OVERCONVERGENT WITT VECTORS

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Abstract. Let A be a finitely generated algebra over a field K of characteristic $p > 0$. We introduce a subring $W^\dagger(A) \subset W(A)$, which we call the ring of overconvergent Witt vectors and prove its basic properties. In a subsequent paper we use the results to define an overconvergent de Rham-Witt complex for smooth varieties over K whose hypercohomology is the rigid cohomology.

Introduction

Overconvergent Witt vectors were used by de Jong in his proof of Tate's conjecture on homomorphisms of p-divisible groups [2] and in Kedlaya's work on the Crew conjecture. In [1] we define a de Rham-Witt complex over the ring of overconvergent Witt vectors which computes the rigid cohomology of smooth varieties of a perfect field of characterisitc $p > 0$. For this it is necessary to consider overconvergent Witt vectors in a more general setting.

Let A be a finitely generated algebra of a field K of characteristic p. Let $W(A)$ be the ring of Witt vectors with respect to p. We define a subring $W^\dagger(A) \subset W(A)$ which we call the ring of overconvergent Wittvectors. Let $A = K[T_1, \ldots, T_d]$ be the polynomial ring. We say that a Witt vector $(f_0, f_1, f_2, \ldots) \in W(A)$ is overconvergent, if there is a real number $\varepsilon > 0$ and a real number C such that

$$m - \varepsilon p^{-m} \deg f_m \geq C, \quad \text{for all } m \geq 0.$$

The overconvergent Witt vectors form a subring $W^\dagger(A) \subset W(A)$.

There is a natural morphism from the ring of restricted power series

$$W(K)\{T_1, \ldots, T_d\} \to W(A),$$

which maps T_i to its Teichmüller representative $[T_i]$.

The inverse image of $W^\dagger(A)$ is the set of those power series which converge in some neighborhood of the unit ball. This is the weak completion $A^\dagger$ of $W(K)[T_1, \ldots, T_d]$ in the sense of Monsky and Washnitzer. We note that the bounded Witt vectors used by Lubkin [7] are different from the overconvergent Witt vectors.

If $A \to B$ is a surjection of finitely generated K-algebras, we obtain by definition a surjection of the rings of overconvergent Witt vectors

$$W^\dagger(A) \to W^\dagger(B).$$

We prove here basic properties of overconvergent Witt vectors which we use in [1]:

Let $A \subset B$ be two smooth K-algebras. Then

$$W^\dagger(A) = W(A) \cap W^\dagger(B)$$

(see: Proposition 2.16).

Further we show (see Corollary 2.46): Let A be a finitely generated algebra over K. Let $B = A[T]/(f(T))$ be a finite étale A-algebra, where $f(T) \in A[T]$ is a monic polynomial of degree n, such that $f'(T)$ is a unit in B.

We denote by t the residue class of T in B. Then $W^\dagger(B)$ is finite and étale over $W^\dagger(A)$, and the elements $1, [t], [t]^2 \ldots, [t]^{n-1}$ form a basis of the $W^\dagger(A)$-module $W^\dagger(B)$.

Finally we prove that $W^\dagger(A) \to A$ satisfies Hensel's lemma (see Proposition 2.30).

1. Pseudovaluations

We set $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$.

Definition 1.1. *Let A be an abelian group. An order function is a function*

$$\nu : A \to \bar{\mathbb{R}},$$

such that for each $a, b \in A$:

$$\nu(a+b) \geq \min\{\nu(a), \nu(b)\}.$$

Let $\phi : A \to B$ be a surjective homomorphism of abelian groups. Then we define the quotient $\bar{\nu} : B \to \bar{\mathbb{R}}$ by:

$$\bar{\nu}(b) = \sup\{\nu(a) \mid a \in A,\ \phi(a) = b\}. \tag{1.2}$$

This is again an order function.

We define an order function ν^n on the direct sum A^n as follows:

$$\nu^n((a_1, \ldots, a_n)) = \min_i \{\nu(a_i)\}. \tag{1.3}$$

Definition 1.4. *Let A be a ring with 1. A pseudovaluation ν on A is a function:*

$$\nu : A \to \bar{\mathbb{R}}$$

such that the following properties hold

1) $\nu(1) = 0, \qquad \nu(0) = \infty$.
2) $\nu(a) = \nu(-a)$ *for all* $a \in A$.
3) $\nu(a+b) \geq \min\{\nu(a), \nu(b)\}$ *for all* $a, b \in A$.
4) $\nu(ab) \geq \nu(a) + \nu(b)$, *if* $\nu(a) \neq -\infty$ *and* $\nu(b) \neq -\infty$.

We call ν proper if it doesn't take the value $-\infty$. We call ν negative if ν is proper and $\nu(a) \leq 0$ for all $a \in A, a \neq 0$.

If $\phi : A \to B$ is a surjective ring homomorphism then the quotient of a negative pseudovaluation ν is again a negative pseudovaluation. On each ring A we have the trivial pseudovaluation: $\nu(a) = 0$ for $a \neq 0$. If ν is proper and 4) is an equality, ν is called a valuation.

Example 1: Let R be a ring with a negative pseudovaluation μ. Consider the polynomial ring $A = R[T_1, \ldots T_m]$. Let $d_1 > 0, \ldots, d_m > 0$ be real numbers. Then we define a valuation on A as follows: For a polynomial

$$f = \sum_k c_k T_1^{k_1} \cdot \ldots \cdot T_m^{k_m}$$

we set

$$\nu(f) = \inf\{\mu(c_k) - k_1 d_1 - \ldots - k_m d_m\}. \tag{1.5}$$

This is a valuation if μ is a valuation. We often consider the case where R is an integral domain and μ is the trivial valuation. If moreover $d_i = 1$ we call ν the standard degree valuation.

We are interested in pseudovaluations up to equivalence:

Definition 1.6. *Let $\nu_1, \nu_2 : A \to \mathbb{R} \cup \infty$ be two functions such that $\nu_i \neq 0$ for all $a \in A$. We say that they are linearly equivalent, if there are real numbers $c_1 > 0, c_2 > 0, d_1 \geq 0, d_2 \geq 0$ such that for all $a \in A$:*

$$\begin{array}{rcl} \nu_1(a) & \geq & c_2\nu_2(a) - d_2 \\ \nu_2(a) & \geq & c_1\nu_1(a) - d_1 \end{array}$$

In Example 1 (1.5) we obtain for different choices of the numbers d_i linearly equivalent negative pseudovaluations. The equivalence class of ν doesn't change if we replace μ by an equivalent negative pseudovaluation.

Let ν_1 and ν_2 be two negative pseudovaluations on A. If $A \to B$ is a surjective ring homomorphism, then the quotients $\bar{\nu_1}$ and $\bar{\nu_2}$ are again linearly equivalent.

Proposition 1.7. *Let μ be a negative pseudovaluation on a ring R. We consider a surjective ring homomorphism $\phi : R[T_1, \ldots, T_m] \to R[S_1, \ldots, S_n]$. Let ν_T be a pseudovaluation on $R[T_1, \ldots, T_m]$ and let ν_S be a pseudovaluation on $R[S_1, \ldots, S_n]$ as defined by (1.5). Then the quotient of ν_T with respect to ϕ is a pseudovaluation which is linearly equivalent to the valuation ν_S.*

We omit the straightforward proof which is essentially contained in [9]. A proof in a more general context is given in [1].

Definition 1.8. *Let μ be a negative pseudovaluation on a ring R. Let B be a finitely generated R-algebra. Choose an arbitrary surjection $R[T_1, \ldots, T_m] \to B$ and an arbitrary degree valuation ν on $R[T_1, \ldots T_m]$. Then the quotient $\bar{\nu}$ on B, is up to linear equivalence independent of these choices. We call any negative pseudovaluation in this equivalence class admissible.*

Let μ be an admissible pseudovaluation on a finitely generated R-algebra B. Let ν be the pseudovaluation on a polynomial algebra $B[T_1, \ldots, T_m]$ given by Example 1. Then ν is admissible. This it is easily seen, if we write B as a quotient of a polynomial algebra.

Lemma 1.9. *Let (R, μ) be a ring with a negative pseudovaluation. Let A be an R-algebra which is finite and free as an R-module. Let τ be an admissible pseudovaluation on A.*

Choose an R-module isomorphism $R^n \cong A$. With respect to this isomorphism τ is linearly equivalent to the order function μ^n given by (1.3).

Proof. Let $e_1, \ldots, e_n$ be a basis of A as an R-module. Consider the natural surjection:

$$\alpha : R[T_1, \ldots, T_n] \to A$$

such that $\alpha(T_i) = e_i$. We have equations:

$$e_i e_j = \sum_{l=1}^{n} c_{ij}^{(l)} e_l, \quad c_{ij}^{(l)} \in R.$$

We choose a number d, such that for all coefficients $c_{ij}^{(l)}$:

$$\mu(c_{ij}^{(l)}) + d \geq 0.$$

Let $\tilde{\tau}$ be the pseudovaluation (1.5) on $R[T_1, \ldots, T_n]$, such that $\tilde{\tau}(X_i) = -d$. We can take for τ the quotient of $\tilde{\tau}$.

Consider an element $a \in A$. We choose a representative of a:

$$f = \sum_k r_k T_1^{k_1} \cdot \ldots \cdot T_n^{k_n}.$$

We claim that there is a linear polynomial f_1 which maps to a, such that

$$\tilde{\tau}(f_1) \geq \tilde{\tau}(f).$$

Indeed, assume that some of the monomials $r_k T_1^{k_1} \cdot \ldots \cdot T_n^{k_n}$ is divisible by $T_i T_j$. We will pretend in our notation that $i \neq j$, but the other case is the same. We find an equation

$$r_k T^k = \sum_l r_k c_{ij}^{(l)} T^{k(l)},$$

where $|k(l)| = |k| - 1$. We find for any fixed l:

$$\tilde{\tau}(r_k c_{ij}^{(l)} T^{k(l)}) = \mu(r_k c_{ij}^{(l)}) - d(|k| - 1) \geq \mu(a_k) + \mu(c_{ij}^{(l)}) + d - d|k| \geq \tilde{\tau}(f).$$

We conclude that

$$\tau(a) = \sup\{\tilde{\tau}(f) \mid f = r_0 + r_1 T_1 + \ldots + r_n T_n,\ \alpha(f) = b\}. \tag{1.10}$$

By construction a has a unique representative

$$g = \sum_{i=1}^{n} s_i T_i.$$

Clearly $\tilde{\tau}$ restricted to linear forms as above is linearly equivalent to the order function μ^n defined by (1.3). We need to compare $\tilde{\tau}(g)$ and $\tau(a)$.

We have a relation in A:

$$1 = \sum_{i=1}^{n} c_i e_i, \quad c_i \in R.$$

Given a representative f as in (1.10) we find:

$$g = \sum_{i=1}^{n} (r_i + c_i r_0) T_i.$$

Then we find:

$$\begin{aligned} \tilde{\tau}(g) &= \min\{\mu(r_i + c_i r_0) - d\} \geq \min_i\{\min\{\mu(r_i) - d, \mu(c_i) - d + \mu(r_0)\}\} \\ &\geq \tilde{\tau}(f) - d', \end{aligned}$$

where d' is chosen such that $-d' < \mu(c_i) - d$. Since this is true for arbitrary f we find $\tilde{\tau}(g) \geq \tau(b) - d'$. Since τ is the quotient norm we have the obvious inequality

$$\tau(b) \geq \tilde{\tau}(g).$$

This completes the proof. $\square$

Example 2: Let ν be a negative pseudovaluation on A. Let $d > 0$ a real number. Then we have defined a pseudovaluation on the polynomial algebra $A[X]$:

$$\mu(\sum a_i X^i) = \min\{\nu(a_i) - id\}. \tag{1.11}$$

Let $f \in A$, such that f is not nilpotent. Then we define a pseudovaluation ν' on the localization A_f by taking the quotient under the map:

$$A[X] \to A_f,$$

which sends X to f^{-1}. As we remarked above ν' depends only on the linear equivalence class of ν on A.

Let $z \in A_f$. Consider all possible representations of z in the form:

$$z = \sum_l a_l / f^l. \tag{1.12}$$

Then ν' is the supremum over all these representations of the following numbers:

$$\min_l \{\nu(a_l) - ld\}. \tag{1.13}$$

If the supremum is assumed we call the representation optimal.

Lemma 1.14. *Let (A, ν) be a ring with a negative pseudovaluation. Let $f \in A$ be a non-zero divisor. Let ν' be the induced pseudovaluation on A_f which is associated to a fixed number $d > 0$.*

We are going to define a function $\tau : A_f \to \mathbb{R} \cup \{\infty\}$. For $z \in A_f$ we consider the set of all possible representations

$$z = a/(f^m). \tag{1.15}$$

We define $\tau(z)$ to be the maximum of the numbers $\nu(a) - md$ for all possible representations (1.15).

Then there is a real constant $Q > 0$ such that for all $z \in A_f$

$$\nu'(z) \geq \tau(z) \geq Q\nu'(z).$$

Proof. The first of the asserted inequalities is trivial. Consider any representation:

$$z = \sum_{l=0}^{m} u_l/(f^l) \quad \text{such that } u_m \neq 0.$$

We set

$$-C = \min_l \{\nu(u_l) - ld\}.$$

We note that this implies that $-C \leq -md$. We find a representation of the form (1.15):

$$z = (\sum_{l=0}^{m} u_l f^{m-l})/(f^m) = a/f^m.$$

We find:

$$\begin{aligned} \nu(a) - md &= \nu(\textstyle\sum_{l=0}^{m} u_l f^{m-l}) - md \geq \min_l \{\nu(u_l) + (m-l)\nu(f) - md\} \\ &\geq \min_l \{\nu(u_l) - ld - md + m\nu(f) + l(d - \nu(f))\} \\ &\geq \min_l \{-C - C + m\nu(f)\}. \end{aligned}$$

We have further:

$$m\nu(f) = (-dm)\frac{\nu(f)}{-d} \geq (-C)\frac{\nu(f)}{-d}.$$

Together we obtain:

$$\nu(a) - md \geq -C(2 + \frac{\nu(f)}{-d}).$$

This implies:

$$\tau(a/f^m) \geq (2 + \frac{\nu(f)}{-d})\nu'(z).$$

□

The motivation for the following definition is Lemma 2.14 below.

Definition 1.16. *Let (A, ν) be a ring with a negative pseudovaluation. We say that a non-zero divisor $f \in A$ is localizing with respect to ν, if there are real numbers $C > 0$ and $D \geq 0$, such that for all natural numbers n:*

$$\nu(f^n x) \leq C\nu(x) + nD, \quad \textit{for all } x \in A. \tag{1.17}$$

If μ is a negative pseudovaluation on A, which is linearly equivalent to ν then f is localizing with respect to ν, iff it is localizing with respect to μ. Indeed any function ρ linearly equivalent satisfies an inequality (1.17).

It is helpful to remark that making C smaller we may always arrange that D is smaller than any given positive number. It is also easy to see that a unit of the ring A is always localizing.

Let $A = R[T_1, \ldots, T_d]$ be a polynomial ring over an integral domain R with a degree valuation ν. Then we have the equation:

$$\nu(f^n x) = \nu(x) + n\nu(f), \quad x \in A.$$

Therefore (1.17) holds with $C = 1$ and $D = 0$.

More generally, let (B, μ) a ring with a negative pseudovaluation. We endow $A = B[T]$ with the natural extension ν of μ such that $\nu(T) = -d$. Assume that $f = T^m + a_{m-1}T^{m-1} + \ldots + a_0$ is a monic polynomial with $a_i \in B$.

Definition 1.18. *We say that f is regular with respect to T, if*

$$\min_{0 \leq i < m} \{\mu(a_i) - id\} > -md. \tag{1.19}$$

For a regular polynomial we have $\nu(f) = -md$. We remark that each monic polynomial f becomes regular for a suitable choice of d.

Proposition 1.20. *Let $f(T) \in B[T]$ be a regular polynomial (1.18). Then we have for an arbitrary polynomial $g(T) \in B[T]$ that*

$$\nu(f(T)g(T)) = \nu(f(T)) + \nu(g(T)).$$

In particular any monic polynomial in $B[T]$ is localizing.

Proof. We write

$$g = \sum_{k=0}^{n} b_k T^k.$$

Let k_0 be the largest index such that $\nu(g) = \nu(b_{k_0}) - k_0 d$. fg contains the monomial

$$(b_{k_0} + b_{k_0+1}a_{m-1} + \ldots)T^{m+k_0}.$$

We find by (1.19) that

$$\mu(b_{k_0+i}a_{m-i}) \geq \mu(b_{k_0}) + \mu(a_{m-i}) \geq \mu(b_{k_0}) - id.$$

On the other hand we have by the choice of k_0 that

$$\mu(b_{k_0}) - k_0 d < \mu(b_{k_0+i}) - (i + k_0)d.$$

This proves that $\mu(b_{k_0+i}a_{m-i}) > \mu(b_{k_0})$. Therefore we obtain that

$$\mu(b_{k_0} + b_{k_0+1}a_{m-1} + \ldots) = \mu(b_{k_0}).$$

This shows the inequality:

$$\nu(fg) \leq \nu(b_{k_0}) - d(m + k_0) = \nu(g) - md = \nu(g) + \nu(f).$$

The opposite inequality is obvious. The last assertion follows because any monic polynomial is regular for a suitable chosen d. □

Proposition 1.21. *Assume that $f = T^m + a_{m-1}T^{m-1} + \ldots + a_0 \in B[T]$ is a polynomial which is regular with respect to T. Each $z \in A_f$ has a unique representation:*

$$z = \sum_l u_l/f^l, \qquad u_l \in B[T], \tag{1.22}$$

where u_l is for $l > 0$ a polynomial of degree strictly less than $m = \deg f$. Then the representation (1.22) is optimal (compare (1.13)).

Proof. The first assertion follows from the euclidian division. Consider any other representation

$$z = \sum_i v_i/f^i, \qquad v_i \in B[T],$$

Assume that $n = \deg v_i \geq m$ for some $i > 0$. Let $c \in B$ be the highest coefficient of the polynomial v_i and set $t = n - m$. Then we conclude:

$$\nu(cT^t f) \geq \mu(c) + \nu(T^t) + \nu(f) = \mu(c) + \nu(T^t) + \nu(T^m) = \nu(cT^n) \geq \nu(v_i).$$

We write:

$$v_i/f^i = ((v_i - cT^t f)/f^i) - (cT^t/f^{i-1}).$$

If we insert this in the representation (1.12) the number (1.13) becomes bigger because:

$$\nu((v_i - cT^t f) \geq \nu(v_i), \qquad \nu(cT^t) \geq \nu(cT^n) \geq \nu(v_i).$$

Continuing this process proves the lemma. □

The last Proposition applies in particular to a polynomial ring over a field $A = K[T_1, \ldots, T_d]$ with the standard degree valuation. By Noether normalization any polynomial becomes regular with respect to some variable after a coordinate change.

Proposition 1.23. *Let (A, ν) be a ring with a pseudovaluation. Let $f, g \in A$. Then fg is localizing iff f and g are localizing.*

Proof: Assume fg is localizing. Then we find an inequality:

$$\nu(f^n g^n x) \leq C\nu(x) + nD.$$

On the other hand we have the inequality:

$$n\nu(f) + \nu(g^n x) \leq \nu(f^n g^n x).$$

This shows that:

$$\nu(g^n x) \leq C\nu(x) + n(D - \nu(f)).$$

We leave the opposite implication to the reader.

Proposition 1.24. *Let (A, ν) be a ring with a pseudovaluation. Assume that A is an integral domain, such that each non-zero element of A is localizing. Let $A \to B$ be a finite ring homomorphism such that B is a free A-module. Let μ be an admissible pseudovaluation on B. Then any nonzero divisor in B is localizing with respect to μ.*

Proof. We choose an isomorphism of A-modules: $A^r \cong B$. By Lemma 1.9 the order function ν^r on A^r is linearly equivalent to an admissible pseudovaluation on B. Let $f \in A$, $f \neq 0$. Then an inequality (1.17) holds. It follows that for each $z \in A^r$:

$$\nu^r(f^n z) \leq C\nu^r(z) + nD.$$

This shows that f is localizing in B. More generally consider a non-zero divisor $b \in B$. Consider an equation of minimal degree:

$$b^t + a_{t-1}b^{t-1} + \ldots + a_1 b + a_0 = 0, \qquad a_i \in A.$$

Then $a_0 \neq 0$ and therefore localizing. But a_0 is a multiple of b in the ring B. Therefore b is localizing in B by Proposition 1.23. □

Corollary 1.25. *Let $X \to \operatorname{Spec} K$ be a smooth scheme over a field K of characteristic p. Then any point of X has an affine neighbourhood $\operatorname{Spec} A$, such that any non-zero element in A is localizing.*

Proof. This is immediate from a result of [3] which says that each point admits a neighbourhood which is finite and étale over an affine space $\mathbb{A}^n_K$. □

Let us assume that $f \in A$ is localizing with constants C, D given by (1.17). Then we will assume that the constant d used in the definition of ν' on A_f is bigger than D. This can be done with no loss of generality because the equivalence class of ν' doesn't depend on d.

Proposition 1.26. *Let (A, ν) be a ring with a negative pseudovaluation. Let $f \in A$ be a localizing element. Each $z \in A_f$ has a unique representation*

$$z = a/f^m, \quad \text{where } a \in A,\ f \nmid a.$$

We define a real valued function σ on A_f:

$$\sigma(z) = \nu(a) - md.$$

Then there exists a real constant $E > 0$, such that:

$$\nu'(z) \geq \sigma(z) \geq E\nu'(z).$$

In particular, the restriction of ν' to A is linearly equivalent to ν.

Proof. By Lemma 1.14 it suffices to show the last inequality with ν' replaced by τ. All representations (1.15) of z are of the form:

$$af^r/f^{m+r}.$$

Since f is localizing there are real numbers $1 > C > 0$ and $D \geq 0$ such that:

$$\begin{aligned} \nu(af^r) - (m+r)d &\leq C\nu(a) + rD - md - rd \\ &\leq C(\nu(a) - md) + (D-d)r \leq C\sigma(z) + (D-d)r. \end{aligned}$$

We may assume that $d \geq D$. Then the inequality above implies

$$\tau(z) \leq C\sigma(z).$$

□

Corollary 1.27. *Let (B, μ) be an integral domain with a pseudovaluation μ. Assume that each non-zero element is localizing. We endow $B[T]$ with a pseudovaluation of Example 1.*

Then each non-zero element in $B[T]$ is localizing.

Proof. Clearly each $b \in B$, $b \neq 0$ is localizing in $B[T]$. By the Proposition it suffices to find for a given $f \in B[T]$ an element $b \in B$, such that f is localizing in $B_b[T]$. By the remark preceding Proposition 1.20 we may assume that f is a regular polynomial. Then we can apply this proposition. □

The following corollary would allow to prove Corollary 1.25 more generally by considering standard étale neighbourhoods instead of Kedlaya's result.

Corollary 1.28. *Let (A, ν) be a noetherian ring with a negative pseudovaluation. Let $a, f \in A$ be two localizing elements. Then a is localizing in A_f.*

Proof. By the Lemma of Artin-Rees there is a natural number r, such that for $m \geq r$

$$ax \in f^m A \quad \text{implies} \quad x \in f^{m-r}A.$$

Assume that $x \in A$, but $x \notin fA$. Then we conclude that for each $n \in \mathbb{N}$

$$a^n x \in f^m A \quad \text{implies} \quad m \leq nr.$$

Consider a reduced fraction $x/f^m \in A_f$. To show that a is localizing it suffices to find an estimation for

$$\sigma(a^n(x/f^m)),$$

where σ is the function of Proposition 1.26:

$$\sigma(x/f^m) = \nu(x) - md.$$

By the remarks above we may write with $y \notin fA$:

$$\frac{a^n x}{f^m} = \frac{yf^s}{f^m}, \quad s \leq nr.$$

Using this equation we obtain:

$$\begin{aligned} \nu(y) &\leq \nu(yf^s) - \nu(f^s) \leq \nu(a^n x) - s\nu(f) \\ &\leq C\nu(x) + nD - nr\nu(f). \end{aligned}$$

Here $C \leq 1, D$ are positive real constants, which exists because a is localizing in A.

Now it is easy to give an estimation for

$$\sigma(a^n(x/f^m)) = \sigma(yf^s/f^m).$$

We omit the details. □

We reformulate Proposition 1.26 in the case where $A = R[T_1, \ldots, T_d]$ is a polynomial algebra over an integral domain R with the standard degree valuation ν. It extends to a valuation on the quotient field of A which we denote by ν too. Let $f \in A$ be a non-zero element. We define ν' on A_f associated to $d > 0$ as before (1.13).

We define a second pseudovaluation μ on the ring A_f as follows. Let $\vartheta(z)$ be the smallest integer $n \geq 0$ such that $f^n z \in A$. We set:

$$\mu(z) = \min\{\nu'(z), -d\vartheta(z)\} \tag{1.29}$$

Proposition 1.30. *Let A be a polynomial ring with the standard degree valuation ν. Let $f \in A$ be a non constant polynomial. Let us define pseudovaluations ν' resp. μ on A_f by the formulas (1.13) resp. (1.29). Then there are constants Q_1 and Q_2, such that*

$$Q_1\mu \geq \nu' \geq Q_2\mu.$$

Proof. We write an element $z \in A_f$ as a reduced fraction

$$z = (a/f^m),$$

such that $m = \vartheta(z)$. By Proposition 1.26 it is enough to compare μ with the function σ. The inequality $\sigma(z) \leq \mu(z)$ is obvious. We show that for a sufficiently big number $C > 1$:

$$C\mu(z) \leq \nu(a) - md.$$

This is obvious if $-Cmd \leq -md + \nu(a)$. Therefore we can make that assumption:

$$-(C-1)md \geq \nu(a).$$

We have to find C such that the following inequality is satisfied:

$$C(\nu(a) - m\nu(f)) \leq \nu(a) - md.$$

We have by assumption:

$$(C-1)\nu(a) \leq -(C-1)^2 md.$$

Therefore it suffices to show that for big C:

$$-(C-1)^2 md \leq m(C\nu(f) - d).$$

But this is obvious. □

2. Overconvergent Witt vectors

Let us fix a prime number p. We are going to introduce the ring of overconvergent Witt vectors. Let A be a ring with a proper pseudovaluation ν. We assume that $pA = 0$.

Let $W(A)$ be the ring of Witt vectors. For any Witt vector

$$\alpha = (a_0, a_1, a_2, \ldots) \in W(A)$$

we consider the following set $\mathcal{T}(\alpha)$ in the $x-y$-plane:

$$(p^{-i}\nu(a_i), i), \qquad \nu(a_i) \neq \infty.$$

For $\varepsilon, c \in \mathbb{R},\ \varepsilon > 0$ we consider the half plane:

$$H_{\varepsilon,c} = \{(x,y) \in \mathbb{R}^2 \mid y \geq -\varepsilon x + c\}.$$

Moreover we consider for all $c \in \mathbb{R}$ the half plane:

$$H_c = \{(x,y) \in \mathbb{R}^2 \mid x \geq c\}.$$

Let $\mathcal{H}$ the set of all half planes of the two different types above. We define the Newton polygon $\mathrm{NP}(\alpha)$:

$$\mathrm{NP}(\alpha) = \bigcap_{H \in \mathcal{H}, \mathcal{T}(\alpha) \in H} H.$$

Definition 2.1. *We say that a Witt vector α has radius of convergence $\varepsilon > 0$, if there is a constant $c \in \mathbb{R}$, such that*

$$i \geq -\varepsilon p^{-i} \nu(a_i) + c.$$

We denote the set of these Witt vectors by $W^\varepsilon(A)$.

Equivalently one may say that the Newton polygon $\mathrm{NP}(\alpha)$ lies above a line of slope $-\varepsilon$.

We define the Gauss norm $\gamma_\varepsilon : W(A) \to \mathbb{R}$:

$$\gamma_\varepsilon(\alpha) = \inf\{i + \varepsilon p^{-i} \nu(a_i)\} \tag{2.2}$$

Convergence of radius $\varepsilon > 0$ means that $\gamma_\varepsilon(\alpha) \neq -\infty$. We will denote the set of Witt vectors of radius of convergence ε by $W^\varepsilon(A)$.

Proposition 2.3. *Let (A, ν) a ring with a proper pseudovaluation, such that $pA = 0$. Then for any $\varepsilon > 0$ the Gauss norm γ_ε is a pseudovaluation on $W(A)$. In particular $W^\varepsilon(A)$ is a ring.*

If we assume moreover that ν is a valuation, we have the equality for arbitrary $\xi, \eta \in W^\varepsilon(A)$:

$$\gamma_\varepsilon(\xi\eta) = \gamma_\varepsilon(\xi) + \gamma_\varepsilon(\eta). \tag{2.4}$$

Proof. Clearly we may assume $\varepsilon = 1$. We set $\gamma = \gamma_1$. The first two requirements of Definition 1.4 are clear. Consider two Wittvectors:

$$\xi = (a_0, a_1, \ldots) \in W(A), \quad \eta = (b_0, b_1, \ldots) \in W(A).$$

We begin to show the inequality:

$$\gamma(\xi + \eta) \geq \min\{\gamma(\xi), \gamma(\eta)\}.$$

We may assume that there is $g \in \mathbb{R}$, such that

$$i + p^{-i}\nu(a_i) \geq g, \quad i + p^{-i}\nu(b_i) \geq g.$$

We write

$$\xi + \eta = (s_0, s_1, \ldots).$$

Let S_m be the polynomials, which define the addition of the Witt vectors:

$$s_m = S_m(a_0, \ldots, a_m, b_0, \ldots, b_m).$$

We know that S_m is a sum of monomials

$$M = \pm a_0^{e_0} \cdot \ldots \cdot a_m^{e_m} b_0^{f_0} \cdot \ldots \cdot b_m^{f_m},$$

such that
$$\sum_{i=0}^{m} p^i e_i + \sum_{i=0}^{m} p^i f_i = p^m.$$
We have to show that $p^{-m}\nu(m) + m \geq m$. We compute:
$$\begin{aligned}
&p^{-m}\nu(M) + m \\
&\geq p^{-m}(\textstyle\sum_{i=0}^{m} e_i\nu(a_i) + \sum_{i=0}^{m} f_i\nu(b_i) \\
&+ \textstyle\sum_{i=0}^{m} p^i e_i m + \sum_{i=0}^{m} p^i f_i m) \\
&\geq p^{-m}(\textstyle\sum_{i=0}^{m} p^i e_i(p^{-i}\nu(a_i) + i) + \sum_{i=0}^{m} p^i f_i(p^{-i}\nu(b_i) + i)) \\
&\geq p^{-m}(\textstyle\sum_{i=0}^{m} p^i e_i g + \sum_{i=0}^{m} p^i f_i g) \geq g.
\end{aligned}$$
This proves the fourth requirement of Definition 1.4.

Next we prove the inequality:
$$\gamma(\xi\eta) \geq \gamma(\xi) + \gamma(\eta). \tag{2.5}$$
By the inequality already shown we are reduced to the case
$$\xi = {}^{V^i}[a], \quad \text{and } \eta = {}^{V^j}[b].$$
Since by assumption F and V commute on $W(A)$ we find
$$\xi\eta = {}^{V^{i+j}}[a^{p^j} b^{p^i}].$$
We obtain:
$$\begin{aligned}
\gamma(\xi\eta) &= \tfrac{\nu(a^{p^j} b^{p^i})}{p^{i+j}} + i + j \\
&\geq \tfrac{p^j\nu(a) + p^i\nu(b)}{p^{i+j}} + i + j = \gamma(\xi) + \gamma(\eta).
\end{aligned} \tag{2.6}$$
This proves that γ is a pseudovaluation.

Finally we prove the equality (2.4) if ν is a valuation. We remark that (2.6) is an equality in this case. From this we obtain (2.4) in the case where
$$\xi = {}^{V^i}[a] + \xi_1, \quad \eta = {}^{V^j}[b] + \eta_2,$$
where $\xi_1 \in V^{i+1}W(A)$, $\eta \in V^{j+1}W(A)$ and
$$\gamma(\xi_1) \geq \gamma({}^{V^i}[a]), \quad \gamma(\eta_2) \geq \gamma({}^{V^j}[b]).$$
Next we consider the case if there are i and j such that
$$p^{-i}\nu(a_i) + i = \gamma(\xi), \quad p^{-j}\nu(b_j) + j = \gamma(\eta).$$
We assume that i and j are minimal with this property. Then we write
$$\begin{array}{lll}
\xi & = (a_0, \ldots, a_{i-1}, 0, \ldots) + \xi_1 & = \xi' + \xi_1 \\
\eta & = (b_0, \ldots, b_{j-1}, 0, \ldots) + \eta_1 & = \eta' + \eta_1.
\end{array}$$
Then we have by our choice:
$$\gamma(\xi') > \gamma(\xi) = \gamma(\xi_1), \ \gamma(\eta') > \gamma(\eta) = \gamma(\eta_1).$$
By the case already treated we have $\gamma(\xi_1\eta_1) = \gamma(\xi_1) + \gamma(\eta_1)$. Then we obtain:

(2.7)
$$\gamma(\xi\eta) = \gamma(\xi_1\eta_1 + \xi_1\eta' + \xi'\eta_1 + \xi'\eta') \geq \min\{\gamma(\xi_1\eta_1) + \gamma(\xi_1\eta') + \gamma(\xi'\eta_1) + \gamma(\xi'\eta')\}.$$

But by the inequality (2.5) this minimum is assumed only for $\gamma(\xi_1\eta_1)$ and therefore (2.7) is an equality.

Finally if i and j as above don't exist this becomes true if we replace ε by any δ which is a little smaller. If δ approaches ε we obtain the result. □

We have the formulas:

$$\begin{array}{rcl} \gamma_\varepsilon({}^V\alpha) & = & 1+\gamma_{\varepsilon/p}(\alpha) \\ \gamma_\varepsilon({}^F\alpha) & \geq & \gamma_{p\varepsilon}(\alpha) \\ \gamma_\varepsilon(p) & = & 1. \end{array} \tag{2.8}$$

Definition 2.9. *The union of the rings $W^\varepsilon(A)$ for $\varepsilon > 0$ is called the ring of overconvergent Witt vectors $W^\dagger(A)$.*

Corollary 2.10. *Let $\alpha \in W^\dagger(A)$ and let $\delta > 0$ a real number. Then there is an $\varepsilon > 0$ such that $\gamma_\varepsilon(\alpha) > -\delta$.*

Proof. Take some negative line of slope $-\tau$ below the Newton polygon of α. If this line does not meet the negative x-axis we conclude that $\gamma_\tau(\alpha) \geq 0$. In the other case we rotate the line around the intersection point to obtain the desired slope $-\varepsilon$. □

We will from now on assume that the pseudovaluation ν on A is negative. By proposition 2.3 this is a subring. Then we have:

$$W^\delta(A) \subset W^\varepsilon(A), \quad \text{if } \delta > \varepsilon.$$

The ring $W^\dagger(A)$ does not change if we replace ν by a linearly equivalent pseudovaluation. More generally let $f : A \to \mathbb{R} \cup \{\infty\}$ be any function which is linearly equivalent to ν. Then a Witt vector $(x_0, x_1, \ldots) \in W(A)$ is overconvergent with respect to the ν, iff there is an $\varepsilon > 0$ and a constant $C \in \mathbb{R}$ such that for all $i \geq 0$.

$$i + p^i f(x_i)\varepsilon \geq -C.$$

With the notation of Definition 1.8 let A be a finitely generated algebra over (R, μ). Any admissible pseudovaluation on A leads to the same ring $W^\dagger(A)$. Let $\alpha : A \to B$ be a homomorphism of finitely generated R-algebras. Then the induced homomorphism on the rings of Witt vectors respects overconvergent Witt vectors:

$$W(\alpha) : W^\dagger(A) \to W^\dagger(B). \tag{2.11}$$

This is seen by choosing a diagram

$$\begin{tikzcd} R[T_1, \ldots, T_n] \arrow[r] \arrow[d] & R[T_1, \ldots, T_n, S_1, \ldots, S_m] \arrow[d] \\ A \arrow[r] & B. \end{tikzcd}$$

On the truncated Witt vectors we consider the functions $\gamma_\varepsilon[n]$:

$$\gamma_\varepsilon[n] : W_{n+1}(A) \to \mathbb{R} \cup \{\infty\}$$

$$\gamma_\varepsilon[n](\alpha) = \min\{i + \varepsilon p^{-i}\nu(a_i) \mid i \leq n\}.$$

This is the quotient of γ_ε under the natural map $W(A) \to W_{n+1}(A)$ in the sense of (1.2). We conclude that $\gamma_\varepsilon[n]$ is a proper pseudovaluation.

The following is obvious: Let $\sum_{m=0}^{\infty} \alpha_m$ be an infinite sum of Witt vectors $\alpha_m \in W(A)$, which converges in the V-adic topology to $\sigma \in A$. Let $\varepsilon > 0$ and $C \in \mathbb{R}$, such that

$$\gamma_\varepsilon(\alpha_m) \geq C.$$

Then σ is overconvergent, and we have $\gamma_\varepsilon(\sigma) \geq C$.

More generally we can consider families of pseudovaluations $\delta_\varepsilon[n]$ of $W(A)$ which are indexed by real numbers $\varepsilon > 0$ and $n \in \mathbb{N} \cup \{\infty\}$. We write $\delta_\varepsilon = \delta_\varepsilon[\infty]$. We require that

$$\begin{array}{lcll} \delta_{\varepsilon_1}[n] & \geq & \delta_{\varepsilon_2}[n] & \quad \varepsilon_1 \leq \varepsilon_2. \\ \delta_\varepsilon[n] & \geq & \delta_{\varepsilon_2}[m] & \quad n \leq m. \end{array}$$

Definition 2.12. *Two families $\delta_\varepsilon[n]$ and $\delta'_\varepsilon[n]$ as above are called equivalent, if there are constants $c_1, c_2, d_1, d_2 \in \mathbb{R}$, where $c_1 > 0, c_2 > 0$, such that for sufficiently small ε the following inequalities hold:*

$$\begin{array}{lcl} \delta_{c_1\varepsilon}[n] & \geq & \delta'_\varepsilon[n] - d_1 \\ \delta'_{c_2\varepsilon}[n] & \geq & \delta_\varepsilon[n] - d_2. \end{array}$$

Let ν and ν' be negative pseudovaluations on A, which are linearly equivalent. Then the families γ_ε and γ'_ε of Gauss norms defined by (2.2) are equivalent.

We obtain from Lemma 1.9.

Proposition 2.13. *Let (R, μ) be a ring with a negative pseudovaluation. Let A be an R-algebra which is free as an R-module. Let τ an admissible pseudovaluation on A given by Proposition 1.7.*

We transport μ^n to A by an isomorphism $R^n \cong A$. Then a Witt vector $(a_0, a_1, \dots) \in W(A)$ is overconvergent with respect to τ, iff there is an $\varepsilon > 0$ and a constant $C \in \mathbb{R}$ such that:

$$i + p^{-i}\mu^n(a_i) \geq -C.$$

In particular a Witt vector $\underline{r} = (r_0, r_1, \dots) \in W(R)$ is overconvergent iff its image in $W(A)$ is overconvergent.

Proof. Only the last sentence needs a justification. Assume $\underline{r}$ is overconvergent in A. By the first part of the Proposition this means the following:

Let e_i be a basis of the R-module A. we write:

$$1 = \sum_m c_m e_m, \quad c_m \in R.$$

Then overconvergence means that there are constants $\varepsilon > 0$ and $C \in \mathbb{R}$, such that for $1 \leq m \leq n$ and $i \geq 0$

$$i + p^{-i}\mu(c_m r_i)\varepsilon \geq C.$$

By Cohen-Seidenberg it is clear that c_m generate the unit ideal in R:

$$1 = \sum_m c_m u_m.$$

This gives

$$\mu(r_i) \geq \min\{\mu(c_m r_i) + \mu(u_i)\} \geq \min\{\mu(c_m r_i)\} - C'.$$

for some constant C', which depends only on the elements u_m. Therefore we see that $\underline{r} \in W^\dagger(R)$. We leave the inclusion $W^\dagger(R) \subset W^\dagger(A)$ to the reader. □

Lemma 2.14. *Assume that $f \in A$ is localizing. Let $\underline{c} \in W(A)$ be a Witt vector such that $\underline{c} \in W^\dagger(A_f)$. Then $\underline{c} \in W^\dagger(A)$.*

Proof. We write $\underline{c} = (c_0, c_1, c_2, \ldots)$, where $c_i \in A$. By Lemma 1.14 we find representations $c_i = a_i/f^{m_i}$, and real numbers $\varepsilon > 0$ and U, such that

$$i + p^{-i}\varepsilon(\nu(a_i) - m_i d) \geq -U.$$

Since f is localizing we find:

$$\nu(a_i) = \nu(f^{m_i} c_i) \leq C\nu(c_i) + m_i D,$$

and therefore

$$-U \leq i + p^{-i}\varepsilon(C\nu(c_i) + m_i(D - d)) = i + p^{-i}\varepsilon C\nu(c_i) + p^{-i}\varepsilon m_i(D - d).$$

By our choice $D < d$ the last summand is not positive. This shows that $\underline{c} \in W^\dagger(A)$. □

Proposition 2.15. *Let (A, ν) be an integral domain with a negative pseudovaluation, such that any non-zero element is localizing. Let $\alpha : A \to B$ be an injective ring homomorphism of finite type, which is generically finite. Then we have:*

$$W(A) \cap W^\dagger(B) = W^\dagger(A).$$

Proof. Indeed, we find an element $c \in A$, $c \neq 0$ such that $A_c \to B_c$ is finite, and B_c is a free A_c-module. Clearly it suffices to show the Proposition if we replace B by B_c. We consider the maps $A \to A_c \to B_c$ and apply the last Lemma and Proposition 2.13. □

Proposition 2.16. *Let $A \to B$ be a smooth morphism of finitely generated algebras over a field K of characteristic p. We endow them with admissible pseudovaluations. Then we have*

$$W(A) \cap W^\dagger(B) = W^\dagger(A).$$

Proof. By [1] $W^\dagger$ is a sheaf in the Zariski-topology. Therefore the question is local on $\operatorname{Spec} A$. We therefore may assume by Corollary 1.27 that any non-zero element of A is localizing. Obviously the question is local on $\operatorname{Spec} B$. By the definition of smooth we may therefore assume that the morphism factors

$$A \to A[T_1, \ldots, T_d] \to B,$$

where the last arrow is étale and in particular generically finite. We show the Proposition for both arrows separately.

We know by the remark after Definition 1.8 that there is an admissible pseudovaluation on $A[T_1, \ldots, T_d]$, whose restriction to A is an admissible pseudovaluation. This shows the assertion for the first arrow.

For the second arrow we use Proposition 2.15. It is enough to show that any element in $C = A[T_1, \ldots, T_d]$ is localizing. But this is Corollary 1.27. □

Let R be an integral domain and endow it with the trivial valuation. Consider on the polynomial ring $A = R[T_1, \dots, T_d]$ a degree valuation ν, such that $\nu(T_i) = -\delta_i < 0$. Let γ_ε be the associated Gauss norms on $W(A)$. In the following we need the dependence on δ. Therefore we set:

$$\gamma^{(\delta)} = \gamma_1, \quad \text{and then } \gamma^{(\varepsilon\delta)} = \gamma_\varepsilon.$$

Let us denote by $[1, d]$ the set of natural numbers between 1 and d. A weight k is a function $k : [1, d] \to \mathbb{Z}_{\geq 0}[1/p]$. Its values are denoted by k_i. The denominator of k is the smallest number u such that $p^u k$ takes values in $\mathbb{Z}$. We set $\delta(k) = k_1\delta_1 + \ldots + k_d\delta_d$. We write $X_i = T_i$ for the Teichmüller representative and we set $X^k = X_1^{k_1} \cdot \ldots \cdot X_d^{k_d}$.

By [5] any element $\alpha \in W(A)$ has a unique expansion:

$$\alpha = \sum_k \xi_k X^k, \quad \xi_k \in V^u W(R). \tag{2.17}$$

Here u denotes the denominator of k. This series is convergent in the V-adic topology, i.e. for a given $m \in \mathbb{N}$ we have $\xi_k \in V^m W(R)$ for almost all k.

For $\xi \in W(R)$ we define:

$$\operatorname{ord}_V \xi = \min\{m \mid \xi \in V^m W(R)\}.$$

Proposition 2.18. *The Gauss norm of $\gamma^{(\delta)}$ is given by the following formula:*

$$\gamma^{(\delta)}(\alpha) = \inf\{\operatorname{ord}_V \xi_k - \delta(k)\} \tag{2.19}$$

and the truncated Gauss norm is given by:

$$\begin{aligned} \gamma^{(\delta)}[n](\alpha) &= \min\{\infty, \operatorname{ord}_V \xi_k - \delta(k) \mid \xi_k \notin V^{n+1}W(R)\} \\ &= \min_k\{\gamma^{(\delta)}[n](\xi_k X^k)\}. \end{aligned} \tag{2.20}$$

Proof. It is enough to show the equation (2.20). The formula is obvious if $\alpha = \xi_k X^k$ for a particular k. This implies (2.20) if the minimum is attained exactly once on the right hand side.

Let $\delta^{(l)} \in \mathbb{R}^d_{>0}$, $l \in \mathbb{N}$ be a sequence which converges to the given δ. We denote by $\gamma^{(l)}[n]$ the truncated Gauss norm on $W_{n+1}(A)$ associated to numbers $\delta^{(l)}$. We easily see that

$$\lim_{l \to \infty} \gamma^{(l)}[n](\alpha) = \gamma^{(\delta)}[n](\alpha).$$

Clearly the right hand side of (2.20) is also continuous with respect to δ. Therefore it suffices for the proof to construct a sequence $\delta^{(l)}$ such that for each l the minimum

$$\min\{\gamma^{(l)}[n](\xi_k X^k)\}$$

is assumed exactly once. This is the case for $\alpha \neq 0$. Indeed on the right hand side of (2.20) all but finitely many $\gamma_\varepsilon[n](\xi_k X^k)$ are equal to ∞. We denote by g the smallest of these values and by g_1 the next greater value which may be ∞. Let T be the set of weights where the value g is assumed.

The set of linear functions $\eta : \mathbb{R}^d \to \mathbb{R}$ such that

$$\operatorname{ord}_V \xi_k + \eta(k) \neq \operatorname{ord}_V \xi_{k'} + \eta(k')$$

for two different weights involved of T is dense. We find an η in this set whose matrix has positive entries. Moreover we may assume that $\eta(k) < (q_1 - q)/2$ if $\gamma^{(\delta)}[n](\xi_k X^k) \neq \infty$. Then $\delta(l) = \delta + l^{-1}\eta$ meets our requirements. $\square$

Remark: In the case of a polynomial algebra A it is useful to consider a stronger version of overconvergence, which makes only sense for rings of Witt vectors. With the notations above we define:

$$\breve{\gamma}_\varepsilon(\alpha) = \inf\{\mathrm{ord}_V\, \xi_k - \varepsilon|k| - u\}. \tag{2.21}$$

This is clearly a pseudovaluation for each ε. If this inf is not $-\infty$ we call α overconvergent with respect to $\breve{\gamma}_\varepsilon$. One easily verifies:

$$\begin{array}{rcl} \breve{\gamma}_\varepsilon(\alpha) & \leq & \gamma_\varepsilon(\alpha) \\ \breve{\gamma}_\varepsilon(\alpha^r) & \geq & (r-1)\gamma_\varepsilon(\alpha) + \breve{\gamma}_\varepsilon(\alpha). \end{array} \tag{2.22}$$

It is important to note that the Teichmüller representative $[f]$ of an element $f \in A$ is $\breve{\gamma}_\varepsilon$-overconvergent. This is an immediate consequence of the following

Lemma 2.23. *Let R be a $\mathbb{Z}_p$-algebra. Let A be an R-algebra. Let $x_1, \ldots, x_n \in R$ and $t_1, \ldots, t_d \in A$ be elements. We denote by $k = (k_1, \ldots, k_d) \in \mathbb{Z}_{\geq 0}[1/p]$ a weight. Then we have in $W(A)$ the following relation:*

$$[x_1 t_1 + \ldots + x_d t_d] = \sum_{k, |k|=1} \alpha_k [t_1]^{k_1} \cdot \ldots \cdot [t_d]^{k_d}, \tag{2.24}$$

where $\alpha_k \in V^u W(R)$ and p^u is the denominator of k.

Proof. Clearly it is enough to show this Lemma in the case, where $x_1 = 1, \ldots, x_d = 1$. Moreover we may restrict to the case where $R = \mathbb{Z}_p$ and A is the polynomial algebra over $\mathbb{Z}_p$ in the variables $t_1, \ldots, t_d$. Then $W(A)$ is a $\mathbb{Z}_{\geq 0}[1/p]$-graded, such that the monomial $[t_1]^{k_1} \cdot \ldots \cdot [t_d]^{k_d}$ has degree $|k|$ (note that this monomial is in general not in $W(A)$.) More precisely a Witt vector of polynomials $(p_0, p_1, p_2, \ldots) \neq 0$ is homogeneous of degree $m \in \mathbb{Z}_{\geq 0}[1/p]$ if each polynomial p_i has degree $p^i m$, if $p_i \neq 0$. In the case where $p^i m$ is not an integer the condition says that $p_i = 0$.

Since $[t_1 + \ldots + t_d]$ is homogeneous of degree 1 the Lemma follows from [5] Prop.2.3. $\square$

Lemma 2.25. *Let (A, ν) a ring with a proper pseudovaluation. Let $\alpha \in VW(A)$. Assume that $\gamma_\varepsilon(\alpha) \geq 0$. Then the element $1 - \alpha$ is a unit in $W(A)$ and we have*

$$\gamma_\varepsilon(1-\alpha)^{-1} \geq 0. \tag{2.26}$$

Assume moreover that $A = R[T_1, \ldots, T_d]$ is a polynomial ring with a degree valuation.. Then

$$\breve{\gamma}_\varepsilon(1-\alpha)^{-1} \geq \min\{0, \breve{\gamma}_\varepsilon(\alpha)\}.$$

In particular $(1-\alpha)^{-1}$ is $\breve{\gamma}_\varepsilon$-overconvergent if α is.

Proof. We write $\alpha = {}^V\eta$. We find $\gamma_{\varepsilon/p}(\eta) > -1$. We have in $W(A)$ the identity:

$$(1 - {}^V\eta)^{-1} = 1 + \sum_{i>0} p^{i-1}\, {}^V(\eta^i) = \sum_{i\geq 0} \alpha^i.$$

The middle term shows that the series converges V-adically and the last sum proves the inequality (2.26). The last assertion is obvious from (2.22). □

Proposition 2.27. *Let (A,ν) a ring with a proper pseudovaluation. Let $\mathbf{w}_n : W(A) \to A$ denote the Witt polynomials. An element $\alpha \in W^\dagger(A)$ is a unit, iff $\mathbf{w}_0(\alpha)$ is a unit in A.*

Assume moreover that $A = R[T_1, \ldots, T_d]$ is a polynomial ring with a degree valuation. If α is $\breve{\gamma}_\varepsilon$-overconvergent, then α^{-1} is $\breve{\gamma}_\delta$-overconvergent for some $\delta > 0$.

Proof. We write $\alpha = [a] + {}^V\eta$, with $a \in A$ and $\eta \in W(A)$. To prove the first assertion we may assume that $a = 1$. Applying Corollary 2.10 we assume that $\gamma_\varepsilon({}^V\eta) > 0$. Then the assertion follows from Lemma 2.25.

Now we prove the second assertion: Since every Teichmüller representative is $\breve{\gamma}_\varepsilon$-overconvergent, it suffices to show that the inverse of $1+[a^{-1}]\,{}^V\eta = 1 + {}^V([a^{-p}]\eta)$ is $\breve{\gamma}_\varepsilon$-overconvergent. Since $\breve{\gamma}_\varepsilon$ is a pseudovaluation we see that ${}^V([a^{-p}]\eta)$ is $\breve{\gamma}_\varepsilon$-overconvergent too. By Corollary 2.10 we find ε/p such that

$$\gamma_{\varepsilon/p}([a^{-p}]\eta) > -1.$$

Therefore we may apply the Lemma 2.25. □

Proposition 2.28. *Let A be an algebra over a perfect field K. Let ν be an admissible pseudovaluation on A. Then $W^\dagger(A)$ is an algebra over the complete local ring $W(K)$.*

The $W(K)$-algebra $W^\dagger(A)$ is weakly complete in the sense of [9].

Proof. Let $z_1, \ldots, z_r \in W^\dagger(A)$. Consider an infinite series

$$\sum a_k z^k, \quad a_k \in W(K), \quad z^k = z_1^{k_1} \cdot \ldots \cdot z_r^{k_r}. \tag{2.29}$$

We assume that there are real numbers $\delta > 0$, and c, such that

$$\operatorname{ord}_p a_k \geq \delta|k| + c.$$

This implies that the series (2.29) converges in $W(A)$. We have to show that the series converges to an element $W^\dagger(A)$. We choose a common radius ε of convergence for $z_1, \ldots, z_r$. Making ε smaller we may assume that:

$$\gamma_\varepsilon(z_i) \geq -\delta.$$

Then we find:

$$\gamma_\varepsilon(a_k a^k) \geq \operatorname{ord}_p a_k - \delta|k| \geq c.$$

Therefore (2.29) converges to an element of $W^\dagger(A)$. □

We will point out that by Monsky and Washnitzer the last proposition implies Hensel's Lemma for the overconvergent Witt vectors:

Proposition 2.30. *Let A be an algebra over a perfect field K. Let ν be an admissible pseudovaluation on A. Let $f(T) \in W^\dagger(A)[T]$ be a polynomial. We consider the homomorphism $\mathbf{w}_0 : W^\dagger(A) \to A$.*

Let $a \in A$ be an element, such that

$$f(a) = 0 \quad \text{and } f'(a) \text{ is a unit in } A.$$

Then there is a unique $\alpha \in W^\dagger(A)$ such that $f(\alpha) = 0$ and such that $a \equiv \alpha \bmod VW^\dagger(A)$.

Proof. The kernel of the natural morphism $W^\dagger(A)/pW^\dagger(A) \to A$ is an ideal whose square is zero. Therefore there is an $\bar{\alpha} \in W^\dagger(A)/pW^\dagger(A)$ which reduces to a and such that $f(\bar{\alpha}) = 0$. The rest of the proof is a general fact about weakly complete algebras explained below. □

For the explanation we follow the notations of [9]: Let (R, I) be a complete noetherian ring. Let A be a weakly complete finitely generated (w.c.f.g.) algebra over (R, I). We write $\bar{A} = A/IA$. Let $A \to B$ be a morphism of w.c.f.g. algebras, such that $\bar{B} = \bar{A}[X_1, \dots, X_n]/(\bar{F}^{(1)} \dots \bar{F}^{(s)})$, $s \leq n$ and the $s \times s$ subdeterminants of $(\frac{\partial F^{(i)}}{\partial X_j})$ generate the unit ideal in $\bar{B}$. Then by [9] p. 195 the morphism $A \to B$ is very smooth. As an example we may take for B the weak completion of

$$A[X, T]/(f(X), 1 - f'(X)T),$$

where $f(X) \in A[X]$ is a polynomial.

Proposition 2.31. *Let C be a weakly complete (not necessarily finitely generated but p-adically separated) algebra over (R, I). Let $f(X) \in C[X]$ be a polynomial and let $\bar{\gamma} \in \bar{C}$ be an element, such that $f(\bar{\gamma}) = 0$ and $f'(\bar{\gamma})$ is a unit in $\bar{C}$. Then there is a unique element $\gamma \in C$, such that $f(\gamma) = 0$ and $\gamma \equiv \bar{\gamma} \mod IC$.*

Proof. By Hensel's Lemma applied to the completion of C the uniqueness of the solution is clear.

For the existence we write $f(X) = s_d X^d + s_{d-1} X^{d-1} + \cdots + s_1 X + s_0$, where $s_i \in C$.

Let $A = R[S_d, \dots, S_0]^\dagger$ be the weak completion of the polynomial algebra. We set

$$F(X) = S_d X^d + \cdots + S_1 X + S_0 \in A[X]$$

and we let B be the weak completion of

$$A[X, T]/(F(X), 1 - TF'(X)).$$

Let $A \to C$ be the homomorphism defined by $S_i \mapsto s_i$. The solution $\bar{\gamma}$ defines a homomorphism

$$R/I[S_d, \dots, S_0, X, T]/(\bar{F}(X), 1 - T\bar{F}'(X)) \to \bar{C}$$

where $S_i \mapsto s_i \mod IC$ and $X \mapsto \bar{\gamma}, T \mapsto f'(\bar{\gamma})^{-1}$.

Hence we obtain a commutative diagramm

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & \bar{C}. \end{array} \tag{2.32}$$

Since $A \to B$ is very smooth by the example above, we find a morphism $B' \to C$ making (2.32) commutative. The image of X is the desired solution $\gamma \in C$. □

We will now study the behaviour of overconvergent Witt vectors in finite étale extensions. Let A be a finitely generated K-algebra. Let B a finite étale A-algebra which is free as an A-module. Let e_i, $1 \leq i \leq r$ be a basis of the A-module B. Then the natural map

$$W(A)^r \to W(B), \tag{2.33}$$

which maps the standard basis of the free module $W(A)^r$ to the Teichmüller representatives $[e_i]$ is an isomorphism. Moreover $W(B)$ is an étale algebra over $W(A)$.

Indeed, by [5] A8 the $W_n(A)$-algebra $W_n(B)$ is étale for each n. We set $I_n = VW_{n-1}(A) \subset W_n(A)$. Then by loc.cit. we have $I_n W_n(B) \subset VW_n(B)$. ¿From this we conclude by the lemma of Nakayama that:

$$W_n(A)^r \to W_n(B),$$

is an isomorphism. Taking the projective limit we obtain (2.33). If we tensor (2.33) with $A \otimes_{\mathbf{w}_0}$ we obtain that $A \otimes_{\mathbf{w}_0} W(B) = B$.

We will now assume that B is monic

$$B = A[T]/f(T)A[T],$$

where

$$f(T) = T^m - c_{m-1}T^{m-1} - \ldots - c_1 T - c_0. \tag{2.34}$$

Let ν be a negative pseudovaluation on A. We endow B with the equivalence class of admissible pseudovaluations defined by Proposition 1.7.

Lemma 2.35. *Let $d \in \mathbb{R}$, such that $d > \nu(c_i)$ for $i = 1, \ldots, m$. An element $b \in B$ has a unique representation*

$$b = \sum_{i=0}^{m-1} a_i T^i.$$

We set

$$\tilde{\nu}(b) = \min_{i=1,\ldots,m-1} \{\nu(a_i) - id\}. \tag{2.36}$$

Then $\tilde{\nu}$ is an admissible pseudovaluation on B.

Proof. We consider on $A[T]$ the pseudovaluation μ (1.11). We will show that with d as above $\tilde{\nu}$ is the quotient of μ.

Let

$$\tilde{b} = \sum_{j=0}^{s} u_j T^j,$$

be an arbitrary representative of b. We need to show that $\mu(\tilde{b})$ is smaller than the right hand side of (2.36). We prove this by induction on s. For $s < m$ there is nothing to show. For $s \geq m$ we obtain another representative of b:

$$\tilde{b}' = \sum_{j=0}^{m-1} u_j T^j + \sum_{k \geq m} u_k (\sum_{l=0}^{m-1} c_l T^l) T^{k-m}. \tag{2.37}$$

On the right hand side there is a polynomial of degree at most $s-1$. Therefore it suffices by induction to show that

$$\mu(\tilde{b}') \geq \mu(\tilde{b}).$$

The last inequality is a consequence of the following:

$$\begin{array}{lcll} \mu(u_j T^j) & \geq & \mu(\tilde{b}), & \text{for } j = 0, \dots, m-1 \\ \mu(u_k c_l T^{k-m+l}) & \geq & \mu(\tilde{b}) & \text{for } k \geq m,\ 0 \leq l \leq m-1. \end{array} \tag{2.38}$$

The first set of these inequalities is trivial. For the second set we compute

$$\begin{array}{rcl} \mu(u_k c_l T^{k-m+l}) & \geq & \nu(u_k) + \nu(c_l) - kd + (m-l)d \\ & \geq & \mu(u_k) - kd \geq \mu(\tilde{b}). \end{array}$$

The last equation holds because by the choice of d:

$$\nu(c_l) + (m-l)d \geq 0.$$

This shows the second set of inequalities. □

Because $\tilde{\nu}$ restricted to A coincides with ν we simplify the notation by setting $\tilde{\nu} = \nu$. The Gauss norms (2.2) induced by the pseudovaluation ν on $W(B)$ and $W(A)$ will be also denoted by the same symbols γ_ε.

Lemma 2.39. *With the notations of Lemma 2.35 we assume that B is ètale over A. We will denote the residue class of T in B by t.*

Then there is a constant $G \in \mathbb{R}$ with the following property: Each $b \in B$ has for each integer $n \geq 0$ a unique representation

$$b = \sum_{i=0}^{m-1} a_{ni} t^{ip^n}.$$

Then we have the following estimates for the pseudovaluations of a_{ni}:

$$\nu(a_{ni}) \geq \nu(b) - p^n G. \tag{2.40}$$

Proof. Since B is étale over A the elements

$$1, t^{p^n}, t^{2p^n}, \dots, t^{(m-1)p^n}$$

are for each n a basis of the A-module B. We write

$$t^i = \sum_{j=0}^{m-1} u_{ji} t^{jp}. \tag{2.41}$$

We introduce the matrix $U = (u_{ji})$ and we set for matrices

$$\nu(U) = \min_{i,j}\{\nu(u_{ji})\}.$$

We deduce the relation:

$$a_{1j} = \sum_i u_{ji} a_{0i}.$$

We will write the last equality in matrix notation:

$$a(1) = Ua(0).$$

Let $U^{(p^n)}$ the matrix obtained form U by raising all entries of U in the p^n-th power. Then we obtain with the obvious notation:

$$a(n+1) = U^{(p^n)} a(n).$$

It is obvious that for two matrices U_1, U_2 with entries in B

$$\nu(U_1U_2) \geq \nu(U_1) + \nu(U_2).$$

We choose a constant C such that

$$\nu(U) \geq -C.$$

Therefore we obtain:

$$\begin{array}{rcl} \nu(a(n)) & = & \nu(U^{(p^{n-1})} \cdot \ldots \cdot Ua(0)) \\ & \geq & -(p^{n-1}C + \ldots + pC + C) + \nu(a(0)). \end{array}$$

By Lemma 2.33 we have

$$\nu(b) = \min\{\nu(a_{0i}) - id\} \leq \nu(a(0)).$$

Therefore we obtain:

$$\nu(a(n)) \geq -p^n \frac{C}{p-1} + \nu(b).$$

We therefore found the desired constant. □

Proposition 2.42. *Let $B = A[t]$ be a finite étale A-algebra as in Lemma 2.39. Let $G > 0$ be the constant of this Lemma. Let $x = [t] \in W(B)$ be the Teichmüller representative. By (2.33) $1, x, \ldots, x^{m-1}$ is a basis of the $W(A)$-module $W(B)$. We write an element $\eta \in W(B)$*

$$\eta = \sum_{i=0}^{m-1} \xi_i x^i, \quad \xi_i \in W(A).$$

There is a real number $\delta > 0$, such that for $\varepsilon \leq \delta$ an inequality

$$\gamma_\varepsilon(\eta) \geq -C \quad \textit{implies} \quad \gamma_\varepsilon(\xi_i) \geq -C - \varepsilon G.$$

Proof. We choose a constant $G' > 0$ such that

$$\nu(t^i) \geq -G', \quad \text{for } i = 0, \ldots, m-1.$$

We choose δ such that $\delta(G + G') \leq 1$. We write

$$\xi_i = \sum_{s \geq 0} {}^{V^s}[a_{s,i}] \quad \text{with } a_{s,i} \in A.$$

We define

$$\zeta_i(n) = \sum_{s \geq n} {}^{V^{s-n}}[a_{s,i}].$$

We will show by induction on n the following two assertions:

$$\gamma_\varepsilon(\sum_{i=0}^{m-1} {}^{V^n}\zeta_i(n)x^i) \geq -C. \tag{2.43}$$

$$\gamma_\varepsilon({}^{V^n}[a_{n,i}]) \geq -C - \varepsilon G. \tag{2.44}$$

We begin to show that the first inequality for a given n implies the second. We set:

$$\theta(n) = \sum_{i=0}^{m-1} {}^{V^n}\zeta_i(n)x^i.$$

The first non-zero component of this Witt vector is

$$y_n = \sum_{i=0}^{m-1} a_{n,i} t^{ip^n}.$$

in place $n+1$. We conclude that

$$n + \varepsilon p^{-n} \nu(y_n) \geq \gamma_\epsilon(\theta(n)) \geq -C,$$

where the last inequality is (2.43). This shows that:

$$\nu(y) \geq -\varepsilon p^n (C+n).$$

We conclude by Lemma 2.39 that

$$\nu(a_{n,i}) \geq -(p^n/\varepsilon)(C+n) - p^n G, \tag{2.45}$$

and therefore

$$\gamma_\varepsilon({}^{V^n}[a_{n,i}]) = n + \varepsilon p^{-n} \nu(a_{n,i}) \geq -C - \varepsilon G.$$

Therefore the Proposition follows if we show the assertion (2.43) by induction. The assertion is trivial for $n = 0$ and we assume it for n. With the notation above we write:

$$\begin{array}{rcl} \theta(n+1) & = & \theta(n) - \sum_{i=0}^{m-1} {}^{V^n}[a_{n,i}]x^i \\ & = & (\theta(n) - {}^{V^n}[y_n]) - (\sum_{i=0}^{m-1} {}^{V^n}[a_{n,i}]x^i - {}^{V^n}[y_n]). \end{array}$$

The Witt vector in the first brackets has only entries which also appear in $\theta(n)$ and therefore has Gauss norm $\gamma_\varepsilon \geq -C$. The assertion follows if we show the same inequality for the Witt vector in the second brackets:

$${}^{V^{n+1}}\tau = (\sum_{i=0}^{m-1} {}^{V^n}[a_{n,i}]x^i - {}^{V^n}[y_n]).$$

We set

$$[y_n] = \sum_{i=0}^{m-1} [a_{n,i} t^{ip^n}] = (s_0, s_1, s_2, \ldots).$$

Then we find

$${}^V\tau = (0, s_1, s_2, \ldots).$$

We know that s_l is a homogeneous polynomial of degree p^l in the variables $a_{n,i} t^{ip^n}$ for $i = 1, \ldots m-1$. By the choice of G' we find $\nu(t^{ip^n}) \geq p^n \nu(t^i) \geq -p^n G'$. Using (2.45) we find:

$$\nu(s_l) \geq -p^l((p^n/\varepsilon)(C+n) + p^n G) - p^l p^n G' = -p^{n+l}((1/\varepsilon)(C+n) + G + G').$$

We have

$${}^{V^{n+1}}\tau = \sum_{l \geq 1} {}^{V^{n+l}}[s_l].$$

For the Gauss norms of the entries of this vector we find for $l \geq 1$:

$$\begin{array}{rcl} \gamma_\varepsilon({}^{V^{n+l}}[s_l]) & = & n + l + \varepsilon p^{-n-l} \nu(s_l) \geq n + l - \varepsilon((1/\varepsilon)(C+n) + G + G') \\ & = & l - C - \varepsilon(G + G') \geq -C. \end{array}$$

The last inequality follows since $l \geq 1$ by the choice of δ. We conclude that

$$\gamma_\varepsilon({}^{V^{n+1}}\tau) \geq -C.$$

□

Corollary 2.46. *Let A be a finitely generated algebra over K. Let $B = A[T]/(f(T))$ be a finite étale A-algebra, where $f(T) \in A[T]$ is a monic polynomial of degree n. We denote by t the residue class of T in B. We set $x = [t] \in W^{\dagger}(B)$.*

Then $W^{\dagger}(B)$ is finite and étale over $W^{\dagger}(A)$ with basis $1, x \ldots, x^{n-1}$.

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