# Game Form

A *game form* $\Gamma$ consists of

- players $N = \{1, \ldots, n\}$
- actions/plans/pure strategies $A_i$
  Notation
  - $A = A_1 \times \ldots \times A_n$
  - $A_{-i} = A_1 \times \ldots A_{i-1} \times A_{i+1} \times \ldots \times A_n$
  - $a = (a_1, \ldots, a_n) \in A$
  - $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \in A_{-i}$
- consequences $C$
- outcome function $g : A \to C$

# Game in Strategic Form

If we append *utility functions* $v_i : C \to \mathbb{R}$ we obtain a *game in strategic form* $G$

- players $N = \{1, \ldots, n\}$
- actions/plans/pure strategies $A_i$
  Notation
  - $A = A_1 \times \ldots \times A_n$
  - $A_{-i} = A_1 \times \ldots A_{i-1} \times A_{i+1} \times \ldots \times A_n$
  - $a = (a_1, \ldots, a_n) \in A$
  - $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \in A_{-i}$
- utility functions $u_i = v_i \circ g : C \to \mathbb{R}$
  utility mapping $u = (v_1, \ldots, v_n) \circ g : C \to \mathbb{R}^N$

Interpretation: $C = A$; $g = \mathrm{identity}$

# Discussion

What are actions?

- literal actions
- complete plans of play $=$ *pure strategies*
- instructions

# Examples

- Prisoner's Dilemma
- Battle of the Sexes
- Coordination
- Matching Pennies
- Hawk/Dove

# Prisoner's Dilemma

Payoffs in jail time

| | C | D |
|---|---|---|
| C | -1,-1 | -4,0 |
| D | 0,-4 | -3,-3 |

# Prisoner's Dilemma

Payoffs in jail time

| | C | D |
|---|---|---|
| C | -1,-1 | -4,0 |
| D | 0,-4 | -3,-3 |

Payoffs in Utility

| | C | D |
|---|---|---|
| C | 3,3 | 0,4 |
| D | 4,0 | 1,1 |

# Battle of the Sexes

| | Bach | Stravinsky |
|---|---|---|
| Bach | 2,1 | 0,0 |
| Stravinsky | 0,0 | 1,2 |

# Coordination

| | Mozart | Mahler |
|---|---|---|
| Mozart | $2, 2$ | $0, 0$ |
| Mahler | $0, 0$ | $1, 1$ |

# Matching Pennies

| | H | T |
|---|---|---|
| H | $+1, -1$ | $-1, +1$ |
| T | $-1, +1$ | $+1, -1$ |

# Hawk/Dove

| | Dove | Hawk |
|---|---|---|
| Dove | 2,2 | 0,4 |
| Hawk | 4,0 | $-1, -1$ |

# What do Players Do?

Assumptions

- payoffs are *utilities*
- players behave *rationally*
  $\rightarrow$ players optimize $=$ maximize payoffs

# Implications of Optimization

- Prisoner's Dilemma
  D is strictly best for Row (Col) no matter what Col (Row) does; D is a dominant strategy
  $$\text{rationality} \implies \textit{both} \text{ players play D}$$
- Battle of the Sexes
  ?
- Coordination
  ?
- Matching Pennies
  ?
- Hawk/Dove
  ?

# Savage Rationality

Rationality $\rightsquigarrow$ players have beliefs & optimize given those beliefs

What does this imply?

- player's action choice must be a best reply to *some beliefs*
- rules out actions that are never best replies

What should discipline beliefs?

What should discipline beliefs?
Minimum discipline (Pearce, Whinston)

- Common Knowledge of Game
- Common Knowledge of Rationality

# Rationalizability

What should discipline beliefs?
Minimum discipline (Pearce, Whinston)

- Common Knowledge of Game
- Common Knowledge of Rationality

(We will allow mixed beliefs.)

# Rationalizability: Example 1

| | C | D | X |
|---|---|---|---|
| C | 3,3 | 0,4 | 5,2 |
| D | 4,0 | 1,1 | 0,0 |

# Rationalizability is Weak: Example 2

| | H | T |
|---|---|---|
| H | $+1, -1$ | $-1, +1$ |
| T | $-1, +1$ | $+1, -1$ |

Claim (H,T) is rationalizable

- Row plays H because she believes Col will play H (she is wrong of course)
- It would be rational for Col to play H if he believed Row would play T
- It would be rational for Row to play T if she believed Col would play T which would be rational if Col believe Row would play H
- It would be rational for Col to play T if he believed Row would play H which would be rational if she believed Col would play T which would be rational if he believed Row would play H
- etc

# Nash Equilibrium

Stronger (strongest) restriction on beliefs = correctness

**Definition** A profile $(a_1, \ldots, a_n)$ is a *Nash Equilibrium* if: for all $i \in N$ and all $b_i \in A_i$ we have

$$u_i(a_i, a_{-i}) \geq u_i(b_i, a_{-i})$$

Equivalently:

- Each player $i$ optimizes against beliefs about players $j \neq i$
- These beliefs are *correct*

Sometimes we speak about *deviations*: NE means:

*no player has a profitable deviation*

.

Sometimes we speak about *best responses*: NE means:

*each player's strategy is a best response to the strategies of others.*

# Interpretations of Nash Equilibrium

- Steady state of repeated interactions with no links between past, present, future
- Result of *learning process*; players *learn* their way to NE
- Result of *eductive process*; players *think* their way to NE
- Self-enforcing agreement: no player wants to deviate unilaterally

# Prisoner's Dilemma

| | C | D |
|---|---|---|
| C | 3,3 | 0,4 |
| D | 4,0 | 1,1 |

Dominance $\Rightarrow$ unique NE: (D,D)

# Battle of the Sexes

| | B | S |
|---|---|---|
| B | 2,1 | 0,0 |
| S | 0,0 | 1,2 |

Two NE: (B,B) and (S,S)
(One NE in mixed strategies)

# Coordination

| | Moz | Mah |
|---|---|---|
| Moz | $2, 2$ | $0, 0$ |
| Mah | $0, 0$ | $1, 1$ |

Two NE: (Moz,Moz), (Mah,Mah)
(One NE in mixed strategies)

# Matching Pennies

| | H | T |
|---|---|---|
| H | $+1, -1$ | $-1, +1$ |
| T | $-1, +1$ | $+1, -1$ |

No NE (in pure strategies)
(One NE in mixed strategies)

# Hawk/Dove

| | D | H |
|---|---|---|
| D | 2,2 | 0,4 |
| H | 4,0 | $-1, -1$ |

Two NE: (H,D), (D, H)

Neither of these makes sense for animals of the *same species*:
How would pigeons know which is Row and which is Col?
("Right" NE is in mixed strategies)

**Definition** A *mixed strategy* for player $i$ is a probability distribution $\mu_i \in \Delta(A_i)$

Interpretations

- Active randomization
  Football, Jackrabbits
- Correlation with unobserved phenomenon
  "Hunt for Red October"
- Confusion in mind of other players (Aumann)
  Why should all other players have the *same* confusion?
- Population with random meetings
  Evolutionarily Stable Strategies

**Definition** A NE in mixed strategies is a profile $\mu = (\mu_1, \ldots, \mu_n) \in \Delta(A_1) \times \ldots, \Delta(A_n))$ such that for all $i$ and all $\nu_i \in \Delta(A_i)$ we have

$$Eu_i(\mu_i, \mu_{-i}) \geq Eu_i(\nu_i, \mu_{-i})$$

- Important: players randomize *independently* (correlation $\rightsquigarrow$ *Correlated Equilibrium*)
- NE means: for all $i$, $\mu_i$ is a *best response* to $\mu_{-i}$
- linearity of expectation operator $\to$

$$\mu_i \text{ is a best response to } \mu_{-i}$$
$$\Updownarrow$$
$$a_i \text{ is a best response to } \mu_{-i} \text{ for all } a_i \text{ such that } \mu_i(a_i) > 0$$

# Computing Mixed NE

The last point makes it easier to compute mixed NE.
Fix game $G$, NE $\mu = (\mu_1, \ldots, \mu_n)$

- $a_i \in supp(\mu_i) \Leftrightarrow \mu_i(a_i) > 0$
- If $\mu_i$ is a best response to $\mu_{-i}$ then
  $$a_i \in supp(\mu_i) \implies a_i \text{ is a best response to } \mu_{-i}$$
- $\implies$ other players' choices of $\mu_{-i}$ makes $i$ indifferent over $\mathrm{supp}(\mu_i)$

# Example: Matching Pennies

| | H | T |
|---|---|---|
| H | $+1, -1$ | $-1, +1$ |
| T | $-1, +1$ | $+1, -1$ |

Look for NE in mixed strategies.
$\mu_{Col} = \lambda H + (1-\lambda)T$; must make Row indifferent between H, T

$$u_{Row}(H, \lambda H + (1-\lambda)T) = \lambda \cdot (+1) + (1-\lambda) \cdot (-1)$$
$$u_{Row}(T, \lambda H + (1-\lambda)T) = \lambda \cdot (-1) + (1-\lambda) \cdot (+1)$$

Solving: $\lambda = 1/2 \to$ Row plays (1/2)H + (1/2)T
Symmetry $\to$ Both play (1/2)H + (1/2)T; expected payoff $= 0$
Unique NE in mixed strategies

# Example: Hawk/Dove

| | D | H |
|---|---|---|
| D | 2,2 | 0,4 |
| H | 4,0 | $-1, -1$ |

Look for NE in mixed strategies.
$\mu_{Col} = \gamma D + (1-\gamma)H$; must make Row indifferent between D, H

$$u_{Row}(D, \gamma D + (1-\gamma)H) = \gamma \cdot (+2) + (1-\gamma) \cdot (0)$$
$$u_{Row}(H, \lambda D + (1-\lambda)H) = \gamma \cdot (+4) + (1-\gamma) \cdot (-1)$$

Solving: $\gamma = 1/3 \rightarrow$ Row plays (1/3)D + (2/3)D
Symmetry $\implies$ All play (1/3)D + (2/3)H; expected payoff = 2/3
Unique NE in mixed strategies

Suppose the *population* of pigeons consists of 1/3 who are Dove-ish (passive) and 2/3 who are Hawk-ish (aggressive)and pigeons meet at random. Payoffs are not in utility but in *evolutionary fitness*.

At this mixture, in expectation, Doves and Hawks get the same payoffs – no neither does better than the other.

Pigeons don't "think" about deviating – but Nature does (mutations). What would happen if a small fraction $\varepsilon$ of Doves became Hawks (i.e. their offspring had a mutation to the Hawk gene)? Then the population would be $(1/3 - \varepsilon)$ Doves and $(2/3 + \varepsilon)$ Hawks – and now the Doves would be doing better than the Hawks (this is because Hawks do badly when they meet other Hawks – check), so the mutation would tend to die out! This defines an Evolutionarily Stable Strategy (ESS). Notice that an ESS must be a NE; however, not all NE are ESS.

**Theorem** Every finite game admits a NE in mixed strategies.

Proof relies on

**Kakutani Fixed Point Theorem** Let $X \subset \mathbb{R}^M$ be a compact (closed, bounded) convex set, and $F : X \to X$ a correspondence that satisfies

- For each $x \in X$ the set $F(x)$ is closed (hence compact) and convex.
- $F$ is upper-hemi-continuous (uhc): If $x_n \to x$, $y_n \in F(x_n) \to y$ then $y \in F(x)$.

The $F$ has at least one fixed point; i.e. there is some $x^* \in X$ such that $x^* \in F(x^*)$

# Proof of Existence of NE

(1) Set $X = \Delta(A_1) \times \cdots \times \Delta(A_n)$.
Check that $X \subset \mathbb{R}^M$ is compact convex set

(2) For each $\mu = (\mu_1, \ldots, \mu_n) \in X$ set

$$BR_i(\mu) = \{\nu_i \in \Delta(A_i) : \nu_i \text{ is a best response to } \mu_{-i}$$

(3) Define $BR : X \to X$ by

$$BR(\mu) = BR_1(\mu) \times \cdots BR_n(\mu)$$

Check that $BR$ has closed convex values.
Check that $BR$ is uhc

(4) Kakutani $\to$ there is some $\mu^* \in X$ such that $\mu^* \in BR(\mu^*)$.
Check that this $\mu^*$ is a NE

QED

# Homework: Kakutani Fixed Point Theorem

In each of the following settings, find a set $X$ and a correspondence $F : X \rightarrow X$ that satisfies all the hypotheses of the Kakutani Fixed Point Theorem *except* the one listed such that $F$ does not have a fixed point.

- $X$ is not closed
- $X$ is not bounded
- $F$ does not have closed values
- $F$ does not have convex values
- $F$ is not UHC

# Correlated Equilibrium: Motivation

This is Battle of the Sexes

| | B | S |
|---|---|---|
| B | 2,1 | 0,0 |
| S | 0,0 | 1,2 |

In an ongoing relationship, players might mix between (B,S) and (S,B). But what about one-shot? To be fair players might *flip a coin* to decide which NE to play. But what if players are not in the same place?

They can do something equivalent to flipping a coin by correlating on something they can both see: play (B,S) if the 1's digit of the DJIA is odd at close of trading; play (S,B) if the 1's digit is even.

Note: neither play has a unilateral incentive to deviate *conditional on the signal and strategy of others*.

In the example of Battle of the Sexes, the players are correlating with perfect observability of the correlating device; but allowing for imperfect observability is even more helpful.

**Definition** Fix game $\Gamma = (N, \{A_i\}, \{u_i\})$. A *correlated equilibrium (CE)* is a probability distribution $\lambda$ on $A$ with the following property: If $r = (r_1, \ldots, r_n) \in A$ is drawn according to the distribution $\lambda$ and each player $i$ knows $\lambda$ but observes *only* $r_i$ (the *recommendation*) then for all $j$, $r_j$ is a best response for player $j$ assuming that other players play $r_{-j}$. [Note: We could allow for a bigger signal space, but that would reduce to the same thing.]

Important: Player $j$ observes $r_j$ and knows $\lambda$, so believes the distribution of other players' actions is the *marginal* $\lambda(\cdot|r_j)$ on $A_{-j}$.

# Correlated Equilibrium: Example

The game of *Chicken*

| | Don't Swerve | Swerve |
|---|---|---|
| Don't Swerve | 0,0 | 2,7 |
| Swerve | 7,2 | 6,6 |

This game has three NE: (DS,S), (S,DS) and $(\frac{1}{3}DS + \frac{2}{3}S)$. There are also CE; e.g. $\frac{1}{3}(DS, S) + \frac{1}{3}(S, DS) + \frac{1}{3}(S, S)$. [Check this]. Expected utilities to this CE are (5,5); this is socially better than any NE. [Note: This CE is *not* a convex combination of NE and the vector of expected utilities is *not* in the convex hull of the vectors of utilities of NE.)

# Homework: Hawk/Dove Game

If the population begins with 1/4 Hawks and 3/4 Doves, then the Hawks will be out-performing the Doves. If the payoff of the game translates into reproductive fitness then Hawks will reproduce faster than Doves and population will move toward 1/3Hawks and 2/3 Doves *over time*.

- Write down a model of population change over time in which the reproduction rate is proportional to payoffs. (It will be easier to do this in discrete time.) If necessary, look up "replicator dynamics".
- Using your model, will the population proportions converge to 1/3 Hawks and 2/3 Doves in the long run? Give a proof.

By definition, a strategy $\sigma$ is an *evolutionarily stable strategy (ESS)* if it has the properties

(i) $E(\sigma|\sigma) \geq E(\tau|\sigma)$ for all $\tau \in \Delta(H, D)$
(ii) $E(\sigma|\sigma) \geq E(\tau|\tau)$ for all $\tau \in \Delta(H, D)$

$\sigma$ is a *strong ESS* if in addition

(iii) $E(\sigma|\sigma) > E(\tau|\tau)$ for all $\tau \in \Delta(H, D)$ with $\tau \neq \sigma$

*$\sigma$ is an ESS* means that any behavioral mutation $\tau$ in the population will do *no better* than $\sigma$ in a population of non-mutants *and* mutants and so will not grow; *$\sigma$ is a strong ESS* means that any behavioral mutation $\tau$ in the population will do *worse* than $\sigma$ in a population of non-mutants *and* mutants and so will tend to die out. [The weaker notion of ESS is consistent with the possibility that the population *drifts*; the stronger notion of ESS is not.]

- Check that $\sigma = (1/3)H + (2/3)D$ in the Hawk/Dove Game is a strong ESS.
- Find a symmetric game for which some symmetric NE is *not* an ESS.
- Does an ESS always exist? Prove or find counterexample. [You can restrict your attention to 2-player games.]

- Check that $\lambda_1 = \frac{1}{3}(DS, S) + \frac{1}{3}(S, DS) + \frac{1}{3}(S, S)$ is a CE for the game of Chicken and that expected utilities are (5,5).
- Check that $\lambda_2 = \frac{1}{4}(DS, S) + \frac{1}{4}(S, DS) + \frac{1}{2}(S, S)$ is a CE for the game of Chicken and that expected utilities are 5.25.
- Find *all* the CE for the game of Chicken.
- Show that $\lambda_2$ is the *socially best* CE for the game of Chicken (i.e. it yields this highest sum of expected utilities).
- What is the *socially worst* CE for the game of Chicken?