Chris Ackerman 201B Midterm

(1) a) Set arbitrary market size = 1

$$u_i = q_i\left(1 - \sum_{i=1}^{I} q_i\right) - q_i c_i$$

$$BR_i(q_{-i}) \in \begin{cases} \phi \\ \frac{\partial u_i}{\partial q_i}(q_{-i}, BR_i(q_{-i})) = 0 \end{cases}$$

$c_i \geq 1 \Rightarrow$ no production is a NE

suppose $c_i < 1$ for some i

$$\frac{\partial u_i}{\partial q_i} = 1 - 2q_i - \sum_{j \neq i}^{I} q_j - c_i$$

$$0 = 1 - 2q_i - \sum_{j \neq i}^{I} q_j^* - c_i$$

$$1 = 2q_i + \sum q_j^* + c_i$$

$$q_i = \frac{1 - \sum q_j^* - c_i}{2}$$

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This optimality condition defines a system of N equations in N unknowns.

$$q_i = \frac{1-(n-1)q_i - c_i}{2}$$

$$2q_i = 1 - nq_i + q_i - c_i$$

$$(n+1)q_i = 1 - c_i$$

$$q_i^* = \frac{1-c_i}{n+1}$$

<u>Symmetric pure strategy NE:</u>

$$q_i^* = \max\left\{0, \frac{1-c_i}{n+1}\right\} \quad \forall\, i$$

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b) Yes, there are. Any firm i can threaten to produce more than q_i^* in the previous case. If other firms think this is a credible threat, they will decrease their production accordingly, and the BR definition from above will hold.

c) For mixed strategies, a firm needs to be indifferent between playing the strategies they are mixing over.

$$E[\pi(q_i)] = q_i\left(1 - \sum_{j=1}^{I} q_j\right) - q_i c_i$$

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Firms only produce when

$$q_i\left(1-\sum_{j=1}^{I} q_j\right) \geq q_i c_i .$$

If c_i is the same for all firms and $n \to \infty$, $p = c_i \Rightarrow E[\pi(q_i)] = 0$.

If there are infinitely many firms with identical marginal cost, the free entry condition ensures that firms are indifferent between producing and not producing, so this symmetric mixed strategy is a Nash equilibrium. Since there are positive profits for $n < \infty$, there is no symmetric mixed strategy equilibrium with finite firms

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d) $p^*(n) = 1 - nq^*(n)$

$$= 1 - \cancel{n} \cdot \frac{1-c_i}{\cancel{n+1}}$$

$\lim_{n\to\infty}$

$$= 1 - 1 + c_i$$

$$\underline{\lim_{n\to\infty} p^*(n) = c_i}$$

$$\Pi_0^*(n) = p_n^* \cdot q_n - c_i q_n$$

$\lim\ n\to\infty$

$$= c_i \cdot 0 - c_i \cdot 0$$

$$\lim_{n\to\infty} \Pi_0^*(n) \underline{= 0}$$

$$\Pi^*(n) = n\, p^*(n)\, q_n - n c_i q_i$$

$$= \frac{1-c_i}{\cancel{n+1}} \cdot \cancel{n} \cdot (1-Q_n) - \frac{1-c_i}{\cancel{n+1}} \cdot \cancel{n} \cdot c_i$$

$$= 1 - Q_n - 1 - Q_n c_i - c_i + c_i^2$$

$$= -Q_n - Q_n c_i - c_i + c_i^2$$

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② a) The k firms with the lowest index (i.e. the lowest marginal cost) make positive profits, $\pi_i(q^*) > 0$ for $i \in \{1, 2, \ldots, k\}$.

b)

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Start w/ $n=2$, $k=1$. Firm 1 can play q_1^* such that $p^* = c_2^*$. Firm 2 can play

$$(\sigma_2) = \begin{cases} 0 & \text{if } q_1 \text{ s.t. } p^* = c_2 \\ 1 & \text{otherwise} \end{cases}$$

Firm 1 is doing as well as possible given firm 2's "threat", and firm 2 makes negative profit if they enter, since $c_1 < c_2$, firm 1 makes positive profits at $p^* = c_2$. To generalize this case, suppose all firms with index $\leq k$ play $\frac{1}{k} Q^*$, where Q^* is chosen such that $p^*(Q^*) = c_{k+1}$. Suppose also that all firms with index $> k$ play $\begin{cases} 0 & \text{if } Q \text{ s.t. } p^* = c_{k+1} \\ 1 & \text{otherwise.} \end{cases}$ Ackerman

Again, the first k firms make positive profits, but are punished by deviating, and firms with higher indexes cannot enter to get themselves positive profits, so this is a NE.

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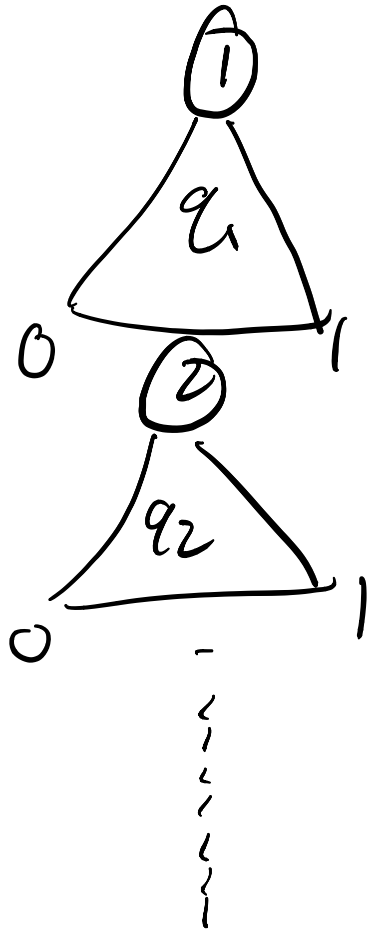
3) a) This is a variant of Stackelberg.



Firm n can condition strategies on $\{q_1, q_2, \ldots, q_{n-1}\}$, but in practice only cares about $s_n\left(\sum_{i=1}^{n-1} q_i\right)$.

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For $n=2$, $k=1$. Essentially the first firm faces a monopolist's problem and chooses

$$q_1 = \frac{1-c_1}{2}.$$

Firm 2 has to take q_1 as given, and then chooses

$$q_2 = \frac{1-q_1-c_1}{2}.$$

The process continues on but each new firm faces the effective market $1-\sum_{i=1}^{n-1} q_i$. Since the effective market (and hence expected profits) is weakly decreasing in the index i (since the sum is weakly increasing) and the marginal cost is strictly increasing for each new firm (by assumption), once firm $k+1$

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finds it unprofitable to enter the market, so does $k+2$, $k+t$ $\forall t$ s.t. $k+t \leq n$.

Thus the k firms with the lowest index and marginal cost make positive profits.

b) We can't use the arbitrary threats from question (2) because they violate subgame perfection. Suppose $n \geq 2$, $k=1$.

$$q_1^* = \frac{1-c_1}{2}$$

But for some values of $1 > c_2 > c_1$,

$$q_2^* = \frac{1-q_1-c_1}{2}$$ yields positive profits.

Thus there is no SPNE with exactly $k=1$ firms making positive profits. (ii) $\rightarrow$ no such array exists

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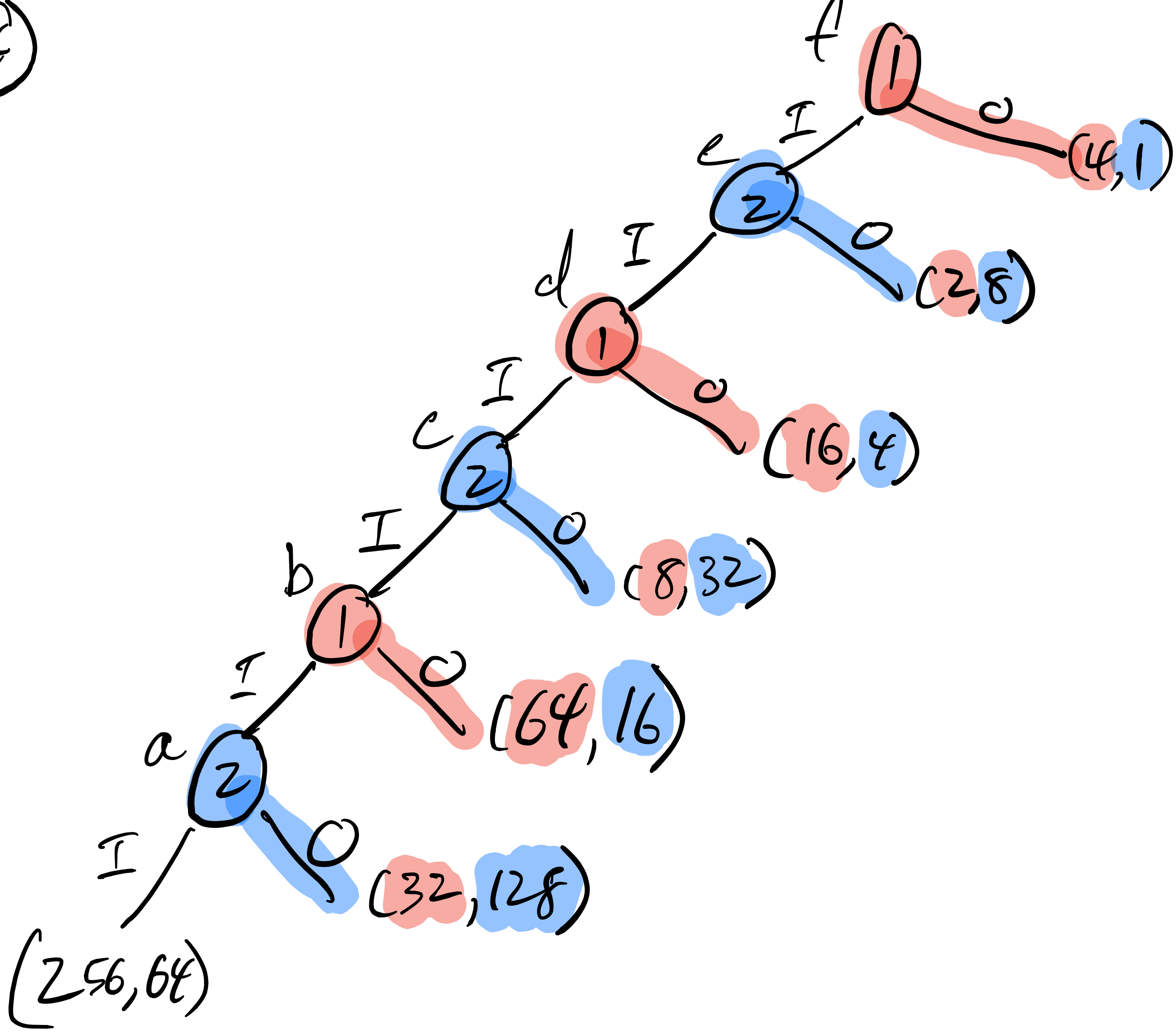
(4)



a) Starting @ node a and working back to node f, I've highlighted the optimal action for each player in each subgame. The unique SPNE of this game is

$s_1 = \{out, out, out\}$, $s_2 = \{out, out, out\}$.

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The definition of subgame perfect equilibrium requires each player's strategy to constitute a Nash equilibrium in each subgame. Since each subgame has a unique Nash equilibrium, the SPNE is unique.

b) NE, unlike SPGE, does not require off-path play to be optimal. Thus

$s_1 = \{out, out, in\}$, $s_2 = \{out, out, out\}$

$s_1 = \{out, out, out\}$ $s_2 = \{out, out, in\}$

$s_1 = \{out, out, in\}$ $s_2 = \{out, out, in\}$

are all PSNE. All of these

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strategies yield payoffs (4,1), and neither player can increase their payoffs by deviating, so these sets of strategy profiles are all NE.

c) Suppose not. Then Player 2 will have a chance to play at node e in my picture.

$$\pi_2(out_e) = 8$$

From the SGPE argument earlier, we know that player 1 will want to play out at node d. Thus Player 2's expected payoff from playing in at node e,

$$\pi_2(in_e) = \pi_2(in_e, out_d) = 4 < \pi_2(out_e) = 8$$

So player 2 <u>always</u> wants to play

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out at node e. Player 1's decision at the first node is then between

$$a_1 \in \{out, in\}.$$

$$\pi_1(out_f) = 4$$

$$\pi_1(in_f) = \pi_1(in_f, out_e) = 2$$

$$\pi_1(in_f) < \pi_1(out_f) \qquad (*)$$

Thus Player 1 must <u>always</u> play out at the first node to satisfy the definition of a NE. Since (*) is a strict inequality, player 1 cannot mix at the first node.

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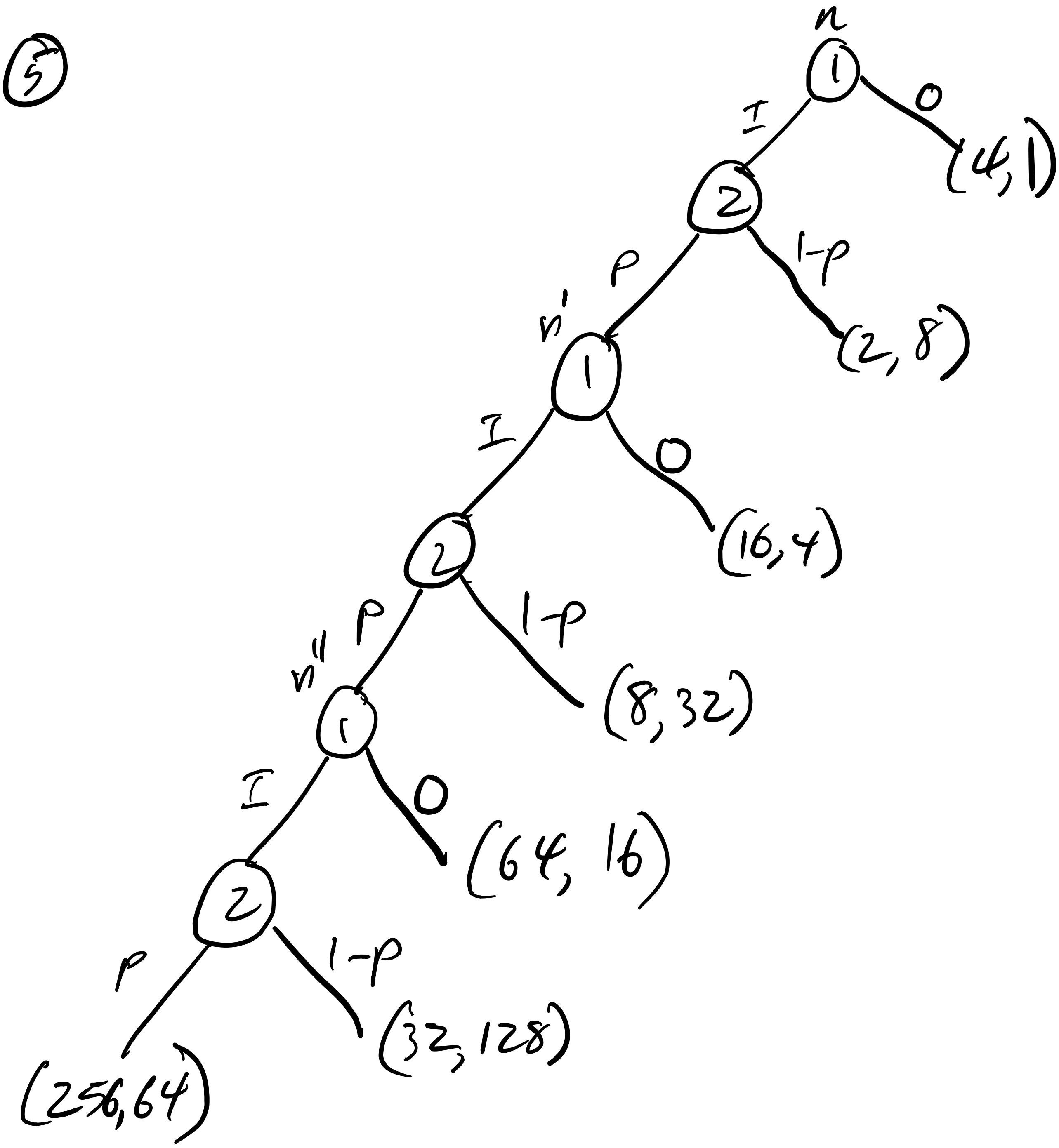

If Player 1 always plays In, expected payoffs are:

$$p^3 256 + p^2(1-p)32 + p(1-p)8 + (1-p)2$$

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See what p we need to induce In, then check if player 9 ever wants to deviate.

$$E[\pi(in, in, in)] \geq E[\pi(out; \cdot)]$$

$$p^3 256 + p^2(1-p)32 + p(1-p)8 + (1-p)2 \geq 4$$

$$64p^3 + 8p^2(1-p) + 2p(1-p) + \frac{1-p}{2} - 1 \geq 0$$

$$\Rightarrow p \geq \frac{1}{7}$$

Now check the condition for person 1's second action, the subgame starting at n'

$$E[\pi(in, in, in)] \geq E[\pi(in, out; \cdot)]$$

$$p^2 \cdot 256 + p(1-p)32 + (1-p)8 \geq 16$$

$$\Rightarrow p \geq \frac{1}{7}$$

Now check the subgame at n''.

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$$E[\pi(in, in, in)] \geq E[\pi(in, in, out)]$$

$$256p + (1-p)32 \geq 64$$

$$\Rightarrow p \geq \frac{1}{7}$$

Optimal strategy:

$$s_1 = \begin{cases} \{in, in, in\} & \text{if } x > \frac{1}{7} \\ \{out, out, out\} & \text{if } x \leq \frac{1}{7} \end{cases}$$

s_1 is an optimal strategy, but there are others. For $x = \frac{1}{7}$, any strategy is optimal. For $x < \frac{1}{7}$, $\{out, out, in\}$ and $\{out, in, in\}$ are also optimal because they yield identical payoffs.

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b) i) Type 0 machine is the same as $S_2 = \{out, out, out\}$. In this case,

$$S_2 = p = 0 \; ; \; S_1 = \begin{cases} \{out, out, out\} \\ \{out, out, in\} \\ \{out, in, in\} \end{cases}$$

S_2, along with one of the elements of S_1, constitute all pure strategy Nash equilibrium where player 2 chooses $p = 0$. The reasoning here is similar to 4(b). In any NE where player 2 always plays out, player 1 wants to play out at node 1. The set of strategies for player 1 that satisfy the definition of PSNE are all S_1 with "out" as the first action.

Now consider $p=1$. This is the same as $s_2 = \{in, in, in\}$. We want to find $s_1 \in BR_1(s_2)$.

$E[\pi(in,in,in)] = 256$, which is the maximum payoff in this game. Now we need to check $s_2 = p=1 \in BR_2(s_1)$.

$$E[\pi(s_2)|s_1] = p^3 64 + p^2(1-p)128 + p(1-p)32 + (1-p)8$$

$$p=1 \Rightarrow \quad = 64$$

$$\frac{d}{dp} = -24(8p^2 - 8p - 1)$$

$$\text{at } p=1 = -24(8-8-1)$$

$$= 24$$

$$> 0$$

Player 2's expected payoff is increasing in p at $p=1$, so the only profitable deviation is to $p > 1$. But since $p \in [0,1]$, this is impossible. Thus

$s_2 = p = 1$, $s_1 = \{in, in, in\}$ is a NE.

ii) We know $s_1 = \{in, in, in\}$ if $p > \frac{1}{7}$, and player 2's expected payoff is increasing in p over $p \in (\frac{1}{7}, 1]$, so $\nexists$ PSNE with $p \in (\frac{1}{7}, 1)$. Now let's check $p < \frac{1}{7}$. We can fix $s_1 = \{out, out, out\}$. Then $E[\Pi_2(p=0)] = 1$.

Let's check for profitable deviations.

$$E[\Pi_2(p) \mid s_1 = \{out, out, out\}] = 1 \ \forall p \in [0, \tfrac{1}{7})$$

since player 2 never gets to play.

Thus $s_1 = \{out, out, out\}$, $s_2 = p \in [0, \frac{1}{7})$ is a NE $\forall\, p \in [0, 1/7)$.

$p = 1/7$ cannot be a NE because

$p' = 1/7 + \varepsilon \Rightarrow s_1 = \{in, in, in\} \Rightarrow$

$E[\pi(p')] > E[\pi(p)]$.

This exhausts all possible pure strategies, so I have found all PSNE in this game.