Chris Ackerman Econ 203 B HW2

① Chapter 2, Q4

Consider $\bar{X}_n' \hat{\beta}_n$ (1)

<u>Consistency</u>

LLN $\Rightarrow$ $\bar{X}_n' \xrightarrow{p} E[X]'$

Given our OLS assumptions,

$\hat{\beta}_n \xrightarrow{p} \beta_0$ (2)

(1)+(2) : $(\bar{X}_n', \hat{\beta}_n) \xrightarrow{p} (E[X]', \beta_0)$

now apply CMT $\Rightarrow$

$$\bar{X}_n' \hat{\beta}_n \xrightarrow{p} E[X]' \beta_0$$

Distribution

WTS: $\sqrt{n}(\bar{X}_n \hat{\beta}_n - E[X]'\beta_0) \xrightarrow{d} N(0, \sigma^2)$

$U_i \equiv Y_i - X_i'\beta_0$

$\sqrt{n}(\bar{X}_n' \hat{\beta}_n - E[X]'\beta_0) =$

$$\sqrt{n}(\bar{X}_n'(\mathbb{X}_n'\mathbb{X}_n)^{-1}\mathbb{X}_n'\mathbb{Y}_n - E[X]'\beta_0) + o_p(1)$$

$= \sqrt{n}(\bar{X}_n'(\mathbb{X}_n'\mathbb{X}_n)^{-1}\mathbb{X}_n'(\mathbb{X}_n\beta_0 + \mathbb{U}_n) - E[X]'\beta_0) + o_p(1)$

$= \sqrt{n}(\bar{X}_n' - E[X]')\beta_0 + \sqrt{n}(\bar{X}_n'(\mathbb{X}_n'\mathbb{X}_n)^{-1}\mathbb{X}_n\mathbb{U}_n) + o_p(1)$

$= \sqrt{n}(\bar{X}_n' - E(X)')\beta_0 + \bar{X}_n'(\frac{1}{n}\mathbb{X}_n'\mathbb{X}_n)^{-1}\sqrt{n}\frac{1}{n}(\mathbb{X}_n'\mathbb{U}_n) + o_p(1)$

$$\text{CMT} \Rightarrow \sqrt{n}\begin{bmatrix} \bar{X}_n' - E[X]' \\ \frac{1}{n}(\mathbb{X}_n'\mathbb{U}_n) \end{bmatrix} \xrightarrow{d} N(0, \Sigma)$$

LLN & CMT $\Rightarrow$

$$\begin{bmatrix} \beta_0 \\ \bar{X}_n'(\frac{1}{n}\mathbb{X}_n'\mathbb{X}_n)^{-1} \end{bmatrix} \xrightarrow{p} \begin{bmatrix} \beta_0 \\ E[X]'E[X'X]^{-1} \end{bmatrix}$$

$\hat{r}$ r

Slutsky's Thm $\Rightarrow$

$$\sqrt{n}\left(\bar{X}_n' - E[X]'\right)\beta_0 + \bar{X}_n'\left(\tfrac{1}{n}\mathbb{X}_n'\mathbb{X}_n\right)^{-1}\sqrt{n}\,\tfrac{1}{n}\left(\mathbb{X}_n'\mathbb{U}_n\right)$$

$$= \hat{r} \cdot \sqrt{n}\begin{bmatrix} X_n' - E[X]' \\ \frac{1}{n}\left(\mathbb{X}_n'\mathbb{U}_n\right] \end{bmatrix}$$

$$\xrightarrow{d} N(0, r'\Sigma r)$$

Thus

$$\sqrt{n}\left(\bar{X}_n'\hat{\beta}_n - E[X]'\beta_0\right) \xrightarrow{d} N(0, r'\Sigma r)$$

② Chapter 2, Q9

We want to test two hypotheses,

$$P \equiv \{N(\mu, 1) : \mu \in \{0, 2\}\}$$

$$P_0 \equiv \{N(0,1)\} \quad P_1 = \{N(2,1)\}$$

Type 1 error:

$$\mathbb{E}_{P_0}[\phi_n] \equiv P_0(\phi_n = 1)$$

$$= P_0(W > c) \; ; \; P_0 = N(0,1)$$

$$P_0(W > c) = 1 - \Phi(c)$$

Type II error:

$$1 - \mathbb{E}_{P_1}[\phi_n] \equiv P_1(\phi_n = 0)$$

$$= P_1(W \leq c) \; ; \; P_1 = N(2,1)$$

$$P_1(W \le c) = P_1(W-2 \le c-2)$$
$$= \Phi(c-2)$$

Now choose optimal value for c

$$c^* \equiv \arg\min P_0(\phi_n = 1) + P_1(\phi_n = 0)$$

$$= \arg\min 1 - \Phi(c) + \Phi(c-2)$$

FOC $\Rightarrow$ $\phi(c) = \phi(c-2)$

$$c-2 = -c$$

$$c^* = 1$$

③ Chapter 2, Q10

(a) $\underline{\text{WTS}}$: $E_p[\phi_n] = P\left(\sqrt{n}\,\frac{\bar{W}_n}{2} > c_{1-\alpha}\right)$

If $P \in P_0$, CMT $\Rightarrow$

$$\sqrt{n}\,\frac{\bar{W}_n - E_p(W)}{\sqrt{Var(W)}} = \sqrt{n}\,\frac{\bar{W}_n}{2}$$

$$\xrightarrow{d} N(0,1)$$

$$\lim_{n\to\infty} E_p[\phi_n] = \lim_{n\to\infty} P\left(\sqrt{n}\,\frac{\bar{W}_n}{2} > c_{1-\alpha}\right)$$
$$\leq \alpha$$

$$\Rightarrow \sup_{P\in P_0} \lim_{n\to\infty} E_p[\phi_n] \leq \alpha$$

(b) No. Take $E_p[W] = -\varepsilon$; $\varepsilon > 0$

$$\lim_{n\to\infty} E_p[\phi_n] = \lim_{n\to\infty} P\left(\frac{\sqrt{n}(\bar{W}_n - E_p(W))}{2} > c_{1-\alpha} + \frac{\sqrt{n}\varepsilon}{2}\right)$$

$$\leq \lim P\left(\frac{\sqrt{n}(\bar{W}_n - E_p(W))}{2} > c_{1-\alpha} + \frac{\varepsilon}{2}\right)$$

$$= 1 - \Phi\left(c_{1-\alpha} + \frac{\varepsilon}{2}\right)$$

$$< 1$$

(c) WTS: $E_p[\phi_n] = P\left(\sqrt{n}\frac{\bar{W}_n}{2} > c_{1-\alpha/2}\right) + P\left(\sqrt{n}\frac{\bar{W}_n}{2} < -c_{1-\alpha/2}\right)$

If $P \in \mathcal{P}_0$, CLT $\Rightarrow$

$$\sqrt{n}\,\frac{\bar{W}_n - E_p(W)}{\sqrt{Var(W)}} = \sqrt{n}\,\frac{\bar{W}_n}{2}$$

$$\xrightarrow{d} N(0,1)$$

$$\lim_{n\to\infty} E_P(\tilde{\phi}_n) = \lim_{n\to\infty} P\left(\sqrt{n}\frac{w_n}{2} > c_{1-\alpha/2}\right) + P\left(\sqrt{n}\frac{w_n}{2} < -c_{1-\alpha/2}\right)$$

$$= P(Z > c_{1-\alpha/2}) + P(Z < -c_{1-\alpha/2})$$

$$\leq \frac{\alpha}{2} + \frac{\alpha}{2}$$

$$= \alpha$$

$$\sup_{P \in \mathbf{P}_0} \lim_{n\to\infty} E_P[\tilde{\phi}_n] \leq \alpha$$

(d) Compute local power

$$\lim_{n\to\infty} E_{P_n}[\phi_n] = \lim_{n\to\infty} P_n\left(\sqrt{n}\frac{\bar{w}_n}{2} > c_{1-\alpha}\right)$$

$$= \lim P_n\left(\frac{\sqrt{n}(\bar{w}_n - \lambda)}{2} > c_{1-\alpha} - \frac{\lambda}{2}\right)$$

$$= 1 - \Phi\left(c_{1-\alpha} - \frac{\lambda}{2}\right)$$

$$\lim E_{P_n}[\tilde{\phi}_n] = \lim P\left(\left|\sqrt{n}\frac{\bar{w}_n}{2}\right| > c_{1-\alpha/2}\right)$$

$$= \lim P\left(\frac{\sqrt{n}(\bar{w}_n - \frac{\lambda}{\sqrt{n}})}{2} > c_{1-\alpha/2} - \frac{\lambda}{2}\right) +$$

$$P\left(-\sqrt{n}\left(\frac{\bar{W}_n - \bar{Z}_n}{2}\right) < -c_{1-\alpha/2} - \frac{\lambda}{2}\right)$$

$$= 2\Phi\left(-c_{1-\alpha/2} - \frac{\lambda}{2}\right)$$

$\tilde{\varphi}_n$ has greater local power when

$$2\Phi\left(-c_{1-\alpha/2} - \frac{\lambda}{2}\right) + \Phi\left(c_{1-\alpha} - \frac{\lambda}{2}\right) - 1 > 0$$

which happens when $\lambda < 0$.

④ Chapter 2, Question 12

OLS II $\Rightarrow \sqrt{n}(\hat{\beta}_n - \beta_0) \xrightarrow{d} N(0, \Sigma_0)$

Since ∇f continuous, Delta method $\Rightarrow$

$$\sqrt{n}(f(\hat{\beta}_n) - f(\beta_0)) \xrightarrow{d} N(0, \nabla f(\beta_0)\Sigma_0(\nabla f(\beta_0))')$$

CMT implies

$$\nabla f(\beta_0)\hat{\Sigma}_n(\nabla f(\beta_0))' \xrightarrow{P} \nabla f(\beta_0)\Sigma_0(\nabla f(\beta_0))'$$

Since $\nabla f(\beta_0)\Sigma_0\nabla f(\beta_0)'$ is invertible

$$\left(\nabla f(\beta_0)\hat{\Sigma}_n(\nabla f(\beta_0))'\right)^{-1/2} \xrightarrow{p} \left[\nabla f(\beta_0)\Sigma_0(\nabla f(\beta_0))'\right]^{-1/2}$$

Use Slutsky's Theorem:

$$\left(\nabla f(\hat{\beta}_n)\hat{\Sigma}_n \nabla f(\hat{\beta}_n)'\right)^{-1/2}\sqrt{n}f(\hat{\beta}_n) \xrightarrow{d}$$

$$N\left(0, \left[\nabla f(\beta_0)\Sigma_0(\nabla f(\beta_0))'\right]^{-1/2}\nabla f(\beta_0)\Sigma_0(\nabla f(\beta_0))'\left[\nabla f(\beta_0)\Sigma_0(\nabla f(\beta_0))'\right]^{-1/2}\right)$$

$$= N(0, I_p)$$

CMT $\Rightarrow$

$$\left\|\left(\nabla f(\hat{\beta}_n)\hat{\Sigma}_n \nabla f(\hat{\beta}_n)'\right)^{-1/2}\sqrt{n}f(\hat{\beta}_n)\right\|^2 \xrightarrow{d} \sum_{j=1}^{p} Z_j^2$$

$$\sim \chi_p^2$$

$f(\beta_0) = 0 \Rightarrow$

$$\lim_{n\to\infty} P\left(\left\|\left(\nabla f(\hat{\beta}_n)\hat{\Sigma}_n \nabla f(\hat{\beta}_n)'\right)^{-1/2}\sqrt{n}f(\hat{\beta}_n)\right\|^2 > c_{1-\alpha}\right) = \alpha$$

⑤ Chapter 2, Q13

(a) iid observations & CLT $\Rightarrow$

$$\sqrt{n}\left(\begin{bmatrix}\bar{Y}_n^A \\ \bar{Y}_n^B\end{bmatrix} - \begin{bmatrix}\mu^A \\ \mu^B\end{bmatrix}\right) \xrightarrow{d} N\left(0, \begin{bmatrix}\sigma_A^2 & 0 \\ 0 & \sigma_B^2\end{bmatrix}\right)$$

Define $f = \frac{\cancel{X}}{Y} - 1$ and assume $\mu_B \neq 0$. Delta Method $\Rightarrow$

$$\sqrt{n}\left(f(\bar{Y}_n^A, \bar{Y}_n^B) - f(\mu_A, \mu_B)\right) = \sqrt{n}\left(\frac{\bar{Y}_n^A}{\bar{Y}_n^B} - 1 - \left(\frac{\mu_A}{\mu_B} - 1\right)\right)$$

$$= \sqrt{n}\left(\hat{\eta}_n - \left(\frac{\mu_A}{\mu_B} - 1\right)\right)$$

$$\xrightarrow{D} N\left(0, \nabla f(\mu_A, \mu_B)\begin{bmatrix}\sigma_A^2 & 0 \\ 0 & \sigma_B^2\end{bmatrix}\nabla f(\mu_A, \mu_B)'\right)$$

$$= N\left(0, \frac{\sigma_A^2}{\mu_B^2} + \frac{\mu_A^2 \sigma_B^2}{\mu_B^4}\right)$$

$$\Rightarrow \hat{\sigma}_n = \frac{\sqrt{\frac{\sigma_A^2}{\mu_B^2} + \frac{\mu_A^2 \sigma_B^2}{\mu_B^4}}}{\sqrt{n}}$$

Plugging in the sample analogs for $\hat{S}_n$, we get $\hat{s}_n = 0.28905$

(b) $\hat{\eta}_n = \dfrac{\bar{Y}_n^A - \bar{Y}_n^B}{\bar{Y}_n^B}$

$= \dfrac{\bar{Y}_n^A}{\bar{Y}_n^B} - 1$

$\xrightarrow{P} \dfrac{\mu^A}{\mu^B} - 1 \equiv \eta$

Define our confidence interval as

$$C_n \equiv \left[\frac{\hat{\eta}_n}{0.25} \pm c_{1-\alpha/2} \frac{\hat{s}_n}{0.25}\right]$$

$$\lim_{n\to\infty} P\left(\frac{\eta}{0.25} \in C_n\right) = \lim P\left(\frac{\hat{\eta}_n}{0.25} - c_{1-\alpha/2}\frac{\hat{s}_n}{0.25} \le \frac{\eta}{0.25} \le \frac{\hat{\eta}_n}{0.25} + c_{1-\alpha/2}\frac{\hat{s}_n}{0.25}\right)$$

$$= \lim P\left(\hat{\eta}_n - c_{1-\alpha/2}\hat{s}_n \le \eta \le \hat{\eta}_n + c_{1-\alpha/2}\hat{s}_n\right)$$

$$= \lim P\left(-c_{1-\alpha/2} \le \frac{\hat{\eta}_n - \eta}{\hat{s}_n} \le c_{1-\alpha/2}\right)$$

$$= 1 - \alpha$$

Take $\alpha = 0.05$, $c_{1-\alpha/2} = 1.96$, $\hat{\eta}_n = 0.3745$

$$\Rightarrow C_n = \left[-0.7682,\ 3.7642 \right]$$