and interest rates* $\{r_t\}_{t=1}^{\infty}$ *constitute a sequential market equilibrium with* $r_t > -1$ *for all* $t \geq 0$*. Then there exists a corresponding Arrow-Debreu equilibrium with allocations* $\tilde{c}_1^0, \{(\tilde{c}_t^t, \tilde{c}_{t+1}^t)\}_{t=1}^{\infty}$ *and prices* $\{p_t\}_{t=1}^{\infty}$ *such that*

$$\begin{aligned} \tilde{c}_t^{t-1} &= \hat{c}_t^{t-1} \text{ for all } t \geq 1 \\ \tilde{c}_t^t &= \hat{c}_t^t \text{ for all } t \geq 1 \end{aligned}$$

Proof. The proof is similar to the infinite horizon counterpart. Given equilibrium Arrow-Debreu prices $\{p_t\}_{t=1}^{\infty}$ define interest rates as

$$\begin{aligned} 1 + r_{t+1} &= \frac{p_t}{p_{t+1}} \\ 1 + r_1 &= \frac{1}{p_1} \end{aligned}$$

and savings

$$\tilde{s}_t^t = e_t^t - \hat{c}_t^t$$

It is straightforward to verify that the allocations and prices so constructed constitute a sequential markets equilibrium.

Given equilibrium sequential markets interest rates $\{r_t\}_{t=1}^{\infty}$ define Arrow-Debreu prices by

$$\begin{aligned} p_1 &= \frac{1}{1 + r_1} \\ p_{t+1} &= \frac{p_t}{1 + r_{t+1}} \end{aligned}$$

Again it is straightforward to verify that the prices and allocations so constructed form an Arrow-Debreu equilibrium. ■

Note that the requirement on interest rates is weaker for the OLG version of this proposition than for the infinite horizon counterpart. This is due to the particular specification of the no-Ponzi condition used. A less stringent condition still ruling out Ponzi schemes would lead to a weaker condtion in the proposition for the infinite horizon economy also.

Also note that with this equivalence we have that

$$\Pi_{\tau=1}^{t}(1+r_\tau)m = \frac{m}{p_t}$$

so that the asset market clearing condition for the sequential markets economy can be written as

$$p_t s_t^t = m$$

i.e. the demand for assets (saving) equals the outside supply of assets, m. Note that the demanders of the assets are the currently young whereas the suppliers are the currently old people. From the equivalence we can also see that the return on the asset (to be interpreted as money) equals

$$\begin{aligned} 1+r_{t+1} &= \frac{p_t}{p_{t+1}} = \frac{1}{1+\pi_{t+1}} \\ (1+r_{t+1})(1+\pi_{t+1}) &= 1 \\ r_{t+1} &\approx -\pi_{t+1} \end{aligned}$$

where π_{t+1} is the inflation rate from period t to $t+1$. As it should be, the real return on money equals the negative of the inflation rate.

8.1.2 Analysis of the Model Using Offer Curves

Unless otherwise noted in this subsection we will focus on Arrow-Debreu equilibria. Gale (1973) developed a nice way of analyzing the equilibria of a two-period OLG economy graphically, using offer curves. First let us assume that the economy is stationary in that $e_t^t = w_1$ and $e_{t+1}^t = w_2$, i.e. the endowments are time invariant. For given $p_t, p_{t+1} > 0$ let by $c_t^t(p_t, p_{t+1})$ and $c_{t+1}^t(p_t, p_{t+1})$ denote the solution to maximizing (8.1) subject to (8.2) for all $t \geq 1$. Given our assumptions this solution is unique. Let the excess demand functions y and z be defined by

$$\begin{aligned} y(p_t, p_{t+1}) &= c_t^t(p_t, p_{t+1}) - e_t^t \\ &= c_t^t(p_t, p_{t+1}) - w_1 \\ z(p_t, p_{t+1}) &= c_{t+1}^t(p_t, p_{t+1}) - w_2 \end{aligned}$$

These two functions summarize, for given prices, all implications that consumer optimization has for equilibrium allocations. Note that from the Arrow-Debreu budget constraint it is obvious that y and z only depend on the ratio $\frac{p_{t+1}}{p_t}$, but not on p_t and p_{t+1} separately (this is nothing else than saying that the excess demand functions are homogeneous of degree zero in prices, as they should be). Varying $\frac{p_{t+1}}{p_t}$ between 0 and ∞ (not inclusive) one obtains a locus of optimal excess demands in (y, z) space, the so called offer curve. Let us denote this curve as

$$(y, f(y)) \tag{8.8}$$

where it is understood that f can be a correspondence, i.e. multi-valued. A point on the offer curve is an optimal excess demand function for *some* $\frac{p_{t+1}}{p_t} \in (0, \infty)$. Also note that since $c_t^t(p_t, p_{t+1}) \geq 0$ and $c_{t+1}^t(p_t, p_{t+1}) \geq 0$ the offer curve obviously satisfies $y(p_t, p_{t+1}) \geq -w_1$ and $z(p_t, p_{t+1}) \geq -w_2$. Furthermore, since the optimal choices obviously satisfy the budget constraint, i.e.

$$\begin{aligned} p_t y(p_t, p_{t+1}) + p_{t+1} z(p_t, p_{t+1}) &= 0 \\ \frac{z(p_t, p_{t+1})}{y(p_t, p_{t+1})} &= -\frac{p_t}{p_{t+1}} \end{aligned} \tag{8.9}$$

Equation (8.9) is an equation in the two unknowns (p_t, p_{t+1}) for a given $t \geq 1$. Obviously $(y, z) = (0, 0)$ is on the offer curve, as for appropriate prices (which we will determine later) no trade is the optimal trading strategy. Equation (8.9) is very useful in that for a given point on the offer curve $(y(p_t, p_{t+1}), z(p_t, p_{t+1}))$ in y-z space with $y(p_t, p_{t+1}) \neq 0$ we can immediately read off the price ratio at which these are the optimal demands. Draw a straight line through the point (y, z) and the origin; the slope of that line equals $-\frac{p_t}{p_{t+1}}$. One should also note that if $y(p_t, p_{t+1})$ is negative, then $z(p_t, p_{t+1})$ is positive and vice versa. Let's look at an example

Example 89 *Let $w_1 = \varepsilon$, $w_2 = 1 - \varepsilon$, with $\varepsilon > 0$. Also let $U(c) = \ln(c)$ and $\beta = 1$. Then the first order conditions imply*

$$p_t c_t^t = p_{t+1} c_{t+1}^t \tag{8.10}$$

Figure 8.1: Offer Curves in OLG Models

and the optimal consumption choices are

$$c_t^t(p_t, p_{t+1}) = \frac{1}{2}\left(\varepsilon + \frac{p_{t+1}}{p_t}(1-\varepsilon)\right) \tag{8.11}$$

$$c_{t+1}^t(p_t, p_{t+1}) = \frac{1}{2}\left(\frac{p_t}{p_{t+1}}\varepsilon + (1-\varepsilon)\right) \tag{8.12}$$

the excess demands are given by

$$y(p_t, p_{t+1}) = \frac{1}{2}\left(\frac{p_{t+1}}{p_t}(1-\varepsilon) - \varepsilon\right) \tag{8.13}$$

$$z(p_t, p_{t+1}) = \frac{1}{2}\left(\frac{p_t}{p_{t+1}}\varepsilon - (1-\varepsilon)\right) \tag{8.14}$$

Note that as $\frac{p_{t+1}}{p_t} \in (0, \infty)$ *varies,* y *varies between* $-\frac{\varepsilon}{2}$ *and* ∞ *and* z *varies between* $-\frac{(1-\varepsilon)}{2}$ *and* ∞*. Solving* z *as a function of* y *by eliminating* $\frac{p_{t+1}}{p_t}$ *yields*

$$z = \frac{\varepsilon(1-\varepsilon)}{4y + 2\varepsilon} - \frac{1-\varepsilon}{2} \quad \textit{for } y \in (-\frac{\varepsilon}{2}, \infty) \tag{8.15}$$

This is the offer curve $(y, z) = (y, f(y))$*. We draw the offer curve in Figure 8*

The discussion of the offer curve takes care of the first part of the equilibrium definition, namely optimality. It is straightforward to express goods market clearing in terms of excess demand functions as

$$y(p_t, p_{t+1}) + z(p_{t-1}, p_t) = 0 \tag{8.16}$$

Also note that for the initial old generation the excess demand function is given by

$$z_0(p_1, m) = \frac{m}{p_1}$$

so that the goods market equilibrium condition for the first period reads as

$$y(p_1, p_2) + z_0(p_1, m) = 0 \tag{8.17}$$

Graphically in (y, z)-space equations (8.16) and (8.17) are straight lines through the origin with slope -1. All points on this line are resource feasible. We therefore have the following procedure to find equilibria for this economy for a given initial endowment of money m of the initial old generation, using the offer curve (8.8) and the resource feasibility constraints (8.16) and (8.17).

1. Pick an initial price p_1 (note that this is NOT a normalization as in the infinitely lived agent model since the value of p_1 determines the real value of money $\frac{m}{p_1}$ the initial old generation is endowed with; we have already normailzed the price of money). Hence we know $z_0(p_1, m)$. From (8.17) this determines $y(p_1, p_2)$.

2. From the offer curve (8.8) we determine $z(p_1, p_2) \in f(y(p_1, p_2))$. Note that if f is a correspondence then there are multiple choices for z.

3. Once we know $z(p_1, p_2)$, from (8.16) we can find $y(p_2, p_3)$ and so forth. In this way we determine the entire equilibrium consumption allocation

$$\begin{aligned} c_1^0 &= z_0(p_1, m) + w_2 \\ c_t^t &= y(p_t, p_{t+1}) + w_1 \\ c_{t+1}^t &= z(p_t, p_{t+1}) + w_2 \end{aligned}$$

4. Equilibrium prices can then be found, given p_1 from equation (8.9). Any initial p_1 that induces, in such a way, sequences $c_1^0, \{(c_t^t, c_{t+1}^t), p_t\}_{t=1}^{\infty}$ such that the consumption sequence satisfies $c_t^{t-1}, c_t^t \geq 0$ is an equilibrium for given money stock. This already indicates the possibility of a lot of equilibria for this model, a fact that we will demonstrate below.

This algorithm can be demonstrated graphically using the offer curve diagram. We add the line representing goods market clearing, equation (8.16). In the (y, z)-plane this is a straight line through the origin with slope -1. This line intersects the offer curve at least once, namely at the origin. Unless we have the degenerate situation that the offer curve has slope -1 at the origin, there is (at least) one other intersection of the offer curve with the goods clearing line. These intersection will have special significance as they will represent stationary equilibria. As we will see, there is a load of other equilibria as well. We will first describe the graphical procedure in general and then look at some examples. See Figure 9.

Figure 8.2: Using Offer Curves in OLG Models

Given any m (for concreteness let $m > 0$) pick $p_1 > 0$. This determines $z_0 = \frac{m}{p_1} > 0$. Find this quantity on the z-axis, representing the excess demand of the initial old generation. From this point on the z-axis go horizontally to the goods market line, from there down to the y-axis. The point on the y-axis represents the excess demand function of generation 1 when young. From this point $y_1 = y(p_1, p_2)$ go vertically to the offer curve, then horizontally to the z-axis. The resulting point $z_1 = z(p_1, p_2)$ is the excess demand of generation 1 when old. Then back horizontally to the goods market clearing condition and down yields $y_2 = y(p_2, p_3)$, the excess demand for the second generation and so on. This way the entire equilibrium consumption allocation can be constructed. Equilibrium prices are easily found from equilibrium allocations with (8.9), given p_1. In such a way we construct an entire equilibrium graphically.

Let's now look at some example.

Example 90 *Reconsider the example with isoelastic utility above. We found the offer curve to be*

$$z = \frac{\varepsilon(1-\varepsilon)}{4y + 2\varepsilon} - \frac{1-\varepsilon}{2} \text{ for } y \in (-\frac{\varepsilon}{2}, \infty)$$

The goods market equilibrium condition is

$$y + z = 0$$

Now let's construct an equilibrium for the case $m = 0$, for zero supply of outside money. Following the procedure outlined above we first find the excess demand function for the initial old generation $z_0(m, p_1) = 0$ for all $p_1 > 0$. Then from goods market $y(p_1, p_2) = -z_0(m, p_1) = 0$. From the offer curve

$$\begin{aligned} z(p_1, p_2) &= \frac{\varepsilon(1-\varepsilon)}{4y(p_1, p_2) + 2\varepsilon} - \frac{1-\varepsilon}{2} \\ &= \frac{\varepsilon(1-\varepsilon)}{2\varepsilon} - \frac{1-\varepsilon}{2} \\ &= 0 \end{aligned}$$

and continuing we find $z(p_t, p_{t+1}) = y(p_t, p_{t+1}) = 0$ for all $t \geq 1$. This implies that the equilibrium allocation is $c_t^{t-1} = 1-\varepsilon, c_t^t = \varepsilon$. In this equilibrium every *consumer eats his endowment in each period and no trade between generations takes place. We call this equilibrium the autarkic equilibrium. Obviously we can't determine equilibrium prices from equation* (8.9)*. However, the first order conditions imply that*

$$\frac{p_{t+1}}{p_t} = \frac{c_t^t}{c_{t+1}^t} = \frac{\varepsilon}{1-\varepsilon}$$

For $m = 0$ *we can, without loss of generality, normalize the price of the first period consumption good* $p_1 = 1$*. Note again that only for* $m = 0$ *this normalization is innocuous, since it does not change the real value of the stock of outside money that the initial old generation is endowed with. With this normalization the sequence* $\{p_t\}_{t=1}^{\infty}$ *defined as*

$$p_t = \left(\frac{\varepsilon}{1-\varepsilon}\right)^{t-1}$$

together with the autarkic allocation form an (Arrow-Debreu)-equilibrium. Obviously any other price sequence $\{\bar{p}_t\}$ *with* $\bar{p}_t = \alpha p_t$ *for any* $\alpha > 1$*, is also an equilibrium price sequence supporting the autarkic allocation as equilibrium. This is not, however, what we mean by the possibility of a continuum of equilibria in OLG-model, but rather the usual feature of standard competitive equilibria that the equilibrium prices are only determined up to one normalization. In fact, for this example with* $m = 0$*, the autarkic equilibrium is the unique equilibrium for this economy.*[3] *This is easily seen. Since the initial old generation has no money, only its endowments* $1 - \varepsilon$*, there is no way for them to consume more than their endowments. Obviously they can always assure to consume at least their endowments by not trading, and that is what they do for any* $p_1 > 0$ *(obviously* $p_1 \leq 0$ *is not possible in equilibrium). But then from the resource constraint it follows that the first young generation must consume their endowments when young. Since they haven't saved anything, the best they can do when old is to consume their endowment again. But then the next young generation is forced to consume their endowments and so forth. Trade breaks down completely. For this allocation to be an equilibrium prices must be such that at these prices all generations*

[3]The fact that the autarkic is the only equilibrium is specific to pure exchange OLG-models with agents living for only two periods. Therefore Samuelson (1958) considered three-period lived agents for most of his analysis.

actually find it optimal not to trade, which yields the prices below.[4]

Note that in the picture the second intersection of the offer curve with the resource constraint (the first is at the origin) occurs in the forth orthant. This need not be the case. If the slope of the offer curve at the origin is less than one, we obtain the picture above, if the slope is bigger than one, then the second intersection occurs in the second orthant. Let us distinguish between these two cases more carefully. In general, the price ratio supporting the autarkic equilibrium satisfies

$$\frac{p_t}{p_{t+1}} = \frac{U'(e_t^t)}{\beta U'(e_{t+1}^t)} = \frac{U'(w_1)}{\beta U'(w_2)}$$

and this ratio represents the slope of the offer curve at the origin. With this in mind define the autarkic interest rate (remember our equivalence result from above) as

$$1 + \bar{r} = \frac{U'(w_1)}{\beta u'(w_2)}$$

Gale (1973) has invented the following terminology: when $\bar{r} < 0$ he calls this the Samueson case, whereas when $\bar{r} \geq 0$ he calls this the classical case.[5] As

[4] If you look at Sargent and Ljungquist (1999), Chapter 8, you will see that they claim to construct several equilibria for exactly this example. Note, however, that their equilibrium definition has as feasibility constraint

$$c_t^{t-1} + c_t^t \leq e_t^{t-1} + e_t^t$$

and all the equilibria apart from the autarkic one constructed above have the feature that for $t = 1$

$$c_1^0 + c_1^1 < e_1^0 + e_1^1$$

which violate feasibility in the way we have defined it. Personally I find the free disposal assumption not satisfactory; it makes, however, their life easier in some of the examples to follow, whereas in my discussion I need more handwaving. You'll see.

[5] More generally, the Samuelson case is defined by the condition that savings of the young generation be positive at an interest rate equal to the population growth rate n. So far we have assumed $n = 0$, so the Samuelson case requires saving to be positive at zero interest rate. We stated the condition as $\bar{r} < 0$. But if the interest rate at which the young don't save (the autarkic allocation) is smaller than zero, then at the higher interest rate of zero they will save a positive amount, so that we can define the Samuelson case as in the text, provided that savings are strictly increasing in the interest rate. This in turn requires the assumption that first and second period consumption are strict gross substitutes, so that the offer curve is not backward-bending. In the homework you will encounter an example in which this assumption is not satisfied.

it will turn out and will be demonstrated below autarkic equilibria are not Pareto optimal in the Samuelson case whereas they are in the classical case.

8.1.3 Inefficient Equilibria

The preceding example can also serve to demonstrate our first major feature of OLG economies that sets it apart from the standard infinitely lived consumer model with finite number of agents: competitive equilibria may be not be Pareto optimal. For economies like the one defined at the beginning of the section the two welfare theorems were proved and hence equilibria are Pareto optimal. Now let's see that the equilibrium constructed above for the OLG model may not be.

Note that in the economy above the aggregate endowment equals to 1 in each period. Also note that then the value of the aggregate endowment at the equilibrium prices, given by $\sum_{t=1}^{\infty} p_t$. Obviously, if $\varepsilon < 0.5$, then this sum converges and the value of the aggregate endowment is finite, whereas if $\varepsilon \geq 0.5$, then the value of the aggregate endowment is infinite. Whether the value of the aggregate endowment is infinite has profound implications for the welfare properties of the competitive equilibrium. In particular, using a similar argument as in the standard proof of the first welfare theorem you can show (and will do so in the homework) that if $\sum_{t=1}^{\infty} p_t < \infty$, then the competitive equilibrium allocation for this economy (and in general for any pure exchange OLG economy) is Pareto-efficient. If, however, the value of the aggregate endowment is infinite (at the equilibrium prices), then the competitive equilibrium MAY not be Pareto optimal. In our current example it turns out that if $\varepsilon > 0.5$, then the autarkic equilibrium is not Pareto efficient, whereas if $\varepsilon = 0.5$ it is. Since interest rates are defined as

$$r_{t+1} = \frac{p_t}{p_{t+1}} - 1$$

$\varepsilon \dot{<} 0.5$ implies $r_{t+1} = \frac{1-\varepsilon}{\varepsilon} - 1 = \frac{1}{\varepsilon} - 2$. Hence $\varepsilon < 0.5$ implies $r_{t+1} > 0$ (the classical case) and $\varepsilon \geq 0.5$ implies $r_{t+1} < 0$. (the Samuelson case). Inefficiency is therefore associated with low (negative interest rates). In fact, Balasko and Shell (1980) show that the autarkic equilibrium is Pareto optimal if and only if

$$\sum_{t=1}^{\infty} \prod_{\tau=1}^{t} (1 + r_{\tau+1}) = +\infty$$

where $\{r_{t+1}\}$ is the sequence of autarkic equilibrium interest rates.[6] Obviously the above equation is satisfied if and only if $\varepsilon \leq 0.5$.

Let us briefly demonstrate the first claim (a more careful discussion is left for the homework). To show that for $\varepsilon > 0.5$ the autarkic allocation (which is the unique equilibrium allocation) is not Pareto optimal it is sufficient to find another feasible allocation that Pareto-dominates it. Let's do this graphically in Figure 10. The autarkic allocation is represented by the origin (excess demand functions equal zero). Consider an alternative allocation represented by the intersection of the offer curve and the resource constraint. We want to argue that this point Pareto dominates the autarkic allocation. First consider an arbitrary generation $t \geq 1$. Note that the indifference curve through the origin must lie to the outside of the offer curve (they are equal at the origin, but everywhere else the indifference curve lies below). Why: the autarkic point can be chosen at all price ratios. Thus a point on the offer

[6] Rather than a formal proof (which is quite involved), let's develop some intuition for why low interest rates are associated with inefficiency. Take the autarkic allocation and try to construct a Pareto improvement. In particular, give additional $\delta_0 > 0$ units of consumption to the initial old generation. This obviously improves this generation's life. From resource feasibilty this requires taking away δ_0 from generation 1 in their first period of life. To make them not worse of they have to recieve δ_1 in additional consumption in their second period of life, with δ_1 satisfying

$$\delta_0 U'(e_1^1) = \delta_1 \beta U'(e_2^1)$$

or

$$\begin{aligned} \delta_1 &= \delta_0 \frac{U'(e_1^1)}{\beta U'(e_2^1)} \\ &= \delta_0 (1 + r_2) > 0 \end{aligned}$$

and in general

$$\delta_t = \delta_0 \prod_{\tau=1}^{t} (1 + r_{\tau+1})$$

are the required transfers in the second period of generation t's life to compensate for the reduction of first period consumption. Obviously such a scheme does not work if the economy ends at fine time T since the last generation (that lives only through youth) is worse off. But as our economy extends forever, such an intergenerational transfer scheme is feasible provided that the δ_t don't grow too fast, i.e. if interest rates are sufficiently small. But if such a transfer scheme is feasible, then we found a Pareto improvement over the original autarkic allocation, and hence the autarkic equilibrium allocation is not Pareto efficient.

Figure 8.3: Pareto Optimality in OLG Models

curve was chosen when the autarkic allocation was affordable, and therefore must represent a higher utility. This demonstrates that the alternative point marked in the figure (which is both on the offer curve as well as the resource constraint, the line with slope -1) is at least as good as the autarkic allocation for all generations $t \geq 1$. What about the initial old generation? In the autarkic allocation it has $c_1^0 = 1 - \varepsilon$, or $z_0 = 0$. In the new allocation it has $z_0 > 0$ as shown in the figure, so the initial old generation is strictly better off in this new allocation. Hence the alternative allocation Pareto-dominates the autarkic equilibrium allocation, which shows that this allocation is not Pareto-optimal. In the homework you are asked to make this argument rigorous by actually computing the alternative allocation and then arguing that it Pareto-dominates the autarkic equilibrium.

What in our graphical argument hinges on the assumption that $\varepsilon > 0.5$. Remember that for $\varepsilon \leq 0.5$ we have said that the autarkic allocation is actually Pareto optimal. It turns out that for $\varepsilon < 0.5$, the intersection of the resource constraint and the offer curve lies in the fourth orthant instead of in the second as in Figure 10. It is still the case that every generation $t \geq 1$ at least weakly prefers the alternative to the autarkic allocation. Now, however, this alternative allocation has $z_0 < 0$, which makes the initial old generation worse off than in the autarkic allocation, so that the argument does not work. Finally, for $\varepsilon = 0.5$ we have the degenerate situation that the slope of the offer curve at the origin is -1, so that the offer curve is tangent to the resource line and there is no second intersection. Again the argument does not work and we can't argue that the autarkic allocation is not Pareto optimal. It is an interesting optional exercise to show that for $\varepsilon = 0.5$ the autarkic allocation is Pareto optimal.

Now we want to demonstrate the second and third feature of OLG models that set it apart from standard Arrow-Debreu economies, namely the possibility of a continuum of equilibria and the fact that outside money may have positive value. We will see that, given the way we have defined our equilibria, these two issues are intimately linked. So now let us suppose that $m \neq 0$. In our discussion we will assume that $m > 0$, the situation for $m < 0$ is symmetric. We first want to argue that for $m > 0$ the economy has a continuum of equilibria, not of the trivial sort that only prices differ by a constant, but that allocations differ across equilibria. Let us first look at equilibria that are stationary in the following sense:

Definition 91 *An equilibrium is stationary if $c_t^{t-1} = c^o$, $c_t^t = c^y$ and $\frac{p_{t+1}}{p_t} = a$, where a is a constant.*

Given that we made the assumption that each generation has the same endowment structure a stationary equilibrium necessarily has to satisfy $y(p_t, p_{t+1}) = y$, $z_0(m, p_1) = z(p_t, p_{t+1}) = z$ for all $t \geq 1$. From our offer curve diagram the only candidates are the autarkic equilibrium (the origin) and any other allocations represented by intersections of the offer curve and the resource line. We will discuss the possibility of an autarkic equilibrium with money later. With respect to other stationary equilibria, they all have to have prices $\frac{p_{t+1}}{p_t} = 1$, with p_1 such that $(\frac{m}{p_1}, -\frac{m}{p_1})$ is on the offer curve. For our previous example, for any $m \neq 0$ we find the stationary equilibrium by solving for the intersection of offer curve and resource line

$$
\begin{aligned}
y + z &= 0 \\
z &= \frac{\varepsilon(1-\varepsilon)}{4y + 2\varepsilon} - \frac{1-\varepsilon}{2}
\end{aligned}
$$

This yields a second order polynomial in y

$$-y = \frac{\varepsilon(1-\varepsilon)}{4y + 2\varepsilon} - \frac{1-\varepsilon}{2}$$

whose one solution is $y = 0$ (the autarkic allocation) and the other solution is $y = \frac{1}{2} - \varepsilon$, so that $z = -\frac{1}{2} + \varepsilon$. Hence the corresponding consumption allocation has

$$c_t^{t-1} = c_t^t = \frac{1}{2} \text{ for all } t \geq 1$$

In order for this to be an equilibrium we need

$$\frac{1}{2} = c_1^0 = (1-\varepsilon) + \frac{m}{p_1}$$

hence $p_1 = \frac{m}{\varepsilon - 0.5} > 0$. Therefore a stationary equilibrium (apart from autarky) only exists for $m > 0$ and $\varepsilon > 0.5$ or $m < 0$ and $\varepsilon < 0.5$. Also note that the choice of p_1 is not a matter of normalization: any multiple of p_1 will not yield a stationary equilibrium. The equilibrium prices supporting the stationary allocation have $p_t = p_1$ for all $t \geq 1$. Finally note that this equilibrium, since it features $\frac{p_{t+1}}{p_t} = 1$, has an inflation rate of $\pi_{t+1} = -r_{t+1} = 0$. It is exactly this equilibrium allocation that we used to prove that, for $\varepsilon > 0.5$, the autarkic equilibrium is not Pareto-efficient.

How about the autarkic allocation? Obviously it is stationary as $c_t^{t-1} = 1-\varepsilon$ and $c_t^t = \varepsilon$ for all $t \geq 1$. But can it be made into an equilibrium if $m \neq 0$. If we look at the sequential markets equilibrium definition there is no problem: the budget constraint of the initial old generation reads

$$c_1^0 = 1 - \varepsilon + (1 + r_1)m$$

So we need $r_1 = -1$. For all other generations the same arguments as without money apply and the interest sequence satisfying $r_1 = -1$, $r_{t+1} = \frac{1-\varepsilon}{\varepsilon} - 1$ for all $t \geq 1$, together with the autarkic allocation forms a sequential market equilibrium. In this equilibrium the stock of outside money, m, is not valued: the initial old don't get any goods in exchange for it and future generations are not willing to ever exchange goods for money, which results in the autarkic, no-trade situation. To make autarky an Arrow-Debreu equilibrium is a bit more problematic. Again from the budget constraint of the initial old we find

$$c_1^0 = 1 - \varepsilon + \frac{m}{p_1}$$

which, for autarky to be an equilibrium requires $p_1 = \infty$, i.e. the price level is so high in the first period that the stock of money de facto has no value. Since for all other periods we need $\frac{p_{t+1}}{p_t} = \frac{\varepsilon}{1-\varepsilon}$ to support the autarkic allocation, we have the obscure requirement that we need price *levels* to be infinite with well-defined finite price *ratios.* This is unsatisfactory, but there is no way around it unless we a) change the equilibrium definition (see Sargent and Ljungquist) or b) let the economy extend from the infinite past to the infinite future (instead of starting with an initial old generation, see Geanakoplos) or c) treat money somewhat as a residual, as something almost endogenous (see Kehoe) or d) make some consumption good rather than money the numeraire (with nonmonetary equilibria corresponding to situations in which money has a price of zero in terms of real consumption goods). For now we will accept autarky as an equilibrium even with money and we will treat it as identical to the autarkic equilibrium without money (because indeed in the sequential markets formulation only r_1 changes and in the Arrow Debreu formulation only p_1 changes, although in an unsatisfactory fashion).

8.1.4 Positive Valuation of Outside Money

In our construction of the nonautarkic stationary equilibrium we have already demonstrated our second main result of OLG models: outside money may have positive value. In that equilibrium the initial old had endowment $1-\varepsilon$ but consumed $c_1^0 = \frac{1}{2}$. If $\varepsilon > \frac{1}{2}$, then the stock of outside money, m, is valued in equilibrium in that the old guys can exchange m pieces of intrinsically worthless paper for $\frac{m}{p_1} > 0$ units of period 1 consumption goods.[7] The currently young generation accepts to transfer some of their endowment to the old people for pieces of paper because they expect (correctly so, in equilibrium) to exchange these pieces of paper against consumption goods when they are old, and hence to achieve an intertemporal allocation of consumption goods that dominates the autarkic allocation. Without the outside asset, again, this economy can do nothing else but remain in the possibly dismal state of autarky (imagine $\varepsilon = 1$ and log-utility). This is why the social contrivance of money is so useful in this economy. As we will see later, other institutions (for example a pay-as-you-go social security system) may achieve the same as money.

Before we demonstrate that, apart from stationary equilibria (two in the example, usually at least only a finite number) there may be a continuum of other, nonstationary equilibria we take a little digression to show for the general infinitely lived agent endowment economies set out at the beginning of this section money cannot have positive value in equilibrium.

Proposition 92 *In pure exchange economies with a finite number of infinitely lived agents there cannot be an equilibrium in which outside money is valued.*

Proof. Suppose, to the contrary, that there is an equilibrium $\{(\hat{c}_t^i)_{i\in I}\}_{t=1}^{\infty}, \{\hat{p}_t\}_{t=1}^{\infty}$ for initial endowments of outside money $(m^i)_{i\in I}$ such that $\sum_{i\in I} m^i \neq 0$. Given the assumption of local nonsatiation each consumer in equilibrium satisfies the Arrow-Debreu budget constraint with equality

$$\sum_{t=1}^{\infty} \hat{p}_t \hat{c}_t^i = \sum_{t=1} \hat{p}_t e_t^i + m^i < \infty$$

[7] In finance lingo money in this equilibrium is a “bubble”. The fundamental value of an assets is the value of its dividends, evaluated at the equilibrium Arrow-Debreu prices. An asset is (or has) a bubble if its price does not equal its fundamental value. Obviuosly, since money doesn’t pay dividends, its fundamental value is zero and the fact that it is valued positively in equilibrium makes it a bubble.

Summing over all individuals $i \in I$ yields

$$\sum_{t=1}^{\infty} \hat{p}_t \sum_{i\in I} \left(\hat{c}_t^i - e_t^i\right) = \sum_{i\in I} m^i$$

But resource feasibility requires $\sum_{i\in I} \left(\hat{c}_t^i - e_t^i\right) = 0$ for all $t \geq 1$ and hence

$$\sum_{i\in I} m^i = 0$$

a contradiction. This shows that there cannot exist an equilibrium in this type of economy in which outside money is valued in equilibrium. Note that this result applies to a much wider class of standard Arrow-Debreu economies than just the pure exchange economies considered in this section. ■

Hence we have established the second major difference between the standard Arrow-Debreu general equilibrium model and the OLG model.

Continuum of Equilibria

We will now go ahead and demonstrate the third major difference, the possibility of a whole continuum of equilibria in OLG models. We will restrict ourselves to the specific example. Again suppose $m > 0$ and $\varepsilon > 0.5$.[8] For any p_1 such that $\frac{m}{p_1} < \varepsilon - \frac{1}{2} > 0$ we can construct an equilibrium using our geometric method before. From the picture it is clear that all these equilibria have the feature that the equilibrium allocations over time converge to the autarkic allocation, with $z_0 > z_1 > z_2 > \ldots z_t > 0$ and $\lim_{t\to\infty} z_t = 0$ and $0 > y_t > \ldots y_2 > y_1$ with $\lim_{t\to\infty} y_t = 0$. We also see from the figure that, since the offer curve lies below the -45^0-line for the part we are concerned with that $\frac{p_1}{p_2} < 1$ and $\frac{p_t}{p_{t+1}} < \frac{p_{t-1}}{p_t} < \ldots < \frac{p_1}{p_2} < 1$, implying that prices are increasing with $\lim_{t\to\infty} p_t = \infty$. Hence all the nonstationary equilibria feature inflation, although the inflation rate is bounded above by $\pi_\infty = -r_\infty = 1 - \frac{1-\varepsilon}{\varepsilon} = 2 - \frac{1}{\varepsilon} > 0$. The real value of money, however, declines to zero in the limit.[9] Note that, although all nonstationary equilibria so constructed in the limit converge to the same allocation (autarky), they differ

[8] You should verify that if $\varepsilon \leq 0.5$, then $\bar{r} \geq 0$ and the only equilibrium with $m > 0$ is the autarkic equilibrium in which money has no value. All other possible equilibrium paths eventually violate nonnegativity of consumption.

[9] But only in the limit. It is crucial that the real value of money is not zero at finite t, since with perfect foresight as in this model generation t would anticipate the fact that money would lose all its value, would not accept it from generation $t-1$ and all monetary equilibria would unravel, with only the autarkic euqilibrium surviving.

in the sense that at any finite t, the consumption allocations and price ratios (and levels) differ across equilibria. Hence there is an entire continuum of equilibria, indexed by $p_1 \in (\frac{m}{\varepsilon-0.5}, \infty)$. These equilibria are arbitrarily close to each other. This is again in stark contrast to standard Arrow-Debreu economies where, generically, the set of equilibria is finite and all equilibria are locally unique.[10] For details consult Debreu (1970) and the references therein.

Note that, if we are in the *Samuelson case* $\bar{r} < 0$, then (and only then) all these equilibria are Pareto-ranked.[11] Let the equilibria be indexed by p_1. One can show, by similar arguments that demonstrated that the autarkic equilibrium is not Pareto optimal, that these equilibria are Pareto-ranked: let $p_1, \hat{p}_1 \in (\frac{m}{\varepsilon-0.5}, \infty)$ with $p_1 > \hat{p}_1$, then the equilibrium corresponding to $\hat{p}_1$ Pareto-dominates the equilibrium indexed by p_1. By the same token, the *only* Pareto optimal equilibrium allocation is the nonautarkic stationary monetary equilibrium.

8.1.5 Productive Outside Assets

We have seen that with a positive supply of an outside asset with no intrinsic value, $m > 0$, then in the Samuelson case (for which the slope of the offer curve is smaller than one at the autarkic allocation) we have a continuum of equilibria. Now suppose that, instead of being endowed with intrinsically useless pieces of paper the initial old are endowed with a Lucas tree that yields dividends $d > 0$ in terms of the consumption good in each period. In a lot of ways this economy seems a lot like the previous one with money. So it should have the same number and types of equilibria!? The definition of equilibrium (we will focus on Arrow-Debreu equilibria) remains the same, apart from the resource constraint which now reads

$$c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t + d$$

[10] Generically in this context means, for almost all endowments, i.e. the set of possible values for the endowments for which this statement does not hold is of measure zero. Local uniquenes means that in for every equilibrium price vector there exists ε such that any ε-neighborhood of the price vector does not contain another equilibrium price vector (apart from the trivial ones involving a different normalization).

[11] Again we require the assumption that consumption in the first and the second period are strict gross substitutes, ruling out backward-bending offer curves.

Figure 8.4: Productive Outside Assets in the OLG Model

and the budget constraint of the initial old generation which now reads

$$p_1 c_1^0 \leq p_1 e_1^0 + d \sum_{t=1}^{\infty} p_t$$

Let's analyze this economy using our standard techniques. The offer curve remains completely unchanged, but the resource line shifts to the right, now goes through the points $(y, z) = (d, 0)$ and $(y, z) = (0, d)$. Let's look at Figure 11.

It appears that, as in the case with money $m > 0$ there are two stationary and a continuum of nonstationary equilibria. The point (y_1, z_0) on the offer curve indeed represents a stationary equilibrium. Note that the constant equilibrium price ratio satisfies $\frac{p_t}{p_{t+1}} = \alpha > 1$ (just draw a ray through the origin and the point and compare with the slope of the resource constraint which is -1). Hence we have, after normalization of $p_1 = 1$, $p_t = \left(\frac{1}{\alpha}\right)^{t-1}$ and therefore the value of the Lucas tree in the first period equals

$$d \sum_{t=1}^{\infty} \left(\frac{1}{\alpha}\right)^{t-1} < \infty$$

How about the other intersection of the resource line with the offer curve, (y_1', z_0')? Note that in this hypothetical stationary equilibrium $\frac{p_t}{p_{t+1}} = \gamma < 1$, so that $p_t = \left(\frac{1}{\gamma}\right)^{t-1} p_1$. Hence the period 0 value of the Lucas tree is infinite and the consumption of the initial old exceed the resources available in the economy in period 1. This obviously cannot be an equilibrium. Similarly all equilibrium paths starting at some point z_0'' converge to this stationary point, so for all hypothetical nonstationary equilibria we have $\frac{p_t}{p_{t+1}} < 1$ for t large enough and again the value of the Lucas tree remains unbounded, and these paths cannot be equilibrium paths either. We conclude that in this economy there exists a unique equilibrium, which, by the way, is Pareto optimal.

This example demonstrates that it is not the existence of a long-lived outside asset that is responsible for the existence of a continuum of equilibria. What is the difference? In all monetary equilibria apart from the stationary nonautarkic equilibrium (which exists for the Lucas tree economy, too) the price level goes to infinity, as in the hypothetical Lucas tree equilibria that

Figure 8.5: Endogenous Cycles in OLG Models

turned out not to be equilibria. What is crucial is that money is intrinsically useless and does not generate real stuff so that it is possible in equilibrium that prices explode, but the real value of the dividends remains bounded. Also note that we were to introduce a Lucas tree with negative dividends (the initial old generation is an eternal slave, say, of the government and has to come up with d in every period to be used for government consumption), then the existence of the whole continuum of equilibria is restored.[12]

8.1.6 Endogenous Cycles

Not only is there a possibility of a continuum of equilibria in the basic OLG-model, but these equilibria need not take the monotonic form described above. Instead, equilibria with cycles are possible. In Figure 12 we have drawn an offer curve that is backward bending. In the homework you will see an example of preferences that yields such a backward bending offer curve, for a rather normal utility function.

Let $m > 0$ and let p_1 be such that $z_0 = \frac{m}{p_1}$. Using our geometric approach we find $y_1 = y(p_1, p_2)$ from the resource line, $z_1 = z(p_1, p_2)$ from the offer curve (ignore for the moment the fact that there are several z_1 will do; this merely indicates that the multiplicity of equilibria is of even higher order than previously demonstrated). From the resource line we find $y_2 = y(p_2, p_3)$ and from the offer curve $z_2 = z(p_2, p_3) = z_0$. After period $t = 2$ the economy repeats the cycle from the first two periods. The equilibrium allocation is of the form

$$\begin{aligned} c_t^{t-1} &= \begin{cases} c^{ol} = z_0 - w_2 \text{ for } t \text{ odd} \\ c^{oh} = z_1 - w_2 \text{ for } t \text{ even} \end{cases} \\ c_t^t &= \begin{cases} c^{yl} = y_1 - w_1 \text{ for } t \text{ odd} \\ c^{yh} = y_2 - w_1 \text{ for } t \text{ even} \end{cases} \end{aligned}$$

[12]Also note that the fact that in the unique equilibrium $\lim_{t\to\infty} p_t = 0$ has to be true (otherwise the Lucas tree cannot have finite value) implies that this equilibrium cannot be made into a monetary equilibrium, since $\lim_{t\to\infty} \frac{m}{p_t} = \infty$ and the real value of money would eventually exceed the aggregate endowment of the economy for any $m > 0$.

with $c^{ol} < c^{oh}$, $c^{yl} < c^{yh}$. Prices satisfy

$$
\begin{aligned}
\frac{p_t}{p_{t+1}} &= \begin{cases} \alpha^h \text{ for } t \text{ odd} \\ \alpha^l \text{ for } t \text{ even} \end{cases} \\
\pi_{t+1} &= -r_{t+1} = \begin{cases} \pi^l < 0 \text{ for } t \text{ odd} \\ \pi^h > 0 \text{ for } t \text{ even} \end{cases}
\end{aligned}
$$

Consumption of generations fluctuates in a two period cycle, with odd generations eating little when young and a lot when old and even generations having the reverse pattern. Equilibrium returns on money (inflation rates) fluctuate, too, with returns from odd to even periods being high (low inflation) and returns being low (high inflation) from even to odd periods. Note that these cycles are purely endogenous in the sense that the environment is completely stationary: nothing distinguishes odd and even periods in terms of endowments, preferences of people alive or the number of people. It is not surprising that some economists have taken this feature of OLG models to be the basis of a theory of endogenous business cycles (see, for example, Grandmont (1985)). Also note that it is not particularly difficult to construct cycles of length bigger than 2 periods.

8.1.7 Social Security and Population Growth

The pure exchange OLG model renders itself nicely to a discussion of a pay-as-you-go social security system. It also prepares us for the more complicated discussion of the same issue once we have introduced capital accumulation. Consider the simple model without money (i.e. $m = 0$). Also now assume that the population is growing at constant rate n, so that for each old person in a given period there are $(1 + n)$ young people around. Definitions of equilibria remain unchanged, apart from resource feasibility that now reads

$$c_t^{t-1} + (1+n)c_t^t = e_t^{t-1} + (1+n)e_t^t$$

or, in terms of excess demands

$$z(p_{t-1}, p_t) + (1+n)y(p_t, p_{t+1}) = 0$$

This economy can be analyzed in exactly the same way as before with noticing that in our offer curve diagram the slope of the resource line is not -1 anymore, but $-(1 + n)$. We know from above that, without any government intervention, the unique equilibrium in this case is the autarkic equilibrium. We now want to analyze under what conditions the introduction of a pay-as-you-go social security system in period 1 (or any other date) is welfare-improving. We again assume stationary endowments $e_t^t = w_1$ and $e_{t+1}^t = w_2$ for all t. The social security system is modeled as follows: the young pay social security taxes of $\tau \in [0, w_1)$ and receive social security benefits b when old. We assume that the social security system balances its budget in each period, so that benefits are given by

$$b = \tau(1+n)$$

Obviously the new unique competitive equilibrium is again autarkic with endowments $(w_1 - \tau, w_2 + \tau(1+n))$ and equilibrium interest rates satisfy

$$1 + r_{t+1} = 1 + r = \frac{U'(w_1 - \tau)}{\beta U'(w_2 + \tau(1+n))}$$

Obviously for any $\tau > 0$, the initial old generation receives a windfall transfer of $\tau(1+n) > 0$ and hence unambiguously benefits from the introduction. For all other generations, define the equilibrium lifetime utility, as a function of the social security system, as

$$V(\tau) = U(w_1 - \tau) + \beta U(w_2 + \tau(1+n))$$

The introduction of a small social security system is welfare improving if and only if $V'(\tau)$, evaluated at $\tau = 0$, is positive. But

$$\begin{aligned} V'(\tau) &= -U'(w_1 - \tau) + \beta U'(w_2 + \tau(1+n))(1+n) \\ V'(0) &= -U'(w_1) + \beta U'(w_2)(1+n) \end{aligned}$$

Hence $V'(0) > 0$ if and only if

$$n > \frac{U'(w_1)}{\beta U'(w_2)} - 1 = \bar{r}$$

where $\bar{r}$ is the autarkic interest rate. Hence the introduction of a (marginal) pay-as-you-go social security system is welfare improving if and only if the population growth rate exceeds the equilibrium (autarkic) interest rate, or, to use our previous terminology, if we are in the Samuelson case where autarky is not a Pareto optimal allocation. Note that social security has the same function as money in our economy: it is a social institution that transfers resources between generations (backward in time) that do not trade among each other in equilibrium. In enhancing intergenerational exchange not provided by the market it may generate allocations that are Pareto superior to the autarkic allocation, in the case in which individuals private marginal rate of substitution $1+\bar{r}$ (at the autarkic allocation) falls short of the social intertemporal rate of transformation $1+n$.

If $n > \bar{r}$ we can solve for optimal sizes of the social security system analytically in special cases. Remember that for the case with positive money supply $m > 0$ but no social security system the unique Pareto optimal allocation was the nonautarkic stationary allocation. Using similar arguments we can show that the sizes of the social security system for which the resulting equilibrium allocation is Pareto optimal is such that the resulting autarkic equilibrium interest rate is at least equal to the population growth rate, or

$$1+n \leq \frac{U'(w_1-\tau)}{\beta U'(w_2+\tau(1+n))}$$

For the case in which the period utility function is of logarithmic form this yields

$$\begin{aligned} 1+n &\leq \frac{w_2+\tau(1+n)}{\beta(w_1-\tau)} \\ \tau &\geq \frac{\beta}{1+\beta}w_1 - \frac{w_2}{(1+\beta)(1+n)} = \tau^*(w_1,w_2,n,\beta) \end{aligned}$$

Note that τ^* is the unique size of the social security system that maximizes the lifetime utility of the representative generation. For any smaller size we could marginally increase the size and make the representative generation better off and increase the windfall transfers to the initial old. Note, however, that any $\tau > \tau^*$ satisfying $\tau \leq w_1$ generates a Pareto optimal allocation, too: the representative generation would be better off with a smaller system, but the initial old generation would be worse off. This again demonstrates the weak requirements that Pareto optimality puts on an allocation. Also note that the "optimal" size of social security is an increasing function of first period income w_1, the population growth rate n and the time discount factor β, and a decreasing function of the second period income w_2.

So far we have assumed that the government sustains the social security system by forcing people to participate.[13] Now we briefly describe how such a system may come about if policy is determined endogenously. We make the following assumptions. The initial old people can decide upon the size of the social security system $\tau_0 = \tau^{**} \geq 0$. In each period $t \geq 1$ there is a majority vote as to whether the current system is to be kept or abolished. If the majority of the population in period t favors the abolishment of the system, then $\tau_t = 0$ and no payroll taxes or social security benefits are paid. If the vote is in favor of the system, then the young pay taxes τ^{**} and the old receive $(1+n)\tau^{**}$. We assume that $n > 0$, so the current young generation determines current policy. Since current voting behavior depends on expectations about voting behavior of future generations we have to specify how expectations about the voting behavior of future generations is determined. We assume the following expectations mechanism (see Cooley and Soares (1999) for a more detailed discussion of justifications as well as shortcomings for this specification of forming expectations):

$$\tau_{t+1}^e = \begin{cases} \tau^{**} \text{ if } \tau_t = \tau^{**} \\ 0 \text{ otherwise} \end{cases} \tag{8.18}$$

that is, if young individuals at period t voted down the original social security system then they expect that a newly proposed social security system will be voted down tomorrow. Expectations are rational if $\tau_t^e = \tau_t$ for all t. Let $\tau = \{\tau_t\}_{t=0}^{\infty}$ be an arbitrary sequence of policies that is feasible (i.e. satisfies $\tau_t \in [0, w_1)$)

Definition 93 *A rational expectations politico-economic equilibrium, given our expectations mechanism is an allocation rule $\hat{c}_1^0(\tau), \{(\hat{c}_t^t(\tau), \hat{c}_{t+1}^t(\tau))\}$, price rule $\{\hat{p}_t(\tau)\}$ and policies $\{\hat{\tau}_t\}$ such that*[14]

1. *for all $t \geq 1$, for all feasible τ, and given $\{\hat{p}_t(\tau)\}$,*

$$\begin{aligned} (\hat{c}_t^t, \hat{c}_{t+1}^t) &\in \arg \max_{(c_t^t, c_{t+1}^t) \geq 0} V(\tau_t, \tau_{t+1}) = U(c_t^t) + \beta U(c_{t+1}^t) \\ s.t.\ p_t c_t^t + p_{t+1} c_{t+1}^t &\leq p_t (w_1 - \tau_t) + p_{t+1} (w_2 + (1+n)\tau_{t+1}) \end{aligned}$$

[13] This section is not based on any reference, but rather my own thoughts. Please be aware of this and read with caution.

[14] The dependence of allocations and prices on τ is implicit from now on.

2. *for all feasible* τ, *and given* $\{\hat{p}_t(\tau)\}$,

$$\begin{aligned} \hat{c}_1^0 &\in \arg\max_{c_1^0 \geq 0} V(\tau_0, \tau_1) = U(c_1^0) \\ s.t.\ p_1 c_1^0 &\leq p_1(w_2 + (1+n)\tau_1) \end{aligned}$$

3.

$$c_t^{t-1} + (1+n)c_t^t = w_2 + (1+n)w_1$$

4. *For all* $t \geq 1$

$$\hat{\tau}_t \in \arg \max_{\theta \in \{0, \tau^{**}\}} V(\theta, \tau_{t+1}^e)$$

where τ_{t+1}^e *is determined according to* (8.18)

5.

$$\hat{\tau}_0 \in \arg \max_{\theta \in [0, w_1)} V(\theta, \hat{\tau}_1)$$

6. *For all* $t \geq 1$

$$\tau_t^e = \hat{\tau}_t$$

Conditions 1-3 are the standard economic equilibrium conditions for any arbitrary sequence of social security taxes. Condition 4 says that all agents of generation $t \geq 1$ vote rationally and sincerely, given the expectations mechanism specified. Condition 5 says that the initial old generation implements the best possible social security system (for themselves). Note the constraint that the initial generation faces in its maximization: if it picks θ too high, the first regular generation (see condition 4) may find it in its interest to vote the system down. Finally the last condition requires rational expectations with respect to the formation of policy expectations.

Political equilibria are in general very hard to solve unless one makes the economic equilibrium problem easy, assumes simple voting rules and simplifies as much as possible the expectations formation process. I tried to do all of the above for our discussion. So let find an (the!) political economic equilibrium. First notice that for any policy the equilibrium allocation will be autarky since there is no outside asset. Hence we have as equilibrium allocations and prices for a given policy τ

$$\begin{aligned} c_t^{t-1} &= w_2 + (1+n)\tau_t \\ c_t^t &= w_1 - \tau_t \\ p_1 &= 1 \\ \frac{p_t}{p_{t+1}} &= \frac{U'(w_1 - \tau_t)}{\beta U'(w_2 + (1+n)\tau_t)} \end{aligned}$$

Therefore the only equilibrium element to determine are the optimal policies. Given our expectations mechanism for any choice of $\tau_0 = \tau^{**}$, when would generation t vote the system τ^{**} down when young? If it does, given the expectation mechanism, it would not receive benefits when old (a newly installed system would be voted down right away, according to the generations' expectation). Hence

$$V(0, \tau_{t+1}^e) = V(0,0) = U(w_1) + \beta U(w_2)$$

Voting to keep the system in place yields

$$V(\tau^{**}, \tau_{t+1}^e) = V(\tau^{**}, \tau^{**}) = U(w_1 - \tau^{**}) + \beta U(w_2 + (1+n)\tau^{**})$$

and a vote in favor requires

$$V(\tau^{**}, \tau^{**}) \geq V(0,0) \tag{8.19}$$

But this is true for all generations, including the first regular generation. Given the assumption that we are in the Samuelson case with $n > \bar{r}$ there exists a $\tau^{**} > 0$ such that the above inequality holds. Hence the initial old generation can introduce a positive social security system with $\tau_0 = \tau^{**} > 0$ that is not voted down by the next generation (and hence by no generation) and creates positive transfers for itself. Obviously, then, the optimal choice is to maximize $\tau_0 = \tau^{**}$ subject to (8.19), and the equilibrium sequence of policies satisfies $\hat{\tau}_t = \tau^{**}$ where $\tau^{**} > 0$ satisfies

$$U(w_1 - \tau^{**}) + \beta U(w_2 + (1+n)\tau^{**}) = U(w_1) + \beta U(w_2)$$

Note that since the offer curve lies everywhere above the indifference curve through the no-social security endowment point (w_1, w_2), we know that the indifference curve thorugh that point intersects the resource line to the northwest of the intersection of resource line and offer curve (in the Samuelson case). But this implies that $\tau^{**} > \tau^*$ (which was defined as the level of social security that maximizes lifetime utility of a typical generation). Consequently the politico-equilibrium social security tax rate is bigger than the one maximizing welfare for the typical generation: by having the right to set up the system first the initial old can steer the economy to an equilibrium that is better for them (and worse for all future generations) than the one implied by tax rate τ^*.

8.2 The Ricardian Equivalence Hypothesis

How should the government finance a given stream of government expenditures, say, for a war? There are two principal ways to levy revenues for a government, namely to tax current generations or to issue government debt in the form of government bonds the interest and principal of which has to be paid later.[15] The question then arise what the macroeconomic consequences of using these different instruments are, and which instrument is to be preferred from a normative point of view. The Ricardian Equivalence Hypothesis claims that it makes no difference, that a switch from one instrument to the other does not change real allocations and prices in the economy. Therefore this hypothesis, is also called Modigliani-Miller theorem of public finance.[16] It's origin dates back to the classical economist David Ricardo (1772-1823). He wrote about how to finance a war with annual expenditures of £20 millions and asked whether it makes a difference to finance the £20 millions via current taxes or to issue government bonds with infinite maturity (so-called consols) and finance the annual interest payments of £1 million in all future years by future taxes (at an assumed interest rate of 5%). His conclusion was (in "Funding System") that

> in the point of the economy, there is no real difference in either of the modes; for twenty millions in one payment [or] one million per annum for ever ... are precisely of the same value

Here Ricardo formulates and explains the equivalence hypothesis, but immediately makes clear that he is sceptical about its empirical validity

[15] I will restrict myself to a discussion of real economic models, in which fiat money is absent. Hence the government cannot levy revenue via seignorage.

[16] When we discuss a theoretical model, Ricardian equivalence will take the form of a theorem that either holds or does not hold, depending on the assumptions we make. When discussing whether Ricardian equivalence holds empirically, I will call it a hypothesis.

> ...but the people who pay the taxes never so estimate them, and therefore do not manage their affairs accordingly. We are too apt to think, that the war is burdensome only in proportion to what we are at the moment called to pay for it in taxes, without reflecting on the probable duration of such taxes. It would be difficult to convince a man possessed of £20,000, or any other sum, that a perpetual payment of £50 per annum was equally burdensome with a single tax of £1,000.

Ricardo doubts that agents are as rational as they should, according to "in the point of the economy", or that they rationally believe not to live forever and hence do not have to bear part of the burden of the debt. Since Ricardo didn't believe in the empirical validity of the theorem, he has a strong opinion about which financing instrument ought to be used to finance the war

> war-taxes, then, are more economical; for when they are paid, an effort is made to save to the amount of the whole expenditure of the war; in the other case, an effort is only made to save to the amount of the interest of such expenditure.

Ricardo thought of government debt as one of the prime tortures of mankind. Not surprisingly he strongly advocates the use of current taxes. We will, after having discussed the Ricardian equivalence hypothesis, briefly look at the long-run effects of government debt on economic growth, in order to evaluate whether the phobia of Ricardo (and almost all other classical economists) about government debt is in fact justified from a theoretical point of view. Now let's turn to a model-based discussion of Ricardian equivalence.

8.2.1 Infinite Lifetime Horizon and Borrowing Constraints

The Ricardian Equivalence hypothesis is, in fact, a theorem that holds in a fairly wide class of models. It is most easily demonstrated within the Arrow-Debreu market structure of infinite horizon models. Consider the simple infinite horizon pure exchange model discussed at the beginning of the section. Now introduce a government that has to finance a given exogenous stream of government expenditures (in real terms) denoted by $\{G_t\}_{t=1}^{\infty}$. These government expenditures do not yield any utility to the agents (this assumption is not at all restrictive for the results to come). Let p_t denote the Arrow-Debreu price at date 0 of one unit of the consumption good delivered at period t. The government has initial outstanding real debt[17] of B_1 that is held by the public. Let b_1^i denote the initial endowment of government bonds of agent i. Obviously we have the restriction

$$\sum_{i\in I} b_1^i = B_1$$

Note that b_1^i is agent i's entitlement to period 1 consumption that the government owes to the agent. In order to finance the government expenditures the government levies lump-sum taxes: let τ_t^i denote the taxes that agent i pays in period t, denoted in terms of the period t consumption good. We define an Arrow-Debreu equilibrium with government as follows

Definition 94 *Given a sequence of government spending $\{G_t\}_{t=1}^{\infty}$ and initial government debt B_1 and $(b_1^i)_{i\in I}$ an Arrow-Debreu equilibrium are allocations $\{(\hat{c}_t^i)_{i\in I}\}_{t=1}^{\infty}$, prices $\{\hat{p}_t\}_{t=1}^{\infty}$ and taxes $\{(\tau_t^i)_{i\in I}\}_{t=1}^{\infty}$ such that*

1. *Given prices $\{\hat{p}_t\}_{t=1}^{\infty}$ and taxes $\{(\tau_t^i)_{i\in I}\}_{t=1}^{\infty}$ for all $i \in I$, $\{\hat{c}_t^i\}_{t=1}^{\infty}$ solves*

$$\max_{\{c_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} U(c_t^i) \tag{8.20}$$
$$\text{s.t.} \quad \sum_{t=1}^{\infty} \hat{p}_t(c_t + \tau_t^i) \leq \sum_{t=1}^{\infty} \hat{p}_t e_t^i + \hat{p}_1 b_1^i$$

2. *Given prices $\{\hat{p}_t\}_{t=1}^{\infty}$ the tax policy satisfies*

$$\sum_{t=1}^{\infty} \hat{p}_t G_t + \hat{p}_1 B_1 = \sum_{t=1}^{\infty} \sum_{i\in I} \hat{p}_t \tau_t^i$$

3. *For all $t \geq 1$*

$$\sum_{i\in I} \hat{c}_t^i + G_t = \sum_{i\in I} e_t^i$$

[17] I.e. the government owes real consumption goods to its citizens.

In an Arrow-Debreu definition of equilibrium the government, as the agent, faces a single intertemporal budget constraint which states that the total value of tax receipts is sufficient to finance the value of all government purchases plus the initial government debt. From the definition it is clear that, with respect to government tax policies, the only thing that matters is the total value of taxes $\sum_{t=1}^{\infty} \hat{p}_t \tau_t^i$ that the individual has to pay, but not the timing of taxes. It is then straightforward to prove the Ricardian Equivalence theorem for this economy.

Theorem 95 *Take as given a sequence of government spending* $\{G_t\}_{t=1}^{\infty}$ *and initial government debt* $B_1, (b_1^i)_{i\in I}$. *Suppose that allocations* $\{(\hat{c}_t^i)_{i\in I}\}_{t=1}^{\infty}$, *prices* $\{\hat{p}_t\}_{t=1}^{\infty}$ *and taxes* $\{(\tau_t^i)_{i\in I}\}_{t=1}^{\infty}$ *form an Arrow-Debreu equilibrium. Let* $\{(\hat{\tau}_t^i)_{i\in I}\}_{t=1}^{\infty}$ *be an arbitrary alternative tax system satisfying*

$$\sum_{t=1}^{\infty} \hat{p}_t \tau_t^i = \sum_{t=1}^{\infty} \hat{p}_t \hat{\tau}_t^i \text{ for all } i \in I$$

Then $\{(\hat{c}_t^i)_{i\in I}\}_{t=1}^{\infty}$, $\{\hat{p}_t\}_{t=1}^{\infty}$ *and* $\{(\hat{\tau}_t^i)_{i\in I}\}_{t=1}^{\infty}$ *form an Arrow-Debreu equilibrium.*

There are two important elements of this theorem to mention. First, the sequence of government expenditures is taken as fixed and exogenously given. Second, the condition in the theorem rules out redistribution among individuals. It also requires that the new tax system has the same cost to each individual *at the old equilibrium prices* (but not necessarily at alternative prices).

Proof. This is obvious. The budget constraint of individuals does not change, hence the optimal consumption choice at the old equilibrium prices does not change. Obviously resource feasibility is satisfied. The government budget constraint is satisfied due to the assumption made in the theorem. ■

A shortcoming of the Arrow-Debreu equilibrium definition and the preceding theorem is that it does not make explicit the substitution between current taxes and government deficits that may occur for two equivalent tax systems $\{(\tau_t^i)_{i\in I}\}_{t=1}^{\infty}$ and $\{(\hat{\tau}_t^i)_{i\in I}\}_{t=1}^{\infty}$. Therefore we will now reformulate this economy sequentially. This will also allow us to see that one of the main assumptions of the theorem, the absence of borrowing constraints is crucial for the validity of the theorem.

As usual with sequential markets we now assume that markets for the consumption good and one-period loans open every period. We restrict ourselves to government bonds and loans with one year maturity, which, in this environment is without loss of generality (note that there is no risk) and will not distinguish between borrowing and lending between two agents an agent an the government. Let r_{t+1} denote the interest rate on one period loans from period t to period $t+1$. Given the tax system and initial bond holdings each agent i now faces a sequence of budget constraints of the form

$$c_t^i + \frac{b_{t+1}^i}{1+r_{t+1}} \leq e_t^i - \tau_t^i + b_t^i \tag{8.21}$$

with b_1^i given. In order to rule out Ponzi schemes we have to impose a no Ponzi scheme condition of the form $b_t^i \geq -a_t^i(r, e^i, \tau)$ on the consumer, which, in general may depend on the sequence of interest rates as well as the endowment stream of the individual and the tax system. We will be more specific about the exact from of the constraint later. In fact, we will see that the exact specification of the borrowing constraint is crucial for the validity of Ricardian equivalence.

The government faces a similar sequence of budget constraints of the form

$$G_t + B_t = \sum_{i\in I} \tau_t^i + \frac{B_{t+1}}{1+r_{t+1}} \tag{8.22}$$

with B_1 given. We also impose a condition on the government that rules out government policies that run a Ponzi scheme, or $B_t \geq -A_t(r, G, \tau)$. The definition of a sequential markets equilibrium is standard

Definition 96 *Given a sequence of government spending $\{G_t\}_{t=1}^\infty$ and initial government debt $B_1, (b_1^i)_{i\in I}$ a Sequential Markets equilibrium is allocations $\{\left(\hat{c}_t^i, \hat{b}_{t+1}^i\right)_{i\in I}\}_{t=1}^\infty$, interest rates $\{\hat{r}_{t+1}\}_{t=1}^\infty$ and government policies $\{(\tau_t^i)_{i\in I}, B_{t+1}\}_{t=1}^\infty$ such that*

1. *Given interest rates $\{\hat{r}_{t+1}\}_{t=1}^\infty$ and taxes $\{(\tau_t^i)_{i\in I}\}_{t=1}^\infty$ for all $i \in I$, $\{\hat{c}_t^i, \hat{b}_{t+1}^i\}_{t=1}^\infty$ maximizes* (8.20) *subject to* (8.21) *and $b_{t+1}^i \geq -a_t^i(\hat{r}, e^i, \tau)$ for all $t \geq 1$.*

2. *Given interest rates $\{\hat{r}_{t+1}\}_{t=1}^\infty$, the government policy satisfies* (8.22) *and $B_{t+1} \geq -A_t(\hat{r}, G)$ for all $t \geq 1$*

3. *For all* $t \geq 1$

$$\begin{aligned}\sum_{i\in I}\hat{c}_t^i + G_t &= \sum_{i\in I} e_t^i \\ \sum_{i\in I}\hat{b}_{t+1}^i &= B_{t+1}\end{aligned}$$

We will particularly concerned with two forms of borrowing constraints. The first is the so called natural borrowing or debt limit: it is that amount that, at given sequence of interest rates, the consumer can maximally repay, by setting consumption to zero in each period. It is given by

$$an_t^i(\hat{r}, e, \tau) = \sum_{\tau=1}^{\infty} \frac{e_{t+\tau}^i - \tau_{t+\tau}^i}{\prod_{j=t+1}^{t+\tau-1}(1+\hat{r}_{j+1})}$$

where we define $\prod_{j=t+1}^{t}(1+\hat{r}_{j+1}) = 1$. Similarly we set the borrowing limit of the government at its natural limit

$$An_t(\hat{r}, \tau) = \sum_{\tau=1}^{\infty} \frac{\sum_{i\in I}\tau_{t+\tau}^i}{\prod_{j=t+1}^{t+\tau-1}(1+\hat{r}_{j+1})}$$

The other form is to prevent borrowing altogether, setting $a0_t^i(\hat{r}, e) = 0$ for all i, t. Note that since there is positive supply of government bonds, such restriction does not rule out saving of individuals in equilibrium. We can make full use of the Ricardian equivalence theorem for Arrow-Debreu economies one we have proved the following equivalence result

Proposition 97 *Fix a sequence of government spending* $\{G_t\}_{t=1}^{\infty}$ *and initial government debt* $B_1, (b_1^i)_{i\in I}$. *Let allocations* $\{(\hat{c}_t^i)_{i\in I}\}_{t=1}^{\infty}$, *prices* $\{\hat{p}_t\}_{t=1}^{\infty}$ *and taxes* $\{(\tau_t^i)_{i\in I}\}_{t=1}^{\infty}$ *form an Arrow-Debreu equilibrium. Then there exists a corresponding sequential markets equilibrium with the natural debt limits* $\{\left(\tilde{c}_t^i, \tilde{b}_{t+1}^i\right)_{i\in I}\}_{t=1}^{\infty}$, $\{\tilde{r}_t\}_{t=1}^{\infty}$, $\{(\tilde{\tau}_t^i)_{i\in I}, \tilde{B}_{t+1}\}_{t=1}^{\infty}$ *such that*

$$\begin{aligned}\hat{c}_t^i &= \tilde{c}_t^i \\ \tau_t^i &= \tilde{\tau}_t^i \text{ for all } i, \text{ all } t\end{aligned}$$

Reversely, let allocations $\{\left(\hat{c}_t^i, \hat{b}_{t+1}^i\right)_{i\in I}\}_{t=1}^{\infty}$, *interest rates* $\{\hat{r}_t\}_{t=1}^{\infty}$ *and government policies* $\{(\tau_t^i)_{i\in I}, B_{t+1}\}_{t=1}^{\infty}$ *form a sequential markets equilibrium with* *natural debt limits. Suppose that it satisfies*

$$\begin{aligned}
\hat{r}_{t+1} &> -1, \text{ for all } t \geq 1 \\
\sum_{t=1}^{\infty} \frac{e_t^i - \tau_t^i}{\prod_{j=1}^{t-1}(1+\hat{r}_{j+1})} &< \infty \text{ for all } i \in I \\
\sum_{\tau=1}^{\infty} \frac{\sum_{i\in I} \tau_{t+\tau}^i}{\prod_{j=t+1}^{t+\tau}(1+\hat{r}_{j+1})} &< \infty
\end{aligned}$$

Then there exists a corresponding Arrow-Debreu equilibrium $\{(\tilde{c}_t^i)_{i\in I}\}_{t=1}^{\infty}$, $\{\tilde{p}_t\}_{t=1}^{\infty}$, $\{(\tilde{\tau}_t^i)_{i\in I}\}_{t=1}^{\infty}$ *such that*

$$\begin{aligned}
\hat{c}_t^i &= \tilde{c}_t^i \\
\tau_t^i &= \tilde{\tau}_t^i \text{ for all } i, \text{ all } t
\end{aligned}$$

Proof. The key to the proof is to show the equivalence of the budget sets for the Arrow-Debreu and the sequential markets structure. Normalize $\hat{p}_1 = 1$ and relate equilibrium prices and interest rates by

$$1 + \hat{r}_{t+1} = \frac{\hat{p}_t}{\hat{p}_{t+1}} \tag{8.23}$$

Now look at the sequence of budget constraints and assume that they hold with equality (which they do in equilibrium, due to the nonsatiation assumption)

$$\begin{aligned}
c_1^i + \frac{b_2^i}{1+\hat{r}_2} &= e_1^i - \tau_1^i + b_1^i && (8.24) \\
c_2^i + \frac{b_3^i}{1+\hat{r}_3} &= e_2^i - \tau_2^i + b_2^i && (8.25) \\
&\vdots \\
c_t^i + \frac{b_{t+1}^i}{1+\hat{r}_{t+1}} &= e_t^i - \tau_t^i + b_t^i && (8.26)
\end{aligned}$$

Substituting for b_2^i from (8.25) in (8.24) one gets

$$c_1^i + \tau_1^i - e_1^i + \frac{c_2^i + \tau_2^i - e_2^i}{1+\hat{r}_2} + \frac{b_3^i}{(1+\hat{r}_2)(1+\hat{r}_3)} = b_1^i$$

and in general

$$\sum_{t=1}^{T} \frac{c_t^i + \tau_t^i - e_t^i}{\prod_{j=1}^{t-1}(1+\hat{r}_{j+1})} + \frac{b_{T+1}^i}{\prod_{j=1}^{T}(1+\hat{r}_{j+1})} = b_1^i$$

Taking limits on both sides gives, using (8.23)

$$\sum_{t=1}^{\infty} \hat{p}_t(c_t^i + \tau_t^i - e_t^i) + \lim_{T\to\infty} \frac{b_{T+1}^i}{\prod_{j=1}^{T}(1+\hat{r}_{j+1})} = b_1^i$$

Hence we obtain the Arrow-Debreu budget constraint if and only if

$$\lim_{T\to\infty} \frac{b_{T+1}^i}{\prod_{j=1}^{T}(1+\hat{r}_{j+1})} = \lim_{T\to\infty} \hat{p}_{T+1} b_{T+1}^i \geq 0$$

But from the natural debt constraint

$$\begin{aligned} \hat{p}_{T+1} b_{T+1}^i &\geq -\hat{p}_{T+1} \sum_{\tau=1}^{\infty} \frac{e_{t+\tau}^i - \tau_{t+\tau}^i}{\prod_{j=t+1}^{t+\tau-1}(1+\hat{r}_{j+1})} = -\sum_{\tau=T+1}^{\infty} \hat{p}_t(e_\tau^i - \tau_\tau^i) \\ &= -\sum_{\tau=1}^{\infty} \hat{p}_t(e_\tau^i - \tau_\tau^i) + \sum_{\tau=1}^{T} \hat{p}_t(e_\tau^i - \tau_\tau^i) \end{aligned}$$

Taking limits with respect to both sides and using that by assumption $\sum_{t=1}^{\infty} \frac{e_t^i - \tau_t^i}{\prod_{j=1}^{t-1}(1+\hat{r}_{j+1})} = \sum_{t=1}^{\infty} \hat{p}_t(e_\tau^i - \tau_\tau^i) < \infty$ we have

$$\lim_{T\to\infty} \hat{p}_{T+1} b_{T+1}^i \geq 0$$

So at equilibrium prices, with natural debt limits and the restrictions posed in the proposition a consumption allocation satisfies the Arrow-Debreu budget constraint (at equilibrium prices) if and only if it satisfies the sequence of budget constraints in sequential markets. A similar argument can be carried out for the budget constraint(s) of the government. The remainder of the proof is then straightforward and left to the reader. Note that, given an Arrow-Debreu equilibrium consumption allocation, the corresponding bond holdings for the sequential markets formulation are

$$b_{t+1}^i = \sum_{\tau=1}^{\infty} \frac{\hat{c}_{t+\tau}^i + \tau_{t+\tau}^i - e_{t+\tau}^i}{\prod_{j=t+1}^{t+\tau-1}(1+\hat{r}_{j+1})}$$

■

As a straightforward corollary of the last two results we obtain the Ricardian equivalence theorem for sequential markets with natural debt limits (under the weak requirements of the last proposition).[18] Let us look at a few examples

Example 98 *(Financing a war) Let the economy be populated by $I = 1000$ identical people, with $U(c) = \ln(c)$, $\beta = 0.5$*

$$e_t^i = 1$$

and $G_1 = 500$ (the war), $G_t = 0$ for all $t > 1$. Let $b_1 = B_1 = 0$. Consider two tax policies. The first is a balanced budget requirement, i.e. $\tau_1 = 0.5$, $\tau_t = 0$ for all $t > 1$. The second is a tax policy that tries to smooth out the cost of the war, i.e. sets $\tau_t = \tau = \frac{1}{3}$ for all $t \geq 1$. Let us look at the equilibrium for the first tax policy. Obviously the equilibrium consumption allocation (we restrict ourselves to type-identical allocations) has

$$\hat{c}_t^i = \begin{cases} 0.5 \text{ for } t = 1 \\ 1 \text{ for } t \geq 1 \end{cases}$$

and the Arrow-Debreu equilibrium price sequence satisfies (after normalization of $p_1 = 1$) $p_2 = 0.25$ and $p_t = 0.25 * 0.5^{t-2}$ for all $t > 2$. The level of government debt and the bond holdings of individuals in the sequential markets economy satisfy

$$B_t = b_t = 0 \text{ for all } t$$

Interest rates are easily computed as $r_2 = 3$, $r_t = 1$ for $t > 2$. The budget constraint of the government and the agents are obviously satisfied. Now consider the second tax policy. Given resource constraint the previous equilibrium allocation and price sequences are the only candidate for an equilibrium

[18] An equivalence result with even less restrictive assumptions can be proved under the specification of a bounded shortsale constraint

$$\inf_t b_t^i < \infty$$

instead of the natural debt limit. See Huang and Werner (1998) for details.

under the new policy. Let's check whether they satisfy the budget constraints of government and individuals. For the government

$$\begin{aligned}\sum_{t=1}^{\infty}\hat{p}_t G_t + \hat{p}_1 B_1 &= \sum_{t=1}^{\infty}\sum_{i\in I}\hat{p}_t\tau_t^i \\ 500 &= \frac{1}{3}\sum_{t=1}^{\infty}1000\hat{p}_t \\ &= \frac{1000}{3}(1+0.25+\sum_{t=3}^{\infty}0.25*0.5^{t-2}) \\ &= 500\end{aligned}$$

and for the individual

$$\begin{aligned}\sum_{t=1}^{\infty}\hat{p}_t(c_t+\tau_t^i) &\leq \sum_{t=1}^{\infty}\hat{p}_t e_t^i + \hat{p}_1 b_1^i \\ \frac{5}{6}+\frac{4}{3}\sum_{t=2}^{\infty}\hat{p}_t &\leq \sum_{t=1}^{\infty}\hat{p}_t \\ \frac{1}{3}\sum_{t=2}^{\infty}\hat{p}_t &= \frac{1}{6}\leq\frac{1}{6}\end{aligned}$$

Finally, for this tax policy the sequence of government debt and private bond holdings are

$$B_t = \frac{2000}{3}, b_2 = \frac{2}{3} \text{ for all } t \geq 2$$

i.e. the government runs a deficit to finance the war and, in later periods, uses taxes to pay interest on the accumulated debt. It never, in fact, retires the debt. As proved in the theorem both tax policies are equivalent as the equilibrium allocation and prices remain the same after a switch from tax to deficit finance of the war.

The Ricardian equivalence theorem rests on several important assumptions. The first is that there are perfect capital markets. If consumers face binding borrowing constraints (e.g. for the specification requiring $b_{t+1}^i \geq 0$), or if, with risk, not a full set of contingent claims is available, then Ricardian equivalence may fail. Secondly one has to require that all taxes are lump-sum. Non-lump sum taxes may distort relative prices (e.g. labor income taxes distort the relative price of leisure) and hence a change in the timing of taxes may have real effects. All taxes on endowments, whatever form they take, are lump-sum, not, however consumption taxes. Finally a change from one to another tax system is assumed to not redistribute wealth among agents. This was a maintained assumption of the theorem, which required that the total tax bill that each agent faces was left unchanged by a change in the tax system. In a world with finitely lived overlapping generations this would mean that a change in the tax system is not supposed to redistribute the tax burden among different generations.

Now let's briefly look at the effect of borrowing constraints. Suppose we restrict agents from borrowing, i.e. impose $b_{t+1}^i \geq 0$, for all i, all t. For the government we still impose the old restriction on debt, $B_t \geq -An_t(\hat{r}, \tau)$. We can still prove a limited Ricardian result

Proposition 99 *Let* $\{G_t\}_{t=1}^{\infty}$ *and* $B_1, (b_1^i)_{i\in I}$ *be given and let allocations* $\{\left(\hat{c}_t^i, \hat{b}_{t+1}^i\right)_{i\in I}\}_{t=1}^{\infty}$, *interest rates* $\{\hat{r}_{t+1}\}_{t=1}^{\infty}$ *and government policies* $\{(\tau_t^i)_{i\in I}, B_{t+1}\}_{t=1}^{\infty}$ *be a Sequential Markets equilibrium with no-borrowing constraints for which* $\hat{b}_{t+1}^i > 0$ *for all* i, t. *Let* $\{(\tilde{\tau}_t^i)_{i\in I}, \tilde{B}_{t+1}\}_{t=1}^{\infty}$ *be an alternative government policy such that*

$$\tilde{b}_{t+1}^i = \sum_{\tau=t+1}^{\infty} \frac{\hat{c}_\tau^i + \tilde{\tau}_\tau^i - e_\tau^i}{\prod_{j=t+2}^{\tau}(1+\hat{r}_{j+1})} \geq 0 \tag{8.27}$$

$$G_t + \tilde{B}_t = \sum_{i\in I} \tilde{\tau}_t^i + \frac{\tilde{B}_{t+1}}{1+\hat{r}_{t+1}} \text{ for all } t \tag{8.28}$$

$$\tilde{B}_{t+1} \geq -An_t(\hat{r}, \tau) \tag{8.29}$$

$$\sum_{\tau=1}^{\infty} \frac{\tilde{\tau}_\tau^i}{\prod_{j=1}^{\tau-1}(1+\hat{r}_j)} = \sum_{\tau=1}^{\infty} \frac{\tau_\tau^i}{\prod_{j=1}^{\tau-1}(1+\hat{r}_{j+1})} \tag{8.30}$$

Then $\{\left(\hat{c}_t^i, \tilde{b}_{t+1}^i\right)_{i\in I}\}_{t=1}^{\infty}$, $\{\hat{r}_{t+1}\}_{t=1}^{\infty}$ *and* $\{(\tilde{\tau}_t^i)_{i\in I}, \tilde{B}_{t+1}\}_{t=1}^{\infty}$ *is also a sequential markets equilibrium with no-borrowing constraint.*

The conditions that we need for this theorem are that the change in the tax system is not redistributive (condition (8.30)), that the new government policies satisfy the government budget constraint and debt limit (conditions (8.28) and (8.29)) and that the new bond holdings of each individual that are required to satisfy the budget constraints of the individual at old consumption allocations do not violate the no-borrowing constraint (condition (8.27)).

Proof. This proposition to straightforward to prove so we will sketch it here only. Budget constraints of the government and resource feasibility are obviously satisfied under the new policy. How about consumer optimization? Given the equilibrium prices and under the imposed conditions both policies induce the same budget set of individuals. Now suppose there is an i and allocation $\{\bar{c}_t^i\} \neq \{\hat{c}_t^i\}$ that dominates $\{\hat{c}_t^i\}$. Since $\{\bar{c}_t^i\}$ was affordable with the old policy, it must be the case that the associated bond holdings under the old policy, $\{\bar{b}_{t+1}^i\}$ violated one of the no-borrowing constraints. But then, by continuity of the price functional and the utility function there is an allocation $\{\check{c}_t^i\}$ with associated bond holdings $\{\check{b}_{t+1}^i\}$ that is affordable under the old policy and satisfies the no-borrowing constraint (take a convex combination of the $\{\hat{c}_t^i, \hat{b}_{t+1}^i\}$ and the $\{\bar{c}_t^i, \bar{b}_{t+1}^i\}$, with sufficient weight on the $\{\hat{c}_t^i, \hat{b}_{t+1}^i\}$ so as to satisfy the no-borrowing constraints). Note that for this to work it is crucial that the no-borrowing constraints are not binding under the old policy for $\{\hat{c}_t^i, \hat{b}_{t+1}^i\}$. You should fill in the mathematical details ■

Let us analyze an example in which, because of the borrowing constraints, Ricardian equivalence fails.

Example 100 *Consider an economy with 2 agents,* $U^i = \ln(c)$, $\beta_i = 0.5$, $b_1^i = B_1 = 0$. *Also* $G_t = 0$ *for all* t *and endowments are*

$$
\begin{aligned}
e_t^1 &= \begin{cases} 2 \text{ if } t \text{ odd} \\ 1 \text{ if } t \text{ even} \end{cases} \\
e_t^2 &= \begin{cases} 1 \text{ if } t \text{ odd} \\ 2 \text{ if } t \text{ even} \end{cases}
\end{aligned}
$$

As first tax system consider

$$
\begin{aligned}
\tau_t^1 &= \begin{cases} 0.5 \text{ if } t \text{ odd} \\ -0.5 \text{ if } t \text{ even} \end{cases} \\
e_t^2 &= \begin{cases} -0.5 \text{ if } t \text{ odd} \\ 0.5 \text{ if } t \text{ even} \end{cases}
\end{aligned}
$$

Obviously this tax system balances the budget. The equilibrium allocation with no-borrowing constraints evidently is the autarkic (after-tax) allocation

$c_t^i = 1.5$, *for all* i, t. *From the first order conditions we obtain, taking account the nonnegativity constraint on* b_{t+1}^i *(here* $\lambda_t \geq 0$ *is the Lagrange multiplier on the budget constraint in period* t *and* μ_{t+1} *is the Lagrange multiplier on the nonnegativity constraint for* b_{t+1}^i*)*

$$\begin{aligned} \beta^{t-1}U'(c_t^i) &= \lambda_t \\ \beta^t U(c_{t+1}^i) &= \lambda_{t+1} \\ \frac{\lambda_t}{1+r_{t+1}} &= \lambda_{t+1} + \mu_{t+1} \end{aligned}$$

Combining yields

$$\frac{U'(c_t^i)}{\beta U'(c_{t+1}^i)} = \frac{\lambda_t}{\lambda_{t+1}} = 1 + r_{t+1} + \frac{(1+r_t)\mu_{t+1}}{\lambda_{t+1}}$$

Hence

$$\begin{aligned} \frac{U'(c_t^i)}{\beta U'(c_{t+1}^i)} &\geq 1 + r_{t+1} \\ &= 1 + r_{t+1} \text{ if } b_{t+1}^i > 0 \end{aligned}$$

The equilibrium interest rates are given as $r_{t+1} \leq 1$, *i.e. are indeterminate. Both agents are allowed to save, and at* $r_{t+1} > 1$ *they would do so (which of course can't happen in equilibrium as there is zero net supply of assets). For any* $r_{t+1} \leq 1$ *the agents would like to borrow, but are prevented from doing so by the no-borrowing constraint, so any of these interest rates is fine as equilibrium interest rates. For concreteness let's take* $r_{t+1} = 1$ *for all* t.[19] *Then the total bill of taxes for the first consumer is* $\frac{1}{3}$ *and* $-\frac{1}{3}$ *for the second agent. Now lets consider a second tax system that has* $\tau_1^1 = \frac{1}{3}$, $\tau_1^2 = -\frac{1}{3}$ *and* $\tau_t^i = 0$ *for all* $i, t \geq 2$. *Obviously now the equilibrium allocation changes to* $c_t^1 = \frac{5}{3}$, $c_1^2 = \frac{4}{3}$ *and* $c_t^i = e_t^i$ *for all* $i, t \geq 2$. *Obviously the new tax system satisfies the government budget constraint and does not redistribute among agents. However, equilibrium allocations change. Furthermore, equilibrium interest rate change to* $r_2 = \frac{3}{2.5}$ *and* $r_t = 0$ *for all* $t \geq 3$. *Ricardian equivalence fails.*[20]

[19] These are the interest rates that would arise under natural debt limits, too.

[20] In general it is very hard to solve for equilibria with no-borrowing constraints analytically, even in partial equilibrium with fixed exogenous interest rates, even more so in gneral equilibrium. So if the above example seems cooked up, it is, since it is about the only example I know how to solve without going to the computer. We will see this more explicitly once we talk about Deaton's (1991) EC piece.

8.2.2 Finite Horizon and Operative Bequest Motives

It should be clear from the above discussion that one only obtains a very limited Ricardian equivalence theorem for OLG economies. Any change in the timing of taxes that redistributes among generations is in general not neutral in the Ricardian sense. If we insist on representative agents within one generation and purely selfish, two-period lived individuals, then in fact any change in the timing of taxes can't be neutral unless it is targeted towards a particular generation, i.e. the tax change is such that it decreases taxes for the currently young only and increases them for the old next period. Hence, with sufficient generality we can say that Ricardian equivalence does not hold for OLG economies with purely selfish individuals.

Rather than to demonstrate this obvious point with another example we now briefly review Barro's (1974) argument that under certain conditions finitely lived agents will behave as if they had infinite lifetime. As a consequence, Ricardian equivalence is re-established. Barro's (1974) article "Are Government Bonds Net Wealth?" asks exactly the Ricardian question, namely does an increase in government debt, financed by future taxes to pay the interest on the debt increase the net wealth of the private sector? If yes, then current consumption would increase, aggregate saving (private plus public) would decrease, leading to an increase in interest rate and less capital accumulation. Depending on the perspective, countercyclical fiscal policy[21] is effective against the business cycle (the Keynesian perspective) or harmful for long term growth (the classical perspective). If, however, the value of government bonds if completely offset by the value of future higher taxes for each individual, then government bonds are not net wealth of the private sector, and changes in fiscal policy are neutral.

Barro identified two main sources for why future taxes are not exactly offsetting current tax cuts (increasing government deficits): a) finite lives of agents that lead to intergenerational redistribution caused by a change in the timing of taxes b) imperfect private capital markets. Barro's paper focuses on the first source of nonneutrality.

Barro's key result is the following: in OLG-models finiteness of lives does not invalidate Ricardian equivalence as long as current generations are connected to future generations by a chain of operational intergenerational, altruistically motivated transfers. These may be transfers from old to young

[21]By fiscal policy in this section we mean the financing decision of the government for a given *exogenous* path of government expenditures.

via bequests or from young to old via social security programs. Let us look at his formal model.[22]

Consider the standard pure exchange OLG model with two-period lived agents. There is no population growth, so that each member of the old generation (whose size we normalize to 1) has exactly one child. Agents have endowment $e_t^t = w$ when young and no endowment when old. There is a government that, for simplicity, has 0 government expenditures but initial outstanding government debt B. This debt is denominated in terms of the period 1 (or any other period) consumption good. The initial old generation is endowed with these B units of government bonds. We assume that these government bonds are zero coupon bonds with maturity of one period. Further we assume that the government keeps its outstanding government debt constant and we assume a constant one-period real interest rate r on these bonds.[23] In order to finance the interest payments on government debt the government taxes the currently young people. The government budget constraint gives

$$\frac{B}{1+r} + \tau = B$$

The right hand side is the old debt that the government has to retire in the current period. On the left hand side we have the revenue from issuing new debt, $\frac{B}{1+r}$ (remember that we assume zero coupon bonds, so $\frac{1}{1+r}$ is the price of one government bond today that pays 1 unit of the consumption good tomorrow) and the tax revenue. With the assumption of constant government debt we find

$$\tau = \frac{rB}{1+r}$$

and we assume $\frac{rB}{1+r} < w$.

Now let's turn to the budget constraints of the individuals. Let by a_t^t denote the savings of currently young people for the second period of their lives and by a_{t+1}^t denote the savings of the currently old people for the next generation, i.e. the old people's bequests. We require bequests to be non-negative, i.e. $a_{t+1}^t \geq 0$. In our previous OLG models obviously $a_{t+1}^t = 0$ was the only optimal choice since individuals were completely selfish. We

[22] I will present a simplified, pure exchange version of his model to more clearly isolate his main point.

[23] This assumption is justified since the resulting equilibrium allocation (there is no money!) is the autarkic allocation and hence the interest rate always equals the autarkic interest rate.

will see below how to induce positive bequests when discussing individuals' preferences. The budget constraints of a representative generation are then given by

$$\begin{aligned} c_t^t + \frac{a_t^t}{1+r} &= w - \tau \\ c_{t+1}^t + \frac{a_{t+1}^t}{1+r} &= a_t^t + a_t^{t-1} \end{aligned}$$

The budget constraint of the young are standard; one may just remember that assets here are zero coupon bonds: spending $\frac{a_t^t}{1+r}$ on bonds in the current period yields a_t^t units of consumption goods tomorrow. We do not require a_t^t to be positive. When old the individuals have two sources of funds: their own savings from the previous period and the bequests a_t^{t-1} from the previous generation. They use it to buy own consumption and bequests for the next generation. The total expenditure for bequests of a currently old individual is $\frac{a_{t+1}^t}{1+r}$ and it delivers funds to her child next period (that has then become old) of a_{t+1}^t. We can consolidate the two budget constraints to obtain

$$c_t^t + \frac{c_{t+1}^t}{1+r} + \frac{a_{t+1}^t}{(1+r)^2} = w + \frac{a_t^{t-1}}{1+r} - \tau$$

Since the total lifetime resources available to generation t are given by $e_t = w + \frac{a_t^{t-1}}{1+r} - \tau$, the lifetime utility that this generation can attain is determined by e. The budget constraint of the initial old generation is given by

$$c_1^0 + \frac{a_1^0}{1+r} = B$$

With the formulation of preferences comes the crucial twist of Barro. He assumes that individuals are altruistic and care about the well-being of their descendant.[24] Altruistic here means that the parents genuinely care about the utility of their children and leave bequests for that reason; it is not that the parents leave bequests in order to induce actions of the children that yield utility to the parents.[25] Preferences of generation t are represented by

$$u_t(c_t^t, c_{t+1}^t, a_{t+1}^t) = U(c_t^t) + \beta U(c_{t+1}^t) + \alpha V_{t+1}(e_{t+1})$$

[24]Note that we only assume that the agent cares only about her immediate descendant, but (possibly) not at all about grandchildren.

[25]This strategic bequest motive does not necessarily help to reestablish Ricardian equivalence, as Bernheim, Shleifer and Summers (1985) show.

where $V_{t+1}(e_{t+1})$ is the maximal utility generation $t+1$ can attain with lifetime resources $e_{t+1} = w + \frac{a_{t+1}^t}{1+r} - \tau$, which are evidently a function of bequests $a_{t+1}^t$from generation t.[26] We make no assumption about the size of α as compared to β, but assume $\alpha \in (0,1)$. The initial old generation has preferences represented by

$$u_0(c_1^0, a_1^0) = \beta U(c_1^0) + \alpha V_1(e_1)$$

The equilibrium conditions for the goods and the asset market are, respectively

$$\begin{aligned} c_t^{t-1} + c_t^t &= w \text{ for all } t \geq 1 \\ a_t^{t-1} + a_t^t &= B \text{ for all } t \geq 1 \end{aligned}$$

Now let us look at the optimization problem of the initial old generation

$$\begin{aligned} V_0(B) &= \max_{c_1^0, a_1^0 \geq 0} \left\{ \beta U(c_1^0) + \alpha V_1(e_1) \right\} \\ \text{s.t. } c_1^0 + \frac{a_1^0}{1+r} &= B \\ e_1 &= w + \frac{a_1^0}{1+r} - \tau \end{aligned}$$

Note that the two constraints can be consolidated to

$$c_1^0 + e_1 = w + B - \tau \tag{8.31}$$

This yields optimal decision rules $c_1^0(B)$ and $a_1^0(B)$ (or $e_1(B)$). Now assume that the bequest motive is operative, i.e. $a_1^0(B) > 0$ and consider the Ricardian experiment of government: increase initial government debt marginally by ΔB and repay this additional debt by levying higher taxes on the first young generation. Clearly, in the OLG model without bequest motives such a change in fiscal policy is not neutral: it increases resources available to the initial old and reduces resources available to the first regular generation. This

[26]To formulate the problem recurively we need separability of the utility function with respect to time and utility of children. The argument goes through without this, but then it can't be clarified using recursive methods. See Barro's original paper for a more general discussion. Also note that he, in all likelihood, was not aware of the full power of recursive techniques in 1974. Lucas (1972) seminal paper was probably the first to make full use of recursive techiques in (macro) economics.

will change consumption of both generations and interest rate. What happens in the Barro economy? In order to repay the ΔB, from the government budget constraint taxes for the young have to increase by

$$\Delta\tau = \Delta B$$

since by assumption government debt from the second period onwards remains unchanged. How does this affect the optimal consumption and bequest choice of the initial old generation? It is clear from (8.31) that the optimal choices for c_1^0 and e_1 do not change as long as the bequest motive was operative before.[27] The initial old generation receives additional transfers of bonds of magnitude ΔB from the government and increases its bequests a_1^0 by $(1+r)\Delta B$ so that lifetime resources available to their descendants (and hence their allocation) is left unchanged. Altruistically motivated bequest motives just undo the change in fiscal policy. Ricardian equivalence is restored.

This last result was just an example. Now let's show that Ricardian equivalence holds in general with operational altruistic bequests. In doing so we will de facto establish between Barro's OLG economy and an economy with infinitely lived consumers and borrowing constraints. Again consider the problem of the initial old generation (and remember that, for a given tax rate and wage there is a one-to-one mapping between e_{t+1} and a_{t+1}^t

$$\begin{aligned}
V_0(B) &= \max_{\substack{c_1^0, a_1^0 \geq 0 \\ c_1^0 + \frac{a_1^0}{1+r} = B}} \left\{\beta U(c_1^0) + \alpha V_1(a_1^0)\right\} \\
&= \max_{\substack{c_1^0, a_1^0 \geq 0 \\ c_1^0 + \frac{a_1^0}{1+r} = B}} \left\{ \beta U(c_1^0) + \alpha \max_{\substack{c_1^1, c_2^1, a_2^1 \geq 0, a_1^1 \\ c_1^1 + \frac{a_1^1}{1+r} = w - \tau \\ c_2^1 + \frac{a_2^1}{1+r} = a_1^1 + a_1^0}} \left\{U(c_1^1) + \beta U(c_2^1) + \alpha V_2(a_2^1)\right\} \right\}
\end{aligned}$$

[27] If the bequest motive was not operative, i.e. if the constraint $a_1^0 \geq 0$ was binding, then by increasing B may result in an increase in c_1^0 and a decrease in e_1.

But this maximization problem can be rewritten as

$$\max_{c_1^0,a_1^0,c_1^1,c_2^1,a_2^1\geq 0,a_1^1} \left\{\beta U(c_1^0)+\alpha U(c_1^1)+\alpha\beta U(c_2^1)+\alpha^2 V_2(a_2^1)\right\}$$

$$\begin{aligned}
\text{s.t. } c_1^0+\frac{a_1^0}{1+r} &= B \\
c_1^1+\frac{a_1^1}{1+r} &= w-\tau \\
c_2^1+\frac{a_2^1}{1+r} &= a_1^1+a_1^0
\end{aligned}$$

or, repeating this procedure infinitely many times (which is a valid procedure only for $\alpha<1$), we obtain as implied maximization problem of the initial old generation

$$\max_{\{(c_t^{t-1},c_t^t,a_t^{t-1})\}_{t=1}^{\infty}\geq 0} \left\{\beta U(c_1^0)+\sum_{t=1}^{\infty}\alpha^t\left(U(c_t^t)+\beta U(c_{t+1}^t)\right)\right\}$$

$$\begin{aligned}
s.t.\ c_1^0+\frac{a_1^0}{1+r} &= B \\
c_t^t+\frac{c_{t+1}^t}{1+r}+\frac{a_{t+1}^t}{(1+r)^2} &= w-\tau+\frac{a_t^{t-1}}{1+r}
\end{aligned}$$

i.e. the problem is equivalent to that of an infinitely lived consumer that faces a no-borrowing constraint. This infinitely lived consumer is peculiar in the sense that her periods are subdivided into two subperiods, she eats twice a period, c_t^t in the first subperiod and c_{t+1}^t in the second subperiod, and the relative price of the consumption goods in the two subperiods is given by $(1+r)$. Apart from these reinterpretations this is a standard infinitely lived consumer with no-borrowing constraints imposed on her. Consequently one obtains a Ricardian equivalence proposition similar to proposition 99, where the requirement of "operative bequest motives" is the equivalent to condition (8.27). More generally, this argument shows that an OLG economy with two period-lived agents and operative bequest motives is formally equivalent to an infinitely lived agent model.

Example 101 *Suppose we carry out the Ricardian experiment and increase initial government debt by ΔB. Suppose the debt is never retired, but the required interest payments are financed by permanently higher taxes. The tax* *increase that is needed is (see above)*

$$\Delta\tau = \frac{r\Delta B}{1+r}$$

Suppose that for the initial debt level $\{(\hat{c}_t^{t-1}, \hat{c}_t^t, \hat{a}_t^{t-1})\}_{t=1}^{\infty}$ *together with* $\hat{r}$ *is an equilibrium such that* $\hat{a}_t^{t-1} > 0$ *for all* t*. It is then straightforward to verify that* $\{(\hat{c}_t^{t-1}, \hat{c}_t^t, \tilde{a}_t^{t-1})\}_{t=1}^{\infty}$ *together with* $\hat{r}$ *is an equilibrium for the new debt level, where*

$$\tilde{a}_t^{t-1} = \hat{a}_t^{t-1} + (1+\hat{r})\Delta B \text{ for all } t$$

i.e. in each period savings increase by the increased level of debt, plus the provision for the higher required tax payments. Obviously one can construct much more complicated tax experiments that are neutral in the Ricardian sense, provided that for the original tax system the non-borrowing constraints never bind (i.e. that bequest motives are always operative). Also note that Barro discussed his result in the context of a production economy, an issue to which we turn next.

8.3 Overlapping Generations Models with Production

So far we have ignored production in our discussion of OLG-models. It may be the case that some of the pathodologies of the OLG-model appear only in pure exchange versions of the model. Since actual economies feature capital accumulation and production, these pathodologies then are nothing to worry about. However, we will find out that, for example, the possibility of inefficient competitive equilibria extends to OLG models with production. The issues of whether money may have positive value and whether there exists a continuum of equilibria are not easy for production economies and will not be discussed in these notes.

8.3.1 Basic Setup of the Model

As much as possible I will synchronize the discussion here with the discrete time neoclassical growth model in Chapter 2 and the pure exchange OLG model in previous subsections. The economy consists of individuals and firms. Individuals live for two periods By N_t^t denote the number of young people in period t, by N_t^{t-1} denote the number of old people at period t. Normalize the size of the initial old generation to 1, i.e. $N_0^0 = 1$. We assume that people do not die early, so $N_t^t = N_{t+1}^t$. Furthermore assume that the population grows at constant rate n, so that $N_t^t = (1+n)^t N_0^0 = (1+n)^t$. The total population at period t is therefore given by $N_t^{t-1} + N_t^t = (1+n)^t(1+\frac{1}{1+n})$.

The representative member of generation t has preferences over consumption streams given by

$$u(c_t^t, c_{t+1}^t) = U(c_t^t) + \beta U(c_{t+1}^t)$$

where U is strictly increasing, strictly concave, twice continuously differentiable and satisfies the Inada conditions. All individuals are assumed to be purely selfish and have no bequest motives whatsoever. The initial old generation has preferences

$$u(c_1^0) = U(c_1^0)$$

Each individual of generation $t \geq 1$ has as endowments one unit of time to work when young and no endowment when old. Hence the labor force in period t is of size N_t^t with maximal labor supply of $1 * N_t^t$. Each member of the initial old generation is endowed with capital stock $(1+n)\bar{k}_1 > 0$.

Firms has access to a constant returns to scale technology that produces output Y_t using labor input L_t and capital input K_t rented from households i.e. $Y_t = F(K_t, L_t)$. Since firms face constant returns to scale, profits are zero in equilibrium and we do not have to specify ownership of firms. Also without loss of generality we can assume that there is a single, representative firm, that, as usual, behaves competitively in that it takes as given the rental prices of factor inputs (r_t, w_t) and the price for its output. Defining the capital-labor ratio $k_t = \frac{K_t}{L_t}$ we have by constant returns to scale

$$y_t = \frac{Y_t}{L_t} = \frac{F(K_t, L_t)}{L_t} = F\left(\frac{K_t}{L_t}, 1\right) = f(k_t)$$

We assume that f is twice continuously differentiable, strictly concave and satisfies the Inada conditions.

8.3.2 Competitive Equilibrium

The timing of events for a given generation t is as follows

1. At the beginning of period t production takes place with labor of generation t and capital saved by the now old generation $t-1$ from the previous period. The young generation earns a wage w_t

2. At the end of period t the young generation decides how much of the wage income to consume, c_t^t, and how much to save for tomorrow, s_t^t. The saving occurs in form of physical capital, which is the only asset in this economy

3. At the beginning of period $t+1$ production takes place with labor of generation $t+1$ and the saved capital of the now old generation t. The return on savings equals $r_{t+1}-\delta$, where again r_{t+1} is the rental rate of capital and δ is the rate of depreciation, so that $r_{t+1}-\delta$ is the real interest rate from period t to $t+1$.

4. At the end of period $t+1$ generation t consumes its savings plus interest rate, i.e. $c_{t+1}^t = (1+r_{t+1}-\delta)s_t^t$ and then dies.

We now can define a sequential markets equilibrium for this economy

Definition 102 *Given $\bar{k}_1$, a sequential markets equilibrium is allocations for households $\hat{c}_1^0, \{(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)\}_{t=1}^\infty$, allocations for the firm $\{(\hat{K}_t, \hat{L}_t)\}_{t=1}^\infty$ and prices $\{(\hat{r}_t, \hat{w}_t)\}_{t=1}^\infty$ such that*

1. *For all $t \geq 1$, given $(\hat{w}_t, \hat{r}_{t+1})$, $(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)$ solves*

$$\begin{aligned} & \max_{c_t^t, c_{t+1}^t \geq 0, s_t^t} U(c_t^t) + \beta U(c_{t+1}^t) \\ s.t.\ c_t^t + s_t^t & \leq \hat{w}_t \\ c_{t+1}^t & \leq (1+\hat{r}_{t+1}-\delta)s_t^t \end{aligned}$$

2. *Given $\bar{k}_1$ and $\hat{r}_1$, $\hat{c}_1^0$ solves*

$$\begin{aligned} & \max_{c_1^0 \geq 0} U(c_1^0) \\ s.t.\ c_1^0 & \leq (1+\hat{r}_1-\delta)(1+n)\bar{k}_1 \end{aligned}$$

3. *For all $t \geq 1$, given $(\hat{r}_t, \hat{w}_t)$, $(\hat{K}_t, \hat{L}_t)$ solves*

$$\max_{K_t, L_t \geq 0} F(K_t, L_t) - \hat{r}_t K_t - \hat{w}_t L_t$$

4. *For all $t \geq 1$*

   (a) *(Goods Market)*

$$N_t^t \hat{c}_t^t + N_t^{t-1} \hat{c}_t^{t-1} + \hat{K}_{t+1} - (1-\delta)\hat{K}_t = F(\hat{K}_t, \hat{L}_t)$$

   (b) *(Asset Market)*

$$N_t^t \hat{s}_t^t = \hat{K}_{t+1}$$

   (c) *(Labor Market)*

$$N_t^t = \hat{L}_t$$

The first two points in the equilibrium definition are completely standard, apart from the change in the timing convention for the interest rate. For firm maximization we used the fact that, given that the firm is renting inputs in each period, the firms intertemporal maximization problem separates into a sequence of static profit maximization problems. The goods market equilibrium condition is standard: total consumption plus gross investment equals output. The labor market equilibrium condition is obvious. The asset or capital market equilibrium condition requires a bit more thought: it states that total saving of the currently young generation makes up the capital stock for tomorrow, since physical capital is the only asset in this economy. Alternatively think of it as equating the total supply of capital in form the saving done by the now young, tomorrow old generation and the total demand for capital by firms next period.[28] It will be useful to single out particular equilibria and attach a certain name to them.

Definition 103 *A steady state (or stationary equilibrium) is $(\bar{k}, \bar{s}, \bar{c}_1, \bar{c}_2, \bar{r}, \bar{w})$ such that the sequences $\hat{c}_1^0, \{(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)\}_{t=1}^\infty$, $\{(\hat{K}_t, \hat{L}_t)\}_{t=1}^\infty$ and $\{(\hat{r}_t, \hat{w}_t)\}_{t=1}^\infty$,*

[28] To define an Arrow-Debreu equilibrium is quite standard here. Let p_t the price of the consumption good at period t, $r_t p_t$ the nominal rental price of capital and $w_t p_t$ the nominal wage. Then the household and the firms problems are in the neoclassical growth model, in the household problem taking into account that agents only live for two periods.

defined by

$$\begin{aligned}
\hat{c}_t^t &= \bar{c}_1 \\
\hat{c}_t^{t-1} &= \bar{c}_2 \\
\hat{s}_t^t &= \bar{s} \\
\hat{r}_t &= \bar{r} \\
\hat{w}_t &= \bar{w} \\
\hat{K}_t &= \bar{k} * N_t^t \\
\hat{L}_t &= N_t^t
\end{aligned}$$

are an equilibrium, for given initial condition $\bar{k}_1 = \bar{k}$.

In other words, a steady state is an equilibrium for which the allocation (per capita) is constant over time, given that the initial condition for the initial capital stock is exactly right. Alternatively it is allocations and prices that satisfy all the equilibrium conditions apart from possibly obeying the initial condition.

We can use the goods and asset market equilibrium to derive an equation that equates saving to investment. By definition gross investment equals $\hat{K}_{t+1} - (1-\delta)\hat{K}_t$, whereas savings equals that part of income that is not consumed, or

$$\hat{K}_{t+1} - (1-\delta)\hat{K}_t = F(\hat{K}_t, \hat{L}_t) - \left(N_t^t \hat{c}_t^t + N_t^{t-1}\hat{c}_t^{t-1}\right)$$

But what is total saving equal to? The currently young save $N_t^t \hat{s}_t^t$, the currently old dissave $\hat{s}_{t-1}^{t-1} N_{t-1}^{t-1} = (1-\delta)\hat{K}_t$ (they sell whatever capital stock they have left).[29] Hence setting investment equal to saving yields

$$\hat{K}_{t+1} - (1-\delta)\hat{K}_t = N_t^t \hat{s}_t^t - (1-\delta)\hat{K}_t$$

or our asset market equilibrium condition

$$N_t^t \hat{s}_t^t = \hat{K}_{t+1}$$

[29]By definition the saving of the old is their total income minus their total consumption. Their income consists of returns on their assets and hence their total saving is

$$\begin{aligned}
& \left[(r_t s_{t-1}^{t-1} - c_t^{t-1}\right] N_{t-1}^{t-1} \\
= \quad & -(1-\delta) s_{t-1}^{t-1} N_{t-1}^{t-1} = -(1-\delta) K_t
\end{aligned}$$

Now let us start to characterize the equilibrium It will turn out that we can describe the equilibrium completely by a first order difference equation in the capital-labor ratio k_t. Unfortunately it will have a rather nasty form in general, so that we can characterize analytic properties of the competitive equilibrium only very partially. Also note that, as we will see later, the welfare theorems do not apply so that there is no social planner problem that will make our lives easier, as was the case in the infinitely lived consumer model (which I dubbed the discrete-time neoclassical growth model in Section 3).

From now on we will omit the hats above the variables indicating equilibrium elements as it is understood that the following analysis applies to equilibrium sequences. From the optimization condition for capital for the firm we obtain

$$r_t = F_K(K_t, L_t) = F_K\left(\frac{K_t}{L_t}, 1\right) = f'(k_t)$$

because partial derivatives of functions that are homogeneous of degree 1 are homogeneous of degree zero. Since we have zero profits in equilibrium we find that

$$w_t L_t = F(K_t, L_t) - r_t K_t$$

and dividing by L_t we obtain

$$w_t = f(k_t) - f'(k_t)k_t$$

i.e. factor prices are completely determined by the capital-labor ratio. Investigating the households problem we see that its solution is completely characterized by a saving function (note that given our assumptions on preferences the optimal choice for savings exists and is unique)

$$\begin{aligned} s_t^t &= s\left(w_t, r_{t+1}\right) \\ &= s\left(f(k_t) - f'(k_t)k_t, f'(k_{t+1})\right) \end{aligned}$$

so optimal savings are a function of this and next period's capital stock. Obviously, once we know s_t^t we know c_t^t and c_{t+1}^t from the household's budget constraint. From Walras law one of the market clearing conditions is redundant. Equilibrium in the labor market is straightforward as

$$L_t = N_t^t = (1+n)^t$$

So let's drop the goods market equilibrium condition.[30] Then the only condition left to exploit is the asset market equilibrium condition

$$\begin{aligned} s_t^t N_t^t &= K_{t+1} \\ s_t^t &= \frac{K_{t+1}}{N_t^t} = \frac{N_{t+1}^{t+1}}{N_t^t} \frac{K_{t+1}}{N_{t+1}^{t+1}} \\ &= (1+n)\frac{K_{t+1}}{L_{t+1}} \\ &= (1+n)k_{t+1} \end{aligned}$$

Substituting in the savings function yields our first order difference equation

$$k_{t+1} = \frac{s\left(f(k_t) - f'(k_t)k_t, f'(k_{t+1})\right)}{1+n} \tag{8.32}$$

where the exact form of the saving function obviously depends on the functional form of the utility function U. As starting value for the capital-labor ratio we have $\frac{K_1}{L_1} = \frac{(1+n)\bar{k}_1}{N_1^1} = \bar{k}_1$. So in principle we could put equation (8.32) on a computer and solve for the entire sequence of $\{k_{t+1}\}_{t=1}^{\infty}$ and hence for the entire equilibrium. Note, however, that equation (8.32) gives k_{t+1} only as an implicit function of k_t as k_{t+1} appears on the right hand side of the equation as well. So let us make an attempt to obtain analytical properties of this equation. Before, let's solve an example.

Example 104 *Let $U(c) = \ln(c)$, $n = 0, \beta = 1$ and $f(k) = k^{\alpha}$, with $\alpha \in (0,1)$. The choice of log-utility is particularly convenient as the income and substitution effects of an interest change cancel each other out; saving is independent of r_{t+1}. As we will see later it is crucial whether the income or substitution effect for an interest change dominates in the saving decision, i.e. whether*

$$s_{r_{t+1}}(w_t, r_{t+1}) \lesseqgtr 0$$

[30] In the homework you are asked to do the analysis with dropping the asset market instead of the goods market equilibrium condition. Keep the present analysis in mind when doing this question.

But let's proceed. The saving function for the example is given by

$$\begin{aligned} s(w_t, r_{t+1}) &= \frac{1}{2} w_t \\ &= \frac{1}{2}\left(k_t^\alpha - \alpha k_t^\alpha\right) \\ &= \frac{1-\alpha}{2} k_t^\alpha \end{aligned}$$

so that the difference equation characterizing the dynamic equilibrium is given by

$$k_{t+1} = \frac{1-\alpha}{2} k_t^\alpha$$

There are two steady states for this differential equation, $k_0 = 0$ and $k^* = \left(\frac{1-\alpha}{2}\right)^{\frac{1}{1-\alpha}}$. The first obviously is not an equilibrium as interest rates are infinite and no solution to the consumer problem exists. From now on we will ignore this steady state, not only for the example, but in general. Hence there is a unique steady state equilibrium associated with k^*. From any initial condition $\bar{k}_1 > 0$, there is a unique dynamic equilibrium $\{k_{t+1}\}_{t=1}^{\infty}$ converging to k^* described by the first order difference equation above.

Unfortunately things are not always that easy. Let us return to the general first order difference equation (8.32) and discuss properties of the saving function. Let, us for simplicity, assume that the saving function s is differentiable in both arguments (w_t, r_{t+1}).[31] Since the saving function satisfies the first order condition

$$U'(w_t - s(w_t, r_{t+1})) = \beta U'((1 + r_{t+1} - \delta)s(w_t, r_{t+1})) * (1 + r_{t+1} - \delta)$$

we use the Implicit Function Theorem (which is applicable in this case) to obtain

$$\begin{aligned} s_{w_t}(w_t, r_{t+1}) &= \frac{U''(w_t - s(w_t, r_{t+1}))}{U''(w_t - s(w_t, r_{t+1})) + \beta U''((1 + r_{t+1} - \delta)s(w_t, r_{t+1}))(1 + r_{t+1} - \delta)^2} \in (0 \\ s_{r_{t+1}}(w_t, r_{t+1}) &= \frac{-\beta U'((1 + r_{t+1} - \delta)s(.,.)) - \beta U''((1 + r_{t+1} - \delta)s(.,.))(1 + r_{t+1} - \delta)s(.,.}{U''(w_t - s(.,.)) + \beta U''((1 + r_{t+1} - \delta)s(.,.))(1 + r_{t+1} - \delta)^2} \end{aligned}$$

[31] One has to invoke the implicit function theorem (and check its conditions) on the first order condition to insure differentiability of the savings function. See Mas-Colell et al. p. 940-942 for details.

Figure 8.6: Capital Dynamics in the OLG Model with Production

Given our assumptions optimal saving increases in first period income w_t, but it may increase or decrease in the interest rate. You may verify from the above formula that indeed for the log-case $s_{r_{t+1}}(w_t, r_{t+1}) = 0$. A lot of theoretical work focused on the case in which the saving function increases with the interest rate, which is equivalent to saying that the substitution effect dominates the income effect (and equivalent to assuming that consumption in the two periods are strict gross substitutes).

Equation (8.32) traces out a graph in (k_t, k_{t+1}) space whose shape we want to characterize. Differentiating both sides of (8.32) with respect to k_t we obtain[32]

$$\frac{dk_{t+1}}{dk_t} = \frac{-s_{w_t}(w_t, r_{t+1})f''(k_t)k_t + s_{r_{t+1}}(w_t, r_{t+1})f'(k_{t+1})\frac{dk_{t+1}}{dk_t}}{1+n}$$

or rewriting

$$\frac{dk_{t+1}}{dk_t} = \frac{-s_{w_t}(w_t, r_{t+1})f''(k_t)k_t}{1+n-s_{r_{t+1}}(w_t, r_{t+1})f''(k_{t+1})}$$

Given our assumptions on f the nominator of the above expression is strictly positive for all $k_t > 0$. If we assume that $s_{r_{t+1}} \geq 0$, then the (k_t, k_{t+1})-locus is upward sloping. If we allow $s_{r_{t+1}} < 0$, then it may be downward sloping.

Figure 13 shows possible shapes of the (k_t, k_{t+1})-locus under the assumption that $s_{r_{t+1}} \geq 0$. We see that even this assumption does not place a lot of restrictions on the dynamic behavior of our economy. Without further assumptions it may be the case that, as in case A there is no steady state with positive capital-labor ratio. Starting from any initial capital-per worker level the economy converges to a situation with no production over time. It may be that, as in case C, there is a unique positive steady state k_C^* and this steady state is globally stable (for state space excluding 0). Or it is possible that there are multiple steady states which alternate in being locally stable (as k_B^*) and unstable (as k_B^{**}) as in case B. Just about any dynamic behavior is possible and in order to deduce further qualitative properties we must either specify special functional forms or make assumptions about endogenous variables, something that one should avoid, if possible.

[32]Again we appeal to the Implicit function theorem that guarantees that k_{t+1} is a differentiable function of k_t with derivative given below.

We will proceed however, doing exactly this. For now let's *assume* that there exists a unique positive steady state. Under what conditions is this steady state locally stable? As suggested by Figure 13 stability requires that the saving locus intersects the 45^0-line from above, provided the locus is upward sloping. A necessary and sufficient condition for local stability at the assumed unique steady state k^* is that

$$\left|\frac{-s_{w_t}(w(k^*), r(k^*))f''(k^*)k^*}{1+n-s_{r_{t+1}}(w(k^*), r(k^*))f''(k^*)}\right| < 1$$

If $s_{r_{t+1}} < 0$ it may be possible that the slope of the saving locus is negative. Under the condition above the steady state is still locally stable, but it exhibits oscillatory dynamics. If we require that the unique steady state is locally stable and that the dynamic equilibrium is characterized by monotonic adjustment to the unique steady state we need as necessary and sufficient condition

$$0 < \frac{-s_{w_t}(w(k^*), r(k^*))f''(k^*)k^*}{1+n-s_{r_{t+1}}(w(k^*), r(k^*))f''(k^*)} < 1$$

The procedure to make sufficient assumptions that guarantee the existence of a well-behaved dynamic equilibrium and then use exactly these assumption to deduce qualitative comparative statics results (how does the steady state change as we change δ, n or the like) is called Samuelson's correspondence principle, as often exactly the assumptions that guarantee monotonic local stability are sufficient to draw qualitative comparative statics conclusions. Diamond (1965) uses Samuelson's correspondence principle extensively and we will do so, too, assuming from now on that above inequalities hold.

8.3.3 Optimality of Allocations

Before turning to Diamond's (1965) analysis of the effect of public debt let us discuss the dynamic optimality properties of competitive equilibria. Consider first steady state equilibria. Let c_1^*, c_2^* be the steady state consumption levels when young and old, respectively, and k^* be the steady state capital labor ratio. Consider the goods market clearing (or resource constraint)

$$N_t^t \hat{c}_t^t + N_t^{t-1}\hat{c}_t^{t-1} + \hat{K}_{t+1} - (1-\delta)\hat{K}_t = F(\hat{K}_t, \hat{L}_t)$$

Divide by $N_t^t = \hat{L}_t$ to obtain

$$\hat{c}_t^t + \frac{\hat{c}_t^{t-1}}{1+n} + (1+n)\hat{k}_{t+1} - (1-\delta)\hat{k}_t = f(k_t) \tag{8.33}$$

and use the steady state allocations to obtain

$$c_1^* + \frac{c_2^*}{1+n} + (1+n)k^* - (1-\delta)k^* = f(k^*)$$

Define $c^* = c_1^* + \frac{c_2^*}{1+n}$ to be total (per worker) consumption in the steady state. We have that

$$c^* = f(k^*) - (n+\delta)k^*$$

Now suppose that the steady state equilibrium satisfies

$$f'(k^*) - \delta < n \tag{8.34}$$

something that may or may not hold, depending on functional forms and parameter values. We claim that this steady state is not Pareto optimal. The intuition is as follows. Suppose that (8.34) holds. Then it is possible to decrease the capital stock per worker marginally, and the effect on per capita consumption is

$$\frac{dc^*}{dk^*} = f'(k^*) - (n+\delta) < 0$$

so that a marginal decrease of the capital stock leads to higher available overall consumption. The capital stock is inefficiently high; it is so high that its marginal productivity $f'(k^*)$ is outweighed by the cost of replacing depreciated capital, δk^* and provide newborns with the steady state level of capital per worker, nk^*. In this situation we can again pull the Gamov trick to construct a Pareto superior allocation. Suppose the economy is in the steady state at some arbitrary date t and suppose that the steady state satisfies (8.34). Now consider the alternative allocation: at date t reduce the capital stock per worker to be saved to the next period, k_{t+1}, by a marginal $\Delta k^* < 0$ to $k^{**} = k^* + \Delta k^*$ and keep it at k^{**} forever. From (8.33) we obtain

$$c_t = f(k_t) + (1-\delta)k_t - (1+n)k_{t+1}$$

The effect on per capita consumption from period t onwards is

$$\begin{aligned}
\Delta c_t &= -(1+n)\Delta k^* > 0 \\
\Delta c_{t+\tau} &= f'(k^*)\Delta k^* + [1-\delta-(1+n)]\Delta k^* \\
&= \left[f'(k^*) - (\delta+n)\right]\Delta k^* > 0
\end{aligned}$$

In this way we can increase total per capita consumption in every period. Now we just divide the additional consumption between the two generations alive in a given period in such a way that make both generations better off, which is straightforward to do, given that we have extra consumption goods to distribute in every period. Note again that for the Gamov trick to work it is crucial to have an infinite hotel, i.e. that time extends to the infinite future. If there is a last generation, it surely will dislike losing some of its final period capital (which we assume is eatable as we are in a one sector economy where the good is a consumption as well as investment good). A construction of a Pareto superior allocation wouldn't be possible. The previous discussion can be summarized in the following proposition

Proposition 105 *Suppose a competitive equilibrium converges to a steady state satisfying* (8.34). *Then the equilibrium allocation is not Pareto efficient, or, as often called, the equilibrium is dynamically inefficient.*

When comparing this result to the pure exchange model we see the direct parallel: an allocation is inefficient if the interest rate (in the steady state) is smaller than the population growth rate, i.e. if we are in the Samuelson case. In fact, we repeat a much stronger result by Balasko and Shell that we quoted earlier, but that also applies to production economies. A feasible allocation is an allocation $c_1^0, \{c_t^t, c_{t+1}^t, k_{t+1}\}_{t=1}^{\infty}$ that satisfies all negativity constraints and the resource constraint (8.33). Obviously from the allocation we can reconstruct s_t^t and K_t. Let $r_t = f'(k_t)$ denote the marginal products of capital per worker. Maintain all assumptions made on U and f and let n_t be the population growth rate from period $t-1$ to t. We have the following result

Theorem 106 *Cass (1972)*[33]*, Balasko and Shell (1980). A feasible allocation is Pareto optimal if and only if*

$$\sum_{t=1}^{\infty} \prod_{\tau=1}^{t} \frac{(1 + r_{\tau+1} - \delta)}{(1 + n_{\tau+1})} = +\infty$$

As an obvious corollary, alluded to before we have that a steady state equilibrium is Pareto optimal (or dynamically efficient) if and only if $f'(k^*) - \delta \geq n$.

That dynamic inefficiency is not purely an academic matter is demonstrated by the following example

[33] The first reference of this theorem is in fact Cass (1972), Theorem 3.

Example 107 *Consider the previous example with log utility, but now with population growth* n *and time discounting* β*. It is straightforward to compute the steady state unique steady state as*

$$k^* = \left[\frac{\beta(1-\alpha)}{(1+\beta)(1+n)}\right]^{\frac{1}{1-\alpha}}$$

so that

$$r^* = \frac{\alpha(1+\beta)(1+n)}{\beta(1-\alpha)}$$

and the economy is dynamically inefficient if and only if

$$\frac{\alpha(1+\beta)(1+n)}{\beta(1-\alpha)} - \delta < n$$

Let's pick some reasonable numbers. We have a 2-period OLG model, so let us interpret one period as 30 years. α *corresponds to the capital share of income, so* $\alpha = .3$ *is a commonly used value in macroeconomics. The current yearly population growth rate in the US is about* 1%*, so lets pick* $(1+n) = (1+0.01)^{30}$*. Suppose that capital depreciates at around* 6% *per year, so choose* $(1-\delta) = 0.94^{30}$*. This yields* $n = 0.35$ *and* $\delta = 0.843$*. Then for a yearly subjective discount factor* $\beta_y \geq 0.998$*, the economy is dynamically inefficient. Dynamic inefficiency therefore is definitely more than just a theoretical curiousum. If the economy features technological progress of rate* g*, then the condition for dynamic inefficiency becomes (approximately)* $f'(k^*) < n+\delta+g$*. If we assume a yearly rate of technological progress of* 2%*, then with the same parameter values for* $\beta_y \geq 0.971$ *we obtain dynamic inefficiency. Note that there is a more immediate way to check for dynamic inefficiency in an actual economy: since in the model* $f'(k^*) - \delta$ *is the real interest rate and* $g+n$ *is the growth rate of real GDP, one may just check whether the real interest rate is smaller than the growth rate in long-run averages.*

If the competitive equilibrium of the economy features dynamic inefficiency its citizens save more than is socially optimal. Hence government programs that reduce national saving are called for. We already have discussed such a government program, namely an unfunded, or pay-as-you-go social security system. Let's briefly see how such a program can reduce the capital stock of an economy and hence leads to a Pareto-superior allocation, provided that the initial allocation without the system was dynamically inefficient.

Suppose the government introduces a social security system that taxes people the amount τ when young and pays benefits of $b = (1+n)\tau$ when old. For simplicity we assume balanced budget for the social security system as well as lump-sum taxation. The budget constraints of the representative individual change to

$$\begin{aligned} c_t^t + s_t^t &= w_t - \tau \\ c_{t+1}^t &= (1 + r_{t+1} - \delta)s_t^t + (1+n)\tau \end{aligned}$$

We will repeat our previous analysis and first check how individual savings react to a change in the size of the social security system. The first order condition for consumer maximization is

$$U'(w_t - \tau - s_t^t) = \beta U'((1 + r_{t+1} - \delta)s_t^t + (1+n)\tau) * (1 + r_{t+1} - \delta)$$

which implicitly defines the optimal saving function $s_t^t = s(w_t, r_{t+1}, \tau)$. Again invoking the implicit function theorem we find that

$$\begin{aligned} & -U''(w_t - \tau - s(.,.,.)) \left(1 + \frac{ds}{d\tau}\right) \\ = \quad & \beta U''((1 + r_{t+1} - \delta)s(.,.,.) + (1+n)\tau) * (1 + r_{t+1} - \delta) * \left((1 + r_{t+1} - \delta)\frac{ds}{d\tau} + 1 + n\right) \end{aligned}$$

or

$$\frac{ds}{d\tau} = s_\tau = -\frac{U''() + (1+n)\beta U''(.)(1 + r_{t+1} - \delta)}{U''(.) + \beta U''(.)(1 + r_{t+1} - \delta)^2} < 0$$

Therefore the bigger the pay-as-you-go social security system, the smaller is the private savings of individuals, holding factor prices constant. This however, is only the partial equilibrium effect of social security. Now let's use the asset market equilibrium condition

$$\begin{aligned} k_{t+1} &= \frac{s(w_t, r_{t+1}, \tau)}{1+n} \\ &= \frac{s\left(f(k_t) - f'(k_t)k_t, f'(k_{t+1}, \tau\right)}{1+n} \end{aligned}$$

Now let us investigate how the equilibrium (k_t, k_{t+1})-locus changes as τ changes. For *fixed* k_t, how does $k_{t+1}(k_t)$ changes as τ changes. Again using the implicit function theorem yields

$$\frac{dk_{t+1}}{d\tau} = \frac{s_{r_{t+1}} f''(k_{t+1}) \frac{dk_{t+1}}{d\tau} + s_\tau}{1+n}$$

Figure 8.7: The Dynamics of the Neoclassical Growth Model

and hence

$$\frac{dk_{t+1}}{d\tau} = \frac{s_\tau}{1 + n - s_{r_{t+1}} f''(k_{t+1})}$$

The nominator is negative as shown above; the denominator is positive by our assumption of monotonic local stability (this is our first application of Samuelson's correspondence principle). Hence $\frac{dk_{t+1}}{d\tau} < 0$, the locus (always under the maintained monotonic stability assumption) tilts downwards, as shown in Figure 14.

We can conduct the following thought experiment. Suppose the economy converged to its old steady state k^* and suddenly, at period T, the government *unanticipatedly* announces the introduction of a (marginal) pay-as-you go system. The saving locus shifts down, the new steady state capital labor ratio declines and the economy, over time, converges to its new steady state. Note that over time the interest rate increases and the wage rate declines. Is the introduction of a marginal pay-as-you-go social security system welfare improving? It depends on whether the old steady state capital-labor ratio was inefficiently high, i.e. it depends on whether $f'(k^*) - \delta < n$ or not. Our conclusions about the desirability of social security remain unchanged from the pure exchange model.

8.3.4 The Long-Run Effects of Government Debt

Diamond (1965) discusses the effects of government debt on long run capital accumulation. He distinguishes between government debt that is held by foreigners, so-called external debt, and government debt that is held by domestic citizens, so-called internal debt. Note that the second case is identical to Barro's analysis if we abstract from capital accumulation and allow altruistic bequest motives. In fact, in Diamond's environment with production, but altruistic and operative bequests a similar Ricardian equivalence result as before applies. In this sense Barro's neutrality result provides the benchmark for Diamond's analysis of the internal debt case, and we will see how the absence of operative bequests leads to real consequences of different levels of internal debt.

External Debt

Suppose the government has initial outstanding debt, denoted in real terms, of B_1. Denote by $b_t = \frac{B_t}{L_t} = \frac{B_t}{N_t^t}$ the debt-labor ratio. All government bonds have maturity of one period, and the government issues new bonds[34] so as to keep the debt-labor ratio constant at $b_t = b$ over time. Bonds that are issued in period $t-1$, B_t, are required to pay the same gross interest as domestic capital, namely $1+r_t-\delta$, in period t when they become due. The government taxes the current young generation in order to finance the required interest payments on the debt. Taxes are lump sum and are denoted by τ. The budget constraint of the government is then

$$B_t(1 + r_t - \delta) = B_{t+1} + N_t^t \tau$$

or, dividing by N_t^t, we get, under the assumption of a constant debt-labor ratio,

$$\tau = (r_t - \delta - n)b$$

For the previous discussion of the model nothing but the budget constraint of young individuals changes, namely to

$$\begin{aligned} c_t^t + s_t^t &= w_t - \tau \\ &= w_t - (r_t - \delta - n)b \end{aligned}$$

In particular the asset market equilibrium condition does not change as the outstanding debt is held exclusively by foreigners, by assumption. As before we obtain a saving function $s(w_t - (r_t - \delta - n)b, r_{t+1})$ as solution to the households optimization problem, and the asset market equilibrium condition reads as before

$$k_{t+1} = \frac{s(w_t - (r_t - \delta - n)b, r_{t+1})}{1+n}$$

Our objective is to determine how a change in the external debt-labor ratio changes the steady state capital stock and the interest rate. This can be answered by examining $s()$. Again we will apply Samuelson's correspondence principle. Assuming monotonic local stability of the unique steady state is equivalent to assuming

$$\frac{dk_{t+1}}{dk_t} = \frac{-s_{w_t}(.,.)f''(k_t)(k_t + b)}{1 + n - s_{r_{t+1}}(.,.)f''(k_{t+1})} \in (0,1) \tag{8.35}$$

[34] As Diamond (1965) let us specify these bonds as interest-bearing bonds (in contrast to zero-coupon bonds). A bond bought in period t pays (interst plus principal) $1+r_{t+1}-\delta$ in period $t+1$.

In order to determine how the saving locus in (k_t, k_{t+1}) space shifts we apply the Implicit Function Theorem to the asset equilibrium condition to find

$$\frac{dk_{t+1}}{db} = \frac{-s_{w_t}(.,.)\,(f'(k_t) - \delta - n)}{1 + n - s_{r_{t+1}}(.,.)f''(k_{t+1})}$$

so the sign of $\frac{dk_{t+1}}{db}$ equals the negative of the sign of $f'(k_t) - \delta - n$ under the maintained assumption of monotonic local stability. Suppose we are at a steady state k^* corresponding to external debt to labor ratio b^*. Now the government marginally increases the debt-labor ratio. If the old steady state was not dynamically inefficient, i.e. $f'(k^*) \geq \delta + n$, then the saving locus shifts down and the new steady state capital stock is lower than the old one. Diamond goes on to show that in this case such an increase in government debt leads to a reduction in the utility level of a generation that lives in the new rather than the old steady state. Note however that, because of transition generations this does not necessarily mean that marginally increasing external debt leads to a Pareto-inferior allocation. For the case in which the old equilibrium is dynamically inefficient an increase in government debt shifts the saving locus upward and hence increases the steady state capital stock per worker. Again Diamond shows that now the effects on steady state utility are indeterminate.

Internal Debt

Now we assume that government debt is held exclusively by own citizens. The tax payments required to finance the interest payments on the outstanding debt take the same form as before. Let's assume that the government issues new government debt so as to keep the debt-labor ratio $\frac{B_t}{L_t}$ constant over time at $\tilde{b}$. Hence the required tax payments are given by

$$\tau = (r_t - \delta - n)\tilde{b}$$

Again denote the new saving function derived from consumer optimization by $s(w_t - (r_t - \delta - n)\tilde{b}, r_{t+1})$. Now, however, the equilibrium asset market condition changes as the savings of the young not only have to absorb the supply of the physical capital stock, but also the supply of government bonds newly issued. Hence the equilibrium condition becomes

$$N_t^t s(w_t - (r_t - \delta - n)\tilde{b}, r_{t+1}) = K_{t+1} + B_{t+1}$$

or, dividing by $N_t^t = L_t$, we obtain

$$k_{t+1} = \frac{s(w_t - (r_t - \delta - n)\tilde{b}, r_{t+1})}{1+n} - \tilde{b}$$

To determine the shift in the saving locus in (k_t, k_{t+1}) we again implicitly differentiate to obtain

$$\frac{dk_{t+1}}{d\tilde{b}} = \frac{-s_{w_t}(.,.)(r_t - \delta - n) + s_{r_{t+1}} f''(k_{t+1}) \frac{dk_{t+1}}{d\tilde{b}}}{1+n} - 1$$

and hence

$$\frac{dk_{t+1}}{d\tilde{b}} = \frac{-s_{w_t}(.,.)(f'(k_t) - \delta - n) - (1+n)}{(1+n) - s_{r_{t+1}} f''(k_{t+1})}.$$

Now we again assume that

$$\frac{dk_{t+1}}{dk_t} = \frac{-s_{w_t}(.,.)f''(k_t)(k_t + \tilde{b})}{1 + n - s_{r_{t+1}}(.,.)f''(k_{t+1})} = \phi(k_t, k_{t+1}, \tilde{b}) \in (0,1)$$

to assure monotonic stability of the steady state. Then

$$\begin{aligned} \frac{dk_{t+1}}{d\tilde{b}} &= \frac{[-s_{w_t}(.,.)(f'(k_t) - \delta - n) - (1+n)]}{(1+n) - s_{r_{t+1}} f''(k_{t+1})} \\ &= \left[\frac{\phi(k_t, k_{t+1}, \tilde{b})}{-f''(k_t) s_{w_t}(.,.)(k_t + \tilde{b})} \right] \cdot [s_{w_t}(.,.)(\delta + n - f'(k_t)) - (1+n)] \end{aligned}$$

The term in the first brackets is positive since $\phi \in (0,1), s_w \in (0,1)$ and $-f''(k_t)(k_t + \tilde{b}) > 0$. The term in the second brackets is negative since

$$s_{w_t}(.,.)(\delta + n - f'(k_t)) \leq s_{w_t}(.,.)(\delta + n) < 1 + n.$$

Thus

$$\frac{dk_{t+1}}{d\tilde{b}} < 0.$$

The $k_{t+1}(k_t)$ curve unambiguously shifts down with an increase in internal debt $\tilde{b}$, leading to a decline in the steady state capital stock per worker. Diamond, again only comparing steady state utilities, shows that if the initial steady state was dynamically efficient, then an increase in internal debt leads to a reduction in steady state welfare, whereas if the initial steady state was dynamically inefficient, then an increase in internal government debt leads to a increase in steady state welfare. Here the intuition is again clear: if the economy has accumulated too much capital, then increasing the supply of alternative assets leads to a interest-driven "crowding out" of demand for physical capital, which is a good thing given that the economy possesses too much capital. In the efficient case the reverse logic applies. In comparison with the external debt case we obtain clearer welfare conclusions for the dynamically inefficient case. For external debt an increase in debt is not necessarily good even in the dynamically inefficient case because it requires higher tax payments, which, in contrast to internal debt, leave the country and therefore reduce the available resources to be consumed (or invested). This negative effect balances against the positive effect of reducing the inefficiently high capital stock, so that the overall effects are indeterminate. In comparison to Barro (1974) we see that without operative bequests the level of outstanding government bonds influences real equilibrium allocations: Ricardian equivalence breaks down.